On Plane Strain Problems in Magneto-Thermo-Visco-Elasticity

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Summary – Plane strain problems on magneto-thermo-visco-elastic interactions in a parallel union of the Kelvin and Maxwell bodies are investigated using the basic equations of electrodynamics and thermo-visco-elasticity. Assuming that the applied magnetic field is transverse to the plane of deformation and that the material is a perfect conductor of electricity, it is seen that the heat sources and the potential part of the body forces produce longitudinal waves only and the rotational part of the body forces gives rise to transverse waves only. The effect of deformation on magnetic permeability is equivalent to an anisotropic rescaling of the primary magnetic field. The effect of the applied magnetic field on waves produced by a plane heat source is equivalent to increasing the value of the material constants which results in an increase in the speed of the waves.

1. Introduction

The investigations of magneto-mechanical interactions in electrically conducting thermo-visco-elastic bodies is of recent origin compared to that in thermoelastic and viscoelastic bodies. The study of magneto-thermo-viscoelastic interactions has been initiated recently by the present author [1][2]) by discussing plane waves in a thermo-viscoelastic body representing a parallel union of the Kelvin and Maxwell bodies, in the presence of a steady magnetic field. In this paper the investigation is continued and plane strain dynamic problems have been considered. Using the basic equations developed in [1] it has been found that when the material is infinitely conducting, the heat sources and the potential part of the body forces produce longitudinal waves only and the rotational part of the body forces produce transverse waves only. The effect of deformation on magnetic permeability is taken into account (as in [1]), and it is found that the effect is equivalent to an anisotropic rescaling of the initial magnetic field. The magneto-thermo-viscoelastic interactions due to plane heat sources are considered and it is observed that the applied magnetic field increases the speed of wave propagation, without affecting its modes of propagation.

2. Basic equations

The equations governing magneto-thermo-visco-elastic interactions in the body under consideration, in the notation of cartesian tensors, are [1]

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[2]) Numbers in brackets refer to References, page 104.

(a) the Maxwell equations

$$\left.\begin{aligned} \varepsilon_{ijk} H_{k,j} &= J_i \qquad \varepsilon_{ijk} E_{k,j} = -\frac{\partial B_i}{\partial t} \\ B_{i,i} &= 0 \qquad B_i = \mu_{ij} H_j \end{aligned}\right\} \tag{2.1}$$

(b) the generalized Ohm's Law

$$J_i = \sigma \left[E_i + \varepsilon_{ijk} \frac{\partial u_j}{\partial t} B_k \right] - K_0 T_{,i} \tag{2.2}$$

(c) The field equations of thermo-visco-elasticity

$$\left(1 + m_1 \frac{\partial}{\partial t}\right) s_{ij} = 2\mu \left(1 + m_2 \frac{\partial}{\partial t}\right) e_{ij} \tag{2.3}$$

with

$$\left.\begin{aligned} s_{ij} &= \sigma_{ij} - \tfrac{1}{3} s\, \delta_{ij}\,; \qquad s = 3K(\varepsilon_{kk} - 3aT) \\ e_{ij} &= \varepsilon_{ij} - \tfrac{1}{3} \varepsilon_{kk}\, \delta_{ij} \\ 2\varepsilon_{ij} &= u_{i,j} + u_{j,i} \end{aligned}\right\} \tag{2.4}$$

and

$$\sigma_{ji,j} + F_i = \varrho \frac{\partial^2 u_i}{\partial t^2} \tag{2.5}$$

and (d) the equation of heat conduction

$$q \nabla^2 T + Q = \varrho c_v \frac{\partial T}{\partial t} + 3aKT_0 \frac{\partial \varepsilon_{kk}}{\partial t} + \pi_0 J_{i,i}\,. \tag{2.6}$$

All the symbols in equations (2.1)–(2.6) except F_i in (2.5), are as explained in [1] and F_i stands for the total body force acting on the material. In a problem on magneto-mechanical interactions F_i is the sum of the electromagnetic body force (Lorentz force) $\varepsilon_{ijk} J_j B_k$ and other external force f_i if any. Accordingly if we put $F_i = \varepsilon_{ijk} J_j B_k + f_i$ and eliminate s_{ij}, e_{ij} and σ_{ij}, ε_{ij} from equations (2.3)–(2.5) we get the equation of motion for displacements in the form [1]:

$$\left.\begin{aligned} &\left(\mu + \mu' \frac{\partial}{\partial t}\right) u_{i,jj} + \left[\lambda + \mu + \left(K' + \tfrac{1}{3}\mu'\right) \frac{\partial}{\partial t}\right] u_{k,ki} \\ &\quad - \left(1 + m_1 \frac{\partial}{\partial t}\right)\left(3aKT_{,i} - f_i - \varepsilon_{ijk} J_j B_k + \varrho \frac{\partial^2 u_i}{\partial t^2}\right) = 0 \end{aligned}\right\} \tag{2.7}$$

where $\mu' = \mu m_2$ and $K' = K m_1$. Equations (2.1), (2.2), (2.6) and (2.7) determine all the field variables of our problem.

