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Brief note

EXACT SOLUTIONS FOR THE INCOMPRESSIBLE ELECTRICALLY CONDUCTING VISCOUS FLOW BETWEEN TWO MOVING PARALLEL DISKS IN UNSTEADY MAGNETO HYDRODYNAMIC AND STABILITY ANALYSIS

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The main interest of the present investigation is to generate exact solutions to the steady Navier-Stokes equations for the incompressible Newtonian viscous electrically conducting fluid flow motion and stability due to disks moving towards each other or in opposite directions with a constant velocity. Making use of the analytic solution, the description of possible conditions of motion is based on the exact solutions of the Navier-Stokes equations. Both stationary and transient cases have been considered. The stability of motion is analyzed for different initial perturbations. Different types of stability were found according to whether the disks moved towards or away from each other.

Key words: viscous flows, magneto hydrodynamic, stability analysis, Riabouchinsky flow, Hermite's differential equation.

1. Introduction

The cases when an exact solution for the Navier-Stokes equations can be obtained are of particular interest in investigations to describe fluid motion of the viscous fluid flows. However, since the Navier-Stokes equations are non-linear in character, there is no known general method to solve the equations in full not does the superposition principle for non-linear partial differential equations work. Exact solutions, on the other hand, are very important for many reasons. They provide a standard for checking the accuracy of the results which can be established only by a comparison with an exact solution

There is a large class of processes which can be considered from the mathematical point of view as the motion of a liquid between two parallel disks, moving towards each other or in opposite directions with a constant velocity. These include such processes as the motion of underground water that can also be described with a help of the current model. In Fig.1 these two applicases are presented. It should be noted that in spite of the different types of hydro dynamical problem at first sight, the mathematical descriptions are the same.

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The purpose of the current work is to present details of some new two dimensional solutions of the Navier-stokes equations governing the steady-state stationary viscous flow of an incompressible Newtonian electrically conducting fluid associated with the movement of disks. The disks are supposed to be non-electrically conducting under the influence of an external magnetic field of constant strength applied normal to the disks.

This work deals with a description of the types of possible instability of such motion. Craik and Criminale (1986) described a procedure for finding classes of exact solutions of the Navier-Stokes equations. These solutions consist of a 'basic flow' with spatially uniform rates of strain and a 'disturbance' of a planar form: the disturbance is continuously distorted by the basic flow but nevertheless remains planar at all times. A somewhat similar formulation was given by Lagnado *et al*. (1984), but was restricted to two-dimensional basic flows and the authors were unaware that their liberalized approximation is in fact an exact solution for single plane wave modes.

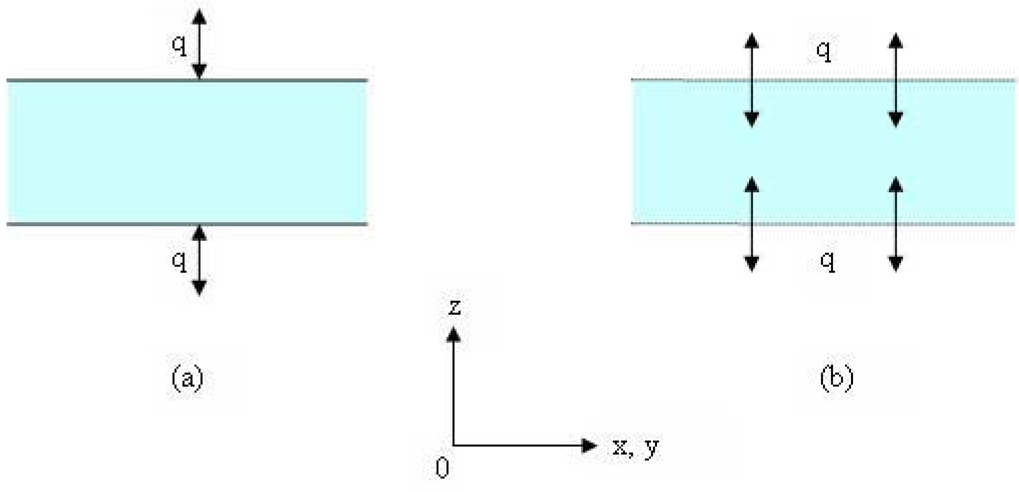

Fig.1 (a) Moving impermeable disks and (b) Moving permeable disks.

