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MODULAR EQUATIONS FOR THE RATIOS OF RAMANUJAN'S THETA FUNCTION ψ AND EVALUATIONS

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Abstract. In this paper, we establish several new modular equations of degree 9 using Ramanujan's mixed modular equations. We also establish several general formulas for explicit evaluations of ratios of Ramanujan's theta function.

1. Introduction

In Chapter 16, of his Second notebook [**16**] Ramanujan has defined his theta function as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.1}$$

Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.4}$$

$$\chi(q) := (-q;q^2)_\infty, \tag{1.5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1.$$

In [**5**] and [**17**], the authors have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} := \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4} e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(-e^{-\pi\sqrt{nk}})}, \tag{1.6}$$

and

$$l'_{k,n} := \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4} e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(e^{-\pi\sqrt{nk}})}. \tag{1.7}$$

They have established several properties and some explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k. They also listed some applications of

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these parameters for the Rogers-Ramanujan continued fraction and Ramanujan's cubic continued fraction. In [**11**], M. S. Mahadeva Naika and S. Chandankumar have established several new modular equations of degree 2 and also general formulas for the explicit evaluation of ratios of Ramanujan's theta function φ. In [**15**], Mahadeva Naika, K. S. Bairy and M. Manjunatha have established several new modular equations of degree 4 and also general formulas for the explicit evaluations of ratios of Ramanujan's theta function φ. For more details see [**1**], [**2**], [**8**], [**9**], [**12**], [**14**].

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$$ {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, $$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $| x |< 1$. Let

$$ z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{1.8} $$

and

$$ q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{1.9} $$

where $0 < x < 1$.

Let r denote a fixed natural number and assume that the following relation holds:

$$ r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{1.10} $$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.10). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In Section 2, we collect the identities which are useful in proving our main results. In Section 3, we establish some new modular equations of degree 9 which are analogous to Ramanujan's modular equations. In Section 4, we establish general formulas for explicit evaluations of $l_{9,n}$ and also establish relations among $l_{9,n}$ and $l'_{9,n}$.

2. Preliminary Results

In this section, we collect some identities which are useful in proving our main results.

Lemma 2.1. [**10**, Theorem 2.2] *We have*

$$ \frac{f^3(-q)}{f^3(-q^9)} = \frac{\psi(q)}{\psi(q^9)}\left(\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}\right)^2, \tag{2.1} $$

$$ \frac{f^3(-q^2)}{f^3(-q^{18})} = \frac{\psi^2(q)}{\psi^2(q^9)}\frac{\psi(q)-3q\psi(q^9)}{\psi(q)-q\psi(q^9)}. \tag{2.2} $$

Lemma 2.2. [**6**, Ch. 17, Entry 10(i) and Entry 11(ii), pp. 122–123] *If β, γ and δ are of degrees r, s and rs over α respectively, then*

$$\varphi(q) = \sqrt{z_1}, \tag{2.3}$$
$$\varphi(q^r) = \sqrt{z_r}, \tag{2.4}$$
$$\varphi(q^s) = \sqrt{z_s}, \tag{2.5}$$
$$\varphi(q^{rs}) = \sqrt{z_{rs}}, \tag{2.6}$$
$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z_1}\{\alpha(1-\alpha)\}^{1/8}, \tag{2.7}$$
$$\sqrt{2}q^{r/8}\psi(-q^r) = \sqrt{z_r}\{\beta(1-\beta)\}^{1/8}, \tag{2.8}$$
$$\sqrt{2}q^{s/8}\psi(-q^s) = \sqrt{z_s}\{\gamma(1-\gamma)\}^{1/8}, \tag{2.9}$$
$$\sqrt{2}q^{rs/8}\psi(-q^{rs}) = \sqrt{z_{rs}}\{\delta(1-\delta)\}^{1/8}. \tag{2.10}$$

Lemma 2.3. [**6**, Ch.20, Entry 1(ii), p. 345] *We have*

$$1+3q\frac{\psi(-q^9)}{\psi(-q)} = \left(1+9q\frac{\psi^4(-q^3)}{\psi^4(-q)}\right)^{1/3}. \tag{2.11}$$

Lemma 2.4. [**6**, Ch. 20, Entry 11 (viii), (ix), p. 384] *Let α, β, γ and δ be of the first, third, fifth and fifteenth degrees respectively. Let m denote the multiplier connecting α, β and m' be the multiplier relating γ, δ, then*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}}, \tag{2.12}$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{2.13}$$

Lemma 2.5. [**6**, Ch. 20, Entry 13 (i) and (ii), p. 401] *Let α, β, γ and δ be of the first, third, seventh and twenty first degrees respectively. Let m denote the multiplier connecting α, β and m' be the multiplier relating γ, δ, then*

$$\begin{aligned}\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} &+ \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4} \\ &+ 4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6} = \frac{m}{m'},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4} &+ \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4} \\ &+ 4\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/6} = \frac{m'}{m}.\end{aligned} \tag{2.15}$$

Lemma 2.6. [**6**, Ch. 20, Entry 14 (i) and (ii), p. 408] *Let α, β, γ and δ be of the first, third, eleventh and thirty third degrees respectively. Let m denote the multiplier connecting α, β and m' be the multiplier relating γ, δ, then*

$$\begin{aligned}\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} &+ \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} - \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8} \\ &- 2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12} = \sqrt{mm'},\end{aligned} \tag{2.16}$$

$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8} -4\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12}=\frac{3}{\sqrt{mm'}}. \tag{2.17}$$

Lemma 2.7. [**6**, Ch. 20, Entry 19 (iv), p. 426] *If β, γ and δ are of degrees 3, 13 and 39 respectively over α, then*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} +2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12}=\sqrt{\frac{m'}{m}}, \tag{2.18}$$

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} +2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}=\sqrt{\frac{m}{m'}}. \tag{2.19}$$

Lemma 2.8. [**5**, Theorem 4.1(ii)] *and* [**17**, Theorem 2.3(ii)] *We have*

$$l_{k,n}l_{k,1/n}=1. \tag{2.20}$$

Lemma 2.9. [**7**, Ch. 25, Entry 56, p. 210] *If $M=\dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $N=\dfrac{f(-q^2)}{q^{2/3}f(-q^{18})}$, then*

$$M^3+N^3=M^2N^2+3MN. \tag{2.21}$$

Lemma 2.10. [**3**, Theorem 5.1] *If $P=\dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ and $Q=\dfrac{\varphi(q)}{\varphi(q^3)}$, then*

$$Q^4+P^4Q^4=9+P^4. \tag{2.22}$$

3. Modular Equations of Degree 9

In this section, we establish some new modular equations of degree 9 for the ratios of Ramanujan's theta function.

