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Linear and nonlinear electroconvection under AC electric field

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Linear and non-linear stability analyses of electroconvection under an *AC* electric field are investigated using the normal mode method and truncated representation of Fourier series respectively. The principle of exchange of stabilities is shown to be valid and subcritical instability is ruled out. Several qualitative results on stability are discussed on the governing linear autonomous system, and also by using the concept of a self-adjoint operator. Spectral analysis of electroconvection is also made to provide information on the relative dominance of various modes on convection. The quantification of heat transfer is done on the Nusselt number–Rayleigh number plane for steady finite amplitude convection and through time series plots of the Nusselt number for unsteady finite amplitude convection. The effect of the electric number on stream line pattern and Nusselt number is delineated. Time series plots of the amplitudes of thermal conduction and convection are also presented. It is found that the effect of increasing the electric number is to enhance the amplitudes and thereby the heat transport. The sensitive dependence of the solution of the Lorenz system of electroconvection to the choice of initial conditions points to the possibility of chaos.

1. Introduction

The occurrence of cellular convection in Newtonian liquid layers heated from below is generally ascribed to two different mechanisms: the buoyancy and surface tension mechanisms. The buoyancy driven convection is popularly known as "Rayleigh–Bénard Convection (*RBC*)" while the surface-tension driven convection is referred to as "Marangoni Convection (*MC*)". In the case of dielectric liquids, thermally and electrically induced gradients of polarization also contribute to the convective motion besides the two aforesaid candidates pertaining to Newtonian liquids. The Rayleigh–Bénard convection in dielectric liquids, using the classical linear stability theory, has been exhaustively studied by Takashima [1], Takashima and Ghosh [2], Takashima and Hamabata [3], Oliveri and Atten [4], Agrait and Castellanos [5], Ko and Kim [6], Stiles [7], Maekawa et al. [8], Stiles and Kagan [9], Stiles et al. [10], El Adawi et al. [11], Othman and Zaki [12] and Siddheshwar [13].

The study of finite amplitude convection (Veronis [14]) using a truncated Fourier representation, has gained momentum in recent years owing to its simplicity and nonlinear complexity of the solution. It is found handy by the researchers at least for four reasons: It can be used (i) to determine the plan-forms of cellular motion that can occur in the fluid, (ii) to explicate the convective processes of many non-isothermal situations of practical interest, (iii) to quantify the heat transfer and (iv) to advance a bit closer to the challenging problem of the onset of chaotic motion.

The reported works on nonlinear convection in dielectric liquids are very scant owing to the involvedness of both the governing equations and the solution procedure. Ko and Kim [6] have studied electrohydrodynamic convective instability in a

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horizontal fluid layer with temperature gradient. Nonlinear evolution of disturbances near the onset of convection is considered.

Stiles et al. [10] studied the problem of convective heat transfer through polarized dielectric liquids. It is shown that for a critical voltage, as the gravitational Rayleigh number becomes increasingly negative, the critical wave number at the onset of convection becomes very large. As the temperature drop between the plates increases the fraction of the heat transfer associated with convection is found to pass through a maximum value when the critical horizontal wave number is close to 4 times its value when gravity is absent.

Haque and Arajs [15] have examined convective specific heat transfer in liquids in the presence of non-uniform electric fields. The heat transfer coefficient has been evaluated under the influence of *ac* and *dc* electric fields, and the efficiency obtained in a *dc* field is found to be higher than in the *ac* field.

We note that the study of finite amplitude Rayleigh–Bénard convection and heat transport in a dielectric liquid by means of a minimal Fourier series representation does not seem to have been undertaken. Accordingly, in this paper we concentrate on a weakly nonlinear local stability analysis of thermal convection in a dielectric liquid permeated by a vertical, uniform AC electric field.

1.1. Mathematical formulation and solution

Consider an infinite horizontal layer of a Boussinesquian dielectric liquid of depth 'h' that supports a temperature gradient ΔT and an AC electric field in the vertical direction. The upper and lower boundaries are maintained at constant temperatures T_o and $T_o + \Delta T$ ($\Delta T > 0$) respectively. The schematic of the same is shown in Fig. 1. For mathematical tractability we confine ourselves to two-dimensional rolls so that all physical quantities are independent of y, a horizontal co-ordinate. Further, the boundaries are assumed to be free and perfect conductors of heat. We assume the dynamic viscosity μ of the dielectric liquid to be a constant.

The governing equations describing the Rayleigh–Bénard instability situation in a constant viscosity dielectric liquid are:
Continuity equation

$$\nabla \cdot \vec{q} = 0. \tag{1}$$

Conservation of linear momentum

$$\rho_o \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = -\nabla p - \rho g \hat{k} + (\overrightarrow{P} \cdot \nabla)\overrightarrow{E} + \mu \nabla^2 \vec{q}. \tag{2}$$

Conservation of energy

$$\rho_o C_{VE} \left[\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T \right] = k_1 \nabla^2 T. \tag{3}$$

Density equation of state

$$\rho = \rho_o [1 - \alpha_t (T - T_o)]. \tag{4}$$

The effects of heat source and radiation are assumed to be negligible in writing the energy equation (3).