There are two aims of this paper. The first is to generalize the results of Craik (1989) in a case of plane-wave superposition. The other is to find the possible forms of the jet solutions which are generated as a result of the instability development.

In the present work, a weakly nonlinear magnetic field is introduced between two disks. The flow of a viscous fluid is analysed and the differential equation governing the fluid motion is based on the hydromagnetic flow induced in the fluid in the presence of a uniform magnetic field which accounts for the drag exerted by the magnetic effect. The governing nonlinear differential equations are solved analytically using separation of variables method. Furthermore, an instability analysis has been performed.

2. Mathematical formulation

Consider the motion of a viscous incompressible liquid induced by two parallel disks moving towards each other in the case when $h << l$ (where h is the distance between the disks, and L is the length of the disks). Let us assume that the horizontal velocity does not depend on the vertical coordinate, whereas the vertical velocity depends linearly on the distance between the disks. In this case, the Navier-Stokes equations have the following form (Craik 1989; Craik and Criminale, 1986; Lagnado *et al*., 1984)

From the continuity equation $\nabla \cdot \boldsymbol{q} = 0$ $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 2q$

From the momentum equations $\frac{\partial \boldsymbol{q}}{\partial t}+(\boldsymbol{q}\cdot\nabla)\boldsymbol{q}=-\nabla \boldsymbol{p}+\frac{1}{\rho}\hat{j}\times\boldsymbol{B}+\nu\nabla^2\boldsymbol{q}$.

Since we consider the hydromagnetic flow induced in the fluid in the presence of a uniform magnetic field B_0 normal to the plate therefore we come across the term $\frac{1}{\rho}\hat{j}\times\boldsymbol{B}$ in the momentum equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}-\frac{\sigma}{\rho}B_0^2u+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial x}-\frac{\sigma}{\rho}B_0^2v+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

where $\quad u=u(x,y,t),\quad v=v(x,y,t),\quad w=-2qz,\quad p=p(x,y,t)-2q^2z^2$.

Further, p is the pressure divided by the liquid density and q is the relative velocity of disks, assumed here to be constant. It should be noted also that the equation for the vertical velocity coordinate w is identically equal to zero. We consider the hydromagnetic flow induced in the fluid in the presence of a uniform magnetic field B_0 normal to the plate.

The above equations reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2q\,, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}-\frac{\sigma}{\rho}B_0^2u+\nu\Delta u\,, \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}-\frac{\sigma}{\rho}B_0^2v+\nu\Delta v \tag{2.3}$$

where $\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

3. Method of analysis

For convenience of analysis let us select the potential components from the horizontal components of the velocity and introduce the flow function.

$$u=qx+\frac{\partial\psi}{\partial y}\,, \tag{3.1}$$

$$v=qy-\frac{\partial\psi}{\partial x} \tag{3.2}$$

where ψ is the stream function. Now Eq.(2.1) is satisfied identically and Eqs (2.2) and (2.3) together, after elimination of the pressure and introduction of the vorticity $w = \nabla \times u$ will give the equations of motion in the following way.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]+\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]=$$

$$=-\frac{\sigma}{\rho}B_0^2\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\nu\Delta\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$

since $$\omega=\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=\Delta\psi\,, \tag{3.3}$$

$$\frac{\partial \omega}{\partial t}\{\psi,\omega\}=-q\left\{\frac{\partial}{\partial x}(x\omega)+\frac{\partial}{\partial y}(y\omega)\right\}-\frac{\sigma}{\rho}B_0^2\omega+\nu\Delta\omega \tag{3.4}$$

where $\{\psi,\omega\}$ denotes the Poisson brackets.

