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Uniqueness of Meromorphic Functions Whose Certain Differential Polynomials Have Two Pseudo Common Values

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Abstract. In this paper, we study the uniqueness question of meromorphic functions whose certain differential polynomials having two pseudo common values, and obtain some results which improve and generalize the related results due to S.S. Bhoosnurmath and R.S. Davanal [1], P. Sahoo [5], J. Xia and Y. Xu [8] and C. Wu, C. Mu and J. Li [6].

Keywords: Meromorphic functions; Uniqueness; Differential polynomials; Sharing value.

1. Introduction

In this paper, by meromorphic function we will always mean meromorphic function in complex plane. We adopt the standard notations of Nevanlinna theory of meromorphic function as explained in [2], [9] and [10]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying $S(r, h) = o\{T(r, h)\}$, as $r \to \infty$ and $r \notin E$.

Let f and g be two non-constant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [10]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (see [10]). Let l be a positive integer or ∞. Next we denote by $E_{l)}(a;f)$ the set of those a-points of f in the complex plane, where each point is of multiplicity $\leq l$ and counted according to its multiplicity. By $\overline{E}_{l)}(a;f)$ we denote the reduced form of $E_{l)}(a;f)$. If $\overline{E}_{l)}(a;f) = \overline{E}_{l)}(a;g)$, we say that a is a l-order pseudo common value of f and g (see [3]). Obviously, if $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ $\left(\overline{E}_{\infty)}(a;f) = \overline{E}_{\infty)}(a;g) resp.\right)$, then f and g share a CM (IM, resp.).

Recall that S.S. Bhoosmurmath and R.S. Davanal [2]in 2007 proved the following two theorems. Also, it is noted that the problem of meromphic function having three weighted values and some examples of best possible of the above results were given in [4].

Theorem 1.1. [2] *Let f and g be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k+8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.*

Theorem 1.2. [2] *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{3}{n+1}$ and let n, k be two positive integers with $n > 3k + 13$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2010, P. Sahoo [5] proved the following theorem.

Theorem 1.3. [5] *Let f and g be two transcendental meromorphic functions, and let $n \geq 1$, $k \geq 1$ and $m \geq 0$ be three positive integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 IM. Then one of the following holds:*

(i) *when $m = 0$, if $f \neq \infty$, $g \neq \infty$ and $n > 9k + 14$, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;*

(ii) *when $m = 1$, $n > 9k + 20$ and $\Theta(\infty, f) > \frac{2}{n}$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$;*

(iii) *when $m \geq 2$ and $n > 9k + 4m + 16$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$. The possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ does not arise for $k = 1$.*

In 2011, J.Xia and Y. Xu [8] proved the following three theorems.

Theorem 1.4. [8] *Let n, k and m be three positive integers, and f and g be two non-constant meromorphic functions such that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM. If $m > k$ and $n > 3k + m + 8$, and $\Theta(\infty, f) > 2m(m+n)/\left[(n+m)^2 - 4k^2\right]$ or $\Theta(\infty, g) > 2m(m+n)/\left[(n+m)^2 - 4k^2\right]$, then either*

$f \equiv g$, or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y)=x^n(x-1)^m - y^n(y-1)^m$.

Theorem 1.5. [8] *Let n, k and m be three positive integers, and f and g be two non-constant meromorphic functions such that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM. If $m \le k$ and $n > 3k+m+8$, and*

$$\Theta(\infty, f) + \Theta_{[\frac{k}{m}])}(1,f) > 1 + 2m(m+n)/\left[(n+m)^2 - 4k^2\right]$$

or

$$\Theta(\infty, g) + \Theta_{[\frac{k}{m}])}(1,g) > 1 + 2m(m+n)/\left[(n+m)^2 - 4k^2\right]$$

then the conclusion of Theorem 1.4 holds.

Theorem 1.6. [8] *Let n, k and m be three positive integers such that $n > 3k+m+8$, and f and g be two non-constant meromorphic functions such that $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM. If f and g have the same poles (not necessary with the same multiplicity) then the conclusion of Theorem 1.4 holds.*

In 2011, C. Wu, C.Mu and J.Li [6] proved the following theorem.