3. *Plane deformation*

In what follows we consider the deformation parallel to xy-plane when the applied magnetic field is parallel to the z-axis. Accordingly we take $u_3 = 0$ and assume that all field variables are independent of z coordinate. The magnetic field H_i may be taken as $H_i = (h_1, h_2, H + h_3)$ where H is the strength of the applied uniform magnetic field and h_i is the perturbed field. We assume that h_i and u_i are so small that their squares and products are negligible.

To study the effect of deformation on magnetic permeability we assume the tensor permeability μ_{ij} in the form [1]

$$\mu_{ij} = \mu_e(1 + \alpha\, u_{k,k})\,\delta_{ij} - \beta\,\mu_e(u_{i,j} + u_{j,i}) \tag{3.1}$$

where μ_e is the magnetic permeability of the body in the unstrained state and α, β are parameters depending on the nature of the effect of deformation on permeability. For the present problem (3.1) reduces to

$$\mu_{ij} = \mu_e \begin{pmatrix} 1 + (\alpha - 2\beta)\dfrac{\partial u_1}{\partial x} + \alpha\dfrac{\partial u_2}{\partial y} & -\beta e & 0 \\ -\beta e & 1 + \alpha\dfrac{\partial u_1}{\partial x} + (\alpha - 2\beta)\dfrac{\partial u_2}{\partial y} & 0 \\ 0 & 0 & 1 + \alpha e \end{pmatrix} \tag{3.2}$$

where

$$e = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

Substituting (3.2) in (2.1)^4 we get

$$\left.\begin{aligned} (B_0)_i &= (0, 0, \mu_e H) \\ b_i &= \mu_e(h_1, h_2, h_3 + \alpha H e) \end{aligned}\right\} \tag{3.3}$$

where $(B_0)_i$ and b_i are respectively the initial and perturbed magnetic induction vectors.

If we suppose that the body is a perfect conductor of electricity ($\sigma \to \infty$), equation (2.2) gives

$$E_i = -\mu_e H\left(\frac{\partial u_2}{\partial t}, -\frac{\partial u_1}{\partial t}, 0\right). \tag{3.4}$$

Equations (2.1)^2 and (3.4) yield with the help of (3.3) the relations

$$h_i = -H(1 + \alpha)\,[0, 0, e]. \tag{3.5}$$

Equations (2.1)^1 and (3.5) yield

$$J_i = -H(1 + \alpha)\left[\frac{\partial e}{\partial x}, \frac{\partial e}{\partial y}, 0\right]. \tag{3.6}$$

Equations (3.3) and (3.6) give

$$\varepsilon_{ijk} J_j B_k = \mu_e H^2(1+\alpha)\left[\frac{\partial e}{\partial x}, \frac{\partial e}{\partial y}, 0\right]. \tag{3.7}$$

The equation of motion (2.7) then reduces to

$$\left.\begin{aligned}\left(\mu+\mu'\frac{\partial}{\partial t}\right)u_{i,jj} + \Big[\lambda+\mu+(K'+\tfrac{1}{3}\mu')\frac{\partial}{\partial t}\\ +\left(1+m_1\frac{\partial}{\partial t}\right)\mu_e H^2(1+\alpha)\Big]e_{,i}\\ -\left(1+m_1\frac{\partial}{\partial t}\right)\left[3\,a\,K\,T_{,i}-f_i+\varrho\frac{\partial^2 u_i}{\partial t^2}\right]=0\end{aligned}\right\} \tag{3.8}$$

where $i, j=1, 2$.

From equation (3.8), it is clear that the effect of deformation on magnetic permeability is equivalent to replacing H^2 by $H^2(1+\alpha)$.

