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Numerical Integration over Curved Domains Using Convex Quadrangulations And Gauss Legendre Quadrature Rules

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ABSTRACT: This paper presents a numerical integration formula for the evaluation of $II_{\Omega_N}(f) = \iint_{\Omega_N} f(x,y)dxdy$ where $f \epsilon C\ (\Omega_N)$ and Ω_N is any curved domain in $\mathbb{R}^2$. That is a closed domain with boundary composed of N oriented piecewise curved segments $C_{i,k}$ $(k = i+1,$ $i = 1,2,..N)$ with end points (x_i, y_i), (x_k, y_k) and $(x_1, y_1) = ((x_{N+1}, y_{N+1})$. We Join each of these curved segments $C_{i,k}$ to a reference point (x_r, y_r) interior to the domain Ω_N. This creates N triangles Δ_i^c $(i = 1,2,\ldots N)$ in Ω_N and each of these triangles have one curved side and two straight sides. We transform each Δ_i^c into a standard triangle T which also transforms the integrand $f = f(x,y)$ to f_i=$f_i(x,y)$. We then divide T into m^2 right isosceles triangles T_j^m $(j = 1,2,\ldots..m^2)$ of side lengths 1/m units. These triangles T_j^m $(j = 1,2,\ldots..m^2)$ will be finally divided into three special quadrilaterals Q_k $(i = 1,2,\ldots\ldots N;\ n = 1,2,\ldots..m^2;\ k = 1,2,3)$. This process can be expressed as

$$II_{\Omega_N}(f) = \sum_{i=1}^{N} II_{\Delta_i^c}(f) = \sum_{i=1}^{N} II_T(f_i) = \sum_{i=1}^{N}\sum_{j=1}^{m^2} II_{T_j}(f_{i,j}) = \sum_{i=1}^{N}\sum_{j=1}^{m^2}\sum_{k=1}^{3} II_{Q_k}(f_{i,j,k})$$

where $f_i, f_{i,j}, f_{i,j,k}$ represent the transformed forms of the integrand f over the domains T, T_j and Q_k. We approximate the curved segments $c_{i,k}$ by a parabolic arc which passes through the four points of the curved segment, the two end points (x_i, y_i), (x_k, y_k) and the two intermediate points of $C_{i,k}$. Proposed numerical integration formula is applied to integrate over a curved domain in the shape of lunar model for complicated integrands.

1. INTRODUCTION

The finite element method has proved superior to other numerical methods due to its better adaptability to any complex geometry**.** In a recent study[8], finite element method is applied to the numerical integration over polygonal domains using convex quadrangulation and Gauss Legendre quadrature rules**.** The domain of real problems often contains curved boundaries. In classical finite element applications curved boundaries are discretised by extremely refined meshes because simplifying the curved domains by polygonal domains may cause global changes in the physical solution of the problem. Curved boundaries are often more accurately modeled by curved finite elements than by straight edged elements, as straight sides are perfectly satisfactory if the domain has a polygonal boundary. The use of curved elements can model the complex geometry by fewer elements and this result in faster convergence to the desired solution. Curved elements are applied in the literature to solve some boundary value problems with complex geometries in [4 -11]. Curved triangular element with one curved side and two straight sides are found very useful in the solution of two dimensional boundary value problems. In this paper for the first time, the curved triangular finite elements are applied to integrate some complicated functions i.e. the well known Franke test functions over the curved domain in the shape of a lunar model. The curved edge can be interpolated by quadratic polynomial which must pass through at least three points on the curved boundary. This could be modeled by a 6-node quadratic triangular element. But this representation is not very efficient. It is shown in [4,6] that a better accuracy is obtained on representing the boundary side of this curved triangular element by a quadratic polynomial which passes through four points, the two end points and two intermediate points of this boundary side. In section 2, we consider the derivation of a curved triangular element with two straight sides and one curved side. The curved side is modeled by a quadratic curve passing through four points of the original curved side. In sections 3-4, we first explain the numerical integration scheme for curved domains and then proceed to the development of the composite integration formula for a curved domain which is fully discretised by special convex quadrilaterals. In section 5, we apply the numerical scheme of the previous sections to integrate complicated functions over curved domain in the shape of a lunar model. We have compared the computed values of these integral with analytical values and they are summarized in Tables. The importance of the present numerical scheme is its future scope for applications in boundary value problems governed by two dimensional partial differential equations. The relevant MATLAB codes are appended.

2 Cubic Curved Triangular Element

We first consider the ten node triangular element in which all the three sides are curved. The transformation which maps such a general curved triangular element in Cartesian space (x, y) into a right isosceles with sides of 1 unit in local parametric (ξ, η) is shown in Fig 1a, 1b.

The necessary transformation for this purpose is given by

$$x = \sum_{i=1}^{10} x_i \ N_i(\xi, \eta), \qquad y = \sum_{i=1}^{10} y_i \ N_i(\xi, \eta), \qquad 1 = \xi + \eta + \zeta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

where (x_i, y_i) are the cartesian coordinates of ith node and

$$N_1(\xi,\eta,\zeta) = \frac{1}{2}(3\xi-1)(3\xi-2)\xi \qquad N_4(\xi,\eta,\zeta) = \frac{9}{2}\xi\eta(3\eta-1) \qquad N_7(\xi,\eta,\zeta) = \frac{9}{2}\zeta\eta(3\eta-1)$$

$$N_2(\xi,\eta,\zeta) = \frac{1}{2}(3\eta-1)(3\eta-2)\eta \qquad N_5(\xi,\eta,\zeta) = \frac{9}{2}\xi\eta(3\xi-1) \qquad N_8(\xi,\eta,\zeta) = \frac{9}{2}\xi\zeta(3\xi-1)$$

$$N_3(\xi,\eta,\zeta) = \frac{1}{2}(3\zeta-1)(3\zeta-2)\zeta \qquad N_6(\xi,\eta,\zeta) = \frac{9}{2}\zeta\eta(3\zeta-1) \qquad N_9(\xi,\eta,\zeta) = \frac{9}{2}\xi\zeta(3\zeta-1)$$

$$N_{10}(\xi,\eta,\zeta) = 27\xi\zeta\eta$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If nodes 6, 7, 8 and 9 are at trisection point of two straight sides as shown in Fig 2a-2b, then eqn (1) reduces to:

$$t = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + \frac{9}{2}\xi\eta(-t_1 - t_2 - 2t_3 - t_4 - t_5 + 6t_{10})$$
$$+\frac{9}{2}\xi^2\eta(t_2 + 2t_3 + 3t_4 - 6t_{10}) + \frac{9}{2}\xi\eta^2(t_1 + 2t_3 + 3t_5 - 6t_{10})$$
$$(t = x, y)$$

…………..(3)

which shows that the curve of eqn (3) passing through the points $(x_1, y_1), (x_2, y_2), (x_4, y_4), (x_5, y_5)$ is a cubic curve. This can be shown by substituting from $\xi + \eta - 1 = 0$ and eliminating one of the variables $\xi \ or \ \eta$ in eqn (3). In general, it is shown that cubic curve is not desirable as an approximation to a simple smooth curve [12,13,14]. However if we choose,

$$x_5 = x_4 - \tfrac{1}{3}(x_1 - x_2), \ \ y_5 = y_4 - \tfrac{1}{3}(y_1 - y_2) \text{ and } t_{10} = \tfrac{1}{12}(t_1 + t_2 + 4t_3 + 3t_4 + 3t_5), \ \ (t = x, y)$$

… … … … … . (4)

The equation (3) for $\xi + \eta - 1 = 0$, is a cubic curve through $(x_1, y_1), (x_2, y_2), (x_4, y_4), (x_5, y_5)$ degenerates in a unique parabola through the four points $(x_1, y_1), (x_4, y_4)$, $(x_5, y_5) \equiv \left(x_4 - \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2, y_4 - \tfrac{1}{3}y_1 + \tfrac{1}{3}y_2\right)$ and (x_2, y_2) and hence the transformation equation in eq(3) reduces to

$$t = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + \tfrac{9}{4}\xi\eta(t_4 + t_5 - t_1 - t_2) \quad (t = x, y) \quad \text{……..…………} \quad (5a)$$

Equations (5a) can be written as

$$x = a_{00} + a_{10}\xi + a_{01}\eta + a_{11}\xi\eta, \ \ y = b_{00} + b_{10}\xi + b_{01}\eta + b_{11}\xi\eta \quad \text{……………………..}(5b)$$

where $a_{00} = x_3, \ a_{10} = x_1 - x_3, \ a_{01} = x_2 - x_3, \ a_{11} = \tfrac{9}{4}(x_4 + x_5 - x_1 - x_2),$

$b_{00} = y_3, \ b_{10} = y_1 - y_3, \ b_{01} = y_2 - y_3, \ b_{11} = \tfrac{9}{4}(y_4 + y_5 - y_1 - y)$

… … … … … … … (5c)

3. Integration over Curved Domain

We consider the following integral

$$II_{\Omega_N}(f) = \iint_{\Omega_N} f(x, y)dxdy \quad \text{… … … … … … … … . (6)}$$

Where $f \epsilon C\ (\Omega_N)$ and Ω_N is any curved domain in $\mathbb{R}^2$. That is a closed domain with continuous and smooth boundary composed of N-oriented piecewise $N - 1$ curved segments: $c_{i,k}$ $(k = i + 1,$ $i = 1,2, . . N - 1)$ with end points $i\ (x_i, y_i),\ k(x_k, y_k)$ and the Nth curved segments is $C_{N,1}$ and joins the points (x_N, y_N) and (x_1, y_1). We join each of these curved segments to a reference point $(x_r, y_r) \equiv (x_{N+1}, y_{N+1})$. This creates N curved triangles, each having two straight sides and one curved side. Thus the nodal points joining $< i, i + 1, N + 1 >$ is the i-th curved triangle Δ_i^c for $i = 1, 2, \dots N - 1$ and the N-th curved triangle Δ_N^c is obtained by joining the nodal point $< N, 1, N + 1 >$. We then transform each triangle Δ_i^c $(i = 1, 2, \dots N - 1)$into a standard triangle T (a right isosceles triangle with sides of 1 unit). We next divide the standart triangle T into m^2 right isosceles triangles $T_{j,m}(j = 1, 2, \dots\dots, m^2)$of side length (1/m) unit. We finally divide each $T_{j,m}$ into three special quadrilaterals $Q_{i,j,k}$ $(i = 1, 2, \dots N;$ $j = 1, 2, \dots m^2;\ k = 1, 2, 3)$ which can be obtained by joining the centroid of T_j, m to its mid- points. This will be explained further in the next section.We have shown the division of T into right isosceles triangles $T_{j,m}$(j=1,2,3,…,m^2),for m=1,2,3 as sample figures in Figs.2c,2d,2e.We have also developed a MATLAB code which divide each T into $T_{j,m}(j = 1,2, \dots, m^2)$ for any value of m.

4. Derivation of the composite Integration Formula:

Following the procedure proposed in the previous section we have from equations (5) and (6)

$$II_{\Omega_N}(f) = \iint_{\Omega_N} f(x,y)dxdy = \sum_{i=1}^{N} \iint_{\Delta_i^c} f(x,y)dxdy = \sum_{i=1}^{N} II_{\Delta_i^c}(f)$$

$$= \sum_{i=1}^{N} \iint_T f\left(x^i(\xi,\eta), y^i(\xi,\eta)\right) \frac{\partial(x^i, y^i)}{\partial(\xi,\eta)} d\xi d\eta$$

……………………………..(7)

Where $x^i(\xi,\eta) = x_3^i + (x_1^i - x_3^i)\xi + (x_2^i - x_3^i)\eta + \frac{9}{4}\xi\eta(x_4^i + x_5^i - x_1^i - x_2^i),$

$$y^i(\xi,\eta) = y_3^i + (y_1^i - y_3^i)\xi + (y_2^i - y_3^i)\eta + \frac{9}{4}\xi\eta(y_4^i + y_5^i - y_1^i - y_2^i)$$

……………….$(8a)$

and $((x_{k,}^i, y_k^i),\ k = 1,2,3,4,5)$ are the cartesian co-ordinates of the i-th curved triangle Δ_i^c and T is the standard right isosceles triangle in the (ξ,η) space.

From equations (5b) and (5c) we write equation in an alternative way as

$$x^i(\xi,\eta) = a_{00}^i + a_{10}^i\xi + a_{01}^i\eta + a_{11}^i\xi\eta$$

$$y^i(\xi,\eta) = b_{00}^i + b_{10}^i\xi + b_{01}^i\eta + b_{11}^i\xi\eta \qquad \text{…………………………..}(8b)$$

Where

$$a_{00}^i = x_3^i, \quad a_{10}^i = x_1^i - x_3^i, \quad a_{01}^i = x_{2,}^i - x_3^i, \quad a_{11}^i = \frac{9}{4}(x_4^i + x_5^i - x_1^i - x_2^i)$$

$$b_{00}^i = y_3^i, \quad b_{10}^i = y_1^i - y_3^i, \quad b_{01}^i = y_{2,}^i - y_3^i, \quad b_{11}^i = \frac{9}{4}(y_4^i + y_5^i - y_1^i - y_2^i)$$

$\dots\dots\dots..(8c)$

From equations (8a, b, c), we find that the determinant of Jacobian matrix of co-ordinate transformations it can be shown that

$$\frac{\partial(x^i,y^i)}{\partial(\xi,\eta)} = \alpha_0^i + \alpha_0^i\xi + \alpha_0^i\eta \qquad \text{…………………..}\ (9a)$$

Where $\alpha_0^i = a_{10}^i b_{01}^i - a_{01}^i b_{10}^i,$

$$\alpha_1^i = a_{10}^i b_{11}^i - a_{11}^i b_{10}^i,$$

$$\alpha_2^i = a_{11}^i b_{01}^i - a_{01}^i b_{11}^i \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9b)$$

from equation (7) we further write

$$II_{\Omega_N}(f) = \sum_{i=1}^{N}\sum_{j=1}^{m^2}\iint_{T_j} f\left(x^i(\xi^j,\eta^j), y^i(\xi^j,\eta^j)\right)\frac{\partial(x^i,y^i)}{\partial(\xi^j,\eta^j)}\, d\xi^j d\eta^j$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{m^2}\iint_{T} f\left(x^i(\xi^j,\eta^j), y^i(\xi^j,\eta^j)\right)\frac{\partial(x^i,y^i)}{\partial(\xi^j,\eta^j)}\,\frac{\partial(\xi^j,\eta^j)}{\partial(u,v)} dudv \qquad \ldots\ldots\ldots\ldots\ldots.(10\,a)$$

Where, $\xi^j = \xi_3^j + (\xi_1^j - \xi_3^j)u + (\xi_2^j - \xi_3^j)v$

$$\eta^j = \eta_3^j + (\eta_1^j - \eta_3^j)u + (\eta_2^j - \eta_3^j)v \qquad \ldots\ldots\ldots(10b)$$

and $((\xi_k^j, \eta_k^j), k = 1,2,3)$ are the co-ordinates of jth right isosceles triangle T_j of area $\frac{1}{2m^2}$ and T is standard right isosceles triangle in the (u, v) space. The division of T into m^2 right isosceles triangles T_j, j=1, 2…m of equal size, $\frac{1}{2m^2}$ is shown in Fig 3 (a) mapping each T_j into a standard triangle T is done by use of transformations in equations and it is shown in Fig 3(b).

