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On Pseudo Projective Curvature Tensor in Sasakian Manifolds

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Abstract. We consider pseudo projective curvature tensor in Sasakian manifolds. We show that every pseudo projectively flat and pseudo projective semi symmetric Sasakian manifolds are locally isomorphic to unit sphere. Next we show that in a Sasakian space time manifold with conservative pseudo projective curvature tensor the integral curves of the characteristic vector field are geodesics.

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1. Introduction

In 2002 B.Prasad [3] defined and studied a tensor field $\overline{P}$ on a Riemannian manifold of dimension $n > 2$ which includes the projective curvature tensor P. This tensor field $\overline{P}$ is known as Pseudo projective curvature tensor. In [1, 2, 11], the authors C.S.Bagewadi, Venkatesha, D.G.Prakasha and others have extended this notion to Kenmotsu and LP-Sasakian manifolds and obtained the conditions for these manifolds to be of Einstein, η- Einstein and pseudo projectively flat. Further J.P.Jaiswal and R.H.Ojha studied weakly

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pseudo projective symmetric and ricci-symmetric manifolds and investigated the nature of scalar curvature in these manifolds. Motivated by the above studies, in this paper we study pseudo projective curvature tensor in Sasakian manifolds. Section 2 contains preliminaries. In section 3 we prove Sasakian manifolds with condition $R \cdot \bar{P} = 0$ and we prove pseudo projectively flat Sasakian manifolds are locally isometric to unit sphere. Then we prove that in a Sasakian space time manifold with conservative pseudo projective curvature tensor, the integral curves of the characteristic vector field are geodesics.In section 4, we consider pseudo projective ricci tensor and pseudo projective ricci flat spaces.

2. Prelimanaries

A $2n+1$ dimensional Riemannian manifold (M^n, g) is said to be an almost contact metric manifold if there exist on M^n, a $(1,1)$ tensor field φ, a vector field ξ and a 1-form η such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X and Y on M. An almost contact metric structure of M is a contact metric manifold if

$$d\eta(X,Y) = g(X, \varphi Y) \tag{2.3}$$

The almost contact metric structure of M is said to be normal if $[\varphi, \varphi](X,Y) = -2d\eta(X,Y)\xi$ for any X,Y, where $[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$.

A normal contact metric manifold is called a Sasakian manifold. It is well known that an almost contact metric manifold is Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.4}$$

for any X, Y. In a Sasakian manifold the following relations hold.

$$\varphi\xi = 0 \tag{2.5}$$

$$\eta \circ \varphi = 0 \tag{2.6}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.7}$$

$$g(X,\xi) = \eta(X) \tag{2.8}$$

$$\nabla_X \xi = -\varphi X \tag{2.9}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.10}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X \tag{2.11}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.12}$$

3. Pseudo projective curvture tensor in Sasakian manifolds

Let (M^n, g) be an n-dimensional Sasakian manifold. The pseudo projective curvature tensor is defined as follows [3]:

(3.1)

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y]$$

Pseudo projectively flat Sasakian manifolds

If (M^n, g) is pseudo-projectively flat, then (3.1) assumes the form

(3.2)

$$R(X,Y)Z = \frac{b}{a}[S(X,Z)Y - S(Y,Z)X] - \frac{r}{na}\left(\frac{a}{n-1} + b\right)[g(X,Z)Y - g(Y,Z)X]$$

Contracting the above equation with U, we obtain

$$\begin{aligned} {}'R(X,Y,Z,U) &= \frac{b}{a}\left[S(X,Z)g(Y,U) - S(Y,Z)g(X,U)\right] \\ &-\frac{r}{na}(\frac{a}{n-1} + b)\left[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)\right] \end{aligned} \tag{3.3}$$

where ${}'R(X,Y,Z,U) = g(R(X,Y)Z,U)$.
Setting $U = \xi$ in (3.3), we obtain

$$\begin{aligned} {}'R(X,Y,Z,\xi) &= \frac{b}{a}[S(X,Z)\eta(Y) - S(Y,Z)\eta(X)] \\ &-\frac{r}{na}\left(\frac{a}{n-1} + b\right)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \end{aligned} \tag{3.4}$$

