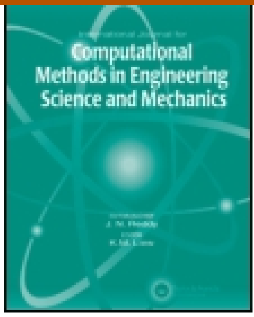

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Gauss Legendre Quadrature Formulae for Tetrahedra

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In this paper we consider the Gauss Legendre quadrature method for numerical integration over the standard tetrahedron: $\{(x, y, z) \mid 0 \leq x, y, z \leq 1, x + y + z \leq 1\}$ in the Cartesian three-dimensional (x, y, z) space. The mathematical transformation from the (x, y, z) space to (ξ, η, ζ) space is described to map the standard tetrahedron in (x, y, z) space to a standard 2-cube: $\{(\xi, \eta, \zeta) \mid -1 \leq \xi, \eta, \zeta \leq 1\}$ in the (ξ, η, ζ) space. This overcomes the difficulties associated with the derivation of new weight co-efficients and sampling points. The effectiveness of the formulae is demonstrated by applying them to the integration of three nonpolynomial and three polynomial functions.

Keywords Finite Element Method, Numerical Integration, Gauss Legendre Quadrature, Tetrahedral Finite Elements, Standard 2-Cube

INTRODUCTION

The integration theory is extended from real line to the plane and three-dimensional space by the introduction of multiple integrals. Integration is of fundamental importance in both pure and applied mathematics as well as in several areas of science and engineering. Most such integrals cannot be evaluated explicitly or analytically and, with many others, it is often faster to integrate them numerically rather than evaluate them exactly using the complicated antiderivatives of the integrands. However, it is not practical to obtain specific integration formulae for all regions of interest. Hence it is desirable to obtain integration formulae over two-simplex, the unit right angled triangle and the three-simplex, the unit orthogonal tetrahedron that may be used in principle to approximate the plane or three-dimensional region of any physical space.

In recent years, we have been witnessing finite element method (FEM) gaining importance due to the most obvious reason that it can provide solutions to many complicated problems that would be intractable by other numerical techniques [1, 2]. In FEM it may be possible to perform some of the integrations analytically, particularly if constant or linear elements are used to discretize the surface or boundary curve of the given region. However, with higher order elements or for more complex distorted elements the integrals become too complicated for analytical integrations and the numerical integration is essential, among various integration schemes. Gauss Legendre quadrature, which can evaluate exactly the (2n−1)th order polynomial with n Gaussian points, is most commonly used in view of the accuracy and efficiency of calculations [3]. The triangular and tetrahedral elements are very widely used in finite element analysis. The versatility of these elements can be further enhanced by improved numerical integration schemes. Mathematically the problem can be defined as the evaluation of the following integrals:

$$II = \int_0^1 \int_0^{1-L_1} F(L_1, L_2, L_3)\, dL_2 dL_1 \qquad [1]$$

where L_1, L_2, L_3 are the well known area co-ordinates and

$$III = \int_0^1 \int_0^{1-L_1} \int_0^{1-L_1-L_2} G(L_1, L_2, L_3, L_4)\, dL_3 dL_2 dL_1 \qquad [2]$$

Where L_1, L_2, L_3, L_4 are the well known volume co-ordinates.

The basic problem of integrating an arbitrary function of two variables over the surface of the triangle was first given by Hammer, Marlowe and Stroud [4], and Hammer and Stroud