Theorem 3.1. *If $P=\dfrac{\psi(q)}{q\psi(q^9)}$ and $Q=\dfrac{\psi(-q)}{q\psi(-q^9)}$, then*

$$\frac{P}{Q}+\frac{Q}{P}+\frac{3}{P}+P=\frac{3}{Q}+Q+4. \tag{3.1}$$

Proof. Using the equation (2.2) by changing q to $-q$ and the equation (2.1) in the equation (2.21), we deduce that

$$M^3=P\left(\frac{P-3}{P-1}\right)^2 \quad\text{and}\quad N^3=Q^2\left(\frac{Q+3}{Q+1}\right). \tag{3.2}$$

The equation (2.21) can be rewritten as

$$a^2+3a-A=0, \tag{3.3}$$

where $a = MN$ and $A = M^3 + N^3$.
Solving the equation (3.3) for a and cubing both sides, we find that

$$8M^3N^3 = (-3+m)^3, \tag{3.4}$$

where $m = \pm\sqrt{9+4A}$.
Eliminating m from the above equation (3.4), we deduce that

$$\begin{aligned}&(P+Q)(Q^2P^2+2P^2Q+P^2-6P+9-4PQ+6Q-2Q^2P+Q^2)\\&(P^2Q+P^2-3P-4PQ+3Q-Q^2P+Q^2)(P^4-Q^4P^3-4P^3Q^3\\&-6P^3Q^2-8P^3Q-9P^3+3Q^4P^2+12Q^3P^2+24Q^2P^2+36P^2Q\\&+27P^2-3Q^4P-8Q^3P-18Q^2P-36PQ-27P+Q^4)=0.\end{aligned} \tag{3.5}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the third factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem third factor vanishes identically. This completes the proof. □

Theorem 3.2. *If* $P = \dfrac{\psi(q)}{q\psi(q^9)}$ *and* $Q = \dfrac{\psi(q^2)}{q^2\psi(q^{18})}$, *then*

$$\frac{P}{Q} + \frac{Q}{P} + 2 = \frac{3}{P} + P. \tag{3.6}$$

Proof. Replacing q with q^2 in the equation (2.1) and equating with the equation (2.2), we find that

$$Q\left(\frac{Q-3}{Q-1}\right)^2 = P^2\left(\frac{P-3}{P-1}\right). \tag{3.7}$$

On factorizing the above equation, we deduce that

$$(QP - Q + 3 - P)(P^2Q - P^2 - 2QP - Q^2 + 3Q) = 0. \tag{3.8}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas the first factor is not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.3. *If* $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ *and* $Q = \dfrac{\psi(-q^2)}{q^2\psi(-q^{18})}$, *then*

$$\begin{aligned}\frac{P^2}{Q^2}+\frac{Q^2}{P^2}+\frac{P}{Q}+\frac{Q}{P} &= PQ+\frac{9}{PQ}-\left(\sqrt{PQ}+\frac{3}{\sqrt{PQ}}\right)\left[\left(\sqrt{\frac{P^3}{Q^3}}+\sqrt{\frac{Q^3}{P^3}}\right)\right.\\&\left.-3\left(\sqrt{\frac{P}{Q}}+\sqrt{\frac{Q}{P}}\right)\right]+12.\end{aligned} \tag{3.9}$$

Proof. Using the equations (3.1) and (3.6), we arrive at the equation (3.9). □

Theorem 3.4. *If* $P = \dfrac{\psi(-q)\psi(-q^5)}{q^6\psi(-q^9)\psi(-q^{45})}$ *and* $Q = \dfrac{\psi(-q)\psi(-q^{45})}{q^{-4}\psi(-q^9)\psi(-q^5)}$, *then*

$$\begin{aligned} Q^3 + \frac{1}{Q^3} &= 15\left(Q^2 + \frac{1}{Q^2}\right) + 45\left(Q + \frac{1}{Q}\right) + \left(P^2 + \frac{81}{P^2}\right) + 10\left(P + \frac{9}{P}\right) \\ &\times \left[2 + Q + \frac{1}{Q}\right] + 5\left(\sqrt{P^3} + \frac{3^3}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 15\left(\sqrt{P} + \frac{3}{\sqrt{P}}\right) \\ &\times \left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 2\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] + 40. \end{aligned} \tag{3.10}$$

Proof. Using the equations (2.3)–(2.10) and the equations (2.12) and (2.13) with $r = 3$ and $s = 5$, we deduce that

$$\frac{p}{q_1} = \frac{(a+b)}{(b-a)}, \tag{3.11}$$

where

$$p = \frac{\varphi(q)}{\varphi(q^3)}, \quad q_1 = \frac{\varphi(q^5)}{\varphi(q^{15})}, \quad a = \frac{\psi(-q)}{q^{1/4}\psi(-q^3)}, \quad b = \frac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}.$$

Using the equations (3.11) and (2.22), we deduce that

$$\begin{aligned} &9b^3a + 9ba^3 + 5a^5b^3 + 6a^6b^2 + 5a^7b + a^5b^7 \\ &+ a^7b^5 + 5b^7a - 6b^6a^2 + 5b^5a^3 + a^8 - b^8 = 0. \end{aligned} \tag{3.12}$$

By squaring the above equation (3.12), we deduce that

$$\begin{aligned} &105a_4b_4^2b_2a_2 + 180a_4^2b_4 + 180a_4b_4^2 + 20a_4^2b_4^2b_2a_2 + 180b_2a_4a_2b_4 + 14a_4^3b_4 \\ &+ a_2b_4^3b_2a_4^2 + a_4^3a_2b_2b_4^2 + 81b_4b_2a_2 + 81a_4a_2b_2 + 162a_4b_4 + 90a_4^2b_2a_2 \\ &+ 13b_4^3b_2a_2 + 13a_4^3a_2b_2 - a_4^4 - b_4^4 + 210a_4^2b_4^2 + 20a_4^3b_4^2 + 20a_4^2b_4^3 + 2a_4^3b_4^3 \\ &+ 14a_4b_4^3 + 10a_2b_4^3b_2a_4 + 10a_4^3a_2b_2b_4 + 90b_4^2b_2a_2 + 105a_4^2b_2a_2b_4 = 0, \end{aligned} \tag{3.13}$$

