Frontogenesis in turbulent flow through porous media using non-linear Forchheimer equation

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Abstract

We present here both one- and two-dimensional models for turbulent flow through heterogeneous unbounded fluid saturated porous media using non-linear Forchheimer extended Darcy (DF) equation in the presence of gravitational field. The fluid is initially at rest and sets in motion due to a uniform horizontal density gradient. It is shown that a purely horizontal motion develops satisfying non-linear DF equation. Analytical solutions of this non-linear Initial Value Problem are obtained and limiting solutions valid for the Darcy regime in the case of laminar flow are derived. A measure of the stability of the flow is discussed briefly using Richardson number. The comparison between the nature of the solutions satisfying the non-linear and linear initial value problems are made. We found that even in the case of turbulent flow the vertical density gradient varies continuously both with space z and time t but the horizontal density gradient remains unchanged. The existence and uniqueness theorem of the Initial Value Problem is proved. The stability of these solutions are discussed and it is shown that the solutions are qualitatively and quantitatively different for $z < \frac{1}{4}$ and $z > \frac{1}{4}$ in the upper and lower half of the region. In particular, we have shown that the solution which is stable for infinitesimal perturbations is also stable for arbitrary perturbations both in time and space.

In the case of two-dimensional motion, a piecewise initial density gradient with continuous distribution of density, stream function formulation is used and the solutions are obtained using time-series analysis. In this case solution shows crowding of the density profiles in the lower-half of the channel reflecting an increase in density gradient and incipient of frontogenesis there, because of the increase in circulation of the flow due to piecewise initial density gradient.

1. Introduction

A proper understanding of the fundamental mechanisms affecting contaminant transport in a porous medium is of paramount importance in the study of many contemporary groundwater problems. The contaminations may be due to municipal, agricultural, industrial or nuclear wastes particularly radio active waste and other volatile organic substances such as gasoline leaked into the groundwater which can pose a long-term threat to the quality of groundwater. These waste materials are often sufficiently solvable leading to the variation of density in the fluid saturated porous rocks making not only

the groundwater undrinkable but also pose a threat to certain agricultural crops, aquaculture, chemical and food processing industries. The presence of a density gradient due to the mixture of these contaminants normal to the direction of a vertical gravitational field in a porous matrix generates the velocity instantaneously no matter how small the density gradient. The resulting motion advects fluid in a porous rock matrix where the effects of Darcy resistance and Forchheimer quadratic drag may either decrease or increase the density gradient. In the extreme case the density gradient may increase to such an extent that an effective discontinuity or front may develop as in a non-viscous fluid in the absence of a porous medium discussed by Simpson [1], Linden and Simpson [2] (hereafter called LS), Simpson and Linden [3] (hereafter referred to as SL) and in the presence of a porous medium discussed by Rudraiah [4] under

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the assumption of laminar flow. Rudraiah [4] has defined fronts as regions in a heterogeneous fluid-saturated porous rock matrix where the mixing of the contaminant transport is observed to occur. Such a mixing process in a porous rock matrix having velocity produce turbulent flow due to quadratic drag resulting from the curvature of the complicate geometry and large length scales involved in porous rock matrix. Recently Rudraiah [4] has investigated this effect of mixing process on front in a porous medium using laminar flow approximation. However, the mixing process in contaminated ground water in porous rocks, laminar flow approximation is very restrictive because of large velocities, large length scales and complicated geometry of the flow that prevails in such a high porosity and less permeability porous rock. Such length scales and complicated geometry lead to high Reynolds number $R = VD/v$ of the order of 4 for length scale based on average grain diameter ranging from 1 to 5 mm and having porosity ε of 38% (see [5]). Here V is the specific discharge, v the kinematic viscosity of the fluid and D is the average grain diameter. Lindquist [5] in his experiments also found that for this medium the upper limit of R, above which there is always turbulence, (called R critical) was greater than 180. At that high Reynolds number the flow in the porous media, generated instantaneously by horizontal density gradient, is turbulent no matter how small it is. Hence one has to study the mixing process using turbulent flow because the turbulent motion affects the velocity of fluid in the porous media, while the Darcy resistance and Forchheimer drag may either increase or decrease the density gradient. In the extreme case the density gradient may increase to such an extent that an effective discontinuity (see [4]) or front, may develop. We note that fronts are regions in heterogeneous turbulent fluid-saturated large porosity porous media where mixing occurs. Such a mixing process in turbulent flow through a large porosity porous medium is important not only in ground water pollution but also in many industrial problems, particularly in the chemical, food processing and paper industries. It is also of importance in the manufacture of polymer materials, involving heterogeneous fluid, by solidification process. This process produces a mushy layer namely solid–fluid mixture region regarded as a porous layer. To manufacture such polymer materials free from impurities it is important to understand the nature of fronts. The fronts in a turbulent fluid flow through a porous medium has not been given much attention to our knowledge, in spite of its importance in many applications discussed above. The study of it is the main object of this paper. We concentrate on the study of the effect of turbulence on the heterogeneous fluid saturated porous layer with the objective of knowing whether frontogenesis occurs in a densely packed heterogeneous fluid saturated porous medium using Reynolds averaging procedure. The turbulent flow in the absence of porous media has been extensively investigated both analytically and numerically using direct integration of Navier–Stokes equations. The work on turbulent flow in a porous medium is very sparse. Recently Rudraiah et al. [6–8] and Takatsu and Masuoka [9] have studied the turbulent flow in a porous medium using Darcy–Lapwood equation. We note that the use of Darcy–Lapwood equation to study flow through a porous medium poses the problem of under specified

system (see [10]) when the basic flow is non-quiescent. This problem can be overcomed by using Darcy–Forchheimer (hereafter called DF) equation (see [11]). Therefore, the objective of this paper is to use the DF equation to know the existence of frontogenesis in turbulent flow through a porous medium. To achieve the objective of this paper we follow the following plan of work.

In Section 2 on formulation of the problem, the basic equations including the non-Darcian effects are considered. The effects of constant horizontal density gradient on a heterogeneous fluid through an unbounded porous medium governed by DF equation is considered. In this section the Reynolds decomposition is used to predict the effect of turbulence. In the closure problem we use the gradient diffusion model together with the volume average procedure. The calculations show that the effect of Darcy resistance is to make the vertical shear uniform in space but decaying exponentially with time. Physically this implies that velocity attenuates due to viscosity, eddy viscosity and permeability and varies with the vertical height z as $t \rightarrow \infty$. On the other hand the effect of Forchheimer inertia is to make the vertical shear non-uniform and decays exponentially both with space and time. Sections 3 and 4 are devoted to study the effect of a constant horizontal density gradient and in Section 5 a measure of stability of flow is discussed using the gradient Richardson's number. In Section 6 the existence and uniqueness theorems of the non-linear initial value problem are discussed. The evolution of a piecewise constant density gradient is calculated using the DF equation and the conditions for the existence of front are discussed in Section 8. Some important conclusions are drawn in the final section.

