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A Note on the Characterization of Zero-Inflated Poisson Model

G. Nanjundan, Sadiq Pasha

Department of Statistics, Bangalore University, Bangalore, India
Email: nanzundan@gmail.com

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Abstract

Zero-Inflated Poisson model has found a wide variety of applications in recent years in statistical analyses of count data, especially in count regression models. Zero-Inflated Poisson model is characterized in this paper through a linear differential equation satisfied by its probability generating function [1] [2].

Keywords

Zero-Inflated Poisson Model, Probability Generating Function, Linear Differential Equation

1. Introduction

A random variable X is said to have a zero-inflated Poisson distribution if its probability mass function is given by

$$p(x;\theta,\varphi)=\begin{cases}\varphi+(1-\varphi)\mathrm{e}^{-\theta}, & x=0\\ (1-\varphi)\dfrac{\mathrm{e}^{-\theta}\theta^{x}}{x!}, & x=1,2,\cdots\end{cases} \tag{1}$$

$$=\varphi p_0(x)+(1-\varphi)p_1(x),\quad 0<\varphi<1$$

where $p_0(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0\end{cases}$ and $p_1(x)=\dfrac{\mathrm{e}^{-\theta}\theta^{x}}{x!}$, $x=0,1,2,\cdots$, $\theta>0$.

Thus, the distribution of X is a mixture of a distribution degenerate at zero and a Poisson distribution with mean θ.

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2. Probability Generating Function

The probability generating function (pgf) of X is given by

$$\begin{aligned} f(s) &= E\left(s^X\right) \\ &= \sum_{k=0}^{\infty} p_k s^k \\ &= \varphi + (1-\varphi)\mathrm{e}^{-\theta} + (1-\varphi)\mathrm{e}^{-\theta}\sum_{k=1}^{\infty}\frac{(\theta s)^k}{k!}, \quad 0 < s < 1. \end{aligned}$$

$$f(s) = \varphi + (1-\varphi)\mathrm{e}^{\theta(s-1)}.$$

3. Characterization

Let X be a non-negative integer valued random variable with $0 < P(X = 0) < 1$ and the pgf $f(s)$. Then, the distribution of X is zero-inflated Poisson if and only if $f(s) = a + bf'(s)$, where $0 < a < 1$, b are constants and $f'(s)$ is the first derivative of $f(s)$.

Proof:

1) Suppose that X has a zero-inflated Poisson distribution specified in (1.1). Then the pgf of X is given by

$$f(s) = \varphi + (1-\varphi)\mathrm{e}^{\theta(s-1)}$$

On differentiation, we get

$$\begin{aligned} f'(s) &= (1-\varphi)\mathrm{e}^{\theta(s-1)}\theta \\ &= \theta\left\{f(s) - \varphi\right\}, \end{aligned}$$

$$f(s) = \varphi + \frac{1}{\theta}f'(s).$$

Hence $f(s)$ satisfies the linear differential equation

$$f(s) = a + bf'(s). \tag{2}$$

2) Suppose that the pgf $f(s)$ of X satisfies

$$f(s) = a + bf'(s).$$

If $b = 0$, then $f(s) = a$ and in turn $f(0) = f(1) = a$. By the property of the pgf, $f(1) = 1 = a$. But $f(0) = P(X = 0) = a$, which is not possible because $P(X = 0) < 1$.

Therefore $b \neq 0$.

3) The Linear Differential Equation

The linear differential equation $f(s) = a + bf'(s)$ is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

where P and Q are functions of x.

Then its solution is given by

$$y\mathrm{e}^{\int P\mathrm{d}x} = \int Q\,\mathrm{e}^{\int P\mathrm{d}x}\mathrm{d}x + c,$$

where c is an arbitrary constant.

Here

$$bf'(s) - f(s) = -a$$

$$\Rightarrow f'(s) - \frac{1}{b} f(s) = -\frac{a}{b}.$$

Hence $P = -\frac{1}{b}$, $Q = -\frac{a}{b}$.

Therefore the solution of the Equation (2) is given by

$$f(s) = a + c\mathrm{e}^{s/b}.$$

We now extract the probabilities $P(X = k) = p_k$, $k = 0,1,2,\cdots$ using the above solution.

Since $f(s)$ is a pgf, $p_k = \frac{f^{(k)}(0)}{k!}$, where $f^{(k)}(s)$ is the k-th derivative of $f(s)$.

We get

$$p_0 = a + c,\quad p_1 = c\frac{1}{b},\quad p_2 = c\frac{1}{2!}\frac{1}{b^2}, \text{ and so on.}$$

Now,

$$f(s) = \sum_{k=0}^{\infty} p_k = a + c\mathrm{e}^{1/b}$$

Since $f(1) = 1$, it is easy to see that $c = (1-a)\mathrm{e}^{-1/b}$,

We have

$$p_k = \begin{cases} a + (1-a)\mathrm{e}^{-1/b}, & k = 0; \\ (1-a)\dfrac{\mathrm{e}^{-1/b}(1/b)^k}{k!}, & k = 1,2,3,\cdots. \end{cases}$$

with $\varphi = a$ and $\theta = 1/b$.

Therefore X has the pgf specified in Equation (1). □

References

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