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The Dual Neighborhood Number of a Graph

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Abstract

A set $S \subseteq V(G)$ is a neighborhood set of a graph $G = (V, E)$, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of a graph G induced by v and all vertices adjacent to v. The dual neighborhood number $\eta^{+2}(G) = \text{Min. } \{|S_1| + |S_2| : S_1, S_2$ are two disjoint neighborhood set of $G\}$. In this paper, we extended the concept of neighborhood number to dual neighborhood number and its relationship with other neighborhood related parameters are explored.

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1 Introduction

All the graph considered here are finite and undirected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. The private neighborhood $PN(v, X)$ of $v \in X$ is defined by $PN(v, X) = N[v] - N[X - \{v\}]$. Let $deg(v)$ be the degree of vertex v and usual $\delta(G)$ the minimum degree and $\triangle(G)$ the maximum degree. $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a(an) vertex (edge) cover of G. $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G. For a real number $x > 0$, let$\lceil x \rceil$ be the least integer not less than x and $\lfloor x \rfloor$ be the greatest integer not greater than x. For graph-theoretical terminology and notation not defined here we follow [4].

A set $S \subseteq V$ is a neighborhood set of G, if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of a graph G, see [11]. A neighborhood set $S \subseteq V$ is a minimal neighborhood set, if $S - v$ for all $v \in S$, is not a neighborhood set of G. The nomatic number of G, $N(G)$ is the largest number of sets in a partition of V into disjoint minimal neighborhood sets of a graph G, see [7]. Further, a neighborhood set $S \subseteq V$ is called an independent neighborhood set, if $\langle S \rangle$ is an independent and neighborhood set of G, see [9] /paired neighborhood set, if $\langle S \rangle$ contains at least one perfect matching, see [12] /maximal neighborhood set, if $V - S$ does not contain a neighborhood set of G, see [13] /inverse neighborhood set, if $V - S$ contain a neighborhood set of G, see [8]. The minimum cardinality taken over all independent / maximal / inverse neighborhood set in G is called an independent / paired / maximal / inverse neighborhood number of G and is denoted by $\eta_i(G)$ /$\eta_{pr}(G)$ / $\eta_m(G)$ /$\eta^{-1}(G)$, respectively.

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Further, the dual domination number of a graph G is the minimum cardinality of the union of two disjoint dominating sets in G. The dual domination number $\gamma^{+2}(G) =$ Min.$\{|S_1| + |S_2| : S_1, S_2$ are two disjoint dominating set of $G\}$, see [6]&[10]. For complete review of domination theory, see [5] & [14].

Analogously, we now define dual neighborhood number as follows: A graph G having k- disjoint neighborhood set (kDN-set) with $k \geq 2$ is called a k-disjoint neighborhood graph (abbreviated kDN-graph), where k is a positive integer. Note that, if $k = 1$, then G having a 1- neighborhood set and the 1- neighborhood number $\eta(G)$ of a graph G is the usual neighborhood set and neighborhood number of a graph G, respectively. In fact, if $k = 2$, then G having a 2-disjoint neighborhood set (2DN-set). The dual neighborhood number $\eta^{+2}(G) =$ Min. $\{|S_1| + |S_2| : S_1, S_2$ are 2DN-set of $G\}$. A graph G for which kDN-set with $k \geq 2$ is called a kDN-graph. A neighborhood set S with minimum cardinality is called η - set of G. Similarly, the other sets can be expected. For more details on neighborhood number and its related parameters, [1], [3] & [7].

2 Existing Results

We make use of the following results in sequel.

Theorem 2.1 [4] *A graph is bipartite if and only if all its cycles are even.*

Theorem 2.2 [11] *For any non trivial graph G of order p, $\eta(G) = 1$ if and only if G has a vertex of degree $p-1$. Thus $\eta(G)$ of each of the following graph is one* (i) K_p; (ii) $K_{1,p-1}$; (iii) W_p.
Further, if G is bipartite graph without isolates, with bipartition $\{V_1, V_2\}$ of $V(G)$, then $\eta(G) = Min.\ \{V_1, V_2\}$.

