Heat and Mass Transfer 33 (1998) 287–293 © Springer-Verlag 1998

A weak nonlinear stability analysis of double diffusive convection with cross-diffusion in a fluid-saturated porous medium

N. Rudraiah, P. G. Siddheshwar

Abstract The effect of "Cross Diffusion" on the linear and nonlinear stability of double diffusive convection in a fluid-saturated porous medium has been studied analytically. In the case of linear theory, the normal mode technique has been used and the condition for the maintenance of "finger" and "diffusive" instabilities have been obtained. It has been found that fingers can form by taking cross diffusion terms of appropriate sign and magnitude even though both components make stabilizing contributions to the net vertical density gradient. It has also been shown that "finger" and "diffusive" instabilities can never occur simultaneously. The nonlinear theory is based on the truncated representation of Fourier series and it has been found that the finite amplitude convection may occur when both initial property gradients are stabilizing. Further, the region of finite amplitude instability always encloses the region of infinitesimal oscillatory instability. The effects of permeability and cross-diffusion terms on the heat and mass transports have also been clearly brought out.

1
Introduction

Convection induced by temperature and concentration differences or by concentration differences of two species, known as double – diffusive convection, has attracted considerable interest in the last several decades (see Rudraiah and Malashetty 1986). The unconstrained/constrained double-diffusive convection in porous media has also become increasingly important in recent years because of its many applications, particularly in hydrothermal growth and solar pond.

Double-diffusive convection in a porous medium occurs under the presence of opposing gradients of two properties and is characterized by well-mixed convecting layers which are separated by relatively sharp density steps. These steps may be of the "finger" or "diffusive" kind. "Fingers" will form when warm salty water overlies cooler fresher water and the "diffusive interfaces" will form when warm salty water underlies cooler fresher water. In other words, in a two-component system, in the absence of cross-diffusion, instabilities can occur only if, atleast, one of the components is destabilizing. However, if the terms describing the diffusion of one property along the gradient of another (i.e. cross-diffusion) is incorporated in the species transport equation, then the situation will be quite different [6, 7, 1, 2, 5]. This problem of double-diffusive convection is not well investigated because of the interaction of the fluid with the porous matrix and also due to the lack of accurate values of the cross-diffusion coefficients.

Rudraiah and Patil [6, 7], Brand and Steinberg [1, 2] and Taslim and Narusawa [9] performed a linear and nonlinear stability analyses of the problem in a densely packed porous medium but with only Soret effect. Recently, Rudraiah and Malashetty [5] performed a linear and non-linear analyses of the problem in a densely-packed porous medium with both cross-diffusion effects. Their linear analysis is based on normal mode and the non-linear study is based on the assumption that the motion is only in the vertical direction (i.e. long slender fingers). Through the latter analysis they just obtained a condition for the maintenance of 'fingers'. Their analysis ignored the possibility of steady finite-amplitude convection that might occur in the system and also the mass (heat) transports. Since finite-amplitude effects have not been given much attention so far, the main object of the present paper is to perform a non-linear stability analysis of the problem using the minimal representation of Fourier series with the object of computing the heat and mass transports.

Received on 14 March 1997

N. Rudraiah, P. G. Siddheshwar
UGC-DSA centre in Fluid Mechanics,
Department of Mathematics, Bangalore University,
Bangalore – 560 001, India
and
National Research Institute for Applied Mathematics (NRIAM)
7th Cross, 7th Block (West), Jayanagar,
Bangalore – 560 082, India

Correspondence to: N. Rudraiah

2
Mathematical formulation

The physical configuration considered in this shown schematically in Fig. 1, consists of a horizontal fluid-saturated porous medium of thickness d and of infinite horizontal extent. The system is subject to concentrations gradients $\Delta C_1/d$ and $\Delta C_2/d$. Here C_1 and C_2 may both represent solute concentrations or may represent temperature and solute concentrations. The property C_1 is assumed to have a larger diffusivity than that of C_2. The governing equations are:

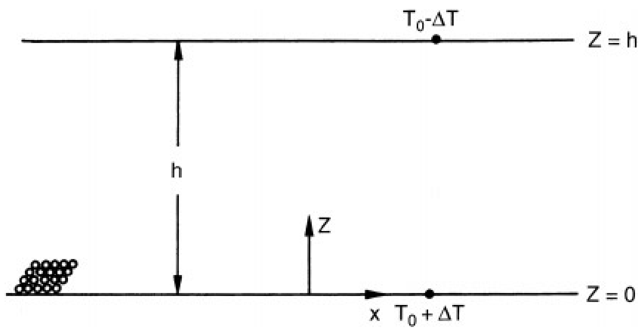

Fig. 1. Schematic illustration of a porous layer

$$\frac{1}{\epsilon}\frac{\partial \vec{q}}{\partial t} + \frac{1}{\epsilon^2}(\vec{q}\cdot\nabla)\vec{q} = -\frac{1}{\rho_0}\nabla_P + \frac{\rho}{\rho_0}\vec{g} + \gamma\nabla^2\vec{q} - \frac{\gamma}{K}\vec{q} \tag{1}$$