It should be observed that the nonlinear terms $(\vec{q} \cdot \nabla)\vec{q}$, $\vec{q} \cdot \nabla T$ and $(\overrightarrow{P} \cdot \nabla)\overrightarrow{E}$ are to be retained in the considered nonlinear stability analysis and are the overriding objects of interest in so far as the finite amplitude theory is concerned.

Electric field equations, simplified for a dielectric liquid under an AC electric field, take the form

$$\nabla . \overrightarrow{D} = 0, \quad \nabla \times \overrightarrow{E} = \vec{0}, \tag{5}$$

$$\overrightarrow{D} = \varepsilon_o \overrightarrow{E} + \overrightarrow{P}, \quad \overrightarrow{P} = \varepsilon_o (\varepsilon_r - 1) \overrightarrow{E}, \tag{6}$$

$$\varepsilon_r = \varepsilon_r^o - e(T - T_o), \tag{7}$$

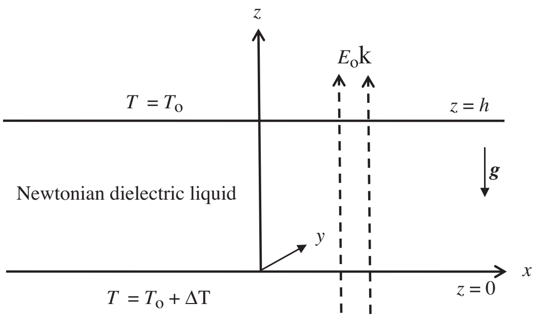


Fig. 1. Configuration of the problem.

where $\overrightarrow{E}$ is an *AC* electric field, which is assumed to oscillate sufficiently rapidly so as to make the body force on any free charges in the liquid inconsequential and the rest of the quantities have their usual meaning. It is expedient to write $\varepsilon_r^o = (1+\chi_e)$, where χ_e is the electric susceptibility, for it enables us to arrive at the conventional definition $\overrightarrow{P} = \varepsilon_o \chi_e \overrightarrow{E}$ in the absence of the temperature dependence of ε_r, that is, when $e = 0$. In writing Eq. (7) we have assumed that ε_r varies with the electric field strength quite insignificantly [9].

The electric boundary conditions are that the normal component of the electric displacement $\overrightarrow{D}$ and tangential component of the electric field $\overrightarrow{E}$ are continuous across the boundaries.

Taking the components of polarization and electric field in the basic state to be $[0, P_b(z)]$ and $[0, E_b(z)]$, we obtain the quiescent state solution

$$\left.\begin{aligned}&\vec{q}_b = (0,0), T_b = T_o - \tfrac{\Delta T}{h} z, \quad \rho_b = \rho_o\left[1 + \alpha_t \tfrac{\Delta T}{h} z\right],\\ &\overrightarrow{E}_b = \left[\tfrac{(1+\chi_e)E_o}{(1+\chi_e)+\frac{e\Delta T}{h}z}\right]\hat{k}, \quad P_b = \varepsilon_o E_o (1+\chi_e)\left[1 - \tfrac{1}{(1+\chi_e)+\frac{e\Delta T}{h}z}\right]\hat{k},\end{aligned}\right\} \tag{8}$$

where E_o is the root mean square value of the electric field at the lower surface. On this basic state we superpose finite amplitude perturbations of the form

$$\left.\begin{aligned}&\vec{q} = \vec{q}_b + (u', w'), \quad T = T_b + T', \quad p = p_b + p', \quad \rho = \rho_b + \rho',\\ &\overrightarrow{P} = \overrightarrow{P}_b + (P_1', P_3'), \quad \overrightarrow{E} = \overrightarrow{E}_b + (E_1', E_3'),\end{aligned}\right\} \tag{9}$$

where the prime denotes perturbation. The second of Eq. (6) now leads to

$$\left.\begin{aligned}&P_1' = \varepsilon_o \chi_e E_1' - e\varepsilon_o T' E_1',\\ &P_3' = \varepsilon_o \chi_e E_3' - e\varepsilon_o E_o T' - e\varepsilon_o T' E_3',\end{aligned}\right\} \tag{10}$$

where it has been assumed that $e\Delta T \ll (1+\chi_e)$. Since we consider only two-dimensional disturbances, we introduce the stream function ψ'

$$u' = \frac{\partial \psi'}{\partial z}, \quad w' = -\frac{\partial \psi'}{\partial x}, \tag{11}$$

which satisfy the continuity equation (1) in the perturbed state. Introducing the perturbed electric potential Φ' through the relation $\overrightarrow{E}' = \nabla \Phi'$, eliminating the pressure p in Eq. (2), incorporating the quiescent state solution, we obtain the dimensionless form of the vorticity and heat transport equations as