Theorem 2.3 [11]
(i) *$\eta(G) = \alpha_0(G)$, provided G has no triangles.*
(ii)*Let G be any graph and S be any subset of $V(G)$. Then S is an η-set of G if and only if every edge in $\langle V - S\rangle$ belongs to $\langle N[u]\rangle$ for some $u \in S$.*

Theorem 2.4 [13]. *A neighborhood set S of a graph G is a maximal neighborhood set of G if and only if there exist two adjacent vertices $u, v \in S$ such that every vertex $w \in V - S$ is adjacent to at most one of u and v.*

Theorem 2.5 [12]. *If G has no isolated vertices, then*
(i) $\eta_{pr}(G) \geq Max.([p/\triangle(G)], [2p/\triangle(G) + 1])$
(ii) $\eta_{pr}(G) \geq (4p - 2q)/3$
(iii) $\eta_{pr}(G) \leq \eta G)$.

Theorem 2.6 [7]. *For any graph G,*
(i) $N(G) \leq \delta(G) + 1$,
(ii) $\eta(G) + \eta(\overline{G}) \leq p + 1$, *and equality holds if and only if $G \approx K_p$ or $\overline{K_p}$,*
(iii) $\eta(G) + N(G) \leq p + 1$, *and equality holds if and only if $G \approx K_p$ or $\overline{K_p}$,*
(iv) *For any graph G, $N(G) = 1$ if and only if $G \approx \overline{K_p}$ or $C_{2r+1}; r \geq 2$, and $N(G) = p$ if and only if $G \approx K_p$.*

3 Main Results

These easily computed values of $\eta^{+2}(G)$ are stated without proof.

Proposition 3.1 .
(i) *For any complete graph K_p with $p \geq 2$ vertices, $\eta^{+2}(K_p) = 2$*
(ii) *For any wheel graph W_p with $p \geq 4$ vertices, $\eta^{+2}(W_p) = \lfloor p/2 \rfloor + 1$*
(iii) *For any cycle C_{2n} with $n \geq 2$, path P_p with $p \geq 2$ and complete bipartite graph $K_{r,s}$ with $1 \geq r \leq s$ vertices, $\eta^{+2}(C_{2n}) = \eta^{+2}(P_p) = \eta^{+2}(K_{r,s}) = p$.*

Let G be a graph having more than one minimal neighborhood set. Then multiple neighborhood set is a union of all minimal neighborhood set of G and the cardinality of multiple neighborhood set is called multiple neighborhood number and is denoted by $\eta^{+k}(G) = \sum \mid S_i \mid$, where S_i $(1 \leq i \leq k)$ is a minimal neighborhood set of G.

Proposition 3.2 .
(i) $\eta^{+k}(K_p) = \eta^{+k}(P_p) = \eta^{+k}(C_{2n}) = \eta^{+k}(K_{r,s}) = p$, *if* $\{p, n\} \geq 2$ *and* $\{r, s\} \geq 1$.
(ii) $\eta^{+k}(C_{2n+1}) = \eta(C_{2n+1}) = \lceil p/2 \rceil$, *if* $n \geq 2$.

A graph G for which k-independent neighborhood set ($kIN - set$) with $k \geq 2$ is called a kIN-graph. Also, here we consider an invariant to both $\eta^{+2}(G)$ and $\eta_i^{+2}(G)$, namely, the minimum cardinality of the disjoint union of minimum neighborhood set S and an independent neighborhood set S_i, which we will denote $\eta\eta_i(G)$. We will call such a pair of neighborhood sets (S, S_i) a $\eta\eta_i$-pair (or simply, a mixed η - set). We note that every graph G with no isolates has a $\eta\eta_i$-pair, which can be found by letting S_i be any maximal independent set, and then noting that complement $V - S_i$ is a neighborhood set, and there fore contains a minimal neighborhood set, say S.