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[5, 6]. Cowper [7] provided a table of Gaussian quadrature formulae with symmetrically placed integration points. Lyness and Jespersen [8] made an elaborate study of symmetric quadrature rules by formulating the problem in polar co-ordinates. Lannoy [9] discussed the symmetric four-point integration formula, which is presented in [7]. Laurie [10] derived a seven-point integration rule and discussed the numerical error in integrating some functions. Laursen and Gellert [11] gave a table of symmetric integration formulae up to a precision of degree ten. Lether and Hillion [12, 13] derived the formulae for triangles as product of one-dimensional Gauss Legendre and Gauss Jacobi quadrature rules. The precision of these formulae is up to degree seven. This is because the zeros and weight co-efficients of Gauss Jacobi orthogonal polynomials with weight functions x, x^2, x^3 were available for polynomials of degree up to six only. Even today the zeros and weights for integral $\int_0^1 x^r f(x)\ dx$, $r = 1, 2, 3, \ldots\ldots$ are not available beyond a formula of order-eight as documented in Abramowicz and Stegun [14]. Reddy [15] and Reddy and Shippy [16] derived three-point, four-point, six-point, seven-point of precision 3, 4, 6 and 7 respectively, which gave improved accuracy. Since the precision of all the formulae derived by the authors [4–16] is limited to a precision of degree ten and it is not likely that the techniques can be extended much further to give a greater accuracy, which may be demanded in future, Lague and Baldur [17] proposed product formulae based only on the roots and weight co-efficients of Gauss Legendre quadrature rules. By the proposed method, this restriction is removed and one can now obtain numerical integration of very high degree of precision as the derivations now rely on standard Gauss Legendre quadarature rules [17]. However, Lague and Baldur [17] have not worked out explicit weight co-efficients and sampling points for application to integrals over a triangular surface [17]. Rathod et al. provided this information in a systematic manner in their recent works [18–20]. For tetrahedral regions, four volume coordinates L_1, L_2, L_3, L_4 are involved and we have to compute numerically the integral *III* stated in Eq. (2). Numerical integration formulae of *III* with a degree of precision $d = 1, 2, 3$ are listed in Zienkiewicz [1] and these are based on reference [4]. Numerical integration formulae of precision higher than cubic are not available in the current literature and hence we propose here the derivation of higher order formulae for tetrahedral regions.

1. FORMULATION OF INTEGRALS OVER A TETRAHEDRON

The finite element method for three-dimensional problems with tetrahedron element requires the numerical integration of shape functions and their derivatives on a tetrahedron. Since an affine transformation makes it possible to transform any tetrahedron into the three-dimensional tetrahedron T with coordinates $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in Cartesian three-dimensional space, say (x, y, z), we thus have to consider numerical integration on T. The numerical integration of an arbitrary function f over the tetrahedron T is given by

$$I = \int_0^1 dy \int_0^{1-y} dx \int_0^{1-x-y} f(x, y, z)dz \qquad [3]$$

It is now required to find the value of the integral by a quadrature formula:

$$I = \sum_{m=1}^{N} c_m f(x_m, y_m, z_m) \qquad [4]$$

where c_m are the weights associated with the sampling points (x_m, y_m, z_m) and N is the number of pivotal points related to the required precision.

The integral I of Eq. (3) can be transformed into an integral over the cube: $\{(u, v, w) \mid 0 \le u, v, w \le 1\}$ by the substitution

$$x = uvw, \quad y = uv(1-w), \quad z = u(1-v) \qquad [5]$$

Then the determinant of the Jacobian and the differential area are:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = -u^2 v \quad \text{and}$$

$$dxdydz = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw = u^2 v\ dudvdw \qquad [6]$$

Then on using Eqs. (5) and (6) in Eq. (3), we have

$$\begin{aligned} I &= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} f(x, y, z)dz \right) dy \right) dx \\ &= \int_0^1 \int_0^1 \int_0^1 f(uvw, uv(1-w), u(1-v)) \times u^2 v\ dwdvdu \end{aligned} \qquad [7]$$

The integral I of Eq. (7) can be further transformed into an integral over the standard 2-cube: $\{(\xi, \eta, \zeta) \mid -1 \le \xi, \eta, \zeta \le 1\}$ by the substitution

$$u = \frac{(1+\xi)}{2}, \quad v = \frac{(1+\eta)}{2}, \quad w = \frac{(1+\zeta)}{2} \qquad [8]$$

Then clearly the determinant of the Jacobian and the differential area are:

$$\frac{\partial(u, v, w)}{\partial(\xi, \eta, \zeta)} = \frac{1}{8} \quad \text{and } dudvdw = \frac{\partial(u, v, w)}{\partial(\xi, \eta, \zeta)} d\xi d\eta d\zeta = \frac{1}{8} d\xi d\eta d\zeta \qquad [9]$$

Now on using Eqs. (8) and (9) in Eq. (7), we have

$$I = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} f(x, y, z) dz \right) dy \right) dx$$
$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f\left(\frac{(1+\xi)(1+\eta)(1+\zeta)}{8}, \frac{(1+\xi)(1+\eta)(1-\zeta)}{8}, \frac{(1+\xi)(1-\eta)}{4} \right) \times \frac{(1+\xi)^2(1+\eta)}{64} d\xi d\eta d\zeta \quad [10]$$

Equation (10) represents an integral over the standard 2-cube: $\{(\xi, \eta, \zeta) \mid -1 \leq \xi, \eta, \zeta \leq 1\}$. Efficient quadrature co-efficients are readily available in the literature so that any desired accuracy can be readily obtained [14].