$$\frac{\partial C_1}{\partial t} + (\vec{q}\cdot\nabla)C_1 = D_{11}\nabla^2 C_1 + D_{12}\nabla^2 C_2 \tag{2}$$

$$\frac{\partial C_2}{\partial t} + (\vec{q}\cdot\nabla)C_2 = D_{22}\nabla^2 C_2 + D_{21}\nabla^2 C_1 \tag{3}$$

$$\nabla\cdot\vec{q} = 0 \tag{4}$$

$$\rho = \rho_0\left[1 + \sum_{i=1}^{2}\alpha_i(C_i - C_{i0})\right] \tag{5}$$

By operating curl on Eq. (1) we eliminate pressure and obtain

$$\left(\frac{\partial}{\epsilon\partial t} + \nabla^2 + \frac{\gamma}{K}\right)\nabla^2\psi = \alpha_1 g\frac{\partial C_1}{\partial x} + \alpha_2 g\frac{\partial C_2}{\partial x} + \frac{1}{\epsilon^2}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} \tag{6}$$

where $\vec{q} = \left(\frac{\partial\psi}{\partial z}, 0, -\frac{\partial\psi}{\partial x}\right)$ and the Jacobian$\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}(\nabla^2\psi) - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}(\nabla^2\psi)$ represents the vorticity advection term. By introducing the following dimensionless variables

$$\psi^* = \frac{\psi}{D_{11}};\; C_1^* = \frac{C_1}{\Delta C_1};\; C_2^* = \frac{C_2}{\Delta C_2}$$
$$(x^*, z^*) = \frac{1}{d}(x,z);\; t^* = \frac{t}{(d^2/D_{11})} \tag{7}$$

into the Eqs. (2), (3) and (6), and dropping the asterisks, we arrive at the following dimensionless equations:

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)C_1 - (\mathrm{DUFOUR})\frac{R_{c2}}{R_{c1}}\nabla^2 C_2 + \frac{\partial\psi}{\partial x} = \frac{\partial(\psi, C_1)}{\partial(x,z)} \tag{8}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)C_2 - (\mathrm{SORET})\frac{R_{c1}}{R_{c2}}\nabla^2 C_1 + \frac{\partial\psi}{\partial x} = \frac{\partial(\psi, C_2)}{\partial(x,z)} \tag{9}$$

$$\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t} - \nabla^2 + \frac{1}{\mathrm{Da}}\right)\nabla^2\psi + R_{c1}\frac{\partial C_1}{\partial x} + R_{c2}\frac{\partial C_2}{\partial x} = \frac{1}{\epsilon\mathrm{Pr}}\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} \tag{10}$$

where DUFOUR $= \frac{D_{12}\alpha_1}{D_{11}\alpha_2}$, SORET $= \frac{D_{21}\alpha_2}{D_{11}\alpha_1}$, and τ, R_{c1}, R_{c2}, Pr and Da have their pre-defined meaning.

We consider stress-free permeable (with respect to C_1 and C_2) flat boundaries and therefore Eqs. (8)–(10) have to be solved subject to the following boundary conditions:

$$\psi = \frac{\partial^2\psi}{\partial z^2} = C_1 = C_2 = 0 \quad \text{at } z = 0, 1\ . \tag{11}$$

3
Linear stability theory

In this section we discuss the linear stability analysis considering both 'Diffusive' and 'Finger' regimes. To this end we seek solutions to the linearized form of Eqs. (8)–(10) in the form:

$$\begin{bmatrix}\psi\\ C_1\\ C_2\end{bmatrix} = e^{qt}\begin{bmatrix}\psi_0\ \sin\pi ax\\ C_{10}\ \cos\pi ax\\ C_{20}\ \cos\pi ax\end{bmatrix}\sin\pi z\ , \tag{12}$$

which satisfy (11). Substituting (12) into Eqs. (8)–(10) and simplifying we obtain the dispersion relation

$$q^3 + k^2[(\tau+1) + \mathrm{Pr}\ \eta]q^2 + \left[\tau k^4 + \mathrm{Pr}(\tau+1)k^4\eta + \mathrm{Pr}\frac{(\pi a)^2}{k^2}\left\{R_{c1}^e + R_{c2}^e\right\}\right]q + \mathrm{Pr}\left[\tau k^6\eta + (\pi a)^2\left\{\tau R_{c1}^e + R_{c2}^e\right\}\right] = 0 \tag{13}$$

where

$$\eta = 1 + \frac{1}{k^2\mathrm{Da}}\ ,$$

$$k^2 = \pi^2(a^2+1),\ \ R_{c1}^e = R_{c1} + \frac{B-A}{1-\tau}\ ,$$

$$R_{c2}^e = R_{c2} + \frac{A-B}{1-\tau},\ \ A = \frac{\mathrm{Pr}\ k^6}{(\pi a)^2}(\mathrm{SORET})(\mathrm{DUFOUR})\ ,$$

$$B = \frac{k^6\eta}{(\pi a)^2}(\mathrm{SORET})(\mathrm{DUFOUR}) + (\mathrm{DUFOUR})R_{c1} + (\mathrm{SORET})R_{c2}\ ,$$

R^e and R_s^e are called the equivalent Rayleigh numbers which reduce to the usual Rayleigh numbers (when cross-diffusion effects are absent) when $D_{12} = D_{21} = 0$, i.e.

$$R^e = R \text{ and } R_s^e = R_s\ .$$

We note that the dispersion relation (13) is so general that the results of viscous flow (absence of porous matrix) and Darcy flow regimes can be obtained in the limiting process of Da $\to\infty$ and Da $\to 0$ respectively.