By the definitions of $\eta(G)$ / $\eta_i(G)$ /$\eta_{pr}(G)$ / $\eta_m(G)$ /$\eta^{-1}(G)$/ $\gamma^{+2}(G)$/ $\eta^{+2}(G)$, we have the following inequalities, since their proofs are immediate, they are omitted.

Proposition 3.3 *Let G be a kIN- graph with no isolated vertices. Then,*
(i) $\eta(G) \leq \eta_i(G) \leq \eta^{+2}(G)$
(ii) $2 \leq \gamma_{pr}(G) \leq \eta^{+2}(G) \leq p$
(iii) $2 \leq \eta^{+2}(G) \leq \eta(G) + \beta_0(G)$
(iv) $\eta(G) \leq \eta^{-1}(G) \leq p - \eta(G) \leq \eta^{+2}(G)$
(v) $\eta(G) + 1 \leq \eta^{+2}(G) \leq \eta(G) + \eta^{-1}(G)$
(vi) $2\eta(G) \leq \eta^{+2}(G) \leq \eta\eta_i(G) \leq \eta_i^{+2}(G)$
(vii) $\gamma(G) + 1 \leq \gamma^{+2}(G) \leq \eta^{+2}(G)$.

Theorem 3.1 *A graph G with no isolated vertices has $V(G)$ as its 2DN-set if and only if G is a bipartite graph.*

Proof. Clearly, a graph is bipartite if and only if each of its components is bipartite. So, without loss of generality, we assume that G is connected. Let G be a bipartite graph with $V = V_1 \cup V_2$, so that every edge of G joins a vertex of V_1 with the vertex of V_2. Then V_1 and V_2 have independent set of $V(G)$, and the minimum and maximum cardinality of V_1 and V_2 have a η - set and η^{-1}-set of G, respectively. Thus $\eta^{+2}(G) = p$. This proves the necessity. Assume that $\eta^{+2}(G) = p$ and G is not a bipartite graph. Then there exist at least three vertices u, v and w such that u and v are adjacent and w is adjacent to both u and v, which is form a odd cycle and by Theorem 2.1, this implies that $\{V - w\}$ is a 2DN-set of G, which is a contradiction. Thus the sufficiency is proved.

Theorem 3.2 *Let G be a kDN-graph with no isolated vertices. Then $\eta^{+2}(G) = 2$ if and only if there exist two adjacent vertices $u, v \in V(G)$ such that $deg(u) = deg(v) = p - 1$.*

Proof. Suppose $\eta^{+2}(G) = 2$ holds. On contrary, suppose the graph G not satisfies the above condition, then there exist at least three vertices u, v and w such that u and v are adjacent and w is adjacent to at most one of u and v, suppose v is adjacent to w, then v is a vertex of the minimal neighborhood set S and whose complement $(V - \{v\})$ is also a neighborhood set of a graph G. This implies that $\eta^{+2}(G) > 2$, which is a contradiction. This proves necessity, sufficiency is obvious.

Theorem 3.3 *Let G be a graph with no isolated vertices. Then $\eta_{pr}(G) = 2\eta(G)$ if and only if every η - set of G is an η_i - set of G.*

