Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 47, 2327 - 2333

On Weakly Symmetric and Weakly Conformally Symmetric Spaces admitting Veblen identities

Y. B. Maralabhavi

Department of Mathematics
Bangalore University
Bangalore, India
ybmbub@yahoo.co.in

Hari Baskar R.

Department of Mathematics
Christ University
Bangalore, India

Shivaprasanna G. S.

Department of Mathematics
Amruta Institute of Engineering and Management Sciences
Bangalore, India

Abstract

In the present paper some properties involving curvature tensor, conformal curvature tensor, Ricci tensor and scalar curvature, on weakly symmetric, weakly conformally symmetric and pseudo symmetric spaces are obtained.

Mathematics Subject Classification: 53A40

Keywords: Weakly Symmetric, Weakly Conformally symmetric

1 Introduction

L.Tamassy and T.Q.Binh [3] have introduced the notion of weakly symmetric and weakly projective symmetric spaces. Based on this work, U.C.De and S.Bandyopadhyaa [7] introduced the notion of weakly conformally symmetric spaces and investigated some properties of such spaces. We consider these spaces admitting Veblen identities and conformal Veblen identities [4] and determine some other properties.

Let M_n be a Riemannian n-dimensional space covered by system of coordinate neighbourhoods (U, x_i). Suppose g_{ij}, R_{hijk} and R_{ij} denote the local components of the metric tensor, the curvature tensor and the Ricci tensor, respectively, and let R denote the scalar curvature. The non-flat Riemannian space M_n $(n > 2)$ is called weakly symmetric space if the curvature tensor R_{hijk} satisfies the condition [3]

$$R_{hijk,l} = a_1 R_{hijk} + b_h R_{lijk} + d_i R_{hljk} + e_j R_{hilk} + f_k R_{hijl} \ , \tag{1.1}$$

where a, b, d, e, f are 1-forms (non-zero simultaneously) and the ',' denotes the covariant differentiation with respect to the metric tensor of the space. An n-dimensional weakly symmetric space M_n is denoted by $(WS)_n$. Such spaces are studied by M.Pranovic [5], T.Q.Binh [6] and others. U.C.De and S.Bandyopadhyay [8] proved that the associated 1-forms d and f in (1.1) are identical with b and e, respectively. Hence the condition (1.1) of $(WS)_n$ becomes

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + b_i R_{hljk} + e_j R_{hilk} + e_k R_{hijl} \ . \tag{1.2}$$

Further, the space M_n is called pseudo symmetric if

$$R_{hijk,l} = 2a_l R_{hijk} + a_h R_{lijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hijl} \ .$$

An n-dimensional non conformally flat Riemannian space M_n $(n > 3)$ is called weakly conformally symmetric if its conformal curvature tensor C_{hijk}, given by

$$C_{hijk} = R_{hijk} + \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \tag{1.3}$$

satisfies the condition

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lijk} + d_i C_{hljk} + e_j C_{hilk} + f_k C_{hijl} \tag{1.4}$$

where a, b, d, e, f are associated 1-forms (non zero simultaneously). A weakly conformally symmetric space M_n is denoted by $(WCS)_n$. As in case of $(WS)_n$ it is proved that d and f are identical with b and e respectively. So, (1.4) reduces to

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lijk} + b_i C_{hljk} + e_j C_{hilk} + e_k C_{hijl} \tag{1.5}$$

for weakly conformally symmetric spaces. The space M_n is called pseudo conformally symmetric if

$$C_{hijk,l} = 2a_l C_{hijk} + a_h C_{lijk} + a_i C_{hljk} + a_j C_{hilk} + a_k C_{hijl} \ .$$