Theorem 1.7. [6] *Let f and g be two non-constant meromorphic functions, and let $n \ge 1$, $k \ge 1$ and $m \ge 0$ be three positive integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 IM. Then one of the following holds:*

(i) *when $m = 0$ and $n > 9k+14$, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t with $t^n = 1$;*

(ii) *when $m = 1$, $n > 9k+18$ and $\Theta(\infty, f) > \frac{2}{n}$, then $f \equiv g$;*

(iii) *when $m \ge 2$ and $n > 9k+4m+14$, then $f \equiv g$ or f and g satisfy the algebraic equation $R(x,y) = x^n(x-1)^m - y^n(y-1)^m = 0$.*

One may ask the following question which is the motivation of the paper: Is it possible to relax the nature of the sharing value in Theorem 1.7 ?

In this paper, we give positive answers to the above question by establishing the following two theorems, which improves Theorems 1.1–1.7.

Theorem 1.8. *Let f and g be two non-constant meromorphic functions, and let $n \ge 1$, $k \ge 1$ and $m \ge 0$ be three positive integers. If $\overline{E}_{l)}(1;[f^n(f-1)^m]^{(k)}) = \overline{E}_{l)}(1;[g^n(g-1)^m]^{(k)})$ and $E_{1)}(1;[f^n(f-1)^m]^{(k)}) = E_{1)}(1;[g^n(g-1)^m]^{(k)})$, where $l \ge 3$ is an integer. Then one of the following holds:*

(i) *If $m = 0$, if $f \ne \infty$, $g \ne \infty$ and $n > \frac{13k+28}{3}$, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;*

(ii) *If* $m = 1$, $n > \frac{13k+41}{3}$ *and* $\Theta(\infty, f) > \frac{2}{n}$, *then* $f \equiv g$;
(iii) *If* $m \geq 2$ *and* $n > \frac{13k+5m+36}{3}$, *then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(x, y) = x^n(x-1)^m - y^n(y-1)^m = 0$.

Theorem 1.9. *Let* f *and* g *be two non-constant meromorphic functions, and let* $n \geq k+1$, $k \geq 1$ *and* $m \geq k+1$ *be three positive integers. If* $\overline{E}_{l)}(1; [f^n(f-1)^m]^{(k)}) = \overline{E}_{l)}(1; [g^n(g-1)^m]^{(k)})$ *and* $E_{2)}(1; [f^n(f-1)^m]^{(k)}) = E_{2)}(1; [g^n(g-1)^m]^{(k)})$, *where* $l \geq 4$ *is an integer. Then one of the following holds:*

(i) *If* $m = 0$, *if* $f \neq \infty$, $g \neq \infty$ *and* $n > \frac{3k+8}{3}$, *then either* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$;
(ii) *If* $m = 1$, $n > \frac{3k+9}{3}$ *and* $\Theta(\infty, f) > \frac{2}{n}$, *then* $f \equiv g$;
(iii) *If* $m \geq 2$ *and* $n > \frac{3k-m+10}{3}$, *then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(x, y) = x^n(x-1)^m - y^n(y-1)^m = 0$.

Remark 1.10. Theorem 1.8 and Theorem 1.9 extend Theorem 1.3 and Theorem 1.7.

Remark 1.11. Theorem 1.9 extends Theorem 1.1 for $m = 0$ and Theorem 1.2 for $m = 1$.

Remark 1.12. Theorem 1.9 extends Theorem 1.4, Theorem 1.5 and Theorem 1.6.

2. Lemmas

In this section, we present some lemmas which are needed in the sequel.

Lemma 2.1. [9] *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + ... + a_n f^n$,*where* a_0, a_1,...,a_n *are constants and* $a_n \neq 0$. *Then* $T(r, P(f)) = nT(r, f) + S(r, f)$.

Lemma 2.2. [7] *Let* $\overline{E}_{l)}(1; [F^*]^{(k)}) = \overline{E}_{l)}(1; [G^*]^{(k)}), E_{1)}(1; [F^*]^{(k)}) = E_{1)}(1; [G^*]^{(k)})$ *and* $H^* \neq 0$, *where* $l \geq 3$. *Then*

$$\begin{aligned} T(r, F^*) \leq & \left(\frac{8}{3} + \frac{2}{3}k\right) \overline{N}(r, \infty; F^*) + \frac{5}{3}\overline{N}(r, 0; F^*) + \frac{2}{3}N_k(r, 0; F^*) \\ & + N_{k+1}(r, 0; F^*) + (k+2)\overline{N}(r, \infty; G^*) + \overline{N}(r, 0; G^*) \\ & + N_{k+1}(r, 0; G^*) + S(r, F^*) + S(r, G^*) \end{aligned}$$

where

$$H^* \equiv \left[\frac{(F^*)^{(k+2)}}{(F^*)^{(k+1)}} - \frac{2(F^*)^{(k+1)}}{(F^*)^{(k)}-1}\right] - \left[\frac{(G^*)^{(k+2)}}{(G^*)^{(k+1)}} - \frac{2(G^*)^{(k+1)}}{(G^*)^{(k)}-1}\right].$$

