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On Some New Modular Equations and their Applications to Continued Fractions

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Abstract

In this paper, we obtain some new modular equations of degree 2. We obtain several general formulas for the explicit evaluations of the Ramanujan's theta–function. As an application, we establish some new modular relations for Ramanujan–Göllnitz–Gordon continued fraction $H(q)$ with $H(q^n)$, Ramanujan–Selberg continued fraction $V(q)$ with $V(q^n)$ and Eisenstein continued fraction $E(q)$ with $E(q^n)$ for $n = 6, 10, 14$ and 16. We also establish their explicit evaluations.

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1 Introduction

In Chapter 16, of his second notebook [19], Ramanujan has defined his theta-function as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1}$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty \tag{4}$$

and

$$\chi(q) := (-q;q^2)_\infty, \tag{5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1-aq^n), \qquad |q| < 1.$$

Now we define a modular equation in brief. Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k_1, $k' := \sqrt{1-k_1^2}$, l and $l' := \sqrt{1-l^2}$ respectively, where $0 < k_1,\ l < 1$. For a fixed positive integer n, suppose that

$$n\frac{K'}{K} = \frac{L'}{L}. \tag{6}$$

Then a modular equation of degree n is a relation between k_1 and l induced by (6). Following Ramanujan, we set $\alpha = k_1^2$ and $\beta = l^2$. Then we say β is of degree n over α.

In [22], J. Yi introduced two parameterizations $h_{k,n}$ and $h'_{k,n}$ as follows:

$$h_{k,n} = \frac{\varphi(e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(e^{-\pi\sqrt{nk}})}, \tag{7}$$

$$h'_{k,n} = \frac{\varphi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}\varphi(-e^{-\pi\sqrt{nk}})} \tag{8}$$

and established several properties as well as explicit evaluations of $h_{k,n}$ and $h'_{k,n}$ for some positive rational values of n and k.

In [12], M. S. Mahadeva Naika and S. Chandankumar, have established several new modular equations of degree 2 and established general formulas for explicit evaluations of $h_{2,n}$. The authors have also established several new explicit evaluations for Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan-Selberg continued fraction and a continued fraction of Eisenstein. In [8], Mahadeva Naika, K. Sushan Bairy and Chandankumar have established some new modular equations of degree 2. In [9], Mahadeva Naika, Bairy and M. Manjunatha have established several new modular equations of degree 4 and established general formulas for explicit evaluations of $h_{4,n}$. In [14], Mahadeva Naika, Bairy and Chandankumar have established some new modular equations of degree 3. The authors have also established new explicit evaluations of ratios of ramanujan's theta functions. In [13], [7] Mahadeva Naika, Bairy and Chandankumar have established several new modular equations of degree 9, and also established several general formulas for explicit evaluations for ratios of Ramanujan's theta functions.

2 Preliminary Section

Lemma 2.1. *For* $0 < x' < 1$,

1. *[5, Entry 10(i), p. 122] We have*

$$\varphi(e^{-y'}) = \sqrt{z}. \tag{9}$$

2. *[5, Entry 10(iii), p. 122] We have*

$$\varphi(e^{-2y'}) = \sqrt{z}\left(\frac{1}{2}(1+\sqrt{1-x'})\right)^{1/2}. \tag{10}$$

3. *[5, Entry 10(vi), p. 122] We have*

$$\varphi(e^{-y'/2}) = \sqrt{z}(1+\sqrt{x'})^{1/2}. \tag{11}$$

4. *[5, Entry 12(ii), p. 124] We have*

$$f(e^{-y'}) = \sqrt{z}2^{-1/6}\{x'(1-x')e^{y'}\}^{1/24}. \tag{12}$$

5. *[5, Entry 12(iv), p. 124] We have*

$$f(-e^{-4y'}) = \sqrt{z}2^{-2/3}(1-x')^{1/24}\{x'e^{y'}\}^{1/6}. \tag{13}$$

6. *[5, Entry 12(v), Ch.17, p. 124] We have*

$$\chi(e^{-y'}) = 2^{1/6}\{x'(1-x')e^{y'}\}^{-1/24}. \tag{14}$$

7. *[5, Entry 12(vi), Ch.17, p. 124] We have*

$$\chi(-e^{-y'}) = 2^{1/6}(1-x')^{1/12}(x'e^{y'})^{-1/24}, \tag{15}$$

where

$$z := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x'\right),$$

$$y' := \pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x'\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x'\right)}.$$

8. *[5, Entry 5(ii), p. 230] If* β *has degree 3 over* α, *then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{16}$$

9. [5, Entry 13(i), p. 280] If β has degree 5 over α, then

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \quad (17)$$

10. [5, Entry 19(i), p. 314] If β has degree 7 over α, then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \quad (18)$$

11. [2, Entry 11.3.3, p. 275] If β has degree 8 over α, then

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt{2}\sqrt[8]{\beta(1-\alpha)}. \quad (19)$$

12. [22, Theorem 2.2(ii)] We have

$$h_{k,n}h_{k,1/n} = 1. \quad (20)$$

13. [22, Theorem 4.6] We have

$$\sqrt{2}\left(h_{2,n}h_{2,4n} + \frac{1}{h_{2,n}h_{2,4n}}\right) = 2 + \frac{h_{2,4n}}{h_{2,n}}. \quad (21)$$

In Section 3, we establish some new modular equations of degree 2. In Section 4, we establish general formulas for explicit evaluations of $h_{2,n}$. In Section 5, we establish the relation between Ramanujan–Göllnitz–Gordon continued fraction $H(q)$ and other four continued fractions $H(q^6)$, $H(q^{10})$, $H(q^{14})$ and $H(q^{16})$. In Section 6, we establish the relation between Ramanujan–Selberg continued fraction $V(q)$ and other four continued fractions $V(q^6)$, $V(q^{10})$, $V(q^{14})$ and $V(q^{16})$. In Section 7, we establish the relation between Eisenstein continued fraction $E(q)$ and other four continued fractions $E(q^6)$, $E(q^{10})$, $E(q^{14})$ and $E(q^{16})$.

3 Modular equations of degree two

In this section, we establish some new modular equations of degree 2.

Theorem 3.1. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^6)}{\varphi(q^{12})}$, *then*

$$\begin{aligned}&Q^8(P^8+1)+4(3Q^7-4Q^5)P^7+4(32Q^2-Q^8-48Q^4+18Q^6)P^6\\&+4(-5Q^7+6Q^5)P^5+2(292Q^4+3Q^8-108Q^6-192Q^2)P^4\\&+4(12Q^3-16Q-7Q^7+12Q^5)P^3+4(38Q^6-108Q^4-Q^8+72Q^2)P^2\\&+4(9Q^7-14Q^5+24Q-20Q^3)P-32Q^2+24Q^4-8Q^6+16=0.\end{aligned} \tag{22}$$

Proof. Using the equations (9), (10) and (11) in the equation (16), we deduce that

$$\begin{aligned}&4-4s^2+s^4-4s^4ab+12s^2a_2-6s^4a_2\\&-8s^2aba_2+4s^4aba_2-p^4s^4+2p^2s^4=0,\end{aligned} \tag{23}$$

where

$$a=(1-\alpha)^{1/4},\quad a_2=(1-\alpha)^{1/2},\quad b=(1-\beta)^{1/4},\quad p=\frac{\varphi(q^{1/2})}{\varphi(q)}\ \text{ and }\ s=\frac{\varphi(q^3)}{\varphi(q^6)}.$$

Collecting the terms containing b on one side of the above equation (23) and squaring both sides, we deduce that

$$\begin{aligned}&16-32s^2+24s^4-8s^6+96s^2a_2-4s^8a_2p^4+8s^8a_2p^2+40s^6a_2\\&+72s^6a_2p^4-144s^6a_2p^2+24p^4s^6-24p^4s^4+48p^2s^4-48p^2s^6\\&+4s^8a_2-2s^8p^4+384s^4a_2p^2+s^2+12s^8p^2+p^8s^8-4p^6s^8\\&-256s^2a_2p^2+128s^2a_2p^4-192s^4a_2p^4-144s^4a_2=0.\end{aligned} \tag{24}$$

Collecting the terms containing a_2 on one side of the above equation (24) and squaring both sides, we find that

$$\begin{aligned}&(16-32s^2+24s^4-8s^6-384p^4s^2+288p^2s^2-216p^4s^6+584p^4s^4\\&-432p^2s^4+152p^2s^6+6s^8p^4-4s^8p^2+p^8s^8-4p^6s^8-80ps^3-64p^3s\\&+s^8-20p^5s^7+48p^3s^3+24s^5p^5+128p^6s^2-56s^5p+48s^5p^3+36s^7p\\&-28s^7p^3-192p^6s^4+72p^6s^6-16p^7s^5+96ps+12p^7s^7)(16-32s^2\\&-384p^4s^2+p^8s^8-216p^4s^6+584p^4s^4-432p^2s^4+152p^2s^6+6s^8p^4\\&-4s^8p^2-4p^6s^8-48s^5p^3+80ps^3-96ps+64p^3s+20p^5s^7-48p^3s^3\\&-24s^5p^5+128p^6s^2+56s^5p-36s^7p+28s^7p^3-192p^6s^4+72p^6s^6\\&+16p^7s^5-12p^7s^7+288p^2s^2+s^8+24s^4-8s^6)=0.\end{aligned} \tag{25}$$

