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Some New Modular Equations of Degree 2 Akin to Ramanujan

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Abstract. In this paper, we obtain some new modular equations of degree 2 for the ratios of Ramanujan's theta-function φ and also establish the general formulas for their explicit evaluations. As an application, we establish some new modular relations for Ramanujan–Göllnitz–Gordon continued fraction $H(q)$ with $H(q^{n/2})$, Ramanujan–Selberg continued fraction $V(q)$ with $V(q^{n/2})$ and Eisenstein continued fraction $E(q)$ with $E(q^{n/2})$ for $n = 3, 5$ and 7.

Keywords: Modular equation; Theta-function; Continued fractions.

1. Introduction

First we define modular equation in brief. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right),$$

where $0<k<1$ and ${}_2F_1$ is the ordinary or Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}z^n, \quad 0\le |z|<1,$$

where

$$(a)_0=1, \quad (a)_n = a(a+1)\cdots(a+n-1) \ \text{ for } \ n \text{ a positive integer,}$$

a, b and c are complex numbers such that $c\neq 0,-1,-2,\ldots$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1). Following S. Ramanujan, set $\alpha=k^2$ and $\beta=l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{2}$$

The Ramanujan theta function $f(a,b)$ is defined as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab|<1. \tag{3}$$

The special cases of Ramanujan's theta-function (3) are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{6}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

In [23], J. Yi introduced parameterization $h_{k,n}$ as follows:

$$h_{k,n} := \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \tag{7}$$

and established several properties as well as explicit evaluations of $h_{k,n}$ for different positive rational values of n and k.

In [13], M. S. Mahadeva Naika and S. Chandankumar, have established several new modular equations of degree 2 and established general formulas for explicit evaluations of $h_{2,n}$. The authors have also established several new explicit evaluations for Ramanujan-Göllnitz-Gordon continued fraction. In [10], Mahadeva Naika, K. S. Bairy and M. Manjunatha have established several new modular equations of degree 4 and established general formulas for explicit evaluations of $h_{4,n}$. In [9], Mahadeva Naika, Bairy and Chandankumar have established several new modular equations of degree 9, and also established several general formulas for explicit evaluations of $h_{9,n},\ h'_{9,n}$. In [15], Mahadeva Naika, Chandankumar and Bairy have established some new modular equations for the ratios of Ramanujan's theta functions and their explicit evaluations.

2. Preliminary Section

Lemma 2.1. *We have*

(1) [5, Entry 10(i), (iii), (vi), p. 122] *For* $0<x'<1$,

$$\varphi(e^{-y'}) = \sqrt{z}, \tag{8}$$

$$\varphi(e^{-2y'}) = \sqrt{z}\left(\frac{1}{2}(1+\sqrt{1-x'})\right)^{1/2}, \tag{9}$$

$$\varphi(e^{-y'/2}) = \sqrt{z}(1+\sqrt{x'})^{1/2}, \tag{10}$$

where $z := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x'\right)$, *and* $y' := \pi\dfrac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x'\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x'\right)}$.

(2) [5, Entry 5(ii), p. 230] *If* β *has degree 3 over* α, *then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{11}$$

(3) [5, Entry 13(i), p. 280] *If* β *has degree 5 over* α, *then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{12}$$

(4) [5, Entry 19(i), p. 314] *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{13}$$

(5) [23, Theorem 2.2(ii)] *We have*

$$h_{k,n}h_{k,1/n} = 1. \tag{14}$$

(6) [23, Theorem 4.6] *We have*

$$\sqrt{2}\left(h_{2,n}h_{2,4n} + \frac{1}{h_{2,n}h_{2,4n}}\right) = 2 + \frac{h_{2,4n}}{h_{2,n}}. \tag{15}$$

3. Modular Equations of Degree Two

In this section, we establish some new modular equations of degree 2.

Theorem 3.1. *If $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ and $Q := \dfrac{\varphi(q^{3/2})}{\varphi(q^3)}$, then*

$$\begin{aligned}&288P^2Q^2 - 384P^2Q^4 - 32P^2 + 24P^4 + 584Q^4P^4 - 432Q^2P^4 + Q^8P^8\\ &- 216P^6Q^4 + 152P^6Q^2 + 6P^8Q^4 - 4P^8Q^2 - 4Q^6P^8 + 72Q^6P^6\\ &+ 16 + 80P^3Q - 96PQ - 48Q^3P^3 - 8P^6 + P^8 - 192P^4Q^6\\ &+ 128P^2Q^6 + 64Q^3P + 16Q^7P^5 - 24Q^5P^5 - 48Q^3P^5 + 56QP^5\\ &- 12Q^7P^7 + 20Q^5P^7 + 28Q^3P^7 - 36QP^7 = 0.\end{aligned} \tag{16}$$

Proof. Using equations (8), (9), (10) and (11), we deduce that

$$\begin{aligned}&4 - 4P^2 + P^4 - 4P^4ab + 12P^2b_2 - 6P^4b_2\\ &\quad - 8P^2abb_2 + 4P^4abb_2 - Q^4P^4 + 2Q^2P^4 = 0,\end{aligned} \tag{17}$$

where $a = (1-\alpha)^{1/4}$, $b = (1-\beta)^{1/4}$ and $b_2 = b^2$. Collecting the terms containing ab on one side of the equation (17) and squaring, we obtain that

$$\begin{aligned}&-16 + 32P^2 - 24P^4 - 96P^2b_2 + 24Q^4P^4 - 48Q^2P^4 - 24P^6Q^4 - P^8\\ &- 40P^6b_2 - 4P^8b_2 + 2P^8Q^4 - 12P^8Q^2 + 4Q^6P^8 + 8P^6 + 4P^8b_2Q^4\\ &- 8P^8b_2Q^2 - 72P^6b_2Q^4 + 144P^6b_2Q^2 + 192P^4b_2Q^4 - 384P^4b_2Q^2\\ &- Q^8P^8 + 256P^2b_2Q^2 - 128P^2b_2Q^4 + 48P^6Q^2 + 144P^4b_2 = 0.\end{aligned} \tag{18}$$

Again, collecting the terms containing b_2 on one side of the equation (18) and squaring, we find that

$$\begin{aligned}&(16 - 384P^2Q^4 + P^8 + 288P^2Q^2 - 32P^2 + 24P^4 + 584Q^4P^4 - 432Q^2P^4\\ &- 216P^6Q^4 + 152P^6Q^2 + 6P^8Q^4 - 4P^8Q^2 + Q^8P^8 - 4Q^6P^8 + 72Q^6P^6\\ &+ 80P^3Q - 96PQ - 48Q^3P^3 - 192P^4Q^6 - 36QP^7 + 128P^2Q^6 + 64Q^3P\\ &- 8P^6 + 16Q^7P^5 - 24Q^5P^5 - 48Q^3P^5 + 56QP^5 - 12Q^7P^7 + 20Q^5P^7\end{aligned}$$

$$
\begin{aligned}
&+28Q^3P^7)(16-384P^2Q^4+288P^2Q^2-32P^2+24P^4+584Q^4P^4\\
&-432Q^2P^4+P^8-216P^6Q^4+152P^6Q^2+6P^8Q^4-4P^8Q^2+Q^8P^8\\
&-4Q^6P^8+72Q^6P^6-80P^3Q+96PQ+48Q^3P^3-8P^6-192P^4Q^6\\
&+128P^2Q^6-64Q^3P-16Q^7P^5+24Q^5P^5+48Q^3P^5-56QP^5\\
&+12Q^7P^7-20Q^5P^7-28Q^3P^7+36QP^7)=0.
\end{aligned}
\tag{19}
$$