where $a^4 = a_4,\ b^4 = b_4,\ a^2 = a_2$ and $b^2 = b_2$.
Isolating the terms containing a_2b_2 on one side of the equation (3.13), squaring both sides

and by using the equation (2.11), we deduce that

$$\begin{aligned}
&(-40A^3B^3 - 15BA^5 - 45BA^4 - 90BA^3 - 135BA^2 - 81BA - 10B^3A^5\\
&- 30B^3A^4 - 90B^3A^2 - 90B^3A - 15B^2A^5 - 45B^2A^4 - 90B^2A^3 - 180B^2A^2\\
&- 135B^2A - 5B^4A^5 - 20B^4A^4 - 30B^4A^3 - 45B^4A^2 - 45B^4A - B^5A^5\\
&- 5B^5A^4 - 10B^5A^3 - 15B^5A^2 - 15B^5A + B^6 + A^6)(64068165A^6B^3\\
&+ 64068165A^3B^6 + 55801305A^3B^3 + 90655524A^6B^6 + 7282710A^9B^6\\
&+ 7282710A^6B^9 + 617490A^9B^9 + 5234220A^9B^3 + 5234220A^3B^9\\
&+ 15849A^6B^{12} + 810A^9B^{12} + A^{12}B^{12} + 21870A^3B^{12} + 3159A^{10}B^{11}\\
&+ 15849A^{12}B^6 + 810A^{12}B^9 + 21870A^{12}B^3 + 98415BA^8 + 59049BA^7\\
&+ 15812010B^3A^8 + 36183915B^3A^7 + 87392520B^3A^5 + 87687765B^3A^4\\
&+ 15943230B^3A^2 + 4166235B^2A^8 + 9310059B^2A^7 + 16533720B^2A^6\\
&+ 23029110B^2A^5 + 23914845B^2A^4 + 15943230B^2A^3 + 4782969B^2A^2\\
&+ 27479655B^4A^8 + 62985600B^4A^7 + 110454435B^4A^6 + 147819330B^4A^5\\
&+ 143784315B^4A^4 + 87687765B^4A^3 + 23914845B^4A^2 + 22956210B^6A^8\\
&+ 52623594B^6A^7 + 118032390B^6A^5 + 110454435B^6A^4 + 16533720B^6A^2\\
&+ 29590110B^5A^8 + 67884480B^5A^7 + 118032390B^5A^6 + 155561310B^5A^5\\
&+ 147819330B^5A^4 + 87392520B^5A^3 + 23029110B^5A^2 + 6070140B^8A^8\\
&+ 13603140B^8A^7 + 22956210B^8A^6 + 29590110B^8A^5 + 27479655B^8A^4\\
&+ 15812010B^8A^3 + 4166235B^8A^2 + 98415B^8A + 13603140B^7A^8\\
&+ 30913974B^7A^7 + 52623594B^7A^6 + 67884480B^7A^5 + 62985600B^7A^4\\
&+ 36183915B^7A^3 + 9310059B^7A^2 + 59049B^7A + 405405B^{10}A^8\\
&+ 922185B^{10}A^7 + 1620810B^{10}A^6 + 2194290B^{10}A^5 + 2125035B^{10}A^4\\
&+ 1290330B^{10}A^3 + 393660B^{10}A^2 + 32805B^{10}A + 1925775B^9A^8\\
&+ 4287735B^9A^7 + 9528030B^9A^5 + 8966700B^9A^4 + 1443420B^9A^2\\
&+ 65610B^9A + 26730A^{12}B^4 + 324405A^{11}B^4 + 2125035A^{10}B^4\\
&+ 8966700A^9B^4 + 229149A^{11}B^6 + 1620810A^{10}B^6 + 23814A^{12}B^5\\
&+ 322704A^{11}B^5 + 2194290A^{10}B^5 + 9528030A^9B^5 + 7938A^{12}B^7\\
&+ 123363A^{11}B^7 + 922185A^{10}B^7 + 4287735A^9B^7 + 15525A^{11}B^9\\
&+ 129195A^{10}B^9 + 2970A^{12}B^8 + 51030A^{11}B^8 + 405405A^{10}B^8\\
&+ 1925775A^9B^8 + 18A^{12}B^{11} + 369A^{11}B^{11} + 211410A^3B^{11}\\
&+ 15525A^9B^{11} + 51030A^8B^{11} + 123363A^7B^{11} + 153A^{12}B^{10}\\
&+ 3159A^{11}B^{10} + 26919A^{10}B^{10} + 129195A^9B^{10} + 1290330A^{10}B^3\\
&+ 153A^{10}B^{12} + 2970A^8B^{12} + 7938A^7B^{12} + 229149A^6B^{11} + 18A^{11}B^{12}\\
&+ 211410A^{11}B^3 + 393660B^2A^{10} + 1443420B^2A^9 + 32805BA^{10}
\end{aligned}$$

$$\begin{aligned}&+23814A^5B^{12}+26730A^4B^{12}+322704A^5B^{11}+324405A^4B^{11}\\&+729A^{12}+729B^{12}+4374A^{12}B+12393A^{12}B^2+76545A^{11}B^2\\&+4374B^{12}A+12393B^{12}A^2+10935B^{11}A+76545B^{11}A^2\\&+10935BA^{11}+65610BA^9)=0.\end{aligned}\tag{3.14}$$

By observing the above factors of the equation (3.14) near $q=0$, it can be seen that there is a neighborhood about the origin where the first factor is zero, whereas the second factor is not zero. Hence by the Identity Theorem the first factor vanishes identically and then by setting $P=AB$ and $Q=\dfrac{A}{B}$, we arrive at the equation (3.10). □

Theorem 3.5. *If* $P=\dfrac{\psi(-q)\psi(-q^7)}{q^8\psi(-q^9)\psi(-q^{63})}$ *and* $Q=\dfrac{\psi(-q)\psi(-q^{63})}{q^{-6}\psi(-q^9)\psi(-q^7)}$, *then*

$$\begin{aligned}&Q^4+\frac{1}{Q^4}=35\left(Q^3+\frac{1}{Q^3}\right)+413\left(Q^2+\frac{1}{Q^2}\right)+1379\left(Q+\frac{1}{Q}\right)+1694\\&+\left(P^3+\frac{9^3}{P^3}\right)+7\left(P^2+\frac{9^2}{P^2}\right)\left[7+3\left(Q+\frac{1}{Q}\right)\right]+21\left(P+\frac{9}{P}\right)\\&\times\left[21+14\left(Q+\frac{1}{Q}\right)+3\left(Q^2+\frac{1}{Q^2}\right)\right]+7\left(\sqrt{P^5}+\frac{3^5}{\sqrt{P^5}}\right)\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)\\&+63\left[\sqrt{P}+\frac{3}{\sqrt{P}}\right]\left[7\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)+14\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)+\left(\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right)\right]\\&+21\left(\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right)\left[2\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)+7\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)\right].\end{aligned}\tag{3.15}$$

The proof of (3.15) is similar to the proof of the equation (3.10), except that in place of results (2.12) and (2.13), the results (2.14) and (2.15) with $r=3$ and $s=7$ are used.