By examining the behaviour of the factors of the equation (25) near $q = 0$, we can find a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. By changing q to q^2 in the first factor of the equation (25) and set $p = P$ and $s = Q$, we obtain (22). □

Theorem 3.2. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{10})}{\varphi(q^{20})}$, *then*

$$\begin{aligned} &Q^{12}(P^{12}+1) + 8P^{11}Q^7(20Q^2 - 16 - 5Q^4) + P^{10}Q^2(8192 - 20480Q^2 \\ &- 6Q^{10} - 6400Q^6 + 17920Q^4 + 800Q^8) + 40P^9Q^5(48Q^2 - 18Q^4 - 32 \\ &+ Q^6) + 5P^8Q^2(20480Q^2 - 8192 + 3Q^{10} - 17920Q^4 + 6428Q^6 - 796Q^8) \\ &+ 16P^7Q^3(100Q^6 - 160 - 700Q^4 + 720Q^2 + 43Q^8) + 4P^6Q^2(-44800Q^2 \\ &+ 17920 - 5Q^{10} - 15820Q^6 + 2300Q^8 + 40384Q^4) + 16P^5Q(480Q^2 - 64 \\ &- 1400Q^4 + 1180Q^6 - 103Q^{10} - 90Q^8) + 5P^4Q^2(25712Q^2 - 25312Q^4 \\ &- 2344Q^8 - 10240 + 3Q^{10} + 12184Q^6) + 40P^3Q(19Q^{10} - 144Q^2 + 64 \\ &- 72Q^6 + 160Q^4 - 28Q^8) + 2P^2Q^2(2856Q^8 - 15920Q^2 + 18400Q^4 \\ &- 3Q^{10} - 11720Q^6 + 6400) + 8PQ(25Q^{10} - 824Q^6 + 80Q^2 + 190Q^8 \\ &- 160 + 688Q^4) + 64 + 240Q^4 - 192Q^2 - 12Q^{10} - 160Q^6 + 60Q^8 = 0. \end{aligned} \tag{26}$$

Proof. Using the equations (9), (10) and (11) in the equation (17), we deduce that

$$\begin{aligned} &s^6 - 52ba_2s^4p^2 + 26ba_2s^6p^2 + 24a_2 - 3s^6p^4 + 6s^6p^2 - 26ba_2s^6 + 3bs^6 \\ &+ 3bs^6p^2 + 6a_2s^4 + 12s^4 + 48p^2s^2 + 3a_2s^6 - 24p^4s^2 - 48p^2s^4 - 36bs^4p^2 \\ &+ 52ba_2s^4 + 24s^4p^4 - 12s^2 - 48p^4a_2s^2 + 4bs^4 + 36bs^2p^2 - 36a_2s^2 \\ &- 4bs^2 - 36p^2a_2s^4 + 2p^2a_2s^6 + 48bp^4s^4 - 9bp^4s^6 + 96p^2a_2s^2 \\ &+ 18p^4a_2s^4 - 48bs^2p^4 - p^4a_2s^6 + 16bs^2p^6 - 16bp^6s^4 + 3bp^6s^6 \\ &- 64a_2p^2 + 32a_2p^4 = 0, \end{aligned} \tag{27}$$

where

$$a_2 = \sqrt{1-\alpha}, \quad b_1 = \sqrt{\beta}, \quad p = \frac{\varphi(q^{1/2})}{\varphi(q)} \quad \text{and} \quad s = \frac{\varphi(q^5)}{\varphi(q^{10})}.$$

Isolating the terms containing a_2 on one side of the above equation (27) and squaring both sides, we deduce that

$$\begin{aligned} &64 + 81920s^6p^6 - 101920s^6p^4 + 40000s^6p^2 + 192b_1s^6 - 12480b_1s^6p^2 \\ &- 20000p^2s^4 + 6240b_1s^4p^2 + 50960s^4p^4 - 96b_1s^4 - 60s^{10} - 80b_1s^{10} \end{aligned}$$

$$
\begin{aligned}
&- 40960p^6s^4 - 25360s^8p^2 + 5360s^{10}p^2 + 64120s^8p^4 - 13160s^{10}p^4 \\
&- 6b_1s^{12} + 6400b_1s^{10}p^8 - 26880b_1s^8p^8 + 10240p^8s^4 - 51440p^6s^8 \\
&- 20480p^8s^6 + 10480p^6s^{10} + 20s^{12}p^6 - 2620s^{10}p^8 - 15s^{12}p^8 + 6s^{12}p^{10} \\
&- 25600b_1p^4s^4 + 51200b_1p^4s^6 + 240b_1s^{10}p^2 + 35840b_1p^6s^4 - s^{12}p^{12} \\
&+ 260b_1s^{12}p^4 + 4880b_1s^{10}p^4 - 30480b_1s^8p^4 + 46000b_1s^8p^6 - s^{12} \qquad (28)\\
&- 354b_1s^{12}p^2 - 20480b_1p^8s^4 + 40960b_1p^8s^6 + 4096b_1p^{10}s^4 - 8192b_1p^{10}s^6 \\
&+ 70b_1s^{12}p^{10} - 350b_1s^{12}p^8 - 15s^{12}p^4 + 5376b_1s^8p^{10} - 1280b_1s^{10}p^{10} \\
&- 180s^8 + 6000b_1s^8p^2 + 6s^{12}p^2 + 416s^6 - 192s^2 - 16b_1s^8 + 12860p^8s^8 \\
&- 71680b_1p^6s^6 - 48s^4 - 10160b_1s^{10}p^6 + 380b_1s^{12}p^6 = 0.
\end{aligned}
$$

Isolating the terms containing b_1 on one side of the above equation (28) and squaring both sides, we deduce that

$$
\begin{aligned}
&(64 + 161536s^6p^6 - 126560s^6p^4 + 36800s^6p^2 + 240s^4 + 12800p^2s^2 - 179200p^6s^4 \\
&- 51200p^4s^2 - 31840p^2s^4 + 128560s^4p^4 - 192s^2 - 12s^{10} + 71680s^2p^6 - 160s^6 \\
&+ 640s^3p - 1280ps - 23440s^8p^2 + 5712s^{10}p^2 + 60920s^8p^4 - 11720s^{10}p^4 - 6s^{12}p^2 \\
&+ 15s^{12}p^4 + 102400p^8s^4 - 63280p^6s^8 - 89600p^8s^6 + 32140p^8s^8 - 40960p^8s^2 \\
&+ 9200p^6s^{10} - 20s^{12}p^6 - 3980s^{10}p^8 + 15s^{12}p^8 + 17920s^6p^{10} + 8192s^2p^{10} + s^{12} \\
&- 20480p^{10}s^4 - 6400p^{10}s^8 + 800p^{10}s^{10} - 6s^{12}p^{10} + 60s^8 + s^{12}p^{12} - 5760s^3p^3 \\
&+ 2560sp^3 + 7680s^3p^5 - 1024sp^5 + 6400p^3s^5 + 5504s^5p - 22400p^5s^5 - 6592s^7p \\
&- 2880p^3s^7 + 18880p^5s^7 + 11520s^5p^7 - 11200s^7p^7 - 2560s^3p^7 - 1280p^9s^5 \\
&+ 1920p^9s^7 + 1520s^9p - 1120s^9p^3 - 1440s^9p^5 + 1600s^9p^7 - 720p^9s^9 + 200s^{11}p \\
&+ 760s^{11}p^3 - 1648s^{11}p^5 + 688s^{11}p^7 + 40s^{11}p^9 - 128p^{11}s^7 + 160p^{11}s^9 - 40p^{11}s^{11}) \\
&(161536s^6p^6 - 126560s^6p^4 - 12s^{10} + 36800s^6p^2 + 240s^4 - 160s^6 + 12800p^2s^2 \\
&- 51200p^4s^2 - 31840p^2s^4 + 128560s^4p^4 + 71680s^2p^6 + 1280p^9s^5 - 179200p^6s^4 \\
&- 640s^3p + 1280ps - 23440s^8p^2 + 5712s^{10}p^2 + 60920s^8p^4 - 11720s^{10}p^4 - 6s^{12}p^2 \\
&+ 15s^{12}p^4 + 102400p^8s^4 - 63280p^6s^8 - 89600p^8s^6 + 32140p^8s^8 - 40960p^8s^2 \\
&+ 9200p^6s^{10} - 20s^{12}p^6 - 3980s^{10}p^8 + 15s^{12}p^8 + 17920s^6p^{10} + 8192s^2p^{10} + s^{12} \\
&- 20480p^{10}s^4 - 6400p^{10}s^8 + 800p^{10}s^{10} - 6s^{12}p^{10} + s^{12}p^{12} + 60s^8 - 1920p^9s^7 \\
&+ 5760s^3p^3 - 2560sp^3 - 7680s^3p^5 + 1024sp^5 - 6400p^3s^5 - 5504s^5p + 22400p^5s^5 \\
&+ 6592s^7p + 2880p^3s^7 - 18880p^5s^7 - 11520s^5p^7 + 11200s^7p^7 + 2560s^3p^7 - 192s^2 \\
&- 1520s^9p + 1120s^9p^3 + 1440s^9p^5 - 1600s^9p^7 + 720p^9s^9 - 200s^{11}p - 760s^{11}p^3 \\
&+ 1648s^{11}p^5 - 688s^{11}p^7 - 40s^{11}p^9 + 128p^{11}s^7 - 160p^{11}s^9 + 40p^{11}s^{11} + 64) = 0. \qquad (29)
\end{aligned}
$$

