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Research Article

D_a-Homothetic Deformation of K-Contact Manifolds

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We study D_a-homothetic deformations of K-contact manifolds. We prove that D_a-homothetically deformed K-contact manifold is a generalized Sasakian space form if it is conharmonically flat. Further, we find expressions for scalar curvature of D_a-homothetically deformed K-contact manifolds.

1. Introduction

In 1968 Tanno [1] introduced the notion of D_a-homothetic deformations. Carriazo and Martín-Molina [2] studied D_a-homothetic deformation of generalized (k, μ) space forms and gave several examples for manifolds of dimension 3. De and Ghosh [3] studied D_a-homothetic deformation of almost normal contact metric manifolds and prove that $Q\phi$-ϕQ is invariant under such transformation. Bagewadi and Venkatesha [4] studied concircularly semisymmetric trans-Sasakian manifolds and De et al. [5] studied conharmonically semisymmetric, conharmonically flat, ξ-conharmonically flat, and conharmonically recurrent generalized Sasakian space forms. Several authors [6–11] studied K-contact manifolds and proved conditions for these manifolds to be of ξ-conformally flat, ϕ-conformally flat, quasi-conharmonically flat, and ξ-conharmonically flat. Motivated by the above studies, in this paper we study D_a-homothetic deformations of K-contact manifolds by considering conharmonic and projective curvature tensor. The paper is organized as follows. After Preliminaries, we give a brief account of information of D_a-homothetic deformation of K-contact manifolds in Section 3. In Section 4, we study conharmonically flat, semisymmetric, ϕ-conharmonically flat, quasi-conharmonically flat, and ξ-conharmonically flat K-contact manifolds with respect to D_a-homothetic deformation. In the last section, we consider Weyl projective curvature in K-contact manifolds with respect to D_a-homothetic deformation.

2. Preliminaries

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost contact metric manifold [12], consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η, and Riemannian metric g. Then

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \phi X) = 0, \qquad g(X, \xi) = \eta(X), \tag{3}$$

for all $X, Y \in TM$. If ξ is a Killing vector field, then M is called a K-contact Riemannian manifold [13]. A K-contact Riemannian manifold is called Sasakian [12], if the relation

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y) X \tag{4}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g.

If M^{2n+1} is a K-contact Riemannian manifold, then besides (1), (2), (3), and (4) the following relations hold [14]:

$$\nabla_X \xi = -\phi X, \tag{5}$$

$$(\nabla_X \eta)(Y) = -g(\phi X, Y), \tag{6}$$

$$S(X,\xi) = g(QX,\xi) = 2n\eta(X), \tag{7}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{9}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{10}$$

for any vector fields X and Y, where R and S denote, respectively, the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$.

Definition 1. A contact metric manifold M is said to be η-Einstein if $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$, where α and β are smooth functions on M.

3. D_a-Homothetic Deformation of K-Contact Manifolds

Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional almost contact metric manifold. A D_a-homothetic deformation is defined by

$$\overline{\phi} = \phi, \qquad \overline{\xi} = \frac{1}{a}\xi, \qquad \overline{\eta} = a\eta, \qquad \overline{g} = ag + a(a-1)\eta\otimes\eta, \tag{11}$$

with a being a positive constant [1].

It is clear that the $(\overline{M},\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is also an almost contact metric manifold.

If (M,ϕ,ξ,η,g) is a K-contact manifold with Riemannian connection ∇, the connection $\overline{\nabla}$ of the D_a-deformed K-contact manifold $(\overline{M},\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ can be calculated from ∇ and $\overline{g}$. Using Koszul's formula and (5), (6), and (11), $\overline{\nabla}$ of $\overline{g}$ is given by

$$\overline{\nabla}_X Y = \nabla_X Y - a(a-1)\left[\eta(Y)\phi X + \eta(X)\phi Y\right]. \tag{12}$$

Using (12), we obtain

$$\left(\overline{\nabla}_X\phi\right)Y = (\nabla_X\phi)Y + (a-1)\eta(Y)\phi^2 X. \tag{13}$$

The curvature tensor $\overline{R}$ of $(M,\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is given by

$$\begin{aligned}
\overline{R}&(X,Y)Z \\
&= R(X,Y)Z - (a-1) \\
&\quad\times\big(g(\phi Y,Z)\phi X + g(\phi Z,X)\phi Y + 2g(\phi Y,X)\phi Z \\
&\qquad + \left[g(X,Z)\xi - \eta(Z)X\right]\eta(Y) \\
&\qquad - \left[g(Y,Z)\xi - \eta(Z)Y\right]\eta(X) \\
&\qquad + a\left[\eta(Y)X - \eta(X)Y\right]\eta(Z)\big).
\end{aligned} \tag{14}$$

Using (9), (10), and (14), we have

$$\begin{aligned}
\overline{R}(X,Y)\overline{\xi} &= (2-a)\left[\eta(Y)X - \eta(X)Y\right], \\
\overline{R}\left(\overline{\xi},Y\right)Z &= \left[g(Y,Z)\xi - \eta(Z)Y\right] \\
&\quad - (a-1)\left[\eta(Y)\xi - Y\right]\eta(Z), \\
\overline{R}\left(\overline{\xi},Y\right)\overline{\xi} &= \frac{(2-a)}{a}\left[\eta(Y)\xi - Y\right], \\
\overline{\eta}\left(\overline{R}(X,Y)Z\right) &= a^2\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right].
\end{aligned} \tag{15}$$

From (14), we get

$$\begin{aligned}
\overline{S}&(Y,Z) \\
&= aS(Y,Z) - a(a-1) \\
&\quad\times\left((2-a)g(Y,Z) + [2n(a-1)+a-2]\eta(Y)\eta(Z)\right),
\end{aligned} \tag{16}$$

where $\overline{S}$ and S are the Ricci tensors of $(\overline{M},\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ and (M,ϕ,ξ,η,g), respectively.