By examining the behaviour of the factors of the equation (19) as $q \to 0$, it can be seen that the first factor vanishes rapidly than the second factor for q sufficiently small. By the Identity Theorem first factor vanishes identically. This completes the proof. ■

Theorem 3.2. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{5/2})}{\varphi(q^5)}$, *then*

$$
\begin{aligned}
&64+128560P^4Q^4-31840P^4Q^2-3980P^{10}Q^8+9200P^{10}Q^6-63280P^8Q^6\\
&-192P^2+32140P^8Q^8+161536P^6Q^6-126560P^6Q^4+36800P^6Q^2\\
&-160P^6+40Q^{11}P^{11}+240P^4+12800P^2Q^2+60920P^8Q^4-11720P^{10}Q^4\\
&-23440P^8Q^2+5712P^{10}Q^2+15P^{12}Q^4-6P^{12}Q^2+P^{12}Q^{12}+15P^{12}Q^8\\
&-20P^{12}Q^6-6P^{12}Q^{10}+P^{12}+60P^8-12P^{10}-6400P^8Q^{10}-89600P^6Q^8\\
&+102400P^4Q^8+2560Q^7P^3-7680Q^5P^3+5760Q^3P^3-640QP^3-51200P^2Q^4\\
&+17920P^6Q^{10}-20480P^4Q^{10}+800P^{10}Q^{10}-40960Q^8P^2+8192Q^{10}P^2\\
&+71680Q^6P^2+1024Q^5P-2560Q^3P+1280QP-11520Q^7P^5\\
&+1280Q^9P^5+22400Q^5P^5-6400Q^3P^5-5504QP^5+11200Q^7P^7\\
&-1920Q^9P^7-18880Q^5P^7+128Q^{11}P^7+2880Q^3P^7+6592QP^7-1600Q^7P^9\\
&+720Q^9P^9+1440Q^5P^9-160Q^{11}P^9+1120Q^3P^9-1520QP^9-688Q^7P^{11}\\
&-40Q^9P^{11}+1648Q^5P^{11}-760Q^3P^{11}-200QP^{11}-179200P^4Q^6=0.
\end{aligned}
\tag{20}
$$

The proof of (20) is similar to the proof of the equation (16), except that in place of result (11), the result (12) is used.

Theorem 3.3. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{7/2})}{\varphi(q^7)}$, *then*

$$
\begin{aligned}
&256-1024P^2-3211264P^2Q^4+401408P^2Q^2+1792P^4-1792P^6\\
&+15122688Q^4P^4-1286656Q^2P^4-29512448P^6Q^4+3358208P^6Q^2\\
&-5412736P^8Q^2+29447488P^8Q^4-66351488Q^6P^8+77310048Q^8P^8\\
&+1120P^8+4080384P^{10}Q^2-15895936P^{10}Q^4+30551808Q^6P^{10}\\
&-33175744Q^8P^{10}-448P^{10}+7168P^3Q+28P^{16}Q^4-56P^{16}Q^6+P^{16}\\
&-8P^{16}Q^2-1242080P^{12}Q^2+101600P^{14}Q^2+70P^{16}Q^8+4672304P^{12}Q^4
\end{aligned}
$$

$$
\begin{aligned}
&-621040P^{14}Q^4-7947968P^{12}Q^6+7361872P^{12}Q^8+1020096P^{14}Q^6\\
&-676592P^{14}Q^8-3689056Q^{10}P^{12}+945168Q^{12}P^{12}+209888P^{14}Q^{10}\\
&-40208P^{14}Q^{12}+112P^{12}-16P^{14}-56P^{16}Q^{10}+28P^{16}Q^{12}\\
&-8Q^{14}P^{16}+Q^{16}P^{16}-211456P^5Q-9877504Q^3P^5-96051200Q^8P^6\\
&+365312P^7Q+17269504P^7Q^3+77600768Q^6P^6-65151744P^7Q^5\\
&+19400192P^{10}Q^{10}-5562368P^{10}Q^{12}+14336PQ+1870848Q^3P^3\\
&+41972224P^5Q^5-57344PQ^3+59006976P^4Q^8-44498944P^4Q^6\\
&+9633792P^2Q^6-13762560P^2Q^8+6272Q^{14}P^{14}+96687360Q^7P^7\\
&+10092544Q^{10}P^2-3670016Q^{12}P^2+14751744Q^{12}P^8-48025600Q^{10}P^8\\
&+62160896Q^{10}P^6+57344Q^5P-40140800P^4Q^{10}-20070400Q^{12}P^6\\
&+13647872P^4Q^{12}-9719808Q^5P^3-100352Q^{14}P^{12}+602112Q^{14}P^{10}\\
&-1720320Q^{14}P^8-1835008P^4Q^{14}+2523136P^6Q^{14}+524288P^2Q^{14} \qquad (21)\\
&-16384Q^7P+17690624Q^7P^3-14909440Q^9P^3+5963776Q^{11}P^3\\
&-917504Q^{13}P^3-67192832Q^7P^5+52211712Q^9P^5-19898368Q^{11}P^5\\
&+2981888Q^{13}P^5-71550976Q^9P^7+26105856Q^{11}P^7-3727360Q^{13}P^7\\
&-161152QP^9-12504576Q^3P^9+45887744Q^5P^9-66977792Q^7P^9\\
&+48343680Q^9P^9-16798208Q^{11}P^9+2211328Q^{13}P^9-1024Q^{15}P^9\\
&-41664QP^{11}+3569216Q^3P^{11}-14824320Q^5P^{11}+22943872Q^7P^{11}\\
&-16287936Q^9P^{11}+5246528Q^{11}P^{11}-607488Q^{13}P^{11}+1792Q^{15}P^{11}\\
&+26656QP^{13}-284032Q^3P^{13}+22832Q^9P^{15}+58464Q^{13}P^{13}\\
&+1784608Q^5P^{13}-3126144Q^7P^{13}+2158688Q^9P^{13}-617344Q^{11}P^{13}\\
&-896Q^{15}P^{13}+784QP^{15}+13328Q^3P^{15}-10416Q^5P^{15}-20144Q^7P^{15}\\
&-6608Q^{11}P^{15}+112Q^{13}P^{15}+112Q^{15}P^{15}=0.
\end{aligned}
$$

The proof of (20) is similar to the proof of the equation (16), except that in place of result (11), the result (13) is used.

4. General Formulas for Explicit Evaluations of $h_{2,n}$

In this section, we establish some general formulas for explicit evaluations of $h_{2,n}$.