2. Formulation of the problem

We consider a two-dimensional turbulent motion of an unbounded fluid saturated porous medium in the (x, z)-plane with x-axis horizontal having the velocity u and z-axis vertical, anti-parallel to gravity g, having the velocity w. The basic equations for this incompressible heterogeneous Boussinesq two-dimensional fluid through a porous medium are the DF equations

$$\frac{\partial q_i}{\partial t} + \frac{C_b}{\sqrt{k}} |q_i| q_i = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - \frac{\rho}{\rho_0} g \delta_{i3} - \frac{v}{k} q_i, \qquad (2.1)$$

and the equation of continuity for heterogeneous fluid

$$\frac{\partial q_i}{\partial x_i} = 0, \qquad (2.2)$$

$$\frac{\partial \rho}{\partial t} + q_i \frac{\partial \rho}{\partial x_i} = 0. \qquad (2.3)$$

Here q_i is the Darcian velocity, ρ, p and g are density, pressure and gravitational constant, v is the kinematic viscosity of fluid, k and C_b are the permeability and the drag coefficients of a porous medium. We derive the equations for turbulent flow using Reynolds decomposition, where the velocity, density and pressure are expressed as the sums of the mean and fluctuations

of the form

$$q_i = \bar{q}_i + q'_i, \quad \rho = \bar{\rho} + \rho', \quad p = \bar{p} + p', \tag{2.4}$$

where the bar denotes the mean and prime denotes the fluctuation.

Now,

$$|q_i|q_i = |\bar{q}_i + q'_i|(\bar{q}_i + q'_i). \tag{2.5}$$

We know, from triangular inequality, that

$$|\bar{q}_i + q'_i| \leqslant |\bar{q}_i| + |q'_i|. \tag{2.6}$$

In this paper, following Rudraiah et al. [12] we force equality which will be valid when $|q'_i| = \lambda_i \bar{q}_i$ where $\lambda_i > 0$ or when q'_i and $\bar{q}_i$ both have the same sign.

Then, Eq. (2.5) becomes

$$|q_i|q_i = \{|\bar{q}_i| + |q'_i|\}(\bar{q}_i + q'_i),$$

i.e.

$$|q_i|q_i = |\bar{q}_i|\bar{q}_i + |\bar{q}_i|q'_i + |q'_i|\bar{q}_i + |q'_i|q'_i. \tag{2.7}$$

Applying Reynolds averaging on this we get

$$\overline{|q_i|q_i} = \overline{|\bar{q}_i|\bar{q}_i} + \overline{|\bar{q}_i|q'_i} + \overline{|q'_i|\bar{q}_i} + \overline{|q'_i|q'_i}$$

$$= |\bar{\bar{q}}_i|\bar{\bar{q}}_i + |\bar{\bar{q}}_i|\overline{q'_i} + \overline{|q'_i|}\bar{\bar{q}}_i + \overline{|q'_i|q'_i}. \tag{2.8}$$

By definition $\bar{\bar{q}}_i = \bar{q}_i$, and $\overline{q'_i} = \overline{q_i - \bar{q}_i} = \bar{q}_i - \bar{\bar{q}}_i = \bar{q}_i - \bar{q}_i = 0$, so that Eq. (2.8) becomes

$$\overline{|q_i|q_i} = |\bar{q}_i|\bar{q}_i + \overline{|q'_i|q'_i}. \tag{2.9}$$

For the Closure problem, we use the Gradient Diffusion Model namely

$$\overline{|q'_i|q'_i} = -k_m \nabla \bar{q}_i. \tag{2.10}$$

This, after using volume average, becomes

$$\overline{|q'_i|q'_i} = \frac{k_m}{\sqrt{k}}\bar{q}_i, \tag{2.11}$$

where k_m is the eddy viscosity and k is the effective permeability of a porous medium.

The positive sign on the right-hand side of Eq. (2.11) is taken to ensure that turbulence increases the flow.

Then Eq. (2.9), using Eq. (2.11), becomes

$$\overline{|q_i|q_i} = |\bar{q}_i|\bar{q}_i + \frac{k_m}{\sqrt{k}}\bar{q}_i. \tag{2.12}$$

3. Solutions for uniform horizontal density gradient

We consider a two-dimensional turbulent motion of a heterogeneous fluid saturated unbounded porous medium in the (x, z)-plane with the x-axis horizontal and the z-axis vertical and parallel to gravity $\vec{g}$. The fluid is initially at rest so that

$q_i = (u, w) = (0, 0)$ and the initial density is given by

$$\rho = \rho_0[1 - \alpha x - \beta z], \quad \alpha > 0, \quad \beta > 0, \tag{3.1}$$

where $\alpha > 0$ implies the fluid is heavy to the left and $\beta > 0$ ensures static stability. The initial value of θ, the angle that the isopyenals make with the vertical, is given by

$$\tan \theta = \frac{\beta}{\alpha}. \tag{3.2}$$

The equations governing the Boussinesq heterogeneous turbulent fluid saturated with high porosity and low permeability (for example porous rock matrix in the interior of the earth) from Eqs. (2.1) to (2.2), under the assumption of unidirectional horizontal flow (i.e. $w = 0$) because a constant horizontal density gradient sets up a horizontal flow and using Reynolds rule of averages, are

$$\frac{\partial \bar{u}}{\partial t} + \frac{\bar{v}}{k}\bar{u} + \frac{C_b}{\sqrt{k}}|\bar{u}|\bar{u} = \frac{-1}{\rho_0}\frac{\partial \bar{p}}{\partial x}, \tag{3.3}$$

$$-\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial z} - \frac{\bar{\rho}}{\rho_0}g = 0, \tag{3.4}$$

$$\frac{\partial \bar{\rho}}{\partial t} = -\bar{u}\frac{\partial \bar{\rho}}{\partial x}, \tag{3.5}$$

$$\frac{\partial \bar{u}}{\partial x} = 0, \tag{3.6}$$

where $\bar{v} = v[1 + C_b(k_m/v)]$ is the modified viscosity due to turbulence.