Theorem 3.6. *If* $P=\dfrac{\psi(-q)\psi(-q^{11})}{q^{12}\psi(-q^9)\psi(-q^{99})}$ *and* $Q=\dfrac{\psi(-q)\psi(-q^{99})}{q^{-10}\psi(-q^9)\psi(-q^{11})}$, *then*

$$\begin{aligned}&\frac{1}{11}\left(Q^6+\frac{1}{Q^6}\right)-51\left(\sqrt{Q^9}+\frac{1}{\sqrt{Q^9}}\right)\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)-\left(\sqrt{Q^7}+\frac{1}{\sqrt{Q^7}}\right)\\&\times\left[1263\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)+110\left(\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right)\right]-2\left(\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right)\\&\times\left[5271\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)+739\left(\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right)+35\left(\sqrt{P^5}+\frac{3^5}{\sqrt{P^5}}\right)\right]\\&-2\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)\left[18929\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)+3276\left(\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right)\right.\\&\left.+272\left(\sqrt{P^5}+\frac{3^5}{\sqrt{P^5}}\right)+8\left(\sqrt{P^7}+\frac{3^7}{\sqrt{P^7}}\right)\right]-\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)\end{aligned}$$

$$\times\left[67074\left(\sqrt{P}+\frac{3}{\sqrt{P}}\right)+12982\left(\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right)+1348\left(\sqrt{P^5}+\frac{3^5}{\sqrt{P^5}}\right)\right.$$
$$\left.+65\left(\sqrt{P^7}+\frac{3^7}{\sqrt{P^7}}\right)+\left(\sqrt{P^9}+\frac{3^9}{\sqrt{P^9}}\right)\right]=132804+100662\left(Q+\frac{1}{Q}\right)$$
$$+630\left(Q^4+\frac{1}{Q^4}\right)+8077\left(Q^3+\frac{1}{Q^3}\right)+41403\left(Q^2+\frac{1}{Q^2}\right)+15\left(Q^5+\frac{1}{Q^5}\right)$$
$$+\left(P+\frac{9}{P}\right)\left[25214\left(Q+\frac{1}{Q}\right)+9616\left(Q^2+\frac{1}{Q^2}\right)+1613\left(Q^3+\frac{1}{Q^3}\right)\right.$$
$$\left.+91\left(Q^4+\frac{1}{Q^4}\right)+34186\right]+2\left(P^2+\frac{9^2}{P^2}\right)\left[1711\left(Q+\frac{1}{Q}\right)+511\left(Q^2+\frac{1}{Q^2}\right)\right.$$
$$\left.+50\left(Q^3+\frac{1}{Q^3}\right)+2500\right]+\left(P^4+\frac{9^4}{P^4}\right)+\frac{1}{11}\left(P^5+\frac{9^5}{P^5}\right)+\left(P^3+\frac{9^3}{P^3}\right)$$
$$\times\left[221\left(Q+\frac{1}{Q}\right)+38\left(Q^2+\frac{1}{Q^2}\right)+377\right]. \tag{3.16}$$

The proof of (3.16) is similar to the proof of the equation (3.10), except that in place of results (2.12) and (2.13), the results (2.16) and (2.17) with $r = 3$ and $s = 11$ are used.

Theorem 3.7. *If* $P=\dfrac{\psi(-q)\psi(-q^{13})}{q^{14}\psi(-q^9)\psi(-q^{117})}$ *and* $Q=\dfrac{\psi(-q)\psi(-q^{117})}{q^{-12}\psi(-q^9)\psi(-q^{13})}$, *then*

$$Q^7+\frac{1}{Q^7}-13\left\{26\left[Q^6+\frac{1}{Q^6}\right]-1235\left[Q^5+\frac{1}{Q^5}\right]-11638\left[Q^4+\frac{1}{Q^4}\right]\right.$$
$$-21199\left[Q^3+\frac{1}{Q^3}\right]+73664\left[Q^2+\frac{1}{Q^2}\right]+297231\left[Q+\frac{1}{Q}\right]+419532$$
$$-\left[\sqrt{P^{11}}+\frac{3^{11}}{\sqrt{P^{11}}}\right]\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]-\left[\sqrt{P^9}+\frac{3^9}{\sqrt{P^9}}\right]\left[76\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]\right.$$
$$\left.+23\left[\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right]\right]-\left[\sqrt{P^7}+\frac{3^7}{\sqrt{P^7}}\right]\left[1398\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]+735\left[\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right]\right.$$
$$\left.+144\left[\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right]\right]-3\left[\sqrt{P^5}+\frac{3^5}{\sqrt{P^5}}\right]\left[2037\left[\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right]\right.$$
$$\left.+2434\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]+853\left[\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right]+121\left[\sqrt{Q^7}+\frac{1}{\sqrt{Q^7}}\right]\right]$$
$$-9\left[\sqrt{P^3}+\frac{3^3}{\sqrt{P^3}}\right]\left[859\left[\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right]-1109\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]+1243\left[\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right]\right.$$
$$\left.+424\left[\sqrt{Q^7}+\frac{1}{\sqrt{Q^7}}\right]+41\left[\sqrt{Q^9}+\frac{1}{\sqrt{Q^9}}\right]\right]+27\left[\sqrt{P}+\frac{3}{\sqrt{P}}\right]\left[7176\left[\sqrt{Q}+\frac{1}{\sqrt{Q}}\right]\right.$$
$$+3052\left[\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right]-180\left[\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right]-467\left[\sqrt{Q^7}+\frac{1}{\sqrt{Q^7}}\right]$$

