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SOME IDENTITIES FOR RAMANUJAN - GÖLLNITZ - GORDON CONTINUED FRACTION

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ABSTRACT. In this paper, we obtain certain P–Q eta–function identities, using which we establish identities providing modular relations between Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$ and 55.

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1. INTRODUCTION

In Chapter 16, of his Second notebook [15] Ramanujan has defined his theta-function as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \tag{1.1}$$

Following Ramanujan, we define

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) = \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}} = (q;q)_\infty, \tag{1.4}$$

$$\chi(q) := (-q;q^2)_\infty, \tag{1.5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

On page 229 of his second notebook, Ramanujan recorded the continued fraction which is known as Ramanujan-Göllnitz-Gordon continued fraction along with the two identities as follows:

$$H(q) := \frac{q^{\frac{1}{2}}}{1+q_+}\frac{q^2}{1+q^3{}_+}\frac{q^4}{1+q^5{}_+}\frac{q^6}{1+q^7{}_+}\cdots, \qquad |\, q \,| < 1, \tag{1.6}$$

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{\frac{1}{2}}\psi(q^4)}, \tag{1.7}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{\frac{1}{2}}\psi(q^4)}. \tag{1.8}$$

Proofs of the above identities (1.6), (1.7) and (1.8) can be found in [15, p. 221] and for more details one can see, [8] and [18].

Now we define modular equation in brief. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k,\ l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{1.9}$$

Then a modular equation of degree n is a relation between k and l induced by (1.9). Following Ramanujan, we set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.

Recently, in [10] M. S. Mahadeva Naika, S. Chandankumar and M. Manjunatha have established some new modular relations for Ramanujan–Göllnitz–Gordon continued fraction $H(q)$ with $H(q^n)$ for $n = 6, 10, 14$ and 16 and also established their explicit evaluations.

In this paper, we prove the relations between $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$ and 55.

2. PRELIMINARY SECTION

Lemma 2.1.

$$f(a,b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right). \tag{2.1}$$

For a proof of (2.1) , see [5, Entry 30(ii),(iii), p.46].

Lemma 2.2.

$$\varphi(q) = \sqrt{z} \tag{2.2}$$

and

$$\varphi(-q) = \sqrt{z}(1-t)^{1/4}. \tag{2.3}$$

For the proofs of (2.2) and (2.3), see [5, Entry 10(i),(ii), p.122].

Lemma 2.3. *If β is of degree 2 over α, then*

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}. \tag{2.4}$$

For a proof of (2.4), see [4, Entry 17.3.1, p.385].

Lemma 2.4. *If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.5}$$

For a proof of (2.5), see [5, Entry 5(ii), p.230].

Lemma 2.5. *If β has degree 4 over $alpha$, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1-\alpha)}. \tag{2.6}$$

For a proof of (2.6), see [4, Entry 17.3.2, p.385].

Lemma 2.6. *If β has degree 5 over α, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{2.7}$$

For a proof of (2.7), see [5, Entry 13(i), p.280].

Lemma 2.7. *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.8}$$

For a proof of (2.8), see [5, Entry 19(i), p.314].

Lemma 2.8. *If β has degree 8 over α, then*

$$(1-(1-\alpha)^{1/4})(1-\beta^{1/4}) = 2\sqrt{2}(\beta(1-\alpha))^{1/8}. \tag{2.9}$$

For a proof of (2.9), see [5].

Lemma 2.9. *If β has degree 9 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}. \tag{2.10}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \tag{2.11}$$

For the proofs of (2.10) and (2.11), see [5, Entry 3(x),(xi), p.352].

Lemma 2.10. *If β has degree 11 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{2.12}$$

For a proof of (2.12), see [5, Entry 7, p.363].
Let

$$U = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}, \tag{2.13}$$

$$V = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{2.14}$$

and

$$W = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.15}$$

Lemma 2.11. *Let U, V and W be defined as in* (2.13)–(2.15)*. If β has degree 13 over α, then*

$$\sqrt{U}(U^3 + 8W) = \sqrt{W}(11U^2 + V). \tag{2.16}$$

For a proof of (2.16), see [4, Entry 63, p.387].

Lemma 2.12. *Let α and β has degrees 3, 5 or 1, 15 respectively, then*

$$\begin{aligned}&(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \pm \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\ &= \left\{\frac{1}{2}\left(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right)\right\}^{1/2},\end{aligned} \tag{2.17}$$

where the minus sign is chosen in the first case and the plus sign is selected in the second case.

For a proof of (2.17), see [5, Entry 21, p.435].

Lemma 2.13. *If β has degree 17 over α, then*

$$\begin{aligned} m &= \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{(1-\beta)}{(1-\alpha)}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\\ &\quad -2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}.\end{aligned} \tag{2.18}$$

$$\begin{aligned} \frac{17}{m} &= \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\\ &\quad -2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}\left\{1 + \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}.\end{aligned} \tag{2.19}$$

For the proofs of equations (2.18) and (2.19), see [4, Entry 17.3.26, p.392].
Let

$$X = 1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}, \tag{2.20}$$

$$Y = 16\left(\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{2.21}$$

and

$$Z = 16\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}. \tag{2.22}$$

Lemma 2.14. *Let X, Y and Z be defined as in* (2.20)-(2.22). *If β has degree 19 over α, then*

$$X^5 - 7X^2Z - YZ = 0. \tag{2.23}$$

For a proof of (2.23), see [7, Entry 58, p.386].

Lemma 2.15. *If β has degree 23 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{2.24}$$

For a proof of (2.24), see [5, Entry 15, p.411].

Lemma 2.16. *If β has degree 25 over α, then*

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = \sqrt{m}. \tag{2.25}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{\sqrt{m}}. \tag{2.26}$$

For the proofs of (2.25) and (2.26), see [4, Entry 17.3.27, p.392].

Lemma 2.17. *Let U, V and W be defined as in* (2.13)–(2.15). *If β has degree 29 over α, then*

$$\sqrt{U}(U^2 + 17UW^{1/3} - 9W^{2/3}) - W^{1/6}(9X^2 + Y - 13UW^{1/3} + 15W^{2/3}) = 0. \tag{2.27}$$

For a proof of (2.27), see [7, Entry 65, p.388].
Let

$$L = 1 - \{\alpha\beta\}^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}, \tag{2.28}$$

$$\begin{aligned} M = &\left\{\frac{1}{2}\left(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right)\right\}^{1/2} - \{\alpha\beta\}^{1/8} \\ &- \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}, \end{aligned} \tag{2.29}$$

and

$$N = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{2.30}$$

Lemma 2.18. *Let L, M and N be defined as in* (2.28)–(2.30). *If β has degree 55 over α, then*

$$M(L^2 - M)^2 = N(L^3 - N). \tag{2.31}$$

For a proof of (2.31), see [4, Entry 17.4.4, p.393].

3. MAIN RESULTS

In this section, we establish several new theta function identities which are useful in proving certain identities for $H(q)$.

Lemma 3.1.

$$\frac{\varphi(-\sqrt{q})}{\varphi(\sqrt{q})} = \frac{1 - 2H(q) - H^2(q)}{1 + 2H(q) - H^2(q)}. \tag{3.1}$$

Proof. Putting $a = b = q$ in the equation (2.1), we deduce that

$$f(q,q) = f(q^4,q^4) + qf(1,q^8). \tag{3.2}$$

Replacing q by $-q$ in the equation (3.2), we find that

$$f(-q,-q) = f(q^4,q^4) - qf(1,q^8). \tag{3.3}$$

From the equations (1.7), (3.2) and (3.3), we obtain (3.1). □

Lemma 3.2. *If* $0 < t < 1$, *then*

$$\frac{\varphi(-q)}{\varphi(q)} = (1-t)^{1/4}. \tag{3.4}$$

Proof. From the equations (2.2) and (2.3), we obtain (3.4). □

Set

$$P = \{(1-\alpha)(1-\beta)\}^{1/4} \tag{3.5}$$

and

$$Q = \left\{\frac{1-\alpha}{1-\beta}\right\}^{1/4}. \tag{3.6}$$

where β is of degree n over α.
Using the equations (3.5) and (3.6), we find that

$$\alpha = 1 - P^2Q^2 \tag{3.7}$$

and

$$\beta = 1 - \frac{P^2}{Q^2}. \tag{3.8}$$

Throughout this article we are using the following notations:

$$A = \sqrt[4]{1-\alpha},\ \ B = \sqrt[4]{\beta},\ \ B_2 = \sqrt{\beta},\ \ a_1 = \sqrt[8]{\alpha\beta},\ \ a_2 = \sqrt[4]{\alpha\beta},\ \ a_4 = \sqrt{\alpha\beta},$$

$$b_1 = \sqrt[8]{(1-\alpha)(1-\beta)},\ \ b_2 = \sqrt[8]{\frac{1-\alpha}{1-\beta}},\ \ c_1 = \sqrt[8]{\frac{\alpha}{\beta}},\ \ c_2 = \sqrt[4]{\frac{\alpha}{\beta}},\ \ c_3 = \sqrt{\frac{\alpha}{\beta}}.$$

Lemma 3.3. *Let P and Q be defined as in* (3.5) *and* (3.6). *If* β *is of degree 2 over* α, *then*

$$\frac{2Q}{P} = \left(\sqrt{PQ} + \frac{1}{\sqrt{PQ}}\right). \tag{3.9}$$

Proof. The equation (2.4) can be rewritten as

$$\beta = \left[\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right]^2. \tag{3.10}$$

Employing the equations (3.7) and (3.8) in the above equation (3.10), we deduce that

$$1 - \frac{P^2}{Q^2} = \left(\frac{1-PQ}{1+PQ}\right)^2. \tag{3.11}$$

On simplification, we obtain the equation (3.9). □

Lemma 3.4. *Let P and Q be defined as in* (3.5) *and* (3.6). *If* β *is of degree 3 over* α, *then*

$$Q^2 + \frac{1}{Q^2} + 6 = 4\left(P + \frac{1}{P}\right). \tag{3.12}$$

Proof. Employing the equations (3.5), (3.7) and (3.8) in the equation (2.5), we find that

$$(1-P^2Q^2)\left(1 - \frac{P^2}{Q^2}\right) = (1-P)^4. \tag{3.13}$$

On simplification, we obtain the required equation (3.12). □

Lemma 3.5. *Let P and Q be defined as in* (3.5) *and* (3.6). *If β is of degree 4 over α, then*

$$64(1+PQ)^2\left(1-\frac{Q^2}{P^2}\right)+4\frac{P}{Q}=\frac{P}{Q}\left(PQ+\frac{1}{PQ}\right)\left[4-\frac{P}{Q}\left(PQ+\frac{1}{PQ}\right)\right]. \tag{3.14}$$

Proof. The equation (2.6) can be rewritten as

$$\beta=\left[\frac{1-\sqrt[4]{1-\alpha}}{1+\sqrt[4]{1-\alpha}}\right]^4. \tag{3.15}$$

Employing the equations (3.7) and (3.8) in the above equation (3.15), we find that

$$1-\frac{P^2}{Q^2}=\left(\frac{1-A}{1+A}\right)^4. \tag{3.16}$$

Isolating the terms having A on one side of the equation (3.16) and then squaring both sides, we obtain the equation (3.14). □

Lemma 3.6. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 5 over α, then*

$$Q^3+\frac{1}{Q^3}+10\left(Q^2+\frac{1}{Q^2}\right)+15\left(Q+\frac{1}{Q}\right)+20=16\left(P^2+\frac{1}{P^2}\right). \tag{3.17}$$

Proof. From the equations (3.5) and (2.7), we find that

$$(32a_4P)-(1-a_4-P^2)^3=0. \tag{3.18}$$

Isolating the terms having a_4 on one side of the equation (3.18) and then squaring both sides, we find that

$$\begin{aligned}&(16P^4Q^3-P^2-10P^2Q-15P^2Q^2+16Q^3\\ +\;&20P^2Q^3-15P^2Q^4-10P^2Q^5-Q^6P^2)\\ &(16P^4Q^3+P^2-10P^2Q+16Q^3+20P^2Q^3\\ +\;&15P^2Q^4-10P^2Q^5+Q^6P^2+15P^2Q^2)=0.\end{aligned} \tag{3.19}$$

By examining the behaviour of first factor near $q=0$, it can be seen that there is a neighbourhood about the origin, where the first factor is zero, whereas the second factor is not zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. This completes the proof. □

Lemma 3.7. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 7 over α, then*

$$\begin{aligned}&Q^4+\frac{1}{Q^4}+140\left(Q^2+\frac{1}{Q^2}\right)+70\\ =\;&64\left(P^3+\frac{1}{P^3}\right)+112\left(P^2+\frac{1}{P^2}\right)+56\left(P+\frac{1}{P}\right)\left[2-\left(Q^2+\frac{1}{Q^2}\right)\right].\end{aligned} \tag{3.20}$$

Proof. Employing the equations (3.5) and (3.6) in the equation (2.7), we deduce that

$$(1-P^2Q^2)\left(1-\frac{P^2}{Q^2}\right)=(1-b_1)^8. \tag{3.21}$$

On simplification, we find that

$$\begin{aligned}&-P^2-P^2Q^4+8Q^2b_1-28PQ^2+56Q^2Pb_1\\ -\;&70P^2Q^2+56Q^2P^2b_1-28P^3Q^2+8Q^2P^3b_1=0.\end{aligned} \tag{3.22}$$

Isolating the terms having b_1 on one side of the equation (3.22), and then squaring both sides, and employing the equation (3.5), we arrive at the equation (3.20). □

Lemma 3.8. *Let P and Q be defined in* (3.5) *and* (3.6), *If β is of degree 8 over α, then*

$$\begin{aligned}(3.23)\quad & P^{15}Q^8-65536P^7Q^{12}-32768P^4Q^9-32768P^6Q^{11}\\ &+131072P^5Q^{10}-471040P^7Q^8-242688P^6Q^7-242688P^8Q^9-26624P^9Q^{10}\\ &+259072P^8Q^5+405504P^9Q^6+259072P^{10}Q^7+59392P^{11}Q^8+16384Q^{13}P^4\\ &-16384Q^{15}P^6+32768Q^{13}P^8+32768Q^{14}P^5-19456Q^{11}P^{10}+3072Q^9P^{12}\\ &-65536Q^{12}P^3-16384Q^9+16384Q^{11}P^2+32768Q^{10}P+70P^{11}Q^4-56P^{10}Q^3\\ &-56P^{12}Q^5+28P^9Q^2+28P^{13}Q^6-8P^8Q-8P^{14}Q^7+59392P^7Q^4-65536P^3Q^8\\ &+32768P^2Q^7+P^7-19456P^4Q^5-26624P^5Q^6+3072P^6Q^3=0.\end{aligned}$$

Proof. Using the equations (3.5) and (3.6) in the equation (2.9), we deduce that

$$(3.24)\qquad 1-2B+B_2-2A-4AB-2AB_2+PQ-2PQB+PQB_2=0.$$

Isolating the terms having A on one side of the equation (3.24), and then squaring both sides, we find that

$$\begin{aligned}(3.25)\quad & 4PQ^3+24PQ^3B+12PQ^3B_2+24PQ^3BB_2-2P^3Q-2Q^2+4P^2Q^4BB_2+P^2\\ -\quad & 6Q^2B_2+4Q^2BB_2-2P^2Q^4+4Q^2B-6P^2Q^4B_2+4P^2Q^4B+P^4Q^2=0.\end{aligned}$$

Isolating the terms having B on one side of the equation (3.25), and then squaring both sides, we deduce that

$$\begin{aligned}(3.26)\quad & 8P^2Q^2-4P^2Q^2B_2+976P^2Q^6+544PQ^5+544P^3Q^7-8P^4Q^8-544P^5Q^5\\ &-976P^4Q^4-P^4+1072P^2Q^6B_2+4P^7Q^3+8P^6Q^6-6P^6Q^2+4P^5Q+8Q^4B_2\\ &+8P^4Q^8B_2+480PQ^5B_2-8Q^4-240P^5Q^5B_2-536P^4Q^4B_2-4P^6Q^6B_2-P^8Q^4\\ &-240P^3Q^3B_2-544P^3Q^3+480P^3Q^7B_2=0.\end{aligned}$$

Isolating the terms having B_2 on one side of the equation (3.26), and then squaring both sides, we arrive at (3.23). □

Lemma 3.9. *Let P and Q be defined in* (3.5) *and* (3.6), *If β is of degree 9 over α, then*

$$\begin{aligned}(3.27)\quad & Q^6+\frac{1}{Q^6}-44\left(Q^5+\frac{1}{Q^5}\right)+514\left(Q^4+\frac{1}{Q^4}\right)-1212\left(Q^3+\frac{1}{Q^3}\right)\\ &+111\left(Q^2+\frac{1}{Q^2}\right)-2328\left(Q+\frac{1}{Q}\right)+1308=256\left(P^4+\frac{1}{P^4}\right)\left[\left(Q+\frac{1}{Q}\right)+1\right]\\ &-96\left(P^2+\frac{1}{P^2}\right)\left[3\left(Q^3+\frac{1}{Q^3}\right)-2\left(Q^2+\frac{1}{Q^2}\right)+13\left(Q+\frac{1}{Q}\right)+4\right].\end{aligned}$$

Proof. From the equations (2.10) and (2.11), we deduce that

$$(3.28)\qquad -c_2+c_1-Q+b_1+c_1Q+c_2b_1=0.$$

Isolating the terms having b_1 on one side of the equation (3.28) and then squaring both sides, we deduce that

$$(3.29)\qquad Q+Qc_3-c_3+2c_2c_1+2c_2c_1Q-c_2+2c_1Q-2Qc_2-Q^2+2c_1Q^2-c_2Q^2=0.$$

Again isolating the terms having c_1 on one side of the equation (3.29) and then squaring both sides, we deduce that

$$\begin{aligned}(3.30)\quad & -6Q^2c_3c_2P^2 - 6Qc_3c_2P^2 - 2Q^3c_3c_2P^2 + 2Q^2c_3c_2 + 6Q^3c_3c_2 - Q^2 + P^2Q^2 \\ & +2Q^3c_2 + 6c_2Q^4 - Q^2c_3 + 6Q^3c_3 + 2Q^3 - 2Q^4 + 6c_3Q^4 + c_3P^2 + 2Q^4P^2 + 6Q^5c_2 \\ & +2c_2Q^6 + 6Q^5c_3 - 2Q^3P^2 + Q^6P^2 - c_3Q^6 - 2Q^5P^2 + 2Q^5c_3c_2 - 6Q^3c_2P^2 \\ & -6Qc_3P^2 - 2Qc_2P^2 - 6c_2Q^2P^2 + 6Q^4c_3c_2 - 2c_2Q^4P^2 - 6Q^2c_3P^2 - 6Q^3c_3P^2 \\ & -2c_3c_2P^2 + c_3Q^4P^2 + 2Q^5 - Q^6 = 0\end{aligned}$$