By observing the factors of the above equation (29) near $q = 0$, we can find a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. By changing q to q^2 in the first factor of the equation (29) and set $p = P$ and $s = Q$, we obtain (26). □

Theorem 3.3. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{14})}{\varphi(q^{28})}$*, then*

$$
\begin{aligned}
&(P^{16}+1)Q^{16} + 16\{P^{15}Q^9(112Q^2 - 64 - 56Q^4 + 7Q^6) + (186368Q^2\\
&+ 7Q^{12} - 37968Q^8 - 232960Q^4 - 57344 + 3654Q^{10} + 138208Q^6)P^{13}Q^3\\
&+ P^{11}Q^3(1631616Q^4 + 327908Q^8 + 372736 - 38584Q^{10} - 1049888Q^6\\
&- 1243648Q^2 - 413Q^{12}) + P^9Q^3(3021480Q^6 - 1017996Q^8 + 3263232Q^2\\
&- 931840 + 134918Q^{10} - 4471936Q^4 + 1427Q^{12}) + P^7Q(6042960Q^6\\
&+ 1433992Q^{10} - 4186112Q^8 - 1259Q^{14} - 4199552Q^4 - 195384Q^{12}\\
&- 1024 + 1105664Q^2) + P^5Q(111538Q^{12} - 607488Q^2 + 2867984Q^8\\
&- 4071984Q^6 + 2623264Q^4 + 3584 - 926520Q^{10} - 651Q^{14}) + (-3584\\
&+ 1079344Q^6 - 781536Q^8 - 617344Q^4 + 116928Q^2 + 833Q^{14} + 223076Q^{10}\\
&- 17752Q^{12})P^3Q + PQ(896 + 1666Q^{12} + 22832Q^6 + 448Q^2 - 13216Q^4\\
&+ 49Q^{14} - 10072Q^8 - 2604Q^{10})\} + 8P^2Q^2\{P^{12}(784Q^{12} - 12544Q^{10}\\
&- Q^{14} + 315392Q^4 + 75264Q^8 - 229376Q^2 - 215040Q^6 + 2^{16})\\
&+ P^8(1261568Q - 7Q^{14} - 6003200Q^6 - 5017600Q^2 + 7770112Q^4\\
&+ 2425024Q^8 + 26236Q^{12} - 461132Q^{10}) + P^4(1204224 - 7Q^{14}\\
&- 5562368Q^2 - 8293936Q^6 + 3818976Q^8 - 993496Q^{10} + 127512Q^{12}\\
&+ 9700096Q^4) + (419776Q^4 - 160832Q^2 - 155260Q^{10} + 510048Q^8\\
&+ 2^{10}49 - Q^{14} - 676592Q^6 + 12700Q^{12})\} + 4P^{12}Q^2(7Q^{14} - 5017600Q^4\\
&+ 3411968Q^2 + 3687936Q^6 - 10052Q^{12} + 236292Q^{10} - 1390592Q^8\\
&- 917504) + 4(3780672Q^4 - 802816Q^2 - 7378112Q^6 - 3973984Q^{10}\\
&+ 7Q^{16} - 155260Q^{14} + 7361872Q^8 + 1168076Q^{12})P^4 + 2(3680936Q^{12}\\
&- 48025600Q^6 + 35Q^{16} - 16587872Q^{10} - 338296Q^{14} + 29503488Q^4\\
&+ 38655024Q^8 - 6881280Q^2)P^8 + 256 + 1792Q^4(1 - Q^2) + 1120Q^8\\
&- 448Q^{10} + 112Q^{12} - 16Q^{14} - 1024Q^2 = 0.
\end{aligned}
\tag{30}
$$

Proof. The proof of (30) is similar to the proof of the equation (22), except that in place of result (16), the result (18) is used. □

Theorem 3.4. *If* $P := \dfrac{\varphi(q)}{\varphi(q^2)}$ *and* $Q := \dfrac{\varphi(q^{16})}{\varphi(q^{32})}$, *then*

$$\begin{aligned}
&256 + [Q^{16} + 16[16 - Q^{14} + 70Q^{8} - 28Q^{10} + 7Q^{12} - 64Q^{2} - 112[Q^{6} - Q^{4}]]]P^{16}\\
&+ 16P^{15}[46Q^{13} - 11Q^{15} - 1288Q^{9} - 3680Q^{5} + 148Q^{11} + 3184Q^{7} + 1984Q^{3}\\
&- 384Q] - 8[Q^{16} - 16[917Q^{12} - 5926Q^{8} + 7904Q^{6} + 122Q^{14} - 3760Q^{4}\\
&+ 640Q^{2} + 120Q^{10} - 16]]P^{14} + 16[260264Q^{9} - 72896Q^{3} - 445872Q^{7}\\
&+ 207Q^{15} + 3968Q - 19078Q^{13} + 295392Q^{5} - 22020Q^{11}]P^{13} + [4[7Q^{16}\\
&+ 16[-1931Q^{14} - 419536Q^{6} + 29569Q^{12} - 204340Q^{10} + 453482Q^{8}\\
&+ 112 - 15040Q^{2} + 157712Q^{4}]]]P^{12} + 16[-3713928Q^{9} - 14720Q\\
&- 571Q^{15} + 590784Q^{3} - 2997344Q^{5} + 951764Q^{11} + 5184368Q^{7}\\
&- 626Q^{13}]P^{11} + [8[16[31616Q^{2} + 783912Q^{10} - 419536Q^{4} - 112\\
&+ 7598Q^{14} - 157805Q^{12} - 1485834Q^{8} + 1240224Q^{6}] - 7Q^{16}]]P^{10}\\
&+ 16[25472Q + 15725160Q^{9} - 19621936Q^{7} + 10368736Q^{5} - 5166564Q^{11}\\
&- 1783488Q^{3} - 1321Q^{15} + 453226Q^{13}]P^{9} + P^{8}[2[16[15274898Q^{8}\\
&- 11886672Q^{6} + 1964469Q^{12} - 109075Q^{14} - 189632Q^{2} - 8682004Q^{10}\\
&+ 560 + 3627856Q^{4}] + 35Q^{16}]] + [16[2082112Q^{3} - 20608Q + 10492764Q^{11}\\
&- 1273094Q^{13} + 31450320Q^{7} + 5831Q^{15} - 14855712Q^{5} - 27882328Q^{9}]]P^{7}\\
&+ [8[16[1920Q^{2} + 3135648Q^{6} - 638733Q^{12} - 4341002Q^{8} - 817360Q^{4}\\
&- 112 + 2619112Q^{10} + 40590Q^{14}] - 7Q^{16}]]P^{6} + [16[4736Q - 3403Q^{15}\\
&- 352320Q^{3} + 7614112Q^{5} - 20666256Q^{7} + 20985528Q^{9} + 1217310Q^{13}\\
&- 8799980Q^{11}]]P^{5} + [4[16[664705Q^{12} + 3928938Q^{8} - 2524880Q^{6} + 112\\
&- 45707Q^{14} + 473104Q^{4} + 58688Q^{2} - 2554932Q^{10}] + 7Q^{16}]]P^{4}\\
&+ 16[3625808Q^{7} - 5092376Q^{9} + 2434620Q^{11} - 10016Q^{5} - 2689Q^{15}\\
&+ 2944Q - 347830Q^{13} - 610496Q^{3}]P^{3} + [8[16[121568Q^{6} + 3034Q^{14}\\
&- 16 - 30896Q^{4} + 7808Q^{2} + 162360Q^{10} - 45707Q^{12} - 218150Q^{8}]\\
&- Q^{16}]]P^{2} - 16P[18272Q^{5} - 46648Q^{9} + 21136Q^{7} + 13612Q^{11} - 13248Q^{3}\\
&+ 1408Q + 5378Q^{13} + 91Q^{15}] - 1024Q^{2} + 1792[Q^{4} - Q^{6}] + 1120Q^{8}\\
&- 448Q^{10} + 112Q^{12} - 16Q^{14} + Q^{16} = 0.
\end{aligned} \tag{31}$$

Proof. The proof of (31) is similar to the proof of the equation (22), except that in place of result (16), the result (19) is used.