Eliminating the pressure p between Eqs. (3.3) and (3.4) and integrating the resulting expression with respect to z and using the constant horizontal density gradient namely $-\alpha\rho_0$ together with

$$\bar{u} = \frac{\partial \bar{u}}{\partial x} = 0, \tag{3.7}$$

we obtain

$$\frac{\partial \bar{u}}{\partial t} + \frac{\bar{v}}{k}\bar{u} + \frac{C_b}{\sqrt{k}}|\bar{u}|\bar{u} = -\alpha g z. \tag{3.8}$$

Eq. (3.8) is made dimensionless using

$$t = \frac{kt'}{\bar{v}}, \quad \bar{u} = \frac{\bar{v}}{C_b\sqrt{k}}\bar{u}', \quad z = \frac{\bar{v}^2 z'}{\alpha g C_b k^{3/2}}, \tag{3.9}$$

and obtain

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} + |\bar{u}|\bar{u} = -z. \tag{3.10}$$

Eq. (3.10) is solved for the following two cases:

Case 1: When

$$|\bar{u}| = -\bar{u}, \quad z > 0. \tag{3.11}$$

Case 2: When

$$|\bar{u}| = \bar{u}, \quad z < 0. \tag{3.12}$$

In case 1, the exact solution of non-linear Eq. (3.10), using $\bar{u} = 0$ at $t = 0$ is

$$\bar{u}(t) = \frac{a_-(1 - e^{-bt})}{1 + |a_-/a_+|e^{-bt}} \quad \text{for } t \to \infty \qquad (3.13)$$

and

$$\bar{u}(t) = \frac{a_+(1 - e^{bt})}{1 + |a_+/a_-|e^{bt}} \quad \text{for } t \to -\infty, \qquad (3.14)$$

where $a_+ = (1 + b)/2$, $a_- = (1 - b)/2$, $b = \sqrt{1 + 4z}$ are the roots of the equation $a^2 - a - z = 0$. Here a_+ corresponds to $t \to \infty$ and a_- corresponds to $t \to -\infty$.

We note that in obtaining the solutions (3.13) and (3.14) we have used the equilibrium (i.e. $\partial \bar{u}/\partial t = 0$) solution $\bar{u} = a$ of Eq. (3.10).

In case 2, the solution of non-linear Eq. (3.10), using Eq. (3.12), is given by

$$\bar{u}(t) = \frac{d_+(1 - e^{-ct})}{1 + |d_+/d_-|e^{-ct}} \quad \text{for } t \to \infty \qquad (3.15)$$

and

$$\bar{u}(t) = \frac{d_-(1 - e^{ct})}{1 + |d_-/d_+|e^{ct}} \quad \text{for } t \to -\infty, \qquad (3.16)$$

where $d_+ = (c - 1)/2$, $d_- = -(c + 1)/2$, $c = \sqrt{1 - 4z}$ are the roots of the equation $d^2 + d + z = 0$. Here d_+ corresponds to $t \to -\infty$ and d_- corresponds to $t \to \infty$.

4. Solution for Darcy regime

In this case, Eq. (3.8), using $C_b \to 0$ and $\bar{v} \to v$, reduces to the time-dependent linear Darcy equation

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} = -z. \qquad (4.1)$$

Solving this we get

$$\bar{u} = z(e^{-1} - 1). \qquad (4.2)$$

If we transform this to dimensional form, using Eq. (3.9) and taking, $\bar{v} \to v$ we get

$$\bar{u} = \frac{\alpha g k z}{v}\left(e^{\frac{vt}{k}} - 1\right). \qquad (4.3)$$

This coincides with the solution of Rudraiah [4] for laminar case. We note that in contrast to solution (4.3), solutions (3.13)–(3.16) reveal that the effect of turbulence is to generate non-uniform vertical shear that decays exponentially with time.

Solving Eq. (3.5) after making it dimensionless and using Eq. (3.14) we get

$$\bar{\rho} = 1 - \frac{x}{\sigma_1^*} - \frac{zy}{\sigma_1^*} + \sigma a_+ t - \sigma \log\left[1 + \frac{a_+}{b}(e^{bt} - 1)\right], \quad (4.4)$$

where $\sigma = \alpha\sqrt{k}/C_b$, $\sigma_1^* = gC_b k^{3/2}/\bar{v}^2$, $\gamma = \beta/\alpha$, $a_+ = (1 + b)/2$, $b = \sqrt{1 + 4z}$, we note that for Darcy regime,

$-\sigma \log[1 + \frac{a_+}{b}(e^{bt} - 1)] - \sigma z(e^{-t} - 1)$, so that Eq. (4.4) tends to

$$\bar{\rho} = 1 - \frac{x}{\sigma_1^*} - \left(\frac{\gamma}{\sigma_1^*}\right)z + \sigma z(1 - t - e^{-t}), \qquad (4.5)$$

which is the density distribution for Darcy regime. It is easily seen that ρ given by Eq. (4.5) coincides with the laminar case given by Rudraiah [4] in the limit $C_b \to 0$. Further, the isopyenals in the Darcy regime rotate towards the horizontal with an angle θ given by

$$\tan\theta = \gamma + \sigma_2^*(e^{-t} + t - 1), \qquad (4.6)$$

where

$$\sigma_2^* = \frac{\alpha g k^2}{\bar{v}^2}.$$

We note that as $\bar{v} \to v$, and $C_b \to 0$, Eq. (4.6) tends to laminar case given by

$$\tan\theta = \gamma + \sigma_2(e^{-t} + t - 1). \qquad (4.7)$$

Here σ_2 is the value of σ_2^* for $\bar{v} \to v$ and $C_b \to 0$.

5. Stability analysis

In the case of heterogeneous fluids, a measure of the stability of the flow is provided by the gradient Richardson number, R_i, defined by

$$R_i = \frac{g(\partial\bar{\rho}/\partial z)}{\rho_0(\partial\bar{u}/\partial z)^2}. \qquad (5.1)$$

This, using Eqs. (4.2) and (4.5), becomes

$$R_i = \frac{\gamma/\sigma_2^* + t - (1 - e^{-t})}{(1 - e^{-t})^2}. \qquad (5.2)$$

This equation, valid for Darcy regime ($C_b \to 0$) shows that R_i decreases exponentially with time (at the initial instant it is infinite) and approaches to $\frac{1}{2}$ as $t \to \infty$. This result suggests, as in the case of laminar flow discussed by Rudraiah [4], that the flow will be linearly stable.

Even in this non-linear case, the horizontal density gradient remains at its original value $-\rho_0\alpha$ while the vertical stratification varies continuously because a_+ and hence b are non-linear in z and decays exponentially both in space and time. In this non-linear variation in z, the angle at which isopyenals rotate towards the horizontal, is given by

$$\tan\theta = \gamma - \frac{\sigma_2^* t}{b} + \frac{\sigma_2^*[1 + e^{bt}(tb^2 + tb - 1)]}{b^3 + a_+ b^2(e^{bt} - 1)}.$$

Even in the non-linear case frontogenesis does not occur because of uniform initial horizontal density gradient.