From Eqs. (4) and (10), we find that

$$I = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 f\left(\frac{(1+\xi)(1+\eta)(1+\zeta)}{8}, \frac{(1+\xi)(1+\eta)(1-\zeta)}{8}, \frac{(1+\xi)(1-\eta)}{4} \right) \times \frac{(1+\xi)^2(1+\eta)}{64} d\xi d\eta d\zeta$$
$$= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} \frac{\left(1+\xi_i^{(\alpha)}\right)^2 \left(1+\eta_j^{(\beta)}\right)}{64} \mathcal{W}_i^{(\alpha)} \mathcal{W}_j^{(\beta)} \mathcal{W}_k^{(\gamma)} \times f\left(\frac{\left(1+\xi_i^{(\alpha)}\right)\left(1+\eta_j^{(\beta)}\right)\left(1+\zeta_k^{(\gamma)}\right)}{8}, \times \frac{\left(1+\xi_i^{(\alpha)}\right)\left(1+\eta_j^{(\beta)}\right)\left(1-\zeta_k^{(\gamma)}\right)}{8}, \frac{\left(1+\xi_i^{(\alpha)}\right)\left(1-\eta_j^{(\beta)}\right)}{4} \right)$$
$$= \sum_{m=1}^{N=(\alpha \times \beta \times \gamma)} c_m f(x_m, y_m, z_m) \quad [11]$$

where, it is obvious that

$$c_m = \frac{\left(1+\xi_i^{(\alpha)}\right)^2 \left(1+\eta_j^{(\beta)}\right)}{64} \mathcal{W}_i^{(\alpha)} \mathcal{W}_j^{(\beta)} \mathcal{W}_k^{(\gamma)},$$
$$x_m = \frac{\left(1+\xi_i^{(\alpha)}\right)\left(1+\eta_j^{(\beta)}\right)\left(1+\zeta_k^{(\gamma)}\right)}{8},$$
$$y_m = \frac{\left(1+\xi_i^{(\alpha)}\right)\left(1+\eta_j^{(\beta)}\right)\left(1-\zeta_k^{(\gamma)}\right)}{8},$$
$$z_m = \frac{\left(1+\xi_i^{(\alpha)}\right)\left(1-\eta_j^{(\beta)}\right)}{4} \quad [12]$$

in which $\xi_i^{(\alpha)}, \eta_j^{(\beta)}, \zeta_k^{(\gamma)}$ are the sampling points and $\mathcal{W}_i^{(\alpha)}, \mathcal{W}_j^{(\beta)}, \mathcal{W}_k^{(\gamma)}$ are the corresponding weight co-efficients of Gauss Legendre quadrature of order α, β, γ respectively. In Eq. (10), let us now assume that $f(x, y, z) = x^p y^q z^r$, then we obtain

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^p y^q z^r dz dy dx$$
$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left(\frac{(1+\xi)(1+\eta)(1+\zeta)}{8} \right)^p \times \left(\frac{(1+\xi)(1+\eta)(1-\zeta)}{8} \right)^q \left(\frac{(1+\xi)(1-\eta)}{4} \right)^r \left(\frac{(1+\xi)^2(1+\eta)}{64} \right) d\xi d\eta d\zeta$$
$$= \int_{-1}^1 \left(\sum_{l=0}^{p+q+r+2} A_l \xi^l \right) d\xi \int_{-1}^1 \left(\sum_{m=0}^{q+r+1} B_m \eta^m \right) \times d\eta \int_{-1}^1 \left(\sum_{p=0}^{r} C_p \zeta^p \right) d\zeta \quad [13]$$

From Eq. (13), we further infer that appropriate Gauss Legendre quadrature rules have to be applied in variates α, β and γ to obtain the desired accuracy.

1.1. Product Formulae with Odd Degree of Precision

Let us now first assume that the degree of precision $d = p + q + r = 2n - 1, n = 1, 2, 3.........$; then we find from Eq.(13) that I will take values from 0 to $2n + 1$ and m will take values from 0 to $2n$ and p take values from 0 to $2n - 1$. This suggests that we have to choose $(n+1)(n+1)(n)$th order Gauss Legendre quadrature in α, β and γ directions, respectively. This is demonstrated for the monomial X^2Y over the arbitrary linear tetrahedron in XYZ-space. It is seen that in this case $3 \times 3 \times 2$th order Guass Legendre quadrature rule is adequate to integrate the monomial X^2Y.