Eliminating c_2 and c_3 in the above equation (3.30) , we deduce that

$$\begin{aligned}(3.31)\quad & (1+Q)^4(192Q^4P^2 - 288Q^3P^2 - 384Q^6P^2 - 1248Q^5P^2 + 256Q^5 + 256Q^6 \\ & +256Q^7 - 288Q^9P^2 - 1308Q^6P^4 - 111Q^4P^4 + 2328Q^5P^4 + 2328Q^7P^4 \\ & -514Q^2P^4 - P^4 - 514Q^{10}P^4 + 1212Q^9P^4 - Q^{12}P^4 + 44Q^{11}P^4 + 44QP^4 \\ & -1248Q^5P^6 + 192Q^8P^6 - 288Q^9P^6 - 384Q^6P^6 + 192Q^4P^6 - 288Q^3P^6 \\ & +256Q^5P^8 + 256Q^6P^8 + 256Q^7P^8 - 1248Q^7P^2 - 111Q^8P^4 - 1248Q^7P^6 \\ & +192Q^8P^2 + 1212Q^3P^4) = 0.\end{aligned}$$

Since $(1+Q)^4 \neq 0$, it completes the proof. □

Lemma 3.10. *Let P and Q be defined in* (3.5) *and* (3.6), *If β is of degree 11 over α, then*

$$\begin{aligned}(3.32)\quad & Q^6 + \frac{1}{Q^6} + 1298\left(Q^4 + \frac{1}{Q^4}\right) + 1915\left(Q^2 + \frac{1}{Q^2}\right) - 7876 \\ = \; & 1024\left(P^5 + \frac{1}{P^5}\right) + 5632\left(P^4 + \frac{1}{P^4}\right) - 704\left(P^3 + \frac{1}{P^3}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) - 23\right] \\ & -176\left(P^2 + \frac{1}{P^2}\right)\left[25\left(Q^2 + \frac{1}{Q^2}\right) - 106\right] \\ & +44\left(P + \frac{1}{P}\right)\left[9\left(Q^4 + \frac{1}{Q^4}\right) - 372\left(Q^2 + \frac{1}{Q^2}\right) + 422\right] = 0.\end{aligned}$$

Proof. Employing the equation (3.5) in the equation (2.12), we find that

$$(3.33)\qquad 16a_2P = (1 - a_2 - P)^3.$$

Isolating the terms having a_2 on one side of the equation (3.33) and then squaring both sides, we find that

$$\begin{aligned}(3.34)\quad & 38P^3 + 4Q^2 + 18P^2Q^2 + a_4P^2 - 84a_4P^3Q^2 - 82a_4P^2Q^2 - 4a_4P^4Q^2 \\ & +a_4P^2Q^4 + 38P^3Q^4 - 84a_4PQ^2 - 3P^4 - 4a_4Q^2 - 44PQ^2 - 3P^2 \\ & -20P^3Q^2 + 18P^4Q^2 - 44P^5Q^2 + 4P^6Q^2 - 3P^2Q^4 - 3P^4Q^4 = 0.\end{aligned}$$

Isolating the terms having a_4 on one side of the equation (3.34) and then squaring both sides, we obtain the equation (3.32). □

Lemma 3.11. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 13 over α, then*

$$\begin{aligned} (3.35)\quad & Q^7+\frac{1}{Q^7}+130\left(Q^6+\frac{1}{Q^6}\right)+5083\left(Q^5+\frac{1}{Q^5}\right)+62036\left(Q^4+\frac{1}{Q^4}\right)\\ &+195689\left(Q^3+\frac{1}{Q^3}\right)+289822\left(Q^2+\frac{1}{Q^2}\right)+289822\left(Q^2+\frac{1}{Q^2}\right)\\ &+149435\left(Q+\frac{1}{Q}\right)-11752\\ =\ & 4096\left(P^6+\frac{1}{P^6}\right)-6656\left(P^4+\frac{1}{P^4}\right)\left[\left(Q^2+\frac{1}{Q^2}\right)-4\left(Q+\frac{1}{Q}\right)-10\right]\\ &+208\left(P^2+\frac{1}{P^2}\right)\left[13\left(Q^4+\frac{1}{Q^4}\right)-104\left(Q^3+\frac{1}{Q^3}\right)-20\left(Q^2+\frac{1}{Q^2}\right)\right.\\ &\left.+128\left(Q+\frac{1}{Q}\right)+1422\right]. \end{aligned}$$

Proof. Employing the equation (3.5) in the equations (2.13), (2.14) and (2.15), we find that

$$(3.36)\qquad U=1-a_4-P^2,\quad V=64(a_4+P^2-a_4P^2),\quad \text{and}\quad W=32a_4P^2.$$

Employing the equation (3.36) in the equation (2.16), we find that

$$\begin{aligned} (3.37)\quad & 7P^6-7P^8-a_4P^6-67616a_4P^8Q^4+11345a_4P^6Q^4+24a_4P^4Q^2\\ &-67616a_4P^4Q^4-80a_4P^2Q^4-20528P^4Q^6+64a_4Q^6+147300P^6Q^6\\ &-147300P^8Q^6+20528P^{10}Q^6-63456P^{12}Q^6+64P^{14}Q^6+63456P^2Q^6\\ &-95184P^4Q^4+31742P^6Q^2-95184P^4Q^8-52137P^8Q^4+112Q^4P^2-56Q^2P^4\\ &+112Q^8P^2+52137Q^4P^6-56Q^{10}P^4+52137Q^8P^6-52137Q^8P^8+56P^{10}Q^2\\ &-112P^{12}Q^4-64Q^6-112P^{12}Q^8+95184P^{10}Q^4+95184P^{10}Q^8-31742Q^2P^8\\ &-31742Q^{10}P^8+31742P^6Q^{10}+7P^6Q^{12}-67616a_4P^4Q^8+67616a_4P^2Q^6\\ &+20432a_4P^4Q^6+196212a_4P^6Q^6+20432a_4P^8Q^6+67616a_4P^{10}Q^6-80a_4P^{10}Q^4\\ &-80a_4P^{10}Q^8-67616a_4P^8Q^8+24a_4P^8Q^2+11345a_4P^6Q^8+24a_4P^8Q^{10}\\ &+56P^{10}Q^{10}-7P^8Q^{12}+3318a_4P^6Q^2+3318a_4P^6Q^{10}-80a_4Q^8P^2+24a_4Q^{10}P^4\\ &-a_4P^6Q^{12}+64a_4P^{12}Q^6=0. \end{aligned}$$

Isolating the terms having a_4 on one side of the equation (3.37) and then squaring both sides, we find that

$$\begin{aligned} (3.38)\quad & (168064P^4Q^6-130P^6Q^{13}-149435P^6Q^6+168064P^8Q^6+26624P^{10}Q^6\\ &+26624P^2Q^6-21632P^4Q^4-5083P^6Q^2+168064P^4Q^8-21632P^8Q^4\\ &+26624Q^8P^2-195689Q^4P^6-21632Q^{10}P^4-149435Q^8P^6+168064Q^8P^8\\ &+26624P^{10}Q^8-21632Q^{10}P^8-195689P^6Q^{10}-5083P^6Q^{12}+2704P^4Q^3\\ &-62036P^6Q^3-6656Q^5P^2-4160Q^5P^4-Q^{14}P^6+2704P^8Q^3-130P^6Q\\ &-289822P^6Q^5+66560Q^7P^2+295776Q^7P^4+11752P^6Q^7-4160P^8Q^5 \end{aligned}$$

$$+295776P^8Q^7-289822P^6Q^9-6656P^{10}Q^5+66560P^{10}Q^7-4160P^8Q^9-4160P^4Q^9$$
$$-62036P^6Q^{11}+4096P^{12}Q^7-6656P^{10}Q^9+2704Q^{11}P^8-6656Q^9P^2+2704Q^{11}P^4$$
$$-130P^6Q^{13}+4096Q^7)(P^6-168064P^4Q^6+149435P^6Q^6-168064P^8Q^6$$
$$-26624P^{10}Q^6-26624P^2Q^6+21632P^4Q^4+5083P^6Q^2-168064P^4Q^8+21632P^8Q^4$$
$$-26624Q^8P^2+195689Q^4P^6+21632Q^{10}P^4+149435Q^8P^6-168064Q^8P^8$$
$$-26624P^{10}Q^8+21632Q^{10}P^8+195689P^6Q^{10}+5083P^6Q^{12}+2704P^4Q^3-62036P^6Q^3$$
$$-6656Q^5P^2-4160Q^5P^4-289822P^6Q^9+2704P^8Q^3-130P^6Q-289822P^6Q^5$$
$$+295776Q^7P^4+11752P^6Q^7-4160P^8Q^5+295776P^8Q^7-6656P^{10}Q^5+2704Q^{11}P^4$$
$$+Q^{14}P^6-4160P^8Q^9-4160P^4Q^9-62036P^6Q^{11}+4096P^{12}Q^7-6656P^{10}Q^9$$
$$+2704Q^{11}P^8-6656Q^9P^2+66560Q^7P^2+66560P^{10}Q^7+4096Q^7-P^6)=0.$$

By examining the behaviour of second factor near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. This completes the proof. □

Lemma 3.12. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 15 over α, then*

$$(3.39)\quad Q^{12}+\frac{1}{Q^{12}}+8100\left(Q^{10}+\frac{1}{Q^{10}}\right)+1630210\left(Q^8+\frac{1}{Q^8}\right)$$
$$-\quad 6072300\left(Q^6+\frac{1}{Q^6}\right)+27556590\left(Q^4+\frac{1}{Q^4}\right)-29064120\left(Q^2+\frac{1}{Q^2}\right)+75698716$$
$$=\quad 65536\left(P^8+\frac{1}{P^8}\right)-16384\left(P^7+\frac{1}{P^7}\right)\left[\left(Q^4+\frac{1}{Q^4}\right)+45\right]$$
$$+4096\left(P^6+\frac{1}{P^6}\right)\left[45\left(Q^4+\frac{1}{Q^4}\right)-40\left(Q^2+\frac{1}{Q^2}\right)+991\right]$$
$$+10240\left(P^5+\frac{1}{P^5}\right)\left[3\left(Q^6+\frac{1}{Q^6}\right)+99\left(Q^4+\frac{1}{Q^4}\right)+226\left(Q^2+\frac{1}{Q^2}\right)-1092\right]$$
$$-256\left(P^4+\frac{1}{P^4}\right)\left[1764\left(Q^6+\frac{1}{Q^6}\right)-11415\left(Q^4+\frac{1}{Q^4}\right)+45160\left(Q^2+\frac{1}{Q^2}\right)-67210\right]$$
$$-64\left(P^3+\frac{1}{P^3}\right)\left[255\left(Q^8+\frac{1}{Q^8}\right)+32740\left(Q^6+\frac{1}{Q^6}\right)\right.$$
$$\left.+280170+99180\left(Q^4+\frac{1}{Q^4}\right)+434364\left(Q^2+\frac{1}{Q^2}\right)\right]$$
$$+80\left(P^2+\frac{1}{P^2}\right)\left[3243\left(Q^8+\frac{1}{Q^8}\right)-55544\left(Q^6+\frac{1}{Q^6}\right)\right.$$
$$\left.+168116\left(Q^4+\frac{1}{Q^4}\right)-455244\left(Q^2+\frac{1}{Q^2}\right)+367554\right]$$
$$+8\left(P+\frac{1}{P}\right)\left[245\left(Q^{10}+\frac{1}{Q^{10}}\right)-135698\left(Q^8+\frac{1}{Q^8}\right)\right.$$
$$\left.+726705\left(Q^6+\frac{1}{Q^6}\right)-2845080\left(Q^4+\frac{1}{Q^4}\right)+4373850\left(Q^2+\frac{1}{Q^2}\right)-7244460\right].$$

Lemma 3.13. *Let P and Q be defined in* (3.5) *and* (3.6), *If β is of degree 17 over α, then*

$$\begin{aligned}(3.40)\quad & Q^9+\frac{1}{Q^9}-306\left(Q^8+\frac{1}{Q^8}\right)+28441\left(Q^7+\frac{1}{Q^7}\right)-793968\left(Q^6+\frac{1}{Q^6}\right)\\ &+2120308\left(Q^5+\frac{1}{Q^5}\right)+12298888\left(Q^4+\frac{1}{Q^4}\right)+33457156\left(Q^3+\frac{1}{Q^3}\right)\\ &+27917808\left(Q^2+\frac{1}{Q^2}\right)+9441902\left(Q+\frac{1}{Q}\right)\end{aligned}$$

$$\begin{aligned}=\ & 65536\left(P^8+\frac{1}{P^8}\right)-835584\left(P^6+\frac{1}{P^6}\right)\left[\frac{1}{6}\left(Q^2+\frac{1}{Q^2}\right)+\left(Q+\frac{1}{Q}\right)+3\right]\\ &-835584\left(P^6+\frac{1}{P^6}\right)\times\left[\frac{1}{6}\left(Q^2+\frac{1}{Q^2}\right)+\left(Q+\frac{1}{Q}\right)+3\right]\\ &+4352\left(P^4+\frac{1}{P^4}\right)\left[21\left(Q^4+\frac{1}{Q^4}\right)+475\left(Q^3+\frac{1}{Q^3}\right)\right.\\ &\qquad\left.+2298\left(Q^2+\frac{1}{Q^2}\right)+3477\left(Q+\frac{1}{Q}\right)+4866\right]\\ &-544\left(P^2+\frac{1}{P^2}\right)\left[33\left(Q^6+\frac{1}{Q^6}\right)+2212\left(Q^5+\frac{1}{Q^5}\right)+15234\left(Q^4+\frac{1}{Q^4}\right)\right.\\ &\qquad\left.+22388\left(Q^3+\frac{1}{Q^3}\right)+16015\left(Q^2+\frac{1}{Q^2}\right)-35864\left(Q+\frac{1}{Q}\right)-4608\right].\end{aligned}$$

Lemma 3.14. *Let P and Q be defined in* (3.5) *and* (3.6), *If β is of degree 19 over α, then*

$$\begin{aligned}(3.41)\quad & Q^{10}+\frac{1}{Q^{10}}+39710\left(Q^8+\frac{1}{Q^8}\right)+61482057\left(Q^6+\frac{1}{Q^6}\right)\\ &+442721128\left(Q^4+\frac{1}{Q^4}\right)+832992338\left(Q^2+\frac{1}{Q^2}\right)+75505688\end{aligned}$$

$$\begin{aligned}=\ & 262144\left(P^9+\frac{1}{P^9}\right)-2490368\left(P^8+\frac{1}{P^8}\right)-311296\left(P^7+\frac{1}{P^7}\right)\left[2\left(Q^2+\frac{1}{Q^2}\right)-37\right]\\ &+155648\left(P^6+\frac{1}{P^6}\right)\left[3\left(Q^2+\frac{1}{Q^2}\right)-73\right]+19456\left(P^5+\frac{1}{P^5}\right)\left[-1739+25\left(Q^4+\frac{1}{Q^4}\right)\right.\\ &\left.-2046\left(Q^2+\frac{1}{Q^2}\right)\right]+34048\left(P^4+\frac{1}{P^4}\right)\left[230\left(Q^4+\frac{1}{Q^4}\right)+439\left(Q^2+\frac{1}{Q^2}\right)+2302\right]\\ &-2432\left(P^3+\frac{1}{P^3}\right)\times\left[-246014+55\left(Q^6+\frac{1}{Q^6}\right)\right.\\ &\qquad\left.-26313\left(Q^4+\frac{1}{Q^4}\right)-127659\left(Q^2+\frac{1}{Q^2}\right)\right]\\ &-608\left(P^2+\frac{1}{P^2}\right)\times\left[9487\left(Q^6+\frac{1}{Q^6}\right)-57842\left(Q^4+\frac{1}{Q^4}\right)\right.\\ &\qquad\left.-607903\left(Q^2+\frac{1}{Q^2}\right)-1287420\right]\end{aligned}$$

$$+76\left(P+\tfrac{1}{P}\right)\times\left[103\left(Q^8+\tfrac{1}{Q^8}\right)-469880\left(Q^6+\tfrac{1}{Q^6}\right)\right.$$
$$\left.-3161628\left(Q^4+\tfrac{1}{Q^4}\right)-4184264\left(Q^2+\tfrac{1}{Q^2}\right)-747030\right].$$

Lemma 3.15. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 23 over α, then*

$$\begin{aligned}
(3.42)\qquad & Q^{12}+\frac{1}{Q^{12}}+164772\left(Q^{10}+\frac{1}{Q^{10}}\right)+1416790018\left(Q^8+\frac{1}{Q^8}\right)\\
&-2130154476\left(Q^6+\frac{1}{Q^6}\right)-7724991761\left(Q^4+\frac{1}{Q^4}\right)+249131895880\left(Q^2+\frac{1}{Q^2}\right)\\
=\;& 321023061476+4194304\left(P^{11}+\frac{1}{P^{11}}\right)+96468992\left(P^{10}+\frac{1}{P^{10}}\right)\\
&-12058624\left(P^9+\frac{1}{P^9}\right)\left[\left(Q^2+\frac{1}{Q^2}\right)-93\right]\\
&+1507328\left(P^8+\frac{1}{P^8}\right)\times\left[296\left(Q^2+\frac{1}{Q^2}\right)+4397\right]\\
&+1130496\left(P^7+\frac{1}{P^7}\right)\left[11\left(Q^4+\frac{1}{Q^4}\right)-3148\left(Q^2+\frac{1}{Q^2}\right)+21389\right]\\
&+94208\left(P^6+\frac{1}{P^6}\right)\left[5617\left(Q^4+\frac{1}{Q^4}\right)-121400\left(Q^2+\frac{1}{Q^2}\right)+566351\right]\\
&+47104\left(P^5+\frac{1}{P^5}\right)\times\left[101325\left(Q^4+\frac{1}{Q^4}\right)-856348\left(Q^2+\frac{1}{Q^2}\right)+23576361\right]\\
&-58888\left(P^4+\frac{1}{P^4}\right)\left[16164\left(Q^6+\frac{1}{Q^6}\right)-725439\left(Q^4+\frac{1}{Q^4}\right)\right.\\
&\qquad\left.+11202632\left(Q^2+\frac{1}{Q^2}\right)-28284794\right]\\
&+1472\left(P^3+\frac{1}{P^3}\right)\left[593\left(Q^8+\frac{1}{Q^8}\right)-2087612\left(Q^6+\frac{1}{Q^6}\right)\right.\\
&\qquad\left.+14698260\left(Q^4+\frac{1}{Q^4}\right)-93333412\left(Q^2+\frac{1}{Q^2}\right)+189688822\right]\\
&-16\left(P^2+\frac{1}{P^2}\right)\times\left[5258007\left(Q^8+\frac{1}{Q^8}\right)-12214034\left(Q^6+\frac{1}{Q^6}\right)\right.\\
&\qquad\left.-687894396\left(Q^4+\frac{1}{Q^4}\right)-18978170550+10299016728\left(Q^2+\frac{1}{Q^2}\right)\right]\\
&-184\left(P+\frac{1}{P}\right)\left[147\left(Q^{10}+\frac{1}{Q^{10}}\right)-4008318\left(Q^8+\frac{1}{Q^8}\right)+33717559\left(Q^6+\frac{1}{Q^6}\right)\right.\\
&\qquad\left.-196168488\left(Q^4+\frac{1}{Q^4}\right)+1477813558\left(Q^2+\frac{1}{Q^2}\right)-2080998580\right].
\end{aligned}$$