Theorem 4.1. *If* $h := h_{2,n}$ *and* $g := h_{2,9n/4}$, *then*

$$
\begin{aligned}
&(12g^4-8\sqrt{2}g^6-4\sqrt{2}g^2+4g^8+1)h^8+(28\sqrt{2}g^3-36g+40g^5-24\sqrt{2}g^7)h^7\\
&+(144g^6-4\sqrt{2}+152g^2-216\sqrt{2}g^4)h^6+(28\sqrt{2}g-24\sqrt{2}g^5+32g^7-48g^3)h^5
\end{aligned}
$$

$$\begin{aligned}&+(584g^4-192\sqrt{2}g^6+12-216\sqrt{2}g^2)h^4+(40g-24\sqrt{2}g^3)h^3\\&+(144g^2-8\sqrt{2}-192\sqrt{2}g^4+128g^6)h^2+(32g^3-24\sqrt{2}g)h+4=0.\end{aligned}\tag{22}$$

Proof. Employing the equation (14) with $k=2$ in the equation (16), we obtain (22). ■

Corollary 4.2. *We have*

$$h_{2,3/2}=\frac{1}{2}\sqrt{-1+2\sqrt{2}+\sqrt{3}},\tag{23}$$

$$h_{2,2/3}=\left(2\sqrt{2}-1+\sqrt{3}\right)^{1/2}\left(2\sqrt{2}-1-\sqrt{3}\right)\left(1+\sqrt{2}\right)^2,\tag{24}$$

$$h_{2,6}=\left(-1+2\sqrt{2}+\sqrt{3}\right)^{1/2}(\sqrt{6}-2)\tag{25}$$

$$h_{2,1/6}=\frac{1}{4}\left(\sqrt{6}+2\right)\left(\sqrt{2}+1\right)^2\left(-1+2\sqrt{2}+\sqrt{3}\right)^{1/2}(-1+2\sqrt{2}-\sqrt{3}),\tag{26}$$

$$\begin{aligned}h_{2,24}=&\sqrt{598+346\sqrt{3}+244\sqrt{6}+424\sqrt{2}}\\&-\sqrt{578+334\sqrt{3}+236\sqrt{6}+409\sqrt{2}},\end{aligned}\tag{27}$$

$$\begin{aligned}h_{2,1/24}=&\Bigg(\sqrt{598+346\sqrt{3}+244\sqrt{6}+424\sqrt{2}}+\\&\sqrt{578+334\sqrt{3}+236\sqrt{6}+409\sqrt{2}}\Bigg)\left(\frac{8\sqrt{6}-12\sqrt{3}+15\sqrt{2}-20}{2}\right).\end{aligned}\tag{28}$$

Proof. [Proofs of (23) and (24)] Putting $n=2/3$ in the equation (22) and using the fact $h_{2,2/3}h_{2,3/2}=1$, we find that

$$\begin{aligned}&\left(8h_{2,3/2}^4+4h_{2,3/2}^2-8\sqrt{2}h_{2,3/2}^2+3-2\sqrt{2}\right)\\&\left(4h_{2,3/2}^4-4\sqrt{2}h_{2,3/2}^2+1+\sqrt{2}\right)^2=0.\end{aligned}\tag{29}$$

The roots of second factor of the equation (29) are imaginary. Hence

$$8h_{2,3/2}^4+4h_{2,3/2}^2-8\sqrt{2}h_{2,3/2}^2+3-2\sqrt{2}=0.\tag{30}$$

Solving the above equation (30), we obtain (23) and (24). ■

Proof. [Proofs of (25) and (26)] Using (23) in the equation (15), we obtain (25) and (26). ■

Proof. [Proofs of (27) and (28)] Using (25) in the equation (15), we obtain (27) and (28). ■

Theorem 4.3. *If* $h := h_{2,n}$ *and* $g := h_{2,25n/4}$, *then*

$$\begin{aligned}
&8 + (8g^{12} - 40\sqrt{2}g^6 + 30g^4 + 60g^8 - 24\sqrt{2}g^{10} - 6\sqrt{2}g^2 + 1)h^{12} \\
&+ (760\sqrt{2}g^3 - 3296g^5 - 160\sqrt{2}g^{11} + 1376\sqrt{2}g^7 + 160g^9 + 200g)h^{11} \\
&+ (18400g^6 + 3200g^{10} - 7960g^8\sqrt{2} + 5712g^2 - 6\sqrt{2} - 11720\sqrt{2}g^4)h^{10} \\
&+ (640g^{11} - 1440\sqrt{2}g^9 + 760\sqrt{2}g - 1120g^3 + 3200g^7 - 1440\sqrt{2}g^5)h^9 \\
&+ (64280g^8 - 63280\sqrt{2}g^6 - 11720\sqrt{2}g^2 - 12800\sqrt{2}g^{10} + 30 + 60920g^4)h^8 \\
&+ (3840g^9 - 11200\sqrt{2}g^7 - 256\sqrt{2}g^{11} + 18880g^5 - 3296g - 1440\sqrt{2}g^3)h^7 \\
&+ (35840g^{10} - 89600g^8\sqrt{2} - 63280\sqrt{2}g^4 + 18400g^2 - 40\sqrt{2} + 161536g^6)h^6 \\
&+ (-1280\sqrt{2}g^9 + 1376\sqrt{2}g + 3200g^3 + 11520g^7 - 11200\sqrt{2}g^5)h^5 \\
&+ (102400g^8 - 89600\sqrt{2}g^6 + 64280g^4 + 60 - 20480\sqrt{2}g^{10} - 7960\sqrt{2}g^2)h^4 \\
&- (1440\sqrt{2}g^3 - 3840g^5 + 1280\sqrt{2}g^7 - 160g)h^3 - (20480g^8\sqrt{2} - 35840g^6 \\
&+ 24\sqrt{2} - 8192g^{10} - 3200g^2 + 12800\sqrt{2}g^4)h^2 \\
&+ (-256\sqrt{2}g^5 + 640g^3 - 160\sqrt{2}g)h = 0. \qquad (31)
\end{aligned}$$

Proof. Employing the equation (14) with $k = 2$ in the equation (20), we obtain (31). ∎

Corollary 4.4.

$$h^2_{2,5/2} = \frac{1}{4}\left(1 + 2\sqrt{2} - \sqrt{5} + \sqrt{2}\sqrt{\sqrt{5} - 1}\right), \qquad (32)$$

$$h^2_{2,2/5} = \frac{4}{\left(1 + 2\sqrt{2} - \sqrt{5} + \sqrt{2}\sqrt{\sqrt{5} - 1}\right)}, \qquad (33)$$

$$h^2_{2,10} = \left(1 + 2\sqrt{2} - \sqrt{5} + \sqrt{2}\sqrt{\sqrt{5} - 1}\right)\left(\sqrt{\sqrt{5} - 1} - 1\right)^2\left(\sqrt{5} + 2\right)^2, \qquad (34)$$

$$h^2_{2,1/10} = \frac{\left(\sqrt{\sqrt{5} - 1} + 1\right)^2}{\left(1 + 2\sqrt{2} - \sqrt{5} + \sqrt{2}\sqrt{\sqrt{5} - 1}\right)}. \qquad (35)$$