In this non-linear case R_i, defined in Eq. (5.1), takes the form

$$R_i = \frac{f_1}{f_2^2}, \qquad (5.3)$$

$$f_1 = -\frac{\gamma}{\sigma_2^*} b + \frac{2}{b} + t + \frac{[1 + (1 + 2a_+ t)e^{bt}]}{a(e^{bt} - 1) + b},$$

$$f_2 = 1 + \frac{(b + 2a_+ + 2a_+ bt)e^{bt}}{(a_+ - b - a_+ e^{bt})}$$

$$+ \frac{a_+ b[1 + (1 - 2a_+ t)e^{bt}]e^{bt}}{(a_+ - b - a_+ e^{bt})^2}.$$

In contrast to the linear case given by Eq. (5.2), the gradient Richardson number given by Eq. (5.3) reveals that it decays exponentially with both space and time and approaches a constant value as $t \to \infty$.

6. Existence and uniqueness theorems of non-linear initial value problem

In Section 3 we have obtained analytical solutions for the non-linear Eq. (3.3) using the condition (3.7). It is important to know under what conditions these solutions exist. If they exist then we should know under what conditions they are unique. These conditions are established in the following theorems. For convenience, the over bar on the physical quantities are omitted.

Existence theorem: A necessary condition for the existence of a solution of non-linear Eqs. (3.3)–(3.5), is

$$\frac{\partial^2 \rho}{\partial x^2} = 0.$$

Proof. Eliminating the pressure p between Eqs. (3.3) and (3.4) and differentiating the resulting equation with respect to x and using the continuity i.e. Eq. (3.6), we obtain

$$\frac{\partial^2 \rho}{\partial x^2} = 0. \tag{6.1}$$

Hence the theorem. From this theorem, it follows that even in the case of turbulent flow through porous media the frontogenesis is associated with the presence of a velocity and transverse circulation. This also implies that ρ must be linear for $w = 0$ to be a valid assumption.

Uniqueness theorem: Let $z \in R$ and consider the differential equation

$$\frac{\partial u}{\partial t} + u + u|u| = -z, \text{ with the initial condition } u(0) = 0. \tag{6.2}$$

Then the following uniqueness theorem holds:
Eq. (6.2) admits a unique real solution for all fixed z and is stated as follows:
 Case (i): $z > 0$, $|u| = -u$.
 In this case let

$$\left\{ a_- = \frac{1 - \sqrt{1 + 4z}}{2}, \quad a_+ = \frac{1 + \sqrt{1 + 4z}}{2}, \right.$$

$$b = \sqrt{1 + 4z}, \tag{6.3}$$

then

$$u(t) = \frac{a_-(1 - e^{-bt})}{1 + |a_-/a_+|e^{-bt}}. \tag{6.4}$$

 Case (ii): $z < 0$, $|u| = u$.
 In this case let

$$\left\{ d_- = \frac{-1 - \sqrt{1 + 4z}}{2}, \quad d_+ = \frac{-1 + \sqrt{1 - 4z}}{2}, \right.$$

$$c = \sqrt{1 - 4z}, \tag{6.5}$$

then

$$u(t) = \frac{d_+(1 - e^{-ct})}{(1 + |d_-/d_+|e^{-ct})}. \tag{6.6}$$

 Case (iii): $z = 0$, then $u = 0$.

Proof. It is enough to prove case (i). The other cases follow in the same manner. Hence, we assume $z > 0$. From Eq. (6.2), and by continuity, there exists $T_0 > 0$ such that $-z - u(t)|u(t)| < 0$ $\forall t \in [0, T_0]$. This implies that $\partial u / \partial t < 0$ in $[0, T_0]$ and hence u is decreasing. Therefore, $u(t) \leqslant u(0) = 0$ for all $t \in [0, T_0]$.
 Let $T_1 = \max\{T; u$ is decreasing in $[0, T](\text{i.e. } \partial u / \partial t < 0)\}$.
 Next, we allow $T_1 \to \infty$. Now in $[0, T_1]$, $\partial u(t) / \partial t < 0$ and hence Eq. (6.2) becomes

$$\frac{\partial u}{\partial t} = -z - u + u^2 = (u - a_+)(u - a_-),$$

where $a_\pm$ are the roots of the equation $X^2 - X - Z = 0$ and are given by Eq. (6.3).
 Since $a_- < 0 < a_+$ and $\partial u / \partial t < 0$ in $[0, T_1]$, hence $a_- \leqslant u \leqslant 0$. Therefore,

$$\frac{\mathrm{d}u}{(u - a_-)(u - a_+)} = \mathrm{d}t.$$

That is

$$-\frac{1}{b} \left[\frac{1}{a_+ - u} + \frac{1}{u - a_-} \right] \mathrm{d}u = \mathrm{d}t.$$

Integrating this, we obtain for some $c_1 \in R$.

$$\text{Log} \left(\frac{u - a_-}{a_+ - u} \right) = -bt + c_1.$$

That is

$$\left(\frac{u - a_-}{a_+ - u} \right) = -c_2 e^{-bt}, \quad c_2 = e^{c_1}.$$

Now $u(0) = 0$ implies that

$$\left| \frac{a_-}{a_+} \right| = \frac{-a_-}{a_+} = c_2$$

and u is given by

$$u(t) = \frac{a_-(1 - e^{-bt})}{1 + |a_-/a_+|e^{-bt}}.$$

This equation is valid for all $t \in R$ and hence valid as $T_1 \to \infty$ and is the unique solution of (6.2). The other cases can be proved similarly. □

7. Stability of solution of non-linear equation

In the earlier sections we have obtained the analytical solution (3.13) of non-linear initial value problem (3.10) valid for $|u| = -u$. We, however, note that this solution (3.13) is also valid for $|u| = u$ when b in Eq. (3.13) is replaced by $b = \sqrt{1-4z}$. In otherwords the behaviour of the solution of (3.10) with $|u| = -u$ in the upper half of the porous region (i.e. $z > 0$) is the same as the solution of Eq. (3.10) with $|u| = u$ in the lower-half of the porous region (i.e. $z < 0$). The solution (3.13) is quantitatively and qualitatively different for $z \leqslant \frac{1}{4}$ and $z > \frac{1}{4}$ in the upper half of the region (i.e. $z > 0$). It is also different from the linear solution (4.2) of Eq. (4.1) valid for Darcy regime as well as the solution (4.3) valid for laminar flow in the presence of a porous medium.