□

4 General formulas for the explicit evaluations of $h_{2,n}$

In this section, we establish some general formulas for explicit evaluations of $h_{2,n}$.

Theorem 4.1. *If* $h := h_{2,n}$ *and* $h_1 := h_{2,36n}$*, then*

$$
\begin{aligned}
&(4h^8 - 4\sqrt{2}h^2 - 8\sqrt{2}h^6 + 1 + 12h^4)h_1^8 + 4(6\sqrt{2}h^7 - 10h^5 + 9h - 7\sqrt{2}h^3)h_1^7 \\
&+ 4\sqrt{2}\Big[(19\sqrt{2}h^2 - 54h^4 - 1 + 18\sqrt{2}h^6)h_1^6 + (6\sqrt{2}h^3 + 6h^5 - 4\sqrt{2}h^7 \\
&- 7h)\, h_1^5 + (-5\sqrt{2}h + 6h^3)h_1^3\Big] + 4(-54\sqrt{2}h^2 + 3 + 146h^4 - 48\sqrt{2}h^6)h_1^4 \\
&+ 8\sqrt{2}\Big[(-24h^4 + 8\sqrt{2}h^6 - 1 + 9\sqrt{2}h^2)h_1^2 + (3h - 2\sqrt{2}h^3)h_1\Big] + 4 = 0.
\end{aligned} \tag{32}
$$

Proof. Using the equation (22) along with the equation (7) with $k = 2$, we arrive at the equation (32).

□

Corollary 4.2. *We have*

$$h_{2,6} = \sqrt{8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6} - 10}, \tag{33}$$

$$h_{2,1/6} = \frac{\sqrt{1 + 2\sqrt{2} + \sqrt{3}}}{2}, \tag{34}$$

$$h_{2,2/3} = \frac{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} - \sqrt{-1 + 2\sqrt{2} - \sqrt{3}})\sqrt{2}}{\sqrt{3} + 1}, \tag{35}$$

$$h_{2,3/2} = \frac{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} + \sqrt{-1 + 2\sqrt{2} - \sqrt{3}})}{2\sqrt{2}}, \tag{36}$$

$$h_{2,24} = \frac{\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} - \sqrt{10 - 7\sqrt{2} + 6\sqrt{3} - 4\sqrt{6}}}{(\sqrt{2} - 1)^2(\sqrt{3} - \sqrt{2})^2}, \tag{37}$$

$$h_{2,1/24} = \frac{\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} + \sqrt{10 - 7\sqrt{2} + 6\sqrt{3} - 4\sqrt{6}}}{\sqrt{2}}. \tag{38}$$

Proofs of (33) and (34). Using the equations (32) and (20) with $n = 1/6$, we deduce that

$$(h_{2,6}^4 + (20 - 16\sqrt{2})h_{2,6}^2 - 16\sqrt{2} + 24)(h_{2,6}^4 + (8 + 4\sqrt{2})h_{2,6}^2 - 4\sqrt{2} - 4)^2 = 0. \tag{39}$$

The first factor of the equation (39) vanishes for the specific value of $q = e^{-\pi\sqrt{3}}$, but the second factor does not vanish. Hence by solving the first factor and $0 < h_{2,6} < 1$, we obtain the equations (33) and (34).

□

Proofs of (35) and (36). Using the equation (34) in the equation (21), we arrive at the equations (35) and (36).

□

Proofs of (37) and (38). Using the equation (33) in the equation (21), we arrive at the equations (37) and (38).

□

Theorem 4.3. *If $h := h_{2,n}$ and $h_1 := h_{2,100n}$, then*

$$\begin{aligned}
&8h_1^{12}(h^{12}+1)+32(20h_1^9-5\sqrt{2}h_1^{11}-8\sqrt{2}h_1^7)h^{11}+8(400h_1^{10}-3\sqrt{2}h_1^{12}\\
&-2560\sqrt{2}h_1^4+1024h_1^2-1600\sqrt{2}h_1^8+4480h_1^6)h^{10}+160(24h_1^7-9\sqrt{2}h_1^9\\
&-8\sqrt{2}h_1^5+h_1^{11})h^9+20(5120h_1^4-398\sqrt{2}h_1^{10}-4480\sqrt{2}h_1^6-1024\sqrt{2}h_1^2\\
&+3h_1^{12}+3214h_1^8)h^8+32(43\sqrt{2}h_1^{11}-40\sqrt{2}h_1^3+360h_1^5-350\sqrt{2}h_1^7\\
&+100h_1^9)h^7+8(20192h_1^6-11200\sqrt{2}h_1^4+4480h_1^2-7910\sqrt{2}h_1^8-5\sqrt{2}h_1^{12}\\
&+2300h_1^{10})h^6+32(120h_1^3-103h_1^{11}-350\sqrt{2}h_1^5+590h_1^7-45\sqrt{2}h_1^9\\
&-8\sqrt{2}h_1)h^5+2(32140h_1^4-5860\sqrt{2}h_1^{10}-6400\sqrt{2}h_1^2-31640\sqrt{2}h_1^6\\
&+15h_1^{12}+30460h_1^8)h^4+8(400h_1^5-180\sqrt{2}h_1^3-140h_1^9-180\sqrt{2}h_1^7\\
&+95\sqrt{2}h_1^{11}+80h_1)h^3+(3200h_1^2+18400h_1^6-6\sqrt{2}h_1^{12}-11720\sqrt{2}h_1^8\\
&+5712h_1^{10}-7960\sqrt{2}h_1^4)h^2+8(172\sqrt{2}h_1^5+95\sqrt{2}h_1^9-412h_1^7+25h_1^{11}\\
&+20h_1^3-20\sqrt{2}h_1)h+8-24\sqrt{2}h_1^2+30h_1^8+60h_1^4=6\sqrt{2}h_1^{10}+40\sqrt{2}h_1^6.
\end{aligned} \tag{40}$$

Proof. Using the equation (26) along with the equation (7) with $k = 2$, we arrive at the equation (40).

□

Corollary 4.4. *We have*

$$h_{2,10}^2=(\sqrt{2}-1)^2(2\sqrt{2}-1)\sqrt{5}-25+18\sqrt{2}-p_1, \tag{41}$$

$$h_{2,1/10}^2=\frac{1}{4}(-11\sqrt{5}+8\sqrt{5}\sqrt{2}-25+18\sqrt{2}+p_1)(4-\sqrt{2}-\sqrt{5}+\sqrt{5}\sqrt{2}), \tag{42}$$

where $p_1=\sqrt{2454-1736\sqrt{2}+2\sqrt{5}(549-388\sqrt{2})}$.

Proof. Using the equations (40) and (20) with $n = 1/10$, we deduce that

$$\begin{aligned}
&(h_{2,10}^8+(100-72\sqrt{2})h_{2,10}^6+(240-168\sqrt{2})h_{2,10}^4+(176-128\sqrt{2})h_{2,10}^2\\
&+48-32\sqrt{2})(h_{2,10}^4+(48+32\sqrt{2})h_{2,10}^2-32-24\sqrt{2})^2\\
&(h_{2,10}^2+2-2\sqrt{2})^2=0.
\end{aligned} \tag{43}$$

The first factor of the equation (43) vanishes for the specific value of $q = e^{-\pi\sqrt{5}}$, but the other two factors does not vanish. Hence by solving the first factor and $0 < h_{2,10} < 1$, we obtain the equations (41) and (42).