Proof. [Proofs of (32) and (33)] Putting $n = 2/5$ in the equation (31) and using the fact $h_{2,2/5}h_{2,5/2} = 1$, we find that

$$\begin{aligned}
&\left(16h^8_{2,5/2} - 16h^6_{2,5/2}\left[1 + 2\sqrt{2}\right] + 24h^4_{2,5/2}\left[2 + \sqrt{2}\right] - 4h^2_{2,5/2}\left[3 + 4\sqrt{2}\right]\right. \\
&\left.+3 - 2\sqrt{2}\right)\left(16h^4_{2,5/2} - 16\sqrt{2}h^2_{2,5/2} + 4 + 3\sqrt{2}\right)^2 \qquad (36) \\
&\times\left(2h^2_{2,5/2} + 1 - \sqrt{2}\right)^2 = 0.
\end{aligned}$$

First factor of the equation (36) vanishes for the specific value of $q = e^{-\pi\sqrt{5}/2}$; whereas the other factors does not vanish. Hence

$$\begin{aligned} &16h_{2,5/2}^8 - 16h_{2,5/2}^6\left[1+2\sqrt{2}\right] + 24h_{2,5/2}^4\left[2+\sqrt{2}\right] \\ &- 4h_{2,5/2}^2\left[3+4\sqrt{2}\right] + 3 - 2\sqrt{2} = 0. \end{aligned} \tag{37}$$

Solving the above equation (37), we obtain (32) and (33). ∎

Proof. [Proofs of (34) and (35)] Using (32) in the equation (15), we obtain (34) and (35). ∎

Theorem 4.5. *If* $h := h_{2,n}$ *and* $g := h_{2,49n/4}$*, then*

$$\begin{aligned}
&(-224\sqrt{2}g^{10} + 16g^{16} - 8\sqrt{2}g^2 - 64\sqrt{2}g^{14} - 112\sqrt{2}g^6 + 224g^{12} + 56g^4 + 1 \\
&+ 280g^8)h^{16} + (-20832g^5 + 91328g^9 - 40288\sqrt{2}g^7 - 26432\sqrt{2}g^{11} + 784g \\
&+ 896\sqrt{2}g^{15} + 13328\sqrt{2}g^3 + 896g^{13})h^{15} + (2040192g^6 + 50176g^{14} - 8\sqrt{2} \\
&+ 839552g^{10} - 621040\sqrt{2}g^4 - 160832g^{12}\sqrt{2} + 101600g^2 - 1353184\sqrt{2}g^8)h^{14} \\
&+ (13328\sqrt{2}g - 7168g^{15} + 233856\sqrt{2}g^{13} - 2469376g^{11} - 284032g^3 \\
&+ 1784608\sqrt{2}g^5 - 6252288g^7 + 4317376\sqrt{2}g^9)h^{13} + (56 + 14723744g^8 \\
&- 7947968\sqrt{2}g^6 - 7378112\sqrt{2}g^{10} + 4672304g^4 + 3780672g^{12} - 401408\sqrt{2}g^{14} \\
&- 621040\sqrt{2}g^2)h^{12} + (-32575872g^9 + 22943872\sqrt{2}g^7 + 7168\sqrt{2}g^{15} \\
&- 14824320g^5 - 20832g + 10493056\sqrt{2}g^{11} + 1784608\sqrt{2}g^3 - 2429952g^{13})h^{11} \\
&+ (-7947968\sqrt{2}g^4 - 33175744\sqrt{2}g^8 + 30551808g^6 + 2408448g^{14} - 112\sqrt{2} \\
&+ 38800384g^{10} + 2040192g^2 - 11124736g^{12}\sqrt{2})h^{10} + (48343680\sqrt{2}g^9 \\
&- 33596416g^{11} - 6252288g^3 + 4422656\sqrt{2}g^{13} - 40288\sqrt{2}g - 4096g^{15} \\
&- 66977792g^7 + 22943872\sqrt{2}g^5)h^9 + (29503488g^{12} + 14723744g^4 + 280 \\
&- 33175744\sqrt{2}g^6 + 77310048g^8 - 1353184\sqrt{2}g^2 - 48025600\sqrt{2}g^{10} \\
&- 3440640\sqrt{2}g^{14})h^8 + (91328g - 71550976g^9 - 32575872g^5 + 26105856\sqrt{2}g^{11} \\
&+ 48343680\sqrt{2}g^7 - 7454720g^{13} + 4317376\sqrt{2}g^3)h^7 + (839552g^2 + 62160896g^{10} \\
&- 7378112\sqrt{2}g^4 + 5046272g^{14} - 48025600\sqrt{2}g^8 - 224\sqrt{2} - 20070400g^{12}\sqrt{2} \\
&+ 38800384g^6)h^6 + (-26432\sqrt{2}g + 26105856\sqrt{2}g^9 - 19898368g^{11} - 2469376g^3 \\
&+ 2981888\sqrt{2}g^{13} + 10493056\sqrt{2}g^5 - 33596416g^7)h^5 + (3780672g^4 + 224 \\
&- 160832\sqrt{2}g^2 - 11124736\sqrt{2}g^6 + 13647872g^{12} + 29503488g^8 - 1835008\sqrt{2}g^{14} \\
&- 20070400\sqrt{2}g^{10})h^4 + (4422656\sqrt{2}g^7 - 7454720g^9 + 896g - 917504g^{13} \\
&+ 2981888\sqrt{2}g^{11} + 233856\sqrt{2}g^3 - 2429952g^5)h^3 + (50176g^2 - 401408\sqrt{2}g^4 \\
&+ 524288g^{14} + 5046272g^{10} - 64\sqrt{2} + 2408448g^6 - 1835008g^{12}\sqrt{2}
\end{aligned}$$

$$- 3440640\sqrt{2}g^8)h^2 + (-7168g^3 + 7168\sqrt{2}g^5 + 896\sqrt{2}g - 4096g^7)h$$
$$+ 16 = 0. \tag{38}$$

Proof. Employing the equation (14) with $k = 2$ in the equation (21), we obtain (38). ■

Corollary 4.6. *We have*

$$h_{2,7/2} = \frac{1}{4}\sqrt{6 + 8\sqrt{2} - 2\sqrt{7}}, \tag{39}$$

$$h_{2,2/7} = \left(6 + 8\sqrt{2} - 2\sqrt{7}\right)^{1/2}\left(3 + 4\sqrt{2} + \sqrt{7}\right)\left(\sqrt{2} - 1\right)^4, \tag{40}$$

$$h_{2,14} = 2\left(\sqrt{2} - 1\right)^2\sqrt{3 + 4\sqrt{2} - \sqrt{7}}, \tag{41}$$

$$h_{2,1/14} = \frac{1}{4\sqrt{2}}\left(\sqrt{2} - 1\right)^2\left(6 + 8\sqrt{2} - 2\sqrt{7}\right)^{1/2}\left(3 + 4\sqrt{2} + \sqrt{7}\right). \tag{42}$$