It is of interest to note that the solution (4.3) of linear Eq. (4.1) obtained by Rudraiah [4] for laminar flow in the presence of a porous medium is real and establishes a uniform shear which accelerates at a constant rate. In contrast to this, the solution (4.2) of linear Eq. (4.1) valid for Darcy regime $C_b \to 0$ although real and establishes a uniform shear but decreases exponentially with time. However, the solution (3.15) of non-linear Eq. (3.10) behaves differently from the other two solutions (4.2) and (4.3) in the sense that the solutions (4.2) and (4.3) are always real, where as the solution (3.15) may be real or complex depending on $z < \frac{1}{4}$ or $z > \frac{1}{4}$, respectively. For ($z > 0$) solution (3.13) is real for $z < \frac{1}{4}$ and decreases with increasing z. We note that for $z = \frac{1}{4}$, $b = 0$ and Eq. (3.13) (which is also valid for $|u| = u$ when b in Eq. (3.13) is replaced by $\sqrt{1-4z}$) becomes indeterminant and takes the form

$$u = \frac{-t}{2(2+t)}. \tag{7.1}$$

This tends to $-\frac{1}{2}$ as $t \to \infty$ representing stable solution. We note that $z = \frac{1}{4}$ is a stable solution. For $z > \frac{1}{4}$ solution (3.13) becomes complex conjugate. The stability of the equilibrium solution of Eq. (3.10) subject to infinitesimal disturbances is important here because of the oscillatory nature of the solution (3.13) and is discussed below.

The equilibrium solution of (3.10) is a steady solution $u = a$ for which $\partial u / \partial t = 0$ where

$$a = \frac{-1}{2} + \frac{\sqrt{1-4z}}{2} = \frac{-1+b}{2}, \quad b = \sqrt{1-4z}. \tag{7.2}$$

Note that the steady solution is always real and is positive for positive sign and negative for negative sign in Eq. (3.10) in the lower half of the region ($z < 0$). In the upper half of the region ($z > 0$), there are two negative real steady solution for $z < \frac{1}{4}$. No real solution exist for $z > \frac{1}{4}$. To examine the stability

of the steady solution, we write Eq. (3.10) as

$$\frac{du'}{dt} = -(b + u')u', \tag{7.3}$$

where u' is a perturbation without approximation on the equilibrium solution $u = a$ and written as $u = a + u'$. To study the stability of the solution $u = a$ for $b > 0$, we assume u' be infinitesimally small, so that linearising (7.3) we get

$$\frac{du'}{dt} = -bu'. \tag{7.4}$$

This equation has the solution

$$u' = -a e^{-bt}, \tag{7.5}$$

where $-a$ is the given initial condition of u'. If $b > 0$, $u' \to 0$ as $t \to \infty$ for all a. Therefore, all infinitesimal perturbations about the equilibrium point $u = a$ remain infinitesimal for all time and hence the solution is stable. Similarly we can see that for $b < 0$ (i.e. negative root in $b = (1 + 2a)$) the equilibrium solution is unstable because as infinitesimal disturbance, u', grows until it becomes no longer small.

In general, it may not be sufficient to conclude the stability of the solution of the non-linear equation from the stability of the solution corresponding to linear equation because the arbitrary perturbations may make the solution unstable even the solution of linear equation is stable. This is not so in the present problem. The solution which was stable for infinitesimal perturbations is also stable for arbitrary perturbations as shown below. At the critical point $z = \frac{1}{4}$; $b = 0$ and hence Eq. (7.3) becomes

$$\frac{du'}{dt} = -u'^2, \tag{7.6}$$

satisfying the initial condition $u'(0) = \frac{1}{2}$ because $a = -\frac{1}{2}$ in this case. The solution of (7.6) satisfying this initial condition is

$$u'(t) = \frac{1}{2+t}, \tag{7.7}$$

which tends to zero as $t \to \infty$. Therefore, even in the non-linear case the equilibrium solution is stable.

8. Frontogenesis

The results of Section 3 reveal that frontogenesis does not occur when the initial horizontal density gradient is constant. A non-uniform initial horizontal density gradient sets up vertical motion denoted by w in addition to horizontal motion. In this case the governing Eqs. (2.1)–(2.3) under the assumptions described in the previous section and making dimensionless using Eq. (3.9) take the form

$$\frac{\partial u}{\partial t} + \sigma_2^* u \frac{\partial u}{\partial x} + \sigma_2^* w \frac{\partial u}{\partial z} = -\sigma_2^* \frac{\partial p}{\partial x} - u - |\vec{q}|u, \tag{8.1}$$

$$\frac{\partial w}{\partial t} + \sigma_2^* u \frac{\partial w}{\partial x} + \sigma_2^* w \frac{\partial w}{\partial z} = -\sigma_2^* \frac{\partial p}{\partial x} - w - |\vec{q}|w - \sigma_1^* \rho, \tag{8.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{8.3}$$

$$\frac{\partial \rho}{\partial t} + \sigma_2^* u \frac{\partial \rho}{\partial x} + \sigma_2^* w \frac{\partial \rho}{\partial z} = 0, \tag{8.4}$$

where $|\vec{q}| = \sqrt{u^2 + w^2}$,

$$\sigma_2^* = \frac{\alpha g k^2}{\overline{v}^2} = \sigma \sigma_1^*,$$

using Eq. (8.3), we define a stream function ψ as

$$u = -\frac{\partial \psi}{\partial z} = -\psi_z \quad \text{and} \quad w = \frac{\partial \psi}{\partial x} = \psi_x. \tag{8.5}$$

Substituting Eq. (8.5) into Eqs. (8.1) and (8.2) and eliminating the pressure we get

$$-\nabla^2 \psi_t + \sigma_2^* \psi_z \nabla^2 \psi_x - \sigma_2^* \psi_x \nabla^2 \psi_z - \nabla^2 \psi$$
$$- (\psi_x^2 \psi + \psi_z^2)^{1/2} \nabla^2 \psi - (\psi_x^2 + \psi_z^2)^{-1/2}$$
$$\times [2\psi_x \psi_z \psi_{xz} + \psi_x^2 \psi_{xx} + \psi_z^2 \psi_{zz}] - \sigma_1^* \frac{\partial \rho}{\partial x} = 0, \tag{8.6}$$

where $\nabla^2 \psi = \psi_{xx} + \psi_{zz}$.

We consider the time evolution of this system by writing the solution as a power series in t in the form

$$\psi = \psi_0 + \psi_1 t + \psi_2 t^2 + \psi_3 t^3 + \cdots, \tag{8.7}$$

$$\rho = \rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 + \cdots. \tag{8.8}$$

Since the fluid is at rest initially, we have $\psi_0 = 0$ and $\rho_0 = \rho_0(x, z)$ which is the initially specified density distribution.