From equation (10b), we find that

$$\frac{\partial(\xi^j,\eta^j)}{\partial(u,v)} = (\xi_2^j - \xi_1^j)(\eta_3^j - \eta_1^j) - (\xi_3^j - \xi_1^j)(\eta_3^j - \eta_1^j) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(10c)$$

Since the determinant of the Jacobian matrix represent twice the area of the right isosceles triangle, it must be equal to $\frac{1}{m^2}$ i.e, $\frac{\partial(\xi^j,\eta^j)}{\partial(u,v)} = \frac{1}{m^2}$ …………..($10d$)

From equation (10a), and the division of T into three special convex quadrilaterals Q_k $(k = 1,2,3)$ implies the following:

$$II_{\Omega_N}(f) = \sum_{i=1}^{N}\sum_{j=1}^{m^2}\iint_{T} f\left(x^i(\xi^j,\eta^j), y^i(\xi^j,\eta^j)\right)\frac{\partial(x^i,y^i)}{\partial(\xi^j,\eta^j)}\,\frac{\partial(\xi^j,\eta^j)}{\partial(u,v)} dudv$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{m^2}\sum_{k=1}^{3}\iint_{Q} f\left(x^i(\xi^j(u^k,v^k),\eta^j(u^k,v^k)), y^i(\xi^j(u^k,v^k),\eta^j(u^k,v^k))\right)$$

$$\times \frac{\partial(x^i,y^i)}{\partial(\xi^j,\eta^j)}\,\frac{\partial(\xi^j,\eta^j)}{\partial(u^k,v^k)} \quad du^k, dv^k \qquad \ldots\ldots\ldots..(11a)$$

Now finally from equation (11a) and proposed division of the standard triangle into three special quadrilaterals Q_k and the transformation of each Q into a standard 2-square implies the following:

$$II_{\Omega_N}(f)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{m^2}\sum_{k=1}^{3}\int_{-1}^{1}\int_{-1}^{1} f\left(x^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s))\right), y^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s))\right)\right)$$

$$\times \frac{\partial(x^i,y^i)}{\partial(\xi^j,\eta^j)}\,\frac{\partial(\xi^j,\eta^j)}{\partial(u^k,v^k)}\frac{\partial(u^k,v^k)}{\partial(r,s)}\, drds \qquad \ldots\ldots\ldots\ldots\ldots.(11b)$$

Where $\left(u^k(r,s), v^k(r,s)\right), k = 1,2,3)$ are the bilinear transformations

$$u^k(r,s) = \sum_{i=1}^{4} u_i^k\, N_i(r,s), \qquad v^k(r,s) = \sum_{i=1}^{4} v_i^k\, N_i(r,s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(12a)$$

which map the special quadrilateral of the standard triangle. The special quadrilaterals $Q_1, Q_2\ and\ Q_3$ are mapped to the 2-square as shown in Fig 4(a) – 4(b)

$$[(u_i^1, v_i^1), i = 1,2,3,4] = \left[\left(\frac{1}{3},\frac{1}{3}\right),\left(0,\frac{1}{2}\right),(0,0),\left(\frac{1}{2},0\right)\right]$$

$$[(u_i^2, v_i^2), i = 1,2,3,4] = \left[\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{2},0\right),(1,0),(\frac{1}{2},\frac{1}{2})\right]$$

$$[(u_i^3, v_i^3), i = 1,2,3,4] = \left[\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{2}\right),(0,1),(0,\frac{1}{2})\right]$$

$$N_1 = \tfrac{1}{4}(1-r)(1-s) \qquad N_2 = \tfrac{1}{4}(1+r)(1-s)$$

$$N_3 = \tfrac{1}{4}(1+r)(1+s) \qquad N_4 = \tfrac{1}{4}(1-r)(1+s)$$

$$u^1 = \tfrac{1}{3}N_1 + \tfrac{1}{2}N_4 = \lambda_1(r,s) \qquad v^1 = \tfrac{1}{3}N_2 + \tfrac{1}{2}N_2 = \lambda_2(r,s)$$

$$u^2 = \tfrac{1}{3}N_1 + \tfrac{1}{2}N_2 + N_3 + \tfrac{1}{2}N_4 = \lambda_3(r,s) \qquad v^2 = \tfrac{1}{3}N_1 + \tfrac{1}{2}N_4 = \lambda_1(r,s)$$

$$u^3 = \tfrac{1}{3}N_1 + \tfrac{1}{2}N_2 = \lambda_2(r,s) \qquad v^3 = \tfrac{1}{3}N_1 + \tfrac{1}{2}N_2 + N_3 + \tfrac{1}{2}N_4 = \lambda_3(r,s)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(12b)$$

where,

$$\lambda_1(\mathrm{r,s}) = \frac{1}{24}(1-\mathrm{r})(5+\mathrm{s})$$

$$\lambda_2(\mathrm{r,s}) = \frac{1}{24}(1-\mathrm{s})(5+\mathrm{r})$$

$$\lambda_3(\mathrm{r,s}) = \tfrac{1}{12}(7+2\mathrm{r}+2\mathrm{s}+\mathrm{rs}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12c)$$

From equations (12a, b), we find that

$$\frac{\partial(u^k, v^k)}{\partial(r,s)} = \frac{4+r+s}{96}, (\mathrm{k} = 1,2,3) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(12d)$$

From equations (9a), (10d) and (12b), we now obtain

$$II_{\Omega_N}(f) = \sum_{i=1}^{N}\sum_{j=1}^{m^2}\sum_{k=1}^{3}\int_{-1}^{1}\int_{-1}^{1} f\left(x^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s)\right),y^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s))\right)\right)$$

$$\times\ \left[\alpha_0^i + \alpha_1^i\xi^j(u^k(r,s),v^k(r,s) + \alpha_2^i\eta^j(u^k(r,s),v^k(r,s)\right](\frac{1}{m^2})(\frac{4+r+s}{96})drds$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{m^2} \sum_{k=1}^{3} \int_{-1}^{1} \int_{-1}^{1} f_{i,j,k}(r,s) \;\; drds \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \qquad (13a)$$

Where

$$f_{i,j,k}(r,s) = f\left(x^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s)\right),y^i\left(\xi^j(u^k(r,s),v^k(r,s)),\eta^j(u^k(r,s),v^k(r,s)\right)\right)$$

$$\times\ \left[\alpha_0^i + \alpha_1^i \xi^j(u^k(r,s),v^k(r,s) + \alpha_2^i \eta^j(u^k(r,s),v^k(r,s)\right]\left(\frac{4+r+s}{96m^2}\right) \qquad \dots\dots\dots\dots(13b)$$

We apply Gauss Legendre quadrature rule of order say ,L to evaluate the integrals of equation (13a-b). this gives us the following composite integration formula :

$$II_{\Omega_N}(f) = \sum_{i=1}^{N} \sum_{j=1}^{m^2} \sum_{k=1}^{3} \left[\sum_{p=1}^{L} \sum_{q=1}^{L} f_{i,j,k}\left({r_p}^L,{s_q}^L\right) w_p^L w_q^L\right] \qquad \dots\dots\dots\dots\dots\dots\dots\dots.(14a)$$

where

$$\left(r_p^L,\ \ s_q^L,\quad p,q=1,2,\dots L\right) \, and \ \ \left(w_p^L,\ \ w_q^L\ ,\ \ p,q=1,2,\dots L\right) \qquad \dots\dots\dots.\dots\dots\dots\dots(14b)$$

are the sampling points and weight coefficients of the Lth order Gauss Legendre Quadrature rule.

5. Application Example: A Lunar Model

We shall now apply the theoretical developments of the previous sections to compute the integral $II_{\Omega_N}(f)$ of equation (6) over a typical curved domain Ω_N in the shape of a lunar model. The boundary of Ω_N is composed of two circular arcs. The outer circular arc satisfies the equation $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$ and the outer arc satisfies the equation $x^2+y^2=\frac{1}{4}$. This is shown in Fig.4. The numerical integration formula developed in equation (14 a – b) of the previous section 2.3 requires the generation of of data among the boundary of the lunar model Ω_N. We find it most suitable to generate this boundary data by using the following explicit form of parametric equations.

5.1 Explicit form of parametric equations

The following parametric relations can be immediately obtained,

(1) Outer Circular arc

In Fig .4, it is shown that the outer circular arc is the boundary curve over the fourth, first and the second quadrants and this part of the boundary is described by the equation

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. \qquad (15)$$

Hence the parametric equations for the outer arc of lunar model are given by

$$x = \frac{1}{2}+\frac{1}{2}cos\theta, \quad y = \frac{1}{2}+\frac{1}{2}sin\theta, \quad \theta\,\epsilon\left(\left[0,\frac{\pi}{2}\right],\left[\frac{\pi}{2},\pi\right],\left[\frac{3\pi}{2},2\pi\right]\right) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \qquad (16)$$

(2) Inner Circular arc

We again refer to Fig.4, in which we have shown that the inner circular arc is the boundary curve over the third quadrant and this part of the boundary is described by the relation

$$x^2 + y^2 = \frac{1}{4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Hence the parametric equations for the inner arc of lunar model are given by

$$x = \frac{1}{2}cos\theta, \quad y = \frac{1}{2}sin\theta, \quad \theta \,\epsilon \left[\pi, \frac{3\pi}{2}\right] \qquad ------ \qquad (18)$$

5.2 Parabolic Interpolants as parametric equations

The parametric equations described above can also be replaced by parabolic arcs over each piece of curved boundary. One simple representation is to generate the boundary data for the outer arc and the inner arc of the lunar model shown in Fig.4. The lunar model consists of four quadrants. We suppose that divisions of equal area in the form of circular sectors are created in each quadrant. This requires $(d+1)$ nodes for each quadrant. Thus we require $(4d+1)$ nodes to cover the lunar model which is made up of four circular arcs and is equivalent to $4d$ circular sectors. Note that $(4d+1)$th node is an interior node in the lunar model. The lunar model has outer circular arc covering the first, the second and fourth quadrants which is described by the equation $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$ and the lunar model has one inner circular arc which covers the third quadrant and is described by the equation $x^2 + y^2 = \frac{1}{4}$. This is clear from the lunar model shown in Fig.4. We compute the value of integrals over each circular sector and then add their contributions to obtain the integral $II_{\Omega_N}(f_i)$ where Ω_N is the curved domain in the form of a lunar model and $f_i = f_i(x,y)$ (i=1,2,3,4,5,6,7) are some typical integrands. Now Ω_N consists of four circular arcs and each circular is further divided into d circular sectors. Thus Ω_N is discretised into $4d$ circular sectors. Let the end points along the circular boundary of a ith circular sector be (x_1^i, y_1^i), (x_2^i, y_2^i). We join these end points by straight lines to an interior point of the lunar model, say (x_3^i, y_3^i). This gives us a curved triangle with two straight sides and one curved side (boundary piece of the lunar model). Note that $(x_3^i, y_3^i) = \left(\frac{1}{2}, \frac{1}{2}\right)$ for all i in the present case.

We further explain the discretisation for the entire lunar model into $4d$ curved triangles. As said above, we have $(4d+1)$ nodes covering the entire lunar model. The boundary nodes for end points of the quadrants are:

(1) Fourth quadrant end points : $x_1^1 = \frac{1}{2}, \quad y_1^1 = 1, \quad x_2^d = 1, \quad y_2^d = \frac{1}{2}$

(2) First quadrant end points : $x_1^{d+1} = 1, \quad y_1^{d+1} = \frac{1}{2}, \quad x_2^{2d} = \frac{1}{2}, \quad y_2^{2d} = 1$

(3) Second quadrant end points : $x_1^{2d+1} = \frac{1}{2}, \quad y_1^{2d+1} = 1, \quad x_2^{3d} = 0, \quad y_2^{3d} = \frac{1}{2}$

(4) Third quadrant end points : $x_1^{3d+1} = 0, \quad y_1^{3d+1} = \frac{1}{2}, \quad x_2^{4d} = \frac{1}{2}, \quad y_2^{4d} = 1$

Note that $x_2^{4d} = x_1^1, \quad y_2^{4d} = y_1^1$ and $x_3^i = \frac{1}{2}, y_3^i = \frac{1}{2}, i = 1,2, \ldots .4d$

We can generate the remaining end points on all circular sectors by using the parametric equations of the outer circular arc and inner circular arc. This can be further explained as:

(5) Over the fourth quadrant, the kth circular sector has the end points

$$x_1^k = \frac{1}{2} + \frac{1}{2}cos\left(\frac{3\pi}{2} + \frac{k\pi}{2d}\right), \quad x_2^k = x_1^{k+1}$$
$$y_1^k = \frac{1}{2} + \frac{1}{2}sin\left(\frac{3\pi}{2} + \frac{k\pi}{2d}\right), \quad y_2^k = y_1^{k+1} \qquad k = 1,2, \dots . d - 1$$

(6) Over the first quadrant, the kth circular sector has the end points

$$x_1^k = \frac{1}{2} + \frac{1}{2}cos\left(\frac{k\pi}{2d}\right), \quad x_2^k = x_1^{k+1}$$
$$y_1^k = \frac{1}{2} + \frac{1}{2}sin\left(\frac{k\pi}{2d}\right), \quad y_2^k = y_1^{k+1} \qquad k = d + 1, d + 2, \dots .2d - 1$$

(7) Over the second quadrant, the kth circular sector has the end points

$$x_1^k = \frac{1}{2} + \frac{1}{2}cos\left(\frac{\pi}{2} + \frac{k\pi}{2d}\right), \quad x_2^k = x_1^{k+1}$$
$$y_1^k = \frac{1}{2} + \frac{1}{2}sin\left(\frac{\pi}{2} + \frac{k\pi}{2d}\right), \quad y_2^k = y_1^{k+1} \qquad k = 2d + 1, 2d + 2, \dots .3d - 1$$

(8) Over the third quadrant, the kth circular sector has the end points

$$x_1^{4d+1-k} = \frac{1}{2}cos\left(\frac{k\pi}{2d}\right), \quad x_2^{4d-k} = x_1^{4d+1-k}$$
$$y_1^{4d+1-k} = \frac{1}{2}sin\left(\frac{k\pi}{2d}\right), \quad y_2^{kd-k} = y_1^{4d+1-k} \qquad k = 1,2, \dots . d - 1$$

We have written a Matlab program to generate this data.We have further shown in Figs.5-9

the division of lunar model (see Fig.4) into 4,8,12,16 and 20 circular sectors.

Given the end points of the curved triangle elements with arc end points (x_1^i, y_1^i) and (x_2^i, y_2^i) and the point $(x_3^i, y_3^i) = (\frac{1}{2}, \frac{1}{2})$. We have a curved triangle with circular sector joining end points $((x_j^i, y_j^i)\ j = 1,2$ and the two straight sides joining points $((x_1^i, y_1^i), (x_3^i, y_3^i)$ and $(x_2^i, y_2^i), (x_3^i, y_3^i)$. We now approximate the circular arc joining points $((x_j^i, y_j^i),\ j = 1,2$ by a parabolic arc which has same end points but in addition it has to pass through two intermediate points of the circular arc. Let these intermediate points be denoted by $((x_4^i, y_4^i), (x_5^i, y_5^i))$. These points are computed by the theory developed in section 2. This follows from eqns (3)-(4). The points $((x_4^i, y_4^i), (x_5^i, y_5^i))$ must pass through the circular arc and in addition they satisfy the condition that

$$(x_5^i - x_4^i) = -\frac{1}{3}(x_1^i - x_2^i)\ and\ (y_5^i - y_4^i) = -\frac{1}{3}(y_1^i - y_2^i)$$

……………………………………….(19)

and the nodal coordinate (x_{10}^i, y_{10}^i) are computed by the formula

$t_{10}^i = \frac{1}{12}(t_1^i + t_2^i + 4t_3^i + 3t_4^i + 3t_5^o), (t^i = x^i, y^i)$ …………………………………………..(20)

This is also incorporated in the MATLAB programs appended here.

5.3 Examples of Integrals

We shall now consider the following integrals $II_{\Omega_N}(f)=\iint_{\Omega_N} f_i(x,y)dxdy$, Ω_N: lunar domain, i=1,2,3,4,5,6 where the six bivariate test functions are

$$f_1(x,y)=(x+y)^{19}$$

$$f_2(x,y)=\sqrt{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2}$$

$$f_3(x,y)=e^{-[(x-0.5)^2+(y-0.5)^2]}$$

$$f_4(x,y)=e^{-100\,[(x-0.5)^2+(y-0.5)^2]}$$

$$f_5(x,y)=\frac{3}{4}e^{-\frac{1}{4}[(9x-2)^2+(9y-2)^2]}+\frac{3}{4}e^{-\frac{1}{49}\left[(9x+1)^2-\frac{1}{10}(9y+1)^2\right]}$$

$$+\frac{1}{2}e^{-\frac{1}{4}[(9x-7)^2+(9y-3)^2]}-\frac{1}{5}e^{-[(9x-4)^2+(9y-7)^2]}$$

$$f_6(x,y)=cos[20(x+y)]$$

………………………………..(21)

and the exact values of integrals $(II_{\Omega_N}(f_i), i=1,2,3,4,5,6)$ *are* reported below as given in [9-12] :

$II_{\Omega_N}(f_5)=0.210503814662865$

$II_{\Omega_N}(f_2)=0.20646770293563$

$II_{\Omega_N}(f_1)=638.55743274702$

$II_{\Omega_N}(f_3)=0.57263720432530$

$II_{\Omega_N}(f_4)=0.03137185199242$

$II_{\Omega_N}(f_6)=0.0062895812195655$

…………………………………………………………….(22)

which are in order the well known Franke test functions, the distance function from (0.5, 0.5), a polynomial of degree 19, two Gaussians centered at (0.5, 0.5) with different variance parameters and a (moderately) oscillating function, the area of the curved domain (a lune) whose boundary is given by two circular arcs (the sides). The exact value of the area is $\frac{1}{4}+\frac{\pi}{8}\approx 0.6426990816987241$. This corresponds the integral$II_{\Omega_N}(f_7)$ where $f_7=1$. The computed values of above integrals $(II_{\Omega_N}(f_i), i=1,2,3,4,5,6,7)$ using the numerical scheme of sections 2-4 are summarized in Tables A-G. The present scheme is shown to evaluate the integrals complicated test functions to any desired accuracy which is

obvious from the absolute errors in the differences of exact values and computed values

CONCLUSIONS

The purpose of this paper is to develop an efficient numerical integration scheme based on curved triangular finite elements. The present study concentrates on such a phenomenon which may require integrating complicated functions over arbitrary curved domains. We can divide the curved domain into either triangles or quadrilaterals. But in the vicinity of the boundary area, we must fill the domain with triangles or quadrilateral having at least one curved side. In this paper, we have proposed to divide the curved domain into curved triangles having two straight sides and one curved side. The curved side is approximated by a parametric form of parabolic interpolant which passes through four points of the curved side.We first map the curved triangle into a right isosceles triangle of side length one unit which is well known as a standard triangle.This standard triangle is then divided into n^2 right isosceles triangles each of side length (1/n) unit and each of which is then divided into three convex quadrilaterals by joining the centroid to the midpoint of sides(we have referred these as special convex quadrilaterals in this paper),this divides the standard into $3n^2$ special convex quadrilaterals(spqd).We then transform these spqd into 2-square and the domain is now integrable by use of Gauss Legendre Quadrature rules.We illustrate the numerical scheme of this by choosing a curved domain in the shape of a lunar model. The division of the lunar model is explained in detail in section 5 of this paper.We call this domain as Ω_N.We have choosen the complicated test functions,which are well known as Franke test functions (f_i(x,y),i=1(1)7). The computed values of integrals $(II_{\Omega_N}(f_i), i=1,2,3,4,5,6,7)$ using the numerical scheme of sections 2-4 are summarized in Tables A-G. The necessary and relevant MATLAB codes to implement the proposed numerical scheme are appended.The list of MATLAB code as m-files is given below:

(1)specialconvexquadrilateralcurvedomaingausslegendrequadrature0.m,

(2)masterelementnodescoordinates.m, (3) globalnodalcoordinate.m, (4)glsampleptsweights

(5)coordinates_stdtriangle.m, (6)nodaladdresses_special_convex_quadrilaterals.m

We hope that the proposed numerical scheme will be useful and holds promise of further scope

in applications to two dimensional boundary value problems governed by partial differential equations.