Putting $Y = \xi$ in (3.4), we obtain

(3.5)

$$'R(X,\xi,\xi,Z) = \frac{b}{a}[S(X,Z) - (n-1)\eta(Z)\eta(X)] - \frac{r}{na}\left(\frac{a}{n-1} + b\right)[g(X,Z) - \eta(Z)\eta(X)]$$

In view of (2.11), the equation (3.5) assumes the form

$$S(X,Z) = Ag(X,Z) + B\eta(X)\eta(Z), \tag{3.6}$$

where

$$A = \frac{r}{nb}\left((\frac{a}{n-1} + b) - \frac{a}{b}\right) \tag{3.7}$$

$$B = \left((n-1) - \frac{r}{bn}(\frac{a}{n-1} + b) + \frac{a}{b}\right) \tag{3.8}$$

This shows that (M^n, g) is η-Einstein. Let $\{e_i\}, 1 \leq i \leq n$ be an orthonormal basis of the tangent space at any point. Contracting the equation (3.6), we obtain

$$r = nA + B \tag{3.9}$$

By virtue of (3.6), (3.7) and (3.9), we get

$$r = n(n-1)$$

Substituting this in (3.7) and (3.9) and using (3.6), we obtain

$$S(X,Z) = (n-1)g(X,Z) \tag{3.10}$$

i.e. (M^n, g) is an Einstein manifold.
From (3.3) and (3.10), we obtain

$$'R(X,Y,Z,U) = [g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \tag{3.11}$$

i.e. (M^n, g) is locally isometric with unit sphere $S^n(1)$. Conversely suppose the space (M^n, g) is locally isometric with $S^n(1)$, then (M^n, g) becomes pseudo projectively flat. Thus we can state that

Theorem 3.1. *A Sasakian manifold is Pseudo projectively flat if and only if it is locally isometric to unit sphere $S^n(1)$.*

Sasakian manifolds satisfying $R(X,Y).\overline{P}=0$

If (M^n,g) satisfies $R(X,Y).\overline{P}=0$, then we have

(3.12)

$$0=R(X,Y)\overline{P}(U,V)W-\overline{P}(R(X,Y)U,V)W-\overline{P}(U,R(X,Y)V)W-\overline{P}(U,V)R(X,Y)W$$

Putting $X=\xi$ in (3.12), we obtain

(3.13)

$$0=R(\xi,Y)\overline{P}(U,V)W-\overline{P}(R(\xi,Y)U,V)W-\overline{P}(U,R(\xi,Y)V)W-\overline{P}(U,V)R(\xi,Y)W$$

In view of (2.11), the above equation takes the form

(3.14)

$$\begin{aligned}0=g(\overline{P}(U,V)W,Y)-\eta(Y)\eta(\overline{P}(U,V)W)-g(U,Y)\eta(\overline{P}(\xi,V)W)+\eta(U)\eta(\overline{P}(Y,V)W)\\ -g(V,Y)\eta(\overline{P}(U,\xi)W)+\eta(V)\eta(\overline{P}(U,Y)W)+\eta(W)\eta(\overline{P}(U,V)Y)\end{aligned}$$

Setting $Y=U$ in the above equation we obtain

(3.15)

$$0=g(\overline{P}(U,V)W,U)-g(U,U)\eta(\overline{P}(\xi,V)W)+-g(V,U)\eta(\overline{P}(U,\xi)W)+\eta(W)\eta(\overline{P}(U,V)U)$$

Let $\{e_i\},1\le i\le n$ be an orthonormal basis of the tangent space at any point. Taking $U=e_i$ and summing over $1\le i\le n$, in view of (3.1),(2.11) and (2.12) we obtain

$$S(Y,Z)=(n-1)g(Y,Z)+\frac{r-n(n-1)b}{a}\eta(Y)\eta(Z) \tag{3.16}$$

Contracting the above equation we obtain

$$r=n(n-1) \tag{3.17}$$

Plugging this in (3.16), we obtain

$$S(Y,Z)=(n-1)g(Y,Z) \tag{3.18}$$

i.e.M^n is an Einstein manifold. Now from equations (3.1), (2.11), (2.12) and (3.14) we obtain

$$'R(X,Y,Z,U)=[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)]$$

Thus we have proved that a Sasakian manifold (M^n,g) is locally isometric to unit sphere $S^n(1)$ even if the space (M^n,g) satisfies the weaker conditon $R\cdot\overline{P}=0$. Since a pseudo projectively flat space is always pseudo projective semi-symmetric, the converse follows from theorem 3.1. Thus we have