We substitute Eqs. (8.7) and (8.8) into Eq. (8.6) and equate the coefficients of t under the assumption that $w \ll u$ so that $|\vec{q}|u = u^2$ and $|\vec{q}|w = uw$ to obtain the following governing equations without the effect of inertia. The terms of order t^2 will give the effect of inertia.

At $O(t^0)$ we have

$$\nabla^2 \psi_1 = -\sigma_1^* \rho_{0x}, \tag{8.9}$$

$$\rho_1 = 0. \tag{8.10}$$

At $O(t^1)$, we have

$$2\nabla^2 \psi_2 + \nabla^2 \psi_1 + \sigma_1^* \rho_{1x} = 0.$$

This, using Eqs. (8.9) and (8.10), becomes

$$\nabla^2 \psi_2 = \frac{\sigma_1^* \rho_{0x}}{2}. \tag{8.11}$$

From Eq. (8.10) we see that, the derivative in Eq. (8.4) may be approximated by

$$\rho_{xt} = -\sigma_2^* (u\rho_x)_x. \tag{8.12}$$

This implies that the time evolution of the density gradient is determined by the sign of $(u\rho_x)_x$. From this it follows that an increase in the horizontal density gradient, and hence the frontogenesis, will occur only when the horizontal motion in the region of stronger density gradient is towards the region of the weaker density gradient.

To know the effect of inertia we have to go up to third order, namely up to the term involving t^2. Then from Eq. (8.6), equating term of order t^2 to zero, we get

$$\nabla^2 \psi_3 = \tfrac{1}{3}[-\sigma_2^* \psi_{1x} \nabla^2 \psi_{1z} + \sigma_2^* \psi_{1z} \nabla^2 \psi_{1x} - \nabla^2 \psi_2 - \sigma_1 \rho_{2x}$$
$$+ 2\psi_{1z}\psi_{1zz} + \psi_{1z}\psi_{1xx} + \psi_{1x}\psi_{1xz}]. \tag{8.13}$$

Further, from Eq. (8.4), we have

$$\rho_1 = 0, \quad \rho_2 = \tfrac{1}{2}(\psi_{1z}\rho_{0x} - \psi_{1x}\rho_{0z}),$$

$$\rho_3 = \tfrac{1}{3}(\psi_{2z}\rho_{0x} - \psi_{2z}\rho_{0z}) \tag{8.14}$$

and hence

$$\rho = \rho_0 + \tfrac{1}{2}(\psi_{1z}\rho_{0x} - \psi_{1x}\rho_{0z})t^2$$
$$+ \tfrac{1}{3}(\psi_{2z}\rho_{0x} - \psi_{2x}\rho_{0z})t^3 + \cdots. \tag{8.15}$$

Section 3 revealed that frontogenesis cannot occur for uniform horizontal initial density gradient. Hence, as in the case of laminar flow discussed by Rudraiah [4], we have to consider a non-uniform horizontal density gradient to show whether frontogenesis occurs or not. Therefore, we assume, for simplicity the piecewise initial density gradient of the form

$$\rho = \begin{cases} 1 - \gamma_1 \sigma_1^* x, & x < 0, \\ 1 - \gamma_2 \sigma_1^* x, & x > 0, \end{cases} \tag{8.16}$$

where $\gamma_1 = \alpha_1/\alpha$, $\gamma_2 = \alpha_2/\alpha$, α_1 and α_2 are the values of α in $x < 0$ and $x > 0$, respectively. This shows a discontinuity in the density gradient at $x = 0$ even though the density is continuous there. Substituting Eq. (8.16) into Eqs. (8.9), (8.11) and (8.13), we have

$$\nabla^2 \psi_1 \begin{cases} \gamma_1, & x < 0, \\ \gamma_2, & x > 0, \end{cases} \tag{8.17}$$

$$\nabla^2 \psi_2 \begin{cases} -\dfrac{\gamma_1}{2}, & x < 0, \\ -\dfrac{\gamma_2}{2}, & x > 0, \end{cases} \tag{8.18}$$

$$\nabla^2 \psi_3 = \frac{\gamma_1}{6}[1 + \psi_{1zx}\sigma_2] + \frac{1}{3}[2\psi_{1z}\psi_{1zz}$$
$$+ \psi_{1z}\psi_{1xx} + \psi_{1x}\psi_{1xz}] \quad \text{for } x < 0 \tag{8.19}$$

and

$$\nabla^2 \psi_3 = \frac{\gamma_2}{6}[1 + \psi_{1zx}\sigma_2] + \frac{1}{3}[2\psi_{1z}\psi_{1zz}$$

$$+ \psi_{1z}\psi_{1xx} + \psi_{1x}\psi_{1xz}] \quad \text{for } x > 0. \tag{8.20}$$

Eqs. (8.17)–(8.20) are the system of Poisson equations which are of elliptic type. Solving them, we get

$$\psi_1 = \begin{cases} \dfrac{1}{2}\gamma_1 z^2 + \dfrac{1}{8}\gamma_2 h^2 + 2\dfrac{(\gamma_1 - \gamma_2)h^2}{\pi^3}\displaystyle\sum_{i=0}^{\infty} a_i, & x < 0, \\[3mm] \dfrac{1}{2}\gamma_2 z^2 + \dfrac{1}{8}\gamma_1 h^2 - 2\dfrac{(\gamma_1 - \gamma_2)h^2}{\pi^3}\displaystyle\sum_{i=0}^{\infty} b_i, & x < 0, \end{cases}$$
$$\tag{8.21}$$

$$\psi_2 = \begin{cases} \dfrac{1}{4}\gamma_1 z^2 + \dfrac{1}{16}\gamma_2 h^2 + 2\dfrac{(\gamma_1 - \gamma_2)h^2}{\pi^3}\displaystyle\sum_{i=0}^{\infty} a_i, & x < 0, \\[3mm] \dfrac{1}{4}\gamma_2 z^2 + \dfrac{1}{16}\gamma_1 h^2 - 2\dfrac{(\gamma_1 - \gamma_2)h^2}{\pi^3}\displaystyle\sum_{i=0}^{\infty} b_i, & x < 0. \end{cases}$$
$$\tag{8.22}$$

Similarly the solution of (8.19) is determined and the expression is avoided here as it is lengthy. In computing $\psi = \psi_1 t + \psi_2 t^2 + \psi_3 t^3$ the effects ψ_3 is included and the results are represented graphically and discussed in the final section. In Eqs. (8.21) and (8.22), a_i and b_i are given by

$$a_i = \frac{(-1)^i}{(2i+1)^3}\cos[(2i+1)\pi z]e^{(2i+1)\pi x} \quad (x < 0),$$

$$b_i = \frac{(-1)^i}{(2i+1)^3}\cos[(2i+1)\pi z]e^{-(2i+1)\pi x} \quad (x > 0).$$