Lemma 3.16. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 25 over α, then*

$$(3.43)\qquad Q^{15}+\frac{1}{Q^{15}}-1374\left(Q^{14}+\frac{1}{Q^{14}}\right)+624051\left(Q^{13}+\frac{1}{Q^{13}}\right)$$
$$-105877724\left(Q^{12}+\frac{1}{Q^{12}}\right)+5588665101\left(Q^{11}+\frac{1}{Q^{11}}\right)-6149701233\left(Q^{10}+\frac{1}{Q^{10}}\right)$$
$$+216547952495\left(Q^{9}+\frac{1}{Q^{9}}\right)-274685537880\left(Q^{8}+\frac{1}{Q^{8}}\right)+316427310845\left(Q^{7}+\frac{1}{Q^{7}}\right)$$
$$-103388902030\left(Q^{6}+\frac{1}{Q^{6}}\right)+1839454577495\left(Q^{5}+\frac{1}{Q^{5}}\right)-2319422065380\left(Q^{4}+\frac{1}{Q^{4}}\right)$$
$$+2016536239145\left(Q^{3}+\frac{1}{Q^{3}}\right)-1982303356330\left(Q^{2}+\frac{1}{Q^{2}}\right)+3840832608595\left(Q+\frac{1}{Q}\right)$$
$$-4110726704080$$
$$=16777216\left(P^{12}+\frac{1}{P^{12}}\right)\times\left[\left(Q^{2}+\frac{1}{Q^{2}}\right)+\left(Q+\frac{1}{Q}\right)+1\right]$$
$$-10485760\left(P^{10}+\frac{1}{P^{10}}\right)\left[5\left(Q^{4}+\frac{1}{Q^{4}}\right)-80\left(Q^{3}+\frac{1}{Q^{3}}\right)\right.$$
$$\left.+195\left(Q^{2}+\frac{1}{Q^{2}}\right)+110\left(Q+\frac{1}{Q}\right)\right]$$
$$+327680\left(P^{8}+\frac{1}{P^{8}}\right)\left[185\left(Q^{6}+\frac{1}{Q^{6}}\right)-10545\left(Q^{5}+\frac{1}{Q^{5}}\right)\right.$$
$$+87090\left(Q^{4}+\frac{1}{Q^{4}}\right)-158098\left(Q^{3}+\frac{1}{Q^{3}}\right)+231157\left(Q^{2}+\frac{1}{Q^{2}}\right)$$
$$\left.-16013\left(Q+\frac{1}{Q}\right)+99552\right]$$
$$+40960\left(P^{6}+\frac{1}{P^{6}}\right)\left[-13095078+765\left(Q^{8}+\frac{1}{Q^{8}}\right)-99460\left(Q^{7}+\frac{1}{Q^{7}}\right)\right.$$
$$+1478858\left(Q^{6}+\frac{1}{Q^{6}}\right)-4593412\left(Q^{5}+\frac{1}{Q^{5}}\right)+6465168\left(Q^{4}+\frac{1}{Q^{4}}\right)$$
$$\left.-4589852\left(Q^{3}+\frac{1}{Q^{3}}\right)+7082838\left(Q^{2}+\frac{1}{Q^{2}}\right)-7205212\left(Q+\frac{1}{Q}\right)\right]$$
$$+2560\left(P^{4}+\frac{1}{P^{4}}\right)\left[548799216+2655\left(Q^{10}+\frac{1}{Q^{10}}\right)-466519\left(Q^{9}+\frac{1}{Q^{9}}\right)\right.$$
$$+17418696\left(Q^{8}+\frac{1}{Q^{8}}\right)-83433599\left(Q^{7}+\frac{1}{Q^{7}}\right)+155764931\left(Q^{6}+\frac{1}{Q^{6}}\right)$$
$$-207597164\left(Q^{5}+\frac{1}{Q^{5}}\right)+294547776\left(Q^{4}+\frac{1}{Q^{4}}\right)-312051804\left(Q^{3}+\frac{1}{Q^{3}}\right)$$
$$\left.+453415006\left(Q^{2}+\frac{1}{Q^{2}}\right)-516256354\left(Q+\frac{1}{Q}\right)\right]$$
$$+320\left(P^{2}+\frac{1}{P^{2}}\right)\left[5665640828+1353\left(Q^{12}+\frac{1}{Q^{12}}\right)+872568\left(Q^{11}+\frac{1}{Q^{11}}\right)\right.$$

$$+33229428\left(Q^{10}+\frac{1}{Q^{10}}\right)-262420952\left(Q^{9}+\frac{1}{Q^{9}}\right)+843710418\left(Q^{8}+\frac{1}{Q^{8}}\right)$$
$$-1770861112\left(Q^{7}+\frac{1}{Q^{7}}\right)+2370486948\left(Q^{6}+\frac{1}{Q^{6}}\right)-2314319272\left(Q^{5}+\frac{1}{Q^{5}}\right)$$
$$+4574672432\left(Q^{4}+\frac{1}{Q^{4}}\right)-7350748752\left(Q^{3}+\frac{1}{Q^{3}}\right)$$
$$+8589856488\left(Q^{2}+\frac{1}{Q^{2}}\right)-8207902832\left(Q+\frac{1}{Q}\right)\Bigg].$$

Lemma 3.17. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 29 over α, then*

$$\begin{aligned}(3.44)\qquad & Q^{15}+\frac{1}{Q^{15}}+2610\left(Q^{14}+\frac{1}{Q^{14}}\right)+2290739\left(Q^{13}+\frac{1}{Q^{13}}\right)\\
&+773868132\left(Q^{12}+\frac{1}{Q^{12}}\right)+85133704333\left(Q^{11}+\frac{1}{Q^{11}}\right)\\
&+1722106886726\left(Q^{10}+\frac{1}{Q^{10}}\right)+387763344751\left(Q^{9}+\frac{1}{Q^{9}}\right)\\
&+20840144814888\left(Q^{8}+\frac{1}{Q^{8}}\right)+59747787942141\left(Q^{7}+\frac{1}{Q^{7}}\right)\\
&+191351235325442\left(Q^{6}+\frac{1}{Q^{6}}\right)+324184390230999\left(Q^{5}+\frac{1}{Q^{5}}\right)\\
&+597721984006620\left(Q^{4}+\frac{1}{Q^{4}}\right)+1021834090771369\left(Q^{3}+\frac{1}{Q^{3}}\right)\\
&+1302069184703878\left(Q^{2}+\frac{1}{Q^{2}}\right)+1179308291242323\left(Q+\frac{1}{Q}\right)\\
&+914875168239920\\
=\;& 268435456\left(P^{14}+\frac{1}{P^{14}}\right)\\
&-486539264\left(P^{12}+\frac{1}{P^{12}}\right)\left[2\left(Q^{2}+\frac{1}{Q^{2}}\right)-51\left(Q+\frac{1}{Q}\right)-180\right]\\
&+30408704\left(P^{10}+\frac{1}{P^{10}}\right)\Bigg[237763+45\left(Q^{4}+\frac{1}{Q^{4}}\right)-1348\left(Q^{3}+\frac{1}{Q^{3}}\right)\\
&\quad-28020\left(Q^{2}+\frac{1}{Q^{2}}\right)+130092\left(Q+\frac{1}{Q}\right)\Bigg]\\
&-1900544\left(P^{8}+\frac{1}{P^{8}}\right)\Bigg[490\left(Q^{6}+\frac{1}{Q^{6}}\right)21592+\left(Q^{5}+\frac{1}{Q^{5}}\right)+1530060\left(Q^{4}+\frac{1}{Q^{4}}\right)\\
&\quad+4143553\left(Q^{3}+\frac{1}{Q^{3}}\right)+10062832\left(Q^{2}+\frac{1}{Q^{2}}\right)+3649399\left(Q+\frac{1}{Q}\right)+4054860\Bigg]\\
&+237568\left(P^{6}+\frac{1}{P^{6}}\right)\times\Bigg[811380268+1304\left(Q^{8}+\frac{1}{Q^{8}}\right)+434044\left(Q^{7}+\frac{1}{Q^{7}}\right)\\
&\quad+13788577\left(Q^{6}+\frac{1}{Q^{6}}\right)+10411592\left(Q^{5}+\frac{1}{Q^{5}}\right)+105951138\left(Q^{4}+\frac{1}{Q^{4}}\right)\end{aligned}$$

$$+99161472\left(Q^3+\frac{1}{Q^3}\right)+177440463\left(Q^2+\frac{1}{Q^2}\right)+563974332\left(Q+\frac{1}{Q}\right)\Big]$$
$$-3712\left(P^4+\frac{1}{P^4}\right)\Big[14846975376+11856\left(Q^{10}+\frac{1}{Q^{10}}\right)+11599844\left(Q^9+\frac{1}{Q^9}\right)$$
$$+448544664\left(Q^8+\frac{1}{Q^8}\right)-763704252\left(Q^7+\frac{1}{Q^7}\right)+5875300048\left(Q^6+\frac{1}{Q^6}\right)$$
$$+88316609\left(Q^5+\frac{1}{Q^5}\right)+25770706848\left(Q^4+\frac{1}{Q^4}\right)+57741632656\left(Q^3+\frac{1}{Q^3}\right)$$
$$+80052643040\left(Q^2+\frac{1}{Q^2}\right)+52701143288\left(Q+\frac{1}{Q}\right)\Big]$$
$$+464\left(P^2+\frac{1}{P^2}\right)\times\Big[3753\left(Q^{12}+\frac{1}{Q^{12}}\right)-6120856\left(Q^{11}+\frac{1}{Q^{11}}\right)$$
$$+772337076\left(Q^{10}+\frac{1}{Q^{10}}\right)-2864084968\left(Q^9+\frac{1}{Q^9}\right)+22104272594\left(Q^8+\frac{1}{Q^8}\right)$$
$$+27146779640\left(Q^7+\frac{1}{Q^7}\right)+141012884964\left(Q^6+\frac{1}{Q^6}\right)+2782862132604$$
$$+319845340808\left(Q^5+\frac{1}{Q^5}\right)+656499614663\left(Q^4+\frac{1}{Q^4}\right)$$
$$+821445490576\left(Q^3+\frac{1}{Q^3}\right)+1184962932840\left(Q^2+\frac{1}{Q^2}\right)$$
$$+2247200396144\left(Q+\frac{1}{Q}\right)\Big].$$

Lemma 3.18. *Let P and Q be defined in* (3.5) *and* (3.6)*, If β is of degree 55 over α, then*

$$Q^{12}+\frac{1}{Q^{12}}+164772\left(Q^{10}+\frac{1}{Q^{10}}\right)+1416790018\left(Q^8+\frac{1}{Q^8}\right) \tag{3.45}$$
$$-2130154476\left(Q^6+\frac{1}{Q^6}\right)-7724991761\left(Q^4+\frac{1}{Q^4}\right)+249131895880\left(Q^2+\frac{1}{Q^2}\right)$$
$$-321023061476$$
$$=4194304\left(P^{11}+\frac{1}{P^{11}}\right)+96468992\left(P^{10}+\frac{1}{P^{10}}\right)$$
$$-12058624\left(P^9+\frac{1}{P^9}\right)\left[\left(Q^2+\frac{1}{Q^2}\right)-93\right]$$
$$-1507328\left(P^8+\frac{1}{P^8}\right)\times\left[296\left(Q^2+\frac{1}{Q^2}\right)-4397\right]$$
$$+1130496\left(P^7+\frac{1}{P^7}\right)\left[11\left(Q^4+\frac{1}{Q^4}\right)-3148\left(Q^2+\frac{1}{Q^2}\right)-21389\right]$$
$$+94208\left(P^6+\frac{1}{P^6}\right)\left[5617\left(Q^4+\frac{1}{Q^4}\right)-121400\left(Q^2+\frac{1}{Q^2}\right)-566351\right]$$

$$-47104\left(P^5+\frac{1}{P^5}\right)\times\left[115\left(Q^6+\frac{1}{Q^6}\right)-101325\left(Q^4+\frac{1}{Q^4}\right)\right.$$
$$\left.+856348\left(Q^2+\frac{1}{Q^2}\right)-23576361\right]$$
$$-5888\left(P^4+\frac{1}{P^4}\right)\times\left[16164\left(Q^6+\frac{1}{Q^6}\right)-725439\left(Q^4+\frac{1}{Q^4}\right)\right.$$
$$\left.+11202632\left(Q^2+\frac{1}{Q^2}\right)-28284794\right]$$
$$+1472\left(P^3+\frac{1}{P^3}\right)\times\left[593\left(Q^8+\frac{1}{Q^8}\right)-2087612\left(Q^6+\frac{1}{Q^6}\right)\right.$$
$$\left.+14698260\left(Q^4+\frac{1}{Q^4}\right)-93333412\left(Q^2+\frac{1}{Q^2}\right)+189688822\right]$$
$$-368\left(P^2+\frac{1}{P^2}\right)\left[228609\left(Q^8+\frac{1}{Q^8}\right)-5311208\left(Q^6+\frac{1}{Q^6}\right)\right.$$
$$\left.-29908452\left(Q^4+\frac{1}{Q^4}\right)+447783336\left(Q^2+\frac{1}{Q^2}\right)-825137850\right]$$
$$-184\left(P+\frac{1}{P}\right)\left[147\left(Q^{10}+\frac{1}{Q^{10}}\right)-4008318\left(Q^8+\frac{1}{Q^8}\right)\right.$$
$$+33717559\left(Q^8+\frac{1}{Q^8}\right)-196168488\left(Q^4+\frac{1}{Q^4}\right)$$
$$\left.+1477813558\left(Q^2+\frac{1}{Q^2}\right)-2080998580\right].$$

Proofs of the identities (3.39)–(3.45) are similar to the proof of the identity (3.35) given above except that in place of result (2.16), result (2.17) is used for proving (3.39); results (2.18) and (2.19) are used for proving (3.40); result (2.23) is used for proving (3.41); result (2.24) is used for proving (3.42); results (2.25) and (2.26) are used for proving (3.43); result (2.31) is used for proving (3.45).

4. MODULAR RELATIONS BETWEEN $H(q)$ AND $H(q^n)$

In this section, we establish several modular relations between the Ramanujan–Göllnitz–Gordon continued fractions $H(q)$ and $H(q^n)$ using the identities established in section 3, for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$ and 55.

Theorem 4.1. *If $x = H(q)$ and $y = H(q^2)$, then*

$$y^2 - y + x^2y + x^2 = 0. \tag{4.1}$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$P = \frac{\varphi(-q)\varphi(-q^2)}{\varphi(q)\varphi(q^2)} \quad\text{and}\quad Q = \frac{\varphi(-q)\varphi(q^2)}{\varphi(q)\varphi(-q^2)}. \tag{4.2}$$

Replacing q by $\sqrt{q}$ in the equation (4.2) and from the equation (3.1), we deduce that

$$P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad\text{and}\quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}. \tag{4.3}$$

Employing the equation (4.3) in the equation (3.9), we deduce that

$$(4.4)\qquad \begin{aligned}&(y^2-y+x^2y+x^2)(-1+2x+x^2)(-2xy^2+y^2+x^2y^2+1+4xy+2x+x^2)\\&(x^2y^2+y+1-x^2y)(x^2y^2+2xy^2+y^2+1-4xy-2x+x^2)=0.\end{aligned}$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q)$ as $q \to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Remark: For a different proof of the equation (4.1), see [8].

Theorem 4.2. *If $x = H(q)$ and $y = H(q^3)$, then*

$$(4.5)\qquad y^3x^4+x^3+3yx^2-y-3y^2x^3+3y^2x-3y^3x^2-y^4x=0.$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$(4.6)\qquad P=\frac{\varphi(-q)\varphi(-q^3)}{\varphi(q)\varphi(q^3)}\quad\text{and}\quad Q=\frac{\varphi(-q)\varphi(q^3)}{\varphi(q)\varphi(-q^3)}.$$

Replacing q by $\sqrt{q}$ in the equation (4.6) and from the equation (3.1), we deduce that

$$(4.7)\qquad P=\frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)}\quad\text{and}\quad Q=\frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}.$$

Employing the equation (4.7) in the equation (3.12), we deduce that

$$(4.8)\qquad \begin{aligned}&(y^3x^4+x^3+3yx^2-y-3y^2x^3+3y^2x-3y^3x^2-y^4x)(-1+2x+x^2)\\&(yx^4+x-3yx^2+3y^2x^3-3y^2x+3y^3x^2-y^3-y^4x^3)=0.\end{aligned}$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^{3/2})$ as $q \to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Remark: For a different proofs of the equation (4.5), see [8] and [18].

Theorem 4.3. *If $x = H(q)$ and $y = H(q^4)$, then*

$$(4.9)\qquad -y+4x^2y-4x^2y^3+x^4y+x^4y^2+x^4y^3+y^2+x^4-y^3+y^4=0.$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.10)\qquad P=\frac{\varphi(-q)\varphi(-q^4)}{\varphi(q)\varphi(q^4)}\quad\text{and}\quad Q=\frac{\varphi(-q)\varphi(q^4)}{\varphi(q)\varphi(-q^4)}.$$

Replacing q by $\sqrt{q}$ in the equation (4.10) and from the equation (3.1), we deduce that

$$(4.11)\qquad P=\frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)}\quad\text{and}\quad Q=\frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}.$$

Employing the equation (4.11) in the equation (3.14), we deduce that

$$(4.12)\qquad (-y+4x^2y-4x^2y^3+x^4y+x^4y^2+x^4y^3+y^2+x^4-y^3+y^4)$$
$$(-1+2x+x^2)^3(1+4x+4y+8xy+8xy^3-4xy^4-8x^2y+12x^2y^2+8x^2y^3$$
$$+6x^2y^4+8x^3y+8x^3y^3-4x^3y^4+4x^4y+2x^4y^2-4x^4y^3+x^4y^4+6x^2+2y^2$$
$$+4x^3+x^4-4y^3+y^4)(1+y+4x^2y-4x^2y^3-x^4y+x^4y^2-x^4y^3+x^4y^4+y^2+y^3)$$
$$(1-4x+4y-8xy-8xy^3+4xy^4-8x^2y+12x^2y^2+8x^2y^3+6x^2y^4-8x^3y-8x^3y^3$$
$$+4x^3y^4+4x^4y+2x^4y^2-4x^4y^3+x^4y^4+6x^2+2y^2-4x^3+x^4-4y^3+y^4)=0.$$

From the definitions of x and y, we have $x=o(q^{1/2})$ and $y=o(q^2)$ as $q\to 0$, it can be seen that the first factor of the equation (4.12) vanishes for q sufficiently small whereas as other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Remark: For a different proof of the equation (4.9), see [8].