Marginal state

If q is real, then marginal stability occurs when $q = 0$, i.e. when

$$-R_{C1}^e = \frac{R_{C2}^e}{\tau} + \frac{K^6\eta}{(\pi a)^2} \tag{14}$$

and $a^2(= x)$ satisfies the cubic equation

$$2x^3 + \left(\frac{1}{\pi^2 \mathrm{Da}} + 3\right)x^2 - \left(\frac{1}{\pi^2 \mathrm{Da}} + 1\right) = 0 \ . \qquad (15)$$

Equation (15) implies that the critical wave number is independent of R_{C2}, τ, D_{12} and D_{21} and dependent only on the Darcy number Da. Equation (15) is the same for double diffusive convection (without cross-diffusion) in porous media (see Rudraiah et al. [8]).

In the case of double-diffusive convection, with cross-diffusion, through a porous medium the principle of exchange of stability may not be valid and an explanation for the same is provided below.

Due to the cross-diffusion effects each property gradient has a significant influence on the flux of the other property. As a result of this a flux of one property gradient is caused by a spatial gradient of the other and vice-versa. Therefore time-dependent motions of various types can exist in double-diffusive convection with cross-diffusion in a porous medium because the coupled fluxes of the two properties, arising due to irreversible thermodynamics, act as a restoring mechanism. Also, the linear operator associated with the linearized form of Eqs. (8)–(10) is non-Hermitian and as a result oscillatory instabilities can also be mathematically expected. In view of this we consider overstable motions in the next section.

3.1 Oscillatory motions

The condition for infinitesimal overstable motion is that the product of the coefficient of q^2 and q in (13) must be equal to the coefficient of q^0 (i.e. the constant term) and is given by

$$-R^{eo}_{c1} = \frac{\tau + \Pr\,\eta}{1 + \Pr\,\eta} R^e_{c2} + \frac{k^6(\tau+1)\{\tau + \Pr\,\eta[(\tau+1) + \Pr\,\eta]\}}{(\pi a)^2 \Pr\,(1 + \Pr\,\eta)} \ . \qquad (16)$$

The analytical determination of the minimum Rayleigh number for overstable motions is mathematically tedious and also accurate values for the cross-diffusion coefficients are not available. The type of bifurcation, whether direct or Hopf, also depends on the afore-mentioned fact and we therefore obtain, in the next section, the condition for existence of 'fingers' and 'diffusive interfaces'.

3.2 Finger instability

Following Rudraiah and Malashetty [5], the condition for the onset of 'finger' instability can be written, for large Rayleigh numbers, in the form:

$$R^e_{c1} > 0;\ R^e_{c2} < 0;\ R^e_{c2} > \tau R^e_{c1} + \frac{\tau k^6 \eta}{(\pi a_c)^2} \qquad (17)$$

where a_c is the critical wave number obtained from Eq. (15). Expressing the third inequality in Eq. (17) in terms of physical Rayleigh numbers, we obtain

$$-R_{c2} > \tau R_{c1} + \frac{\tau k^6 \eta}{(\pi a_c)^2} - B \ . \qquad (18)$$

Now, for $R_{c1} > 0$, that is for the density gradient statically stable with respect to the faster diffusing component, $\alpha_1 C_{1,z}$ is negative. Inequality (18) thus takes the form

$$[\tau(\mathrm{SORET}) - 1] + \frac{1}{\tau}\frac{\alpha_2 C_{2,z}}{\alpha_1 C_{1,z}}[(\mathrm{DUFOUR}) - 1] > \frac{k^6\eta}{(\pi a_c)^2}\frac{1}{R_{c1}}[1 - (\mathrm{SORET})(\mathrm{DUFOUR})] \qquad (19)$$

where $\alpha_2 C_{2,z}/\alpha_1 C_{1,z}$ is equal to R_{c2}/R_{c1}. For $R_{c1} < 0$, that is $\alpha_1 C_{1,z} > 0$, the above inequality is reversed. The hydrostatic stability is assumed by $R_{c1} + R_{c2} > 0$, i.e. $\alpha_1 C_{1,z} + \alpha_2 C_{2,z} < 0$. For large R_{c1}, compared to $k^6\eta/(\pi a_c)^2$, the right hand side of (19) is negligible and so the condition for the formation of fingers in the presence of cross-diffusion terms is

$$\alpha_1 C_{1,z}[\tau(\mathrm{SORET}) - 1] + \alpha_2 C_{2,z}[(\mathrm{DUFOUR}) - 1] > 0 \qquad (20)$$

with $\alpha_1 C_{1,z} < 0$ and $\alpha_2 C_{2,z} > 0$. Equation (20) implies that the condition for the existence of fingers, at high Rayleigh numbers, is independent of the Darcy number and is the same as that obtained by McDougall [3] and Rudraiah and Malashetty [5].