1.2. Product Formulae with Even Degree of Precision

Let us now first assume that the degree of precision $d = p + q + r = 2n - 2, n = 1, 2, 3.........$; then we find from Eq. (13) that I will take values from 0 to $2n$ and m will take values from 0 to $2n - 1$ and p take values from 0 to $2n - 2$. This suggests that we have to choose Gauss Legendre quadrature of order $(n + 1)(n)(n)$ in variates α, β and γ, respectively. This is demonstrated for the monomials X^2Y^2 and X^4Y^4 over the arbitrary linear tetrahedron in XYZ-space. It is seen that in this case $4 \times 3 \times 3$th and $6 \times 5 \times 5$th order Gauss Legendre quadrature rules are adequate to integrate the monomials X^2Y^2 and X^4Y^4.

We confirm this for the above examples that is for monomials X^2Y, X^2Y^2 and X^4Y^4 over the arbitrary linear tetrahedron in XYZ-space and it is seen that $3 \times 3 \times 3$th, $4 \times 4 \times 4$th and $6 \times 6 \times 6$th order Gauss Legendre quadrature rules are highly suitable. We note from the above discussion on the product formulae for odd and even degree that we require $(n+1)(n+1)(n)$th order or $(n+1)(n)(n)$th order Gauss Legendre quadrature rules. But in either case one can always obtain the desired accuracy by using $(n+1)(n+1)(n+1)$th order rule. We hope that this serves the general purpose.

TABLE 1
Numerical results for integrals of example 1

Order of integration $(m \times m \times m)$	I_1	I_2	I_3
$(2 \times 2 \times 2)$	0.143229713697729	0.199386992166663	0.341460942304256
$(3 \times 3 \times 3)$	0.142876998237370	0.199906205971895	0.388804992651775
$(4 \times 4 \times 4)$	0.142859954536681	0.199975192505890	0.408895274066160
$(5 \times 5 \times 5)$	0.142857772149151	0.199991224470968	0.419255490092258
$(6 \times 6 \times 6)$	0.142857328526509	0.199996269763999	0.425279305858844
Exact result	0.142857142857143	0.200000000000000	0.440686793509772

We have appended a sample of the derived formulae in Appendix A ($2 \times 2 \times 2, 3 \times 3 \times 3$ and $4 \times 4 \times 4$) and Appendix B ($3 \times 3 \times 2$ and $4 \times 3 \times 3$) for immediate reference.

2. SOME NUMERICAL RESULTS

We consider some typical integrals with known exact values:

Example 1: Let us consider the following multiple integrals which are generalized to three-dimensions from Reddy and Shippy. [16]

$$I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{(x+y+z)}dzdydx = 0.142857142857143 \quad [14]$$

$$I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dzdydx}{\sqrt{(x+y+z)}} = 0.200000000000000 \quad [15]$$

$$I_3 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} [(1-x-y)^2 + z^2]^{\frac{-1}{2}} dzdydx = 0.440686793509772 \quad [16]$$

Example 2: We now consider the following multiple integrals of the type considered in Rathod and Govinda Rao [19].

$$III_v^{\alpha\beta\gamma} = \int\int_v\int X^\alpha Y^\beta Z^\gamma \, dXdYdZ$$

where v is the tetrahedron in (X, Y, Z) space with vertices spanning the points $\langle(5, 5, 0), (10, 10, 0), (8, 7, 8), (10, 5, 0)\rangle$.

On using the following transformations

$$X(x, y, z) = 10 - 5x - 2z, \; Y(x, y, z) = 5 + 5y + 2z, \; Z(x, y, z) = 8z \quad [17]$$

we obtain,

$$III_v^{\alpha\beta\gamma} = \int\int_v\int X^\alpha Y^\beta Z^\gamma \, dXdYdZ = 200 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (10-5x-2z)^\alpha \times (5+5y+2z)^\beta (8z)^\gamma \, dxdydz \quad [18]$$

We have evaluated the above integral for $\alpha = 2, \beta = 1, \gamma = 0; \alpha = 2, \beta = 2, \gamma = 0 \& \alpha = 4, \beta = 4, \gamma = 0$.