Theorem 4.4. *If $x=H(q)$ and $y=H(q^5)$, then*

$$(4.13)\qquad y-5xy^4-5x^2y+10x^2y^3-10x^3y^2+10x^3y^4+5x^4y^5$$
$$+5x^5y^2-x^6y^5-10x^4y^3-x^5+xy^6=0.$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.14)\qquad P=\frac{\varphi(-q)\varphi(-q^5)}{\varphi(q)\varphi(q^5)}\quad\text{and}\quad Q=\frac{\varphi(-q)\varphi(q^5)}{\varphi(q)\varphi(-q^5)}.$$

Replacing q by $\sqrt{q}$ in the equation (4.14) and then from equation (3.1) , we deduce that

$$(4.15)\qquad P=\frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)}\quad\text{and}\quad Q=\frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}.$$

Employing the equation (4.15) in the equation (3.17), we deduce that

$$(4.16)\qquad (-y+5xy^4+5x^2y-10x^2y^3+10x^3y^2-10x^3y^4+10x^4y^3-5x^4y^5$$
$$-5x^5y^2+x^6y^5+x^5-xy^6)\times(x-5xy^2-10x^2y^3+5x^2y^5+10x^3y^2$$
$$-10x^3y^4-5x^4y+10x^4y^3+5x^5y^4+x^6y-y^5-x^5y^6)(-1+2x+x^2)^2=0.$$

From the definitions of x and y, we have $x=o(q^{1/2})$ and $y=o(q^{5/2})$ as $q\to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Remark: For a different proof of the equation (4.13), see [18].

Theorem 4.5. *If $x=H(q)$ and $y=H(q^7)$, then*

$$(4.17)\qquad x^8+7x^7y^5-7yx^7-7x^7y^3-y^7x^7+28x^6y^2-7yx^5$$
$$+7y^7x^5-49x^5y^5-7x^5y^3+70x^4y^4+7yx^3-7x^3y^5$$
$$-49x^3y^3-7y^7x^3+28x^2y^6-xy-7xy^5+7xy^3-7xy^7+y^8=0.$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.18)\qquad P=\frac{\varphi(-q)\varphi(-q^7)}{\varphi(q)\varphi(q^7)}\quad\text{and}\quad Q=\frac{\varphi(-q)\varphi(q^7)}{\varphi(q)\varphi(-q^7)}.$$

Replacing q by $\sqrt{q}$ in the equation (4.18) and then from equation (3.1) , we deduce that

$$P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}. \tag{4.19}$$

From the equations (4.19) and (3.20), we deduce that

$$\begin{aligned}(1 &- 7xy^5 + 7xy + 7xy^3 + xy^7 + 28x^2y^2 + 7yx^3 - 7yx^5 + yx^7 + 70x^4y^4 \\ &+49x^3y^5 + 7x^5y^5 + 7x^7y^5 + 28x^6y^6 + 7x^3y^3 + 49x^5y^3 - 7x^7y^3 - 7y^7x^3 \\ &+7y^7x^5 + 7y^7x^7 + y^8x^8)(x^8 + 7x^7y^5 - 7yx^7 - 7x^7y^3 - y^7x^7 + 28x^6y^2 \\ &-7yx^5 + 7y^7x^5 - 49x^5y^5 - 7x^5y^3 + 70x^4y^4 + 7yx^3 - 7x^3y^5 - 49x^3y^3 \\ &-7y^7x^3 + 28x^2y^6 - xy - 7xy^5 + 7xy^3 - 7xy^7 + y^8)(-1 + 2x + x^2)^3 = 0.\end{aligned} \tag{4.20}$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^{7/2})$ as $q \to 0$, it can be seen that the second factor vanishes for q sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, second factor vanishes identically. Hence the proof. □

Remark: For a different proof of (4.17), see [18].

Theorem 4.6. *If $x = H(q)$ and $y = H(q^8)$, then*

$$\begin{aligned}&-3y^2 + 12x^4y^5 + 16x^4y^6 + 12x^4y^7 - 12x^4y - 16x^6y^2 \\ &+ 8x^6y^3 + 8x^6y^5 + 16x^6y^6 - 8x^6y^7 - 8x^6y + 7x^8y^2 + 3x^8y^3 \\ &+ 3x^8y^4 - x^8y^5 - 3x^8y^6 + x^8y^7 + 5x^8y + 16x^2y^2 - 8x^2y^3 - \\ &8x^2y^5 - 16x^2y^6 + 8x^2y^7 + 8x^2y + 16x^4y^2 - 12x^4y^3 - 32x^4y^4 \\ &+ x^8 + y^3 + 3y^4 - 3y^5 + 7y^6 - 5y^7 + y^8 - y = 0.\end{aligned}$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$P = \frac{\varphi(-q)\varphi(-q^8)}{\varphi(q)\varphi(q^8)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^8)}{\varphi(q)\varphi(-q^8)}. \tag{4.21}$$

Replacing q by $\sqrt{q}$ in the equation (4.21) and then from equation (3.1) , we deduce that

$$P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}. \tag{4.22}$$

Employing the equation (4.22) in the equation (3.23), we deduce that

$$\begin{aligned}&(-3y^2 + 12x^4y^5 + 16x^4y^6 + 12x^4y^7 - 12x^4y - 16x^6y^2 + 8x^6y^3 + 8x^6y^5 \\ &+16x^6y^6 - 8x^6y^7 - 8x^6y + 7x^8y^2 + 3x^8y^3 + 3x^8y^4 - x^8y^5 - 3x^8y^6 + x^8y^7 \\ &+5x^8y + 16x^2y^2 - 8x^2y^3 - 8x^2y^5 - 16x^2y^6 + 8x^2y^7 + 8x^2y - 5y^7 + 16x^4y^2 \\ &-12x^4y^3 - 32x^4y^4 + x^8 + y^3 + 3y^4 - 3y^5 + 7y^6 + y^8 - y)(8x - 8y + 48x^2y^2 \\ &+400x^3y + 48x^3y^3 + 48x^3y^5 - 688x^3y^6 + 400x^3y^7 - 56x^3y^8 + 688x^3y^2 + 208x^4y \\ &+208x^4y^3 + 228x^4y^4 - 208x^4y^5 + 376x^4y^6 - 208x^4y^7 + 70x^4y^8 + 376x^4y^2 + 400yx^5 \\ &+160yx^6 - 16yx^7 - 8yx^8 + 688x^5y^2 + 48x^6y^2 - 48x^7y^2 + 20x^8y^2 + 56x^3 + 70x^4 \\ &-8y^3 - 26y^4 + 8y^5 + 20y^6 + 8y^7 + y^8 - 16xy + 80xy^3 + 80xy^5 + 48xy^6 - 16xy^7\end{aligned} \tag{4.23}$$

$$-8xy^8+1-48xy^2+160x^2y+160x^2y^3+296x^2y^4-160x^2y^5+48x^2y^6-160x^2y^7$$
$$+28x^2y^8+28x^2+20y^2+48x^5y^5+28x^6+8x^7+x^8+160x^6y^3+80x^7y^3+48x^5y^3$$
$$-8x^8y^3+296x^6y^4-26x^8y^4-160x^6y^5+80x^7y^5+56x^5+8x^8y^5+48x^6y^6$$
$$+48x^7y^6-688x^5y^6+20x^8y^6-160x^6y^7-16x^7y^7+400x^5y^7+8x^8y^7+28x^6y^8$$
$$-8x^7y^8-56x^5y^8+x^8y^8)(70x^4y^8-8x-8y+48x^2y^2-400x^3y-48x^3y^3$$
$$-48x^3y^5+688x^3y^6-400x^3y^7+56x^3y^8-688x^3y^2+208x^4y+208x^4y^3$$
$$+228x^4y^4-208x^4y^5+376x^4y^6-208x^4y^7+376x^4y^2-400yx^5-160x^2y^5+16yx^7$$
$$-8yx^8-688x^5y^2+48x^6y^2+y^8+20x^8y^2-56x^3+70x^4-8y^3-26y^4+8y^5+20y^6$$
$$+160yx^6+48x^7y^2+16xy-80xy^3-80xy^5-48xy^6+16xy^7+8xy^8+48xy^2$$
$$+160x^2y+160x^2y^3+296x^2y^4+8y^7+48x^2y^6-160x^2y^7+28x^2y^8+28x^2+20y^2$$
$$-56x^5+28x^6+20x^8y^6+x^8+160x^6y^3-80x^7y^3-48x^5y^3-8x^7-8x^8y^3$$
$$+296x^6y^4-26x^8y^4-160x^6y^5-80x^7y^5-48x^5y^5+8x^8y^5+48x^6y^6-48x^7y^6$$
$$+688x^5y^6-160x^6y^7+16x^7y^7-400x^5y^7+8x^8y^7+28x^6y^8+8x^7y^8+56x^5y^8$$
$$+x^8y^8)(1+5y-16x^2y^2-12x^4y-12x^4y^3-32x^4y^4+12x^4y^5+16x^4y^6+12x^4y^7$$
$$+16x^4y^2+8yx^6-yx^8+16x^6y^2-3x^8y^2+3y^3+3y^4-y^5-3y^6+y^7-8x^2y$$
$$+8x^2y^3+8x^2y^5+16x^2y^6-8x^2y^7+7y^2-8x^6y^3+x^8y^3+3x^8y^4-8x^6y^5$$
$$-3x^8y^5-16x^6y^6+7x^8y^6+8x^6y^7-5x^8y^7+x^8y^8)(-1+2x+x^2)^7=0.$$

From the definitions of x and y, we have $x=o(q^{1/2})$ and $y=o(q^4)$ as $q\to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. ☐

Theorem 4.7. *If $x=H(q)$ and $y=H(q^9)$, then*

$$y^{10}-yx+9yx^3-18yx^5-6yx^7+9yx^9-8y^2x^2+48y^2x^4-3y^2x^8-12y^2x^{10}$$
$$+y^2x^{12}+9y^3x+x^{10}-73y^3x^3+114y^3x^5-66y^3x^7-33y^3x^9+9y^3x^{11}$$
$$+48y^4x^2-288y^4x^4+210y^4x^6+48y^4x^8-y^{11}x^{11}-18y^5x+114y^5x^3-288y^5x^5$$
$$+312y^5x^7-66y^5x^9-6y^5x^{11}+210y^6x^4-504y^6x^6+210y^6x^8-66y^7x^3$$
$$+312y^7x^5-288y^7x^7+114y^7x^9-18y^7x^{11}-3y^8x^2+48y^8x^4+210y^8x^6-288y^8x^8$$
$$+9y^9x-33y^9x^3-66y^9x^5+114y^9x^7-73y^9x^9+9y^9x^{11}-12y^{10}x^2-3y^{10}x^4$$
$$+48y^{10}x^8+y^{12}x^2+48y^8x^{10}-y^{11}x-3y^4x^{10}+9y^{11}x^3-6y^7x-8y^{10}x^{10}$$
$$-6y^{11}x^5-yx^{11}-18y^{11}x^7+9y^{11}x^9=0. \tag{4.24}$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$P=\frac{\varphi(-q)\varphi(-q^9)}{\varphi(q)\varphi(q^9)} \quad \text{and} \quad Q=\frac{\varphi(-q)\varphi(q^9)}{\varphi(q)\varphi(-q^9)}. \tag{4.25}$$

Replacing q by $\sqrt{q}$ in the equation (4.25) and then from equation (3.1) , we deduce that

$$P=\frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q=\frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}. \tag{4.26}$$

From the equations (4.26) and (3.27), we deduce that

$$
\begin{aligned}
(4.27)\quad &(y^{10} - yx + 9yx^3 - 18yx^5 - 6yx^7 + 9yx^9 - yx^{11} - 8y^2x^2 + 48y^2x^4 - 3y^2x^8 \\
&-12y^2x^{10} + y^2x^{12} + 9y^3x - 73y^3x^3 + 114y^3x^5 - 66y^3x^7 - 33y^3x^9 \\
&+9y^3x^{11} + 48y^4x^2 - 288y^4x^4 + 210y^4x^6 + 48y^4x^8 - 3y^4x^{10} - 18y^5x \\
&+114y^5x^3 - 288y^5x^5 + 312y^5x^7 - 66y^5x^9 - 6y^5x^{11} + 210y^6x^4 - 504y^6x^6 \\
&+210y^6x^8 - 6y^7x - 66y^7x^3 + 312y^7x^5 - 288y^7x^7 + 114y^7x^9 - 18y^7x^{11} \\
&-3y^8x^2 + 48y^8x^4 + 210y^8x^6 - 288y^8x^8 + 9y^9x - 33y^9x^3 - 66y^9x^5 + 114y^9x^7 \\
&-73y^9x^9 + 9y^9x^{11} - 12y^{10}x^2 - 3y^{10}x^4 + 48y^{10}x^8 - 8y^{10}x^{10} \\
&+9y^{11}x^3 - 6y^{11}x^5 - 18y^{11}x^7 + 9y^{11}x^9 - y^{11}x^{11} + y^{12}x^2 \\
&+x^{10})(x^2 + y^2 + yx - 9yx^3 + 6yx^5 + 18yx^7 - 9yx^9 + yx^{11} - 12y^2x^2 - 3y^2x^4 \\
&+48y^2x^8 - 8y^2x^{10} - 9y^3x + 33y^3x^3 + 66y^3x^5 - 114y^3x^7 + 73y^3x^9 \\
&-9y^3x^{11} - 3y^4x^2 + 48y^4x^4 + 210y^4x^6 - 288y^4x^8 + 48y^4x^{10} + 6y^5x + 66y^5x^3 \\
&-312y^5x^5 + 288y^5x^7 - 114y^5x^9 + 18y^5x^{11} + 210y^6x^4 - 504y^6x^6 + 210y^6x^8 \\
&+18y^7x - 114y^7x^3 + 288y^7x^5 - 312y^7x^7 + 66y^7x^9 + 6y^7x^{11} + 48y^8x^2 - 288y^8x^4 \\
&+210y^8x^6 + 48y^8x^8 - 3y^8x^{10} - 9y^9x + 73y^9x^3 - 114y^9x^5 + 66y^9x^7 \\
&+33y^9x^9 - 9y^9x^{11} - 8y^{10}x^2 + 48y^{10}x^4 - 3y^{10}x^8 - 12y^{10}x^{10} \\
&+y^{10}x^{12} + y^{11}x - 9y^{11}x^3 + 18y^{11}x^5 + 6y^{11}x^7 - 9y^{11}x^9 \\
&+y^{11}x^{11} + y^{12}x^{10} - y^{11}x + 48y^8x^{10}) = 0.
\end{aligned}
$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^{9/2})$ as $q \to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Theorem 4.8. *If $x = H(q)$ and $y = H(q^{11})$, then*

$$
\begin{aligned}
(4.28)\quad &-y - 11xy^2 + 33xy^4 + 11xy^6 - 33xy^8 + 11xy^{10} - xy^{12} + 11x^2y - 66x^2y^3 \\
&+66x^2y^5 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2 - 231x^3y^4 - 55x^3y^6 + 396x^3y^8 \\
&-66x^3y^{10} - 33x^4y + 396x^4y^3 - 528x^4y^5 - 198x^4y^7 + 231x^4y^9 - 33x^4y^{11} \\
&+198x^5y^4 + 396x^5y^6 - 528x^5y^8 + 66x^5y^{10} + 11x^6y - 55x^6y^3 + 396x^6y^5 \\
&-396x^6y^7 + 55x^6y^9 - 11x^6y^{11} - 66x^7y^2 + 528x^7y^4 - 396x^7y^6 - 198x^7y^8 \\
&+33x^8y - 231x^8y^3 + 198x^8y^5 + 528x^8y^7 - 396x^8y^9 + 33x^8y^{11} + 66x^9y^2 \\
&-396x^9y^4 + 55x^9y^6 + 231x^9y^8 - 66x^9y^{10} - 11x^{10}y + 66x^{10}y^3 \\
&-66x^{10}y^7 + 66x^{10}y^9 - 11x^{10}y^{11} - 11x^{11}y^2 + 33x^{11}y^4 \\
&-11x^{11}y^6 - 33x^{11}y^8 + 11x^{11}y^{10} + x^{12}y^{11} + x^{11} = 0.
\end{aligned}
$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$
(4.29)\qquad P = \frac{\varphi(-q)\varphi(-q^{11})}{\varphi(q)\varphi(q^{11})} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^{11})}{\varphi(q)\varphi(-q^{11})}.
$$

Replacing q by $\sqrt{q}$ in the equation (4.29) and then from equation (3.1), we deduce that

$$P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}. \tag{4.30}$$

Employing the equation (4.30) in the equation (3.32), we deduce that

$$\begin{aligned}
&(-y - 11xy^2 + 33xy^4 + 11xy^6 - 33xy^8 + 11xy^{10} - xy^{12} + 11x^2y - 66x^2y^3\\
&+66x^2y^5 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2 - 231x^3y^4 - 55x^3y^6 + 396x^3y^8\\
&-66x^3y^{10} - 33x^4y + 396x^4y^3 - 528x^4y^5 - 198x^4y^7 + 231x^4y^9\\
&-33x^4y^{11} + 198x^5y^4 + 396x^5y^6 - 528x^5y^8 + 66x^5y^{10} + 11x^6y\\
&-55x^6y^3 + 396x^6y^5 - 396x^6y^7 + 55x^6y^9 - 11x^6y^{11} - 66x^7y^2\\
&+528x^7y^4 - 396x^7y^6 - 198x^7y^8 + 33x^8y - 231x^8y^3 + 198x^8y^5\\
&+528x^8y^7 - 396x^8y^9 + 33x^8y^{11} + 66x^9y^2 - 396x^9y^4 + 55x^9y^6\\
&+231x^9y^8 - 66x^9y^{10} - 11x^{10}y + 66x^{10}y^3 - 66x^{10}y^7 + 66x^{10}y^9\\
&-11x^{10}y^{11} - 11x^{11}y^2 + 33x^{11}y^4 - 11x^{11}y^6 - 33x^{11}y^8\\
&+11x^{11}y^{10} + x^12y^{11} + x^{11})(x - 11xy^2 + 33xy^4 - 11xy^6 - 33xy^8\\
&+11xy^{10} + 11x^2y - 66x^2y^3 + 66x^2y^7 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2\\
&-396x^3y^4 + 55x^3y^6 + 231x^3y^8 - 66x^3y^{10} - 33x^4y + 231x^4y^3 - 198x^4y^5\\
&-528x^4y^7 + 396x^4y^9 - 33x^4y^{11} - 66x^5y^2 + 528x^5y^4 - 396x^5y^6 - 198x^5y^8\\
&-11x^6y + 55x^6y^3 - 396x^6y^5 + 396x^6y^7 - 55x^6y^9 + 11x^6y^{11} + 198x^7y^4\\
&+396x^7y^6 - 528x^7y^8 + 66x^7y^{10} + 33x^8y - 396x^8y^3 + 528x^8y^5 + 198x^8y^7\\
&-231x^8y^9 + 33x^8y^{11} + 66x^9y^2 - 231x^9y^4 - 55x^9y^6 + 396x^9y^8 - 66x^9y^{10}\\
&-11x^{10}y + 66x^{10}y^3 - 66x^{10}y^5 + 66x^{10}y^9 - 11x^{10}y^{11} - 11x^{11}y^2\\
&+33x^{11}y^4 + 11x^{11}y^6 - 33x^{11}y^8 + 11x^{11}y^{10} - x^{11}y^{12} + x^{12}y - y^{11}) = 0.
\end{aligned} \tag{4.31}$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^{11/2})$ as $q \to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Remark: For a different proof of the equation (4.28), see [18].