The normal conditions (i.e. in the absence of cross-diffusion) for fingers to form is readily recovered from Eq. (20). In this case $\alpha_2 C_{2,z} > 0$, $\alpha_1 C_{1,z} < 0$ and the well-known condition is $\left|\alpha_2 C_{2,z}/\alpha_1 C_{1,z}\right| > \tau$. The presence of the cross-diffusion terms changes this criterion to

$$\left|\frac{\alpha_2 C_{2,z}}{\alpha_1 C_{1,z}}\right|(1 - \mathrm{DUFOUR}) > (1 - \mathrm{SORET}\ \tau), \quad \alpha_2 C_{2,z} > 0, \alpha_1 C_{1,z} < 0 \ . \qquad (20a)$$

From inequality (20a), it is clear that a positive D_{21} (which means that the flux of more slowly diffusing component C_2 is augmented by the C_1-gradient) encourages the formation of fingers, whereas a positive D_{12} discourages their appearance.

Now consider the case where the two horizontal boundaries are impervious to the C_2 property, but a constant difference of the C_1 property is maintained. We study the stability of the system when the vertical flux of C_2 is zero at all depths and so R_{c1} and R_{c2} are related by $R_{c2}/R_{c1} = -\alpha_2 D_{21}/\alpha_1 D_{22}$. Using this in (20), we obtain

$$\tau\ (\mathrm{SORET}) > \frac{\tau}{1+\tau}[1 + (\mathrm{SORET})(\mathrm{DUFOUR})] + \frac{k^6\eta}{(\pi a_c)^2 R_{c1}}[1 - (\mathrm{SORET})(\mathrm{DUFOUR})] \ . \qquad (22)$$

If both C_1 and C_2 are stably stratified, i.e. $\alpha_1 C_{1,z} < 0$ and $\alpha_2 C_{2,z} < 0$, then we see from Equation (20) that a necessary condition for the formation of fingers is either τ (SORET) > 1 or DUFOUR > 1.

3.3 'Diffusive' instability

By setting the growth rate q to be purely imaginary we obtain the conditions required for the 'diffusive' instability in the form:

$$R^e_{c1} < 0 \text{ and } R^e_{c2} > 0$$

$$-R^e_{c1} > \frac{\tau + \text{Pr}\ \eta}{1 + \text{Pr}\ \eta} R^e_{c2} + \frac{k^6 \eta}{(\pi a)^2} \frac{\tau + \text{Pr}\ \eta}{\text{Pr}(1 + \text{Pr}\ \eta)} (1 + \tau) \ . \quad (23)$$

In terms of physical Rayleigh numbers, the last inequality in (23) takes the form:

$$-R_{c1} > \frac{\tau + \text{Pr}\ \eta}{1 + \text{Pr}\ \eta} R_{c2} + \frac{\tau + \text{Pr}\ \eta}{1 + \text{Pr}\ \eta} \frac{\tau A - B}{1 - \tau} + \frac{B - A}{1 - \tau} + \frac{k^6 \eta}{(\pi a_c)^2} \frac{\tau + \text{Pr}\ \eta}{\text{Pr}(1 + \text{Pr}\ \eta)} (1 + \tau) \ . \quad (24)$$

For large R_{c1} compared to $k^6\eta/(\pi a_c)^2$ and when $\alpha_1 C_{1,z}$ and $\alpha_2 C_{2,z}$ are assigned independent values, we obtain the condition for the 'diffusive' instability in the form

$$\tau \alpha_1 C_{1,z} \left[\tau(\text{SORET}) + \frac{1 + \text{Pr}\ \eta}{\tau} \right] + \alpha_2 C_{2,z} [(\text{DUFOUR}) + \tau + \text{Pr}\ \eta] < 0 \ . \quad (25)$$

When C_1 and C_2 are stably stratified (i.e. $\alpha_1 C_{1,z} < 0$, $\alpha_2 C_{2,z} < 0$), Eq. (25) requires

$$\tau(\text{SORET}) < -\frac{1 + \text{Pr}\ \eta}{\tau}, \ \text{DUFOUR} < -(\tau + \text{Pr}\ \eta) \ . \quad (26)$$

Inequalities (26) show that for small values of the Darcy number large values of negative cross-diffusion coefficients are required to have 'diffusive' instabilities. In contrast to what was reported for 'finger' instability, we see from Eq. (25) that the Darcy number has a significant effect on the condition for the maintenance of 'diffusive' instability even when the Rayleigh number is high.

The linear theory discussed in the previous sections predicts only the condition for the onset of convection. It is silent about the possibility of subcritical motions and also about heat and mass transfers. These aspects are the realm of non-linear theory and are discussed in the subsequent sections.

4 Non-linear theory

The finite-amplitude analysis is carried out here via Fourier series representation for velocity and concentration distributions. Once a set of partial differential equations has been converted to a system of ordinary differential equations via a Fourier transformation it is logical to use the observed fact that laboratory and geophysical systems often exhibit flows dominated by a few spatial harmonics to truncate the system as far as possible. The primary advantage of creating these truncated spectral models is that their steady states and temporally periodic solutions can be obtained analytically in many cases. Although the relationship between the solutions of the governing partial differential equations and the corresponding severely truncated ordinary differential system has not been established, these low-order spectral models may reproduce qualitatively the convective phenomena observed in the full system. The result from such a simple analysis also serve as a starting value in solving a general non-linear convection problem (see [4]).