References:

(1)E. I Ergatoudis, B M Irons, O C Zienkiewicz, curved isoparametric quadrilateral elements for finite element analysis, Int. J. Solids and Structures 4, 31-42, **(1968)**

(2)O C Zienkiewicz, The Finite Element Method, Mc Graw Hill, London**,(1977)**

(3)K J Bathe, Finite Element Procedures, Prentice Hall, Englewood Cliffs, N J. **(1996)**

(4)H T Rathod and **Md. Shajedul Karim,** Synthetic division based integration of rational functions of bivariate polynomial numerators with linear denominators over a unit triangle $\{ 0 \le \xi, \eta \le 1, \ \xi + \eta \le 1\}$, Computer Methods in Applied Mechanics and Engineering vol. 181, PP 191 – 235 **(2000).**

(5)H T Rathod and **Md. Shafiqul Islam,** Some pre-computed universal numeric arrays for linear convex quadrilateral finite elements, Finite Elements in Analysis and Design 38, 113-136, **(2001).**

(6)H T Rathod and **Md. Shajedul Karim,** An explicit integratic scheme based on recursion for the curved triangular finite elements, Computers and Structures, Vol. 80 PP 45-76 **(2002)**

(7)A Sommariva, M Vianello, Product Gauss Cubature based on Green's integration formula, BIT Numerical Mathematics, 47, PP 441-453 **(2007)**

(8)H T Rathod, B Venkatesh, K T Shivaram and T M Mamatha, Numerical Integration over Polygonal Domains using convex quadrangulations and Gauss Legendre Quadrature rules, International Journal of Engineering and Computer Science, Volume 2, Issue 8, 2576 – 2610 **(2013)**

(9)A Sommariva, M Vianello, Gauss Green cubature over spline curvilinear polygons, Applied Mathematics and Computation, Vol. 183 No.2, PP 1098-1107, **(2006)**

(10)A Sommariva, M Vianello, Gauss Green cubature andmoment computation over arbitrary geometries, Journal of Computational and Applied Mathematics,Vol.231,pp886-896**(2009)**

(11)G.Santin, **A Sommariva,**and **M Vianello,**An algebraic cubature formula on curvilinear polygons, Applied Mathematics and Computation, Vol. 217, No.24, PP 10003-10016, **(2011)**

(12)Michel A.R,Basis functions forcurved elements in the mathematical theory of finite elementsIn:Whiteman JR,editor.The mathematics of finite elements and applications II, London:Academic Press,pp43-58,**(1976)**

(13)McLeod R.J.Y,Michel A.R,The use of parabolic arcs in matching curved boundaries in the finite element method,J.Inst.Math.Appl.Vol.16,pp239-246**(1975)**

(14)McLeod R. Overcoming loss of accuracy when using curved finite elements,In:Whiteman JR,editor.The mathematics of finite elements and applications II, London:Academic Press,pp59-66, **(1976)**

TABLES A to G

Table-A

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN

Gauss Legendre rule=5X5, divisions of parabolic arc= 2

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	Absolute error
	1	638.557367595899	6.515112102079e-005
	2	0.20646469363349	3.00930213950612e-006
	3	0.572637200240612	4.08468769919068e-009
100	4	0.0313611257286559	1.07262637640895e-005
(2)	5	0.210503798066369	1.6596496232868e-008
[5X5]	6	0.00627910542873189	1.04757908336054e-005
	7	0.642699076031249	5.66747504482379e-009
	1	638.557406717765	2.60292546272467e-005
	2	0.206464705743591	2.99719203888937e-006
	3	0.572637204071438	2.53862264543159e-010
200	4	0.031361133093959	1.0718898460986e-005
(2)	5	0.210503798227234	1.6435631244871e-008
[5X5]	6	0.00627913192109247	1.04492984730259e-005
	7	0.642699081344527	3.54196894036818e-010
	1	638.557408831661	2.39153595202879e-005
	2	0.206464706314111	2.99662151911906e-006
	3	0.572637204276356	4.89436269290877e-011
300	4	0.0313611345239282	1.0717468491836e-005
(2)	5	0.210503798224162	1.64387028156465e-008
[5X5]	6	0.00627913677953103	1.04444400344707e-005
	7	0.642699081628761	6.99634794543158e-011
	1	638.557409192748	2.35542717064163e-005
	2	0.206464706035875	2.99689975491457e-006
	3	0.572637204310837	1.4462653297187e-011
400	4	0.031361135029184	1.07169632360266e-005
(2)	5	0.210503798220665	1.64421995740849e-008
[5X5]	6	0.00627913847674292	1.04427428225798e-005
	7	0.642699081676588	2.21366258656985e-011
	1	638.557409293391	2.34536290690812e-005
	2	0.20646470750944	2.99542618989457e-006
	3	0.57263720432026	5.03974639798344e-012
500	4	0.0313611352638779	1.07167285421303e-005
(2)	5	0.210503798218622	1.64442428840506e-008
[5X5]	6	0.00627913926175107	1.04419578144327e-005
	7	0.642699081689656	9.06774655362597e-012
	1	638.557409330423	2.34165966048749e-005
	2	0.206464706698119	2.99623751109923e-006
	3	0.572637204323643	1.6565637750432e-012
600	4	0.0313611353915863	1.07166008337378e-005
(2)	5	0.210503798217399	1.64454657114455e-008
[5X5]	6	0.00627913968802817	1.04415315373288e-005
	7	0.642699081694352	4.37228031557879e-012
	1	638.557409346753	2.3400267082252e-005

	2	0.206464707620669	2.99531496109062e-006
	3	0.572637204325095	2.04836148043341e-013
700	4	0.0313611354686653	1.07165237547419e-005
(2)	5	0.210503798216624	.6446241341006e-008
[5X5]	6	0.00627913994501011	1.0441274555394e-005
	7	0.642699081696363	2.36100028416786e-012
	1	638.557409354936	2.33920843584201e-005
	2	0.206464707702655	2.99523297478399e-006
	3	0.572637204325799	4.98823204964083e-013
800	4	0.0313611355187227	1.07164736973114e-005
(2)	5	0.210503798216105	1.64467600372031e-008
[5X5]	6	0.00627914011178099	1.04411077845095e-005
	7	0.642699081697339	1.38533629012727e-012
	1	638.557409359447	2.33875732646993e-005
	2	0.206464707718057	2.9952175728265e-006
	3	0.572637204326174	8.73634498077536e-013
900	4	0.0313611355530556	1.07164393643722e-005
(2)	5	0.210503798215742	1.64471229691099e-008
[5X5]	6	0.0062791402261097	1.04409934557972e-005
	7	0.642699081697861	8.62865334738672e-013
	1	638.557409362127	2.33848926427527e-005
	2	0.206464708116496	2.99481913354493e-006
	3	0.572637204326387	1.08724140801542e-012
1000	4	0.0313611355776205	1.07164147994734e-005
(2)	5	0.210503798215479	1.64473857311442e-008
[5X5]	6	0.00627914030788361	1.04409116818919e-005
	7	0.642699081698157	5.67212943280992e-013
	1	638.557409363815	2.33832046205862e-005
	2	0.2064647079173	2.99501832956572e-006
	3	0.572637204326518	1.2179146580138e-012
1100	4	0.0313611355957995	1.07163966204527e-005
(2)	5	0.210503798215283	1.64475820740861e-008
[5X5]	6	0.00627914036838461	1.04408511808866e-005
	7	0.642699081698337	3.87245790989255e-013
	1	638.557409364932	2.33820878747792e-005
	2	0.206464708127003	2.99480862703261e-006
	3	0.572637204326599	1.29873889420651e-012
1200	4	0.0313611356096282	1.071638279182e-005
(2)	5	0.210503798215132	1.64477325648171e-008
[5X5]	6	0.00627914041439951	1.04408051659916e-005
	7	0.64269908169845	2.73780997872564e-013
	1	638.557409365699	2.33813207159983e-005
	2	0.206464708093399	2.99484223131863e-006
	3	0.572637204326654	.35425004543777e-012
1300	4	0.0313611356203915	1.07163720285408e-005
(2)	5	0.210503798215015	1.64478504149912e-008
[5X5]	6	0.00627914045020888	1.04407693566162e-005
	7	0.642699081698526	1.98063787593128e-013

	1	638.557409366246	2.33807741096825e-005
	2	0.206464708071072	2.99486455765385e-006
	3	0.57263720432669	1.38966615992331e-012
1400	4	0.0313611356289324	1.07163634875743e-005
(2)	5	0.210503798214921	1.64479441733256e-008
[5X5]	6	0.00627914047862184	1.04407409436584e-005
	7	0.642699081698576	1.47659662275146e-013
	1	638.557409366645	2.33803754099426e-005
	2	0.206464708206848	2.99472878212415e-006
	3	0.572637204326716	1.41586742330446e-012
1500	4	0.0313611356358234	1.07163565965795e-005
(2)	5	0.210503798214845	1.64480198072692e-008
[5X5]	6	0.00627914050154386	1.04407180216426e-005
	7	0.642699081698611	1.13353770814228e-013
	1	638.55740936695	2.33800698197228e-005
	2	0.206464708123683	2.99481194743212e-006
	3	0.572637204326734	1.43407508090831e-012
1700	4	0.0313611356414635	1.07163509565217e-005
(2)	5	0.210503798214783	1.64480824516033e-008
[5X5]	6	0.00627914052030348	1.0440699262022e-005
	7	0.642699081698636	8.82627304576999e-014
	1	638.557409367179	2.33798409681185e-005
	2	0.206464708221722	2.99471390838302e-006
	3	0.572637204326746	1.44628753417919e-012
1700	4	0.0313611356461382	1.07163462818166e-005
(2)	5	0.210503798214731	1.64481344377965e-008
[5X5]	6	0.00627914053585096	1.04406837145386e-005
	7	0.642699081698657	6.7390537594747e-014
	1	638.557409367362	2.33796575912493e-005
	2	0.206464708201042	2.99473458767463e-006
	3	0.572637204326757	1.45705669751806e-012
1800	4	0.0313611356500556	1.07163423644085e-005
(2)	5	0.210503798214687	1.64481777087389e-008
[5X5]	6	0.00627914054887985	1.04406706856481e-005
	7	0.64269908169867	5.39568389967826e-014
	1	638.557409367504	2.33795165058837e-005
	2	0.206464708193751	2.99474187939741e-006
	3	0.572637204326764	1.46438416948058e-012
	4	0.0313611356533712	1.07163390487622e-005
1900	5	0.21050379821465	1.644821465141e-008
(2)	6	0.00627914055990597	1.04406596595271e-005
[5X5]	7	0.64269908169868	4.44089209850063e-014
	1	638.557409367624	2.33793961115225e-005
	2	0.206464708253525	2.9946821049065e-006
	3	0.572637204326771	1.47093448532587e-012
2000	4	0.031361135656202	1.07163362179988e-005
(2)	5	0.210503798214619	1.64482459596993e-008
[5X5]	6	0.00627914056931987	1.04406502456338e-005
	7	0.642699081698692	3.25295346215171e-014

Table-B

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN

Gauss Legendre rule=5X5, divisions of parabolic arc= 4

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	Absolute error
	1	638.557391166565	4.15804548765664e-005
	2	0.206467546554011	1.56381618626744e-007
	3	0.572637200239111	4.08618916480918e-009
100	4	0.0313721285130299	2.76520609854281e-007
(4)	5	0.210503814350231	3.1263433508677e-010
[5X5]	6	0.00628959214911356	1.09295480634625e-008
	7	0.642699076031249	5.66747515584609e-009
	1	638.557430128664	2.61835577930469e-006
	2	0.206467567886061	1.35049568805723e-007
	3	0.572637204069936	2.55363730161662e-010
200	4	0.0313721287898978	2.76797477750945e-007
(4)	5	0.210503814600028	6.28373464373055e-011
[5X5]	6	0.00628959077332675	9.55376124739393e-009
	7	0.642699081344529	3.54195339724583e-010
	1	638.557432212958	5.34061655343976e-007
	2	0.206467564732199	1.38203430977146e-007
	3	0.572637204274856	5.04439823245662e-011
300	4	0.031372128841071	2.76848650969386e-007
(4)	5	0.21050381461337	4.94949081719653e-011
[5X5]	6	0.00628959068233114	9.46276563704163e-009
	7	0.64269908162876	6.9964256610433e-011
	1	638.557432563681	1.83338670467492e-007
	2	0.206467566190587	1.36745042539355e-007
	3	0.572637204309337	1.59630086926654e-011
400	4	0.0313721288589767	2.76866556743471e-007
(4)	5	0.210503814615614	4.72511474391979e-011
[5X5]	6	0.00628959066257438	9.44300888438415e-009
	7	0.642699081676587	2.21374030218158e-011
	1	638.557432659527	8.74930492500425e-008
	2	0.20646756594327	1.36992359678345e-007
	3	0.572637204318759	6.54076792727665e-012
500	4	0.0313721288672639	2.76874843926911e-007
(4)	5	0.210503814616227	4.66381933073023e-011
[5X5]	6	0.00628959065555252	9.43598701865639e-009
	7	0.642699081689658	9.06619224139149e-012
	1	638.557432693955	5.30646957486169e-008
	2	0.206467565884285	1.37051344856198e-007

	3	0.572637204322143	3.15736325973148e-012
600	4	0.0313721288717653	2.76879345312286e-007
(4)	5	0.210503814616448	4.64173421921288e-011
[5X5]	6	0.0062895906523012	9.43273569756903e-009
	7	0.64269908169435	4.37416769472065e-012

	1	638.557432708715	3.83048472940573e-008
	2	0.20646756578931	1.37146319689752e-007
	3	0.572637204323594	1.70596869963902e-012
700	4	0.0313721288744796	2.76882059571659e-007
(4)	5	0.210503814616542	4.63232230352162e-011
[5X5]	6	0.0062895906505324	9.43096689978717e-009
	7	0.642699081696364	2.35989006114323e-012

	1	638.557432715876	3.11440544464858e-008
	2	0.206467565925606	1.37010024159334e-007
	3	0.572637204324298	1.00242036893405e-012
800	4	0.0313721288762409	2.76883820933549e-007
(4)	5	0.210503814616587	4.62781479804164e-011
[5X5]	6	0.00628959064946122	9.42989571931646e-009
	7	0.642699081697341	1.38333788868295e-012

	1	638.557432719689	2.73305431619519e-008
	2	0.20646756589371	1.37041920256209e-007
	3	0.572637204324675	6.24944540561501e-013
900	4	0.0313721288774488	2.76885028828444e-007
(4)	5	0.210503814616611	4.62535287848453e-011
[5X5]	6	0.00628959064876215	9.42919665090913e-009
	7	0.642699081697861	8.63642490855909e-013

	1	638.557432721869	2.51510527959908e-008
	2	0.206467565866727	1.37068903144355e-007
	3	0.572637204324886	4.13780121277796e-013
1000	4	0.0313721288783127	2.76885892658285e-007
(4)	5	0.210503814616626	4.62392624189789e-011
[5X5]	6	0.00628959064827948	9.42871397711237e-009
	7	0.642699081698157	5.6665783176868e-013