From Eqs. (8.21) and (8.22) it follows that in the absence of horizontal discontinuity, that is, for a uniform horizontal density gradient for $x < 0$ and $x > 0$, the terms multiplying the summations (Σ) become zero and ψ will be a pure function of z. Therefore, $w_i = 0$ ($i = 1, 2$) and $u_1 = -\gamma_1 z$ ($x > 0$) and $u_2 = \gamma_2 z$ ($x < 0$) as found in Eq. (4.2) of Section 3. In the presence of the discontinuity in the density gradient i.e. $\gamma_1 \neq \gamma_2$, it produces an additional circulation expressed as the sum in Eqs. (8.21) and (8.22). We also see that the effect of increase in the eddy viscosity $\bar{v}$ of the fluid is to increase this circulation while the effect of an increase in effective permeability k of the medium is to decrease this circulation. We also note that u is positive, and increases with a decrease in effective permeability k and is greater if $\gamma_1 > \gamma_2$ for $x < 0$ than for $x > 0$. This is the result of effective permeability of the medium and the steeper density gradient on the left-hand side of the channel. The excess flux at $x = 0$ from $x < 0$ result in developing upward velocity as seen from Eq. (8.22). This upward velocity provides the maximum flux towards $x \to -\infty$ required in the upper half of the channel. If $\gamma_2 > \gamma_1$ the sense of circulation is reversed.

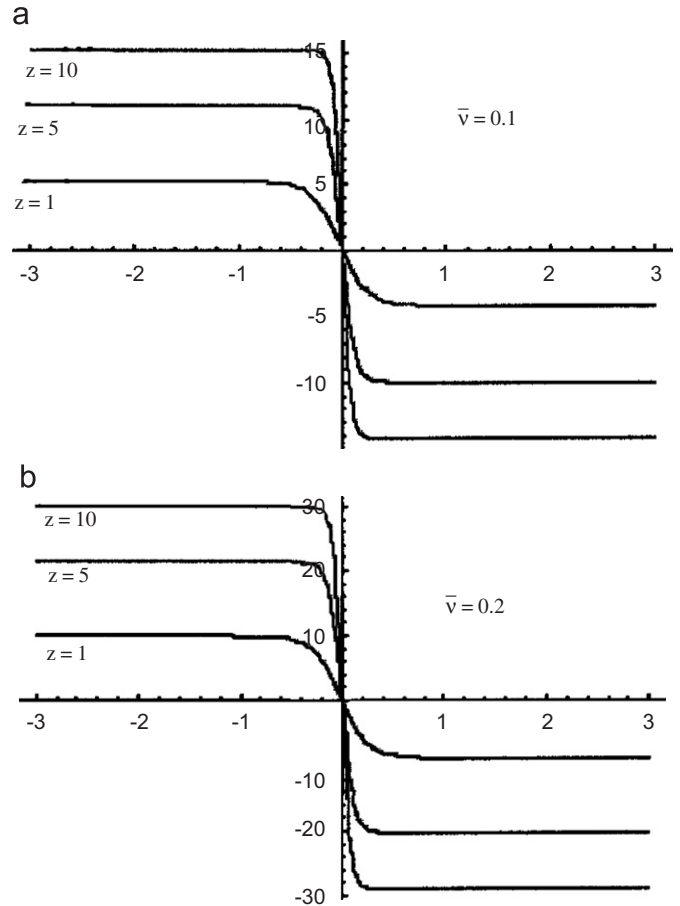Fig. 1. (a), (b) Velocity profiles for $z = 1, 5, 10$.

The advection of density ρ is given by Eq. (8.15). From this, we get

$$\left. \begin{aligned} \rho &= (1 - \gamma_1\sigma_1^*x) - (1+t)t^2\left[\frac{1}{2}\gamma_1^2 z - \frac{(\gamma_1 - \gamma_2)\gamma_1 h}{\pi^2}\right] \\ &\quad \times \sum_{i=0}^{\infty} C_i e^{d_i x} \quad (x < 0) \\[2mm] \rho &= (1 - \gamma_2\sigma_2^*x) - (1+t)t^2\left[\frac{1}{2}\gamma_1^2 z - \frac{(\gamma_1 - \gamma_2)\gamma_2}{\pi^2}\right] \\ &\quad \times \sum_{i=0}^{\infty} C_i e^{d_i x} \quad (x > 0) \end{aligned} \right\},$$
$$\tag{8.23}$$

where

$$C_i = \frac{(-1)^i \pi^3}{d_i^2}\sin d_i z, \quad d_i = (2i+1)\pi.$$

Frontogenesis occurs when there is an increase in ρ_x. The maximum value of ρ_x occurs at $x = 0$ and from (8.23) it follows that as $x \to 0$:

$$\rho_x = -\sigma_1^*\gamma_2 + \frac{(1+t)t^2}{2}(\gamma_1 - \gamma_2)\gamma_2\pi \log\left[\tan\left(\frac{\pi}{4}(1+2z)\right)\right].$$
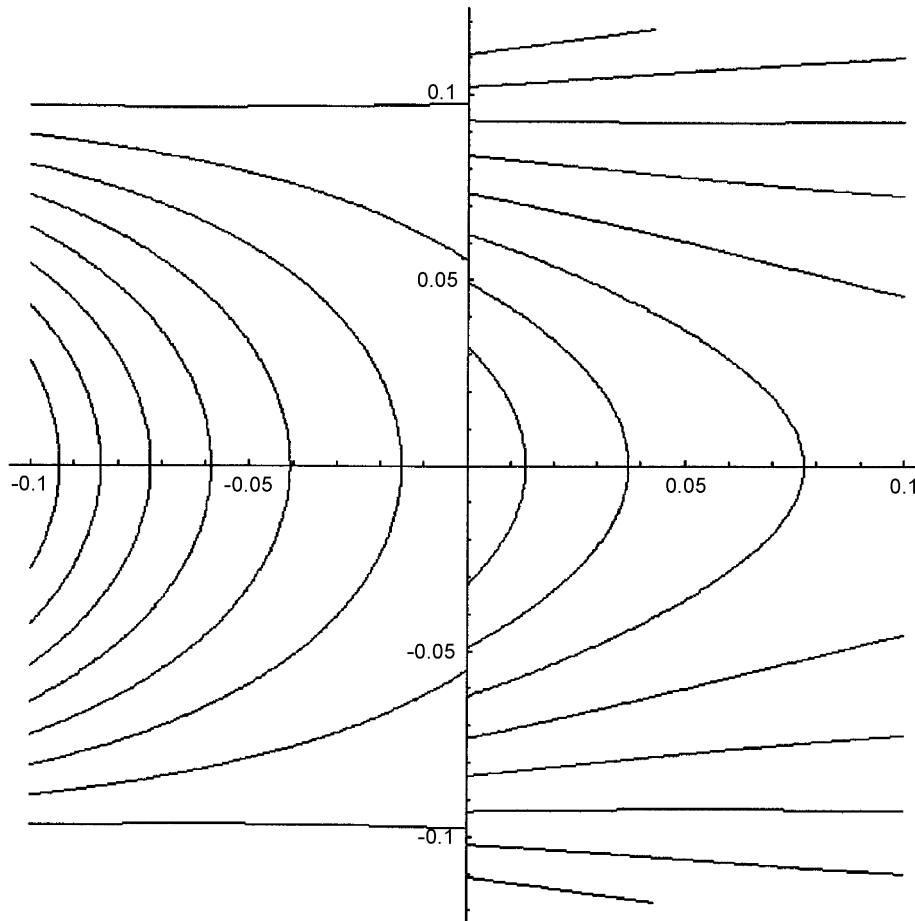$$\tag{8.24}$$