Taking the divergence of equation (3.8) we get

$$\Box_1^2\, e = \frac{1}{\varrho}\left(1+m_1\frac{\partial}{\partial t}\right)\left(3\,a\,K\,\nabla^2 T - \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right) \tag{3.9}$$

where

$$\left.\begin{aligned}&\Box_1^2 = \left[\left(1+m_1\frac{\partial}{\partial t}\right)(c_1^2+V_h^2)+c_2^2\,m_3\frac{\partial}{\partial t}\right]\nabla^2 - \left(1+m_1\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2},\\ &c_1^2 = \frac{\lambda+2\mu}{\varrho};\quad c_2^2=\frac{\mu}{\varrho};\\ &V_h^2 = \frac{\mu_e H^2(1+\alpha)}{\varrho},\quad m_3=\tfrac{4}{3}(m_2-m_1)\quad\text{and}\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.\end{aligned}\right\} \tag{3.10}$$

Taking Curl of equation (3.8) we get

$$\Box_2^2\,\Omega = -\frac{1}{\varrho}\left(1+m_1\frac{\partial}{\partial t}\right)\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right) \tag{3.11}$$

where

$$\left.\begin{aligned}&\Box_2^2 = c_2^2\left(1+m_2\frac{\partial}{\partial t}\right)\nabla^2 - \left(1+m_1\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}\quad\text{and}\\ &\Omega = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.\end{aligned}\right\} \tag{3.12}$$

Decomposing u_i and f_i into potential and rotational parts by the relations

$$\left.\begin{aligned}u&=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial y}; & f_1&=\frac{\partial\chi}{\partial x}-\frac{\partial\xi}{\partial y}\\ v&=\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}; & f_2&=\frac{\partial\chi}{\partial y}+\frac{\partial\xi}{\partial x}\end{aligned}\right\} \tag{3.13}$$

and introducing these into equations (3.9) and (3.11) we get

$$\Box_1^2 \phi = \frac{1}{\varrho}\left(1 + m_1 \frac{\partial}{\partial t}\right)(3\, a\, K\, T - \chi) \tag{3.14}$$

$$\Box_2^2 \psi = -\frac{1}{\varrho}\left(1 + m_1 \frac{\partial}{\partial t}\right)\xi. \tag{3.15}$$

The heat equation (2.6) reduces to

$$\Box_3^2 T + Q = 3\, a\, K\, T_0\, \nabla^2 \left(\frac{\partial \phi}{\partial t}\right) \tag{3.16}$$

where

$$\Box_3^2 = q\, \nabla^2 - \varrho\, c_v \frac{\partial}{\partial t}. \tag{3.17}$$

From equations (3.14)–(3.16) we observe that ϕ is linked with χ and Q, and ψ is linked only with ξ. Accordingly the potential part of the body force (viz χ) and the heat source Q produces longitudinal waves only and the rotational part of the body force (viz. ξ) produces transverse waves only. Further, the applied magnetic field has no influence on the transverse waves.

Eliminating T from equations (3.14) and (3.16) we get

$$\left.\begin{aligned}\Box_1^2\, \Box_3^2\, \phi = \frac{3\, a\, K}{\varrho}\left(1 + m_1 \frac{\partial}{\partial t}\right)\left(3\, a\, K\, T_0\, \nabla^2\left(\frac{\partial \phi}{\partial t}\right) - Q\right)&\\ - \frac{1}{\varrho}\left(1 + m_1 \frac{\partial}{\partial t}\right)\chi.&\end{aligned}\right\} \tag{3.18}$$

It is readily seen from equation (3.18) that the longitudinal waves are subjected to damping and dispersion.

The electromagnetic variables J_i, h_i and E_i given by (3.4)–(3.6) now reduce with the help of (3.13), to

$$\left.\begin{aligned} E_i &= -\mu_e H(1 + \alpha)\frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x}, 0\right]\\ h_i &= -H(1 + \alpha)\,[0, 0, \nabla^2 \phi]\\ J_i &= -H(1 + \alpha)\,\nabla^2\left[\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x}, 0\right].\end{aligned}\right\} \tag{3.19}$$

Thus all the field variables (mechanical, thermal and electromagnetic) are expressible in terms of the functions ϕ and ψ satisfying the equations (3.15) and (3.16). Accordingly the solution of our problem reduces to solving the equations (3.15) and (3.18) under prescribed initial and boundary conditions. If there are no body forces apart from the Lorentz force, there exist only longitudinal waves and in such cases we can take $\psi = 0$.

In what follows we illustrate the theory by considering a plane heat source in an infinite solid.

4. *Plane heat source in an infinite body*

Suppose an infinite body contains a plane heat source $Q = Q_0\, \delta(x)\, e^{i\omega t}$ where Q_0 is a constant and $\delta(x)$ is dirac delta function. Then by symmetry, all field variables depend on x and t only. If there are no forces apart from the Lorentz force, we have $\chi = \xi = \psi = 0$ and ϕ is given by (vide equation (3.18))