$$-91\left[\sqrt{Q^9}+\frac{1}{\sqrt{Q^9}}\right]-4\left[\sqrt{Q^{11}}+\frac{1}{\sqrt{Q^{11}}}\right]\Bigg]\Bigg\}=P^6+\frac{9^6}{P^6}+13\Bigg\{\left[P^5+\frac{9^5}{P^5}\right][13$$
$$+6\left[Q+\frac{1}{Q}\right]\Bigg]+\left[P^4+\frac{9^4}{P^4}\right]\left[420+277\left[Q+\frac{1}{Q}\right]+65\left[Q^2+\frac{1}{Q^2}\right]\right]$$
$$+\left[P^3+\frac{9^3}{P^3}\right]\left[3973+3307\left[Q+\frac{1}{Q}\right]+1530\left[Q^2+\frac{1}{Q^2}\right]+255\left[Q^3+\frac{1}{Q^3}\right]\right]$$
$$+3\left[P^2+\frac{81}{P^2}\right]\Bigg[2065+3099\left[Q+\frac{1}{Q}\right]+3042\left[Q^2+\frac{1}{Q^2}\right]+1153\left[Q^3+\frac{1}{Q^3}\right]$$
$$+138\left[Q^4+\frac{1}{Q^4}\right]\Bigg]-9\left[P+\frac{9}{P}\right]\Bigg[8640+4987\left[Q+\frac{1}{Q}\right]-362\left[Q^2+\frac{1}{Q^2}\right]$$
$$-1344\left[Q^3+\frac{1}{Q^3}\right]-383\left[Q^4+\frac{1}{Q^4}\right]-27\left[Q^5+\frac{1}{Q^5}\right]\Bigg]\Bigg\}. \tag{3.17}$$

The proof of (3.17) is similar to the proof of the equation (3.10), except that in place of results (2.12) and (2.13), the results (2.18) and (2.19) with $r = 3$ and $s = 13$ are used.

4. General Formulas for Explicit Evaluations of $l_{9,n}$

In this section, we establish some general formulas for explicit evaluations of $l_{9,n}$. We also establish relations among $l_{9,n}$ and $l'_{9,n}$.

Theorem 4.1. *If $X = l_{9,n}$ and $Y = l_{9,4n}$, then*

$$\begin{aligned}\frac{X^2}{Y^2}+\frac{Y^2}{X^2}+\frac{X}{Y}+\frac{Y}{X}&=3\left(XY+\frac{1}{XY}\right)-\sqrt{3}\left(\sqrt{XY}+\frac{1}{\sqrt{XY}}\right)\\&\times\left[\left(\sqrt{\frac{X^3}{Y^3}}+\sqrt{\frac{Y^3}{X^3}}\right)-\left(\sqrt{\frac{X}{Y}}+\sqrt{\frac{Y}{X}}\right)\right]+12.\end{aligned} \tag{4.1}$$

Proof. Using the equation (3.6) along with the equation (1.6), we arrive at the equation (4.1). □

Corollary 4.1. *We have*

$$l_{9,2}=\frac{\sqrt{3}+1}{\sqrt{2}}, \tag{4.2}$$
$$l_{9,1/2}=\frac{\sqrt{3}-1}{\sqrt{2}}, \tag{4.3}$$
$$l_{9,4}=\frac{3+\sqrt{3}+\sqrt{2}+\sqrt{2(3+\sqrt{3})(\sqrt{3}+\sqrt{2})}}{2\sqrt{2}}, \tag{4.4}$$
$$l_{9,1/4}=\frac{3+\sqrt{3}+\sqrt{2}-\sqrt{2(3+\sqrt{3})(\sqrt{3}+\sqrt{2})}}{2\sqrt{2}}, \tag{4.5}$$

$$l_{9,8}=\frac{\sqrt{13+5\sqrt{6}+7\sqrt{3}+9\sqrt{2}}}{\sqrt{2}}+\frac{\sqrt{15+6\sqrt{6}+9\sqrt{3}+12\sqrt{2}}}{\sqrt{2}}, \tag{4.6}$$

$$l_{9,1/8}=\frac{\sqrt{13+5\sqrt{6}+7\sqrt{3}+9\sqrt{2}}}{\sqrt{2}}-\frac{\sqrt{15+6\sqrt{6}+9\sqrt{3}+12\sqrt{2}}}{\sqrt{2}}. \tag{4.7}$$

Proof of (4.2)**.** Putting $n=1/2$ in the equation (4.1) and using the fact that $l_{9,2}l_{9,1/2}=1$, we deduce that

$$(-l_{9,2}^2+2+\sqrt{3})(l_{9,2}^2-2+\sqrt{3})(l_{9,2}^2+\sqrt{3}l_{9,2}+1)^2=0. \tag{4.8}$$

We observe that the first factor of the equation (4.8) vanishes for the specific value of $q=e^{-\pi\sqrt{2/9}}$, but the other two factors does not vanish. Since $l_{9,2}>1$, we arrive at the equation (4.2). ☐

Proof of (4.3)**.** By using the fact that $l_{9,2}l_{9,1/2}=1$, we arrive at the equation (4.3). ☐

Proofs of (4.4) *and* (4.5)**.** Putting $n=1$ in the equation (4.1) and by using the fact that $l_{9,1}=1$, we deduce that

$$l_{9,4}^4-2l_{9,4}^3-3l_{9,4}^2-3\sqrt{3}l_{9,4}^2-2l_{9,4}+1=0. \tag{4.9}$$

The above equation (4.9) can be rewritten as

$$x^2-2x-3\sqrt{3}-5=0. \tag{4.10}$$

where $x=l_{9,4}+\dfrac{1}{l_{9,4}}$.