Lemma 2.3. [7] *Let* $\overline{E}_{l)}(1;[F^*]^{(k)}) = \overline{E}_{l)}(1;[G^*]^{(k)}), E_{1)}(1;[F^*]^{(k)}) = E_{1)}(1;[G^*]^{(k)})$*, where* $l \geq 3$*. If*

$$\begin{aligned}\Delta_{1l} \;=\; & \left(\frac{8}{3}+\frac{2}{3}k\right)\Theta(\infty,F^*) + (k+2)\Theta(\infty,G^*) + \frac{5}{3}\Theta(0,F^*) + \Theta(0,G^*) \\ & +\; \delta_{k+1}(0,F^*) + \delta_{k+1}(0,G^*) + \frac{2}{3}\delta_k(0,F^*) \\ & >\; \frac{5}{3}k+9,\end{aligned}$$

then either $[F^*]^{(k)}[G^*]^{(k)} \equiv 1$ *or* $F^* \equiv G^*$*.*

Lemma 2.4. [7] *Let* $\overline{E}_{l)}(1;[F^*]^{(k)}) = \overline{E}_{l)}(1;[G^*]^{(k)}), E_{2)}(1;[F^*]^{(k)}) = E_{2)}(1;[G^*]^{(k)})$ *and* $H^* \neq 0$*, where* $l \geq 4$*. Then*

$$\begin{aligned}T(r,F^*) + T(r,G^*) \;\leq\; & (k+4)\overline{N}(r,\infty;F^*) + 2\overline{N}(r,0;F^*) \\ & +\; 2N_{k+1}(r,0;F^*) + (k+4)\overline{N}(r,\infty;G^*) + 2\overline{N}(r,0;G^*) \\ & +\; 2N_{k+1}(r,0;G^*) + S(r,F^*) + S(r,G^*)\end{aligned}$$

where H^* *is defined as Lemma* 2.2*.*

Lemma 2.5. [7] *Let* $\overline{E}_{l)}(1;[F^*]^{(k)}) = \overline{E}_{l)}(1;[G^*]^{(k)}), E_{2)}(1;[F^*]^{(k)}) = E_{2)}(1;[G^*]^{(k)})$*, where* $l \geq 4$*. If*

$$\begin{aligned}\Delta_{2l} =& \left(\frac{1}{2}k+2\right)[\Theta(\infty,F^*) + \Theta(\infty,G^*)] + \Theta(0,F^*) + \Theta(0,G^*) \\ & + \delta_{k+1}(0,F^*) + \delta_{k+1}(0,G^*) \\ & > k+5,\end{aligned}$$

then either $[F^*]^{(k)}[G^*]^{(k)} \equiv 1$ *or* $F^* \equiv G^*$*.*

Lemma 2.6. *Let* f *and* g *be two non-constant meromorphic functions, and let* $n \geq k+1$*,* $k \geq 1$ *and* $m \geq k+1$ *be a integers. Then* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv 1$*.*

Proof. Let

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \tag{2.1}$$

Let z_0 be a zero of f of order p_0. From (2.1) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (2.1), we obtain $np_0 - k = nq_0 + mq_0 + k$, i.e., $n(p_0 - q_0) = mq_0 + 2k$. which implies that $q_0 \geq \frac{n-2k}{m}$ and so we have $p_0 \geq \frac{n+m-2k}{m}$.

Let z_1 be a zero of $f-1$ of order p_1, then z_1 is a zero of $[f^n(f-1)^m]^{(k)}$ of order $p_1 - k$. Therefore from (2.1) we obtain $p_1 - k = nq_1 + mq_1 + k$ i.e., $p_1 \geq n + m + 2k$.

Let z_2 be a zero of f' of order p_2 that is not a zero of $f(f-1)$, then from (2.1) z_2 is a pole of g of order q_2. Again by (2.1) we get $p_2 - (k-1) = nq_2 + mq_2 + k$ i.e., $p_2 \geq n + m + 2k - 1$.