That is

$$I_4 = III_v^{2,1,0} = \int\int_v\int X^2 Y \, dXdYdZ = 15721.6666666667 \quad [19]$$

$$I_5 = III_v^{2,2,0} = \int\int_v\int X^2 Y^2 dXdYdZ = 109662.063492063 \quad [20]$$

TABLE 2
Numerical results for integrals of example 2

Order of integration $(m \times n \times p)$	I_4	I_5	I_6
$(2 \times 2 \times 2)$	15738.5352088246	109782.342130943	421208011.702195
$(3 \times 3 \times 2)$	15721.6666666667	109644.831944445	423167160.599982
$(3 \times 3 \times 3)$	15721.6666666667	109661.325000000	426894926.913375
$(4 \times 3 \times 3)$	15721.6666666667	109662.063492064	426891578.868533
$(4 \times 4 \times 4)$	15721.6666666656	109662.063492063	426917342.818093
$(5 \times 5 \times 5)$	15721.6666666666	109662.063492063	426917356.551760
$(6 \times 6 \times 6)$	15721.6666666667	109662.063492064	426917356.623377
Exact result	15721.6666666667	109662.063492063	426917356.623377

$$I_6 = III_v^{4,4,0} = \int\int_v\int X^4Y^4 dXdYdZ$$
$$= 426917356.623377 \qquad [21]$$

Again from Rathod and Govinda Rao [19, 20], we know that $I_4 = 47165/3$, other integrals were computed in a similar way. We tabulate the computed values of I_1, I_2 and I_3 of example 1 and I_4, I_5 and I_6 of example 2 in Tables 1 and 2, respectively.

3. CONCLUSIONS

In this paper, we have derived numerical integration rules of order $N = \alpha \times \beta \times \gamma$ based on classical Gauss Legendre quadrature. We have shown how these formulae can be applied to the arbitrary tetrahedral regions, since an affine transformation makes it possible to transform an arbitrary linear tetrahedron in 3-space (X, Y, Z) into an orthogonal tetrahedron T: $\{(x, y, z) \mid 0 \le x, y, z \le 1, x+y+z \le 1\}$ in the 3-space (x, y, z). The derivation of the proposed formulae over the orthogonal tetrahedron T is made possible by transforming the tetrahedral region T into a standard 2-cube: $\{(\xi, \eta, \zeta) \mid -1 \le \xi, \eta, \zeta \le 1\}$ over the 2-cube, the Gauss Legendre quadrature rules of all orders is applicable. It may be noted that a lot of mathematical effort is needed to derive numerical integration rules over the tetrahedral region T and the integration formulae available at this moment in the literature are confined to a precision of cubic order. By the proposed method this restriction is removed and one can now obtain numerical integration rules of very high degree of precision as the derivations proposed here rely on the standard Gauss Legendre quadrature rules. We have also suggested that when the sum of indices $p+q+r$ in the monomial $x^p y^q z^r$ is odd, say equal to $2n-1$, then the lowest order rule to be chosen is $(n+1)(n+1)(n)$ and if $p+q+r$ is even, say equal to $2n-2$ then the lowest order rule is $(n+1)(n)(n)$. The effectiveness of the derived formulae is further demonstrated by applying them to three nonpolynomial and three polynomial functions over the tetrahedral region in 3-space.

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APPENDIX A

Gauss Legendre Quadrature formulae of order ($m \times m \times m$)