Solving the above equation for x, we deduce that

$$x=\frac{\sqrt{2}+\sqrt{3}+3}{\sqrt{2}},\ \ \frac{\sqrt{2}-\sqrt{3}-3}{\sqrt{2}}. \tag{4.11}$$

Since $x>0$, we deduce that

$$l_{9,4}+\frac{1}{l_{9,4}}=\frac{\sqrt{2}+\sqrt{3}+3}{\sqrt{2}}. \tag{4.12}$$

On solving the above equation (4.12), we arrive at the equations (4.4) and (4.5). ☐

Proofs of (4.6) *and* (4.7)**.** Using the equation (4.2) in the equation (4.1), we obtain the equations (4.6) and (4.7). ☐

Theorem 4.2. *If $X=l'_{9,n}$ and $Y=l_{9,n}$, then*

$$\frac{X}{Y}+\frac{Y}{X}+\sqrt{3}\left[\left(X+\frac{1}{X}\right)-\left(Y+\frac{1}{Y}\right)\right]=4. \tag{4.13}$$

Proof. Using the equation (3.1) along with the equations (1.6) and (1.7) with $k=9$, we arrive at the equation (4.13) which completes the proof. ☐

Corollary 4.2. *We have*

$$l'_{9,3}=\frac{(\sqrt{3}+1)^2+(\sqrt{3}+1)\sqrt[3]{4}+\sqrt[3]{16}}{2\sqrt{3}}, \tag{4.14}$$

$$l'_{9,7}=\frac{3+2\sqrt{7}+4\sqrt{3}+\sqrt{21}+\sqrt{9+2\sqrt{21}}(2+3\sqrt{3}-\sqrt{7})}{4}. \tag{4.15}$$

Proof. Using the values of $l_{9,3}$ [**5, 17**] and $l_{9,7}$ in the above equation (4.13), we arrive at the equations (4.14) and (4.15) respectively. □

Theorem 4.3. *If* $X = l_{9,n}$ *and* $Y = l'_{9,4n}$, *then*

$$\frac{X}{Y} + \frac{Y}{X} - 2 = \sqrt{3}\left(X + \frac{1}{X}\right). \tag{4.16}$$

Proof. Changing q to $-q$ in the equation (3.6) and by using the equations (1.6) and (1.7) with $k = 9$, we arrive at the equation (4.16). □

Corollary 4.3. *We have*

$$l'_{9,4} = 1 + \sqrt{3} + \sqrt{2\sqrt{3} + 3}, \tag{4.17}$$

$$l'_{1/9,4} = 1 + \sqrt{3} - \sqrt{2\sqrt{3} + 3}, \tag{4.18}$$

$$l'_{9,8} = (2 + \sqrt{3})(\sqrt{3} + \sqrt{2}), \tag{4.19}$$

$$l'_{9,12} = \frac{1}{\sqrt{3}}[(2 + \sqrt{3})^2 + \sqrt[3]{2}(5 + 3\sqrt{3}) + [\sqrt[3]{2}(\sqrt{3} + 1)]^2], \tag{4.20}$$

$$l'_{9,28} = 7\sqrt{7} + 11\sqrt{3} + 4\sqrt{21} + 18 + (2 + \sqrt{7})(2 + \sqrt{3})\sqrt{9 + 2\sqrt{21}}. \tag{4.21}$$

Proof. Putting $n = 1, 2, 3, 7$ in Theorem 4.3 and using the values of $l_{9,1}, l_{9,2}, l_{9,3}$ and $l_{9,7}$, we arrive at the equations (4.17), (4.19), (4.20) and (4.21) respectively. This completes the proof. □

Theorem 4.4. *If* $X = l_{9,n} l_{9,25n}$ *and* $Y = \dfrac{l_{9,n}}{l_{9,25n}}$, *then*

$$Y^3 + \frac{1}{Y^3} = 15\left(Y^2 + \frac{1}{Y^2}\right) + 45\left(Y + \frac{1}{Y}\right) + 9\left(X^2 + \frac{1}{X^2}\right) + 30\left(X + \frac{1}{X}\right)$$
$$\times \left[2 + \left(Y + \frac{1}{Y}\right)\right] + 15\sqrt{3}\left(\sqrt{X^3} + \frac{1}{\sqrt{X^3}}\right)\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right) + 15\sqrt{3}\left(\sqrt{X} + \frac{1}{\sqrt{X}}\right)$$
$$\times \left[\left(\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right) + 2\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right)\right] + 40. \tag{4.22}$$

Proof. Employing the equation (3.10) along with the equation (1.6) with $k = 9$, we obtain the equation (4.22). □

Corollary 4.4. *We have*

$$l_{9,5} = \frac{(\sqrt{3} + 1)(\sqrt{5} + \sqrt{3})}{2}, \tag{4.23}$$

$$l_{9,1/5} = \frac{(\sqrt{3} - 1)(\sqrt{5} - \sqrt{3})}{2}, \tag{4.24}$$

$$l_{9,25} = \frac{(7 + 3\sqrt{5})(4 + \sqrt{15})}{2} + \frac{a}{2}, \tag{4.25}$$

$$l_{9,1/25} = \frac{(7 + 3\sqrt{5})(4 + \sqrt{15})}{2} - \frac{a}{2}, \tag{4.26}$$

where $a = \sqrt{2910 + 752\sqrt{15} + 1302\sqrt{5} + 1680\sqrt{3}}$.

Proof of (4.23)**.** Putting $n = 1/5$ in the equation (4.22) and then using the equation (2.20), we find that

$$\begin{aligned}&(-l_{9,5}^2 + (-3+\sqrt{3})l_{9,5} + 2 - \sqrt{3})(-l_{9,5}^2 + (3+\sqrt{3})l_{9,5} + 2 + \sqrt{3})\\&(1+l_{9,5}^2)^2(l_{9,5}^2 + \sqrt{3}l_{9,5} + 1)^2 = 0.\end{aligned} \tag{4.27}$$

Since $l_{9,5} > 1$, we find that

$$l_{9,5}^2 - (3+\sqrt{3})l_{9,5} - (2+\sqrt{3}) = 0. \tag{4.28}$$

Solving the above equation (4.28) , we arrive at (4.23). □

Proof of (4.24)**.** Using the equations (2.20) with $n = 5$, $k = 9$ and (4.23), we obtain (4.24). □

Proofs of (4.25) *and* (4.26)**.** Putting $n = 1$ in the equation (4.22) and using the fact that $l_{9,1} = 1$, we deduce that

$$l_{9,25}^2 + \frac{1}{l_{9,25}^2} - (56+30\sqrt{3})\left(l_{9,25} + \frac{1}{l_{9,25}}\right) + 6 = 0. \tag{4.29}$$