The first effect of non-linearity on free convection is to distort the concentrations fields through the interaction of ψ and C_1 and also ψ and C_2. The distortion of these fields will correspond to a change in the horizontal mean, i.e. a component of the form $\sin 2\pi z$ will be generated. Thus a truncated system which describes finite-amplitude free convection is given by (see [10]):

$$\psi = A(t) \sin \pi \alpha x \sin \pi z \quad (27)$$

$$C_1 = B(t) \cos \pi \alpha x \sin \pi z + C(t) \sin 2\pi z \quad (28)$$

$$C_2 = D(t) \cos \pi \alpha x \sin \pi z + E(t) \sin 2\pi z \quad (29)$$

where the amplitudes A, B, C, D and E are to be determined from the dynamics of the system. The stream function ψ does not contain an x-independent term because the spontaneous generation of large-scale flow has been discounted.

Substituting Eqs. (27)–(29) into Eqs. (8)–(10) and equating the coefficients of like terms we obtain the following non-linear autonomous system (Fifth order Lorenz model) of differential equations:

$$\dot{A} = -\text{Pr}\ k^2 \eta A + \frac{\pi a \text{Pr}}{k^2} [R_{c1} B + R_{c2} D] \quad (30)$$

$$\dot{B} = -k^2 \left[B + \text{DUFOUR}\ \frac{R_{c2}}{R_{c1}} D \right] - \pi a A - AC \quad (31)$$

$$\dot{C} = -4\pi^2 \left[C + \text{DUFOUR}\ \frac{R_{c2}}{R_{c1}} E \right] + \frac{AB}{2} \quad (32)$$

$$\dot{D} = -k^2 \left[\tau D + \text{SORET}\ \frac{R_{c1}}{R_{c2}} B \right] - \pi a E - AE \quad (33)$$

$$\dot{E} = -4\pi^2 \left[\tau E + \text{SORET}\ \frac{R_{c1}}{R_{c2}} C \right] + \frac{AD}{2} \quad (34)$$

It is now known that the solutions of Eqs. (30)–(34) are uniformly bounded in time and minic many properties of the full problem. This set of non-linear ordinary differential equations possesses an important symmetry for it is invariant under the transformation

$$(A, B, C, D, E) \to (-A, -B, -C, -D, -E) \ . \quad (35)$$

Also, the phase-space volume contracts at a uniform rate given by

$$\frac{\partial \dot{A}}{\partial A} + \frac{\partial \dot{B}}{\partial B} + \frac{\partial \dot{C}}{\partial C} + \frac{\partial \dot{D}}{\partial D} + \frac{\partial \dot{E}}{\partial E} = -\left[\text{Pr}\ \eta (1 + k^2) + 4\pi^2 + k^2 \tau + 4\pi^2 \tau \right] \quad (36)$$

which is always negative and therefore the system is bounded and dissipative. As a result, the trajectories are attracted to a set of measure zero in the phase space; in particular they may be attracted to a fixed point, a limit cycle or, perhaps, a strange attractor.

The non-linear system of autonomous differential equations is not amenable to analytical treatment for the general time-dependent variable and we have to solve it using a numerical method. In the case of steady motions, however, these equations can be solved analytically. Such solutions are very useful because they show that a finite-amplitude steady solution to the system is possible for sub-critical values of the Rayleigh number.

Steady solutions of Eqs. (30)–(34) are found by setting their left hand sides to zero. An equation for A, after eliminating B, C, D and E, is got in the form

$$\begin{aligned} Q^2 &+ \big[(1+\tau^2+2\,(\text{SORET})(\text{DUFOUR}) \\ &+ R^*_{c1}(1+(\text{SORET})) \\ &+ R^*_{c2}(\tau+(\text{DUFOUR}))\big]Q \\ &+ \text{DET}\big[\text{DET}+R^*_{c1}(\tau-(\text{SORET})) \\ &+ R^*_{c2}(1-(\text{DUFOUR}))\big] = 0 \end{aligned} \tag{37}$$

where $Q = A^2/8\pi^2k^2$, $R^*_{c1} = R_{c1}/R_c$, $R^*_c = R_{c2}/R_c$,

$$R_c = \frac{\pi^2(a_c^2+1)^3}{a_c^2} + \frac{\pi^2(a_c^2+1)^2}{a_c^2\text{Da}} \tag{38}$$

and DET $= \tau - (\text{DUFOUR})(\text{SORET})$.

For DUFOUR $= 0$ (i.e. $D_{12} = 0$), we have

$$\begin{aligned} Q^2 &+ \big[1+\tau^2+R^*_{c1}(1+\text{SORET})+\tau R^*_{c2}\big]Q \\ &+ \tau\big[\tau+R^*_{c1}(\tau-\text{SORET})+R^*_{c2}\big] = 0 \end{aligned} \tag{39}$$

$$Q = \frac{-L \pm \sqrt{(L^2-4M)}}{2} \tag{40}$$

where

$L = 1+\tau^2+R^*_{c1}(1+\text{SORET})+\tau R^*_{c2}$ and

$M = \tau\big[\tau+R^*_{c1}(\tau-\text{SORET})+R^*_{c2}\big]$.