x_i	y_i	z_i	c_i
	A-1 ($m = 2$)		
0.009437387888358	0.035220811090087	0.166666666666667	0.001179673492382
0.035220811090087	0.009437387888358	0.166666666666667	0.001179673492382
0.035220811090087	0.131445856471988	0.044658198978444	0.004402601409914
0.131445856471988	0.035220811090087	0.044658198978444	0.004402601409914
0.035220810850163	0.131445855576580	0.622008467032738	0.016430731923420
0.131445855576580	0.035220810850163	0.622008467032738	0.016430731923420
0.131445855576580	0.490562611456158	0.166666666666667	0.061320326343747
0.490562611456158	0.131445855576580	0.166666666666667	0.061320326343747
	A-2 ($m = 3$)		
0.001431498841332	0.011270166537926	0.100000000000000	$3.068198819728420 \times 10^{-5}$
0.011270166537926	0.001431498841332	0.100000000000000	$3.068198819728420 \times 10^{-5}$
0.006350832689629	0.006350832689629	0.100000000000000	$4.909118111565470 \times 10^{-5}$
0.006350832689629	0.050000000000000	0.056350832689629	$2.177926162424280 \times 10^{-4}$
0.050000000000000	0.006350832689629	0.056350832689629	$2.177926162424280 \times 10^{-4}$
0.028175416344815	0.028175416344815	0.056350832689629	$3.484681859878840 \times 10^{-4}$
0.011270166537926	0.088729833462074	0.012701665379258	$2.415587821057510 \times 10^{-4}$
0.088729833462074	0.011270166537926	0.012701665379258	$2.415587821057510 \times 10^{-4}$
0.050000000000000	0.050000000000000	0.012701665379258	$3.864940513692010 \times 10^{-4}$
0.006350832689629	0.050000000000000	0.443649167310371	$9.662351284230000 \times 10^{-4}$
0.050000000000000	0.006350832689629	0.443649167310371	$9.662351284230000 \times 10^{-4}$
0.028175416344815	0.028175416344815	0.443649167310371	0.001545976205477
0.028175416344815	0.221824583655185	0.250000000000000	0.006858710562414
0.221824583655185	0.028175416344815	0.250000000000000	0.006858710562414
0.125000000000000	0.125000000000000	0.250000000000000	0.010973936899863
0.050000000000000	0.393649167310371	0.056350832689629	0.007607153074595
0.393649167310371	0.050000000000000	0.056350832689629	0.007607153074595
0.221824583655185	0.221824583655185	0.056350832689629	0.012171444919352
0.011270166537926	0.088729833462074	0.787298334620741	0.001901788268649
0.088729833462074	0.011270166537926	0.787298334620741	0.001901788268649
0.050000000000000	0.050000000000000	0.787298334620741	0.003042861229838
0.050000000000000	0.393649167310371	0.443649167310371	0.013499628508586
0.393649167310371	0.050000000000000	0.443649167310371	0.013499628508586
0.221824583655185	0.221824583655185	0.443649167310371	0.021599405613738
0.088729833462074	0.698568501158667	0.100000000000000	0.014972747367084
0.698568501158667	0.088729833462074	0.100000000000000	0.014972747367084
0.393649167310371	0.393649167310371	0.100000000000000	0.023956395787334
	A-3 ($m = 4$)		
0.000334715714594	0.004486065274832	0.064611063213548	$1.761084870822600 \times 10^{-6}$
0.004486065274832	0.000334715714594	0.064611063213548	$1.761084870822600 \times 10^{-6}$
0.001590903418873	0.003229877570553	0.064611063213548	$3.301615549885100 \times 10^{-6}$
0.003229877570553	0.001590903418873	0.064611063213548	$3.301615549885100 \times 10^{-6}$
0.001590903418873	0.021322263257539	0.046518677526561	$1.569257503335800 \times 10^{-5}$
0.021322263257539	0.001590903418873	0.046518677526561	$1.569257503335800 \times 10^{-5}$
0.007561562178966	0.015351604497447	0.046518677526561	$2.941984830275260 \times 10^{-5}$
0.015351604497447	0.007561562178966	0.046518677526561	$2.941984830275260 \times 10^{-5}$
0.003229877570553	0.043288799956008	0.022913166676413	$3.185931686560010 \times 10^{-5}$
0.043288799956008	0.003229877570553	0.022913166676413	$3.185931686560010 \times 10^{-5}$
0.015351604497447	0.031167073029114	0.022913166676413	$5.972864665122530 \times 10^{-5}$
0.031167073029114	0.015351604497447	0.022913166676413	$5.972864665122530 \times 10^{-5}$