	1	638.557432723185	2.38350139625254e-008
	2	0.206467565895405	1.37040224890139e-007
	3	0.57263720432502	2.79554157600614e-013
	4	0.0313721288789517	2.76886531661025e-007
1100	5	0.210503814616634	4.62314908578065e-011
(4)	6	0.00628959064793166	9.42836616418807e-009
[5X5]	7	0.642699081698336	3.88244991711417e-013

	1	638.557432724022	2.29977104027057e-008
	2	0.206467565868114	1.37067516281508e-007
	3	0.572637204325098	2.02171612784241e-013
1200	4	0.031372128879438	2.76887017973404e-007
(4)	5	0.21050381461664	4.62253013644442e-011
[5X5]	6	0.00628959064767272	9.42810721681064e-009
	7	0.64269908169845	2.73669975570101e-013

	1	638.55743272457	2.24496261580498e-008
	2	0.206467565895276	1.37040353509477e-007
	3	0.572637204325152	1.47659662275146e-013
1300	4	0.0313721288798162	2.76887396226388e-007
(4)	5	0.210503814616643	4.62221649843997e-011
[5X5]	6	0.00628959064747435	9.42790884510963e-009
	7	0.642699081698527	1.96731519963578e-013
	1	638.557432724944	2.20755964619457e-008
	2	0.206467565888795	1.37046835130272e-007
	3	0.572637204325188	1.12354570092066e-013
1400	4	0.0313721288801164	2.76887696375183e-007
(4)	5	0.210503814616645	4.62195837158674e-011
[5X5]	6	0.00628959064731873	9.42775323087286e-009
	7	0.642699081698574	1.50324197534246e-013
	1	638.557432725206	2.18144577956991e-008
	2	0.20646756587535	1.37060279598034e-007
	3	0.572637204325214	8.57092175010621e-014
1500	4	0.0313721288803586	2.76887938598092e-007
(4)	5	0.210503814616647	4.62175853144231e-011
[5X5]	6	0.00628959064719464	9.42762914263318e-009
	7	0.642699081698613	1.11355369369903e-013
	1	638.557432725394	2.16259650187567e-008
	2	0.206467565892324	1.37043306092099e-007
	3	0.572637204325234	6.56141807553468e-014
1600	4	0.0313721288805568	2.7688813684229e-007
(4)	5	0.210503814616648	4.62171412252133e-011
[5X5]	6	0.00628959064709372	9.42752822336024e-009
	7	0.642699081698638	8.57092175010621e-014
	1	638.557432725527	2.14929514186224e-008
	2	0.206467565885054	1.37050575915731e-007
	3	0.572637204325246	5.40678612992451e-014
1700	4	0.031372128880721	2.76888301030398e-007
(4)	5	0.210503814616648	4.6216724891579e-011
[5X5]	6	0.00628959064701089	9.42744539204898e-009
	7	0.642699081698659	6.49480469405717e-014
	1	638.557432725634	2.13863131648395e-008
	2	0.206467565887003	1.37048627085745e-007
	3	0.572637204325254	4.56301663120939e-014
1800	4	0.0313721288808585	2.76888438482947e-007
(4)	5	0.21050381461665	4.62153371127982e-011
[5X5]	6	0.00628959064694204	9.42737654260895e-009
	7	0.642699081698667	5.72875080706581e-014
	1	638.557432725708	2.13123030334827e-008
	2	0.20646756587905	1.37056579946337e-007
	3	0.572637204325266	3.39728245535298e-014
1900	4	0.031372128880975	2.76888555007793e-007
(4)	5	0.21050381461665	4.62152260904958e-011
[5X5]	6	0.00628959064688372	9.4273182186036e-009

	7	0.642699081698683	4.16333634234434e-014
	1	638.55743272577	2.12495478990604e-008
	2	0.206467565897774	1.37037855618694e-007
	3	0.57263720432526	3.24185123190546e-014
2000	4	0.0313721288810745	2.76888654483776e-007
(4)	5	0.21050381461665	4.62147264901347e-011
[5X5]	6	0.00628959064683413	9.42726862632887e-009
	7	0.64269908169869	3.38618022510673e-014

Table-C

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN

Gauss Legendre rule=5X5, divisions of parabolic arc= 6

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	**Absolute error**
	1	638.557391187503	4.155951728535e-005
	2	0.206467595936059	1.06999571453104e-007
	3	0.57263720023911	4.08619016400991e-009
100	4	0.0313718434947439	8.49767605587415e-009
(6)	5	0.210503814398556	2.6430915789355e-010
[5X5]	6	0.0062895818542427	6.3467720228666e-010
	7	0.642699076031249	5.6674753778907e-009
	1	638.55743014941	2.59760963672306e-006
	2	0.206467589995776	1.12939853608163e-007
	3	0.572637204069936	2.55364285273174e-010
200	4	0.0313718434898466	8.5025734439248e-009
(6)	5	0.210503814648367	1.44978751226432e-011
[5X5]	6	0.00628958061794059	6.01624906143572e-010
	7	0.642699081344528	3.54195783813793e-010
	1	638.557432233669	5.13351210429391e-007
	2	0.206467591934129	1.11001501018615e-007
	3	0.572637204274855	5.04445374360785e-011
300	4	0.0313718434898515	8.50256846873787e-009
(6)	5	0.210503814661711	1.15377152276608e-012
[5X5]	6	0.00628958055314282	6.66422684397761e-010
	7	0.64269908162876	6.99643676327355e-011
	1	638.557432584384	1.62636411005224e-007
	2	0.206467592170799	1.10764831195986e-007
	3	0.572637204309333	1.59670054955541e-011
400	4	0.0313718434899204	8.50249957939919e-009
(6)	5	0.210503814663956	1.09109943302599e-012

[5X5]	6	0.00628958054258175	6.76983750308446e-010
	7	0.642699081676586	2.2138180177933e-011
	1	638.557432680222	6.67980657453882e-008
	2	0.206467591999344	1.10936286434882e-007
	3	0.572637204318759	6.54110099418403e-012
500	4	0.0313718434899641	8.50245587130649e-009
(6)	5	0.210503814664569	1.70444214298016e-012
[5X5]	6	0.00628958053982089	6.79744614762168e-010
	7	0.642699081689657	9.06708041981119e-012
	1	638.557432714646	3.23739186569583e-008
	2	0.206467592040838	1.10894792043625e-007
	3	0.572637204322141	3.15913961657088e-012
600	4	0.031371843489991	8.50242901778708e-009
(6)	5	0.21050381466479	1.92504345797317e-012
[5X5]	6	0.00628958053888539	6.80680106029952e-010
	7	0.642699081694351	4.37294644939357e-012
	1	638.557432729401	1.76191861100961e-008
	2	0.206467591989171	1.10946458548034e-007
	3	0.572637204323591	1.70874425720058e-012
700	4	0.0313718434900084	8.50241163585785e-009
(6)	5	0.210503814664884	2.01896832585646e-012
[5X5]	6	0.00628958053851332	6.81052183398867e-010
	7	0.642699081696366	2.35844677121122e-012
	1	638.557432736565	1.04546415968798e-008
	2	0.206467591976452	1.10959178206693e-007
	3	0.572637204324297	1.00308650274883e-012
800	4	0.03137184349002	8.50239999933278e-009
(6)	5	0.21050381466493	2.06495931465156e-012
[5X5]	6	0.00628958053834892	6.81216584877409e-010
	7	0.64269908169734	1.3837819778928e-012
	1	638.557432740377	6.64317667542491e-009
	2	0.206467592039442	1.10896187927034e-007
	3	0.572637204324675	6.24833518259038e-013
900	4	0.0313718434900283	8.5023916657212e-009
(6)	5	0.210503814664955	2.08963402137385e-012
[5X5]	6	0.00628958053827149	6.81294005586142e-010
	7	0.642699081697861	8.63309423948522e-013
	1	638.557432742558	4.46198100689799e-009
	2	0.206467592042888	1.10892741905788e-007
	3	0.572637204324884	4.16000567327046e-013
1000	4	0.0313718434900341	8.50238592031705e-009
(6)	5	0.210503814664968	2.10342854245482e-012
[5X5]	6	0.00628958053823357	6.81331932712859e-010
	7	0.642699081698161	5.63216140392342e-013
	1	638.557432743868	3.15162651531864e-009
	2	0.206467592031947	1.10903682654095e-007
	3	0.572637204325017	2.83217893581877e-013

1100	4	0.0313718434900385	8.50238154187499e-009
(6)	5	0.210503814664977	2.1120882820469e-012
[5X5]	6	0.00628958053821488	6.8135061828678e-010
	7	0.642699081698339	3.84692278032617e-013
	1	638.557432744706	2.31398189498577e-009
	2	0.206467592034473	1.10901157285292e-007
	3	0.572637204325096	2.03503880413791e-013
1200	4	0.031371843490042	8.50237798916131e-009
(6)	5	0.210503814664982	2.11713979680894e-012
[5X5]	6	0.00628958053820624	6.81359258944414e-010
	7	0.642699081698454	2.70006239588838e-013
	1	638.557432745259	1.76100911630783e-009
	2	0.206467592025721	1.10909909201151e-007
	3	0.57263720432515	1.50324197534246e-013
1300	4	0.0313718434900448	8.50237521360375e-009
(6)	5	0.210503814664987	2.12188600023921e-012
[5X5]	6	0.00628958053820259	6.81362913139416e-010
	7	0.642699081698528	1.96287430753728e-013
	1	638.55743274562	1.39971234602854e-009
	2	0.206467592030202	1.10905427563868e-007
	3	0.572637204325189	1.11244347067441e-013
1400	4	0.0313718434900467	8.50237327765235e-009
(6)	5	0.210503814664988	2.12341255689807e-012
[5X5]	6	0.00628958053820152	6.81363982596439e-010
	7	0.64269908169858	1.44551037806195e-013
	1	638.557432745889	1.13118403533008e-009
	2	0.206467592029906	1.10905724159949e-007
	3	0.572637204325213	8.72635297355373e-014
	4	0.0313718434900485	8.50237145272326e-009
1500	5	0.210503814664989	2.12435624646901e-012
(6)	6	0.00628958053820227	6.81363230593812e-010
[5X5]	7	0.642699081698614	1.09912079437891e-013
	1	638.557432746076	9.44169187278021e-010
	2	0.206467592031297	1.1090433335581e-007
	3	0.57263720432523	7.02771174587724e-014
1600	4	0.03137184349005	8.50237000943332e-009
(6)	5	0.210503814664992	2.12693751500126e-012
[5X5]	6	0.00628958053820345	6.81362050981849e-010
	7	0.642699081698635	8.870681966755e-014
	1	638.557432746215	8.05130184744485e-010
	2	0.206467592033702	1.1090192841845e-007
	3	0.572637204325246	5.41788836017076e-014
1700	4	0.031371843490051	8.5023690102326e-009
(6)	5	0.210503814664992	2.12663220366949e-012
[5X5]	6	0.00628958053820519	6.81360308452117e-010
	7	0.642699081698653	7.12763181809351e-014
	1	638.557432746312	7.08041625330225e-010
	2	0.206467592033989	1.10901640593131e-007

	3	0.572637204325258	4.22994972382185e-014
	4	0.0313718434900521	8.50236791388737e-009
1800	5	0.210503814664993	2.12846407166012e-012
(6)	6	0.00628958053820721	6.81358290101353e-010
[5X5]	7	0.64269908169867	5.38458166943201e-014
	1	638.557432746393	6.26869223196991e-010
	2	0.206467592038623	1.10897006577737e-007
	3	0.572637204325264	3.60822483003176e-014
1900	4	0.0313718434900531	8.50236689386996e-009
(6)	5	0.210503814664993	2.12838080493327e-012
[5X5]	6	0.00628958053820904	6.81356463437532e-010
	7	0.642699081698675	4.87387907810444e-014
	1	638.557432746452	5.68093128094915e-010
	2	0.206467592033722	1.10901907740546e-007
	3	0.572637204325268	3.19744231092045e-014
2000	4	0.0313718434900536	8.5023663942696e-009
(6)	5	0.210503814664993	2.12835304935766e-012
[5X5]	6	0.00628958053821064	6.81354863155126e-010
	7	0.642699081698688	3.59712259978551e-014

Table-D

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN
Gauss Legendre rule=5X5, divisions of parabolic arc= 8

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	**Absolute error**
	1	638.5573911878	4.15592197668957e-005
	2	0.206467673943824	2.89918062612582e-008
	3	0.572637200239109	4.08619083014372e-009
100	4	0.0313718514562304	5.36189613542515e-010
(8)	5	0.210503814396297	2.66567851125998e-010
[5X5]	6	0.00628958258863561	1.36907010901688e-009
	7	0.64269907603125	5.66747415664537e-009
	1	638.557430149706	2.59731359619764e-006
	2	0.206467685442903	1.74927268781211e-008
	3	0.57263720406993	2.55363064027847e-010
200	4	0.0313718514493285	5.43091495386339e-010
(8)	5	0.210503814646105	1.67602043354975e-011
[5X5]	6	0.00628958134208864	1.22523138336439e-010
	7	0.642699081344528	3.54195894836096e-010

	1	638.557432233959	5.13061308993201e-007
	2	0.206467685665765	1.72698651768055e-008
	3	0.572637204274855	5.04447594806834e-011
300	4	0.0313718514489627	5.43457334689634e-010
(8)	5	0.210503814659449	3.41576766871299e-012
[5X5]	6	0.00628958127540145	5.58359452423884e-011
	7	0.642699081628762	6.99625912758961e-011
	1	638.557432584674	1.62345600074332e-007
	2	0.206467685676034	1.72595957526056e-008
	3	0.572637204309332	1.59675606070664e-011
400	4	0.0313718514489018	5.43518174911384e-010
(8)	5	0.210503814661693	1.17231224727732e-012
[5X5]	6	0.00628958126417963	4.46141318333204e-011
	7	0.642699081676588	2.21365148433961e-011
	1	638.557432680507	6.65127117827069e-008
	2	0.206467685779535	1.71560950446015e-008
	3	0.572637204318756	6.5438765517456e-012
500	4	0.0313718514488854	5.43534564578785e-010
(8)	5	0.210503814662306	5.58664225991379e-013
[5X5]	6	0.00628958126111287	4.1547367109207e-011
	7	0.642699081689655	9.06963393276783e-012
	1	638.557432714937	3.20825392918778e-008
	2	0.206467685771243	1.71643874946614e-008
	3	0.572637204322141	3.15925063887335e-012
600	4	0.0313718514488797	5.43540289166256e-010
(8)	5	0.210503814662527	3.37674332939741e-013
[5X5]	6	0.00628958126001128	4.04457769359601e-011
	7	0.642699081694349	4.37527791774528e-012
	1	638.557432729692	1.73282614923664e-008
	2	0.206467685752027	1.71836029294603e-008
	3	0.572637204323595	1.704525409707e-012
700	4	0.0313718514488774	5.43542648390183e-010
(8)	5	0.210503814662621	2.43915998510147e-013
[5X5]	6	0.00628958125953918	3.99736823503138e-011
	7	0.642699081696363	2.36122232877278e-012
	1	638.557432736856	1.01638306659879e-008
	2	0.206467685771501	1.71641292290303e-008
	3	0.572637204324297	1.00297548044637e-012
800	4	0.0313718514488763	5.43543716979844e-010
(8)	5	0.210503814662666	1.99340544071447e-013
[5X5]	6	0.00628958125930978	3.97442807831272e-011
	7	0.642699081697338	1.38578037933712e-012
	1	638.557432740669	6.35088781564264e-009
	2	0.206467685774031	1.71615986421791e-008
	3	0.572637204324672	6.27609075820601e-013
900	4	0.0313718514488758	5.43544202702417e-010
(8)	5	0.210503814662692	1.72639680329212e-013
[5X5]	6	0.00628958125918793	3.96224277346424e-011
	7	0.642699081697861	8.62865334738672e-013