□

Theorem 4.5. *If* $h := h_{2,n}$ *and* $h_1 := h_{2,196n}$*, then*

$$
\begin{aligned}
&16 + (1 - 8h^2\sqrt{2} + 16h^{16} - 112h^6\sqrt{2} + 280h^8 + 224h^{12} - 64h^{14}\sqrt{2} + 56h^4\\
&- 224h^{10}\sqrt{2})h_1^{16} + 16\{(49h + 833h^3\sqrt{2} - 1302h^5 - 2518h^7\sqrt{2} + 5708h^9\\
&+ 56h^{15}\sqrt{2} + 56h^{13} - 1652h^{11}\sqrt{2})h_1^{15} + (269836h^9\sqrt{2} + 833h\sqrt{2} + 14616h^{13}\sqrt{2}\\
&- 17752h^3 + 111538h^5\sqrt{2} - 448h^{15} - 390768h^7 - 154336h^{11})h_1^{13} + (127512h^2\\
&- 695296h^{12}\sqrt{2} + 2425024h^{10} + 150528h^{14} + 1909488h^6 - 2073484h^8\sqrt{2} - 7\sqrt{2}\\
&- 496748h^4\sqrt{2})h_1^{10}\} + 8\{(-20104h^{12}\sqrt{2} + 255024h^6 - 77630h^4\sqrt{2} + 104944h^{10}\\
&+ 6272h^{14} + 12700h^2 - 169148h^8\sqrt{2} - \sqrt{2})h_1^{14} + (7 + 472584h^{12} + 584038h^4\\
&+ 1840468h^8 - 50176h^{14}\sqrt{2} - 922264h^{10}\sqrt{2} - 993496h^6\sqrt{2} - 77630h^2\sqrt{2})h_1^{12}\\
&+ (35 - 430080h^{14}\sqrt{2} - 6003200h^{10}\sqrt{2} - 4146968h^6\sqrt{2} + 9663756h^8\\
&- 169148h^2\sqrt{2} + 1840468h^4 + 3687936h^{12})h_1^8\} + 32\{h_1^{11}(224h^{15}\sqrt{2} - 75936h^{13}\\
&+ 327908h^{11}\sqrt{2} - 463260h^5 - 651h + 716996h^7\sqrt{2} + 55769h^3\sqrt{2} - 1017996h^9)\\
&+ (-1259h\sqrt{2} - 195384h^3 + 1510740h^9\sqrt{2} + 138208h^{13}\sqrt{2} - 2093056h^7\\
&- 1049888h^{11} + 716996h^5\sqrt{2} - 128h^{15})h_1^9 + (-627200h^{10}\sqrt{2} + 118146h^4\\
&- 347648h^6\sqrt{2} + 7 - 5026h^2\sqrt{2} - 57344h^{14}\sqrt{2} + 426496h^{12} + 921984h^8)h_1^4\\
&+ (1212512h^6 + 157696h^{14} - 7\sqrt{2} + 1942528h^{10} - 230566h^4\sqrt{2} + 26236h^2\\
&- 627200h^{12}\sqrt{2} - 1500800h^8\sqrt{2})h_1^6\} + 64\{(67459h^3\sqrt{2} + 755370h^7\sqrt{2}\\
&+ 407904h^{11}\sqrt{2} - 508998h^5 - 1117984h^9 + 1427h - 116480h^{13})h_1^7\\
&+ (163954h^5\sqrt{2} + 46592h^{13}\sqrt{2} - 310912h^{11} - 524944h^7 - 38584h^3\\
&- 413h\sqrt{2} + 407904h^9\sqrt{2})h_1^5 + (37632h^6 - 6272h^4\sqrt{2} - 53760h^8\sqrt{2} - \sqrt{2}\\
&+ 784h^2 + 78848h^{10} + 8192h^{14} - 28672h^{12}\sqrt{2})h_1^2\} + 128\{(23296h^{11}\sqrt{2}\\
&- 7168h^{13} - 18984h^5 + 1827h^3\sqrt{2} + 34552h^7\sqrt{2} + 7h - 58240h^9)h_1^3\\
&+ (-32h^7 - 56h^3 + 56h^5\sqrt{2} + 7h\sqrt{2})h_1\} = 0.
\end{aligned}
\tag{44}
$$

Proof. Using the equation (30) along with the equation (7) with $k = 2$, we arrive at the equation (44).

□

Corollary 4.6. *We have*

$$h_{2,14} = 2\sqrt{-45 + 12\sqrt{14} + 32\sqrt{2} - 17\sqrt{7}}, \tag{45}$$

$$h_{2,1/14} = \frac{\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}}}{2\sqrt{2}}, \tag{46}$$

$$h_{2,2/7} = \frac{2(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} - \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}})}{3 + \sqrt{7}}, \tag{47}$$

$$h_{2,7/2} = \frac{(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} + \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}})}{4}. \tag{48}$$

Proofs of (45) and (46). Using the equations (44) and (20) with $n = 1/14$, we deduce that

$$\begin{aligned}&(h_{2,14}^4 + (360 - 256\sqrt{2})h_{2,14}^2 + 544 - 384\sqrt{2})(h_{2,14}^4 + (20 - 16\sqrt{2})h_{2,14}^2\\ &+ 24 - 16\sqrt{2})^2(h_{2,14}^8 + (192 + 136\sqrt{2})h_{2,14}^6 - (680 + 480\sqrt{2})h_{2,14}^4\\ &+ (768 + 544\sqrt{2})h_{2,14}^2 - 272 - 192\sqrt{2})^2 = 0.\end{aligned} \tag{49}$$

The first factor of the equation (49) vanishes for the specific value of $q = e^{-\pi\sqrt{7}}$, but the other two factors does not vanish. Hence by solving the first factor and $0 < h_{2,14} < 1$, we obtain the equations (45) and (46).

□

Proofs of (47) and (48). Using the equation (45) in the equation (21), we arrive at the equations (47) and (48).

□

5 Certain identities for Ramanujan–Göllnitz–Gordon continued fraction

On page 229 of his second notebook, Ramanujan recorded the continued fraction which is known as Ramanujan-Göllnitz-Gordon continued fraction along with the two identities as follows:

$$H(q) := \frac{q^{1/2}}{1+q} {\atop +} \frac{q^2}{1+q^3} {\atop +} \frac{q^4}{1+q^5} {\atop +} \frac{q^6}{1+q^7} {\atop +} \cdots, \qquad |\, q \,| < 1, \tag{50}$$

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{51}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{52}$$

Proofs of the above identities (50), (51) and (52) can be found in [19, p. 221] and for more details one can see [6] and [21]. In [17], the authors have established several modular relations between a Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ and $H(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13,$ 15, 17, 19, 23, 25, 29, 31 and 55 . In [4], the authors have established the following identity

$$H^2(e^{-\pi\sqrt{n/2}}) = \frac{2^{1/4}h_{2,n} - 1}{2^{1/4}h_{2,n} + 1}, \quad \text{for any positive rational number n,} \tag{53}$$

which follows from the identity established by Chan and Huang in [6].

Theorem 5.1. *If $x := H(q)$ and $y := H(q^6)$, then*

$$\begin{aligned} &- y + y^5 - 2y^2 - 5x^2y^4 - 3x^2y^6 + x^2y^8 - 6x^4y^2 + 12x^4y^4 - 6x^4y^6 \\ &- 3x^6y^2 - 5x^6y^4 + 15x^6y^6 + 2x^8y^4 - 2x^8y^6 + x^6 + 2y^4 + 15x^2y^2 \\ &- 6x^2y^3 - 10x^2y^5 - 3x^4y - 3x^4y^3 + 3x^4y^5 + 3x^4y^7 - 2x^6y + 10x^6y^3 \\ &+ 6x^6y^5 - 6x^6y^7 - x^8y^3 + x^8y^7 + 6x^2y + 2y^7x^2 = 0. \end{aligned} \tag{54}$$

Proof. Using the equations (51) and (52) in the equation (22), we deduce that

$$\begin{aligned} &(y^5 + 2y^2 + 5x^2y^4 + 3x^2y^6 - x^2y^8 + 6x^4y^2 - 12x^4y^4 + 6x^4y^6 + 3x^6y^2 \\ &+ 5x^6y^4 - y - 15x^6y^6 - 2x^8y^4 + 2x^8y^6 - x^6 - 2y^4 - 15x^2y^2 - 6x^2y^3 \\ &- 10x^2y^5 - 3x^4y - 3x^4y^3 + 3x^4y^5 + 3x^4y^7 - 2x^6y + 10x^6y^3 + 6x^6y^5 \\ &- 6x^6y^7 - x^8y^3 + x^8y^7 + 6x^2y + 2y^7x^2)(y^5 - y - 2y^2 - 5x^2y^4 - 3x^2y^6 \\ &+ 12x^4y^4 - 6x^4y^6 - 3x^6y^2 - 5x^6y^4 + 15x^6y^6 + 2x^8y^4 - 2x^8y^6 + 2y^4 \\ &+ 15x^2y^2 - 6x^2y^3 - 10x^2y^5 - 3x^4y - 3x^4y^3 + 3x^4y^5 + 3x^4y^7 - 2x^6y \\ &+ 10x^6y^3 + 6x^6y^5 - 6x^6y^7 - x^8y^3 + x^8y^7 + 6x^2y + 2y^7x^2 + x^6 \\ &+ x^2y^8 - 6x^4y^2) = 0. \end{aligned} \tag{55}$$

From the definitions of x and y, we have $x = o(q^{1/2})$ and $y = o(q^3)$ as $q \to 0$, it can be seen that the second factor vanishes for q sufficiently small whereas the first factor does not vanish. Thus by the Identity Theorem, second factor vanishes identically. Hence the proof.