Proof. [Proofs of (39) and (40)] Putting $n = 2/7$ in the equation (38) and using the fact $h_{2,2/7}h_{2,7/2} = 1$, we find that

$$\left(32h^4_{2,7/2} - 8h^2_{2,7/2}\left[3 + 4\sqrt{2}\right] + 17 + 12\sqrt{2}\right)\left(16h^8_{2,7/2} - 32\sqrt{2}h^6_{2,7/2} + 40h^4_{2,7/2}\right.$$
$$\left. -8\sqrt{2}h^2_{2,7/2} - 17 + 12\sqrt{2}\right)^2$$
$$\times\left(8h^4_{2,7/2} - 4h^2_{2,7/2} - 8\sqrt{2}h^2_{2,7/2} + 3 + 2\sqrt{2}\right)^2 = 0. \tag{43}$$

First factor of the equation (43) vanishes for the specific value of $q = e^{-\pi\sqrt{7/2}}$; whereas the other factors does not vanish. Hence

$$32h^4_{2,7/2} - 8h^2_{2,7/2}\left[3 + 4\sqrt{2}\right] + 17 + 12\sqrt{2} = 0. \tag{44}$$

Solving the above equation (44), we obtain (39) and (40). ■

Proof. [Proofs of (41) and (42)] Using (39) in the equation (15), we obtain (41) and (42). ■

5. New Identities for Ramanujan–Göllnitz–Gordon Continued Fraction

On page 229 of his second notebook, Ramanujan recorded the continued fraction which is known as Ramanujan-Göllnitz-Gordon continued fraction along with the two identities as follows:

$$H(q) := \frac{q^{1/2}}{1+q}\,{}_{+}\,\frac{q^2}{1+q^3}\,{}_{+}\,\frac{q^4}{1+q^5}\,{}_{+}\,\frac{q^6}{1+q^7}\,{}_{+}\cdots, \qquad |\, q \,|< 1, \tag{45}$$

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{46}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{47}$$

Proofs of the above identities (45), (46) and (47) can be found in [19, p. 221] and for more details one can see [6]. In [17], the authors have established several modular relations between a Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13,$ 15, 17, 19, 23, 25, 29, 31 and 55. In [4], the authors have established the following identity

$$H^2(e^{-\pi\sqrt{n/2}}) = \frac{2^{1/4}h_{2,n} - 1}{2^{1/4}h_{2,n} + 1}, \quad \text{for any positive rational number n,} \tag{48}$$

which follows from the identity established by Chan and Huang in [6].

Theorem 5.1. *If* $x := H(q)$ *and* $y := H(q^{3/2})$, *then*

$$\begin{aligned}&\left(x^5 + 2x^4 - 2x^2 - x\right) y^8 + \left(x^8 + 2x^7 - 3x^6 - 10x^5 - 5x^4 - 6x^3 + 15x^2 + 6x\right) y^6 \\ &+ \left(3x^7 - 6x^6 + 3x^5 + 12x^4 - 3x^3 - 6x^2 - 3x\right) y^4 + \left(1 - 2x - 3x^2 + 10x^3 - 5x^4 \right.\\ &\left.+6x^5 + 15x^6 - 6x^7\right) y^2 + x^7 - 2x^6 + 2x^4 - x^3 = 0.\end{aligned} \tag{49}$$

Proof. Using the equations (46) and (47) in the equation (16), we deduce that

$$\begin{aligned}&\left(15x^6y^2 - 3x^2y^2 - 6x^2y^4 - 5x^4y^2 + 12x^4y^4 + 15x^2y^6 - 2x^2y^8 - 5x^4y^6 + 2x^4y^8\right.\\ &- 6x^6y^4 - 3x^6y^6 + x^8y^6 + y^2 + 2x^4 - 2x^6 - 10x^3y^2 + 3xy^4 - 6xy^6 + xy^8 + x^3 \\ &- 6x^5y^2 + 6x^7y^2 + 3x^3y^4 - 3x^5y^4 - 3x^7y^4 + 6x^3y^6 + 10x^5y^6 - 2x^7y^6 - x^5y^8 \\ &\left.-x^7 + 2xy^2\right) \left(15x^6y^2 - 3x^2y^2 - 6x^2y^4 - 5x^4y^2 + 12x^4y^4 + 15x^2y^6 - 2x^2y^8\right.\\ &- 5x^4y^6 + 2x^4y^8 - 6x^6y^4 - 3x^6y^6 + x^8y^6 + y^2 + 2x^4 - 2x^6 + 10x^3y^2 \\ &- 3xy^4 + 6xy^6 - xy^8 + 6x^5y^2 - 6x^7y^2 - 3x^3y^4 + 3x^5y^4 + 3x^7y^4 \\ &\left.-6x^3y^6 - 10x^5y^6 + 2x^7y^6 + x^5y^8 - x^3 + x^7 - 2xy^2\right) = 0.\end{aligned} \tag{50}$$

From the definitions of x and y, it can be seen that the second factor vanishes as $q \to 0$, whereas the first factor does not vanish. Thus by the Identity theorem, second factor vanishes identically. This completes the proof. ∎

Theorem 5.2. *If $x := H(q)$ and $y := H(q^{5/2})$, then*

$$\begin{aligned}
&\left(x^7+4x^6+5x^5-5x^3-4x^2-x\right)y^{12}+\left(x^{12}+4x^{11}-20x^9-30x^8-6x^7\right.\\
&\left.-4x^6-30x^5+45x^4+10x^3+20x^2+10x\right)y^{10}+\left(-20x^{11}-40x^{10}+150x^9\right.\\
&\left.+240x^8-145x^7+20x^6-65x^5-240x^4+105x^3+20x^2-25x\right)y^8+\left(10x^{10}\right.\\
&\left.-20x^9+60x^7-20x^6-60x^5+20x^3+10x^2\right)y^6+\left(25x^{11}+20x^{10}-105x^9\right.\\
&\left.-240x^8+65x^7+20x^6+145x^5+240x^4-150x^3-40x^2+20x\right)y^4+\left(-10x^{11}\right.\\
&\left.+20x^{10}-10x^9+45x^8+30x^7-4x^6+6x^5-30x^4+20x^3-4x+1\right)y^2\\
&-4x^{10}-5x^7+5x^9+4x^6-x^5+x^{11}=0. \qquad (51)
\end{aligned}$$

The proof of the equation (51) is similar to the proof of the equation (49) except that in the place of the equation (16), the equation (20) is employed.