In the same manner as above, we have similar results for the zeros of $[g^n(g-1)^m]^{(k)}$.

On other hand, suppose that z_3 is a pole of f. From (2.1), we get that z_3 is the zero of $[g^n(g-1)^m]^{(k)}$.

Thus

$$\begin{aligned}\overline{N}(r,f) \leq& \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right)\\ \leq& \frac{1}{p_0}N\left(r,\frac{1}{g}\right) + \frac{1}{p_1}N\left(r,\frac{1}{g-1}\right) + \frac{1}{p_2}N\left(r,\frac{1}{g'}\right)\\ \leq& \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k} + \frac{2}{n+m+2k-1}\right]T(r,g)\\ &+ S(r,g).\end{aligned} \tag{2.2}$$

By second fundamental theorem and equation (2.2), we have

$$\begin{aligned}T(r,f) \leq& \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}(r,f)\\ \leq& \frac{m}{n+m-2k}N\left(r,\frac{1}{f}\right) + \frac{1}{n+m+2k}N\left(r,\frac{1}{f-1}\right)\\ &+ \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k} + \frac{2}{n+m+2k-1}\right]T(r,g)\\ &+ S(r,g) + S(r,f).\end{aligned}$$

$$\begin{aligned}T(r,f) \leq& \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k}\right]T(r,f)\\ &+ \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k} + \frac{2}{n+m+2k-1}\right]T(r,g)\\ &+ S(r,g) + S(r,f).\end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned}T(r,g) \leq& \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k}\right]T(r,g)\\ &+ \left[\frac{m}{n+m-2k} + \frac{1}{n+m+2k} + \frac{2}{n+m+2k-1}\right]T(r,f)\\ &+ S(r,g) + S(r,f).\end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we get

$$\begin{aligned}
&T(r,f)+T(r,g)\\
&\le \left[\frac{2m}{n+m-2k}+\frac{2}{n+m+2k}+\frac{2}{n+m+2k-1}\right]\{T(r,f)+T(r,g)\}\\
&+\ S(r,g)+S(r,f).
\end{aligned}$$

which is a contradiction. Thus Lemma proved. ■

3. Proof of the Theorem

Proof of Theorem 1.8. Let $F^* = f^n(f-1)^m$, $G^* = g^n(g-1)^m$.

By Lemma 2.1, we get

$$\Theta(0,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}(r,0;F^*)}{T(r,F^*)} \ge \frac{n+m-m^*-1}{n+m} \tag{3.1}$$

where $m^* = \{\begin{smallmatrix} 0 & if & m=0\\ 1 & if & m\ge 1\end{smallmatrix}$

Similarly

$$\Theta(0,G^*) \ge \frac{n+m-m^*-1}{n+m} \tag{3.2}$$

$$\Theta(\infty,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}(r,\infty;F^*)}{T(r,F^*)} \ge \frac{n+m-1}{n+m} \tag{3.3}$$

Similarly

$$\Theta(\infty,G^*) \ge \frac{n+m-1}{n+m} \tag{3.4}$$

$$\delta_{k+1}(0,F^*) = 1-\limsup_{r\to\infty}\frac{\overline{N}_{k+1}(r,0;F^*)}{T(r,F)} \ge \frac{n-k-1}{n+m} \tag{3.5}$$

Similarly

$$\delta_{k+1}(0,G^*) \ge 1-\frac{n-k-1}{n+m},\ \delta_k(0,F^*) \ge \frac{n-k}{n+m},\ \delta_k(0,G^*) \ge \frac{n-k}{n+m} \tag{3.6}$$

From the assumptions of Theorem 1.8, we have $\overline{E}_{l)}(1;[f^n(f-1)^m]^{(k)}) = \overline{E}_{l)}(1;[g^n(g-1)^m]^{(k)})$ and $E_{1)}(1;[f^n(f-1)^m]^{(k)}) = E_{1)}(1;[g^n(g-1)^m]^{(k)})$, where $l\ge 3$.