It follows from (16) that

$$\overline{S}\left(Y,\overline{\xi}\right) = 2na(2-a)\eta(Y), \tag{17}$$

$$\overline{S}(\phi Y,\phi Z) = \overline{S}(Y,Z) - 2na^2(2-a)\eta(Y)\eta(Z). \tag{18}$$

Again contracting (16) over Y, Z, we get

$$\overline{r} = ar - 2na(a-1), \tag{19}$$

where $\overline{r}$ and r are the scalar curvatures of $(\overline{M},\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ and (M,ϕ,ξ,η,g), respectively.

4. Conharmonic Curvature Tensor in D_a-Homothetically Deformed K-Contact Manifolds

The conharmonic tensor of a D_a-homothetically deformed K-contact manifold is defined by [15]

$$\begin{aligned}
\overline{K}(X,Y)Z &= \overline{R}(X,Y)Z - \frac{1}{2n-1} \\
&\quad\times\Big[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y + \overline{g}(Y,Z)\overline{Q}X \\
&\qquad - \overline{g}(X,Z)\overline{Q}Y\Big],
\end{aligned} \tag{20}$$

for X, Y, $Z \in TM$, where $\overline{R}$, $\overline{S}$, and $\overline{Q}$ are the Riemannian curvature tensor, Ricci tensor, and Ricci operator of $(\overline{M},\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$.

Definition 2. An almost contact metric manifold (M,ϕ,ξ,η,g) is said to be

(1) conharmonically flat if

$$K(X,Y)Z = 0, \tag{21}$$

(2) conharmonically semisymmetric if

$$R \cdot K = 0, \tag{22}$$

(3) ϕ-conharmonically flat if

$$g\left(K\left(\phi X, \phi Y\right)\phi Z, \phi W\right) = 0, \tag{23}$$

(4) quasi-conharmonically flat if

$$g\left(K\left(X, Y\right)Z, \phi W\right) = 0, \tag{24}$$

(5) ξ-conharmonically flat if

$$K\left(X, Y\right)\xi = 0, \tag{25}$$

for all vector fields X, Y, and Z.

Assume that $\overline{M}$ is conharmonically flat K-contact manifold with respect to D_a-homothetic deformation. So, we have $\overline{K}(X,Y)Z = 0$.

Then from (20), we have

$$\overline{R}\left(X, Y\right)Z = \frac{1}{2n-1} \times \left[\overline{S}\left(Y, Z\right)X - \overline{S}\left(X, Z\right)Y + \overline{g}\left(Y, Z\right)\overline{Q}X - \overline{g}\left(X, Z\right)\overline{Q}Y\right]. \tag{26}$$

Setting $Z = \overline{\xi}$, contracting (26) with W, and using (7), (9), (14), and (16), we obtain

$$(2-a)(2n-1-2na)\left[\eta\left(Y\right)\overline{g}\left(X, W\right) - \eta\left(X\right)\overline{g}\left(Y, W\right)\right] = \overline{\eta}\left(Y\right)\overline{S}\left(X, W\right) - \overline{\eta}\left(X\right)\overline{S}\left(Y, W\right). \tag{27}$$

Taking $Y = \overline{\xi}$ in (27) and using (1), (7), and (16), it follows that

$$\overline{S}\left(X, W\right) = \frac{(2-a)(2n-1-2na)}{a}\overline{g}\left(X, W\right) + \frac{(2-a)(4na-2n+1)}{a}\overline{\eta}\left(X\right)\overline{\eta}\left(W\right). \tag{28}$$

Thus, $\overline{M}$ is η-Einstein.

Using (28) in (26), we obtain

$$\begin{aligned}\overline{R}\left(X, Y, Z, W\right) &= \frac{2(2-a)(2n-1-2na)}{a(2n-1)} \times \left[\overline{g}\left(Y, Z\right)\overline{g}\left(X, W\right) - \overline{g}\left(X, Z\right)\overline{g}\left(Y, W\right)\right] \\ &+ \frac{(2-a)(4na-2n+1)}{a(2n-1)} \times \left(\left[\overline{g}\left(X, W\right)\overline{\eta}\left(Y\right) - \overline{g}\left(Y, W\right)\overline{\eta}\left(X\right)\right]\overline{\eta}\left(Z\right) + \left[\overline{g}\left(Y, Z\right)\overline{\eta}\left(X\right) - \overline{g}\left(X, Z\right)\overline{\eta}\left(Y\right)\right]\overline{\eta}\left(W\right)\right).\end{aligned} \tag{29}$$