□

Theorem 5.2. *If $x := H^2(q)$ and $y := H(q^{10})$, then*

$$\begin{aligned} &[4(y^6 - y^{10}) + 5(y^9 - y^7) - y^5 + y^{11}]x^6 + [1 + 30(y^7 - y^4) + 45y^8 \\ &- 10(y^{11} + 20(y^3 + y^{10}) + y^9) - 4(y + y^6) + 6y^5]x^5 + [240(y^4 - y^8) \\ &- 150y^3 - 40y^2 + 20(y + y^{10} + y^6) - 105y^9 + 145y^5 + 25y^{11} + 65y^7]x^4 \\ &+ [60(y^7 - y^5) + 20(y^3 - y^6 - y^9) + 10(y^{10} + y^2)]x^3 + [240(y^8 - y^4) \\ &+ 150y^9 - 65y^5 - 25y - 145y^7 - 40y^{10} + 105y^3 - 20(y^{11} - y^2 + y^6)]x^2 \\ &+ [y^{12} - 30(y^8 + y^5) - 6y^7 + 20(y^2 - y^9) + 10(y^3 + y) + 45y^4 \\ &+ 4(y^{11} - y^6)]x - 4(y^2 - y^6) + 5(y^5 - y^3) + y^7 - y = 0. \end{aligned} \tag{56}$$

Proof. The proof of the equation (56) is similar to the proof of the equation (54) except that in the place of the equation (22), the equation (26) is employed. □

Theorem 5.3. *If $x := H^2(q)$ and $y := H(q^{14})$, then*

$$\begin{aligned}
&(28(y^6+y^2)+8(y+y^7)+y^8+70y^4+56(y^5+y^3)+1)x^8+(392y^6+8y^{14}\\
&+231y^5+21y^{13}+y^{15}-112y^4-280y^{10}-56y^2-245y^3+48y^8+363y^7\\
&-259y^9+7y-119y^{11})x^7+(1050y^9-770y^5+1260y^{10}-98y-1512y^6\\
&+112(y^8-y^{14})+854y^3-2282y^7+140y^2+448y^{12}-336y^4-182y^{13}\\
&+1442y^{11}-14y^{15})x^6+(63y^{15}-2072y^6+4123y^9+2520y^{10}-392y^2\\
&-4375y^5+301y^3+2576y^4-4081y^{11}-2688y^{12}-336y^8+3717y^7+392y^{14}\\
&+203y^{13}+49y)x^5+(2100(y^5-y^{11})-280(y^6+y^{10})+924(y^{13}-y^3)+420y^8\\
&+3108(y^7-y^9)+70(y^{12}+y^4)+84(y-y^{15}))x^4+(2576y^{12}-49y^{15}-301y^{13}\\
&-392y^{14}+392y^2+2520y^6+4081y^5-2072y^{10}+4375y^{11}-203y^3-2688y^4\\
&-63y-4123y^7-3717y^9-336y^8)x^3+(-1512y^{10}+448y^4-1050y^7-854y^{13}\\
&+140y^{14}+112(y^8-y^2)+1260y^6+14y+770y^{11}+182y^3-336y^{12}-1442y^5\\
&+2282y^9+98y^{15})x^2+(259y^7-7y^{15}+8y^2+119y^5-280y^6+48y^8+392y^{10}\\
&+245y^{13}-y-363y^9-231y^{11}-56y^{14}-21y^3-112y^{12})x-56(y^{11}+y^{13})\\
&+28(y^{10}+y^{14})-8(y^9+y^{15})+70y^{12}+y^8+y^{16}=0.
\end{aligned} \tag{57}$$

Proof. The proof of the equation (57) is similar to the proof of the equation (54) except that in the place of the equation (22), the equation (30) is employed. □

Theorem 5.4. *If $x := H^2(q)$ and $y := H(q^{16})$, then*

$$\begin{aligned}
&(88(x^6-x^2)-176x^5+x^8-16x^7+3-48x+12x^4+240x^3)y^{15}+(32(x-x^7)\\
&-13+1696(x^5-x^3)+1264x^2-240x^6+5x^8-912x^4)y^{14}+(2576x^3\\
&-3280x^5+4476x^4-39-3848x^2+560x+144x^7-248x^6-5x^8)y^{13}+(53\\
&-448x^6+1152(x^5-x^3)+2496x^2-384(x-x^7)-57x^8-1568x^4)y^{12}\\
&+y^{11}(143-4308x^4+2136x^6-1584x-43x^8+1960x^2-2736x^5+4336x^3\\
&-16x^7)+(153x^8-112x^4-1184(x^7-x)+1872x^6-2896x^2-736(x^5-x^3)\\
&-81)y^{10}+(6072x^2-592x-8772x^4-67+215x^8-5488x^3+2120x^6+7856x^5\\
&-1776x^7)y^9+(5184x^4-2048(x^2+x^6)+67(x^8+1))y^8+(1776x-7856x^3\\
&-2120x^2-215+592x^7+8772x^4+67x^8-6072x^6+5488x^5)y^7+(1872x^2
\end{aligned}$$

$$- 1184(x - x^7) - 736(x^3 - x^5) - 2896x^6 - 81x^8 + 153 - 112x^4)y^6 + (2736x^3 + 16x - 143x^8 - 4336x^5 + 43 + 1584x^7 - 1960x^6 + 4308x^4 - 2136x^2)y^5 + (384(x - x^7) - 57 + 2496x^6 + 53x^8 - 1568x^4 + 1152(x^3 - x^5) - 448x^2)y^4 + (5 - 144x + 248x^2 + 39x^8 - 560x^7 - 2576x^5 - 4476x^4 + 3848x^6 + 3280x^3)y^3 + (5 - 912x^4 - 240x^2 - 13x^8 - 1696(x^5 - x^3) + 32(x^7 - x) + 1264x^6)y^2 + x^8 + (48x^7 - 3x^8 - 12x^4 + 88(x^6 - x^2) - 240x^5 + 16x + 176x^3 - 1)y + y^{16} = 0. \tag{58}$$

Proof. The proof of the equation (58) is similar to the proof of the equation (54) except that in the place of the equation (22), the equation (31) is employed. □

Using the values of $h_{2,n}$, we establish some explicit evaluations of $H(q)$ in the following Theorem.

Theorem 5.5. *We have*

$$H^2(e^{-\pi\sqrt{3}}) = \frac{2^{1/4}\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} - 1}{2^{1/4}\sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}} + 1},$$

$$H^2(e^{-\pi/2\sqrt{3}}) = \frac{2^{1/4}\sqrt{1 + 2\sqrt{2} + \sqrt{3}} - 2}{2^{1/4}\sqrt{1 + 2\sqrt{2} + \sqrt{3}} + 2},$$

$$H^2(e^{-\pi\sqrt{3}/2}) = \frac{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} + \sqrt{2\sqrt{2} - 1 - \sqrt{3}}) - 2^{5/4}}{(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} + \sqrt{2\sqrt{2} - 1 - \sqrt{3}}) + 2^{5/4}},$$

$$H^2(e^{-\pi\sqrt{7}}) = \frac{2^{5/4}\sqrt{-45 + 12\sqrt{14} + 32\sqrt{2} - 17\sqrt{7}} - 1}{2^{5/4}\sqrt{-45 + 12\sqrt{14} + 32\sqrt{2} - 17\sqrt{7}} + 1}$$

$$H^2(e^{-\pi/2\sqrt{7}}) = \frac{\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} - 2^{5/4}}{\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} + 2^{5/4}},$$

$$H^2(e^{-\pi\sqrt{7}/2}) = \frac{2^{1/4}(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} + \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}}) - 4}{2^{5/4}(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} + \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}}) + 4},$$

$$H^2(e^{-\pi/\sqrt{7}}) = \frac{2^{5/4}(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} - \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}}) - (3 + \sqrt{7})}{2^{5/4}(\sqrt{3 + 4\sqrt{2} + 2\sqrt{7}} - \sqrt{-3 + 4\sqrt{2} - 2\sqrt{7}}) + (3 + \sqrt{7})},$$

$$H^2(e^{-\pi/\sqrt{3}}) = \frac{2^{3/4}(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} - \sqrt{2\sqrt{2} - 1 - \sqrt{3}}) - (1 + \sqrt{3})}{2^{3/4}(\sqrt{1 + 2\sqrt{2} + \sqrt{3}} - \sqrt{2\sqrt{2} - 1 - \sqrt{3}}) + (1 + \sqrt{3})},$$

$$H^2(e^{-2\pi\sqrt{3}}) = \frac{2^{1/4}(c - d) - (\sqrt{2} - 1)^2(\sqrt{3} - \sqrt{2})^2}{2^{1/4}(c - d) + (\sqrt{2} - 1)^2(\sqrt{3} - \sqrt{2})^2},$$

$$H^2(e^{-\pi/4\sqrt{3}}) = \frac{c + d - \sqrt{2}}{c + d + \sqrt{2}},$$

where $c = \sqrt{-10 + 8\sqrt{2} - 6\sqrt{3} + 4\sqrt{6}}$ *and* $d = \sqrt{10 - 7\sqrt{2} + 6\sqrt{3} - 4\sqrt{6}}$.

6 Certain identities for Ramanujan–Selberg continued fraction

The continued fraction identity

$$V(q) := \frac{q^{1/8}}{1+}\,\frac{q}{1+}\,\frac{q^2+q}{1+}\,\frac{q^3}{1+}\,\frac{q^4+q^2}{1+}\cdots = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}, \qquad |q|<1, \tag{59}$$

appears as Formula 5 [19, p.290] and was first proved by Selberg [20, eq.(54)]. Other proofs have been given by Ramanathan [18] and Andrews, Berndt, Jacobsen and Lamphere [3]. Adiga, Mahadeva Naika and Ramya Rao [1] have obtained two integral representations for $V(q)$, also derived a relation between $V(q)$ and $V(q^n)$ and some explicit evaluations of $V(q)$. Recently, Mahadeva Naika, Remy Y Denis and Bairy [16] have established several modular relations and explicit evaluations of Ramanujan-Selberg continued fraction. In [11], Mahadeva Naika, Bairy and Manjunatha have established several modular relations and explicit evaluations for a continued fraction of order four.