Proof. Let G be a graph having $\eta_{pr}(G) = 2\eta(G)$. Then we have the followings cases:
Case 1. Suppose that a η-set, say S' is an independent set of G, then the complement $(V - S')$ is contain a another set, say S'', which is also a η - set as well as η_i - set of G, since two disjoint neighborhood set S' and S'' are both η_i - sets of a graph G, hence G is a $2IN$- graph with $v_i \in S'$ and $v_j \in S''$; $i \neq j$. Thus, the collection of all pairs of edges $v_i v_j \in E(G)$ in $S' \cup S''$ form a paired neighborhood set of a graph G and the results desired.
Case 2. Suppose that a η-set S' is not independent. Then, there is an adjacent pair of vertices u and w in S', this form a paired- neighborhood set for G by pairing u and w and pairing each vertex in $S' - \{u, w\}$ with a neighbor in $V - S'$. This is possible since the minimality of S' implies that for each $x \in S'$, either x has a private neighbor $PN(x, S')$ or x is isolated in $\langle S' \rangle$. Let I be the set of isolates in S' without private neighbors. Now each vertex in I must have at least one neighbor in $V - S'$, since G has no isolates. The minimality of S' implies that no two vertices in I have a common neighbor. Hence, each vertex in $V - u, w$ can be paired with a neighbor forming a paired- neighborhood set of order $\eta(G) + \eta(G) - 2 < 2\eta(G)$, that is $\eta_{pr}(G) < 2\eta(G)$, which is a contrary to our hypothesis.

Theorem 3.4 *For any kDN-graph G with no isolates, $\eta_{pr}(G) \leq \eta^{+2}(G)$. Further, the bound is attained if the graph G satisfies one of the following*
(i) *$G \approx mK_2$ or $K_{t,t}$; $t \geq 1$,*
(ii) *There exist at least two vertices u, v such that $deg(u) = deg(v) = p - 1$.*

Proof. Clearly every $2DN$-set is a paired neighborhood set of a graph G, then $\eta_{pr}(G) \leq \eta^{+2}(G)$ follows. Further, by Theorem 3.3, the bound is attained.

Theorem 3.5 *Let G be a kDN-graph with no isolated vertices. Then*
(i) $\eta^{+2}(G) \geq Max.\ \{[p/\Delta(G)], [2p/\Delta(G)+1]\}$, *bound is attained if* $G = mK_2$,
(ii) $\eta^{+2}(G) \geq (4p-2q)/3$, *bound is attained if* $G = K_3$ *or* mK_2 *or* $K_2 + \overline{K_p}$.

Proof. (i) and (ii) follows from Theorem 2.4 and Theorem 3.4.

Theorem 3.6 *For any complete multipartite graph* $G = K_{r_1,r_2,\ldots,r_k}$,
(i) $\eta^{+2}(G) = Min.\{6, r_1 + r_2\}$, *if* $2 \leq r_1 \leq r_2 \leq \ldots \leq r_k$
(ii) $\eta^{+2}(\overline{G}) = 2k$, *if* $2 \leq r_1 \leq r_2 \leq \ldots \leq r_k$
(iii) $\eta^{+2}(G) \leq \eta_m(G)$, *if* $3 \leq r_1 \leq r_2 \leq \ldots \leq r_k$
(iv) $\eta^{+2}(G) \geq \eta_m(G)$, *if* $1 \leq r_1 \leq r_2$
(v) $\eta^{+2}(\overline{G}) = \eta^{+2}(G)$ *if and only if* $G \approx K_{2,2}$ *or* K_{r_1,r_2,r_3} ; $3 \leq r_1 \leq r_2 \leq r_3$.

Proof. Let $G = K_{r_1,r_2,\ldots,r_k}$ be a complete multipartite graph with $2 \leq r_1 \leq r_2 \leq r_3 \leq 3$. Then $V = V_1 \cup V_2 \cup V_3$ with $\langle V_1\rangle, \langle V_2\rangle$ and $\langle V_3\rangle$ are an independent in G and complete in $\overline{G}$, respectively. Thus, $\eta^{+2}(G) = r_1 + r_2$. Also, if $4 \leq r_1 \leq r_2 \leq r_3 \leq k$, then $\eta^{+2}(G) = 6$. Thus (i) holds and hence by Theorem 3.2, (ii) follows.
By the definition of $\eta_m(G)$, if $3 \leq r_1 \leq r_2 \leq \ldots \leq r_k$ vertices, then (iii) follows and if $1 \leq r_1 \leq r_2$, the (iv) follows.
Suppose $\eta^{+2}(\overline{G}) = \eta^{+2}(G)$ holds. On contrary, suppose G is not isomorphic with $K_{2,2}$ or K_{r_1,r_2,r_3} ; $3 \leq r_1 \leq r_2 \leq r_3$. Then there exist at least one of the partite set V_i for $1 \leq i \leq k$, in complete multipartite graph G contains exactly one vertex, thus $\eta^{+2}(G)$ does not exist, which is a contradiction. This proves necessity, sufficiency is obvious and hence (v) follows.