Theorem 4.9. *If $x = H(q)$ and $y = H(q^{13})$, then*

$$\begin{aligned}
&-y - 52xy^4 + 52xy^6 + 52xy^8 - 52xy^{10} + 13xy^{12} - xy^{14} + 13x^2y - 130x^2y^3\\
&+312x^2y^5 - 312x^2y^9 + 52x^2y^{11} - 52x^3y^2 + 572x^3y^4 - 520x^3y^6 - 520x^3y^8\\
&+858x^3y^{10} - 130x^3y^{12} - 52x^4y + 858x^4y^3 - 2327x^4y^5 + 1612x^4y^9 - 572x^4y^{11}\\
&+52x^4y^{13} + 312x^5y^2 - 1612x^5y^4 + 1300x^5y^6 + 2587x^5y^8 - 2327x^5y^{10} + 312x^5y^{12}\\
&+52x^6y - 520x^6y^3 + 2587x^6y^5 - 1716x^6y^7 - 1300x^6y^9 + 520x^6y^{11} - 52x^6y^{13}\\
&+1716x^7y^6 - 1716x^7y^8 + 52x^8y - 520x^8y^3 + 1300x^8y^5 + 1716x^8y^7 - 2587x^8y^9
\end{aligned} \tag{4.32}$$

$$+520x^8y^{11} - 52x^8y^{13} - 312x^9y^2 + 2327x^9y^4 - 2587x^9y^6 - 1300x^9y^8 + 1612x^9y^{10}$$
$$-312x^9y^{12} - 52x^{10}y + 572x^{10}y^3 - 1612x^{10}y^5 + 2327x^{10}y^9 - 858x^{10}y^{11} + 52x^{10}y^{13}$$
$$+130x^{11}y^2 - 858x^{11}y^4 + 520x^{11}y^6 + 520x^{11}y^8 - 572x^{11}y^{10} + 52x^{11}y^{12} - 52x^{12}y^3$$
$$+312x^{12}y^5 - 312x^{12}y^9 + 130x^{12}y^{11} - 13x^{12}y^{13} - 13x^{13}y^2 + 52x^{13}y^4 - 52x^{13}y^6$$
$$-52x^{13}y^8 + 52x^{13}y^{10} + x^{14}y^{13} + x^{13} = 0.$$

Proof. Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.33)\qquad P = \frac{\varphi(-q)\varphi(-q^{13})}{\varphi(q)\varphi(q^{13})} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^{13})}{\varphi(q)\varphi(-q^{13})}.$$

Replacing q by $\sqrt{q}$ in the equation (4.33) and then from equation (3.1), we deduce that

$$(4.34)\qquad P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}.$$

Employing the equation (4.34) in the equation (3.35), we deduce that

$$(4.35)\ (-y - 52xy^4 + 52xy^6 + 52xy^8 - 52xy^{10} + 13xy^{12} - xy^{14} + 13x^2y - 130x^2y^3$$
$$+312x^2y^5 - 312x^2y^9 + 52x^2y^{11} - 52x^3y^2 + 572x^3y^4 - 520x^3y^6 - 520x^3y^8$$
$$+858x^3y^{10} - 130x^3y^{12} - 52x^4y + 858x^4y^3 - 2327x^4y^5 + 1612x^4y^9 - 572x^4y^{11}$$
$$+52x^4y^{13} + 312x^5y^2 - 1612x^5y^4 + 1300x^5y^6 + 2587x^5y^8 - 2327x^5y^{10} + 312x^5y^{12}$$
$$+52x^6y - 520x^6y^3 + 2587x^6y^5 - 1716x^6y^7 - 1300x^6y^9 + 520x^6y^{11} - 52x^6y^{13}$$
$$+1716x^7y^6 - 1716x^7y^8 + 52x^8y - 520x^8y^3 + 1300x^8y^5 + 1716x^8y^7 - 2587x^8y^9$$
$$+520x^8y^{11} - 52x^8y^{13} - 312x^9y^2 + 2327x^9y^4 - 2587x^9y^6 - 1300x^9y^8 + 1612x^9y^{10}$$
$$-312x^9y^{12} - 52x^{10}y + 572x^{10}y^3 - 1612x^{10}y^5 + 2327x^{10}y^9 - 858x^{10}y^{11} + 52x^{10}y^{13}$$
$$+130x^{11}y^2 - 858x^{11}y^4 + 520x^{11}y^6 + 520x^{11}y^8 - 572x^{11}y^{10} + 52x^{11}y^{12} - 52x^{12}y^3$$
$$+312x^{12}y^5 - 312x^{12}y^9 + 130x^{12}y^{11} - 13x^{12}y^{13} - 13x^{13}y^2 + 52x^{13}y^4 - 52x^{13}y^6$$
$$-52x^{13}y^8 + 52x^{13}y^{10} + x^{13} + x^{14}y^{13}) \times (x - 13xy^2 - y^{13} + 52xy^4 - 52xy^6$$
$$-52xy^8 + 52xy^{10} + 52x^2y^3 - 312x^2y^5 + 312x^2y^9 - 130x^2y^{11} + 13x^2y^{13} + 130x^3y^2$$
$$-858x^3y^4 + 520x^3y^6 + 520x^3y^8 - 572x^3y^{10} + 52x^3y^{12} + 52x^4y - 572x^4y^3$$
$$+1612x^4y^5 - 2327x^4y^9 + 858x^4y^{11} - 52x^4y^{13} - 312x^5y^2 + 2327x^5y^4 + 1612x^5y^{10}$$
$$-312x^5y^{12} - 2587x^5y^6 - 1300x^5y^8 - 52x^6y + 520x^6y^3 - 1300x^6y^5 - 1716x^6y^7$$
$$+2587x^6y^9 - 520x^6y^{11} + 52x^6y^{13} - x^13y^{14} + 1716x^7y^6 - 1716x^7y^8 - 52x^8y$$
$$+520x^8y^3 - 2587x^8y^5 + 1716x^8y^7 + 1300x^8y^9 - 520x^8y^{11} + 52x^8y^{13} + 312x^9y^2$$
$$-1612x^9y^4 + 1300x^9y^6 + 2587x^9y^8 - 2327x^9y^{10} + 312x^9y^12 + 52x^{10}y - 858x^{10}y^3$$
$$+2327x^{10}y^5 - 1612x^{10}y^9 + 572x^{10}y^{11} - 52x^{10}y^{13} - 52x^{11}y^2 + 572x^{11}y^4$$
$$-520x^{11}y^6 - 520x^{11}y^8 + 858x^{11}y^{10} - 130x^{11}y^{12} - 13x^{12}y + 130x^{12}y^3$$
$$-312x^{12}y^5 + 312x^{12}y^9 - 52x^{12}y^{11} - 52x^{13}y^4 + 52x^{13}y^6 + 52x^{13}y^8 - 52x^{13}y^{10}$$
$$+13x^{13}y^{12} + x^{14}y)(-1 + 2x + x^2)^6 = 0.$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^{13/2})$ as $q \to 0$, it can be seen that the first factor vanishes for q sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof. □

Theorem 4.10. *If* $x = H(q)$ *and* $y = H(q^{15})$*, then*

$$
\begin{aligned}
(4.36)\quad & 2200x^3y^{13} - 210x^3y^{15} + 15xy^3 + 9380x^4y^{12} - 2850x^4y^{14} - 300x^4y^{16} + 30720x^5y^{11} \\
- \;& 16120x^5y^{13} + 10x^5y^{15} - 57694x^6y^{12} - xy - 75xy^5 + 125xy^7 + 30xy^9 + 15x^2y^2 \\
- \;& 210x^2y^4 + 693x^2y^6 + 300x^2y^8 - 221120x^{10}y^8 + 15x^3y - 330x^3y^3 + 1410x^3y^5 \\
+ \;& 485x^3y^7 - 2610x^3y^9 - 210x^4y^2 + 1720x^4y^4 - 1966x^4y^6 - 5580x^4y^8 - 230x^4y^{10} \\
- \;& 75x^5y + 1410x^5y^3 - 3930x^5y^5 - 11265x^5y^7 + 1350x^5y^9 + 693x^6y^2 - 1966x^6y^4 \\
- \;& 14655x^6y^6 + 5520x^6y^8 + 68225x^6y^{10} - 162xy^{11} + 30xy^{13} + 30xy^{15} - 210x^2y^{12} \\
+ \;& 435x^2y^{14} - 8x^2y^{16} - 948x^3y^{11} - 347150x^{15}y^{11} + 220290x^{15}y^{13} - 261446x^{15}y^{15} \\
+ \;& 132664x^{16}y^{12} - 162x^{11}y + 300x^8y^2 + 68225x^{10}y^6 - 948x^{11}y^3 + 485x^7y^3 \\
- \;& 5580x^8y^4 - 230x^{10}y^4 - 995x^{10}y^2 + 1350x^9y^5 + 30720x^{11}y^5 - 261446x^9y^9 \\
- \;& 995x^2y^{10} - 11265x^7y^5 + 5520x^8y^6 + 113518x^9y^7 - 133770x^{11}y^7 + 8071x^7y^7 \\
+ \;& 135270x^8y^8 - 2610x^9y^3 + 220290x^{11}y^9 + 113518x^7y^9 - 221120x^8y^{10} \\
+ \;& 256960x^{10}y^{10} - 210x^{12}y^2 + 9380x^{12}y^4 - 57694x^{12}y^6 + 132664x^{12}y^8 \\
- \;& 617680x^{12}y^{10} + 2200x^{13}y^3 - 16120x^{13}y^5 + 48890x^{13}y^7 - 347150x^{13}y^9 \\
- \;& 750394x^{11}y^{11} + 30x^{13}y - 2850x^{14}y^4 + 10203x^{14}y^6 - 139368x^{14}y^8 \\
+ \;& 645180x^{14}y^{10} + 30x^{15}y + 10203x^6y^{14} - 12940x^6y^{16} + 125yx^7 + 30yx^9 \\
- \;& 617680x^{10}y^{12} + 1068900x^{12}y^{12} + 132664x^8y^{12} + 945854x^{11}y^{13} - 750394x^{13}y^{13} \\
+ \;& 48890x^7y^{13} - 347150x^9y^{13} + 645180x^{10}y^{14} - 617680x^{12}y^{14} - 139368x^8y^{14} \\
+ \;& 1350x^{15}y^{19} - 347150x^{11}y^{15} + 220290x^{13}y^{15} - 45790x^7y^{15} + 323050x^9y^{15} \\
- \;& 139368x^{10}y^{16} + 132664x^{12}y^{16} + 105800x^8y^{16} + 435x^{14}y^2 + 945854x^{13}y^{11} \\
- \;& 133770x^7y^{11} + 220290x^9y^{11} - 617680x^{14}y^{12} + 256960x^{14}y^{14} - 221120x^{14}y^{16} \\
+ \;& x^{16} - 210x^{15}y^3 + 10x^{15}y^5 - 45790x^{15}y^7 + 323050x^{15}y^9 - 8x^{16}y^2 - 300x^{16}y^4 \\
- \;& 12940x^{16}y^6 + 105800x^{16}y^8 - 139368x^{16}y^{10} - 221120x^{16}y^{14} + 135270x^{16}y^{16} \\
+ \;& 21444x^7y^{17} - 2388x^7y^{19} - 12940x^8y^{18} - 2388x^5y^{17} - 8x^8y^{22} + x^8y^{24} \\
- \;& 45790x^9y^{17} - 210x^9y^{21} + 10x^9y^{19} + 30x^9y^{23} + 10203x^{10}y^{18} - 2850x^{10}y^{20} \\
+ \;& 435x^{10}y^{22} + 48890x^{11}y^{17} + 2200x^{11}y^{21} - 16120x^{11}y^{19} + 30x^{11}y^{23} + x^3y^{21} \\
- \;& 5x^3y^{19} - 120x^4y^{18} + 30x^4y^{20} - 300x^8y^{20} - 5x^5y^{21} + 461x^5y^{19} + 2930x^6y^{18} \\
- \;& 120x^6y^{20} - 57694x^{12}y^{18} + 9380x^{12}y^{20} - 210x^{12}y^{22} - 133770x^{13}y^{17} \\
- \;& 948x^{13}y^{21} + 30720x^{13}y^{19} - 162x^{13}y^{23} + 68225x^{14}y^{18} - 230x^{14}y^{20} - 995x^{14}y^{22} \\
+ \;& 113518x^{15}y^{17} - x^{23}y^{23} - 2610x^{15}y^{21} + 30x^{15}y^{23} + 5520x^{16}y^{18} - 5580x^{16}y^{20} \\
+ \;& 300x^{16}y^{22} - 2388x^{17}y^5 + 21444x^{17}y^7 - 45790x^{17}y^9 + 48890x^{17}y^{11} - 133770x^{17}y^{13} \\
+ \;& 113518x^{17}y^{15} + 8071x^{17}y^{17} + 485x^{17}y^{21} - 11265x^{17}y^{19} + 125x^{17}y^{23} - 120x^{18}y^4 \\
+ \;& 2930x^{18}y^6 - 12940x^{18}y^8 + 10203x^{18}y^{10} - 57694x^{18}y^{12} + 68225x^{18}y^{14}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 5520x^{18}y^{16} - 5x^{19}y^3 - 14655x^{18}y^{18} - 1966x^{18}y^{20} + 693x^{18}y^{22} + 461x^{19}y^5 \\
&- \quad 2388x^{19}y^7 + 10x^{19}y^9 - 16120x^{19}y^{11} + 30720x^{19}y^{13} + 1350x^{19}y^{15} - 11265x^{19}y^{17} \\
&+ \quad 1410x^{19}y^{21} - 3930x^{19}y^{19} - 75x^{19}y^{23} + 30x^{20}y^4 - 120x^{20}y^6 - 300x^{20}y^8 \\
&- \quad 2850x^{20}y^{10} + 9380x^{20}y^{12} - 230x^{20}y^{14} - 5580x^{20}y^{16} - 1966x^{20}y^{18} + 1720x^{20}y^{20} \\
&\quad -210x^{20}y^{22} - 8x^{22}y^8 - 5x^{21}y^5 - 210x^{21}y^9 + 2200x^{21}y^{11} - 948x^{21}y^{13} - 2610x^{21}y^{15} \\
&\quad +485x^{21}y^{17} - 330x^{21}y^{21} + 1410x^{21}y^{19} + y^{16} + 15x^{21}y^{23} + x^{21}y^3 + 435x^{22}y^{10} \\
&\quad -210x^{22}y^{12} - 995x^{22}y^{14} + 300x^{22}y^{16} + 693x^{22}y^{18} - 210x^{22}y^{20} + x^{24}y^8 + 30x^{23}y^9 \\
&\quad +30x^{23}y^{11} - 162x^{23}y^{13} + 30x^{23}y^{15} + 125x^{23}y^{17} + 15x^{23}y^{21} - 75x^{23}y^{19} + 15x^{22}y^{22} = 0.
\end{aligned}
$$

Theorem 4.11. *If $x = H(q)$ and $y = H(q^{17})$, then*

$$
\begin{aligned}
(4.37)\quad & -xy + 17xy^3 - 102xy^5 + 238xy^7 - 85xy^9 - 323xy^{11} + 204xy^{13} + 68xy^{15} - 34xy^{17} \\
& +34x^2y^2 - 442x^2y^4 + 1564x^2y^6 - 748x^2y^8 - 646x^2y^{10} + 1054x^2y^{12} - 952x^2y^{14} \\
& +289x^2y^{16} + x^{18} + y^{18} + 17x^3y - 544x^3y^3 + 4029x^3y^5 - 8432x^3y^7 - 1105x^3y^9 \\
& +9112x^3y^{11} - 3009x^3y^{13} - 952x^3y^{15} + 68x^3y^{17} - 442x^4y^2 + 5406x^4y^4 \\
& -19448x^4y^6 + 15844x^4y^8 + 11254x^4y^{10} - 20298x^4y^{12} + 11696x^4y^{14} - 952x^4y^{16} \\
& -102x^5y + 4029x^5y^3 - 29274x^5y^5 + 53805x^5y^7 + 9690x^5y^9 - 54825x^5y^{11} \\
& +10914x^5y^{13} - 3009x^5y^{15} + 204x^5y^{17} + 1564x^6y^2 - 19448x^6y^4 + 74358x^6y^6 \\
& -75514x^6y^8 - 55624x^6y^{10} + 112472x^6y^{12} - 20298x^6y^{14} + 1054x^6y^{16} + 238x^7y \\
& -8432x^7y^3 + 53805x^7y^5 - 76432x^7y^7 + 1105x^7y^9 + 43928x^7y^{11} - 54825x^7y^{13} \\
& +9112x^7y^{15} - 323x^7y^{17} - 748x^8y^2 + 15844x^8y^4 - 75514x^8y^6 + 40426x^8y^8 \\
& +108766x^8y^{10} - 55624x^8y^{12} + 11254x^8y^{14} - 646x^8y^{16} - 85x^9y - 1105x^9y^3 \\
& +9690x^9y^5 + 1105x^9y^7 - 67830x^9y^9 + 1105x^9y^{11} + 9690x^9y^{13} - 1105x^9y^{15} \\
& -85x^9y^{17} - 646x^{10}y^2 + 11254x^{10}y^4 - 55624x^{10}y^6 + 108766x^{10}y^8 + 204x^{13}y \\
& +40426x^{10}y^{10} - 75514x^{10}y^{12} + 15844x^{10}y^{14} - 748x^{10}y^{16} - 323x^{11}y + 9112x^{11}y^3 \\
& -54825x^{11}y^5 + 43928x^{11}y^7 + 1105x^{11}y^9 - 76432x^{11}y^{11} + 53805x^{11}y^{13} \\
& -8432x^{11}y^{15} + 238x^{11}y^{17} + 1054x^{12}y^2 - 20298x^{12}y^4 + 112472x^{12}y^6 - 55624x^{12}y^8 \\
& -75514x^{12}y^{10} + 74358x^{12}y^{12} - 19448x^{12}y^{14} + 1564x^{12}y^{16} - 3009x^{13}y^3 + 10914x^{13}y^5 \\
& -54825x^{13}y^7 + 9690x^{13}y^9 + 53805x^{13}y^{11} - 29274x^{13}y^{13} + 4029x^{13}y^{15} - 102x^{13}y^{17} \\
& -952x^{14}y^2 + 11696x^{14}y^4 - 20298x^{14}y^6 + 11254x^{14}y^8 + 15844x^{14}y^{10} - 19448x^{14}y^{12} \\
& +5406x^{14}y^{14} - 442x^{14}y^{16} + 68x^{15}y - 952x^{15}y^3 - 3009x^{15}y^5 + 9112x^{15}y^7 - 1105x^{15}y^9 \\
& -8432x^{15}y^{11} + 4029x^{15}y^{13} - 544x^{15}y^{15} + 17x^{15}y^{17} + 289x^{16}y^2 - 952x^{16}y^4 \\
& +1054x^{16}y^6 - 646x^{16}y^8 - 748x^{16}y^{10} + 1564x^{16}y^{12} - 442x^{16}y^{14} + 34x^{16}y^{16} - 34x^{17}y \\
& +68x^{17}y^3 + 204x^{17}y^5 - 323x^{17}y^7 - 85x^{17}y^9 + 238x^{17}y^{11} - 102x^{17}y^{13} + 17x^{17}y^{15} \\
& -x^{17}y^{17} = 0.
\end{aligned}
$$

