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On two-sample test for detecting differences in the IFR property of life distributions

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A test proposed for testing whether one distribution is more Increasing Failure Rate (IFR) than another, based on a measure of IFR is presented in this paper. The asymptotic normality of the proposed test statistic was also established. The asymptotic null variance from the data was estimated since the variance depends on the unknown distribution. The Pitman asymptotic efficacies of the proposed test statistic are computed for various alternative IFR distributions.

Key words: Two-sample test, IFR, Pitman efficacy.

INTRODUCTION

A life is represented by a non-negative random variable X with distribution function F and survival function $\bar{F} = 1 - F$. Classes of life distributions based on some notion of ageing have been introduced in the literature. The increasing failure rate (IFR) is one of the most important classes of lifetime distributions. We define IFR class below. A distribution F is said to be increasing failure rate

(IFR), if $\frac{\bar{F}(x+t)}{\bar{F}(x)}$ is decreasing in $x, t \geq 0$.

Proschan and Pyke (1967) proposed a test for testing exponentiality against IFR alternatives followed by Barlow and Proschan (1969), Bickel and Dorsum (1969), Bickel (1969), Ahmad (2004) and among others.

In practice, one might be interested in comparing two life distributions with respect to their ageing properties, particularly, IFR condition. Kochar (1981) and Cheng (1985) proposed several test procedures for testing equality of failure rates of two distributions. Hollandar et al. (1986) developed a test procedure for testing the null hypothesis that two life distributions F and G are equal versus the alternative hypothesis that F is more NBU than G . Tiwari and Zalkikar (1988) proposed a test for testing the null hypothesis that two life distributions F and G are identical versus the alternative hypothesis that F is 'More increasing failure rate average' than G . Recently,

Lim et al. (2005) developed a class of test procedures for testing the null hypothesis that two life distributions F and G are equal against the alternative that F is 'more NBU at specified age' than G . However, less attention is paid in the literature for testing the null hypothesis that two life distributions F and G are identical against the alternative that F is more IFR than G .

In this paper, we develop a simple test procedure for testing the null hypothesis that two life distribution F and G are equal against the alternative that F is more IFR than G .

The proposed two sample 'more IFR' test

Let X_1, X_2, \dots, X_m and Y_1, Y_2, \dots, Y_n denote two random samples from continuous life distributions F and G respectively. We want to develop test statistic for testing the null hypothesis.

$H_0 : F = G$ (the common distribution is not specified)

Versus

$H_1 : F$ is 'more IFR' than G .

Based on the two independent random samples.

Let μ_1 and μ_2 be the means corresponding to F and G respectively.

Consider the parameter

$$\gamma(F, G) = \gamma(F) - \gamma(G)$$

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Where

$$\gamma(F) = \frac{1}{\mu_1^2} \left(\int_0^\infty \int_0^\infty \bar{F}^2 \left(\frac{x+t}{2} \right) dxdt - \int_0^\infty \int_0^\infty \bar{F}(x)\bar{F}(t) dxdt \right)$$

and

$$\gamma(G) = \frac{1}{\mu_2^2} \left(\int_0^\infty \int_0^\infty \bar{G}^2 \left(\frac{x+t}{2} \right) dxdt - \int_0^\infty \int_0^\infty \bar{G}(x)\bar{G}(t) dxdt \right)$$

Here $\gamma(F)$ and $\gamma(G)$ can be considered as the measure of degree of the IFR-ness. Ahmed (2004) used this measure as basis in the construction of a test statistic for testing exponentiality against the IFR alternatives in one sample setting. If F and G belongs to IFR class, then $\gamma(F) > 0$ ($\gamma(G) > 0$) and $\gamma(F, G)$ can be taken as a measure for deciding whether F is 'more IFR' than G or not. Under H_0 , $\gamma(F, G) = 0$ and it is strictly greater than zero under H_1 .

The following unbiased estimator for $\gamma(F, G)$ is proposed, which is give

$$V = V_1 - V_2$$

Where

$$V_1 = \frac{[m(m-1)]^{-1} \sum \sum [2\{\min(x_i, x_j)\}^2 - x_i x_j]}{\bar{X}^2}$$

$$V_2 = \frac{[n(n-1)]^{-1} \sum_{i \neq j} \sum [2\{\min(y_i, y_j)\}^2 - y_i y_j]}{\bar{Y}^2}$$

Here \bar{X} and \bar{Y} the sample means of X and Y samples respectively.

The asymptotic normality of the test V is presented in the following theorem.