Consider the case where finite-amplitude solutions exist for Rayleigh numbers less than that obtained from linear theory for a given R_{c2}, Pr, τ, Da and D_{21}. The minimum value of the Rayleigh number for which solutions exist is that value of R_{c1}, denoted by R_f, which makes the radical of (40) vanish. This condition leads to

$$\begin{aligned} &(1+\text{SORET})^2R_f^2 + \big[2\big[(1+\tau^2)R_c+\tau R_{c2}\big](1+\text{SORET}) \\ &\quad + 4\tau(\text{SORET}-\tau)R_c\big]R_f + \big[(1-\tau^2)R_c+\tau R_{c2}\big]^2 \\ &\quad - 4\tau R_cR_{c2} = 0 \end{aligned} \tag{41}$$

where R_c is given by (38).

From Eq. (41), we see that finding the finite amplitude Rayleigh number, with just the Soret effect alone, is a mathematically cumbersome procedure. Therefore, as in the case of linear theory we resort to drawing stability boundaries for non-linear case considering both Soret and Dufour effects.

For finite amplitude solutions to exist, we require that the two roots of (37) to be positive, so that the coefficient of Q in (37) must be negative and the constant term must be positive. If the magnitudes of both R^*_{c1} and R^*_{c2} are large, the condition for finite-amplitude convection to exist are DET > 0 and

$$R^*_{c1}[1+(\text{SORET})] + R^*_{c2}[\tau+(\text{DUFOUR})] < 0\ . \tag{42}$$

Equation (42) can now be compared with the conditions found from the linear stability theory (Eqs. (20) and (25)). In the case of C_1 and C_2 stably stratified, Eq. (42) shows that finite-amplitude instability requires

$$(\text{SORET}) < -\tau^{-2} \quad \text{or} \quad (\text{DUFOUR}) < -\tau \tag{43}$$

Conditions (43) are always less restrictive than condition (26). The negative sign of both quantities reflects the fact that a stable gradient of one property must induce a larger unstable gradient of the other property in order to produce finite-amplitude convection. One way of viewing the conditions (43) is to consider what happens if convection is able to completely mix up the fluid so that at some point in time, C_1, C_2 and $\vec{q}$ are all zero. As the gradients of C_1 and C_2 re-establish themselves due to diffusion from the perfectly conducting boundaries, the equation for the diffusion of density can be found by combining Eq. (5) with Eqs. (2) and (3) (where $\vec{q} = 0$) which is of the form

$$\frac{1}{\rho_0}\frac{\partial\rho}{\partial t} = (\alpha_1D_{11}+\alpha_2D_{21})\nabla^2T + (\alpha_1D_{12}+\alpha_2D_{22})\nabla^2S\ . \tag{44}$$

This equation can potentially correspond to the negative diffusion of density (and hence to instability and further convection), if either of the bracketed coefficients on the right of (43) are negative, i.e. if conditions (43) are satisfied.

The stability boundaries implied by equations (20), (25) and (42) may be compared by plotting them for fixed R^*_{c1}/R^*_{c2}, subject to the static stability constraint $R^*_{c1}+R^*_{c2} > 0$. The three stability boundaries are parallel lines through the points $(1,1)$, $(-[\text{Pr}\ \eta+\tau], -[\text{Pr}\ \eta+1]/\tau)$ and $(-\tau, -\tau^{-1})$ which have slope equal to $R_{c2}/\tau R_{c1}$. These are shown in Figs. (2)–(4). These stability boundaries show

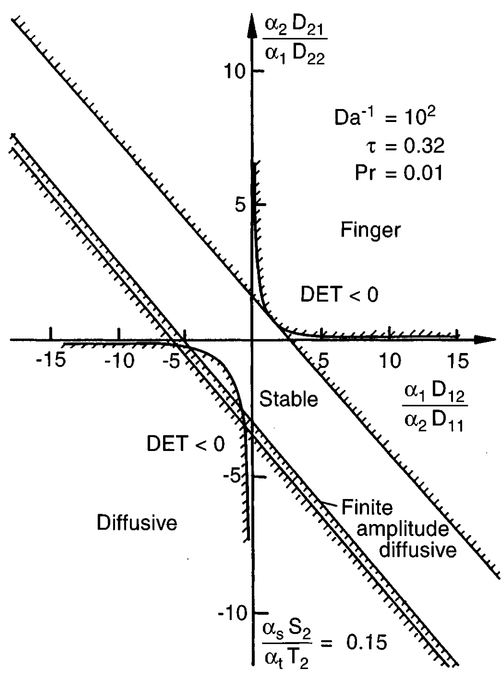

Fig. 2. Stability boundaries as a function of $\alpha_2D_{21}/\alpha_1D_{22}$ and $\alpha_1D_{12}/\alpha_2D_{11}$ for different ranges of R_{c1} and R_{c2}