The conformal curvature tensor satisfies the conditions:

$$C^h_{ijk} + C^h_{jki} + C^h_{kij} = 0 \ , \tag{1.6}$$

$$C^r_{rjk} = C^r_{irk} = C^r_{ijr} = 0 \ , \tag{1.7}$$

and

$$C_{hijk} = -C_{hikj} = C_{ihkj} = C_{kjih} \ . \tag{1.8}$$

A space is said to be quasi conformally flat if

$$C'_{hijk} = 0, \tag{1.9}$$

where

$$C'_{hijk} = aZ_{hijk} + b\left(g_{hk}G_{ij} - g_{jk}G_{ik} + g_{ij}G_{hk} - g_{ik}G_{hj}\right), \tag{1.10}$$

with a, b as arbitrary constants and

$$Z_{hijk} = K_{hijk} - \frac{R}{n\,(n-1)}\left(g_{hk}g_{ij} - g_{ki}g_{hj}\right), \qquad G_{ij} = R_{ij} - \frac{R}{n}g_{ij}.$$

The Veblen identities and conformal Veblen identities in M_n are given by [4]:

$$V^h_{ijkl} = R^h_{ijk,l} + R^h_{kil,j} + R^h_{lkj,i} + R^h_{jli,k} = 0 \tag{1.11}$$

and

$$\begin{aligned} W^h_{ijkl} = {} & C^h_{ijk,l} + C^h_{kil,j} + C^h_{lkj,i} + C^h_{jli,k} \\ & - \frac{1}{(n-3)} g^{hm}\{(g_{jm}C^p_{kil,p} + g_{km}C^p_{jli,p} + g_{im}C^p_{lkj,p} + g_{lm}C^p_{ijk,p}) \\ & - (g_{ik}C^p_{mlj,p} + g_{ij}C^p_{mkl,p} + g_{kl}C^p_{mji,p} + g_{lj}C^p_{mil,p})\} = 0 \ . \end{aligned} \tag{1.12}$$

If the Ricci tensor satisfies the condition

$$R_{ij} = \frac{R}{n}g_{ij} \ , \tag{1.13}$$

then M_n is called Einstein space.

2 Weakly symmetric space $(WS)_n$.

By using (1.2) in the Bianchi identity,

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0 \ , \tag{2.1}$$

we get

$$\beta_l R_{hijk} + \beta_j R_{hikl} + \beta_k R_{hilj} = 0 \ , \tag{2.2}$$

where we have put

$$\beta_l = a_l - 2e_l \tag{2.3}$$

and used the relations

$$R_{hijk} + R_{hjki} + R_{hkij} = 0 \tag{2.4}$$

and

$$R_{hijk} = -R_{ihjk} = R_{ihkj} \tag{2.5}$$

satisfied by the curvature tensor field. By transvecting (2.2) by g^{lh} and using (2.5), we get

$$\beta_h R^h_{ijk} = \beta_k R_{ji} - \beta_j R_{ki}. \tag{2.6}$$

Now transvection of (2.6) with g^{ij} gives

$$\beta_h R^h_k = \frac{\beta_k R}{2}. \tag{2.7}$$

If M_n is an Einstein space, then (2.7) reduces to

$$(n-2)\,\beta_k R = 0. \tag{2.8}$$

Now write the Veblen identity (1.11) in the form

$$R_{hijk,l} + R_{hkil,j} + R_{hlkj,i} + R_{hjli,k} = 0 \tag{2.9}$$

and use (1.2) to get

$$\alpha_i R_{hlkj} + \alpha_j R_{hkil} + \alpha_k R_{hjli} + \alpha_l R_{hijk} = 0, \tag{2.10}$$

where we have put

$$\alpha_i = a_i - (b_i + e_i)\,, \tag{2.11}$$

and used the results (2.4) and (2.5) . Transvecting (2.10) by $g^{kh}$and using (2.5), we get

$$\alpha_h R^h_{jli} = \alpha_i R_{lj} - \alpha_l R_{ij}. \tag{2.12}$$

Now by transvecting (2.12) with g^{il} and using that M_n is an Einstein space,we get

$$(n-2)\,\alpha_i R = 0. \tag{2.13}$$

In view of (2.8) and (2.13) we have

Theorem 2.1 *The scalar curvature of weakly symmetric Einstein Riemannian space M_n is zero provided the 1-form a in (1.2) is neither $2e$ nor $b+e$*

Remark 2.2 *As per G.Herglotz [1] the scalar curvature of Einstein space M_n is constant. But that constant is necessarily zero if M_n is weakly symmetric in which $a \neq 2e$, $b+e$ for the 1-forms a, b , e used in (1.2).*