Theorem 5.3. *If $x := H(q)$ and $y := H(q^{7/2})$, then*

$$\begin{aligned}
&\left(x^8+8x^7+28x^6+56x^5+70x^4+56x^3+28x^2+8x+1\right)y^{16}+\left(x^{15}+8x^{14}\right.\\
&+21x^{13}-119x^{11}-280x^{10}-259x^9+48x^8+363x^7+392x^6+231x^5-112x^4\\
&\left.-245x^3-56x^2+7x\right)y^{14}+\left(-14x^{15}-112x^{14}-182x^{13}+448x^{12}+1442x^{11}\right.\\
&+1260x^{10}+1050x^9+112x^8-2282x^7-1512x^6-770x^5-336x^4+854x^3\\
&\left.+140x^2-98x\right)y^{12}+\left(63x^{15}+392x^{14}+203x^{13}-2688x^{12}-4081x^{11}\right.\\
&+2520x^{10}+4123x^9-336x^8+3717x^7-2072x^6-4375x^5+2576x^4+301x^3\\
&\left.-392x^2+49x\right)y^{10}+\left(-84x^{15}+924x^{13}+70x^{12}-2100x^{11}-280x^{10}-3108x^9\right.\\
&\left.+420x^8+3108x^7-280x^6+2100x^5+70x^4-924x^3+84x\right)y^8+\left(-49x^{15}\right.\\
&-392x^{14}-301x^{13}+2576x^{12}+4375x^{11}-2072x^{10}-3717x^9-336x^8-4123x^7\\
&\left.+2520x^6+4081x^5-2688x^4-203x^3+392x^2-63x\right)y^6+\left(98x^{15}+140x^{14}\right.\\
&-854x^{13}-336x^{12}+770x^{11}-1512x^{10}+2282x^9+112x^8-1050x^7+1260x^6\\
&\left.-1442x^5+448x^4+182x^3-112x^2+14x\right)y^4+\left(-7x^{15}-56x^{14}+245x^{13}-x\right.\\
&-112x^{12}-231x^{11}+392x^{10}-363x^9+48x^8+259x^7-280x^6+119x^5-21x^3\\
&\left.+8x^2\right)y^2+x^{16}\\
=&8x^{15}-28x^{14}+56x^{13}-70x^{12}+56x^{11}-28x^{10}+8x^9-x^8. \qquad (52)
\end{aligned}$$

The proof of the equation (52) is similar to the proof of the equation (49) except that in the place of the equation (16), the equation (21) is employed.

In the following corollary, we establish several explicit evaluations of Ramanujan-Göllnitz-Gordon continued fraction by using the values of $h_{2,n}$ obtained in the Section in the equation (48).

Corollary 5.4. *We have*

$$H^2\left(e^{-\pi\sqrt{3/4}}\right) = \left(1+\sqrt{3}\right)\left(2\sqrt{2} - 2^{3/4}\sqrt{-1+2\sqrt{2}+\sqrt{3}}\right) + 1,$$

$$H^2\left(e^{-\pi/\sqrt{3}}\right) = \sqrt{3393 - 1952\sqrt{3} + 1384\sqrt{6} - 2392\sqrt{2}} - \sqrt{3392 - 1952\sqrt{3} + 1384\sqrt{6} - 2392\sqrt{2}},$$

$$H^2\left(e^{-\pi\sqrt{3}}\right) = \sqrt{3393 + 1952\sqrt{3} + 1384\sqrt{6} + 2392\sqrt{2}} - \sqrt{3392 + 1952\sqrt{3} + 1384\sqrt{6} + 2392\sqrt{2}},$$

$$H^2\left(e^{-\pi/2\sqrt{3}}\right) = \sqrt{33 - 16\sqrt{3} - 4\sqrt{2} + 4\sqrt{6}} - \sqrt{32 - 16\sqrt{3} - 4\sqrt{2} + 4\sqrt{6}},$$

$$H^2\left(e^{-\pi\sqrt{5}/2}\right) = \sqrt{73 + 32\sqrt{5} + 6\sqrt{2} + 2\sqrt{10} + \sqrt{\sqrt{5}-1}\left(4\sqrt{5} + 8 + 20\sqrt{10} + 44\sqrt{2}\right)} - \sqrt{72 + 32\sqrt{5} + 6\sqrt{2} + 2\sqrt{10} + \sqrt{\sqrt{5}-1}\left(4\sqrt{5} + 8 + 20\sqrt{10} + 44\sqrt{2}\right)},$$

$$H^2\left(e^{-\pi\sqrt{7}/2}\right) = \sqrt{513 + 24\sqrt{2} + 8\sqrt{14} + 192\sqrt{7}} - \sqrt{512 + 24\sqrt{2} + 8\sqrt{14} + 192\sqrt{7}},$$

$$H^2\left(e^{-\pi/\sqrt{7}}\right) = \sqrt{1037313 - 277232\sqrt{14} - 392064\sqrt{7} + 733488\sqrt{2}} - \sqrt{1037312 - 277232\sqrt{14} - 392064\sqrt{7} + 733488\sqrt{2}},$$

$$H^2\left(e^{-\pi\sqrt{7}}\right) = \sqrt{1037313 + 277232\sqrt{14} + 392064\sqrt{7} + 733488\sqrt{2}} - \sqrt{1037312 + 277232\sqrt{14} + 392064\sqrt{7} + 733488\sqrt{2}},$$

6. New Identities for Ramanujan–Selberg Continued Fraction

The continued fraction identity

$$V(q) := \frac{q^{1/8}}{1+}\,\frac{q}{1+}\,\frac{q^2+q}{1+}\,\frac{q^3}{1+}\,\frac{q^4+q^2}{1+}\cdots = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1, \tag{53}$$

appears as Formula 5 [19, p. 290] and was first proved by Selberg [21, eq. (54)]. Recently, Mahadeva Naika, Remy Y Denis and Sushan Bairy [16] have established several modular relations and explicit evaluations of Ramanujan-Selberg continued fraction. In [11], Mahadeva Naika, Sushan Bairy and Manjunatha have established several modular relations and explicit evaluations continued fraction of order four.

Lemma 6.1. *We have*

$$\frac{\varphi^2(q^{1/2})}{\varphi^2(q)} = 1 + 4V^4(q). \tag{54}$$

Lemma 6.2. *For any positive rational number n, we have*

$$V^4(e^{-\pi\sqrt{2n}}) = \frac{\sqrt{2}h_{2,n}^2 - 1}{4}. \tag{55}$$

Theorem 6.3. *If $v := V(q)$ and $u := V(q^{3/2})$, then*

$$\begin{aligned} &u^{16}\left(256v^{16}+1\right) + u^4v^2\left(12u^8-1\right)\left(64v^{12}+1\right) - 112u^{12}v^6\left(4v^4+1\right) \\ &+ 2u^8v^4\left(8u^8+3\right)\left(16v^8+1\right) + \left(96u^{16}+48u^8+1\right)v^8 = 0. \end{aligned} \tag{56}$$

Proof. Using the equations (54) and (16), we deduce that

$$\begin{aligned} &-768PQu^4v^4 + 3072PQu^4v^8 - 6144PQu^8v^8 + 8192PQu^4v^{12} - 1024PQu^{12}v^4 \\ &- 20480PQu^{12}v^8 - 16384PQu^8v^{12} - 49152PQu^{12}v^{12} + 128PQu^8 + 256PQu^{12} \\ &+ 128PQv^4 + 512u^4v^4 - 2048u^4v^8 + 1024u^8v^4 + 2048u^8v^8 - 16384u^4v^{12} \\ &- 10240u^{12}v^4 + 24576u^{12}v^8 + 294912u^{12}v^{12} + 4096u^{16}v^4 + 24576u^{16}v^8 \\ &+ 65536u^{16}v^{12} + 65536u^{16}v^{16} - 128v^4 + 256u^{16} \\ &- 256v^8 + 128u^8 + 512u^{12} = 0, \end{aligned} \tag{57}$$

where P and Q are as defined as in (16).
Isolating the terms containing PQ on one side of the equation (57) and squaring, we deduce that

$$\begin{aligned} &(6u^8v^4 + 48u^8v^8 + 96u^8v^{12} + 16u^{16}v^4 + 96u^{16}v^8 + 256u^{16}v^{12} + 256u^{16}v^{16} + v^8 \\ &- u^4v^2 + u^{16} - 64v^{14}u^4 + 768v^{14}u^{12} - 448v^{10}u^{12} - 112v^6u^{12} + 12v^2u^{12}) \\ &(6u^8v^4 + 48u^8v^8 + 96u^8v^{12} + 16u^{16}v^4 + 96u^{16}v^8 + 256u^{16}v^{12} + 256u^{16}v^{16} + v^8 \\ &+ u^4v^2 + u^{16} + 64v^{14}u^4 - 768v^{14}u^{12} \\ &+ 448v^{10}u^{12} + 112v^6u^{12} - 12v^2u^{12}) = 0. \end{aligned} \tag{58}$$