$$\{\psi,\omega\}=\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}. \tag{3.5}$$

One of the solutions of Eq.(3.4) is $\psi = 0$, which corresponds to the liquid potential motion, known as the motion near the stagnation point (other solutions for ψ are given below) is used following the work of Craik (1989) to investigate the stability of this solution. Let us consider the periodical one-dimensional perturbation $\delta\psi$. This perturbation is expressed by the following equation.

$$\begin{gathered}\psi=\psi+\hat{\psi}=\psi+\delta\psi\,,\\ \psi=\delta\psi=k^{-2}A\cos(kx)\,,\end{gathered} \tag{3.6}$$

substituting all this in Eq.(3.4) we get

$$-\frac{\partial A}{\partial t}=\left(2q+\frac{\sigma}{\rho}B_0^2+\nu k^2\right)A\,, \tag{3.7}$$

$$\frac{\partial k}{\partial t}=-qk\,. \tag{3.8}$$

Solving the above equations we get

$$k(t)=k(0)e^{-qt}\,, \tag{3.9}$$

$$A(t)=A(0)\exp\left[-\left\{2q+\frac{\sigma}{\rho}B_0^2\right\}t+\frac{\nu k^2(0)}{2q}\left(-1+e^{-2qt}\right)\right] \tag{3.10}$$

where $k(0)$, $A(0)$ are free constants, which determine the amplitude and wavelength at the initial point of time. The sign of q in Eq.(3.9) determines the stability of the solution $\psi = 0$. When $q < 0$, the solution is stable, the amplitude A is decreasing; otherwise, the solution is unstable, the amplitude A is increasing. However, for $q < 0$ the solution is unstable only until $t = \frac{1}{-2q}\ln\left|\frac{2q + \frac{\sigma}{\rho}B_0^2}{\nu k^2(0)}\right|$ after which the amplitude decreases rapidly, owing to dissipation.