From (29), we get

$$\begin{aligned}\overline{R}\left(X, Y\right)Z &= \frac{2(2-a)(2n-1-2na)}{a(2n-1)} \times \left[\overline{g}\left(Y, Z\right)X - \overline{g}\left(X, Z\right)Y\right] \\ &- \frac{(2-a)(4na-2n+1)}{a(2n-1)} \times \left[\overline{\eta}\left(X\right)\overline{\eta}\left(Z\right)Y - \overline{\eta}\left(Y\right)\overline{\eta}\left(Z\right)X + \overline{g}\left(X, Z\right)\overline{\eta}\left(Y\right)\xi - \overline{g}\left(Y, Z\right)\overline{\eta}\left(X\right)\xi\right].\end{aligned} \tag{30}$$

Hence, it reduces to a generalized Sasakian space form with $f_1 = 2(2-a)(2n-1-2na)/a(2n-1)$, $f_2 = 0$, and $f_3 = -(2-a)(4na-2n+1)/a(2n-1)$. Thus, (30) leads to the following.

Theorem 3. *A conharmonically flat K-contact manifold admitting D_a-homothetic deformation reduces to a generalized Sasakian space form with associated functions $f_1 = 2(2-a)(2n-1-2na)/a(2n-1)$, $f_2 = 0$, and $f_3 = -(2-a)(4na-2n+1)/a(2n-1)$.*

Let us now consider a conharmonically semisymmetric K-contact manifold admitting D_a-homothetic deformation. Then the condition

$$\overline{R}\left(X, Y\right) \cdot \overline{K} = 0 \tag{31}$$

holds on $\overline{M}$ for all vector fields X, Y.

From (8), (14), (17), and (20), we obtain

$$\begin{aligned}\eta\left(\overline{K}\left(X, Y\right)Z\right) &= a^2\left[g\left(Y, Z\right)\eta\left(X\right) - g\left(X, Z\right)\eta\left(Y\right)\right] - \frac{1}{(2n-1)} \\ &\times \left[\left(\overline{S}\left(Y, Z\right)\overline{\eta}\left(X\right) - \overline{S}\left(X, Z\right)\overline{\eta}\left(Y\right)\right) + 2na(2-a)\left(\overline{g}\left(Y, Z\right)\eta\left(X\right) - \overline{g}\left(X, Z\right)\eta\left(Y\right)\right)\right].\end{aligned} \tag{32}$$

Setting $Z = \overline{\xi}$, in (32), we get

$$\eta\left(\overline{K}\left(X, Y\right)\overline{\xi}\right) = 0. \tag{33}$$

Again taking $X = \overline{\xi}$ in (32) and using (17), we obtain

$$\eta\left(\overline{K}\left(\overline{\xi}, Y\right)Z\right) = \frac{-1}{2n-1}\overline{S}\left(Y, Z\right) + \left[1 - \frac{2n(2-a)}{2n-1}\right]\overline{g}\left(Y, Z\right) + \left[-1 + \frac{4n(2-a)}{2n-1}\right]\overline{\eta}\left(Z\right)\overline{\eta}\left(Y\right). \tag{34}$$

Now, (21) yields

$$\overline{R}\left(X, Y\right)\overline{K}\left(U, V\right)Z - \overline{K}\left(\overline{R}\left(X, Y\right)U, V\right)Z - \overline{K}\left(U, \overline{R}\left(X, Y\right)V\right)Z - \overline{K}\left(U, V\right)\overline{R}\left(X, Y\right)Z = 0. \tag{35}$$

Therefore,

$$\overline{g}\left(\overline{R}\left(\overline{\xi},Y\right)\overline{K}(U,V)Z,\overline{\xi}\right)-\overline{g}\left(\overline{K}\left(\overline{R}\left(\overline{\xi},Y\right)U,V\right)Z,\overline{\xi}\right)-\overline{g}\left(\overline{K}\left(U,\overline{R}\left(\overline{\xi},Y\right)V\right)Z,\overline{\xi}\right)-\overline{g}\left(\overline{K}(U,V)\overline{R}\left(\overline{\xi},Y\right)Z,\overline{\xi}\right)=0. \tag{36}$$

From this it follows that

$$\begin{aligned}&-\overline{K}(U,V,Z,Y)+\eta\left(\overline{K}(U,V)Z\right)\eta(Y)\\&+\left[g(Y,U)-(a-1)\eta(U)\eta(Y)\right]\eta\left(\overline{K}(\xi,V)Z\right)\\&+(a-2)\eta(U)\eta\left(\overline{K}(Y,V)Z\right)\\&+\left[g(Y,V)-(a-1)\eta(V)\eta(Y)\right]\eta\left(\overline{K}(U,\xi)Z\right)\\&+(a-2)\eta(V)\eta\left(\overline{K}(U,Y)Z\right)\\&+\left[g(Y,Z)-(a-1)\eta(Z)\eta(Y)\right]\eta\left(\overline{K}(U,V)\xi\right)\\&+(a-2)\eta(Z)\eta\left(\overline{K}(U,V)Y\right)=0,\end{aligned} \tag{37}$$

where

$$\overline{K}(U,V,Z,Y)=\overline{g}\left(\overline{K}(U,V)Z,Y\right). \tag{38}$$