	1	638.55743274285	4.16969214711571e-009
	2	0.206467685769311	1.71663185055682e-008
	3	0.572637204324884	4.15667500419659e-013
1000	4	0.0313718514488753	5.43544688424991e-010
(8)	5	0.210503814662704	1.60566004936413e-013
[5X5]	6	0.00628958125911802	3.95525157764753e-011
	7	0.642699081698154	5.6976645623763e-013
	1	638.557432744166	2.85410806100117e-009
	2	0.206467685768744	1.71668855242224e-008
	3	0.572637204325018	2.82218692859715e-013
1100	4	0.0313718514488754	5.43544619036052e-010
(8)	5	0.210503814662714	1.51045842500253e-013
[5X5]	6	0.00628958125907589	3.95103932210317e-011
	7	0.642699081698337	3.8702374638433e-013
	1	638.557432744999	2.020669853664e-009
	2	0.206467685770624	1.71650061386863e-008
	3	0.572637204325096	2.04392058833491e-013
1200	4	0.031371851448875	5.43545035369686e-010
(8)	5	0.21050381466272	1.45272682772202e-013
[5X5]	6	0.00628958125904933	3.94838302678058e-011
	7	0.642699081698457	2.67230682027275e-013
	1	638.557432745552	1.46758338814834e-009
	2	0.206467685772118	1.71635120560509e-008
	3	0.572637204325155	1.45106149318508e-013
1300	4	0.0313718514488751	5.43544861897338e-010
(8)	5	0.210503814662722	1.42635903088717e-013
[5X5]	6	0.00628958125903167	3.94661707828203e-011
	7	0.642699081698527	1.97397653778353e-013
	1	638.557432745916	1.10378550743917e-009
	2	0.206467685770693	1.71649371383253e-008
	3	0.572637204325192	1.08024700296028e-013
1400	4	0.0313718514488746	5.43545368436593e-010
(8)	5	0.210503814662724	1.40609746068776e-013
[5X5]	6	0.00628958125901963	3.94541318018971e-011
	7	0.642699081698572	1.52211576676109e-013
	1	638.557432746186	8.34234015201218e-010
	2	0.206467685769528	1.71661017345226e-008
	3	0.572637204325215	8.45989944764369e-014
1500	4	0.0313718514488749	5.43545070064155e-010
(8)	5	0.210503814662726	1.39471767468535e-013
[5X5]	6	0.00628958125901139	3.94458866612157e-011
	7	0.642699081698613	1.11244347067441e-013
	1	638.557432746367	6.53358256386127e-010
	2	0.20646768577182	1.71638098456217e-008
	3	0.572637204325237	6.25055562863963e-014
1600	4	0.0313718514488748	5.43545208842033e-010
(8)	5	0.210503814662727	1.37639899477904e-013
[5X5]	6	0.00628958125900526	3.94397622199838e-011

	7	0.642699081698639	8.54871728961371e-014
	1	638.557432746497	5.23073140357155e-010
	2	0.20646768577113	1.71645000435205e-008
	3	0.572637204325246	5.40678612992451e-014
1700	4	0.0313718514488751	5.43544854958444e-010
(8)	5	0.210503814662729	1.36168853970275e-013
[5X5]	6	0.00628958125900084	3.94353378077583e-011
	7	0.642699081698659	6.56141807553468e-014
	1	638.557432746615	4.0517988963984e-010
	2	0.206467685770872	1.71647583646628e-008
	3	0.572637204325258	4.16333634234434e-014
1800	4	0.0313718514488749	5.43545070064155e-010
(8)	5	0.210503814662728	1.36501920877663e-013
[5X5]	6	0.00628958125899764	3.94321398450304e-011
	7	0.642699081698675	4.89608353859694e-014
	1	638.557432746687	3.3276137401117e-010
	2	0.206467685769821	1.71658094128002e-008
	3	0.572637204325261	3.88578058618805e-014
1900	4	0.0313718514488744	5.4354557660341e-010
(8)	5	0.21050381466273	1.34586786160185e-013
[5X5]	6	0.00628958125899501	3.94295082695173e-011
	7	0.642699081698678	4.66293670342566e-014
	1	638.557432746736	2.84330781141762e-010
	2	0.206467685772234	1.71633956214112e-008
	3	0.57263720432527	2.95319324550292e-014
2000	4	0.0313718514488745	5.43545521092259e-010
(8)	5	0.210503814662731	1.34087185799103e-013
[5X5]	6	0.00628958125899325	3.94277457904657e-011
	7	0.64269908169869	3.46389583683049e-014

Table-E

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN
Gauss Legendre rule=5X5, divisions of parabolic arc= 10

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	Absolute error
	1	638.557391187816	4.15592041917989e-005
	2	0.206467680257341	2.267828858038e-008
	3	0.572637200239108	4.08619149627754e-009
100	4	0.0313718520276705	3.52504553324806e-011
(10)	5	0.210503814396428	2.66436983586971e-010
[5X5]	6	0.00628958255188898	1.3323234789725e-009
	7	0.64269907603125	5.66747415664537e-009

	1	638.557430149719	2.59730131801916e-006
	2	0.206467679107035	2.3828595374642e-008
	3	0.572637204069938	2.5536184278252e-010
200	4	0.0313718520207211	2.83010559432029e-011
(10)	5	0.210503814646235	1.66300861970115e-011
[5X5]	6	0.00628958130533712	8.57716241781104e-011
	7	0.642699081344526	3.54197782215238e-010
	1	638.557432233977	5.13043346472841e-007
	2	0.20646767915136	2.37842695538504e-008
	3	0.572637204274851	5.0448756283572e-011
300	4	0.0313718520203462	2.79261891389382e-011
(10)	5	0.210503814659579	3.2856217746513e-012
[5X5]	6	0.00628958123865006	1.90845577188736e-011
	7	0.642699081628758	6.99658109226675e-011
	1	638.557432584685	1.62335027198424e-007
	2	0.206467678994359	2.39412710489884e-008
	3	0.572637204309335	1.59647850495048e-011
400	4	0.0313718520202825	2.78625178484759e-011
(10)	5	0.210503814661824	1.04088959673732e-012
[5X5]	6	0.00628958122742833	7.86282584180897e-012
	7	0.642699081676587	2.21375140441182e-011
	1	638.557432680527	6.64931576466188e-008
	2	0.206467678892079	2.40435508447323e-008
	3	0.57263720431876	6.53954668194956e-012
500	4	0.0313718520202643	2.78442616186148e-011
(10)	5	0.210503814662437	4.28240776173539e-013
[5X5]	6	0.00628958122436169	4.79618515042413e-012
	7	0.642699081689659	9.06485997376194e-012
	1	638.557432714955	3.20653725793818e-008
	2	0.206467678894651	2.40409785134954e-008
	3	0.572637204322142	3.15747428203395e-012
600	4	0.0313718520202581	2.78380582474647e-011
(10)	5	0.210503814662657	2.08388861722142e-013
[5X5]	6	0.00628958122326013	3.69462620219974e-012
	7	0.64269908169435	4.37416769472065e-012
	1	638.557432729712	1.73083662957652e-008
	2	0.206467678913019	2.40226111225539e-008
	3	0.572637204323597	1.7027490528676e-012
700	4	0.0313718520202551	2.7835143912025e-011
(10)	5	0.210503814662753	1.12188036638372e-013
[5X5]	6	0.00628958122278818	3.2226799354107e-012
	7	0.642699081696365	2.35900188272353e-012
	1	638.557432736873	1.01467776403297e-008
	2	0.206467678906764	2.4028866091319e-008
	3	0.572637204324297	1.00275343584144e-012
800	4	0.0313718520202535	2.78334785774881e-011
(10)	5	0.210503814662798	6.69464483848969e-014
[5X5]	6	0.00628958122255877	2.99326882724493e-012

	7	0.64269908169734	1.38455913401003e-012
	1	638.55743274068	6.34008756605908e-009
	2	0.206467678898408	2.40372215465356e-008
	3	0.572637204324675	6.25166585166426e-013
900	4	0.0313718520202527	2.78327430547343e-011
(10)	5	0.210503814662821	4.36317648677687e-014
[5X5]	6	0.00628958122243696	2.87146348365575e-012
	7	0.642699081697859	8.65307825392847e-013
	1	638.557432742861	4.15923295804532e-009
	2	0.206467678902507	2.4033123241507e-008
	3	0.572637204324884	4.16222611931971e-013
1000	4	0.0313718520202523	2.78322989655244e-011
(10)	5	0.210503814662836	2.87270207621759e-014
[5X5]	6	0.006289581222367	2.80150468795481e-012
	7	0.642699081698155	5.68767255515468e-013
	1	638.557432744183	2.83728240901837e-009
	2	0.206467678909537	2.40260930317593e-008
1100	3	0.572637204325013	2.87436741075453e-013
(10)	4	0.0313718520202519	2.78319173263597e-011
[5X5]	5	0.210503814662845	1.97897254139434e-014
	6	0.00628958122232486	2.75936131582943e-012
	7	0.642699081698339	3.84692278032617e-013
	1	638.557432745021	1.9990693544969e-009
	2	0.20646767890325	2.40323796141251e-008
	3	0.572637204325096	2.03614902716254e-013
1200	4	0.0313718520202517	2.78317091595426e-011
(10)	5	0.210503814662852	1.33504318711175e-014
[5X5]	6	0.0062895812222983	2.73280183205049e-012
	7	0.642699081698451	2.72781797150401e-013
	1	638.557432745564	1.45553258334985e-009
	2	0.206467678900115	2.4035515383547e-008
	3	0.572637204325144	1.56319401867222e-013
1300	4	0.0313718520202517	2.78316744650731e-011
(10)	5	0.210503814662853	1.18516307878735e-014
[5X5]	6	0.0062895812222806	2.7151024484251e-012
	7	0.642699081698521	2.02726724296554e-013
	1	638.557432745947	1.07286268757889e-009
	2	0.20646767890616	2.40294701081556e-008
1400	3	0.57263720432519	1.09912079437891e-013
(10)	4	0.0313718520202518	2.78317577317999e-011
[5X5]	5	0.210503814662855	9.88098491916389e-015
	6	0.0062895812222685	2.70300101745669e-012
	7	0.642699081698579	1.45439216225896e-013
	1	638.557432746189	8.31050783745013e-010
	2	0.206467678909351	2.40262792161605e-008
	3	0.572637204325215	8.44879721739744e-014
1500	4	0.0313718520202511	2.78311332313486e-011
(10)	5	0.210503814662856	9.0205620750794e-015

[5X5]	6	0.00628958122226019	2.69468562047459e-012
	7	0.642699081698609	1.15019105351166e-013
	1	638.557432746379	6.41080077912193e-010
	2	0.206467678907098	2.40285320529665e-008
	3	0.572637204325236	6.3504757008559e-014
1600	4	0.0313718520202515	2.78314524204681e-011
(10)	5	0.21050381466286	5.44009282066327e-015
[5X5]	6	0.00628958122225438	2.68888036836223e-012
	7	0.642699081698638	8.62643290133747e-014
	1	638.557432746526	4.94082996738143e-010
	2	0.206467678905239	2.40303913157103e-008
	3	0.572637204325249	5.14033260401447e-014
1700	4	0.0313718520202511	2.78310569035156e-011
(10)	5	0.210503814662858	6.60582699651968e-015
[5X5]	6	0.00628958122225012	2.68462248959045e-012
	7	0.642699081698661	6.29496454962464e-014
	1	638.557432746615	4.04838829126675e-010
	2	0.206467678906183	2.40294474596059e-008
	3	0.572637204325267	3.28626015289046e-014
1800	4	0.0313718520202514	2.78313830315291e-011
(10)	5	0.21050381466286	4.8017145815038e-015
[5X5]	6	0.00628958122224678	2.68128401426093e-012
	7	0.642699081698669	5.52891066263328e-014
	1	638.557432746695	3.24916982208379e-010
	2	0.206467678907743	2.40278873187005e-008
	3	0.572637204325261	3.9190872769268e-014
1900	4	0.0313718520202508	2.78307515921838e-011
(10)	5	0.210503814662862	3.33066907387547e-015
[5X5]	6	0.00628958122224429	2.67879295134943e-012
	7	0.642699081698672	5.22915044598449e-014
	1	638.557432746753	2.67050381808076e-010
	2	0.206467678907023	2.40286074648655e-008
	3	0.572637204325268	3.15303338993544e-014
2000	4	0.0313718520202509	2.78309320034253e-011
(10)	5	0.210503814662861	3.94129173741931e-015
[5X5]	6	0.00628958122224233	2.67682924437462e-012
	7	0.642699081698688	3.59712259978551e-014

Table-F

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN

Gauss Legendre rule=5X5, divisions of parabolic arc=12

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	Absolute error

	1	638.557391187819	4.15592011222543e-005
	2	0.206467694609033	8.32659657890034e-009
	3	0.572637200239111	4.08618938685379e-009
100	4	0.0313718519864146	6.00539062922678e-012
(12)	5	0.21050381439643	2.66434957429951e-010
[5X5]	6	0.00628958254957209	1.33000659183896e-009
	7	0.642699076031251	5.66747326846695e-009
	1	638.55743014972	2.59730040852446e-006
	2	0.206467697571426	5.3642037756152e-009
	3	0.572637204069935	2.55365173451594e-010
200	4	0.0313718519794897	1.29303442952811e-011
(12)	5	0.210503814646237	1.66280045288403e-011
[5X5]	6	0.00628958130302657	8.34610653158152e-011
	7	0.642699081344527	3.5419744914833e-010
	1	638.557432233976	5.13044142280705e-007
	2	0.206467697813437	5.12219278103743e-009
300	3	0.572637204274856	5.04443153914735e-011
(12)	4	0.0313718519791194	1.33006106128875e-011
[5X5]	5	0.210503814659581	3.28415072914368e-012
	6	0.00628958123634071	1.6775205356756e-011
	7	0.642699081628764	6.99605928744518e-011
	1	638.557432584689	1.62331275532779e-007
	2	0.206467697869458	5.06617198192671e-009
	3	0.572637204309332	1.59682267408812e-011
400	4	0.0313718519790571	1.33628524912055e-011
(12)	5	0.210503814661825	1.03955732910777e-012
[5X5]	6	0.00628958122511958	5.55408323499318e-012
	7	0.642699081676587	2.21371809772108e-011
	1	638.557432680533	6.64872459310573e-008
	2	0.20646769784417	5.09145989258109e-009
	3	0.572637204318759	6.54087894957911e-012
500	4	0.0313718519790402	1.33798319645884e-011
(12)	5	0.21050381466244	4.25409707460744e-013
[5X5]	6	0.00628958122205313	2.48762555693505e-012
	7	0.642699081689659	9.06485997376194e-012
	1	638.557432714947	3.20732169711846e-008
	2	0.206467697842359	5.09327091613443e-009
	3	0.572637204322139	3.16091597341028e-012
600	4	0.0313718519790337	1.33862712581312e-011
(12)	5	0.210503814662659	2.05557793009348e-013
[5X5]	6	0.00628958122095173	1.38623054007914e-012
	7	0.642699081694351	4.3728354270911e-012
	1	638.55743272971	1.73099579114933e-008
	2	0.206467697853388	5.0822417385632e-009
	3	0.572637204323593	1.70674585575625e-012
700	4	0.0313718519790315	1.33884708874987e-011
(12)	5	0.210503814662753	1.11716191852906e-013
[5X5]	6	0.00628958122047962	9.1412034192162e-013

	7	0.642699081696365	2.35877983811861e-012
	1	638.557432736863	1.01566683952115e-008
	2	0.206467697847564	5.08806602406153e-009
	3	0.572637204324294	1.005528993403e-012
800	4	0.03137185197903	1.3389969688582e-011
(12)	5	0.210503814662801	6.35880237354058e-014
[5X5]	6	0.00628958122025025	6.84746530310587e-013
	7	0.64269908169734	1.38433708940511e-012
	1	638.557432740689	6.33099261904135e-009
	2	0.206467697852709	5.08292119505427e-009
	3	0.57263720432468	6.20170581555612e-013
900	4	0.0313718519790294	1.33906358223967e-011
(12)	5	0.210503814662824	4.13835632429027e-014
[5X5]	6	0.00628958122012856	5.63056545832552e-013
	7	0.642699081697859	8.6541884769531e-013
	1	638.557432742855	4.16468992625596e-009
	2	0.206467697849764	5.08586556202673e-009
	3	0.572637204324885	4.14779321999958e-013
1000	4	0.0313718519790285	1.33914893063469e-011
(12)	5	0.210503814662838	2.70061750740069e-014
[5X5]	6	0.00628958122005873	4.93231323839272e-013
	7	0.642699081698161	5.63216140392342e-013
	1	638.55743274417	2.84967427433003e-009
	2	0.206467697849788	5.08584183100957e-009
	3	0.572637204325018	2.82107670557252e-013
1100	4	0.0313718519790284	1.33915656341799e-011
(12)	5	0.210503814662848	1.70974345792274e-014
[5X5]	6	0.00628958122001647	4.50971725241001e-013
	7	0.64269908169834	3.83804099612917e-013
	1	638.557432745017	2.00304839381715e-009
	2	0.206467697849564	5.08606629034958e-009
	3	0.572637204325104	1.96287430753728e-013
1200	4	0.031371851979028	1.33920305400714e-011
(12)	5	0.210503814662852	1.27675647831893e-014
[5X5]	6	0.00628958121998995	4.24453007463743e-013
	7	0.642699081698446	2.78443934575989e-013
	1	638.55743274556	1.45996637002099e-009
	2	0.206467697849126	5.08650394026589e-009
	3	0.572637204325147	1.52655665885959e-013
1300	4	0.0313718519790278	1.33921554401617e-011
(12)	5	0.210503814662857	8.38218383591993e-015
[5X5]	6	0.00628958121997239	4.06889799631216e-013
	7	0.642699081698528	1.96509475358653e-013
	1	638.55743274594	1.08036601886852e-009
	2	0.206467697849213	5.08641717633651e-009
	3	0.572637204325192	1.0769163338864e-013
1400	4	0.031371851979028	1.33920166622836e-011
(12)	5	0.210503814662858	6.74460487459783e-015