Fig. 2. Streamlines for $\gamma_1 = 1.05$ and $\gamma_2 = 0.06$.

Since $\gamma_1 > \gamma_2$, it is clear from Eq. (8.24) that ρ_x increases with time for $z < 0$ and decreases for $z > 0$. This implies that Frontogenesis occurs along the lower portion of the channel. This can also be established from the stream function profiles and isopyenal deformation drawn in Figs. 2 and 3.

9. Conclusions

The effects of Darcy resistance, Forchheimer inertia and the variation of initial density both in the horizontal and vertical directions on a turbulent motion of a heterogeneous fluid saturated porous medium are investigated analytically. The main conclusions are as follows.

1. In the case of a uniform horizontal density gradient, discussed in Section 3, a purely horizontal motion develops where the inertial effect comes through the Forchheimer quadratic drag term $|\vec{q}|\vec{q}$ satisfying the non-linear partial differential Eq. (3.10). This non-linear initial value problem is solved for both $|u| = u$ and $|u| = -u$. Its solution for $|u| = -u$ is given by Eq. (3.13). We note that the solution for $|u| = u$ is the same as Eq. (3.13) with b given by $b = \sqrt{1 - 4z}$ with $z < 0$. The solution given by Eq. (3.13) is

computed and the results are drawn in Fig. 1 valid for $t \rightarrow \pm\infty$. For $t \rightarrow \infty$ the velocity decreases and approaches a constant value $-2z/(1 + b)$ for a fixed z and for $t \rightarrow -\infty$ the velocity increases initially and approaches a constant value $2z/(b - 1)$ for a fixed z. Further, the non-linear solution (3.13) is different from the linear solution (4.2) in the sense that the solution (4.2) establishes a uniform shear which decreases exponentially with time whereas the solution (3.13) establishes a non-uniform shear varying both with space z and time t as seen in Fig. 1.

2. Instantaneous streamlines for the flow with $\gamma_1 > \gamma_2$ and for different values of t are drawn in Fig. 2. We see that the streamlines are closer together on the left ($x < 0$) than on the right ($x > 0$) as a result of the more intensive flow produced by the larger density gradient there. The time evolution of the density field given by Eq. (8.23) is shown in Fig. 3 for different dimensionless time t. From this it is clear that the density profiles are crowded at the lower region showing the increase in density gradient and beginning of frontogenesis there. We also see that the density profiles no longer remain straight and curvature develops near $x = 0$. This curvature sets up a circulation in the transverse plane and hence the magnitude of the density gradient increases with dimensionless time.
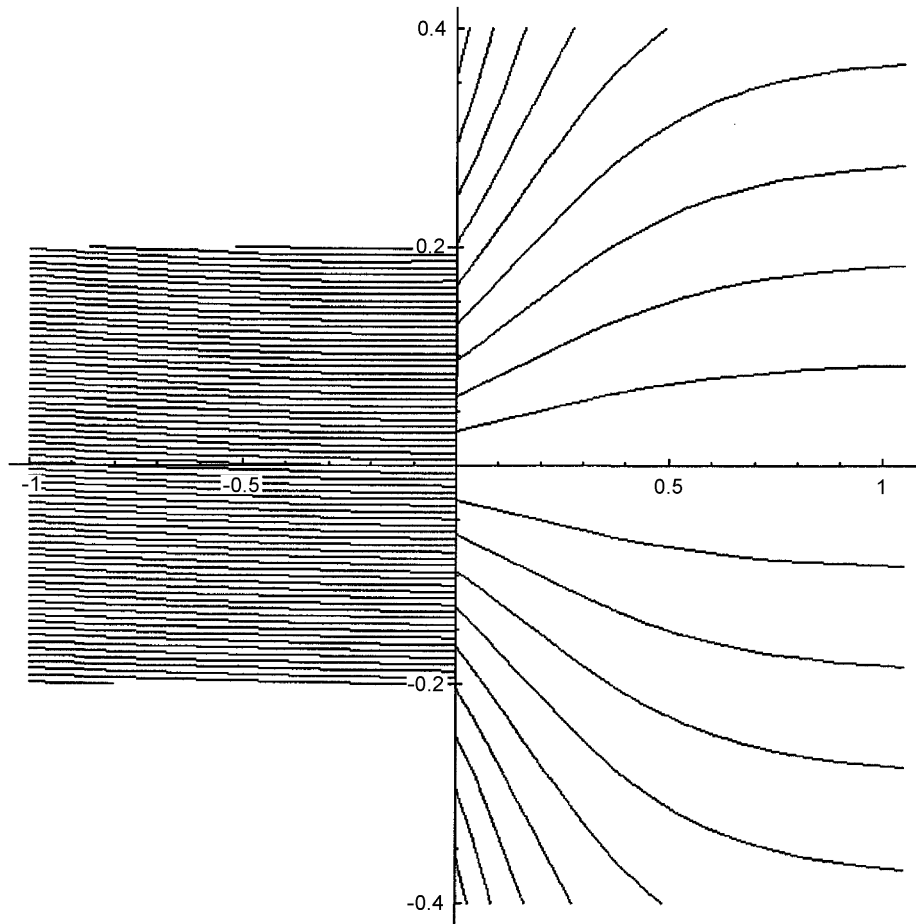
Fig. 3. Isopyenals for $t = 1.4$, $\gamma_1 = 1.05$, $\gamma_2 = 0.06$ and $\sigma_1^* = 4$.

Acknowledgements

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