APPENDIX A

Gauss Legendre Quadrature formulae of order $(m \times m \times m)$

x_i	y_i	z_i	c_i
0.004486065274832	0.060124997938716	0.004820780989426	$2.360313944207820 \times 10^{-5}$
0.060124997938716	0.004486065274832	0.004820780989426	$2.360313944207820 \times 10^{-5}$
0.021322263257539	0.043288799956008	0.004820780989426	$4.425027634907300 \times 10^{-5}$
0.043288799956008	0.021322263257539	0.004820780989426	$4.425027634907300 \times 10^{-5}$
0.001590903418873	0.021322263257539	0.307096311531159	$7.458679166511140 \times 10^{-5}$
0.021322263257539	0.001590903418873	0.307096311531159	$7.458679166511140 \times 10^{-5}$
0.007561562178966	0.015351604497447	0.307096311531159	$1.398325062338110 \times 10^{-4}$
0.015351604497447	0.007561562178966	0.307096311531159	$1.398325062338110 \times 10^{-4}$
0.007561562178966	0.101344693527868	0.221103222500738	$6.646237464725280 \times 10^{-4}$
0.101344693527868	0.007561562178966	0.221103222500738	$6.646237464725280 \times 10^{-4}$
0.035940096619353	0.072966159087481	0.221103222500738	0.001246011553748
0.072966159087481	0.035940096619353	0.221103222500738	0.001246011553748
0.015351604497447	0.205751618003291	0.108906255706834	0.001349329762022
0.205751618003291	0.015351604497447	0.108906255706834	0.001349329762022
0.072966159087481	0.148137063413257	0.108906255706834	0.002529672588768
0.148137063413257	0.072966159087481	0.108906255706834	0.002529672588768
0.021322263257539	0.285774048273620	0.022913166676413	$9.996579230088670 \times 10^{-4}$
0.285774048273620	0.021322263257539	0.022913166676413	$9.996579230088670 \times 10^{-4}$
0.101344693527868	0.205751618003291	0.022913166676413	0.001874121002261
0.205751618003291	0.101344693527868	0.022913166676413	0.001874121002261
0.003229877570553	0.043288799956008	0.623471844265867	$3.074301219528830 \times 10^{-4}$
0.043288799956008	0.003229877570553	0.623471844265867	$3.074301219528830 \times 10^{-4}$
0.015351604497447	0.031167073029114	0.623471844265867	$5.763584072291740 \times 10^{-4}$
0.031167073029114	0.015351604497447	0.623471844265867	$5.763584072291740 \times 10^{-4}$
0.015351604497447	0.205751618003291	0.448887299291690	0.002739430867978
0.205751618003291	0.015351604497447	0.448887299291690	0.002739430867978
0.072966159087481	0.148137063413257	0.448887299291690	0.005135781756688
0.148137063413257	0.072966159087481	0.448887299291690	0.005135781756688
0.031167073029114	0.417720226262576	0.221103222500738	0.005561636370626
0.417720226262576	0.031167073029114	0.221103222500738	0.005561636370626
0.148137063413257	0.300750235878433	0.221103222500738	0.010426746279122
0.300750235878433	0.148137063413257	0.221103222500738	0.010426746279122
0.043288799956008	0.580183044309859	0.046518677526561	0.004120367029080
0.580183044309859	0.043288799956008	0.046518677526561	0.004120367029080
0.205751618003291	0.417720226262576	0.046518677526561	0.007724708831375
0.417720226262576	0.205751618003291	0.046518677526561	0.007724708831375
0.004486065274832	0.060124997938716	0.865957092583479	$3.163437496694650 \times 10^{-4}$
0.060124997938716	0.004486065274832	0.865957092583479	$3.163437496694650 \times 10^{-4}$
0.021322263257539	0.043288799956008	0.865957092583479	$5.930693405649470 \times 10^{-4}$
0.043288799956008	0.021322263257539	0.865957092583479	$5.930693405649470 \times 10^{-4}$
0.021322263257539	0.285774048273620	0.623471844265867	0.002818857915521
0.285774048273620	0.021322263257539	0.623471844265867	0.002818857915521
0.101344693527868	0.205751618003291	0.623471844265867	0.005284688592238
0.205751618003291	0.101344693527868	0.623471844265867	0.005284688592238
0.043288799956008	0.580183044309859	0.307096311531159	0.005722890433136
0.580183044309859	0.043288799956008	0.307096311531159	0.005722890433136
0.205751618003291	0.417720226262576	0.307096311531159	0.010729059318706
0.417720226262576	0.205751618003291	0.307096311531159	0.010729059318706
0.060124997938716	0.805832094644763	0.064611063213548	0.004239832934111
0.805832094644763	0.060124997938716	0.064611063213548	0.004239832934111
0.285774048273620	0.580183044309859	0.064611063213548	0.007948679008092
0.580183044309859	0.285774048273620	0.064611063213548	0.007948679008092

APPENDIX B(Continued)