[5X5]	6	0.00628958121996015	3.9464698869951e-013
	7	0.64269908169858	1.44217970898808e-013
	1	638.557432746204	8.15930434328038e-010
	2	0.206467697848833	5.08679737221129e-009
	3	0.572637204325213	8.65973959207622e-014
1500	4	0.0313718519790279	1.33920929901166e-011
(12)	5	0.21050381466286	4.96824803519758e-015
[5X5]	6	0.00628958121995176	3.86259600693162e-013
	7	0.642699081698614	1.10134124042816e-013
	1	638.557432746386	6.34031493973453e-010
	2	0.206467697849358	5.08627179263144e-009
	3	0.572637204325229	7.105427357601e-014
1600	4	0.0313718519790281	1.33918501288299e-011
(12)	5	0.210503814662862	3.41393580072236e-015
[5X5]	6	0.00628958121994592	3.80421388834762e-013
	7	0.642699081698641	8.27116153345742e-014
	1	638.557432746522	4.980620360584e-010
	2	0.206467697849312	5.08631833873174e-009
	3	0.572637204325244	5.56221735337203e-014
	4	0.0313718519790272	1.33928007572948e-011
1700	5	0.210503814662864	1.49880108324396e-015
(12)	6	0.00628958121994167	3.76173918403833e-013
[5X5]	7	0.642699081698664	5.99520433297585e-014
	1	638.557432746631	3.89150045521092e-010
	2	0.206467697849423	5.08620748296273e-009
	3	0.572637204325262	3.75255382323303e-014
1800	4	0.0313718519790274	1.33925717737959e-011
(12)	5	0.210503814662863	2.35922392732846e-015
[5X5]	6	0.00628958121993839	3.72889219502071e-013
	7	0.642699081698665	5.87307980026708e-014
	1	638.557432746708	3.1161562219495e-010
	2	0.206467697848698	5.08693207001976e-009
	3	0.572637204325262	3.78586051397178e-014
1900	4	0.0313718519790275	1.33925370793264e-011
(12)	5	0.210503814662863	1.63757896132211e-015
[5X5]	6	0.00628958121993559	3.70088508450106e-013
	7	0.642699081698682	4.18554080283684e-014
	1	638.557432746763	2.57387000601739e-010
	2	0.206467697849482	5.08614808603092e-009
	3	0.572637204325265	3.530509218308e-014
2000	4	0.0313718519790275	1.3392495445963e-011
(12)	5	0.210503814662864	5.82867087928207e-016
[5X5]	6	0.00628958121993385	3.68352050250653e-013
	7	0.642699081698693	3.11972669919669e-014
	1	638.557432746972	4.77484718430787e-011
	2	0.20646769784921	5.08642009067195e-009

3000	3	0.572637204325288	1.23234755733392e-014
(12)	4	0.0313718519790264	1.33935501578364e-011
[5X5]	5	0.210503814662866	1.08246744900953e-015
	6	0.00628958121992736	3.61858112918334e-013
	7	0.642699081698716	7.88258347483861e-015
	1	638.557432746985	3.45607986673713e-011
	2	0.206467697849182	5.08644812380332e-009
	3	0.572637204325289	1.0547118733939e-014
4000	4	0.0313718519790249	1.33950836533892e-011
(12)	5	0.210503814662863	1.66533453693773e-015
[5X5]	6	0.00628958121992603	3.60524977927046e-013
	7	0.642699081698719	4.9960036108132e-015
	1	638.557432747018	1.81898940354586e-012
	2	0.206467697849141	5.08648864694372e-009
5000	3	0.572637204325293	7.105427357601e-015
(12)	4	0.0313718519790231	1.33968947046981e-011
[5X5]	5	0.210503814662865	1.11022302462516e-016
	6	0.00628958121992577	3.60266504129125e-013
	7	0.642699081698733	8.43769498715119e-015

Table-G

NUMERICAL VALUES OF INTEGRALS $II_{\Omega_N}(f_i)$ OVER THE LUNAR DOMAIN

Gauss Legendre rule=5X5, divisions of parabolic arc=20

No. of curved triangles (Divisions of arc) [Gauss Legendre rule]	i	$II_{\Omega_N}(f_i)$	**Absolute error**
	1	638.55739118782	4.15592000990728e-005
	2	0.20646769820628	4.72935027251431e-009
100	3	0.572637200239109	4.08619071912142e-009
(20)	4	0.0313718519998232	7.40318223391156e-012
[5X5]	5	0.21050381439643	2.66435040696678e-010
	6	0.00628958254921184	1.32964633487581e-009
	7	0.642699076031248	5.66747615504681e-009
	1	638.557430149721	2.59729904428241e-006
	2	0.206467701403872	1.53175835904129e-009
200	3	0.572637204069932	2.55367504919946e-010
(20)	4	0.0313718519928962	4.76188533049537e-013
[5X5]	5	0.210503814646237	1.6627949017689e-011
	6	0.00628958130266779	8.31022906738754e-011
	7	0.642699081344522	3.54201667995824e-010
	1	638.557432233984	5.13035956828389e-007
	2	0.206467701773629	1.16200074562478e-009
300	3	0.572637204274854	5.04464248152203e-011

(20)	4	0.0313718519925257	1.05720987519931e-013
[5X5]	5	0.210503814659583	3.28237437230428e-012
	6	0.0062895812359822	1.64166961275081e-011
	7	0.642699081628763	6.99612590082666e-011
	1	638.557432584695	1.62324568009353e-007
	2	0.206467701835002	1.10062800540156e-009
400	3	0.572637204309335	1.59652291387147e-011
(20)	4	0.0313718519924636	4.35762537165374e-014
[5X5]	5	0.210503814661827	1.03830832820506e-012
	6	0.00628958122476078	5.19528083747778e-012
	7	0.642699081676585	2.21392904009576e-011
	1	638.557432680526	6.64940671413206e-008
	2	0.206467701831085	1.10454492774359e-009
500	3	0.572637204318763	6.53721521359785e-012
(20)	4	0.031371851992446	2.59653409884208e-014
[5X5]	5	0.210503814662439	4.25603996490054e-013
	6	0.00628958122169469	2.12918571662613e-012
	7	0.642699081689663	9.06164032699053e-012
	1	638.557432714964	3.20558228850132e-008
	2	0.206467701830243	1.10538736497467e-009
600	3	0.572637204322142	3.15747428203395e-012
(20)	4	0.0313718519924396	1.96231919602496e-014
[5X5]	5	0.210503814662661	2.03614902716254e-013
	6	0.00628958122059343	1.02793381445698e-012
	7	0.64269908169435	4.37416769472065e-012
	1	638.557432729717	1.73033640749054e-008
	2	0.206467701834641	1.1009891331959e-009
700	3	0.572637204323587	1.71329617160154e-012
(20)	4	0.031371851992437	1.6979473382861e-014
[5X5	5	0.210503814662754	1.11105569189363e-013
	6	0.00628958122012122	5.5571866552917e-013
	7	0.64269908169637	2.35422792371764e-012
	1	638.557432736875	1.01448449640884e-008
800	2	0.206467701838564	1.09706588258263e-009
(20)	3	0.572637204324306	9.93649607039515e-013
[5X5]	4	0.031371851992435	1.49880108324396e-014
	5	0.210503814662802	6.30606677987089e-014
	6	0.00628958121989198	3.26484499157953e-013
	7	0.642699081697343	1.38100642033123e-012
	1	638.557432740682	6.33826857665554e-009
	2	0.206467701837706	1.09792400171393e-009
900	3	0.572637204324664	6.36046770807752e-013
(20)	4	0.0313718519924343	1.42524880786254e-014
[5X5]	5	0.210503814662823	4.1883163603984e-014
	6	0.00628958121977001	2.04505683221168e-013
	7	0.642699081697857	8.6675111532486e-013
	1	638.557432742857	4.16298462369014e-009
	2	0.206467701836765	1.09886527654979e-009

1000	3	0.572637204324896	4.0423220326602e-013
(20)	4	0.0313718519924339	1.39263600651418e-014
[5X5]	5	0.210503814662839	2.61179966543068e-014
	6	0.00628958121970053	1.35031742731773e-013
	7	0.642699081698167	5.57554002966754e-013
	1	638.557432744194	2.82602741208393e-009
	2	0.206467701836262	1.09936793002419e-009
1100	3	0.572637204325023	2.76778600039052e-013
(20)	4	0.031371851992433	1.301736496373e-014
[5X5]	5	0.210503814662845	2.02893257750247e-014
	6	0.00628958121965763	9.21268270004383e-014
	7	0.642699081698334	3.9013237085328e-013
	1	638.557432745026	1.99372607312398e-009
	2	0.206467701836346	1.09928446900831e-009
1200	3	0.572637204325104	1.95621296938953e-013
(20)	4	0.0313718519924329	1.2864709297844e-014
[5X5]	5	0.210503814662852	1.34892097491957e-014
	6	0.00628958121963153	6.60253102191533e-014
	7	0.642699081698447	2.77555756156289e-013
	1	638.557432745562	1.45780632010428e-009
	2	0.206467701836666	1.09896430844358e-009
	3	0.572637204325158	1.41886502547095e-013
1300	4	0.0313718519924322	1.22055143769728e-014
(20)	5	0.210503814662858	7.16093850883226e-015
[5X5]	6	0.00628958121961441	4.89079263199521e-014
	7	0.642699081698535	1.89070981093664e-013
	1	638.557432745922	1.09798747871537e-009
	2	0.20646770183687	1.09875961107342e-009
	3	0.572637204325183	1.17017506795492e-013
1400	4	0.031371851992432	1.19904086659517e-014
(20)	5	0.210503814662857	8.07687250414801e-015
[5X5]	6	0.00628958121960203	3.65315416805956e-014
	7	0.64269908169859	1.34003919072256e-013
	1	638.5574327462	8.20364220999181e-010
	2	0.206467701837747	1.09788303448433e-009
1500	3	0.572637204325223	7.66053886991358e-014
(20)	4	0.0313718519924305	1.04985464766116e-014
[5X5]	5	0.210503814662862	2.58126853225349e-015
	6	0.0062895812195936	2.81025203108243e-014
	7	0.642699081698611	1.12798659301916e-013
	1	638.557432746374	6.45968611934222e-010
	2	0.206467701837167	1.09846270968106e-009
1600	3	0.572637204325226	7.39408534400354e-014
(20)	4	0.0313718519924305	1.04985464766116e-014
[5X5]	5	0.210503814662859	5.93969318174459e-015
	6	0.00628958121958757	2.20691520613769e-014
	7	0.642699081698634	9.01501095995627e-014
	1	638.557432746519	5.00676833325997e-010

	2	0.206467701836816	1.09881420629065e-009
1700	3	0.572637204325234	6.56141807553468e-014
(20)	4	0.0313718519924306	1.06303854607859e-014
[5X5]	5	0.21050381466286	4.57966997657877e-015
	6	0.00628958121958301	1.75111661282479e-014
	7	0.642699081698655	6.89448498292222e-014
	1	638.55743274662	3.99950295104645e-010
	2	0.206467701837035	1.0985951592879e-009
1800	3	0.572637204325263	3.70814490224802e-014
(20)	4	0.0313718519924302	1.01654795692241e-014
[5X5]	5	0.210503814662861	3.96904731303493e-015
	6	0.00628958121958024	1.47425474605889e-014
	7	0.642699081698675	4.9515946898282e-014
	1	638.557432746692	3.28213900502305e-010
	2	0.206467701836774	1.09885575638735e-009
1900	3	0.572637204325273	2.68673971959288e-014
(20)	4	0.0313718519924285	8.51402282009417e-015
[5X5]	5	0.210503814662864	1.30451205393456e-015
	6	0.0062895812195777	1.21959733978549e-014
	7	0.642699081698678	4.66293670342566e-014
	1	638.557432746737	2.83080225926824e-010
	2	0.206467701837114	1.09851586160836e-009
2000	3	0.572637204325261	3.8746783559418e-014
(20)	4	0.0313718519924287	8.70831184940357e-015
[5X5]	5	0.210503814662868	2.99760216648792e-015
	6	0.00628958121957591	1.04118103028128e-014
	7	0.642699081698689	3.54161144855425e-014
	1	638.55743274695	6.98037183610722e-011
	2	0.206467701837071	1.09855910479517e-009
	3	0.572637204325287	1.34336985979644e-014
3000	4	0.0313718519924256	5.64825963778048e-015
(20)	5	0.210503814662861	3.83026943495679e-015
[5X5]	6	0.00628958121956939	3.88664794792604e-015
	7	0.642699081698704	1.99840144432528e-014
	1	638.557432746986	3.4333424991928e-011
	2	0.206467701837068	1.09856171381928e-009
4000	3	0.572637204325285	1.50990331349021e-014
(20)	4	0.0313718519924197	3.19189119579733e-016
[5X5]	5	0.210503814662866	5.82867087928207e-016
	6	0.00628958121956806	2.56218657401774e-015
	7	0.642699081698722	1.99840144432528e-015
	1	638.557432747004	1.58024704433046e-011
	2	0.206467701837052	1.0985783116535e-009
	3	0.572637204325279	2.07611705604904e-014
5000	4	0.031371851992416	4.01068067645838e-015
(20)	5	0.210503814662862	2.96984659087229e-015
[5X5]	6	0.00628958121956826	2.75821032680312e-015
	7	0.64269908169873	5.6621374255883e-015

Figures 1 to 12:

Figure 1

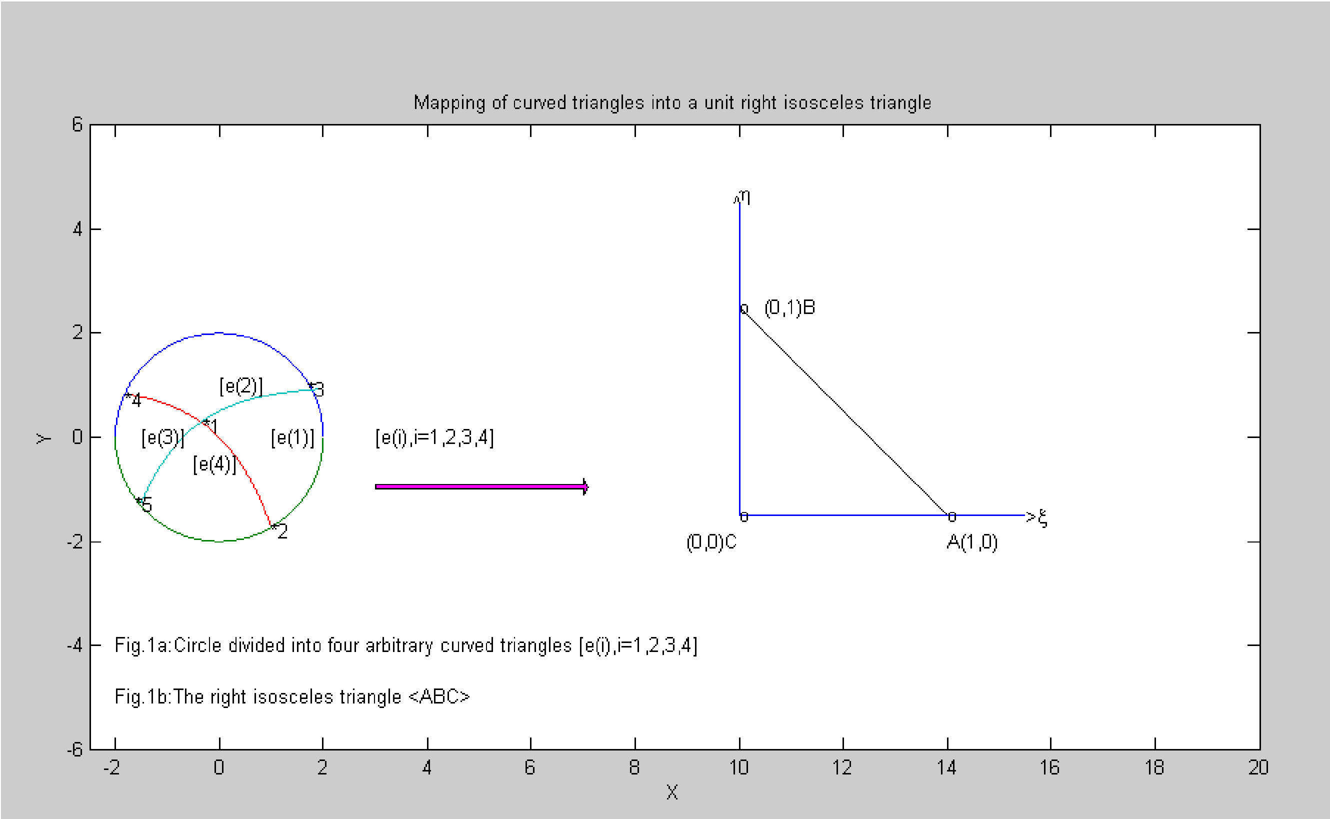


Figure 2

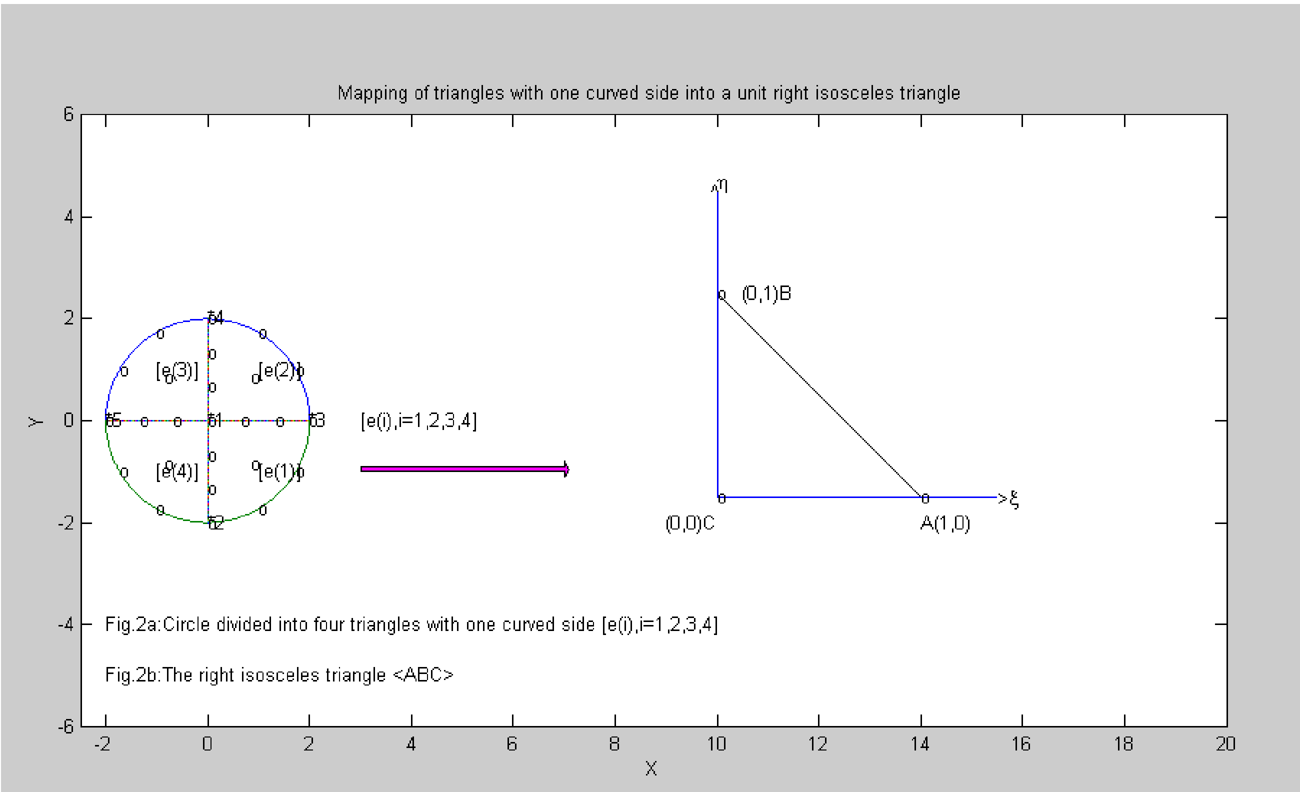


Figure 3

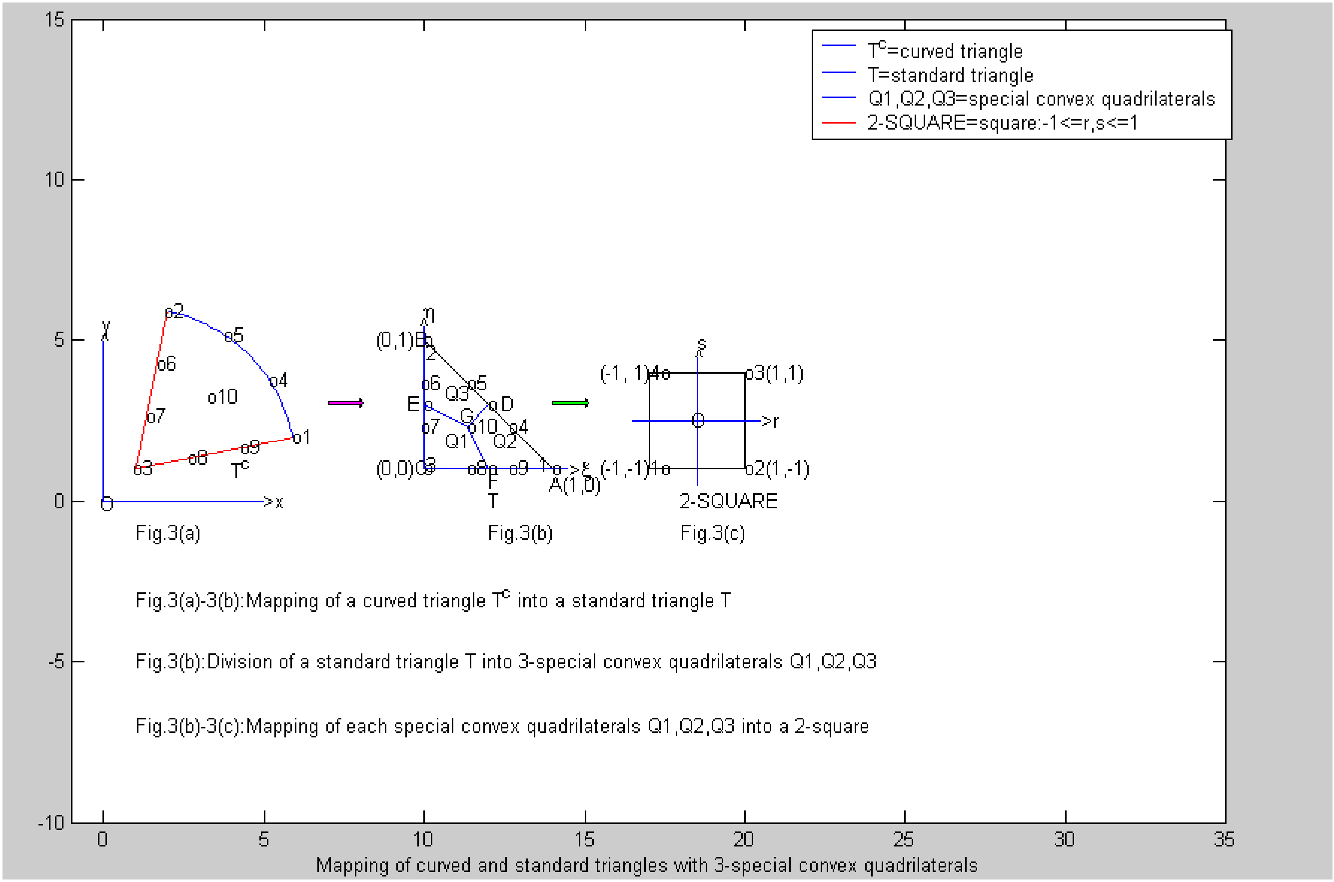


Figure 4

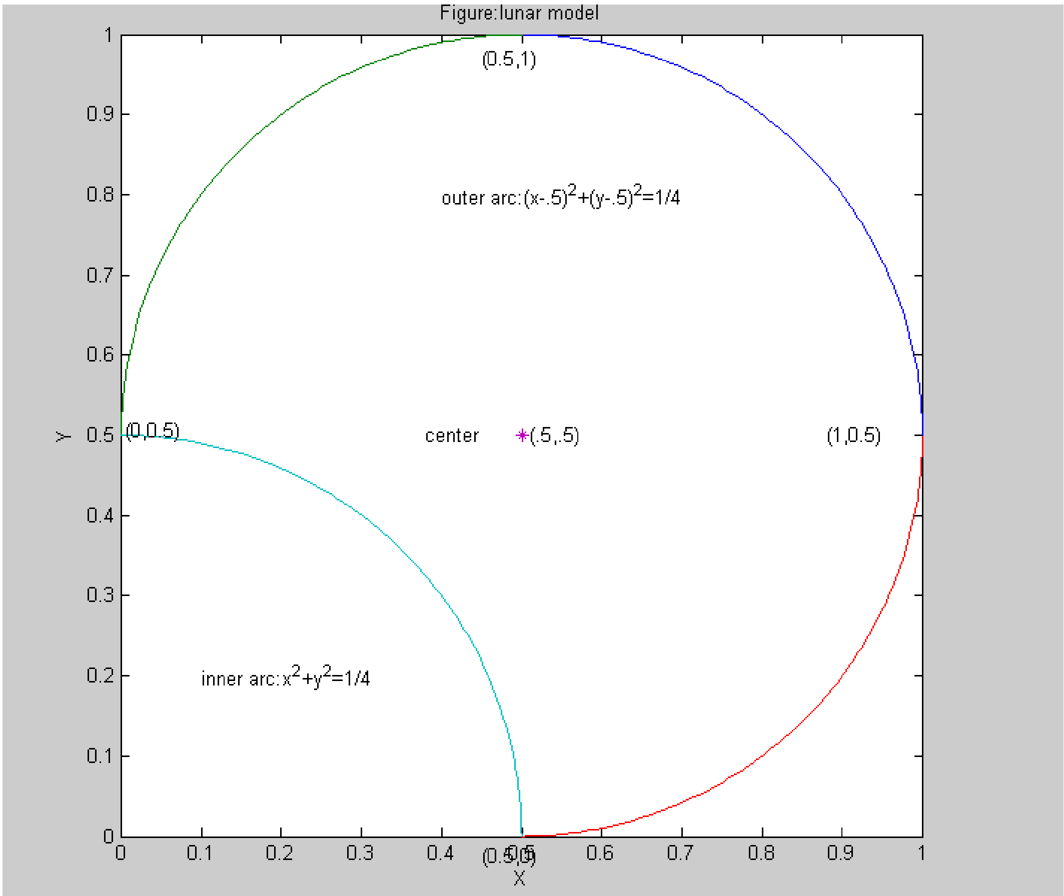


Figure 5

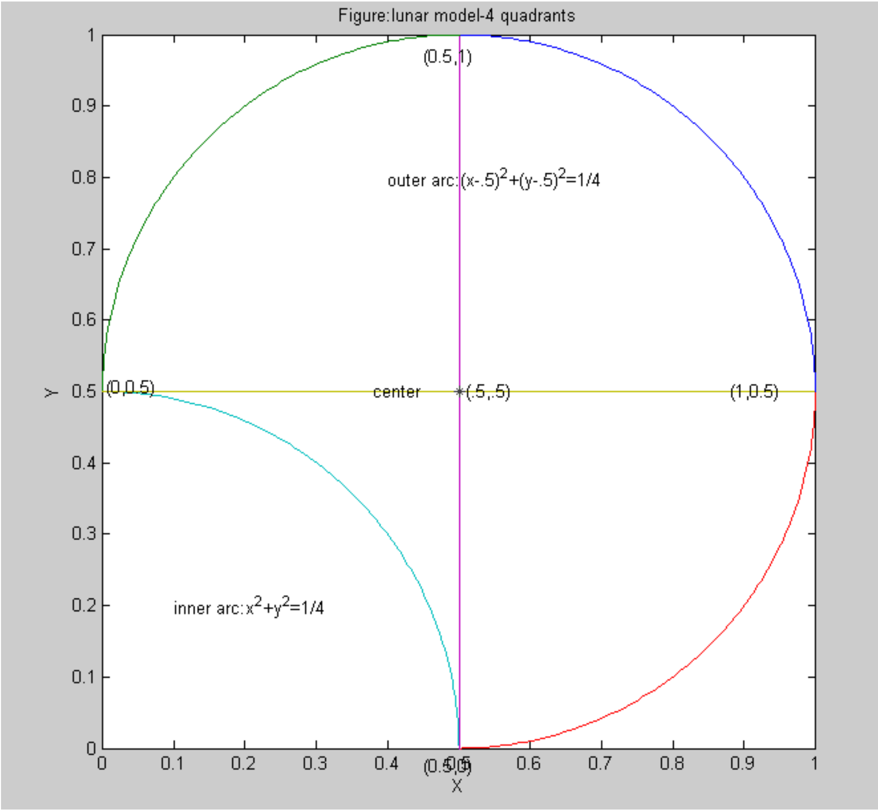


Figure 6

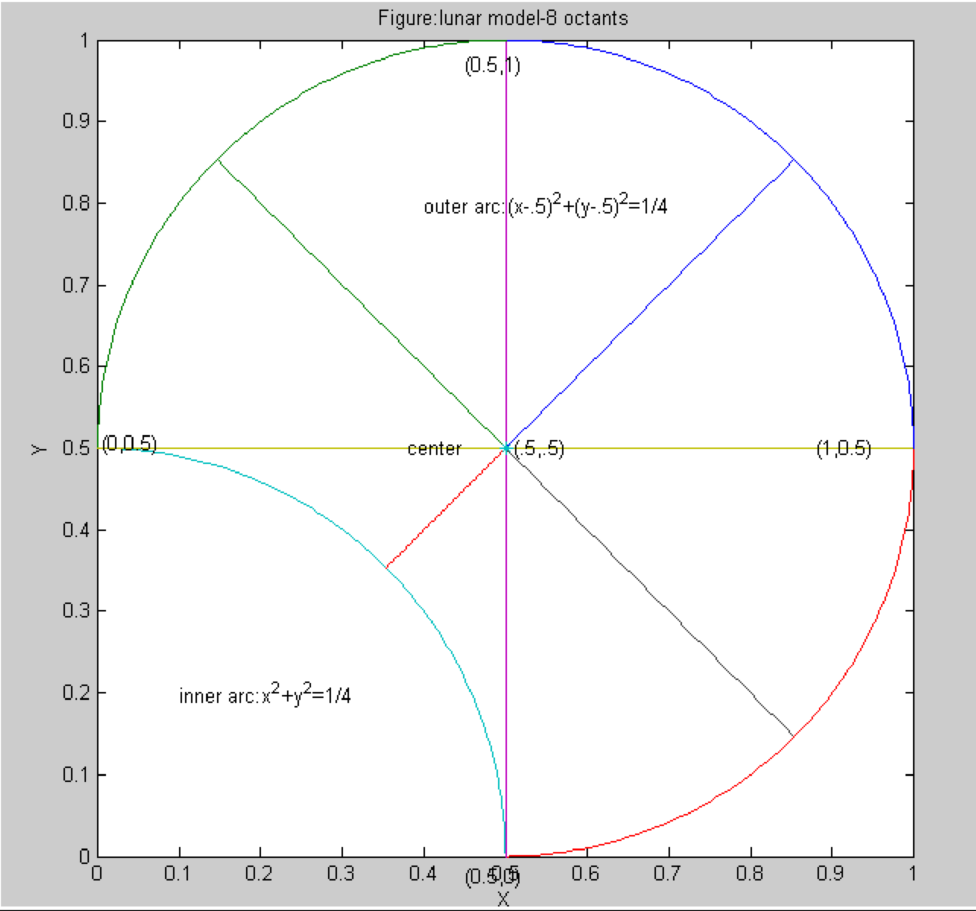


Figure 7

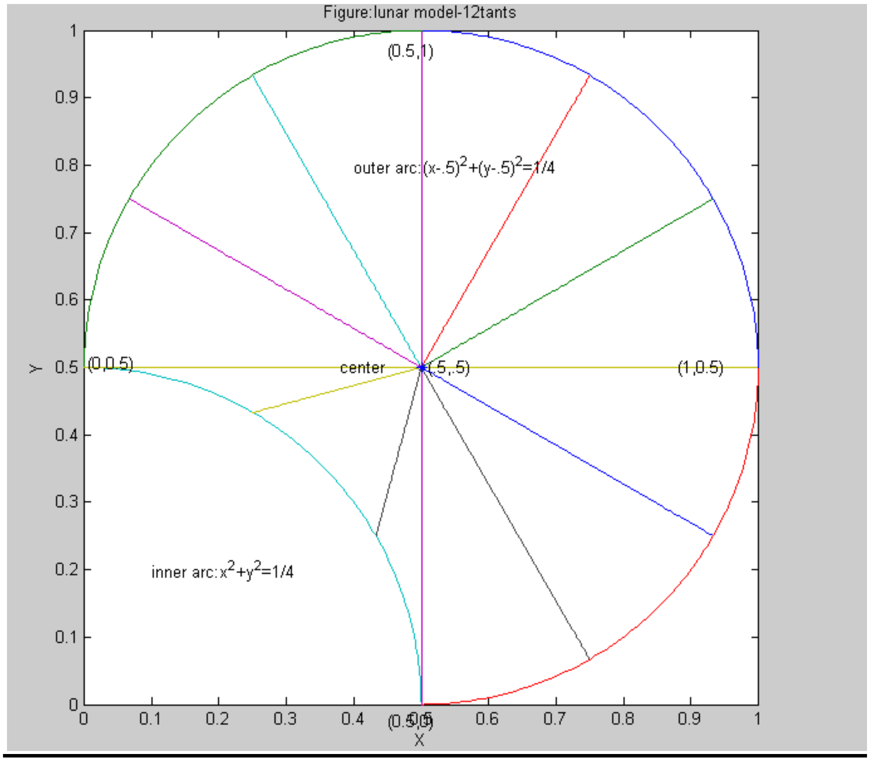


Figure 8

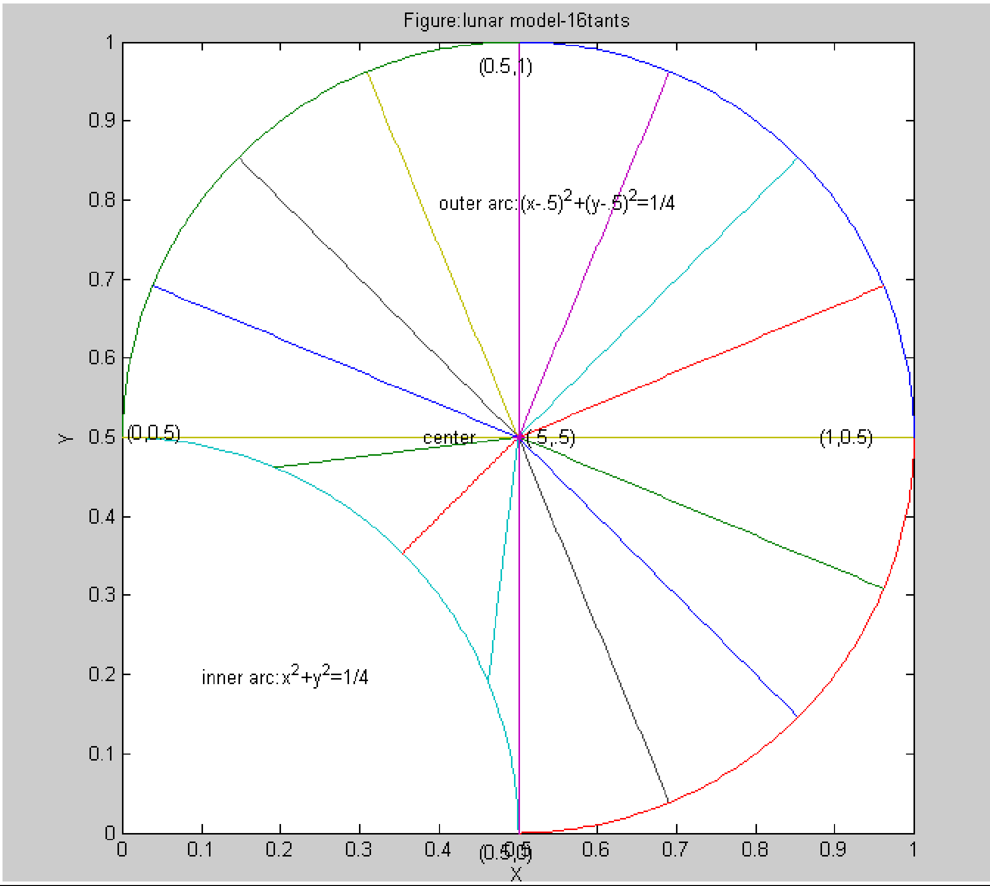


Figure 9

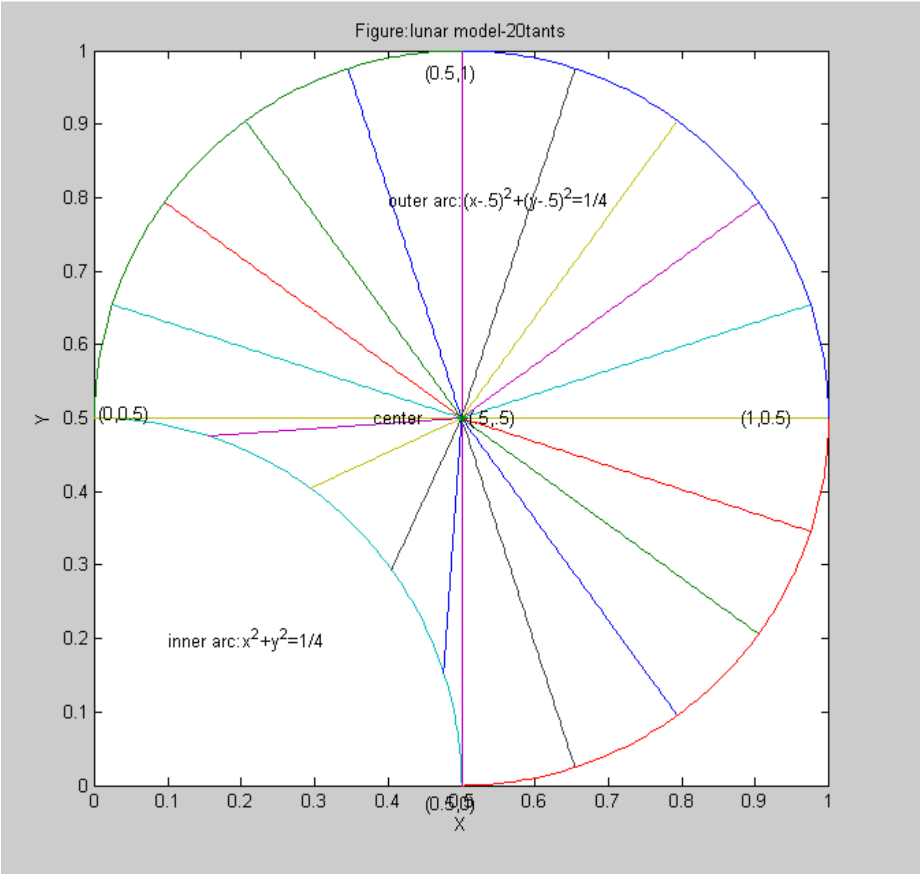