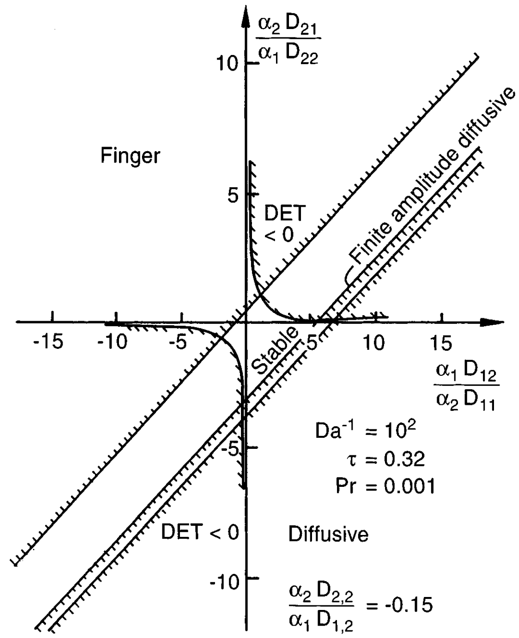

Fig. 3. For text see Fig. 2

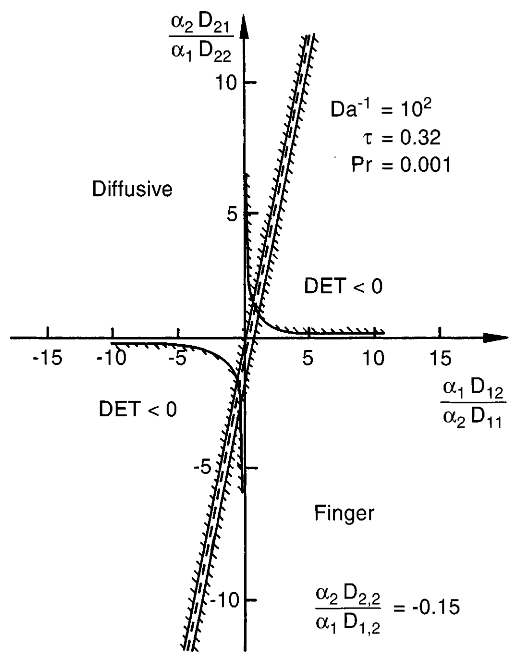

Fig. 4. For text see Fig. 2

that the two types of double-diffusive instability never occur together even though both types of instability can occur in concentration gradients that are normally conducive to the other type of instability. In addition, these show that the region of finite-amplitude instability always encloses the region of infinitesimal oscillatory diffusive instability. Furthermore, 'finger' or 'diffusive' instability may occur even when both components have stabilizing effects.

5
Heat/mass transports

Heat/mass transport in a double-diffusive system with cross-diffusion in a porous medium depend on the imposed temperature/concentration difference, on the cross diffusion coefficients and on the Darcy number. This is more specifically expressed as a functional relation between the Nusselt number Nu_i and the Rayleigh number R_{ci}. The Nusselt number is the ratio of total mass (heat) transport to the rate at which mass (heat) is transported by diffusion (conduction) alone. It can be expressed as

$$\mathrm{Nu}_i = \frac{-D_{11}\frac{\partial C_i}{\partial z}/_{z=0}}{D_{11}\frac{\Delta C_i}{d}} \quad . \tag{45}$$

When DUFOUR $=0$, Nusselt number, Nu_1, for the component C_1 is

$$\mathrm{Nu}_1 = 1 + \frac{2\pi}{R_{c1}} C_1 \quad . \tag{46}$$

Similarly the Nusselt number, Nu_2, for the component C_2, when SORET $=0$, is

$$\mathrm{Nu}_2 = 1 + \frac{2\pi}{R_{c2}} E \tag{47}$$

where the amplitudes are given by Eqs. (30)–(34). The two cases, one DUFOUR $=0$ and the second SORET $=0$ are of particular interest because the behaviour of the Rayleigh number with respect to these parameters reflects on the Nusselt number. We now consider these two cases separately.

For DUFOUR $=0$ (SORET $\neq 0$), from Eq. (42) we get

$$(-R^*_{c1})_{\text{critical}} > R^*_{c2}/(1+\ \text{SORET})(R^*_{c1} \text{ and } R^*_{c2} \gg 1) \quad . \tag{48}$$

For SORET $=0$ (DUFOUR $\neq 0$), we get from (42)

Table 1. Illustration of the effects of SORET and DUFOUR on the critical Rayleigh number

DUFOUR $=0$		SORET $=0$	
SORET no. Range	Critical Rayleigh number $(-R_{c1})$	DUFOUR no. Range	Critical Rayleigh number $(-R^*_{c1})$
>0	decreased	<0 and $>-\tau$	decreased
<0 and >-1	increased	>0	increased
	Remarks		Remarks
SORET <-1	Convection can occur for stable gradients of R^*_{c1} and R^*_{c2} (Fig. 6)	DUFOUR $<-\tau$	Convection can occur for stable gradients of R^*_{c1} and R^*_{c2} (Fig. 6)