Taking $Y=U$ in (37) and making use of (32) and (33), we obtain

$$\begin{aligned}&-\overline{K}(U,V,Z,U)+(a-1)\eta(U)\eta\left(\overline{K}(U,V)Z\right)\\&+\left[g(U,U)-(a-1)\eta(U)\eta(U)\right]\eta\left(\overline{K}(\xi,V)Z\right)\\&+\left[g(U,V)-(a-1)\eta(U)\eta(V)\right]\eta\left(\overline{K}(U,\xi)Z\right)\\&+(a-2)\eta(Z)\eta\left(\overline{K}(U,V)U\right)=0.\end{aligned} \tag{39}$$

If $\{e_1,e_2,\ldots,e_{2n},\overline{\xi}\}$ is a local orthonormal basis of vector fields in $\overline{M}$, then, from (39), we get

$$\begin{aligned}&\sum_{i=1}^{2n}\overline{K}(e_i,V,Z,e_i)\\&=(a-1)\sum_{i=1}^{2n}\eta(e_i)\eta\left(\overline{K}(e_i,V)Z\right)\\&+\sum_{i=1}^{2n}\left[g(e_i,e_i)-(a-1)\eta(e_i)\eta(e_i)\right]\eta\left(\overline{K}(\xi,V)Z\right)\\&+\sum_{i=1}^{2n}\left[g(e_i,V)-(a-1)\eta(e_i)\eta(V)\right]\eta\left(\overline{K}(e_i,\xi)Z\right)\\&+(a-2)\sum_{i=1}^{2n}\eta\left(\overline{K}(e_i,V)e_i\right)\eta(Z).\end{aligned} \tag{40}$$

From (20), it follows that

$$\begin{aligned}\sum_{i=1}^{2n}\overline{K}(e_i,V,Z,e_i)&=\frac{1}{2n-1}\overline{S}(V,Z)-\left[1-\frac{\overline{r}+2n(2-a)}{2n-1}\right]\overline{g}(V,Z)+\left[1-\frac{4n(2-a)}{2n-1}\right]\overline{\eta}(V)\overline{\eta}(Z),\\
\sum_{i=1}^{2n}\eta(e_i)\eta\left(\overline{K}(e_i,V)Z\right)&=\frac{1-a^2}{a^2(2n-1)}\overline{S}(V,Z)+\left[\frac{a^2-1}{a^2}+\frac{2n(2-a)\left(1-a^2\right)}{a^2(2n-1)}\right]\overline{g}(V,Z)+\frac{1-a^2}{a^2}\overline{\eta}(V)\overline{\eta}(Z),\\
\sum_{i=1}^{2n}\left[g(e_i,V)-(a-1)\eta(e_i)\eta(V)\right]\eta\left(\overline{K}(e_i,\xi)Z\right)&=\frac{1}{2n-1}\overline{S}(V,Z)+\left[-a^2+\frac{2na^2(2-a)}{2n-1}\right]g(V,Z)\\&\quad+\left[a^2-\frac{2na(2-a)(a+1)\left(a^2-a+1\right)}{2n-1}\right]\eta(V)\eta(Z),\\
\sum_{i=1}^{2n}\eta\left(\overline{K}(e_i,V)Z\right)\eta(Z)&=\left[\frac{\overline{r}}{2n-1}+4n(1-a)\right]\eta(V)\eta(Z).\end{aligned} \tag{41}$$

Using (41) in (40), we obtain

$$\begin{aligned}(2n+2-a)\eta\left(\overline{K}\left(\overline{\xi},V\right)Z\right)&=\frac{(a-1)\left(a^2-1\right)}{a^2(2n-1)}\overline{S}(V,Z)+\left[\frac{\overline{r}}{2n-1}+\frac{2na^2-a(1+4n)+1+2n}{a^2(2n-1)}\right]\overline{g}(V,Z)\\&\quad+\left[\frac{(a-2)\overline{r}}{a^2(2n-1)}+\frac{p}{a^2(2n-1)}\right]\overline{\eta}(V)\overline{\eta}(Z),\end{aligned} \tag{42}$$

where

$$\begin{aligned}p=&-2na^5+2na^4+10na^3+a^2\left(8n^2-18n\right)\\&+a\left(14n-24n^2+1\right)+16n^2-6n-1.\end{aligned} \tag{43}$$

In view of (34), (42) yields

$$\overline{S}(V,Z) = -\frac{1}{a^2(2n+1)+1-a}\left[\alpha\overline{g}(V,Z)+\beta\overline{\eta}(V)\overline{\eta}(Z)\right], \tag{44}$$

where

$$\begin{aligned}\alpha &= \overline{r}a^2 + 2na^4 + a^3\left(-4n^2-6n-1\right)\\ &\quad + a^2\left(4n^2+8n+2\right) + a(-1-4n) + 1 + 2n,\\ \beta &= (a-2)\overline{r} - 2na^5 - 2na^4 + a^3\left(8n^2+24n+1\right)\\ &\quad + a^2\left(-4n^2-32n-2\right) + a\left(14n-24n^2+1\right)\\ &\quad + 16n^2 - 6n - 1.\end{aligned} \tag{45}$$

Thus, $\overline{M}$ is η-Einstein.