Lemma 6.1. *We have*

$$\frac{\varphi^2(q^{1/2})}{\varphi^2(q)} = 1 + 4V^4(q). \tag{60}$$

Proof. Using the equations (9), (11), (12), (13) and (59), we arrive at the equation (60). □

Lemma 6.2. *For any positive rational number n, we have*

$$V^4(e^{-\pi\sqrt{2n}}) = \frac{\sqrt{2}h_{2,n}^2 - 1}{4}. \tag{61}$$

Proof. Using the equations (60) and (7) with $k = 2$, we obtain the equation (61). □

Theorem 6.3. *If $v := V^4(q)$ and $u := V^2(q^6)$, then*

$$\begin{aligned} &u^4 + uv(12v^2-1)(64u^6+1) + u^2v^2(8v^2+3)(32u^6+2) \\ &- 112v^3u^3(4u^2+1) + 48v^2(2v^2+1) + v^4(1+(2u)^8) = 0. \end{aligned} \tag{62}$$

Proof. Using the equations (60) and (22), we deduce that

$$\begin{aligned} &6PQc^4d^4 - 24PQd^8c^4 + 48PQd^8c^8 - 64PQd^{12}c^4 + 2c^{16} + 4c^{12} - 128d^{12}c^4 \\ &- d^4 + c^8 + 512c^{16}d^{16} + 16c^8d^8 + 8PQc^{12}d^4 + 160PQc^{12}d^8 + 128PQc^8d^{12} \\ &+ 384PQc^{12}d^{12} + 8c^8d^4 - 2PQc^{12} - PQd^4 - PQc^8 - 2d^8(8c^4-1) + 4d^4c^4 \\ &- 80c^{12}d^4 + 192c^{12}d^8 + 2304c^{12}d^{12} + 32c^{16}d^4 + 192c^{16}d^8 + 512c^{16}d^{12} = 0, \end{aligned} \tag{63}$$

where P and Q are as defined in the equation (22), $c := V(q)$ and $d := V(q^6)$. Isolating the terms containing PQ on one side of the equation (63), we deduce that

$$\begin{aligned}&(d^8 + 6d^4c^8 + 48d^8c^8 + 96c^8d^{12} + 16c^{16}d^4 + 96c^{16}d^8 + 256c^{16}(d^{12} + d^{16})\\&+ c^4d^2 + c^{16} + 64d^{14}c^4 - 12d^2c^{12} + 112d^6c^{12} + 448d^{10}c^{12} - 768d^{14}c^{12})\\&(d^8 + 6d^4c^8 + 48d^8c^8 + 96(c^8d^{12} + c^{16}d^8) + 16c^{16}d^4 + 256c^{16}(d^{12} + d^{16})\\&- c^4d^2 + c^{16} - 64d^{14}c^4 + 12d^2c^{12} - 112c^{12}d^6(1 + 4d^4) + 768d^{14}c^{12}) = 0.\end{aligned} \tag{64}$$

From the definitions of v and u, we have $c = o(q^{1/8})$ and $d = o(q^{3/4})$ as $q \to 0$, it can be seen that the second factor vanishes rapidly than the first factor for q sufficiently small. Thus by the Identity Theorem, second factor vanishes identically. Hence the proof.

□

Theorem 6.4. *If $v := V^4(q)$ and $u := V^2(q^{10})$, then*

$$\begin{aligned}&v^6(1 + 2^{12}u^{12}) - uv(2^{10}u^{10} + 1)[70v^4 - 20v^2 + 1] + u^2v^2(2^8u^8 + 1)\\&[24v^4 + 655v^2 - 40] + 20u^3v^3(2^6u^6 + 1)[186v^2 - 13] + 15u^4v^2(2^4u^4 + 1)\\&[1 + 2^4v^2(v^2 + 1)] - 2u^5v(2^2u^2 + 1)[7648v^4 - 200v^2 - 5] + u^6\\&+ 40v^2u^6(32v^4 - 476v^2 + 35) = 0.\end{aligned} \tag{65}$$

Proof. The proof of the equation (65) is similar to the proof of the equation (62) except that in the place of the equation (22), the equation (26) is employed.

□

Theorem 6.5. *If $v := V^4(q)$ and $u := V^2(q^{14})$, then*

$$\begin{aligned}&(2^{16}u^{16} + 1)v^8 + uv(1 + 2^{14}u^{14})[280v^6 - 210v^4 + 28v^2 - 1] + u^2v^2(2^{12}u^{12} + 1)\\&\times [32v^6 + 13468v^4 - 728v^2 - 7] - 28u^3v^3(2^{10}u^{10} + 1)[1640v^4 - 1396v^2 + 81]\\&+ 14u^4v^2(1 + 2^8u^8)[32v^6 + 9776v^4 - 64 + 421v^2] + 14u^5v(2^6u^6 + 1)[60992v^6\\&+ 3392v^4 - 476v^2 + 1] + 28v^2u^6(2^4u^4 + 1)[128v^6 - 7440v^4 + 456v^2 + 5]\\&+ 32u^7v^3(2^2u^2 + 1)[-85360v^4 + 1883 - 25508v^2]\\&+ 32v^2u^8[560v^6 - 5614v^2 - 135072v^4 + 903] + u^8 = 0.\end{aligned} \tag{66}$$

Proof. The proof of the equation (66) is similar to the proof of the equation (62) except that in the place of the equation (22), the equation (30) is employed.

□

Theorem 6.6. *If $v := V^8(q)$ and $u := V^2(q^{16})$, then*

$$
\begin{aligned}
&v^4(2^{16}u^{16}+1)+u(1+2^{14}u^{14})[-32v^4-304v^2+32v+768v^3-1]\\
&+8u^2(2^{12}u^{12}+1)[7424v^3-208v^2+60v^4-32v+1]+4u^3(2^{10}u^{10}+1)\\
&[-3-1120v^4+96v-17296v^2+264448v^3]+64u^4(2^8u^8+1)[32v-7984v^2\\
&+455v^4-1+123648v^3]+16u^5(2^6u^6+1)[2347776v^3-142896v^2-480v\\
&+15-8736v^4]+128u^6(2^4u^4+1)[4004v^4-65328v^2+1041152v^3-1\\
&32v]+64u^7(2^2u^2+1)[-22880v^4+352v-363792v^2-11+5775616v^3]\\
&+512u^8[-128v+6435v^4+1602560v^3-99136v^2+4]=0.
\end{aligned}
\tag{67}
$$

Proof. The proof of the equation (67) is similar to the proof of the equation (62) except that in the place of the equation (22), the equation (31) is employed. □

Using the values of $h_{2,n}$, we establish some explicit evaluations of $V(q)$ in the following Theorem.

Theorem 6.7. *We have*

$$V^4(e^{-\pi\sqrt{2}})=\frac{\sqrt{2}-1}{4}, \tag{68}$$

$$V^4(e^{-2\pi\sqrt{3}})=\frac{(\sqrt{3}-\sqrt{2})^2(\sqrt{2}-1)^2}{4}, \tag{69}$$

$$V^4(e^{-\pi\sqrt{1/3}})=\frac{\sqrt{3}+1}{8\sqrt{2}}, \tag{70}$$

$$V^4(e^{-2\pi/\sqrt{3}})=\frac{(\sqrt{3}-\sqrt{2})^2(\sqrt{2}+1)^2}{4}, \tag{71}$$

$$V^4(e^{-\pi\sqrt{3}})=\frac{\sqrt{3}-1}{8\sqrt{2}}, \tag{72}$$

$$V^4(e^{-2\pi\sqrt{7}})=\frac{(\sqrt{2}-1)^4(2\sqrt{2}-\sqrt{7})^2}{4}, \tag{73}$$

$$V^4(e^{-2\pi/\sqrt{7}})=\frac{(\sqrt{2}-1)^4(2\sqrt{2}+\sqrt{7})^2}{4}, \tag{74}$$

$$V^4(e^{-\pi/\sqrt{7}})=\frac{3+\sqrt{7}}{16\sqrt{2}}, \tag{75}$$

$$V^4(e^{-\pi\sqrt{7}})=\frac{3-\sqrt{7}}{16\sqrt{2}}. \tag{76}$$

7 Some identities for a continued fraction of Eisenstein

In [10], Mahadeva Naika, Bairy and Manjunatha have established the following continued fraction

$$E(q) := \frac{(q;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1+}\frac{2q}{1-q^2+}\frac{-q^3-q}{1+q^4+}\frac{q^5+q^3}{1-q^6+}\cdots, \quad \text{for } |q|<1. \tag{77}$$

They have also established several modular relations between a continued fraction of Eisenstein $E(q)$ and $E(q^n)$ for $n = 2, 3, 4, 5, 7, 8, 9, 11, 13,$ 15, 17, 19, 23, 25, 29, 31 and 55. They have also established integral representation for $E(q)$ and several explicit evaluations for $E(e^{-\pi\sqrt{n}})$, where n is any positive rational.