Theorem 3.2. *A Sasakian manifold is locally isometric to unit sphere $S^n(1)$ if and only if $R.\overline{P}=0$ holds.*

Conservative pseudo projective curvature tensor

Suppose (M^n, g) is an n-dimensional Sasakian space-time manifold with the Einstein field equations without cosmological constant.Then we have

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y) \tag{3.19}$$

where $\alpha=(kp+\frac{1}{2}r), \beta=k(\sigma+p)$ and k is the gravitational constant, σ is the energy density, p is the pressure and r is the scalar curvature.

Taking the covariant derivative of (3.1), we get

$$\begin{aligned}(\nabla_W\overline{P})(X,Y)Z &= a(\nabla_W R)(X,Y)Z+b[(\nabla_W S)(Y,Z)X-(\nabla_W S)(X,Z)Y]\\ &\quad -\frac{dr(W)}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{3.20}$$

Contracting with respect to W, and from
$divR=(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)$, we obtain

$$\begin{aligned}(div\overline{P})(X,Y)Z &= (a+b)[(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)]\\ &\quad -\frac{1}{n}\left(\frac{a}{n-1}+b\right)[dr(X)g(Y,Z)-dr(Y)g(X,Z)]\end{aligned} \tag{3.21}$$

If the pseudo projective curvature tensor is conservative i.e. $div\overline{P}=0$, then equation (3.21) becomes

$$(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)=\frac{1}{n(a+b)}\left(\frac{a}{n-1}+b\right)[dr(X)g(Y,Z)-dr(Y)g(X,Z)] \tag{3.22}$$

In view of (3.19), (3.21) assumes the form

$$\begin{aligned}(\alpha-\frac{1}{2})(dr(X)g(Y,Z)-dr(Y)g(X,Z)) &= \beta\,[((\nabla_X\eta)Y-(\nabla_Y\eta)X)\eta(Z)+\\ &\quad (\eta(Y)(\nabla_X\eta)Z-\eta(X)(\nabla_Y\eta)Z)]\end{aligned} \tag{3.23}$$

Taking $Y=Z=e_i$ and summing over $1\le i\le n$,

$$(n-1)(\alpha-\frac{1}{2})dr(X)=\beta\left[2(\nabla_X\eta)\xi-(\nabla_\xi\eta)X-\delta\eta\eta(X)\right], \tag{3.24}$$

where $\delta\eta = (\nabla_{e_i}\eta)e_i$. Putting $Y = Z = \xi$ in (3.23), we obtain

(3.25)
$$(\alpha - \frac{1}{2})(dr(X) - dr(\xi)\eta(X)) = \beta\,[(2(\nabla_X\eta)\xi - (\nabla_\xi\eta)X - \eta(X)(\nabla_\xi\eta)\xi)]$$

From (3.24) and (3.25), we obtain

$$(\alpha - \frac{1}{2})((n-2)dr(X) - dr(\xi)\eta(X)) = \beta\,[-\delta\eta\;\eta(X)] \tag{3.26}$$

Setting $X = \xi$ in (3.26), we obtain

$$(\alpha - \frac{1}{2})(n-1)dr(\xi) = -\beta(\delta\eta) \tag{3.27}$$

From (3.26) and (3.27), we obtain

$$dr(X) = dr(\xi)\eta(X), n > 2, \alpha \neq \frac{1}{2} \tag{3.28}$$

Using (3.28) in (3.23) with $Z = \xi$, we obtain

$$\beta\,[(\nabla_X\eta)Y - (\nabla_Y\eta)X] = 0 \tag{3.29}$$

Since $\beta = k(\sigma + p) \neq 0$, we must have $(\nabla_X\eta)Y - (\nabla_Y\eta)X = 0$.
This implies $g(X, (\nabla_\xi\xi)) = 0 \;\; \forall X$.
Hence $(\nabla_\xi\xi) = 0$.
i.e. ξ is a parallel vector field or that the integral curves of ξ are geodesics.
Thus we can state that