If $\{e_1, e_2, \dots, e_{2n}, \overline{\xi}\}$ is a local orthonormal basis of vector fields in $\overline{M}$, then, from (44), we get

$$\begin{aligned}\overline{r} = &\left(-2na^5 + 4n^2a^4 + a^3l + a^2m\right.\\ &\left.+ a\left(-324n^2+8n\right) + 20n^2 + 2n\right)\\ &\times\left(1-2a^2(2n+1)\right)^{-1},\end{aligned} \tag{46}$$

where

$$\begin{aligned}l &= -8n^3 - 8n^2 + 16n,\\ m &= 8n^3 + 16n^2 - 20n.\end{aligned} \tag{47}$$

So, we can state the following.

Theorem 4. *In a $(2n+1)$-dimensional conharmonically semisymmetric K-contact manifold admitting D_a-homothetic deformation, scalar curvature $\overline{r}$ is given by* (46).

Analogous to the definition of ϕ-conharmonically flat K-contact manifolds [8], we define ϕ-conharmonically flat K-contact manifolds with respect to D_a-homothetic deformation. Let us assume that M is a ϕ-conharmonically flat K-contact manifold with respect to D_a-homothetic deformation. It can be easily seen that

$$\overline{g}\left(\overline{K}(\phi X,\phi Y)\phi Z,\phi W\right) = 0, \tag{48}$$

where $X, Y, Z, W \in T\overline{M}$.

Using (20), (48) yields

$$\begin{aligned}&\overline{g}\left(\overline{R}(\phi X,\phi Y)\phi Z,\phi W\right)\\ &\quad= \frac{1}{2n-1}\left(\overline{S}(\phi Y,\phi Z)\overline{g}(\phi X,\phi W)\right.\\ &\qquad - \overline{S}(\phi X,\phi Z)\overline{g}(\phi Y,\phi W)\\ &\qquad + \overline{S}(\phi X,\phi W)\overline{g}(\phi Y,\phi Z)\\ &\qquad \left.-\overline{S}(\phi Y,\phi W)\overline{g}(\phi X,\phi Z)\right),\end{aligned} \tag{49}$$

for all $X, Y, Z, W \in T\overline{M}$.

If $\{e_1, e_2, \dots, e_{2n}, \overline{\xi}\}$ is a local orthonormal basis of vector fields in $\overline{M}$, then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \overline{\xi}\}$ is also a local orthonormal basis. So, using (1), (6), (14), (16), and (18), it can be easily verified that

$$\begin{aligned}\sum_{i=1}^{2n}\overline{R}(\phi e_i,\phi Y,\phi Z,\phi e_i) &= \overline{S}(Y,Z) - ag(Y,Z)\\ &\quad + [a - 2na(2-a)]\eta(Y)\eta(Z),\\ \sum_{i=1}^{2n}\overline{g}(\phi e_i,\phi e_i) &= 2n,\\ \sum_{i=1}^{2n}\overline{S}(\phi e_i,\phi e_i) &= \overline{r} - 2n(2-a),\\ \sum_{i=1}^{2n}\overline{S}(\phi Y,\phi e_i)\overline{g}(\phi e_i,\phi Z) &= \overline{S}(\phi Y,\phi Z).\end{aligned} \tag{50}$$

For a local orthonormal basis $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \overline{\xi}\}$ of vector fields in $\overline{M}$, putting $X = W = e_i$ in (49) and summing up with respect to $i = 1, 2, \dots, 2n+1$, we have

$$\begin{aligned}&\sum_{i=1}^{2n}\overline{g}\left(\overline{R}(\phi e_i,\phi Y)\phi Z,\phi e_i\right)\\ &\quad= \frac{1}{2n-1}\sum_{i=1}^{2n}\left(\overline{S}(\phi Y,\phi Z)\overline{g}(\phi e_i,\phi e_i)\right.\\ &\qquad - \overline{S}(\phi e_i,\phi Z)\overline{g}(\phi Y,\phi e_i)\\ &\qquad + \overline{S}(\phi e_i,\phi e_i)\overline{g}(\phi Y,\phi Z)\\ &\qquad \left.-\overline{S}(\phi Y,\phi e_i)\overline{g}(\phi e_i,\phi Z)\right),\end{aligned} \tag{51}$$

for all $Y, Z \in T\overline{M}$. The previous equation, in view of (50), becomes

$$\begin{aligned}\overline{S}(\phi Y,\phi Z) = &\left[a(2n-1) + a(\overline{r} - 2n(2-a))\right]\\ &\times\left[g(Y,Z) - \eta(Y)\eta(Z)\right],\end{aligned} \tag{52}$$

for all $Y, Z \in T\overline{M}$.

Using (2) and (18), (52) reduces to

$$\begin{aligned}\overline{S}(Y,Z) = &\left[\overline{r} + 2n - 1 - 2n(2-a)\right]\overline{g}(Y,Z)\\ &+ \left[2n(2-a)\left(a^2+2a-1\right)\right.\\ &\left.-(2a-1)(\overline{r}+2n-1)\right]\overline{\eta}(Y)\overline{\eta}(Z).\end{aligned} \tag{53}$$

Setting $Y = Z = e_i$ in (53), summing up with respect to $i = 1, 2, \dots, 2n+1$, and using (19), we obtain

$$\overline{r} = \frac{-2na^3 + a\left(4n^2+8n+2\right) - 4n^2 - 6n - 2}{2a - 2n - 1}. \tag{54}$$

Replacing X by ϕX and Y by ϕY in (53) and using (54), we obtain $\overline{S}(\phi X, \phi Y) = (\overline{r} + 2n - 1 - 2n(2-a))\overline{g}(\phi X, \phi Y)$ for all $X, Y \in T\overline{M}$.