Lemma 7.1. *We have*

$$\frac{\varphi^2(q)}{\varphi^2(q^2)} = \frac{2}{1+E^4(q)}. \tag{78}$$

Proof. Using the equations (9), (10), (14) and (15), we arrive at the equation (78).

□

Lemma 7.2. *For any positive rational number n, we have*

$$E^4(e^{-\pi\sqrt{n/2}}) = \frac{\sqrt{2}-h_{2,n}^2}{h_{2,n}^2}. \tag{79}$$

Proof. Using the equations (78) and (7) with $k = 2$, we arrive at the equation (79).

□

Theorem 7.3. *If $e := E^2(q)$ and $f := E^4(q^6)$, then*

$$\begin{aligned} &f^4(1+e^8) + 8ef(3f^2-4)(1+e^6) + 4e^2f^2(f^2+6)(1+e^4) \\ &- 56f^3e^3(1+e^2) + 6e^4f^2(f^2+8) + (2e)^4 = 0. \end{aligned} \tag{80}$$

Proof. Using the equations (78) and (22), we deduce that

$$\begin{aligned} &8d^8 - 16c^8 + 8d^{12} - 16PQd^4c^8 + 8PQd^4c^4 + 24PQc^8d^{12} + 12PQc^8d^8 \\ &- 4PQc^4d^{12} + 20PQc^4d^8 - 20PQc^{12}d^8 - 40PQc^{12}d^4 + 36PQc^{12}d^{12} \\ &+ 12PQc^8d^{16} - 16PQc^{16}d^4 + 16PQc^{12}d^{16} + 14PQc^{16}d^{12} + 6PQc^{16}d^{16} \\ &+ c^{16}d^{16} - 32d^4c^8 + 32d^4c^4 + 24c^8d^{12} + 8c^8d^8 - 40c^4d^{12} + 16c^4d^8 \\ &+ 72c^{12}d^{12} + 4c^{12}d^{16} + 6c^8d^{16} + 4c^4d^{16} - 16PQc^8 - 64c^{12}d^4 + d^{16} \\ &- 8PQc^{16}d^8 - 2PQd^{16} - 6PQd^{12} - 4PQd^8 - 32c^4 - 16PQc^4 = 0, \end{aligned} \tag{81}$$

where P and Q are as defined in the equation (22), $c := E(q)$ and $d := E(q^6)$. Isolating the terms containing PQ on one side of the equation (81) and squaring both sides, we find that

$$\begin{aligned}&(16c^8 - 56c^6d^{12} + c^{16}d^{16} + 48c^8d^8 + 24c^4d^8 + 4c^{12}d^{16} + 6c^8d^{16} + 4c^4d^{16}\\&+ 24c^{12}d^8 - 32c^2d^4 + 24c^2d^{12} - 32c^{14}d^4 - 56c^{10}d^{12} + 24c^{14}d^{12} + d^{16})\\&(16c^8 + c^{16}d^{16} + 48c^8d^8 + 24c^4d^8 + 4c^{12}d^{16} + 56c^6d^{12} + 6c^8d^{16} + 4c^4d^{16}\\&+ 24c^{12}d^8 + 32c^2d^4 - 24c^2d^{12} + 32c^{14}d^4 + 56c^{10}d^{12} - 24c^{14}d^{12} + d^{16}) = 0.\end{aligned} \tag{82}$$

By examining the behaviour of the factors of the equation (82) near $q \to 0$, it can be seen that the first factor vanishes rapidly than the second factor for q sufficiently small. By the Identity Theorem first factor vanishes identically. This completes the proof.

□

Theorem 7.4. *If* $e := E^2(q)$ *and* $f := E^4(q^{10})$, *then*

$$\begin{aligned}&f^6[e^{12} + 1] + 4ef(1 + e^{10})[2^5(5f^2 - 4) - 35f^4] + 20e^3f^3(1 + e^6)[93f^2 - 104]\\&+ 2e^2f^2(1 + e^8)[3f^4 - 2^85 + 1310f^2] + 15e^4f^2(1 + e^4)[f^3 + 2^4(1 + f^2)]\\&+ 8e^5f(1 + e^2)[20(2 + 5f^2) - 239f^4] + 4e^6[1400f^2 - 1190f^4 + 2^4 + 5f^6] = 0.\end{aligned} \tag{83}$$

Proof. The proof of the equation (83) is similar to the proof of the equation (80) except that in the place of the equation (22), the equation (26) is employed.

□

Theorem 7.5. *If* $e := E^2(q)$ *and* $f := E^4(q^{14})$, *then*

$$\begin{aligned}&2^4\{ef(1 + e^{14})[35f^6 - 2^7(4 - 7f^2) - 420f^4] - e^7f^3(1 + e^2)[5335f^4\\&- 28(1076 - 911f^2)]\} + 112\{e^3f^3(1 + e^{10})[2^3(349f^2 - 18^2) - 205f^4]\\&+ e^5f(1 + e^6)[2^4(4 + 53f^4 - 119f^2) + 953f^6]\} + 28f^2\{2e^6(1 + e^4)\\&\times [912f^2 + f^6 - 930f^4 + 160] + e^4(1 + e^8)[2^3(611f^4 - 2^{10} + 421f^2)\\&+ f^6]\} + 8e^2f^2(1 + e^{12})[f^6 + 6734f^4 - 2^67(13f^2 + 2)] + e^8[70f^8 + 2^8\\&+ 2^67(1032f^2 - 401f^6 - 603f^4)] + f^8(e^{16} + 1) = 0.\end{aligned} \tag{84}$$

Proof. The proof of the equation (84) is similar to the proof of the equation (80) except that in the place of the equation (22), the equation (30) is employed.

□

Theorem 7.6. *If* $e := E^2(q)$ *and* $f := E^8(q^{16})$, *then*

$$\begin{aligned}
&f^4(e^{16}+1)+2^4e(1+e^{14})[-f^4-2^7(19f^2+3f^3+2^5f-2^4)]\\
&+8e^2(1+e^{12})[15f^4+2^{10}[2^4(1-2f)+29f^3-13f^2]]+2^4e^3(1+e^{10})\\
&\times[2^7[2^4(6f-3)-1081f^2+1033f^3]-35f^4]+4e^4(1+e^8)[455f^4\\
&+2^{12}[2^4(2f-1)-499f^2+483f^3]]+16e^5(1+e^6)[2^73[3057f^3-2977f^2\\
&-2^4(10f-5)]-273f^4]+8e^6(1+e^4)[1001f^4+2^{10}[4067f^3-4083f^2\\
&+2^4(2f-1)]]+16e^7(1+e^2)[2^711(2^4(2f-1)-2067f^2+2051f^3)\\
&-715f^4]+2e^8[6435f^4+2^{14}(1565f^3-2^4(2f-1)-1549f^2)]=0.
\end{aligned}\tag{85}$$

Proof. The proof of the equation (85) is similar to the proof of the equation (80) except that in the place of the equation (22), the equation (31) is employed. □

Using the values of $h_{2,n}$, we establish some explicit evaluations of $E(q)$ in the following Theorem.

Theorem 7.7. *We have*

$$E^4(e^{-\pi/\sqrt{2}}) = \sqrt{2}-1, \tag{86}$$

$$E^4(e^{-\pi\sqrt{3}}) = \frac{\sqrt{3}+1}{2\sqrt{2}}, \tag{87}$$

$$E^4(e^{-\pi/2\sqrt{3}}) = (\sqrt{3}-\sqrt{2})^2(\sqrt{2}-1)^2, \tag{88}$$

$$E^4(e^{-\pi/\sqrt{3}}) = \frac{\sqrt{3}-1}{2\sqrt{2}}, \tag{89}$$

$$E^4(e^{-\pi\sqrt{3}/2}) = (\sqrt{3}-\sqrt{2})^2(\sqrt{2}+1)^2, \tag{90}$$

$$E^4(e^{-\pi\sqrt{7}}) = \frac{3+\sqrt{7}}{4\sqrt{2}}, \tag{91}$$

$$E^4(e^{-\pi/\sqrt{7}}) = \frac{3-\sqrt{7}}{4\sqrt{2}}, \tag{92}$$

$$E^4(e^{-\pi/2\sqrt{7}}) = (\sqrt{2}-1)^4(2\sqrt{2}-\sqrt{7})^2, \tag{93}$$

$$E^4(e^{-\pi\sqrt{7}/2}) = (\sqrt{2}-1)^4(2\sqrt{2}+\sqrt{7})^2. \tag{94}$$

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