Theorem: As $N \rightarrow \infty$, $\sqrt{N}[V - \gamma(F, G)]$ is asymptotically normal with mean zero and variance

$$\sigma^2(F, G) = \frac{\sigma_1^2(F)}{\eta} + \frac{\sigma_2^2(G)}{1-\eta}, \text{ where } N=m+n$$

$$\eta = \lim_{N \rightarrow \infty} \frac{m}{N}$$

$$\sigma_1^2(F) = \frac{4}{\mu_1^4} \text{Var} \left(\mu_1 X_1 - 2 \int_0^{X_1} x dF(x) - 2X_1^2 \bar{F}(X_1) \right)$$

and

$$\sigma_2^2(G) = \frac{4}{\mu_2^2} \text{Var} \left(\mu_2 Y_1 - 2 \int_0^{Y_1} y dG(y) - 2Y_1^2 \bar{G}(Y_1) \right)$$

Under H_0 : $\sigma_1^2(F) = \sigma_2^2(G)$

$$\sigma^2 = \frac{\sigma_1^2(F)}{\eta(1-\eta)}$$

Proof: To establish asymptotic normality of the test statistic V , define

$$h(x_1, x_2) = 2\{\min(x_1, x_2)\}^2 - x_1 x_2$$

and set

$$h_1(x_1) = E[h(X_1, X_2) | X_1 = x_1]$$

$$= 2 \int_0^{x_1} x^2 dF(x) - 2x_1^2 \bar{F}(X_1) - \mu_1 x_1$$

$$h_2(x_1, x_2) = E[h(X_1, X_2) | X_1 = x_1, X_2 = x_2]$$

and

$$\xi_k(F) = E(h_1^2(x_1, \dots, x_k)) - \{\gamma(F)\}^2, \quad k = 1, 2$$

where

$$\gamma(F) = \int_0^\infty \int_0^\infty \bar{F}^2 \left(\frac{x+t}{2} \right) dxdt - \int_0^\infty \int_0^\infty \bar{F}(x)\bar{F}(t) dxdt$$

Also,

$$\xi_k(G) = E(h_1^2(y_1, \dots, y_k)) - \{\gamma(G)\}^2, \quad k = 1, 2$$

is defined analogously.

Since X_1, X_2, \dots, X_m and Y_1, Y_2, \dots, Y_n are independent (b assumption) (Serfling, 1980).

$$\text{Var}(V) = \text{Var}(V_1) + \text{Var}(V_2)$$

$$= \binom{m-1}{2} \sum_{k=1}^2 \binom{2}{k} \binom{m-2}{2-k} \xi_k(F) + \binom{n-1}{2} \sum_{k=1}^2 \binom{2}{k} \binom{n-2}{2-k} \xi_k(G).$$

It follows from Hoeffdings (1948) U - statistics theory that the limiting distribution of $\sqrt{N}[V - \gamma(F,G)]$ is asymptotic normal with mean 0 and variance σ^2 , where $N = m + n$ is the combined sample size, $\eta = \lim_{N \rightarrow \infty} \frac{m}{N}$ and

$$= \frac{\sigma_1^2(F)}{\eta} + \frac{\sigma_2^2(G)}{1-\eta}.$$

Under H_0 , the limiting distribution of \sqrt{NV} is normal with mean 0 and variance

$$\frac{\sigma_1^2(F_0)}{\eta(1-\eta)} = \frac{4\xi_1(F_0)}{\eta(1-\eta)},$$

where F_0 is the unspecified common distribution.

Here it is to be noted that the asymptotic mean of \sqrt{NV} is zero, independent of unspecified common distribution F_0 . However, the null asymptotic variance $4\xi_1(F_0)/\eta(1-\eta)$ does depend on F_0 through $\xi_1(F_0)$ and must be estimated from the data. To estimate σ^2 , one possible way is to obtain consistent estimator for σ^2 .

Since $\xi_1(F) = \text{Var}[h_1(x_1)]$, we let

$$\xi_1^*(F) = \frac{1}{m-1} \sum_{i=1}^m [h_1(x_i) - V_1]^2$$

where

$$h_1(x_i) = \frac{1}{(m-1)} \sum_{\substack{j=1 \\ j \neq i}}^m h(x_i, x_j)$$

In similar fashion, we let

$$\xi_1^*(G) = \frac{1}{(n-1)} \sum_{j=1}^n [h_1(y_j) - V_2]$$

where

$$\hat{h}_1(y_j) = \frac{1}{n-1} \sum_{\substack{i=1 \\ j \neq i}}^n h(y_i, y_j)$$

Then it is easily verified that

$$V_1 = \frac{1}{m} \sum_{i=1}^m \hat{h}_1(x_i)$$

and

$$V_2 = \frac{1}{n} \sum_{j=1}^n \hat{h}_1(y_j).$$

$\xi_1^*(F)$ and $\xi_1^*(G)$ are consistent estimators of $\xi_1(F)$ and $\xi_1(G)$, (Puri and Sen, 1971) and thus a consistent estimator $\hat{\sigma}_N^2$ is obtained by replacing $\xi_1(F)$ and $\xi_1(G)$ by $\xi_1^*(F)$ and $\xi_1^*(G)$, respectively in the expression of σ^2 . and hence

$$\hat{\sigma}_N^2 = \left(4N \xi_1^*(F) / m \right) + \left(4N \xi_1^*(G) / n \right).$$

By the consistency of $\hat{\sigma}_N^2$ and Slutsky's theorem $\sqrt{NV}\hat{\sigma}_N^{-1}$ is asymptotically $-N(0,1)$ under H_0 .