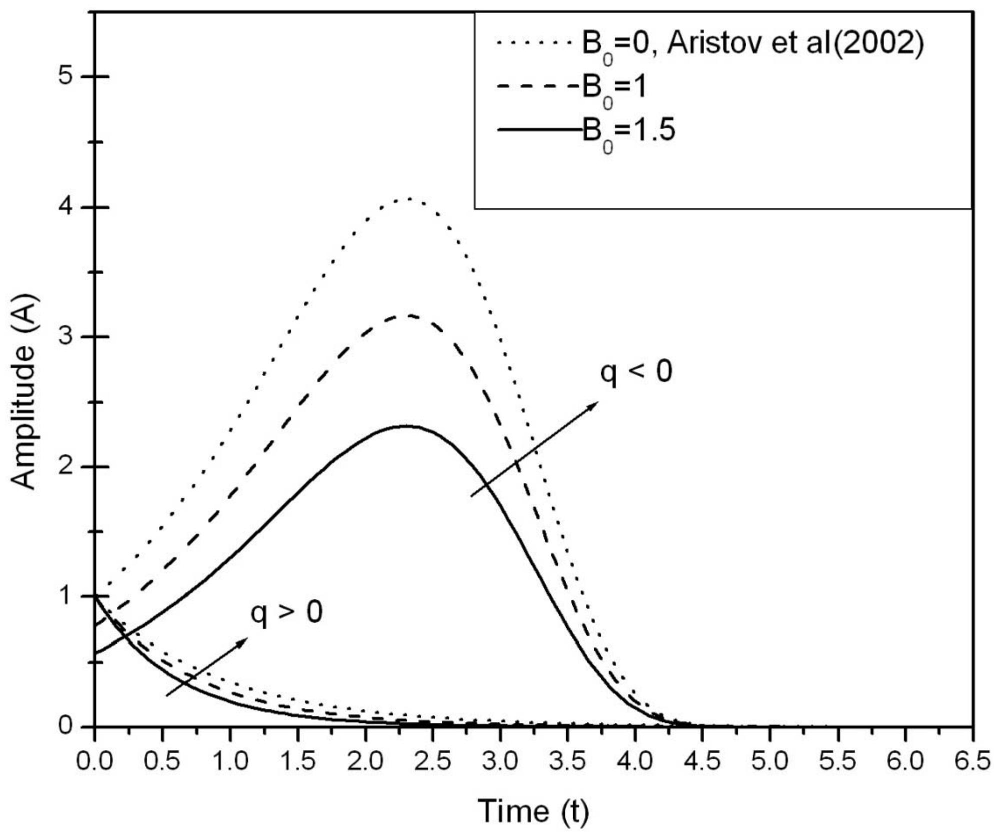

Fig.2. Variation of amplitude versus time for different values of B_0.

4. Results and discussion

4.1. Stability analysis

Let us consider the case when the flow function perturbation has the following form.

$$\delta\psi = \sum_{i=1}^{N} \frac{A_i(t)}{k_i^2(t)} \cos\left(k_{i1}(t)x + k_{i2}(t)y\right), \tag{4.1}$$

provided that $k_{i1}^2 + k_{i2}^2 = k^2 \quad \forall i$

Lemma 1: If $k_{i1}^2 + k_{i2}^2 = k^2 \quad \forall i$ then Eq.(4.1) is the exact solution of Eq.(3.4), with $k(t)$, $A(t)$ defined from expressions (3.9) and (3.10).
Proof: The proof is based on the reduction of Eq.(3.4) to a linear equation (where the principle of superposition is valid). The property of the Poisson brackets is $\{\psi,\omega\} = 0$. We find the vorticity. Since $k_{i1}^2 + k_{i2}^2 = k^2 \quad \forall i$, it is possible to carry out the summation in Eq.(4.1) from

$$\omega = \Delta\psi = -\sum_{i=1}^{N} A_i \cos(k_{i1}(t)x + k_{i2}(t)y) \equiv -k^2(t)\psi , \tag{4.2}$$

$k^2(t)$ does not depend on the spatial coordinate. Therefore

$$\{\psi,\omega\} = \{\psi, -k^2(t)\psi\} = -k^2(t)\{\psi,\psi\} = 0 . \tag{4.3}$$

This proves the lemma.

Remark 1: If $q > 0$, the solution is stable with both amplitude and the wave number k decreasing in the course of time. Otherwise, if $q < 0$, the solution is unstable, however, the amplitude increases until $t = \frac{1}{-2q}\ln\left|\frac{2q + \frac{\sigma}{\rho}B_0^2}{\nu k^2(0)}\right|$, after which owing to dissipation it decreases rapidly. The wave number k increases in the course of time. A new and interesting fact which has been discovered in the course of this research is that the wave number k, corresponding to the time. $t = \sum_{i=1}^{N}\left(\frac{1}{-2q}\ln\left|\frac{2q + \frac{\sigma}{\rho}B_0^2}{\nu k^2(0)}\right|\right)$ is not dependent on the initial conditions and is equal to $k = \sqrt{\frac{-2q}{\nu}}$.

It should be noted that in each of the cases investigated $q > 0$ corresponds to the situation when the disks are moving towards each other and $q < 0$ to the situation when the disks are moving apart.

Remark 2: Note that if in equation *N=1* then the results obtained by Craik (1989) are retrieved. This case corresponds to a perturbation in a form of one plane wave. The case when *N>1* corresponds to plane-wave superposition, which can (for special conditions for the wave number and amplitude (Chandrashekar, 1997)) reduce to the appearance of different space structures.