Now using the previous expression in (49), we obtain

$$\overline{R}(\phi X, \phi Y, \phi Z, \phi W) = \frac{2(\overline{r} + 2n - 1 - 2n(2-a))}{2n-1} \times (\overline{g}(\phi Y, \phi Z)\overline{g}(\phi X, \phi W) - \overline{g}(\phi X, \phi Z)\overline{g}(\phi Y, \phi W)), \quad (55)$$

for all $X, Y, Z, W \in T\overline{M}$.

The converse is obvious. Thus we have the following.

Theorem 5. *A $(2n+1)$-dimensional K-contact manifold is ϕ-conharmonically flat with respect to D_a-homothetic deformation if and only if $\overline{M}$ satisfies* (55).

From (20), we obtain

$$\begin{aligned}\overline{g}(\overline{K}(X,Y)Z, \phi W) &= \overline{R}(X,Y,Z,\phi W) - \frac{1}{2n-1} \\ &\quad \times (\overline{S}(Y,Z)\overline{g}(X,\phi W) - \overline{S}(X,Z)\overline{g}(Y,\phi W) \\ &\quad + \overline{S}(X,\phi W)\overline{g}(Y,Z) - \overline{S}(Y,\phi W)\overline{g}(X,\phi Z)),\end{aligned} \quad (56)$$

for all $X, Y, Z, W \in T\overline{M}$.

Suppose that M is quasi-conharmonically flat K-contact manifold with respect to D_a-homothetic deformation; that is,

$$\overline{g}(\overline{K}(X,Y)Z, \phi W) = 0. \quad (57)$$

Then (56) reduces to

$$\begin{aligned}\overline{R}(X,Y,Z,\phi W) &= \frac{1}{2n-1}(\overline{S}(Y,Z)\overline{g}(X,\phi W) - \overline{S}(X,Z)\overline{g}(Y,\phi W) \\ &\quad + \overline{S}(X,\phi W)\overline{g}(Y,Z) - \overline{S}(Y,\phi W)\overline{g}(X,Z)).\end{aligned} \quad (58)$$

For a local orthonormal basis $\{e_1, e_2, \ldots, e_{2n}, \overline{\xi}\}$ of vector fields in $\overline{M}$, putting $X = \phi e_i$ and $W = e_i$ in (58) and summing up with respect to $i = 1, 2, \ldots, 2n+1$, we obtain

$$\begin{aligned}\sum_{i=1}^{2n}\overline{R}(\phi e_i, Y, Z, \phi e_i) &= \frac{1}{2n-1}\sum_{i=1}^{2n}(\overline{S}(Y,Z)\overline{g}(\phi e_i, \phi e_i) \\ &\quad - \overline{S}(\phi e_i, Z)\overline{g}(Y, \phi e_i) \\ &\quad + \overline{S}(\phi e_i, \phi e_i)\overline{g}(Y,Z) \\ &\quad - \overline{S}(Y, \phi e_i)\overline{g}(\phi e_i, Z)).\end{aligned} \quad (59)$$

Using (2), (10), (14), and (17) in (59), we obtain

$$\begin{aligned}\overline{S}(Y,Z) &= [\overline{r} + 2n - 1 - 2n(2-a)]\overline{g}(Y,Z) \\ &\quad - (2n-1)\overline{\eta}(Y)\overline{\eta}(Z).\end{aligned} \quad (60)$$

Taking $Z = \overline{\xi}$ and using (1) and (17), we obtain

$$\overline{r} = 2n(2-a)(a+1), \quad (61)$$

and using (61) in (19), we obtain

$$r = \frac{4n}{a}. \quad (62)$$

Hence, we can state the following.

Theorem 6. *Let M be a K-contact manifold. Suppose that $\overline{M}$ is obtained from M by D_a-homothetic deformation. If $\overline{M}$ is quasi-conharmonically flat, then the scalar curvatures $\overline{r}$ and r of $\overline{M}$ and M are, respectively, given by* (61) *and* (62).

Suppose that $\overline{M}$ is ξ-conharmonically flat. Then from (20), we have

$$\begin{aligned}\overline{R}(X,Y)\overline{\xi} &= \frac{1}{2n-1} \\ &\quad \times (2na(2-a)[\eta(Y)X - \eta(X)Y] \\ &\quad + [\overline{Q}X\overline{\eta}(Y) - \overline{Q}Y\overline{\eta}(X)]).\end{aligned} \quad (63)$$

Contracting the above equation with respect to W, we obtain

$$\begin{aligned}\overline{R}(X,Y,\overline{\xi},W) &= \frac{1}{2n-1}(2na(2-a)[\eta(Y)\overline{g}(X,W) - \eta(X)\overline{g}(Y,W)] \\ &\quad + [\overline{S}(X,W)\overline{\eta}(Y) - \overline{S}(Y,W)\overline{\eta}(X)]),\end{aligned} \quad (64)$$

for all $X, Y, Z, W \in T\overline{M}$.