The approximate α -level test of H_0 versus H_1 rejects H_0 in favour of H_1 if

$$\sqrt{NV} / \left\{ \left(4N \xi_1^*(F) / m \right) + \left(4N \xi_1^*(G) / n \right) \right\}^{1/2} > z_\alpha$$

where Z_α is the upper α -percentile point of the normal distribution. This ensures the consistency of the two sample IFR test against the class of (F,G) pairs satisfying $\gamma(F) - \gamma(G) > 0$.

Asymptotic efficacies of the test

We study the asymptotic performance of V, for three pairs of distribution $(F_{1,\theta}, G)$ by evaluating Pitman effi-

Table 1. Pitman Efficacies for Weibull distribution.

θ	γ'	σ^2	Efficacy
2	0.299	3.907	0.151
3	0.318	4.553	0.149
4	0.318	5.143	0.140
5	0.308	5.623	0.130
6	0.295	6.012	0.120
7	0.280	6.331	0.111
8	0.265	6.595	0.108
9	0.251	6.817	0.096
10	0.237	7.007	0.090

Table 2. Pitman Efficacies for LFR distribution.

θ	$\gamma'(F, G)$	σ^2	Efficacy
2	0.326	0.0802	1.328
3	0.230	0.263	0.201
4	0.177	0.180	0.174
5	0.143	0.117	0.155

cacy. Here, we assume that G is an exponential distribution with mean one. The different distributions considered here for $(F_{i,\theta})$ are given below.

1. Weibull Distribution

$$\bar{F}_{1,\theta}(x) = \exp\{-x^\theta\}, \theta > 0, x > \theta$$

2. Linear Failure Rate Distribution

$$\bar{F}_{2,\theta}(x) = \exp\left\{-\left(x + \theta \frac{x^2}{2}\right)\right\}, x > 0, \theta \geq 0$$

3. Makeham Distribution

$$\bar{F}_{3,\theta}(x) = \exp\left\{-\left[x + \theta(x + e^{-x} - 1)\right]\right\}, \theta \geq 0, x \geq 0$$

The Pitman asymptotic efficacy is

$$Eff(V, F_{i,\theta}, G) = \sigma_0^{-2}(V) \left\{ \frac{d}{d\theta} \gamma(F_{i,\theta}, G) \right\}_{\theta \rightarrow \theta_0}^2$$

The Pitman asymptotic efficacies of the proposed test for Weibull distribution, Linear Failure Rate distribution and Makeham distribution are $\frac{1.2711}{\lambda(1-\lambda)}, \frac{0.8438}{\lambda(1-\lambda)}$

Table 3. Pitman Efficacies for Makeham distribution.

θ	$\gamma'(F, G)$	σ^2	Efficacy
1	0.0065	0.0075	0.0056
2	0.0042	0.0034	0.0052
3	0.0092	0.0564	0.0150
4	0.0110	0.0055	0.0220
5	0.0120	0.0052	0.0280
6	0.0128	0.0046	0.0355
7	0.0131	0.0038	0.0401
8	0.0130	0.0037	0.0451
9	0.0121	0.0033	0.0450
10	0.0120	0.0029	0.049

and $\frac{1.366}{\lambda(1-\lambda)}$ respectively.

The Pitman efficacies of two sample test based on V is determined by specifying a common distribution with parameter θ in the null hypothesis and by considering sequence of alternatives $(F_{\theta\phi}, F_\theta)$ where

$$\phi = 1 + \frac{a}{\sqrt{N}}, a > 0 \text{ be constant.}$$

The Pitman asymptotic efficacy is

$$Eff(V, F_{i,\theta}, G) = \sigma_0^{-2}(V) \left\{ \frac{d}{d\theta} \gamma(F_{i,\theta}, G) \right\}_{\theta \rightarrow \theta_0}$$

The Pitman efficacies of V for different values of θ for the pair $(F_{1,\theta\phi}, F_{1,\theta}), (F_{2,\theta\phi}, F_{2,\theta})$ and $(F_{3,\theta\phi}, F_{3,\theta})$ are presented in Table 1, 2 and 3 respectively.

Some remarks

- i.) A simple test procedure for testing the null hypothesis that two life time distributions are identical against the alternative that one possesses more IFRA property than another is presented. The test proposed is based on the measure considered by Ahmad (2004) for one sample problem.
- ii.) As far as we know, no test has been found in the literature for the problem stated in this article. Hence, only Pitman efficacies are computed for the few alternatives by specifying the common null distribution to be exponential. However, Hollander et al. (1986), Tiwari and Zalkikar (1988) and Lim et al. (2005) consider similar procedures for testing whether the distribution is, respectively, more new better than used(NBU), increase-ing failure rate average(IFRA) and new better than used of specified age(NBU-t₀) than another.
- iii.) We have also computed Pitman efficacies for those

alternatives with common null distributions being Weibull, Makeham and Linear failure rate distribution.

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