$$\left.\begin{aligned}&\left[\left(A^2 + B^2 \frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2} - \left(1 + m_1 \frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial t^2}\right]\left[q \frac{\partial^2}{\partial x^2} - \varrho\, c_v \frac{\partial}{\partial t}\right]\phi \\ &\qquad = \frac{3\, a\, K}{\varrho}\left(1 + m_1 \frac{\partial}{\partial t}\right)\left[3\, a\, K\, T_0 \frac{\partial^3 \phi}{\partial x^2\, \partial t} - Q_0\, \delta(x)\, e^{i\omega t}\right]\end{aligned}\right\} \tag{4.1}$$

where

$$A^2 = c_1^2 + V_h^2$$
$$B^2 = (c_1^2 + V_h^2)\, m_1 + c_2^2\, m_3 .$$

The solution of (4.1) is given by

$$\phi = \frac{3\, a\, K\, Q_0(1 - m_1\, i\, \omega)}{2\, \varrho(\alpha_1^2 - \alpha_2^2)}\, e^{i\omega t}\left[\frac{e^{-\alpha_1 x}}{\alpha_1} - \frac{e^{-\alpha_2 x}}{\alpha_2}\right], \qquad x > 0 \tag{4.2}$$

where α_1, α_2 are the roots of equation

$$\left.\begin{aligned}\alpha^4 + \alpha^2 &\left[\frac{\omega^2(1 - i\, \omega\, m_1)}{A^2 - B^2\, i\, \omega} + \frac{\varrho\, c_v\, i\, \omega}{q} - \frac{9\, a^2\, K^2\, i\, \omega\, T_0}{q(A^2 - i\, \omega\, B^2)}\right] \\ &\qquad + \frac{(1 - i\, \omega\, m_1)\, \varrho\, c_v\, i\, \omega^3}{q(A^2 - B^2\, i\, \omega)} = 0\end{aligned}\right\} \tag{4.3}$$

and $\mathrm{Re}(\alpha_j) > 0$.

Function ϕ being known, we can determine all field variables from (3.13), (3.14) and (3.19). Introducing

$$\zeta^2 = \frac{\alpha^2\, m^2}{n^2}; \qquad \gamma = \frac{i\, m^2}{n}$$

where

$$m^2 = \frac{\omega^2(1 - i\, m_1\, \omega)}{A^2 - i\, \omega\, B^2}; \qquad n = \frac{\varrho\, c_v\, i\, \omega}{q}$$

equation (4.3) reduces to

$$\zeta^4 - \zeta^2\, \gamma[\gamma + i(1 - \varepsilon)] + i\, \gamma^3 = 0 \tag{4.4}$$

where

$$\varepsilon = \frac{9\, a^2\, K^2\, T_0(1 - i\, \omega\, m_1)}{\varrho\, c_v(A^2 - i\, \omega\, B^2)}.$$

For actually occurring vibrations, we may take [2] $\gamma \ll 1$ and the solution of (4.4) may be written as

$$\left.\begin{aligned} \alpha_1^2 &= \frac{\varrho\, c_v\, \omega}{2\, q}(1-\varepsilon)\left[1+\frac{\varepsilon\,\gamma}{2(1-\varepsilon)^2}\right]^2 \\ \alpha_2^2 &= \tfrac{1}{2}\,\varepsilon(1-\varepsilon)^{-5}\,\frac{\alpha^2\,\gamma^4}{\zeta^2} \\ v_1^2 &= \frac{2\,q\,\omega}{\varrho\, c_v(1-\varepsilon)}\left[1+\frac{\varepsilon\,\gamma}{2(1-\gamma)^2}\right]^{-2}; \quad v_2^2 = (A^2 - B^2\, i\,\omega)(1-\varepsilon) \end{aligned}\right\} \tag{4.5}$$

where $v_i = \omega/\mathrm{Im}(\alpha_i)$ is the phase velocity of the waves. We can easily verify that α_1, v_1 correspond to modified thermal wave and α_2, v_2 correspond to modified visco-elastic wave. Writing v_2^2 explicitly by using expressions for A^2 and B^2 we get

$$v_2^2 = [c_1^2 + V_h^2 - i\,\omega\{(c_1^2 + V_h^2)\, m_1 + c_2^2\, m_3\}]\,[1-\varepsilon]\,. \tag{4.6}$$

We at once see that the effect of the magnetic field on the wave propagation is to change the speed of propagation from c_1 to $c_1(1+V_h^2/c_1^2)^{1/2}$. In the absence of ε (4.6) gives the phase velocity of magneto-visco-elastic wave and in the absence of m_1 and m_2 it gives the phase velocity of a magneto-thermo-elastic wave. In the particular case $m_1 = m_2 = 0$ (with effect of deformation on magnetic permeability neglected) equation (4.6) reduces to the one obtained by Nowacki [3], apart from a slight change in notations.

References

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[2] P. Chadwick and I. N. Sneddon, J. Mech. Phy. Sol. *6* (1958), 223.
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