For a local orthonormal basis $\{e_1, e_2, \ldots, e_{2n}, \overline{\xi}\}$ of vector fields in $\overline{M}$, using (64), we obtain

$$\begin{aligned}\sum_{i=1}^{2n}\overline{R}(e_i, Y, \overline{\xi}, e_i) &= \frac{1}{2n-1}\sum_{i=1}^{2n}(2na(2-a)[\eta(Y)\overline{g}(e_i,e_i) - \eta(e_i)\overline{g}(Y,e_i)] \\ &\quad + [\overline{S}(e_i,e_i)\overline{\eta}(Y) - \overline{S}(Y,e_i)\overline{\eta}(e_i)]).\end{aligned} \quad (65)$$

Therefore,

$$2n\overline{S}(Y,\overline{\xi}) = [4n^2(2-a) + \overline{r}]\overline{\eta}(Y). \quad (66)$$

Using (17) in (66), we obtain

$$\overline{r} = 0. \quad (67)$$

Taking $Y = \bar{\xi}$ in (64) and using (10), (14), and (17), we obtain

$$\begin{aligned}\bar{S}(X,W) = & [2na - 4n - 1]\,\bar{g}(X,W) \\ & + [6n - 4na + 1]\,\bar{\eta}(X)\,\bar{\eta}(W).\end{aligned} \tag{68}$$

From this we can conclude that M is η-Einstein. Thus, we have the following.

Theorem 7. *A D_a-Homothetically deformed ξ-conharmonically flat K-contact manifold is η-Einstein and its scalar curvature vanishes.*

5. D_a-Homothetic Deformation of ξ-Weyl Projectively Flat K-Contact Manifolds

Suppose that, in a $(2n+1)$-dimensional K-contact manifold M with D_a-homothetic deformation, the Ricci tensor vanishes; that is,

$$\bar{S}(X,Y) = 0. \tag{69}$$

Then from (16), we have

$$\begin{aligned}S(Y,Z) = & (a-1)\big(3g(\phi Y,\phi Z) - (1+a)\,g(Y,Z) \\ & + [2n(a-1) + 1 + a]\,\eta(Y)\,\eta(Z)\big).\end{aligned} \tag{70}$$

The Weyl projective curvature tensor of M is given by [16]

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{71}$$

If $W(X,Y)\xi = 0$, then (71) reduces to

$$R(X,Y)\xi = \frac{1}{2n}[S(Y,\xi)X - S(X,\xi)Y]. \tag{72}$$

Using (70) in (72), we obtain

$$R(X,Y)\xi = (a-1)^2[\eta(Y)X - \eta(X)Y]. \tag{73}$$

The Weyl projective curvature tensor of $\overline{M}$ with respect to D_a-homothetic deformation is given by

$$\overline{W}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2n}\left[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\right]. \tag{74}$$

Now using (14) and (69) in (74), we get

$$\overline{W}(X,Y)\bar{\xi} = R(X,Y)\bar{\xi} - \frac{(a-1)^2}{a}[\eta(Y)X - \eta(X)Y]. \tag{75}$$

From (73), the (75) reduces to

$$\overline{W}(X,Y)\bar{\xi} = 0. \tag{76}$$

Thus, we can state the following.

Theorem 8. *Let $\overline{M}$ be obtained from a K-contact manifold M by D_a-homothetic deformation. If the Ricci tensor of $\overline{M}$ vanishes, then it is ξ-Weyl projectively flat.*

References

[1] S. Tanno, "The topology of contact Riemannian manifolds," *Illinois Journal of Mathematics*, vol. 12, pp. 700–717, 1968.

[2] A. Carriazo and V. Martín-Molina, "Generalized (κ,μ)-space forms and D_a-homothetic deformations," *Balkan Journal of Geometry and its Applications*, vol. 16, no. 1, pp. 37–47, 2011.

[3] U. C. De and S. Ghosh, "D-homothetic deformation of normal almost contact metric manifolds," *Ukrainian Mathematical Journal*, vol. 64, no. 10, pp. 1514–1530, 2013.

[4] C. S. Bagewadi and Venkatesha, "Some curvature tensors on a trans-Sasakian manifold," *Turkish Journal of Mathematics*, vol. 31, no. 2, pp. 111–121, 2007.

[5] U. C. De, R. N. Singh, and S. K. Pandey, "On the conharmonic curvature tensor of generalized Sasakian-space-forms," *ISRN Geometry*, vol. 2012, Article ID 876276, 14 pages, 2012.

[6] Z. Guo, "Conformally symmetric K-contact manifolds," *Chinese Quarterly Journal of Mathematics*, vol. 7, no. 1, pp. 5–10, 1992.

[7] G. Zhen, J. L. Cabrerizo, L. M. Fernández, and M. Fernández, "On ξ-conformally flat contact metric manifolds," *Indian Journal of Pure and Applied Mathematics*, vol. 28, no. 6, pp. 725–734, 1997.

[8] J. L. Cabrerizo, L. M. Fernández, M. Fernández, and Z. Guo, "The structure of a class of K-contact manifolds," *Acta Mathematica Hungarica*, vol. 82, no. 4, pp. 331–340, 1999.