Theorem 4.12. *If $x = H(q)$ and $y = H(q^{19})$, then*

$$
\begin{aligned}
(4.38)\quad & -22648x^{12}y^{17}+9310x^{14}y^{17}-1957x^{16}y^{17}+171x^{18}y^{17}-171x^3y^{18}+608x^5y^{18}\\
&-4465x^7y^{18}+4902x^9y^{18}+3667x^{11}y^{18}-5624x^{13}y^{18}+817x^{15}y^{18}+114x^{17}y^{18}\\
&-19x^{19}y^{18}+209x^4y^{19}-19x^6y^{19}-703x^8y^{19}+437x^{10}y^{19}+361x^{12}y^{19}\\
&-399x^{14}y^{19}+133x^{16}y^{19}-19x^{18}y^{19}+x^{20}y^{19}+x^{19}-xy^{20}+19xy^2-209xy^4\\
&+19xy^6+703xy^8-437xy^{10}-361xy^{12}+399xy^{14}-133xy^{16}+19xy^{18}+19x^2y\\
&-171x^2y^3+608x^2y^5-4465x^2y^7+4902x^2y^9+3667x^2y^{11}-5624x^2y^{13}\\
&+817x^2y^{15}+114x^2y^{17}-19x^2y^{19}-252529x^{14}y^{11}+60705x^{16}y^{11}-347529x^8y^{11}\\
&-101023x^{10}y^{11}+420147x^{12}y^{11}-57741x^4y^{11}+2280x^{10}y^{17}-y-32813x^{13}y^{10}\\
&-22325x^{15}y^{10}+2280x^{17}y^{10}+437x^{19}y^{10}-133yx^4+399yx^6-361yx^8\\
&-437yx^{10}+703yx^{12}+19yx^{14}-209yx^{16}+19yx^{18}-114y^2x^3-817y^2x^5\\
&+5624y^2x^7-3667y^2x^9-4902y^2x^{11}+4465y^2x^{13}-608y^2x^{15}+171y^2x^{17}\\
&-19y^2x^{19}+1957x^4y^3-9310x^6y^3+22648x^8y^3-2280x^{10}y^3-22990x^{12}y^3\\
&+11704x^{14}y^3-646x^{16}y^3-114x^{18}y^3-646x^3y^4+21413x^5y^4-72789x^7y^4\\
&+57741x^9y^4+60705x^{11}y^4-77957x^{13}y^4+16473x^{15}y^4-1957x^{17}y^4+133x^{19}y^4\\
&-16473x^4y^5+107521x^6y^5-196289x^8y^5+22325x^{10}y^5+187473x^{12}y^5\\
&-118009x^{14}y^5+21413x^{16}y^5-817x^{18}y^5+11704x^3y^6-118009x^5y^6\\
&+337117x^7y^6-262409x^9y^6-252529x^{11}y^6+398221x^{13}y^6-107521x^{15}y^6\\
&+9310x^{17}y^6-399x^{19}y^6+77957x^4y^7-398221x^6y^7+563445x^8y^7+32813x^{10}y^7\\
&-518225x^{12}y^7+337117x^{14}y^7-72789x^{16}y^7+5624x^{18}y^7-22990x^3y^8+187473x^5y^8\\
&-518225x^7y^8+347529x^9y^8+420147x^{11}y^8-563445x^{13}y^8+196289x^{15}y^8\\
&-22648x^{17}y^8+361x^{19}y^8-60705x^4y^9+252529x^6y^9-420147x^8y^9+101023x^{10}y^9\\
&+347529x^{12}y^9-262409x^{14}y^9+57741x^{16}y^9-3667x^{18}y^9-2280x^3y^{10}+22325x^5y^{10}\\
&+32813x^7y^{10}+101023x^9y^{10}-101023x^{11}y^{10}-4902x^{18}y^{11}+22648x^3y^{12}\\
&-196289x^5y^{12}+563445x^7y^{12}-420147x^9y^{12}-347529x^{11}y^{12}+518225x^{13}y^{12}\\
&-187473x^{15}y^{12}+22990x^{17}y^{12}-703x^{19}y^{12}+72789x^4y^{13}-337117x^6y^{13}\\
&+518225x^8y^{13}-32813x^{10}y^{13}-563445x^{12}y^{13}+398221x^{14}y^{13}-77957x^{16}y^{13}\\
&+4465x^{18}y^{13}-9310x^3y^{14}+107521x^5y^{14}-398221x^7y^{14}+252529x^9y^{14}+262409x^6y^{11}\\
&-337117x^{13}y^{14}+22990x^8y^{17}-11704x^{17}y^{14}-19x^{19}y^{14}-21413x^4y^{15}+118009x^6y^{15}\\
&-187473x^8y^{15}-22325x^{10}y^{15}+196289x^{12}y^{15}-107521x^{14}y^{15}+16473x^{16}y^{15}\\
&-608x^{18}y^{15}+1957x^3y^{16}-16473x^5y^{16}+77957x^7y^{16}-60705x^9y^{16}-57741x^{11}y^{16}\\
&+72789x^{13}y^{16}-21413x^{15}y^{16}+646x^{17}y^{16}+209x^{19}y^{16}+646x^4y^{17}-11704x^6y^{17}\\
&+118009x^{15}y^{14}+262409x^{11}y^{14}=0.
\end{aligned}
$$

Theorem 4.13. *If $x = H(q)$ and $y = H(q^{23})$, then*

$$
\begin{aligned}
(4.39)\quad & 23x^3y - 1610x^3y^3 + 18078x^3y^5 - 68655x^3y^7 + 90620x^3y^9 - 67022x^3y^{11} \\
& -50462x^3y^{13} + 133584x^3y^{15} - 57270x^3y^{17} + 1725x^3y^{19} - 989x^3y^{21} \\
& -46x^3y^{23} + 1012x^4y^2 - 23000x^4y^4 + 170292x^4y^6 - 484748x^4y^8 \\
& +391046x^4y^{10} + 261096x^4y^{12} - 343022x^4y^{14} + 171764x^4y^{16} - 212106x^4y^{18} + \\
& 85514x^4y^{20} - 7222x^4y^{22} - 207x^5y + 18078x^5y^3 - 234646x^5y^5 + 1041739x^5y^7 \\
& -1593532x^5y^9 + 817558x^5y^{11} + 661066x^5y^{13} - 1635484x^5y^{15} \\
& +1123090x^5y^{17} - 243133x^5y^{19} + 1725x^5y^{21} + 1242x^5y^{23} - 7222x^6y^2 \\
& +170292x^6y^4 - 1297292x^6y^6 + 4046252x^6y^8 - 4913329x^6y^{10} - 672750x^6y^{12} \\
& +5037253x^6y^{14} - 3331688x^6y^{16} + 1300443x^6y^{18} - 212106x^6y^{20} \\
& +14743x^6y^{22} + 897x^7y - 68655x^7y^3 + 1041739x^7y^5 - 5600293x^7y^7 \\
& +10840636x^7y^9 - 5288620x^7y^{11} - 5626444x^7y^{13} + 9489340x^7y^{15} \\
& -6199558x^7y^{17} + 1123090x^7y^{19} - 57270x^7y^{21} - 966x^7y^{23} - 46x^2y^2 \\
& +1012x^2y^4 - 7222x^2y^6 + 16928x^2y^8 + 1863x^2y^{10} - 12834x^2y^{12} \\
& -10143x^2y^{14} + 2116x^2y^{16} + 14743x^2y^{18} - 7222x^2y^{20} + 1081x^2y^{22} - xy \\
& +23xy^3 - 207xy^5 + 897xy^7 - 1748xy^9 + 460xy^{11} + 2944xy^{13} - 2576xy^{15} \\
& -966xy^{17} + 1242xy^{19} - 46xy^{21} - 46xy^{23} + x^{24} + y^{24} + 1012x^{22}y^{20} - 46x^{22}y^{22} \\
& -46x^{23}y - 46x^{23}y^3 + 1242x^{23}y^5 + 1081x^{22}y^2 - 7222x^{22}y^4 + 14743x^{22}y^6 \\
& +2116x^{22}y^8 - 10143x^{22}y^{10} - 12834x^{22}y^{12} + 1863x^{22}y^{14} + 16928x^{22}y^{16} \\
& -7222x^{22}y^{18} - 966x^{23}y^7 - 2576x^{23}y^9 + 2944x^{23}y^{11} + 460x^{23}y^{13} - 1748x^{23}y^{15} \\
& +897x^{23}y^{17} - 207x^{23}y^{19} + 23x^{23}y^{21} - x^{23}y^{23} - 46x^{21}y - 989x^{21}y^3 + 1725x^{21}y^5 \\
& -57270x^{21}y^7 + 133584x^{21}y^9 - 50462x^{21}y^{11} - 67022x^{21}y^{13} + 23x^{21}y^{23} \\
& +90620x^{21}y^{15} - 68655x^{21}y^{17} + 18078x^{21}y^{19} - 1610x^{21}y^{21} + 16928x^8y^2 \\
& -484748x^8y^4 + 4046252x^8y^6 - 13628512x^8y^8 + 19391208x^8y^{10} \\
& -2667816x^8y^{12} - 20124816x^8y^{14} + 17344783x^8y^{16} - 3331688x^8y^{18} \\
& +171764x^8y^{20} + 2116x^8y^{22} - 1748x^9y + 90620x^9y^3 - 1593532x^9y^5 \\
& +10840636x^9y^7 - 28673502x^9y^9 + 18981210x^9y^{11} + 19338354x^9y^{13} \\
& -28274406x^9y^{15} + 9489340x^9y^{17} - 1635484x^9y^{19} + 133584x^9y^{21} \\
& -2576x^9y^{23} + 1863x^{10}y^2 + 391046x^{10}y^4 - 4913329x^{10}y^6 + 19391208x^{10}y^8 \\
& -26483212x^{10}y^{10} + 685584x^{10}y^{12} + 28328824x^{10}y^{14} - 20124816x^{10}y^{16} \\
& +5037253x^{10}y^{18} - 343022x^{10}y^{20} - 10143x^{10}y^{22} + 460x^{11}y - 67022x^{11}y^3 \\
& +817558x^{11}y^5 - 5288620x^{11}y^7 + 18981210x^{11}y^9 - 17142866x^{11}y^{11} \\
& -14122322x^{11}y^{13} + 19338354x^{11}y^{15} - 5626444x^{11}y^{17} + 661066x^{11}y^{19} \\
& -50462x^{11}y^{21} + 2944x^{11}y^{23} - 5626444x^{17}y^{11} + 261096x^{12}y^4 - 672750x^{12}y^6
\end{aligned}
$$

$$-2667816x^{12}y^8+685584x^{12}y^{10}+7517596x^{12}y^{12}+685584x^{12}y^{14}$$
$$-2667816x^{12}y^{16}-672750x^{12}y^{18}+261096x^{12}y^{20}-12834x^{12}y^{22}+2944x^{13}y$$
$$-50462x^{13}y^3+661066x^{13}y^5-5626444x^{13}y^7+19338354x^{13}y^9-14122322x^{13}y^{11}$$
$$-17142866x^{13}y^{13}+18981210x^{13}y^{15}-5288620x^{13}y^{17}+817558x^{13}y^{19}$$
$$-67022x^{13}y^{21}+460x^{13}y^{23}-10143x^{14}y^2-343022x^{14}y^4+5037253x^{14}y^6$$
$$-20124816x^{14}y^8+28328824x^{14}y^{10}+685584x^{14}y^{12}-26483212x^{14}y^{14}$$
$$+19391208x^{14}y^{16}-4913329x^{14}y^{18}+391046x^{14}y^{20}+1863x^{14}y^{22}-2576x^{15}y$$
$$+133584x^{15}y^3-1635484x^{15}y^5+9489340x^{15}y^7-28274406x^{15}y^9+19338354x^{15}y^{11}$$
$$+18981210x^{15}y^{13}-28673502x^{15}y^{15}+10840636x^{15}y^{17}-1593532x^{15}y^{19}$$
$$+90620x^{15}y^{21}-1748x^{15}y^{23}+2116x^{16}y^2+171764x^{16}y^4-3331688x^{16}y^6$$
$$+17344783x^{16}y^8-20124816x^{16}y^{10}-2667816x^{16}y^{12}+19391208x^{16}y^{14}$$
$$-13628512x^{16}y^{16}+4046252x^{16}y^{18}-484748x^{16}y^{20}+16928x^{16}y^{22}-966x^{17}y$$
$$-57270x^{17}y^3+1123090x^{17}y^5-6199558x^{17}y^7+9489340x^{17}y^9-5288620x^{17}y^{13}$$
$$+10840636x^{17}y^{15}-5600293x^{17}y^{17}+1041739x^{17}y^{19}-68655x^{17}y^{21}+897x^{17}y^{23}$$
$$+14743x^{18}y^2-212106x^{18}y^4+1300443x^{18}y^6-3331688x^{18}y^8+5037253x^{18}y^{10}$$
$$-672750x^{18}y^{12}-4913329x^{18}y^{14}+4046252x^{18}y^{16}-1297292x^{18}y^{18}+170292x^{18}y^{20}$$
$$-7222x^{18}y^{22}+1242x^{19}y+1725x^{19}y^3-243133x^{19}y^5+1123090x^{19}y^7-1635484x^{19}y^9$$
$$+661066x^{19}y^{11}+817558x^{19}y^13-1593532x^{19}y^{15}+1041739x^{19}y^{17}-234646x^{19}y^{19}$$
$$+18078x^{19}y^{21}-207x^{19}y^{23}-7222x^{20}y^2+85514x^{20}y^4-212106x^{20}y^6+171764x^{20}y^8$$
$$-343022x^{20}y^{10}+261096x^{20}y^{12}+391046x^{20}y^{14}-484748x^{20}y^{16}+170292x^{20}y^{18}$$
$$-23000x^{20}y^{20}+1012x^{20}y^{22}-12834x^{12}y^2=0.$$

Theorem 4.14. *If $x=H(q)$ and $y=H(q^{25})$, then*

$$(4.40)\quad -xy-1050xy^{21}-350xy^{23}+3925xy^{19}+25xy^3-250xy^5+1250xy^7-3075xy^9$$
$$+2355xy^{11}+4000xy^{13}-7200xy^{15}+275xy^{17}+75xy^{25}-xy^{27}-74x^2y^2$$
$$-9150x^2y^{18}-12900x^2y^{22}+29400x^2y^{20}+1350x^2y^4-7500x^2y^6+7125x^2y^8$$
$$+38600x^2y^{10}-40705x^2y^{12}-9000x^2y^{14}+1410x^2y^{16}+1750x^2y^{24}-2175x^3y^9$$
$$-80x^2y^{26}+x^2y^{28}+25x^3y-54865x^3y^{21}+16750x^3y^{23}+199450x^3y^{19}$$
$$-2026x^3y^3+1863100x^{24}y^{20}-67850x^3y^7+95740x^3y^{11}+166955x^3y^{13}+x^{26}$$
$$-8325525x^{24}y^{18}+3464580x^{24}y^{22}+4607920x^{24}y^{14}-37700x^{24}y^4+347840x^{24}y^6$$
$$+1750x^{24}y^2-1060020x^{24}y^8+38135x^{24}y^{10}-1816060x^{24}y^{12}+22075x^3y^5$$
$$+2221160x^{24}y^{16}-1436500x^{24}y^{24}+185555x^{24}y^{26}+y^{26}-7500x^{24}y^{28}$$
$$+75x^{25}y+192522400x^{17}y^{13}+1083170x^{25}y^{23}-1609775x^{25}y^{19}+38400x^{25}y^5$$
$$-235590x^{25}y^7-549000x^{25}y^{21}-87495925x^{17}y^{11}+82506520x^{17}y^{15}$$
$$-274121325x^{17}y^{17}-2475x^{25}y^3-151835x^{17}y^{25}+166955x^{17}y^{27}+4000x^{17}y^{29}$$
$$-9150x^{18}y^2-116535050x^{18}y^{18}+16209500x^{18}y^{22}+40667030x^{18}y^{20}$$
$$+232860x^{18}y^4-1816060x^{18}y^6+6767370x^{18}y^8-9048640x^{18}y^{10}$$
$$-29557375x^{18}y^{12}+66718200x^{18}y^{14}+33346730x^{18}y^{16}-8325525x^{18}y^{24}$$