From the definitions of v and u, we have $v = o(q^{1/8})$ and $u = o(q^{3/4})$ as $q \to 0$, it can be seen that the first factor vanishes rapidly than the second factor for q

sufficiently small. Thus by the Identity theorem, first factor vanishes identically. This completes the proof. ■

Theorem 6.4. *If $v := V(q)$ and $u := V(q^{5/2})$, then*

$$\begin{aligned}
&u^{24}\left(4096v^{24}+1\right)-u^4v^2\left(70u^{16}-20u^8+1\right)\left(1024v^{20}+1\right)\\
&+u^8v^4\left(24u^{16}+655u^8-40\right)\left(256v^{16}+1\right)+20u^{12}v^6\left(186u^8-13\right)\left(64v^{12}+1\right)\\
&+15u^8v^8\left(16u^{16}+16u^8+1\right)\left(16v^8+1\right)-2u^4v^{10}\left(7648u^{16}-200u^8-5\right)\\
&\times\left(4v^4+1\right)+\left(1280u^{24}-19040u^{16}+1400u^8+1\right)v^{12}=0. \qquad (59)
\end{aligned}$$

The proof of the equation (59) is similar to the proof of the equation (56) except that in the place of the equation (16), the equation (20) is used.

Theorem 6.5. *If $v := V(q)$ and $u := V(q^{7/2})$, then*

$$\begin{aligned}
&u^{32}\left(65536v^{32}+1\right)+\left(280u^{28}-210u^{20}+28u^{12}-u^4\right)\left(16384v^{30}+v^2\right)\\
&+\left(32u^{32}+13468u^{24}-728u^{16}-7u^8\right)\left(4096v^{28}+v^4\right)\\
&-\left(1640u^{28}-1396u^{20}+81u^{12}\right)\left(28672v^{26}+28v^6\right)\\
&+\left(32u^{32}+9776u^{24}+421u^{16}-64u^8\right)\left(3584v^{24}+14v^8\right)\\
&+\left(3392u^{20}+u^4+60992u^{28}-476u^{12}\right)\left(896v^{22}+14v^{10}\right)\\
&+\left(128u^{32}-7440u^{24}+456u^{16}+5u^8\right)\left(448v^{20}+28v^{12}\right)\\
&+\left(-25508u^{20}+1883u^{12}-85360u^{28}\right)\left(128v^{18}+32v^{14}\right)\\
&+\left(28896u^8+17920u^{32}+1-4322304u^{24}-179648u^{16}\right)v^{16}=0. \qquad (60)
\end{aligned}$$

The proof of the equation (60) is similar to the proof of the equation (56) except that in the place of the equation (16), the equation (21) is used.

In the following corollary, we establish several explicit evaluations of Ramanujan-Selberg continued fraction by using the values of $h_{2,n}$ obtained in the Section along with the equation (55):

Corollary 6.6. *We have*

$$\begin{aligned}
V^4\left(e^{-\pi\sqrt{3}}\right)&=\frac{\sqrt{3}-1}{8\sqrt{2}},\\
V^4\left(e^{-2\pi/\sqrt{3}}\right)&=\frac{1}{4}\left(3+2\sqrt{2}\right)\left(5-2\sqrt{6}\right),\\
V^4\left(e^{-2\pi\sqrt{3}}\right)&=\frac{1}{4}\left(3-2\sqrt{2}\right)\left(5-2\sqrt{6}\right),\\
V^4\left(e^{-\pi/\sqrt{3}}\right)&=\frac{\sqrt{3}+1}{8\sqrt{2}},
\end{aligned}$$

$$
\begin{aligned}
V^4\left(e^{-4\pi\sqrt{3}}\right) =& \frac{1}{4}\left(\sqrt{11077377+7832880\sqrt{2}+6395520\sqrt{3}+4522320\sqrt{6}}\right.\\
&\left.-\sqrt{11077376+7832880\sqrt{2}+6395520\sqrt{3}+4522320\sqrt{6}}\right),\\
V^4\left(e^{-\pi\sqrt{5}}\right) =& \frac{1}{8\sqrt{2}}\left(1-\sqrt{5}+\sqrt{2}\sqrt{\sqrt{5}-1}\right),\\
V^4\left(e^{-\pi\sqrt{7}}\right) =& \frac{3-\sqrt{7}}{16\sqrt{2}},\\
V^4\left(e^{-\pi/\sqrt{7}}\right) =& \frac{3+\sqrt{7}}{16\sqrt{2}},\\
V^4\left(e^{-2\pi/\sqrt{7}}\right) =& \frac{1}{4}\left(17-12\sqrt{2}\right)\left(15+4\sqrt{14}\right),\\
V^4\left(e^{-2\pi\sqrt{7}}\right) =& \frac{1}{4}\left(17-12\sqrt{2}\right)\left(15-4\sqrt{14}\right),
\end{aligned}
$$

7. New Identities for a Continued Fraction of Eisenstein

In [12], the authors have established the following continued fraction

$$E(q) := \frac{(q;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1+}\,\frac{2q}{1-q^2+}\,\frac{-q^3-q}{1+q^4+}\,\frac{q^5+q^3}{1-q^6+}\cdots, \quad \text{for } |q|<1. \tag{61}$$

They have also established several modular relations between a continued fraction of Eisenstein $E(q)$ and $E(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13$, 15, 17, 19, 23, 25, 29, 31 and 55. They have also established several explicit evaluations for $E(e^{-\pi\sqrt{n}})$, where n is any positive rational.