4.2. Stationary solutions in the form of jets

The solution $\psi = 0$, corresponding to the liquid motion near a stagnation point, has been considered. It is also relevant to find and examine other stationary solutions, such as jets. We consider the flow function in the Riabouchinsky type form

$$\psi(x,y) = xF(y) + \varphi(y) . \tag{4.4}$$

In this case (3.4) takes the following form

$$-\psi_x\omega_y+\psi_y\omega_x=-q(2\omega+y\omega_y+x\omega_x)-\frac{\sigma}{\rho}B_0^2\omega+\nu\Delta\omega, \tag{4.5}$$

and since $\Delta\psi=\omega=xF''(y)+\varphi''(y)$. (4.6)

Equation (4.5) reduces to

$$-F(y)F'''(y)+F'(y)F''(y)=-3qF''(y)+qyF'''(y)-\frac{\sigma}{\rho}B_0^2F''(y)+\nu F'^{\nu}(y), \tag{4.7}$$

$$-F(y)\varphi'''(y)+\varphi'(y)F''(y)=-2q\varphi''(y)-qy\varphi'''(y)-\frac{\sigma}{\rho}B_0^2\varphi''(y)+\nu\varphi'^{\nu}(y). \tag{4.8}$$

We consider the particular case when the analytical solution of Eq.(4.7) is $F=ay$. In this case (4.8) will take the following form

$$-ay\varphi'''(y)+q(2\varphi''(y)+y\varphi'''(y))+\frac{\sigma}{\rho}B_0^2\varphi''(y)=\nu\varphi'^{\nu}(y), \tag{4.9}$$

after some mathematical transformations and integrating twice, we obtain the following equation.

$$\nu\varphi''(y)=(q-a)y\varphi'(y)-2q\varphi+\frac{\sigma}{\rho}B_0^2\varphi, \tag{4.10}$$

which is the form of Hermite's differential equation. When the following two conditions are satisfied: $\frac{q-a}{\nu}=2$ and $\frac{q+\frac{\sigma}{2\rho}}{\nu}$ is a non-negative integer, then the solution of this equation has the following form

$$\varphi=\frac{d^n}{dy^n}\left[A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right] \tag{4.11}$$

where the relation between a and q is

$$a=-\frac{1+n}{3+n}q, \qquad n\in[0,\infty], \tag{4.12}$$

thus the solutions of Eq.(3.4) can be written as

$$\psi=-\frac{1+n}{3+n}qxy+\frac{d^n}{dy^n}\left[A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right], \tag{4.13}$$

in Eq.(4.13), the first term denotes the liquid motion corresponding to the potential flow component and the second term represents the jet behavior (non-potential flow component) since $q<0, n>0, \nu>0$ it can be seen that this second term approaches zero for $y \to \pm\infty$.

5. Conclusions

In this investigation, we came to know that if the disks are moving apart ($q \prec 0$) the electrically conducting viscous fluid is unstable up to a certain time $t=\sum_{i=1}^{N}\left(\frac{1}{-2q}\ln\left|\frac{2q+\frac{\sigma}{\rho}B_0^2}{\nu k^2(0)}\right|\right)$, then it is stable. This is because of the wave number $k(t)$ which increases in the course of time. But in case of the disks moving towards each other ($q \succ 0$) the electrically conducting viscous fluid is stable with both the amplitude $A(t)$ and the wave number $k(t)$ at all time.

The solution for the stream function through Hermit's differential equation is given by $\psi=-\frac{1+n}{3+n}qxy+\frac{d^n}{dy^n}\left[A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right]$. In this equation the first term denotes the liquid motion corresponding to the potential flow component and the second term denotes the jet behavior.

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Nomenclature

A – amplitude
B_0 – uniform magnetic field
h – distance between the disks
$\hat{j}$ – current density
k – wave length
l – length of the disks
p – pressure
q – velocity of disks
u – horizontal velocity component along the x-axis
v – horizontal velocity component along the y-axis
w – vertical velocity component
x, y – horizontal Cartesian coordinates
σ – electric conductivity of the fluid
ρ – density
ν – kinematic viscosity
ψ – stream function
ω – vorticity

Subscript

0 – reference

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