$$+1078360x^{18}y^{26} - 40705x^{18}y^{28} + 3925x^{19}y + 18490695x^{19}y^{21} + 5270050x^{19}y^{23}$$
$$- 109573050x^{19}y^{19} + 199450x^{19}y^{3} - 2004985x^{19}y^{5} + 521210x^{19}y^{7} + 30451505x^{19}y^{9}$$
$$- 44846360x^{19}y^{11} - 87495925x^{19}y^{13} + 121201700x^{19}y^{15} + 69658245x^{19}y^{17} - 1609775x^{19}y^{25}$$
$$+ 95740x^{19}y^{27} + 2355x^{19}y^{29} + 29400x^{20}y^{2} + 40667030x^{20}y^{18} + 16295005x^{20}y^{22}$$
$$- 68113850x^{20}y^{20} - 277150x^{20}y^{4} + 38135x^{20}y^{6} + 5897940x^{20}y^{8} - 12759005x^{20}y^{10}$$
$$- 9048640x^{20}y^{12} - 28632175x^{20}y^{14} + 54510600x^{20}y^{16} + 1863100x^{20}y^{24} - 722775x^{20}y^{26}$$
$$+ 38600x^{20}y^{28} - 24925925x^{21}y^{21} + 7444970x^{21}y^{23} + 18490695x^{21}y^{19} - 54865x^{21}y^{3}$$
$$+ 780095x^{21}y^{5} - 1050x^{21}y - 2482910x^{21}y^{7} - 4219440x^{21}y^{9} + 30451505x^{21}y^{11}$$
$$- 20239885x^{21}y^{13} - 46677200x^{21}y^{15} + 42021150x^{21}y^{17} - 549000x^{21}y^{25} - 2175x^{21}y^{27}$$
$$- 12900x^{22}y^{2} + 16209500x^{22}y^{18} - 15835700x^{22}y^{22} + 16295005x^{22}y^{20} + 194365x^{22}y^{4}$$
$$- 1060020x^{22}y^{6} + 926960x^{22}y^{8} + 5897940x^{22}y^{10} + 6767370x^{22}y^{12} - 7830840x^{22}y^{14}$$
$$- 24696800x^{22}y^{16} + 3464580x^{22}y^{24} - 3075x^{21}y^{29} - 262300x^{22}y^{26} + 7125x^{22}y^{28} - 350x^{23}y$$
$$+ 7444970x^{23}y^{21} - 5618900x^{23}y^{23} + 5270050x^{23}y^{19} + 16750x^{23}y^{3} - 235590x^{23}y^{5}$$
$$+ 1248020x^{23}y^{7} - 2482910x^{23}y^{9} + 521210x^{23}y^{11} + 8747730x^{23}y^{13} - 2424360x^{23}y^{15}$$
$$- 13577550x^{23}y^{17} + 1083170x^{23}y^{25} - 67850x^{23}y^{27} + 1250x^{23}y^{29} - 244900x^{3}y^{15}$$
$$- 128235x^{3}y^{17} - 2475x^{3}y^{25} + 80x^{3}y^{27} - x^{3}y^{29} + 1350x^{4}y^{2} + 232860x^{4}y^{18} + 194365x^{4}y^{22}$$
$$- 277150x^{4}y^{20} - 29274x^{4}y^{4} + 185555x^{4}y^{6} - 262300x^{4}y^{8} - 722775x^{4}y^{10} + 1078360x^{4}y^{12}$$
$$+ 123070x^{4}y^{14} - 482360x^{4}y^{16} - 37700x^{4}y^{24} + 2475x^{4}y^{26} - 80x^{4}y^{28} + x^{4}y^{30} - 250x^{5}y$$
$$+ 780095x^{5}y^{21} - 235590x^{5}y^{23} - 2004985x^{5}y^{19} + 22075x^{5}y^{3} - 282001x^{5}y^{5} + 1083170x^{5}y^{7}$$
$$- 549000x^{5}y^{9} - 1609775x^{5}y^{11} - 151835x^{5}y^{13} + 2808680x^{5}y^{15} + 83110x^{5}y^{17} + 38400x^{5}y^{25}$$
$$- 2475x^{5}y^{27} - 7500x^{6}y^{2} + 1750x^{6}y^{28} - 1816060x^{6}y^{18} - 1060020x^{6}y^{22} + 38135x^{6}y^{20}$$
$$+ 185555x^{6}y^{4} - 1436500x^{6}y^{6} + 3464580x^{6}y^{8} + 75x^{5}y^{29} + 1863100x^{6}y^{10} - 8325525x^{6}y^{12}$$
$$+ 2221160x^{6}y^{14} + 4607920x^{6}y^{16} + 347840x^{6}y^{24} - 37700x^{6}y^{26} + 1250x^{7}y - 2482910x^{7}y^{21}$$
$$+ 1248020x^{7}y^{23} + 521210x^{7}y^{19} - 67850x^{7}y^{3} + 1083170x^{7}y^{5} - 5618900x^{7}y^{7} + 7444970x^{7}y^{9}$$
$$+ 5270050x^{7}y^{11} - 13577550x^{7}y^{13} - 2424360x^{7}y^{15} + 8747730x^{7}y^{17} - 235590x^{7}y^{25}$$
$$- 350x^{7}y^{29} + 7125x^{8}y^{2} + 6767370x^{8}y^{18} + 926960x^{8}y^{22} + 5897940x^{8}y^{20} - 262300x^{8}y^{4}$$
$$+ 3464580x^{8}y^{6} - 15835700x^{8}y^{8} + 16295005x^{8}y^{10} + 16209500x^{8}y^{12} + 16750x^{7}y^{27}$$
$$- 24696800x^{8}y^{14} - 7830840x^{8}y^{16} - 1060020x^{8}y^{24} + 194365x^{8}y^{26} - 12900x^{8}y^{28}$$
$$- 3075x^{9}y - 4219440x^{9}y^{21} - 2482910x^{9}y^{23} - 1050x^{9}y^{29} + 30451505x^{9}y^{19} - 549000x^{9}y^{5}$$
$$+ 7444970x^{9}y^{7} - 24925925x^{9}y^{9} + 18490695x^{9}y^{11} + 42021150x^{9}y^{13} - 46677200x^{9}y^{15}$$
$$- 20239885x^{9}y^{17} + 780095x^{9}y^{25} - 54865x^{9}y^{27} + 38600x^{10}y^{2} - 9048640x^{10}y^{18} - 2175x^{9}y^{3}$$
$$+ 5897940x^{10}y^{22} - 12759005x^{10}y^{20} - 722775x^{10}y^{4} + 1863100x^{10}y^{6} + 16295005x^{10}y^{8}$$
$$- 68113850x^{10}y^{10} + 40667030x^{10}y^{12} + 54510600x^{10}y^{14} - 28632175x^{10}y^{16} + 38135x^{10}y^{24}$$
$$- 277150x^{10}y^{26} + 521210x^{11}y^{23} + 29400x^{10}y^{28} + 30451505x^{11}y^{21} - 44846360x^{11}y^{19}$$
$$+ 95740x^{11}y^{3} - 1609775x^{11}y^{5} - 87495925x^{11}y^{17} + 5270050x^{11}y^{7} + 18490695x^{11}y^{9}$$
$$- 109573050x^{11}y^{11} + 69658245x^{11}y^{13} + 121201700x^{11}y^{15} + 2355x^{11}y + 199450x^{11}y^{27}$$
$$+ 3925x^{11}y^{29} - 40705x^{12}y^{2} - 29557375x^{12}y^{18} + 6767370x^{12}y^{22} - 9048640x^{12}y^{20}$$
$$+ 1078360x^{12}y^{4} - 8325525x^{12}y^{6} + 16209500x^{12}y^{8} + 40667030x^{12}y^{10} - 116535050x^{12}y^{12}$$
$$+ 33346730x^{12}y^{14} + 66718200x^{12}y^{16} - 1816060x^{12}y^{24} + 232860x^{12}y^{26} - 9150x^{12}y^{28}$$
$$- 20239885x^{13}y^{21} + 8747730x^{13}y^{23} - 87495925x^{13}y^{19} + 166955x^{13}y^{3} - 151835x^{13}y^{5}$$

$$-13577550x^{13}y^7 + 42021150x^{13}y^9 + 69658245x^{13}y^{11} - 274121325x^{13}y^{13} + 82506520x^{13}y^{15} + 4000x^{13}y + 192522400x^{13}y^{17} + 83110x^{13}y^{25} - 128235x^{13}y^{27} + 275x^{13}y^{29} - 9000x^{14}y^2 + 66718200x^{14}y^{18} - 7830840x^{14}y^{22} - 28632175x^{14}y^{20} + 123070x^{14}y^4 + 25x^{29}y^{27} + 2221160x^{14}y^6 - 24696800x^{14}y^8 + 54510600x^{14}y^{10} + 33346730x^{14}y^{12} - 187634800x^{14}y^{14} + 1410x^{16}y^2 + 88165480x^{14}y^{16} + 4607920x^{14}y^{24} - 482360x^{14}y^{26} + 1410x^{14}y^{28} - 7200x^{15}y - 46677200x^{15}y^{21} + x^{30}y^4 - 2424360x^{15}y^{23} + 121201700x^{15}y^{19} - 244900x^{15}y^3 + 2808680x^{15}y^5 - 2424360x^{15}y^7 - 46677200x^{15}y^9 + 121201700x^{15}y^{11} + 82506520x^{15}y^{13} - 314920800x^{15}y^{15} + 82506520x^{15}y^{17} + 2808680x^{15}y^{25} - 250x^{29}y^{25} - 244900x^{15}y^{27} + 33346730x^{16}y^{18} - 24696800x^{16}y^{22} + 54510600x^{16}y^{20} - 482360x^{16}y^4 + 4607920x^{16}y^6 - 7200x^{29}y^{15} - 7200x^{15}y^{29} - 7830840x^{16}y^8 - 28632175x^{16}y^{10} + 66718200x^{16}y^{12} + 88165480x^{16}y^{14} - 187634800x^{16}y^{16} + 2221160x^{16}y^{24} - 2004985x^{11}y^{25} + 123070x^{16}y^{26} - 9000x^{16}y^{28} + 42021150x^{17}y^{21} - 13577550x^{17}y^{23} + 69658245x^{17}y^{19} - 128235x^{17}y^3 + 83110x^{17}y^5 + 8747730x^{17}y^7 - 20239885x^{17}y^9 + 780095x^{25}y^9 - 2004985x^{25}y^{11} + 2808680x^{25}y^{15} - 151835x^{25}y^{17} - 282001x^{25}y^{25} + 123070x^{26}y^{16} - 7500x^{28}y^{24} + 22075x^{25}y^{27} - 250x^{25}y^{29} - 80x^{26}y^2 + 1078360x^{26}y^{18} - 262300x^{26}y^{22} - 722775x^{26}y^{20} + 2475x^{26}y^4 - 37700x^{26}y^6 + 194365x^{26}y^8 - 277150x^{26}y^{10} + 232860x^{26}y^{12} - 482360x^{26}y^{14} + 83110x^{25}y^{13} - x^{29}y^{29} + 185555x^{26}y^{24} - 29274x^{26}y^{26} + 1350x^{26}y^{28} - x^{27}y - 2175x^{27}y^{21} - 67850x^{27}y^{23} + 95740x^{27}y^{19} + 80x^{27}y^3 + 16750x^{27}y^7 - 54865x^{27}y^9 + 199450x^{27}y^{11} - 128235x^{27}y^{13} + 166955x^{27}y^{17} - 244900x^{27}y^{15} + 22075x^{27}y^{25} - 2475x^{27}y^5 - 2026x^{27}y^{27} + 25x^{27}y^{29} + x^{28}y^2 - 40705x^{28}y^{18} + 7125x^{28}y^{22} + 38600x^{28}y^{20} - 80x^{28}y^4 + 1750x^{28}y^6 - 12900x^{28}y^8 + 29400x^{28}y^{10} - 9150x^{28}y^{12} + 1410x^{28}y^{14} - 9000x^{28}y^{16} + 275x^{17}y + 1350x^{28}y^{26} - 74x^{28}y^{28} - 3075x^{29}y^{21} + 1250x^{29}y^{23} + 2355x^{29}y^{19} + 4000x^{29}y^{17} + 75x^{29}y^5 - 350x^{29}y^7 - 1050x^{29}y^9 + 3925x^{29}y^{11} + 275x^{29}y^{13} - x^{29}y^3 = 0$$

Theorem 4.15. *If $x = H(q)$ and $y = H(q^{29})$, then*

$$(4.41)\quad 67501676x^6y^{15} + 421618356x^{21}y^{22} + 3084410768x^{21}y^{18} - 1888927296x^{21}y^{20} - 1743406021x^{21}y^{10} + 2854862196x^{21}y^{12} - 1697818572x^{21}y^{14} - 1443012566x^{21}y^{16} + 6206116x^{21}y^4 - 75880008x^{21}y^6 + 2204x^6y - y + 511110471x^{21}y^8 - 35850496x^{21}y^{24} + 1339162x^{21}y^{26} - 96280x^{21}y^{28} + 2204x^{22}y - 74585216x^{22}y^{23} + 6279080x^6y^{23} + 131459929x^6y^{19} - 35850496x^6y^{21} - 191327819x^6y^{11} + 171454554x^6y^{13} - 278690x^{21}y^2 - 211816000x^6y^{17} - 113245x^6y^3 + 1675011x^6y^5 - 14417872x^6y^7 + 75880008x^6y^9 - 1222756x^6y^{25} + 143231x^6y^{27} - 72065x^7y^2 + 74585216x^7y^{22} + 531992037x^7y^{18} - 277293911x^7y^{20} - 424132511x^7y^{10} - 3770x^6y^{29} + 626545029x^7y^{12} - 176717532x^7y^{14} - 418618828x^7y^{16} + 731467x^7y^4 + 1384037992x^8y^{17} + 82733984x^7y^8 - 14417872x^7y^{24} + 2184715x^7y^{26} - 154889x^7y^{28} - 82733984x^8y^{23} - 1888927296x^{10}y^{21} - 7656x^8y + 421618356x^8y^{21} + 1355970429x^8y^{11} - 1305338024x^8y^{13} - 213382116x^8y^{15} - 6279080x^7y^6$$

$$-23925x^8y^3 - 2926448x^8y^5 + 74585216x^8y^7 - 511110471x^8y^9 + 9436310x^8y^{25}$$
$$- 514489x^8y^{27} - 2204x^8y^{29} - 1339162x^9y^4 - 511110471x^9y^{22} - 2854862196x^9y^{18}$$
$$+ 1743406021x^9y^{20} + 1888927296x^9y^{10} + 12760x^{10}y - 3084410768x^9y^{12} + 1443012566x^9y^{14}$$
$$+ 96280x^9y^2 + 1697818572x^9y^{16} + 35850496x^9y^6 - 421618356x^9y^8 + 20764x^{10}y^{29}$$
$$+ 75880008x^9y^{24} - 6206116x^9y^{26} + 278690x^9y^{28} - 1132340351x^8y^{19} + 424132511x^{10}y^{23}$$
$$+ 4284312419x^{10}y^{19} - 4603021694x^{10}y^{11} + 4751192264x^{10}y^{13} + 149551608x^{10}y^{15}$$
$$- 4561890704x^{10}y^{17} + 294669x^{10}y^3 + 13505532x^{10}y^5 - 277293911x^{10}y^7 + 1743406021x^{10}y^9$$
$$- 47304336x^{10}y^{25} + 1994388x^{10}y^{27} + 247660x^{11}y^2 + 1355970429x^{11}y^{22} + 7654749462x^{11}y^{18}$$
$$- 4603021694x^{11}y^{20} - 4284312419x^{11}y^{10} - xy^{30} + 7678466677x^{11}y^{12} - 4571021180x^{11}y^{14}$$
$$- 4040038140x^{11}y^{16} + 3586836x^{11}y^4 - 131459929x^{11}y^6 + 1132340351x^{11}y^8$$
$$- 191327819x^{11}y^{24} - 7656xy^{22} - 1160xy^{18} + 12760xy^{20} - 20764xy^{10} + 10266xy^{12}$$
$$+ 26158xy^{14} - 26680xy^{16} - 754xy^4 + 3770xy^6 + 2204xy^8 + 2204xy^{24} - 348xy^{26}$$
$$+ 29xy^{28} + x^{29} + 29x^2y + 72065x^2y^{23} - 247660x^2y^{19} - 96280x^2y^{21} - 36221x^2y^{11}$$
$$- 514576x^2y^{13} + 368184x^2y^{15} + 313200x^2y^{17} - 2610x^2y^3 + 31784x^2y^5 - 154889x^2y^7$$
$$+ 278690x^2y^9 - 12818x^2y^{25} + 725x^2y^{27} + 20387x^3y^4 - 23925x^3y^{22} + 914109x^3y^{18}$$
$$+ 294669x^3y^{20} - 1994388x^3y^{10} + 2242628x^3y^{12} + 1218348x^3y^{14} - 725x^3y^2 - 2956724x^3y^{16}$$
$$- 143231x^3y^6 + 514489x^3y^8 - 113245x^3y^{24} + 33466x^3y^{26} - 2610x^3y^{28} - 731467x^4y^{23}$$
$$- 149551608x^{15}y^{10} + 1339162x^4y^{21} + 10423267x^4y^{11} - 348x^4y - 5256714x^4y^{13}$$
$$- 8299104x^4y^{15} + 10334672x^4y^{17}33466x^4y^3 - 424937x^4y^5 + 2184715x^4y^7 - 6206116x^4y^9$$
$$+ 189776x^4y^{25} - 20387x^4y^{27} + 12818x^5y^2 - 2926448x^5y^{22} - 43689312x^5y^{18}$$
$$+ 13505532x^5y^{20} + 47304336x^5y^{10} - 61755732x^5y^{12} - 1522964x^5y^{14} + 56288246x^5y^{16}$$
$$- 189776x^5y^4 + 1222756x^5y^6 - 9436310x^5y^8 + 1675011x^5y^{24} + 754x^4y^{29} - 424937x^5y^{26}$$
$$+ 31784x^5y^{28} + 171454554x^{13}y^{24} - 5256714x^{13}y^{26} - 514576x^{13}y^{28} - 26680x^{14}y$$
$$+ 176717532x^{14}y^{23} + 4571021180x^{14}y^{19} - 1443012566x^{14}y^{21} - 4040038140x^{14}y^{11}$$
$$+ 4980578233x^{14}y^{13} - 362465432x^{14}y^{15} - 5225278696x^{14}y^{17} - 2956724x^{14}y^3$$
$$+ 56288246x^{14}y^5 - 418618828x^{14}y^7 + 1697818572x^{14}y^9 + 1522964x^{14}y^{25} - 1218348x^{14}y^{27}$$
$$- 26158x^{14}y^{29} - 368184x^{15}y^2 - 213382116x^{15}y^{22} + 154196016x^{15}y^{18} + 149551608x^{15}y^{20}$$
$$- 3586836x^4y^{19} - 154196016x^{15}y^{12} + 362465432x^{15}y^{14} - 362465432x^{15}y^{16}$$
$$+ 8299104x^{15}y^4 - 67501676x^{15}y^6 + 7656x^{22}y^{29} + 213382116x^{15}y^8 + 67501676x^{15}y^{24}$$
$$- 8299104x^{15}y^{26} + 368184x^{15}y^{28} + 26158x^{16}y + 418618828x^{16}y^{23} + 4040038140x^{16}y^{19}$$
$$- 1697818572x^{16}y^{21} - 4571021180x^{16}y^{11} + 5225278696x^{16}y^{13} + 362465432x^{16}y^{15}$$
$$- 4980578233x^{16}y^{17} + 1218348x^{16}y^3 - 1522964x^{16}y^5 - 176717532x^{16}y^7 + 1443012566x^{16}y^9$$
$$- 2204x^{24}y^{29} - 56288246x^{16}y^{25} + 2956724x^{16}y^{27} + 26680x^{16}y^{29} + 514576x^{17}y^2$$
$$+ 1384037992x^{17}y^{22} + 8440473896x^{17}y^{18} - 4561890704x^{17}y^{20} - 4751192264x^{17}y^{10}$$
$$+ 8739973397x^{17}y^{12} - 4980578233x^{17}y^{14} - 5225278696x^{17}y^{16} + 5256714x^{17}y^4$$
$$- 171454554x^{17}y^6 + 1305338024x^{17}y^8 - 211816000x^{17}y^{24} + 10334672x^{17}y^{26}$$
$$+ x^{30}y^{29} + 313200x^{17}y^{28} + 10266x^{18}y - 531992037x^{18}y^{23} - 7654749462x^{18}y^{19}$$
$$+ 2854862196x^{18}y^{21} - 2204x^{29}y^6 + 7678466677x^{18}y^{11} - 8440473896x^{18}y^{13}$$
$$- 154196016x^{18}y^{15} + 8739973397x^{18}y^{17} + 2242628x^{18}y^3 - 61755732x^{18}y^5$$
$$+ 626545029x^{18}y^7 - 3084410768x^{18}y^9 + 43689312x^{18}y^{25} - 914109x^{18}y^{27} + 1160x^{18}y^{29}$$
$$+ 277293911x^{20}y^{23} - 1132340351x^{19}y^{22} - 7678466677x^{19}y^{18} + 4284312419x^{19}y^{20}$$