From (3.1)-(3.6) and Lemma 2.3, we have

$$\Delta_{1l} \ge \left(\frac{14}{3}+\frac{5}{3}k\right)\frac{n+m-1}{n+m}+\frac{8}{3}\frac{n+m-m^*-1}{n+m}+\frac{2}{3}\frac{n-k}{n+m}+2\frac{n-k-1}{n+m}.$$

It is easily verified that if $n > \frac{13k+5m+8m^*+28}{3}$, then $\Delta_{1l} > \frac{5}{3}k+9$. Since

$$\begin{aligned}
\frac{13k+5m+8m^*+28}{3} &= \frac{13k+28}{3} \quad \text{if} \quad m=0\\
&= \frac{13+41}{3} \quad \text{if} \quad m=1\\
&= \frac{13k+5m+36}{3} \quad \text{if} \quad m\ge 2
\end{aligned}$$

by Lemma 2.3, we have $F^* \equiv G^*$ or $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$. If $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$, i.e.,

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$$

then by Lemma 2.6 we can get a contradiction. Hence, we deduce that $F^* \equiv G^*$, i.e.,

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{3.7}$$

Now we consider following three cases.

Case(i) Let $m = 0$. Then from (3.7) we get $f \equiv tg$ for a constant t such that $t^n = 1$

Case (ii) Let $m = 1$ then from (3.7) we have

$$f^n(f-1) \equiv g^n(g-1). \tag{3.8}$$

Suppose $f \not\equiv g$. Let $h = \frac{f}{g}$ be a constant. Then from (3.8) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = \frac{1-h^n}{1-h^{n+1}} = \mathit{constant}$, a contradiction. So we suppose that h is not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$. From (3.8) we obtain $g = \frac{1-h^n}{1-h^{n+1}}$ and $f = \left(\frac{1-h^n}{1-h^{n+1}}\right)h$. Hence it follows that $T(r,f) = nT(r,h) + S(r,f)$. Again by second fundamental theorem of Nevanlinna, we have $\overline{N}(r,\infty;f) = \sum_{j=1}^{n} \overline{N}(r,\alpha_j;h) \geq (n-2)T(r,h) + S(r,f)$, where $\alpha_j(\neq 1)(j = 1,2,...,n)$ are distinct roots of the equation $h^{n+1} = 1$. So we obtain

$$\Theta(\infty,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,\infty;f)}{T(r,f)} \leq \frac{2}{n}$$

which contradicts the assumption $\Theta(\infty,f) > \frac{2}{n}$. Thus $f \equiv g$.

Case(iii) Let $m \geq 2$. Then from (3.7) we obtain

$$\begin{aligned} &f^n[f^m + ... + (-1)^i C^m_{m-i} f^{m-i} + ... + (-1)^m] \\ =&g^n[g^m + ... + (-1)^i C^m_{m-i} g^{m-i} + ... + (-1)^m]. \end{aligned} \tag{3.9}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.9) we obtain

$$g^{n+m}(h^{n+m}-1)+...+(-1)^i C^m_{m-i} g^{n+m-i}(h^{n+m-i}-1)+...+(-1)^m g^n(h^n-1) = 0,$$

which imply $h = 1$. Hence $f \equiv g$. If h is not a constant, then from (3.9) we can say that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$. ■

Proof of Theorem 1.9. From the condition of Theorem 1.9, we have

$\overline{E}_{l)}(1;[f^n(f-1)^m]^{(k)}) = \overline{E}_{l)}(1;[g^n(g-1)^m]^{(k)})$ and $E_{2)}(1;[f^n(f-1)^m]^{(k)}) = E_{2)}(1;[g^n(g-1)^m]^{(k)})$, where $l \geq 4$.

From (3.1)-(3.6) we have

$$\Delta_{2l} \geq (k+4)\frac{n+m-1}{n+m} + 2\frac{n+m-m^*-1}{n+m} + 2\frac{n-k-1}{n+m}.$$

It is easily verified that if $n > \frac{3k-m+2m^*+8}{3}$, then $\Delta_{2l} > k+5$. Since

$$\begin{aligned}\frac{3k-m+2m^*+8}{3} &= \frac{3k+8}{3} \quad \text{if} \quad m=0\\ &= \frac{3k+9}{3} \quad \text{if} \quad m=1\\ &= \frac{3k-m+10}{3} \quad \text{if} \quad m\geq 2\end{aligned}$$

by Lemma 2.5, we have $F^* \equiv G^*$ or $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$. If $(F^*)^{(k)}(G^*)^{(k)} \equiv 1$, i.e.,

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$$

then by Lemma 2.6 we can get a contradiction. Hence, we deduce that $F^* \equiv G^*$, i.e.,

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

Proceeding as in the proof of Theorem 1.8, we can get the conclusion of Theorem 1.9. ■

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