The above equation (4.29) reduces to

$$a^2 - (56+30\sqrt{3})a + 4 = 0, \tag{4.30}$$

where $a = l_{9,25} + \dfrac{1}{l_{9,25}}$.
On solving the above equation and $a > 1$, we deduce that

$$l_{9,25} + \frac{1}{l_{9,25}} = (7+3\sqrt{5})(4+\sqrt{15}). \tag{4.31}$$

Again by solving the above equation, we obtain (4.25) and (4.26). □

Theorem 4.5. *If* $X = l_{9,n}l_{9,49n}$ *and* $Y = \dfrac{l_{9,n}}{l_{9,49n}}$, *then*

$$\begin{aligned}Y^4 + \frac{1}{Y^4} &= 35\left(Y^3 + \frac{1}{Y^3}\right) + 413\left(Y^2 + \frac{1}{Y^2}\right) + 1379\left(Y + \frac{1}{Y}\right) + 1694\\&+ 27\left(X^3 + \frac{1}{X^3}\right) + 63\left\{\left(X^2 + \frac{1}{X^2}\right)\left[7 + 3\left(Y + \frac{1}{Y}\right)\right] + \left(X + \frac{1}{X}\right)\right.\\&\times\left[21 + 14\left(Y + \frac{1}{Y}\right) + 3\left(Y^2 + \frac{1}{Y^2}\right)\right] + \sqrt{3}\left(\sqrt{X^5} + \frac{1}{\sqrt{X^5}}\right)\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right)\\&+ \sqrt{3}\left[\sqrt{X} + \frac{1}{\sqrt{X}}\right]\left[7\left(\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right) + 14\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right) + \left(\sqrt{Y^5} + \frac{1}{\sqrt{Y^5}}\right)\right]\\&\left.+\sqrt{3}\left(\sqrt{X^3} + \frac{1}{\sqrt{X^3}}\right)\left[2\left(\sqrt{Y^3} + \frac{1}{\sqrt{Y^3}}\right) + 7\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right)\right]\right\}.\end{aligned} \tag{4.32}$$

Proof. Employing the equation (3.15) along with the equation (1.6) with $k = 9$, we obtain the equation (4.32). □

Corollary 4.5. *We have*

$$l_{9,7} = \sqrt{10+2\sqrt{21}} + \sqrt{9+2\sqrt{21}}, \tag{4.33}$$

$$l_{9,1/7} = \sqrt{10+2\sqrt{21}} - \sqrt{9+2\sqrt{21}}. \tag{4.34}$$

Proofs of (4.33) *and* (4.34)**.** Putting $n = 1/7$ in the equation (4.32) and then using the equation (2.20), we find that

$$(l_{9,7}^8 + 3\sqrt{3}l_{9,7}^7 + 17l_{9,7}^6 + 21\sqrt{3}l_{9,7}^5 + 48l_{9,7}^4 + 21\sqrt{3}l_{9,7}^3 + 17l_{9,7}^2 + 3\sqrt{3}l_{9,7} + 1)$$
$$(-l_{9,7}^4 + 4\sqrt{3}l_{9,7}^3 + 14l_{9,7}^2 + 4\sqrt{3}l_{9,7} - 1)(l_{9,7}^4 + \sqrt{3}l_{9,7}^3 + l_{9,7}^2 + \sqrt{3}l_{9,7} + 1) = 0. \tag{4.35}$$

We observe that the first and the last factors are not zero for specific value of $q = e^{-\pi\sqrt{7/9}}$. Hence the second factor is zero,

$$x^2 - 4\sqrt{3}x - 16 = 0, \tag{4.36}$$

where $x = l_{9,7} + \dfrac{1}{l_{9,7}}$.
On solving the above equation (4.36) for x and $x > 1$, we deduce that

$$l_{9,7} + \frac{1}{l_{9,7}} = 2(\sqrt{3}+\sqrt{7}). \tag{4.37}$$

On solving the above equation, we arrive at the equations (4.33) and (4.34) respectively. □

Theorem 4.6. *If* $X = l_{9,n}l_{9,169n}$ *and* $Y = \dfrac{l_{9,n}}{l_{9,169n}}$*, then*

$$\begin{aligned}
&Y^7 + \frac{1}{Y^7} - 13\Bigg\{26\left[Y^6+\frac{1}{Y^6}\right] - 1235\left[Y^5+\frac{1}{Y^5}\right] - 11638\left[Y^4+\frac{1}{Y^4}\right] \\
&- 21199\left[Y^3+\frac{1}{Y^3}\right] + 73664\left[Y^2+\frac{1}{Y^2}\right] + 297231\left[Y+\frac{1}{Y}\right] + 419532 \\
&- 27\sqrt{3}\Bigg(9\left[\sqrt{X^{11}}+\frac{1}{\sqrt{X^{11}}}\right]\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] + 3\left[\sqrt{X^9}+\frac{1}{\sqrt{X^9}}\right]\Bigg[76\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] \\
&+23\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right]\Bigg] + \left[\sqrt{X^7}+\frac{1}{\sqrt{X^7}}\right]\Bigg[1398\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] + 735\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right] \\
&+144\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right]\Bigg] + \left[\sqrt{X^5}+\frac{1}{\sqrt{X^5}}\right]\Bigg[2037\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right] + 2434\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] \\
&+853\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right] + 121\left[\sqrt{Y^7}+\frac{1}{\sqrt{Y^7}}\right]\Bigg] + \left[\sqrt{X^3}+\frac{1}{\sqrt{X^3}}\right]\Bigg[859\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right] \\
&-1109\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] + 1243\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right] + 424\left[\sqrt{Y^7}+\frac{1}{\sqrt{Y^7}}\right] + 41\left[\sqrt{Y^9}+\frac{1}{\sqrt{Y^9}}\right]\Bigg] \\
&- \left[\sqrt{X}+\frac{1}{\sqrt{X}}\right]\Bigg[7176\left[\sqrt{Y}+\frac{1}{\sqrt{Y}}\right] + 3052\left[\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right] - 180\left[\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right] \\
&-467\left[\sqrt{Y^7}+\frac{1}{\sqrt{Y^7}}\right] -91\left[\sqrt{Y^9}+\frac{1}{\sqrt{Y^9}}\right] - 4\left[\sqrt{Y^{11}}+\frac{1}{\sqrt{Y^{11}}}\right]\Bigg]\Bigg)\Bigg\} = 3^6\left[X^6+\frac{1}{X^6}\right]
\end{aligned}$$

$$+351\Bigg\{9\left[X^5+\frac{1}{X^5}\right]\left[13+6\left[Y+\frac{1}{Y}\right]\right]+3\left[X^4+\frac{1}{X^4}\right]\left[420+277\left[Y+\frac{1}{Y}\right]\right.$$
$$\left.+65\left[Y^2+\frac{1}{Y^2}\right]\right]+\left[X^3+\frac{1}{X^3}\right]\left[3973+3307\left[Y+\frac{1}{Y}\right]+1530\left[Y^2+\frac{1}{Y^2}\right]\right.$$
$$\left.+255\left[Y^3+\frac{1}{Y^3}\right]\right]+\left[X^2+\frac{1}{X^2}\right]\left[2065+3099\left[Y+\frac{1}{Y}\right]+3042\left[Y^2+\frac{1}{Y^2}\right]\right.$$
$$\left.+1153\left[Y^3+\frac{1}{Y^3}\right]+138\left[Y^4+\frac{1}{Y^4}\right]\right]-\left[X+\frac{1}{X}\right]\left[8640+4987\left[Y+\frac{1}{Y}\right]\right.$$
$$\left.-362\left[Y^2+\frac{1}{Y^2}\right]-1344\left[Y^3+\frac{1}{Y^3}\right]-383\left[Y^4+\frac{1}{Y^4}\right]-27\left[Y^5+\frac{1}{Y^5}\right]\right]\Bigg\}. \tag{4.38}$$