$$+4603021694x^{19}y^{10} - 7654749462x^{19}y^{12} + 4040038140x^{19}y^{14} + 4571021180x^{19}y^{16}$$
$$- 10423267x^{19}y^{4} + 191327819x^{19}y^{6} - 1355970429x^{19}y^{8} + 131459929x^{19}y^{24}$$
$$- 3586836x^{19}y^{26} - 20764x^{20}y - 247660x^{19}y^{28} - 2184715x^{26}y^{23} + 4603021694x^{20}y^{19}$$
$$- 1743406021x^{20}y^{21} - 4284312419x^{20}y^{11} + 4561890704x^{20}y^{13} - 149551608x^{20}y^{15}$$
$$- 4751192264x^{20}y^{17} - 1994388x^{20}y^{3} + 47304336x^{20}y^{5} - 424132511x^{20}y^{7}$$
$$+ 1888927296x^{20}y^{9} - 1160x^{12}y - 13505532x^{20}y^{25} - 294669x^{20}y^{27} - 12760x^{20}y^{29}$$
$$+ 10423267x^{11}y^{26} - 36221x^{11}y^{28} - 626545029x^{12}y^{23} - 7678466677x^{12}y^{19}$$
$$+ 3084410768x^{12}y^{21} + 7654749462x^{12}y^{11} - 8739973397x^{12}y^{13} + 154196016x^{12}y^{15}$$
$$+ 8440473896x^{12}y^{17} + 914109x^{12}y^{3} - 43689312x^{12}y^{5} - 10266x^{12}y^{29} + 531992037x^{12}y^{7}$$
$$- 2854862196x^{12}y^{9} + 61755732x^{12}y^{25} - 2242628x^{12}y^{27} - 313200x^{13}y^{2} + 514489x^{22}y^{3}$$
$$- 1305338024x^{13}y^{22} - 8739973397x^{13}y^{18} + 4751192264x^{13}y^{20} + 4561890704x^{13}y^{10}$$
$$- 8440473896x^{13}y^{12} + 5225278696x^{13}y^{14} + 4980578233x^{13}y^{16} - 10334672x^{13}y^{4}$$
$$+ 211816000x^{13}y^{6} - 1384037992x^{13}y^{8} - 1355970429x^{22}y^{19} + 511110471x^{22}y^{21}$$
$$+ 1132340351x^{22}y^{11} - 1384037992x^{22}y^{13} + 213382116x^{22}y^{15} + 1305338024x^{22}y^{17}$$
$$- 9436310x^{22}y^{5} + 82733984x^{22}y^{7} - 421618356x^{22}y^{9} + 2926448x^{22}y^{25} + 23925x^{22}y^{27}$$
$$+ 154889x^{23}y^{2} - 82733984x^{23}y^{22} - 626545029x^{23}y^{18} + 424132511x^{23}y^{20}$$
$$+ 277293911x^{23}y^{10} - 531992037x^{23}y^{12} + 418618828x^{23}y^{14} + 176717532x^{23}y^{16}$$
$$- 2184715x^{23}y^{4} + 14417872x^{23}y^{6} - 74585216x^{23}y^{8} + 6279080x^{23}y^{24} - 731467x^{23}y^{26}$$
$$+ 72065x^{23}y^{28} + 3770x^{24}y + 14417872x^{24}y^{23} + 191327819x^{24}y^{19} - 75880008x^{24}y^{21}$$
$$- 131459929x^{24}y^{11} + 211816000x^{24}y^{13} - 67501676x^{24}y^{15} - 171454554x^{24}y^{17}$$
$$- 143231x^{24}y^{3} + 1222756x^{24}y^{5} - 6279080x^{24}y^{7} + 35850496x^{24}y^{9} - 1675011x^{24}y^{25}$$
$$+ 113245x^{24}y^{27} - 31784x^{25}y^{2} + 9436310x^{25}y^{22} + 61755732x^{25}y^{18} - 47304336x^{25}y^{20}$$
$$- 13505532x^{25}y^{10} + 43689312x^{25}y^{12} - 56288246x^{25}y^{14} + 1522964x^{25}y^{16} + 424937x^{25}y^{4}$$
$$- 1675011x^{25}y^{6} + 2926448x^{25}y^{8} - 1222756x^{25}y^{24} + 189776x^{25}y^{26} - 12818x^{25}y^{28}$$
$$- 754x^{26}y + 36221x^{19}y^{2} - 10423267x^{26}y^{19} + 6206116x^{26}y^{21} + 3586836x^{26}y^{11}$$
$$- 10334672x^{26}y^{13} + 8299104x^{26}y^{15} + 5256714x^{26}y^{17} + 20387x^{26}y^{3} - 189776x^{26}y^{5}$$
$$+ 731467x^{26}y^{7} - 1339162x^{26}y^{9} + 424937x^{26}y^{25} - 33466x^{26}y^{27} + 348x^{26}y^{29} + 2610x^{27}y^{2}$$
$$- 514489x^{27}y^{22} - 2242628x^{27}y^{18} + 1994388x^{27}y^{20} - 294669x^{27}y^{10} - 914109x^{27}y^{12}$$
$$+ 2956724x^{27}y^{14} - 1218348x^{27}y^{16} - 33466x^{27}y^{4} + 113245x^{27}y^{6} + 23925x^{27}y^{8}$$
$$+ 143231x^{27}y^{24} + 348x^{29}y^{4} - 20387x^{27}y^{26} + 725x^{27}y^{28} + 154889x^{28}y^{23} + 36221x^{28}y^{19}$$
$$- 278690x^{28}y^{21} + 247660x^{28}y^{11} - 313200x^{28}y^{13} - 368184x^{28}y^{15} + 514576x^{28}y^{17}$$
$$- 725x^{28}y^{3} + 12818x^{28}y^{5} - 72065x^{28}y^{7} + 96280x^{28}y^{9} - 31784x^{28}y^{25} + 2610x^{28}y^{27}$$
$$- 29x^{28}y^{29} - 29x^{29}y^{2} - 2204x^{29}y^{22} - 10266x^{29}y^{18} + 20764x^{29}y^{20} - 12760x^{29}y^{10}$$
$$+ 754x^{29}y^{26} + 26680x^{29}y^{14} - 26158x^{29}y^{16} + 7656x^{29}y^{8} - 3770x^{29}y^{24} + 1160x^{29}y^{12} = 0$$

Theorem 4.16. *If $x = H(q)$ and $y = H(q^{55})$, then*

$$897x^{7}y - 68655x^{7}y^{3} + 1041739x^{7}y^{5} - 5600293x^{7}y^{7} + 10840636x^{7}y^{9} \tag{4.42}$$

$$-5288620x^7y^{11} - 5626444x^7y^{13} + 9489340x^7y^{15} + 23x^3y - 1610x^3y^3 + 18078x^3y^5$$
$$- 68655x^3y^7 + 90620x^3y^9 - 67022x^3y^{11} - 50462x^3y^{13} + 133584x^3y^{15} - 57270x^3y^{17}$$
$$+ 1725x^3y^{19} - 989x^3y^{21} - 46x^3y^{23} + 1012x^4y^2 - 23000x^4y^4 + 170292x^4y^6 - 484748x^4y^8$$
$$+ 391046x^4y^{10} + 261096x^4y^{12} - 343022x^4y^{14} + 171764x^4y^{16} - 212106x^4y^{18} + 85514x^4y^{20}$$
$$- 7222x^4y^{22} - 207x^5y + 18078x^5y^3 - 234646x^5y^5 + 1041739x^5y^7 - 1593532x^5y^9$$
$$- 3331688x^6y^{16} + 817558x^5y^{11} + 661066x^5y^{13} - 1635484x^5y^{15} + 1123090x^5y^{17}$$
$$- 243133x^5y^{19} + 1725x^5y^{21} + 1242x^5y^{23} + 170292x^6y^4 - 1297292x^6y^6 + 4046252x^6y^8$$
$$- 4913329x^6y^{10} - 672750x^6y^{12} + 5037253x^6y^{14} - 7222x^6y^2 + 1300443x^6y^{18} - 212106x^6y^{20}$$
$$+ 14743x^6y^{22} - 46x^2y^2 + 1012x^2y^4 - 7222x^2y^6 + 16928x^2y^8 + 1863x^2y^{10} - 12834x^2y^{12}$$
$$- 10143x^2y^{14} + 2116x^2y^{16} + 14743x^2y^{18} - 7222x^2y^{20} + 1081x^2y^{22} - xy + 23xy^3 - 207xy^5$$
$$+ 897xy^7 - 1748xy^9 + 460xy^{11} + 2944xy^{13} - 2576xy^{15} - 966xy^{17} + 1242xy^{19} - 46xy^{21}$$
$$- 46xy^{23} + x^{24} - 46x^{23}y - 46x^{23}y^3 + 1242x^{23}y^5 - 966x^{23}y^7 - 2576x^{23}y^9 + 2944x^{23}y^{11}$$
$$+ 460x^{23}y^{13} - 1748x^{23}y^{15} + y^{24} + 897x^{23}y^{17} - 6199558x^7y^{17} + 1123090x^7y^{19} - 57270x^7y^{21}$$
$$- 966x^7y^{23} + 16928x^8y^2 - 484748x^8y^4 + 4046252x^8y^6 - 13628512x^8y^8 + 19391208x^8y^{10}$$
$$- 2667816x^8y^{12} - 20124816x^8y^{14} + 17344783x^8y^{16} - 3331688x^8y^{18} + 171764x^8y^{20}$$
$$+ 2116x^8y^{22} - 1748x^9y + 90620x^9y^3 - 1593532x^9y^5 + 10840636x^9y^7 - 28673502x^9y^9$$
$$+ 18981210x^9y^{11} + 19338354x^9y^{13} - 28274406x^9y^{15} + 9489340x^9y^{17} - 1635484x^9y^{19}$$
$$+ 133584x^9y^{21} - 2576x^9y^{23} + 1863x^{10}y^2 + 391046x^{10}y^4 - 4913329x^{10}y^6 + 19391208x^{10}y^8$$
$$- 10143x^{10}y^{22} + 685584x^{10}y^{12} + 28328824x^{10}y^{14} - 20124816x^{10}y^{16} + 5037253x^{10}y^{18}$$
$$- 343022x^{10}y^{20} - 26483212x^{10}y^{10} - 14122322x^{11}y^{13} - 67022x^{11}y^3 + 817558x^{11}y^5$$
$$- 5288620x^{11}y^7 + 18981210x^{11}y^9 - 17142866x^{11}y^{11} + 460x^{11}y + 19338354x^{11}y^{15}$$
$$- 5626444x^{11}y^{17} + 661066x^{11}y^{19} - 50462x^{11}y^{21} + 2944x^{11}y^{23} - 12834x^{12}y^2 + 261096x^{12}y^4$$
$$- 672750x^{12}y^6 - 2667816x^{12}y^8 + 685584x^{12}y^{10} + 7517596x^{12}y^{12} + 685584x^{12}y^{14}$$
$$- 2667816x^{12}y^{16} - 672750x^{12}y^{18} + 261096x^{12}y^{20} - 12834x^{12}y^{22} + 2944x^{13}y - 50462x^{13}y^3$$
$$+ 661066x^{13}y^5 - 5626444x^{13}y^7 + 19338354x^{13}y^9 - 14122322x^{13}y^{11} - 17142866x^{13}y^{13}$$
$$+ 18981210x^{13}y^{15} - 5288620x^{13}y^{17} + 817558x^{13}y^{19} - 67022x^{13}y^{21} + 460x^{13}y^{23} - 10143x^{14}y^2$$
$$- 343022x^{14}y^4 + 5037253x^{14}y^6 - 20124816x^{14}y^8 + 28328824x^{14}y^{10} + 685584x^{14}y^{12}$$
$$- 26483212x^{14}y^{14} + 19391208x^{14}y^{16} - 4913329x^{14}y^{18} + 391046x^{14}y^{20} + 1863x^{14}y^{22}$$
$$- 2576x^{15}y + 133584x^{15}y^3 - 1635484x^{15}y^5 + 9489340x^{15}y^7 - 28274406x^{15}y^9 + 2116x^{16}y^2$$
$$+ 18981210x^{15}y^{13} - 28673502x^{15}y^{15} + 10840636x^{15}y^{17} - 1593532x^{15}y^{19} + 90620x^{15}y^{21}$$
$$- 1748x^{15}y^{23} + 171764x^{16}y^4 - 3331688x^{16}y^6 + 17344783x^{16}y^8 - 20124816x^{16}y^{10}$$
$$+ 19338354x^{15}y^{11} - 966x^{17}y - 2667816x^{16}y^{12} + 19391208x^{16}y^{14} - 13628512x^{16}y^{16}$$
$$+ 4046252x^{16}y^{18} - 484748x^{16}y^{20} + 16928x^{16}y^{22} - 57270x^{17}y^3 + 1123090x^{17}y^5$$
$$- 6199558x^{17}y^7 + 9489340x^{17}y^9 - 5626444x^{17}y^{11} - 5288620x^{17}y^{13} + 10840636x^{17}y^{15}$$
$$- 5600293x^{17}y^{17} + 1041739x^{17}y^{19} - 68655x^{17}y^{21} + 897x^{17}y^{23} + 5037253x^{18}y^{10}$$

$$+14743x^{18}y^2 - 212106x^{18}y^4 + 1300443x^{18}y^6 - 3331688x^{18}y^8 - 672750x^{18}y^{12}$$
$$- 4913329x^{18}y^{14} - 46x^{21}y + 4046252x^{18}y^{16} - 1297292x^{18}y^{18} + 170292x^{18}y^{20} - 7222x^{18}y^{22}$$
$$+ 1242x^{19}y + 1725x^{19}y^3 - 243133x^{19}y^5 + 1123090x^{19}y^7 - 1635484x^{19}y^9 + 661066x^{19}y^{11}$$
$$+ 817558x^{19}y^{13} - 1593532x^{19}y^{15} + 1041739x^{19}y^{17} - 234646x^{19}y^{19} + 18078x^{19}y^{21}$$
$$- 207x^{19}y^{23} - 7222x^{20}y^2 + 85514x^{20}y^4 - 212106x^{20}y^6 + 171764x^{20}y^8 - 343022x^{20}y^{10}$$
$$+ 261096x^{20}y^{12} + 391046x^{20}y^{14} - 484748x^{20}y^{16} + 170292x^{20}y^{18} - 23000x^{20}y^{20} + 1012x^{20}y^{22}$$
$$- 989x^{21}y^3 + 1725x^{21}y^5 - 57270x^{21}y^7 + 133584x^{21}y^9 - 50462x^{21}y^{11} - 67022x^{21}y^{13}$$
$$+ 90620x^{21}y^{15} - 68655x^{21}y^{17} + 18078x^{21}y^{19} - 1610x^{21}y^{21} + 23x^{21}y^{23} + 1081x^{22}y^2$$
$$- 7222x^{22}y^4 + 14743x^{22}y^6 + 2116x^{22}y^8 - 10143x^{22}y^{10} - 12834x^{22}y^{12} + 1863x^{22}y^{14}$$
$$+ 16928x^{22}y^{16} - 7222x^{22}y^{18} + 1012x^{22}y^{20} - 46x^{22}y^{22} - 207x^{23}y^{19} + 23x^{23}y^{21} - x^{23}y^{23} = 0.$$

Proofs of the identities (4.36)–(4.42) are similar to the proof of the identity (4.32) given above except that in place of result (3.35), result (3.39) is used for proving (4.36); result (3.40) is used for proving (4.37); result (3.42) is used for proving (4.39); result (3.43) is used for proving (4.40); result (3.44) is used for proving (4.41); result (3.45) is used for proving (4.42).

REFERENCES

[1] C. ADIGA, T. KIM, M. S. MAHADEVA NAIKA, and H. S. MADHUSUDHAN, On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, *Indian J. Pure Appl. Math.* **35** (9) (2004), pp. 1047-1062.

[2] C. ADIGA, K. R. VASUKI and M. S. MAHADEVA NAIKA, On some new identities involving integrals of theta-functions, *Adv. Stud. Contemp. Math.*, **3** (2) (2001), pp. 1-11.

[3] C. ADIGA, K. R. VASUKI and M. S. MAHADEVA NAIKA, Some new explicit evaluations of Ramanujan's cubic continued fraction, *New Zealand J. Math.*, **31** (2002), pp. 1-6.

[4] G. E. ANDREWS and B. C. BERNDT, *Ramanujan's Lost Notebook, Part I*, New York. Springer 2005.

[5] B. C. BERNDT, *Ramanujan's Notebooks, Part III*, New York. Springer-Verlag 1991.

[6] B. C. BERNDT, *Ramanujan's Notebooks, Part IV*, New York. Springer-Verlag 1994.

[7] B. C. BERNDT, *Ramanujan's Notebooks, Part V*, New York. Springer-Verlag 1998.

[8] H. H. CHAN and S. S. HAUNG, On the Ramanujan-Göllnitz-Gordon continued fraction, *Ramanujan J.*, **1** (1997), pp. 75-90.

[9] M. S. MAHADEVA NAIKA, Some theorems on Ramanujan's cubic continued fraction and related identities, *Tamsui Oxf. J. Math. Sci.* **24** (3) (2008), pp. 243-259.

[10] M. S. MAHADEVA NAIKA, K. S. CHANDANKUMAR and M. MANJUNATHA, On some new modular equations and their applications to continued fractions, *Int. Mat. Forum*, **6** 58, (2011), pp. 99-115.

[11] M. S. MAHADEVA NAIKA, B. N. DHARMENDRA and K. SHIVASHANKARA, A continued fraction of order twelve, *Cent. Eur. J. Math.*, **6** (3) (2008), pp. 393-404.

[12] M. S. MAHADEVA NAIKA and H. S. MADHUSUDHAN, Some integral identities for the Rogers-Ramanujan continued fraction, *Bull. Allahabad Math. Soc.*, **23** (1) (2008), pp. 193-203.

[13] M. S. MAHADEVA NAIKA, M. C. MAHESHKUMAR and B. SUSHAN BAIRY, General formulas for explicit evaluations of Ramanujan's cubic continued fraction, *Kyungpook Math. J.*, **49** (2009), pp. 435-450.

[14] M. S. MAHADEVA NAIKA, R. Y. DENIS, and B. SUSHAN BAIRY, On some Ramanujan-Selberg continued fraction, *Indian J. Math.* **51** (3) (2009), pp. 585-596.

[15] S. RAMANUJAN, *Notebooks* (2 volumes), Bombay, Tata Institute of Fundamental Research 1957.

[16] S. RAMANUJAN, *The 'Lost' Notebook and Other Unpublished Papers*, New Delhi, Narosa 1988.

[17] K. R. VASUKI, and M. S. MAHADEVA NAIKA, Some evaluations of the Rogers-Ramanujan continued fractions, In: *Proc. Inter. Conf. on the works of Ramanujan*, (Eds.: C. Adiga and D. D. Somashekara) 209-217, Mysore, (2000).

[18] K. R. VASUKI and B. R. SRIVATSA, Certain identities for Ramanujan-Göllnitz-Gordon continued fraction, *J. Compu. App. Math.*, **187** (2006), pp. 87-95.