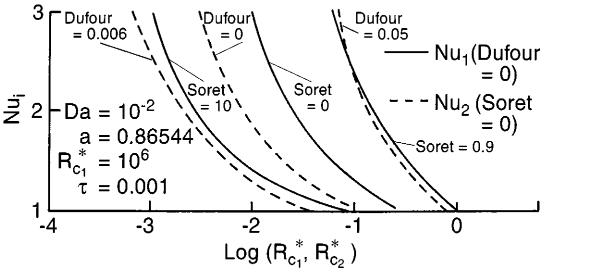

Fig. 5. Nusselt number versus R^*_{c1}/R^*_{c2} for various values of SORET and DUFOUR co-efficients

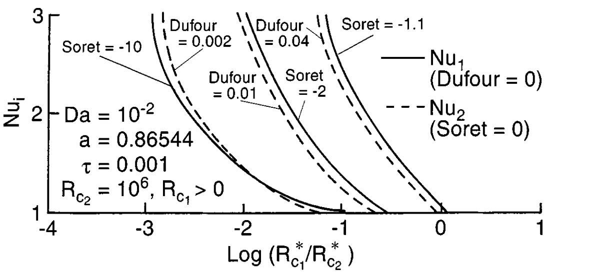

Fig. 6. For text see Fig. 5

$$(-R^*_{c1})_{\text{critical}} > R^*_{c2}/(\tau + \text{ DUFOUR})$$
$$(R^*_{c1} \text{ and } R^*_{c2} \gg 1) \quad . \tag{49}$$

The results of Eqs. (48) and (49) have been summarized in Table 1. The results of mass transports are depicted in Figs. 5 and 6.

6
Discussions and conclusions

The linear stability of the problem has been studied with the help of the usual normal mode analysis and the following conclusions are drawn:

1. Finger and diffusive instabilities never occur together.
2. Choosing the cross-diffusion coefficients of appropriate sign and magnitude it is possible to have finger and diffusive instabilities even when both solutes make stabilizing contributions to the net density gradient.
3. The effect of permeability (through the Darcy number) is to reduce the region of instability. This is because of the fact that large cross-diffusion coefficients are required to cause instability in a porous medium.
4. The condition for the maintenance of diffusive instability is dependent on the Darcy number in contrast to that for fingers.
5. The critical wave number for stationary instability is independent of the cross-diffusion coefficients but depends on the Darcy number.

The non-linear analysis has been performed with the help of truncated representation of Fourier series and the following conclusions are drawn:

1. Finite amplitude convection may occur even when both the properties have stabilizing gradients. In particular, if SORET < -1 or DUFOUR $< -\tau$, finite amplitude convection may occur for stabilizing gradients of the two properties.
2. The region of finite amplitude instability always encloses the region of infinitesimal oscillatory instability.
3. Sub-critical motions are possible (see Figs. 2 to 4).
4. In the absence of DUFOUR effect, for SORET > 0 the critical Rayleigh number is lowered while for SORET < 0 it is increased. Similarly, in the absence of the SORET effect when DUFOUR $< -\tau$ the system is unstable to all destabilizing thermal gradients and will be unstable unless a large enough stabilizing thermal gradient is applied. These findings are clearly reflected in the heat/mass transport curves (see Figs. 5 and 6).
5. The effect of permeability is to decrease the heat/mass transports.
6. The stability boundaries for infinitesimal finger and diffusive instabilities and finite amplitude instability are parallel lines with slope equal to R_{c1}/R_{c2}.
7. Due to lack of accurate values for the cross-diffusion coefficients it is advantageous to resort to stability boundaries rather than find critical Rayleigh numbers for marginal, oscillatory and sub-critical instabilities.

References

1. **Brand, H.R.; Steinberg, V.:** Convective instabilities in binary mixtures in a porous medium. Physica 119A (1983) 327
2. **Brand, H.R.; Steinberg, V.:** Nonlinear effects in the convective instability of a binary mixture in a porous medium near threshold. Physics letters 93A (1983) 333
3. **McDougall, T.J.:** Double-diffusive convection caused by coupled molecular diffusion. J F M 126 (1982) 379
4. **Rudraiah, N.:** Linear and nonlinear magnetoconvection in a porous medium. Proc Ind Acad Sci (Math. Sci.) 93 Nos. 2 and 3 (1984) 117
5. **Rudraiah, N.; Malashetty, M.S.:** The influence of coupled molecular diffusion on double-diffusive convection in a porous medium. J Heat Trans (ASME) 108 (1986) 872
6. **Rudraiah, N.; Patil, P.R.:** Thermal diffusion and convective stability of two component fluid in a porous medium. Proc. V Int. Heat transfer conference, Tokyo, Japan (1974) 79
7. **Rudraiah, N.; Patil, P.R.:** Linear convective stability and thermal diffusion of a horizontal quiescent layer of a two-component fluid in a porous medium. Int J Engg Sci 18 (1980) 1055
8. **Rudraiah, N.; Shivakumara, I.S.; Friedrich, R.:** The effect of rotation on linear and non-linear double-diffusive convection in a sparsely packed porous medium. Int J Heat Mass Trans 29 (1986) 1301
9. **Taslim, M.E.; Narusawa, U.:** Binary fluid convection and double diffusive convection in a porous medium. J Heat Trans (ASME) 108 (1985) 221
10. **Veronis, G.:** On finite amplitude instability in a rotating fluid. J F M 5 (1965) 401