[9] M. K. Dwivedi, L. M. Fernández, and M. M. Tripathi, "The stucture of some classes of contact metric manifolds," *Georgian Mathematical Journal*, vol. 16, no. 2, pp. 295–304, 2009.

[10] M. M. Tripathi and M. K. Dwivedi, "The structure of some classes of K-contact manifolds," *Proceedings of the Indian Academy of Sciences*, vol. 118, no. 3, pp. 371–379, 2008.

[11] M. K. Dwivedi and J.-S. Kim, "On conharmonic curvature tensor in K-contact and Sasakian manifolds," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 34, no. 1, pp. 171–180, 2011.

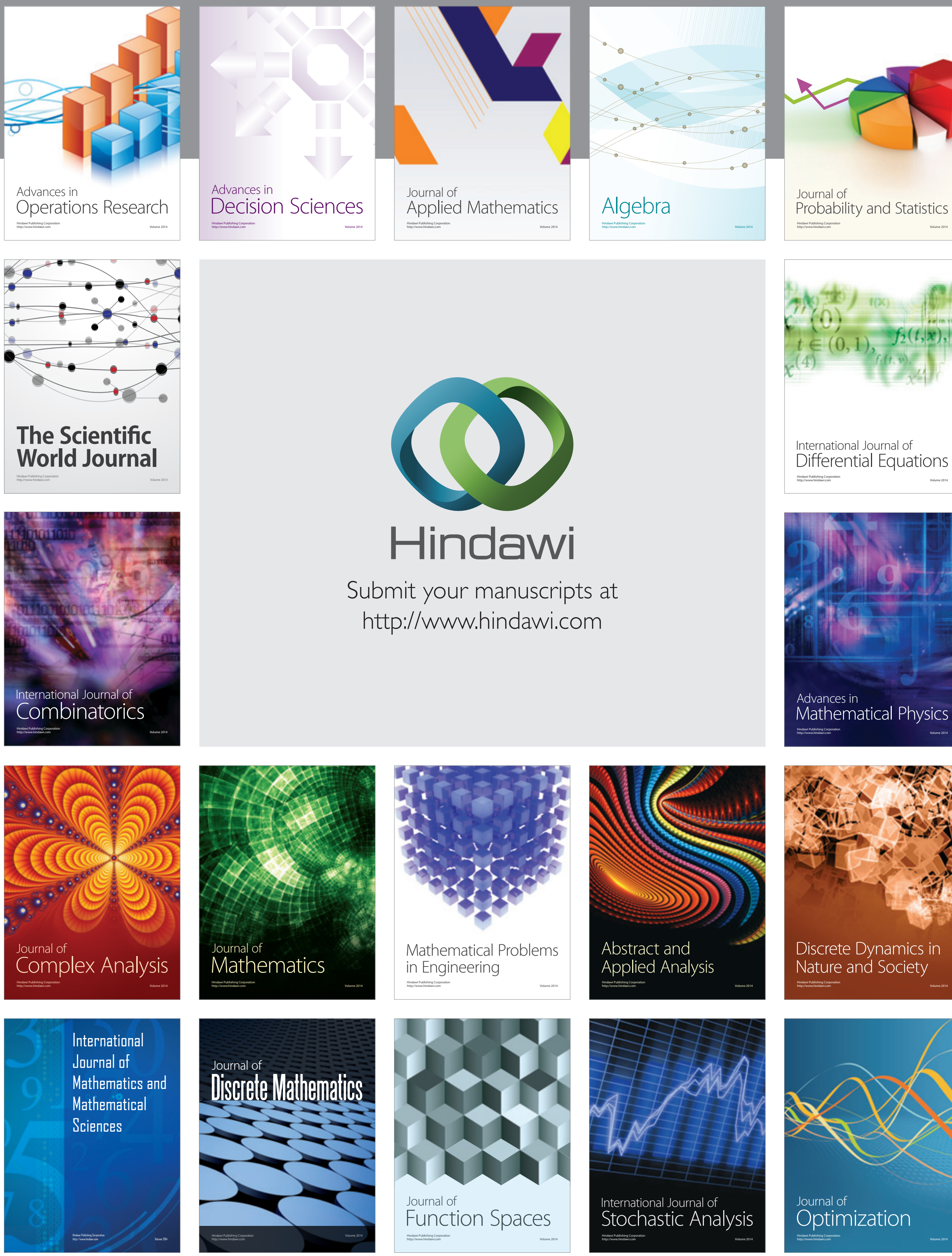
[12] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, vol. 509 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1976.

[13] M. C. Chaki and M. Tarafdar, "On a type of Sasakian manifold," *Soochow Journal of Mathematics*, vol. 16, no. 1, pp. 23–28, 1990.

[14] J. A. Oubiña, "New classes of almost contact metric structures," *Publicationes Mathematicae Debrecen*, vol. 32, no. 3-4, pp. 187–193, 1985.

[15] Y. Ishii, "On conharmonic transformations," *The Tensor Society*, vol. 7, pp. 73–80, 1957.

[16] R. S. Mishra, *Structures on a Differentiable Manifold and Their Applications*, Chandrama Prakashan, 50-A, Bairampur House, Allahabad, India, 1984.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