Gauss Legendre Quadrature formulae of order $3 \times 3 \times 2$ and $4 \times 3 \times 3$

x_i	y_i	z_i	c_i
	B-1 $(3 \times 3 \times 2)$		
0.002684177726694	0.010017487652565	0.100000000000000	$5.522757875511150 \times 10^{-5}$
0.010017487652565	0.002684177726694	0.100000000000000	$5.522757875511150 \times 10^{-5}$
0.011908332133606	0.044442500556023	0.056350832689629	$3.920267092363690 \times 10^{-4}$
0.044442500556023	0.011908332133606	0.056350832689629	$3.920267092363690 \times 10^{-4}$
0.021132486540519	0.078867513459481	0.012701665379258	$4.348058077903510 \times 10^{-4}$
0.078867513459481	0.021132486540519	0.012701665379258	$4.348058077903510 \times 10^{-4}$
0.011908332133606	0.044442500556023	0.443649167310371	0.001739223231161
0.044442500556023	0.011908332133606	0.443649167310371	0.001739223231161
0.052831216351297	0.197168783648703	0.250000000000000	0.012345679012346
0.197168783648703	0.052831216351297	0.250000000000000	0.012345679012346
0.093754100568987	0.349895066741383	0.056350832689629	0.013692875534271
0.349895066741383	0.093754100568987	0.056350832689629	0.013692875534271
0.021132486540519	0.078867513459481	0.787298334620741	0.003423218883568
0.078867513459481	0.021132486540519	0.787298334620741	0.003423218883568
0.093754100568987	0.349895066741383	0.443649167310371	0.024299331315455
0.349895066741383	0.093754100568987	0.443649167310371	0.024299331315455
0.166375714597456	0.620922620023285	0.100000000000000	0.026950945260751
0.620922620023285	0.166375714597456	0.100000000000000	0.026950945260751
	B-2 $(4 \times 3 \times 3)$		
0.000881900051731	0.006943184420297	0.061606759730945	$7.291397835302120 \times 10^{-6}$
0.006943184420297	0.000881900051731	0.061606759730945	$7.291397835302120 \times 10^{-6}$
0.003912542236014	0.003912542236014	0.061606759730945	$1.166623653648340 \times 10^{-5}$
0.003912542236014	0.030803379865473	0.034715922101487	$5.175716125056670 \times 10^{-5}$
0.030803379865473	0.003912542236014	0.034715922101487	$5.175716125056670 \times 10^{-5}$
0.017357961050743	0.017357961050743	0.034715922101487	$8.281145800090660 \times 10^{-5}$
0.006943184420297	0.054663575310648	0.007825084472028	$5.740505372790620 \times 10^{-5}$
0.054663575310648	0.006943184420297	0.007825084472028	$5.740505372790620 \times 10^{-5}$
0.030803379865473	0.030803379865473	0.007825084472028	$9.184808596464990 \times 10^{-5}$
0.004191669964176	0.033000947820757	0.292816860422638	0.000308810767896
0.033000947820757	0.004191669964176	0.292816860422638	0.000308810767896
0.018596308892467	0.018596308892467	0.292816860422638	0.000494097228634
0.018596308892467	0.146408430211319	0.165004739103786	0.002192058240538
0.146408430211319	0.018596308892467	0.165004739103786	0.002192058240538
0.082502369551893	0.082502369551893	0.165004739103786	0.003507293184861
0.033000947820757	0.259815912601881	0.037192617784934	0.002431262032776
0.259815912601881	0.033000947820757	0.037192617784934	0.002431262032776
0.146408430211319	0.146408430211319	0.037192617784934	0.003890019252442
0.008509995415082	0.066999052179243	0.594481474198103	0.001272849118661
0.066999052179243	0.008509995415082	0.594481474198103	0.001272849118661
0.037754523797163	0.037754523797163	0.594481474198103	0.002036558589858
0.037754523797163	0.297240737099051	0.334995260896214	0.009035175225692
0.297240737099051	0.037754523797163	0.334995260896214	0.009035175225692
0.167497630448107	0.167497630448107	0.334995260896214	0.014456280361107
0.066999052179243	0.527482422018860	0.075509047594325	0.010021119913454
0.527482422018860	0.066999052179243	0.075509047594325	0.010021119913454
0.297240737099051	0.297240737099051	0.075509047594325	0.016033791861526
0.011819765327527	0.093056815579703	0.825691574889796	0.001309754100876
0.093056815579703	0.011819765327527	0.825691574889796	0.001309754100876
0.052438290453615	0.052438290453615	0.825691574889796	0.002095606561402
0.052438290453615	0.412845787444898	0.465284077898513	0.009297141059763
0.412845787444898	0.052438290453615	0.465284077898513	0.009297141059763
0.232642038949257	0.232642038949257	0.465284077898513	0.014875425695620
0.093056815579703	0.732634759310093	0.104876580907230	0.010311672223827
0.732634759310093	0.093056815579703	0.104876580907230	0.010311672223827
0.412845787444898	0.412845787444898	0.104876580907230	0.016498675558123