Theorem 3.3. *In a Sasakian space time manifold with conservative pseudo projective curvature tensor, the integral curves of the characteristic vector field are geodesics.*

Suppose $(divR)(X,Y)Z = 0$.
Then from (3.21), $(div\overline{P})(X,Y)Z = 0$ if and only if $dr(X) = 0, \forall X$ or r is a constant. It means that a divergence free Sasakian manifold (M^n, g) is of constant curvature if and only if the pseudo projective curvature tensor is conservative. Thus from the above theorem it follows that

Corollary 3.1. *In a divergence free Sasakian space form, the integral curves of the characteristic vector field are geodesics.*

4. Pseudo projective Ricci tensor and pseudo projective Ricci flat spaces

From the pseudo projective tensor defined as in 3.1, a symmetric tensor of type (0,2) can be obtained as follows:

$$Ric\overline{P}(X,Y) =' P(X,e_i,e_i,Y) \tag{4.1}$$

where $'\overline{P}(X,Y,Z,U) = g(\overline{P}(X,Y)Z,U)$, $\{e_i\}$ is an orthonormal basis of the tangent space at any point and summing over $1 \leq i \leq n$ in (4.1),from (3.1), we have

(4.2)
$$'\overline{P}(X,Y,Z,U) = a\ \ 'R(X,Y,Z,U) + b\,[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)] - \frac{r}{na}(\frac{a}{n-1} + b)\,[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$$

Putting $Y = Z = e_i$ in (4.2) and summing over $1 \leq i \leq n$,we obtain

$$Ric\overline{P}(X,Y) = (a-b)S(X,Y) + \frac{b-a}{n}rg(X,Y) \tag{4.3}$$

From (4.3) it follows that

$S(X,Y) = \frac{r}{n}g(X,Y)$ if and only if $Ric'P(X,Y) = 0$.
i.e. Einstein space is characterised by the vanishing of pseudo projective ricci tensor. Thus the pseudo projective tensor in a Sasakian manifold M^n is a measure of the failure of M^n to be Einstein.
We call (M^n, g) as Ricci pseudo projectively flat if $Ric'P(X,Y)$ vanishes.
It is clear that a pseudo projectively flat space is always Ricci pseudo projectively flat and a Ricci pseudo projectively flat is Einstein.
Suppose (M^n, g) is pseudo projective ricci semi-symmetric.
Then $R.Ric\overline{P} = 0$ or $Ric\overline{P}(R(X,Y)Z,W) + Ric\overline{P}(Z,R(X,Y)W) = 0$.
In view of (4.3), the above equation yields $S(R(X,Y)Z,W)+S(Z,R(X,Y)W) = 0$ or $R.S = 0$.
Converse is also true. Thus we have

Theorem 4.4. *A Sasakian manifold (M^n, g) is pseudo projective ricci semi-symmetric if and only if it is ricci semi-symmetric.*

Suppose in a Sasakian manifold (M^n, g) ,the condition $\overline{P}.Ric\overline{P} = 0$ holds. Then $Ric\overline{P}(\overline{P}(X,Y)Z,W) + Ric\overline{P}(Z,\overline{P}(X,Y)W) = 0$. This with (4.3) gives

(4.4)

$$(a-b)S(\overline{P}(X,Y)Z,W)+\{\frac{(b-a)r}{n}\}'\overline{P}(X,Y,Z,W)+(a-b)S(Z,\overline{P}(X,Y)W)$$
$$+\{\frac{(b-a)r}{n}\}'\overline{P}(X,Y,W,Z)=0$$

Setting $X=W=\xi$ in (4.5) and making use of (2.11),(2.12) and (3.1), we obtain

$$S(Y,Z)=\alpha g(Y,Z)+((n-1)\beta+1)\eta(Y)\eta(Z) \tag{4.5}$$

$$\alpha=\frac{a(a+(n-1)b)(n(n-1)-r)}{n-1}, \tag{4.6}$$

$$\beta=\frac{a(n(n-1)-r)-(n-1)br}{n(n-1)} \tag{4.7}$$

Thus we have

Theorem 4.5. *A Sasakian manifold (M^n,g) satisfying the condition $\overline{P}.Ric\overline{P}=0$ is η- Einstein.*

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