Proof. Employing the equation (3.17) along with the equation (1.6) with $k=9$, we obtain the equation (4.32). □

Corollary 4.6. *We have*

$$l_{9,13}=\sqrt{\frac{77+21\sqrt{13}+44\sqrt{3}+12\sqrt{39}}{2}}+\sqrt{\frac{75+21\sqrt{13}+44\sqrt{3}+12\sqrt{39}}{2}}, \tag{4.39}$$

$$l_{9,1/13}=\sqrt{\frac{77+21\sqrt{13}+44\sqrt{3}+12\sqrt{39}}{2}}-\sqrt{\frac{75+21\sqrt{13}+44\sqrt{3}+12\sqrt{39}}{2}}. \tag{4.40}$$

Proof. Putting $n=1/13$ in the equation (4.38) and using the equation (2.20), we find that

$$({l_{9,13}}^4-12{l_{9,13}}^3-6\sqrt{3}{l_{9,13}}^3-26{l_{9,13}}^2-16\sqrt{3}{l_{9,13}}^2-12l_{9,13}-6\sqrt{3}l_{9,13}+1)$$
$$({l_{9,13}}^4+12{l_{9,13}}^3-6\sqrt{3}{l_{9,13}}^3-26{l_{9,13}}^2+16\sqrt{3}{l_{9,13}}^2+12l_{9,13}-6\sqrt{3}l_{9,13}+1)$$
$$({l_{9,13}}^2-2l_{9,13}+2\sqrt{3}l_{9,13}+1)^2({l_{9,13}}^2+2l_{9,13}+2\sqrt{3}l_{9,13}+1)^2(l_{9,13}-1)^2$$
$$({l_{9,13}}^2+l_{9,13}+\sqrt{3}l_{9,13}+1)^2({l_{9,13}}^2-l_{9,13}+\sqrt{3}l_{9,13}+1)^2(l_{9,13}+1)^2=0. \tag{4.41}$$

We observe that the first factor is zero for the specific value of $q=e^{-\pi\sqrt{13/9}}$, where as the other factors are not zero. Hence we deduce that

$$x^2-(12+6\sqrt{3})x-(28+16\sqrt{3})=0, \tag{4.42}$$

where $x=l_{9,13}+\dfrac{1}{l_{9,13}}$. Solving the above equation for x and $x>0$, we deduce that

$$l_{9,13}+\frac{1}{l_{9,13}}=(2+\sqrt{3})(3+\sqrt{13}). \tag{4.43}$$

On solving the above quadratic equation, we arrive at the equations (4.39) and (4.40). □

Remark 1. By using the values of $l_{9,n}$ and $l'_{9,n}$ established in the earlier section, one can compute the explicit evaluations of Ramanujan's cubic continued fractions. For details see [**3**], [**4**], [**13**].

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References

[1] C. Adiga, M. S. Mahadeva Naika and K. Shivashankara, *On some P-Q eta-function identities of Ramanujan*, Indian J. Math., **44** (2) (2002), 253–267.

[2] C. Adiga, Taekyun Kim and M. S. Mahadeva Naika, *Modular equations in the theory of signature 3 and P-Q identities*, Adv. Stud. Contemp. Math., **7** (1) (2003), 33–40.

[3] C. Adiga, Taekyun Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, *On Ramnujan's cubic continued fraction and explicit evaluations of theta-functions*, Indian J. pure appl. math., **35** (9) (2004), 1047–1062.

[4] C. Adiga, K. R. Vasuki, and M. S. Mahadeva Naika, *Some new explicit evaluations of Ramanujan's cubic continued fraction*, The New Zealand J. Math., **31** (2002), 1–6.

[5] N. D. Baruah and Nipen Saikia, *Two parameters for Ramanujan's theta-functions and their explicit values*, Rocky Mountain J. Math., **37** (6) (2007), 1747–1790.

[6] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.

[7] B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York, 1994.

[8] S. Bhargava, C. Adiga and M. S. Mahadeva Naika, *A new class of modular equations akin to Ramanujan's P-Q eta-function identities and some evaluations there from*, Adv. Stud. Contemp. Math., **5** (1) (2002), 37–48.

[9] M. S. Mahadeva Naika, *P-Q eta-function identities and computation of Ramanujan-Weber class invariants*, J. Indian Math. Soc., **70** (1-4) (2003), 121–134.

[10] M. S. Mahadeva Naika, *Some theorems on Ramanujan's cubic continued fraction and related identities*, Tamsui Oxf. J. Math. Sci. **24** (3) (2008), 243–256.

[11] M. S. Mahadeva Naika and S. Chandankumar, Some new modular equations and their applications, (Communicated).

[12] M. S. Mahadeva Naika, S. Chandankumar and M. Manjunatha, On some new modular equations of degree 9 and their applications, (Communicated).

[13] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, *General formulas for explicit evaluations of Ramanujan's cubic continued fraction*, Kyungpook Math. J., **49** (3) (2009), 435–450.

[14] M. S. Mahadeva Naika, K. Sushan Bairy and S. Chandankumar, On some explicit evaluation of the ratios of Ramanujan's theta-function, (Communicated).

[15] M. S. Mahadeva Naika, K. Sushan Bairy and M. Manjunatha, *Some new modular equations of degree four and their explicit evaluations*, Eur. J. Pure Appl. Math., **3** (6) (2010), 924-947.

[16] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[17] J. Yi, Yang Lee and Dae Hyun Paek, *The explicit formulas and evaluations of Ramanujan's theta-function ψ*, J. Math. Anal. Appl., **321** (2006), 157–181.

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