Suppose $R \neq 0$ in M_n, then from (2.8) and (2.13) we see that $\beta_i = 0$ and $\alpha_i = 0$, which in turn indicate $a = 2e$,$a = b + e$ and hence $b = e$. Hence M_n reduces to pseudosymmetric. So, we state the following:

Theorem 2.3 *A weakly symmetric Einstein space with non zero scalar curvature is pseudo symmetric.*

3 Weakly conformally symmetric space $(WCS)_n$.

By using (1.5) in (1.12), we get

$$a_l C_{hijk} + b_h C_{lijk} + b_i C_{hljk} + e_j C_{hilk} + e_k C_{hijl}$$

$$+ a_j C_{hkil} + b_h C_{jkil} + b_k C_{hjil} + e_i C_{hkjl} + e_l C_{hkij}$$

$$+ a_i C_{hlkj} + b_h C_{ilkj} + b_l C_{hikj} + e_k C_{hlij} + e_j C_{hlki}$$

$$+ a_k C_{hjli} + b_h C_{kjli} + b_j C_{hkli} + e_l C_{hjki} + e_i C_{hjlk}$$

$$- \left(\frac{a_p + b_p}{n-3}\right) [\{ g_{lh} C^p_{ijk} + g_{jh} C^p_{kil} + g_{ih} C^p_{lkj} + g_{kh} C^p_{jli} \}$$

$$-\{g_{kl}C^p_{hji}+g_{lj}C^p_{hik}+g_{ji}C^p_{hkl}+g_{ik}C^p_{hlj}\}]=0\ .$$

In view of (1.6), (1.7) and (1.8) above result reduces to

$$\alpha_l C_{hijk}+\alpha_j C_{hkil}+\alpha_i C_{hlkj}+\alpha_k C_{hjli}-\left(\frac{a_p+b_p}{n-3}\right)[\{g_{lh}C^p_{ijk}+g_{jh}C^p_{kil} +g_{ih}C^p_{lkj}+g_{kh}C^p_{jli}\}-\{g_{kl}C^p_{hji}+g_{lj}C^p_{hik}+g_{ij}C^p_{hkl}+g_{ik}C^p_{hlj}\}]=0\ , \tag{3.1}$$

where $\alpha_l=a_l-(b_l+e_l)$. Contracting (3.1) by g^{hi} and using (1.6), (1.7) and (1.8), we get

$$\lambda_p C^p_{lkj}=0,\ \text{where}\ \lambda_p=2b_p+e_p\ . \tag{3.2}$$

Transvecting (3.1) by λ^l and using (1.7), (1.8) and (3.2), we get

$$\left(\lambda^l\alpha_l\right)C_{hijk}=\left(\frac{a_l+b_l}{n-3}\right)\left[\lambda_h C^l_{ijk}+\lambda_k C^l_{hij}+\lambda_j C^l_{hki}\right]\ . \tag{3.3}$$

Transvecting (3.3) with λ^h and using (3.2), we get

$$\left(\lambda^h\lambda_h\right)\left(\frac{a_l+b_l}{n-3}\right)C^l_{ijk}=0$$

and hence

$$\left(a_l+b_l\right)C^l_{ijk}=0\ . \tag{3.4}$$

Now the equation (3.3), in view of (3.4), reduces to

$$\left(\lambda^l\alpha_l\right)C_{hijk}=0.$$

So we state the following:

Theorem 3.1 *A weakly conformally symmetric space* $(WCS)_n$, $(n>3)$, *is conformally flat if the 1-forms* a,b,c *satisfy the condition* $\lambda^l\alpha_l\neq 0$, *where* $\lambda_l=2b_l+e_l$ *and* $\alpha_l=a_l-(b_l+e_l)$.

Suppose $\lambda^l\alpha_l=0$. Then $C_{hijk}\neq 0$. Now, by using

Proposition A:[2] A quasi-conformally flat space is either conformally flat or Einstein.

We state the following:

Theorem 3.2 *A quasi conformally flat weakly conformally symmetric space* $(WCS)_n$, $(n>3)$ *with* $\lambda^l\alpha_l=0$ *is Einstein space.*

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Received: June, 2011