Figure 10

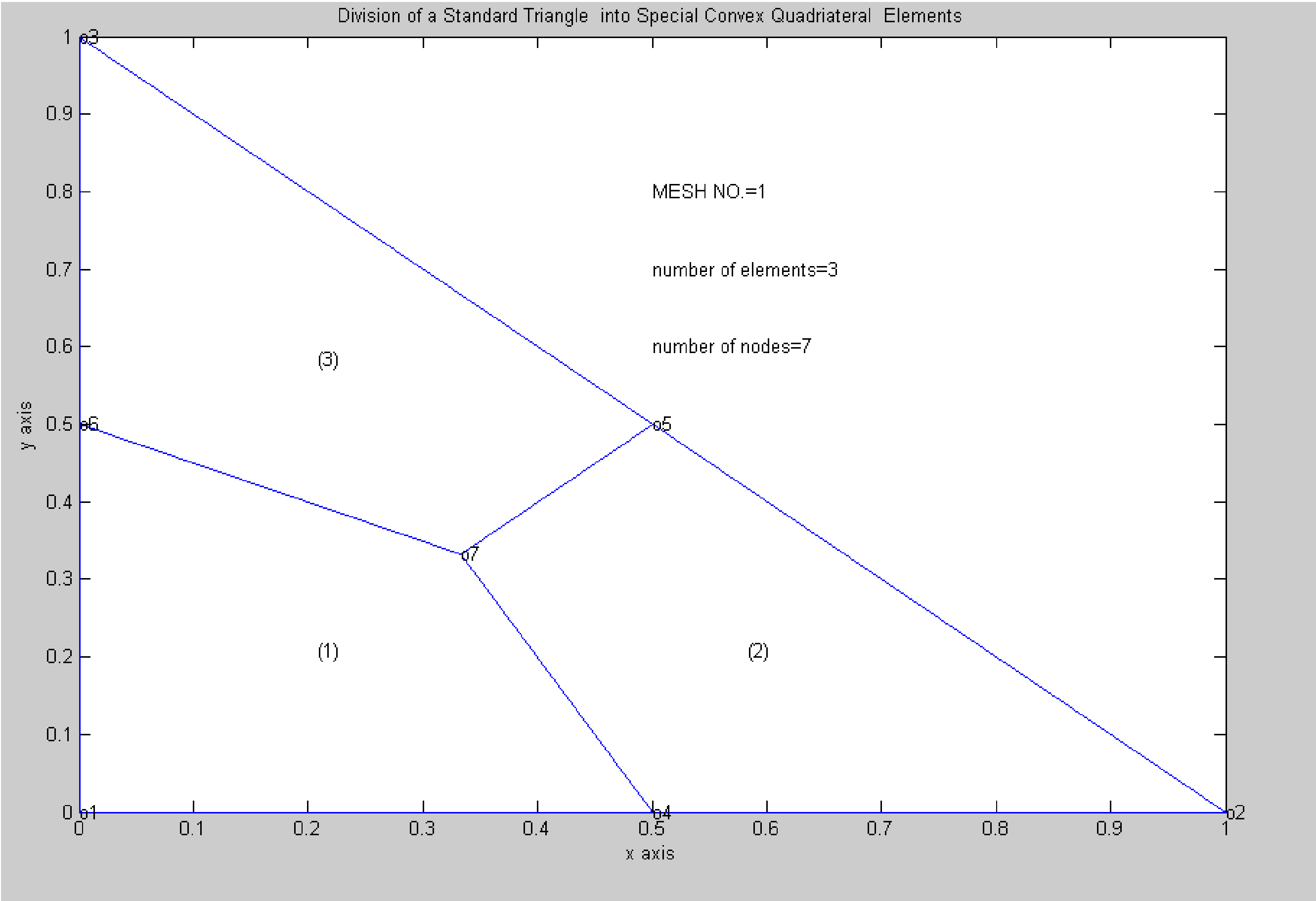


Figure 11

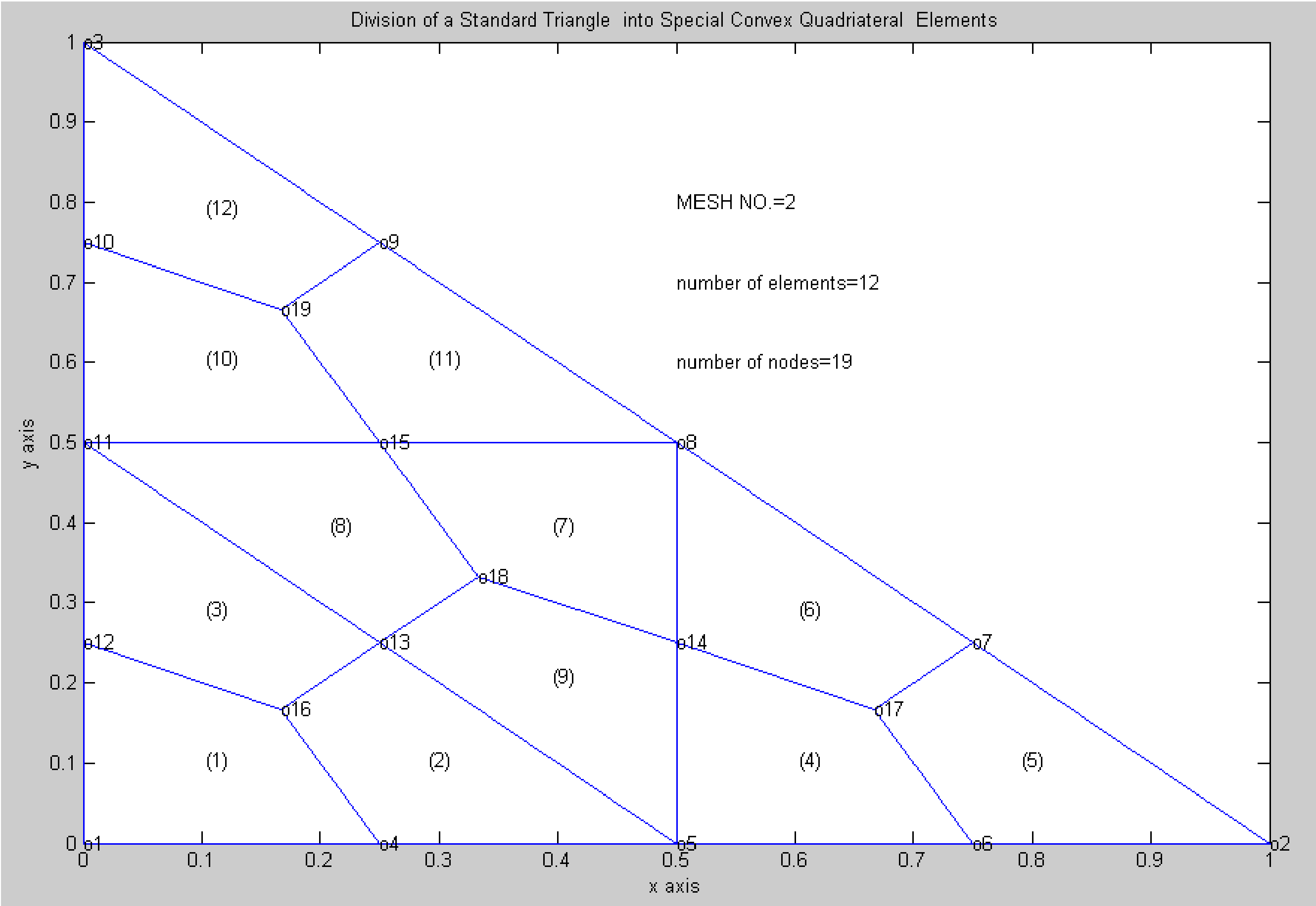
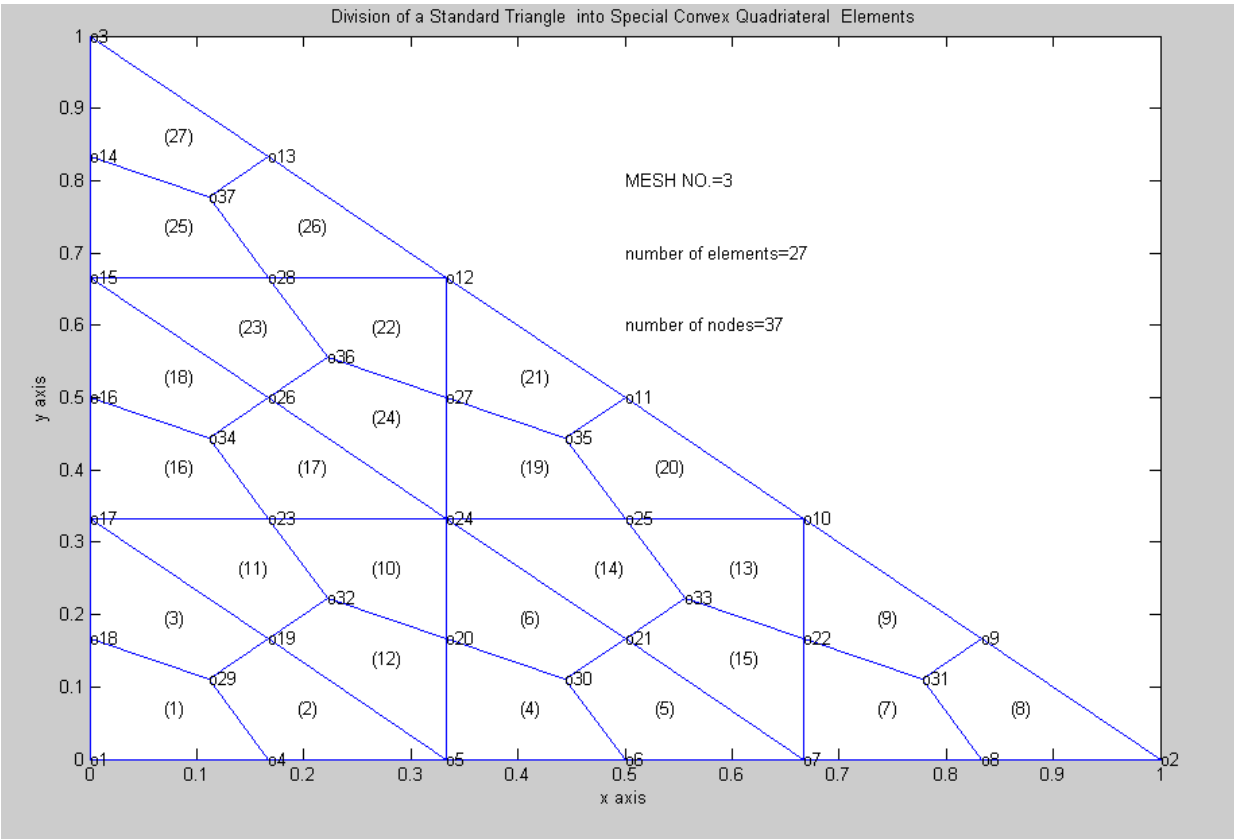

Division of a Standard Triangle into Special Convex Quadriateral Elements
MESH NO.=2
number of elements=12
number of nodes=19
y axis
x axis

Figure 12


Division of a Standard Triangle into Special Convex Quadriateral Elements
MESH NO.=3
number of elements=27
number of nodes=37
y axis
x axis