Lemma 7.1. [12] *We have*

$$E(q) := (1-\alpha)^{1/8} \tag{62}$$

Lemma 7.2. *We have*

$$\frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{2}{1+E^4(q)}. \tag{63}$$

Proof. Using the equations (8), (9) and (62), we arrive at the equation (63). ∎

Lemma 7.3. *For any positive rational number n, we have*

$$E^4(e^{-\pi\sqrt{n/2}}) = \frac{\sqrt{2}-h_{2,n}^2}{h_{2,n}^2}. \tag{64}$$

Proof. Using the equations (63) and (7) with $k = 2$, we arrive at the equation (64). ∎

Theorem 7.4. *If* $e_1 = E^4(q)$ *and* $f_1 = E^2(q^{3/2})$, *then*

$$\begin{aligned} e_1^4\left(f_1^8+1\right) - 8e_1f_1\left(4-3e_1^2\right)\left(f_1^6+1\right) + 4e_1^2f_1^2\left(e_1^2+6\right)\left(f_1^4+1\right) \\ - 56e_1^3f_1^3\left(f_1^2+1\right) + \left(16+48e_1^2+6e_1^4\right)f_1^4 = 0. \end{aligned} \tag{65}$$

Proof. Using the equations (63) and (16), we deduce that

$$\begin{aligned} &-8e^8 - 8e^{12} - e^{16} + 16f^8 - 16PQf^{16}e^4 - 8PQf^{16}e^8 + 14PQf^{16}e^{12} \\ &-32e^4f^4 + 32e^4f^8 - 16f^4e^8 + 40f^4e^{12} - 4f^4e^{16} - 8f^8e^8 - 24f^8e^{12} \\ &-6f^8e^{16} + 64f^{12}e^4 - 72f^{12}e^{12} - 4f^{12}e^{16} - f^{16}e^{16} + 32f^4 + 12PQf^8e^{16} \\ &+16PQf^{12}e^{16} + 6PQf^{16}e^{16} + 8PQe^4f^4 + 20PQf^4e^8 - 4PQf^4e^{12} \\ &-16PQe^4f^8 + 12PQf^8e^8 + 24PQf^8e^{12} - 40PQf^{12}e^4 - 20PQf^{12}e^8 \\ &+36PQf^{12}e^{12} - 4PQe^8 - 6PQe^{12} - 16PQf^4 - 16PQf^8 - 2PQe^{16} = 0, \end{aligned} \tag{66}$$

where P and Q are as defined in (16), $e = E(q)$ and $f = E(q^{3/2})$.
Isolating the terms containing PQ on one side of the equation (66) and squaring both sides, we find that

$$\begin{aligned} &(e^{16} + 16f^8 + 32e^4f^2 - 24f^2e^{12} + 24f^4e^8 + 4f^4e^{16} + 48f^8e^8 + 6f^8e^{16} \\ &+24f^{12}e^8 + 4f^{12}e^{16} + f^{16}e^{16} + 32f^{14}e^4 + 56f^6e^{12} + 56f^{10}e^{12} - 24f^{14}e^{12}) \\ &(e^{16} + 16f^8 - 32e^4f^2 + 24f^2e^{12} + 24f^4e^8 + 4f^4e^{16} + 48f^8e^8 + 6f^8e^{16} \\ &+24f^{12}e^8 + 4f^{12}e^{16} + f^{16}e^{16} - 32f^{14}e^4 \\ &-56f^6e^{12} - 56f^{10}e^{12} + 24f^{14}e^{12}) = 0. \end{aligned} \tag{67}$$

By examining the behaviour of the factors of the equation (67) near $q \to 0$, it can be seen that the second factor vanishes rapidly than the first factor for q sufficiently small. By the Identity Theorem second factor vanishes identically. This completes the proof. ∎

Theorem 7.5. *If* $e_1 = E^4(q)$ *and* $f_1 = E^2(q^{5/2})$, *then*

$$\begin{aligned} &20e_1^3f_1^3\left(93e_1^2-104\right) - 4e_1f_1\left(128-160e_1^2+35e_1^4\right)\left[f_1^{10}+1\right] \\ &\times\left[f_1^6+1\right] + 2e_1^2f_1^2\left(3e_1^4+1310e_1^2-1280\right)\left[f_1^8+1\right] \\ &+15e_1^2f_1^4\left(e_1^4+16e_1^2+16\right)\left[f_1^4+1\right] + 8e_1f_1^5\left(100e_1^2-239e_1^4+40\right) \\ &\times\left[f_1^2+1\right] + e_1^6\left[f_1^{12}+1\right] + 5\left(4e_1^6-1190e_1^4+16+1400e_1^2\right)f_1^6 = 0. \end{aligned} \tag{68}$$

The proof of the equation (68) is similar to the proof of the equation (65) except that in the place of the equation (16), the equation (20) is employed.

Theorem 7.6. *If* $e_1 = E^4(q)$ *and* $f_1 = E^2(q^{7/2})$*, then*

$$\begin{aligned}
&e_1^8\left(f_1^{16}+1\right)+16e_1f_1\left(35e_1^6-420e_1^4+896e_1^2-512\right)\left(f_1^{14}+1\right)\\
&+8e_1^2f_1^2\left(e_1^6+6734e_1^4-5824e_1^2-896\right)\left(f_1^{12}+1\right)\\
&-112e_1^3f_1^3\left(205e_1^4-2792e_1^2+2592\right)\left(f_1^{10}+1\right)\\
&+28e_1^2f_1^4\left(e_1^6+4888e_1^4+3368e_1^2-8192\right)\left(f_1^8+1\right)\\
&+112e_1f_1^5\left(953e_1^6+848e_1^4-1904e_1^2+64\right)\left(f_1^6+1\right)\\
&+56e_1^2f_1^6\left(e_1^6-930e_1^4+912e_1^2+160\right)\left(f_1^4+1\right)\\
&-16e_1^3f_1^7\left(5335e_1^4+25508e_1^2-30128\right)\left(f_1^2+1\right)\\
&+\left(70e_1^8-270144e_1^6-179648e_1^4+462336e_1^2+256\right)f_1^8=0.
\end{aligned} \tag{69}$$

The proof of the equation (69) is similar to the proof of the equation (65) except that in the place of the equation (16), the equation (21) is employed.

In the following corollary, we establish several explicit evaluations of continued fraction of Eisenstein by using the values of $h_{2,n}$ obtained in the Section along with the equation (64).

Corollary 7.7. *We have*

$$\begin{aligned}
E^4\left(e^{-\pi\sqrt{3}/2}\right) &= \left(3+2\sqrt{2}\right)\left(5-2\sqrt{6}\right),\\
E^4\left(e^{-\pi/\sqrt{3}}\right) &= \frac{\sqrt{3}-1}{2\sqrt{2}},\\
E^4\left(e^{-\pi\sqrt{3}}\right) &= \frac{\sqrt{3}+1}{2\sqrt{2}},\\
E^4\left(e^{-\pi/(2\sqrt{3})}\right) &= \left(3-2\sqrt{2}\right)\left(5-2\sqrt{6}\right),\\
E^4\left(e^{-\pi\sqrt{5}/2}\right) &= \sqrt{4965+2220\sqrt{5}+3510\sqrt{2}+1570\sqrt{10}}\\
&\quad-\sqrt{2196\sqrt{5}+4908+1552\sqrt{10}+3472\sqrt{2}},\\
E^4\left(e^{-\pi\sqrt{7}/2}\right) &= \left(17-12\sqrt{2}\right)\left(15+4\sqrt{14}\right),\\
E^4\left(e^{-\pi/\sqrt{7}}\right) &= \frac{3-\sqrt{7}}{4\sqrt{2}},\\
E^4\left(e^{-\pi\sqrt{7}}\right) &= \frac{3+\sqrt{7}}{4\sqrt{2}},\\
E^4\left(e^{-\pi/(2\sqrt{7})}\right) &= \left(17-12\sqrt{2}\right)\left(15-4\sqrt{14}\right),
\end{aligned}$$

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