A set $S \subseteq V(G)$ is a double neighborhood set of G such that for every vertex $v \in V$, $|N[v] \cap S| \geq 2$. The double neighborhood number $\eta_d(G)$ of G is the minimum cardinality of a double neighborhood set in G, see [3].

Observation 3.1 *If vertex v has degree one, then both v and its support must be in double neighborhood set as well as dual neighborhood set of a graph G.*

Theorem 3.7 *For any kDN-graph G with no isolates,* $\eta_d(G) \leq \eta^{+2}(G)$.

Proof. By the definition of $\eta_d(G)$ and $\eta^{+2}(G)$. Clearly, every dual neighborhood set is a double neighborhood set of a graph G. Then $\eta_d(G) \leq \eta^{+2}(G)$ follows.

Theorem 3.8 *Let T be a tree such that both T and $\overline{T}$ having kDN-sets with no isolated vertices. Then,* $\eta^{+2}(T) = \eta^{+2}(\overline{T})$ *if and only if* $T \approx P_4$.

Proof. Suppose $\eta^{+2}(T) = \eta^{+2}(\overline{T})$ holds. On contrary, suppose T is not isomorphic with P_4. Then we consider the following cases:
Case 1. If tree T has at least two adjacent cut vertices u and v with $\{deg(u), deg(v)\} \geq 3$, then by Theorem 3.1, we have V is a dual neighborhood set of T. But the dual neighborhood set of $\overline{T}$ is $V - (u, v)$, since $q(\overline{T}) = (\frac{1}{2}p(p-1)) - (p-1)$ and hence this implies that $\eta^{+2}(\overline{T}) < \eta^{+2}(T) = p$, which is a contradiction.
Case 2. If tree T has at least two non adjacent cut vertices, which form a path of length greater than or equal to 4, then by Theorem 3.1, we have $\eta^{+2}(\overline{T}) < \eta^{+2}(T) = p$, which is again a contradiction. This proves necessity, sufficiency is obvious.

By Theorem 2.5, and the definitions of $\eta(G)$, $\eta^{+2}(G)$ and $N(G)$, we have following results, which are straight forward, hence we omits the proofs.

Theorem 3.9 *Let G be a graph such that both G and $\overline{G}$ have no isolated vertices. Then*
(i) $\eta(G) \leq N(G)$, *provided G does not contain $C_{2r+1}; r \geq 2$,*
(ii) $\eta(G) \leq N(\overline{G})$, *provided G does not contain $C_{2r+1}; r \geq 2$,*
(iii) $\eta^{+2}(G) \leq 2p/N(G)$, *provided G having kDN-sets,*
(iv) $N(G) \leq \eta^{+2}(G)$, *provided G having kDN-sets and which is not contain a $(p-1)$-regular graph.*
(v) $N(T) = N(\overline{T})$ *if and only if $T \approx P_4$ or $K_{1,t}; t \geq 2$, with exactly one subdivided edge.*

4 Conclusions

Being new concepts, dual domination and dual neighborhood are both invariants whose properties are relatively unknown. For more details on the study of the disjoint dominating sets and its related parameters in graphs, see [6]. Many questions are suggested by this research, among them are the following.
1. When $\eta^{+2}(G) = \gamma^{+2}(G)$?
2. When $\eta^{+2}(\overline{G}) = \gamma^{+2}(\overline{G})$?
3. When $\eta^{+2}(G) = \eta_m(G)$?
4. When $\eta^{+2}(G) = \eta_d(G)$?
5. When $\eta\gamma(G) = \eta^{+2}(G)$?

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