$$\frac{1}{Pr}\frac{\partial}{\partial t}(\nabla^2\psi) = -(1+L)\frac{\partial T}{\partial x} + L\frac{\partial^2 \Phi}{\partial x \partial z} + \nabla^4\psi + L\frac{\partial(T, \frac{\partial \Phi}{\partial z})}{\partial(x,z)} + \frac{1}{Pr}\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,z)}, \tag{12}$$

$$\frac{\partial T}{\partial t} = -R_T\frac{\partial \psi}{\partial x} + \nabla^2 T + \frac{\partial(\psi, T)}{\partial(x,z)}, \tag{13}$$

where

$$L = \frac{\varepsilon_0 e^2 E_0^2 \Delta T}{\alpha_t g \rho_0 (1+\chi_e) h} \quad \text{(Electric number)},$$

$$Pr = \frac{\mu}{\rho_o k_1} \quad \text{(Prandtl number)}$$

and

$\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$ is the three dimensional Laplace operator. It can easily be verified that

$$\frac{R_{ET}}{L} = R_T,$$

where

$$R_{ET} = \frac{\varepsilon_0 \alpha_s \Delta T V^2}{\mu_1 k_1} \quad \text{(Electric Rayleigh number)}.$$

In Eqs. (12) and (13), the asterisks have been dropped for simplicity and we continue doing so in the remaining part of the paper. Using Eq. (10) in the first of Eq. (5) and non-dimensionalizing the resulting equation, we obtain

$$\nabla^2 \Phi - \frac{\partial T}{\partial z} = 0, \tag{14}$$

where it has been assumed that $\frac{e\Delta T}{R_T} \ll (1+\chi_e)$.

Eqs. (12)–(14) are solved using the boundary conditions

$$\psi = \nabla^2\psi = T = \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = 0, 1. \tag{15}$$

From Eq. (15) it is clear that the boundaries are taken to be flat, stress-free and perfect conductors of heat. We also note that the boundary condition for the electric potential Φ, which allows periodic solutions in the vertical direction, is tantamount to assuming that the electric susceptibility χ_e with respect to the perturbed field is large at both the boundaries [2]. In the next section, we discuss the linear stability analysis, which is of great utility in the local nonlinear stability analysis to be carried out later on in the paper.

2. Linear stability analysis

In order to study the linear theory we consider the linear version of Eqs. (12)–(14) and assume the solutions to be periodic waves of the form [16]

$$\begin{bmatrix} \psi \\ T \\ \Phi \end{bmatrix} = e^{\omega_1 t} \begin{bmatrix} \psi_o \sin \pi a x \sin \pi z \\ \theta_o \cos \pi a x \sin \pi z \\ \left(\frac{\Phi_o}{\pi}\right) \cos \pi a x \cos \pi z \end{bmatrix}, \tag{16}$$

which satisfy the boundary conditions in Eq. (15). In Eq. (16), $\omega_1 = \omega_r + i\omega$, in which ω_r is the growth rate and ω is the frequency of oscillations, πa is the horizontal wave number and π is the vertical wave number. ψ_o, θ_o and Φ_o are, respectively, amplitudes of the stream function, temperature and the electric potential. Substituting Eq. (16) into the linearized versions of Eqs. (12)–(14), we obtain

$$\left(\frac{\omega_1}{Pr} + \eta_1^2\right)\eta_1^2\psi_o + (1+L)\pi a\theta_o + L\pi a\Phi_o = 0, \tag{17}$$

$$R_T\pi a\psi_o + \left(\omega_1 + \eta_1^2\right)\theta_o = 0, \tag{18}$$

$$\theta_o + (1+a^2)\Phi_o = 0, \tag{19}$$

where $\eta_1^2 = \pi^2(1+a^2)$. For a non-trivial solution for ψ_o, θ_o and Φ_o, we require

$$R_T = \frac{\left(\omega_1 + \eta_1^2\right)\left(\frac{\omega_1}{Pr} + \eta_1^2\right)\eta_1^2(1+a^2)}{\pi^2a^2[1+(1+L)a^2]}. \tag{20}$$

The onset of convection in dielectric liquids can occur in one of the following ways:

(i) marginal stationary convection (steady convection),
(ii) marginal oscillatory convection (unsteady convection).

In the marginal state the real part of ω_1 is equal to zero.

The thermal Rayleigh number R_T is the eigenvalue of the problem that throws light on the stability or otherwise of the system. The critical value of R_T, *i.e.*, R_{Tc} signifies the onset of convection via one of the above modes. R_{Tc} of stationary is different from R_{Tc} of oscillatory. If R_{Tc} of stationary convection is less than that of oscillatory convection, then we say the "*Principle of Exchange of Stabilities* (*PES*)" is valid. We now move over to the discussion on the stationary instability followed by that on the validity or otherwise of the *PES*.

2.1. Marginal stationary state

If ω_1 is real, then the marginal stationary convection occurs when $\omega_1 = 0$. This gives the stationary thermal Rayleigh number [2]

$$R_T^s = \frac{(1+a^2)\eta_1^6}{\pi^2a^2[1+(1+L)a^2]}. \tag{21}$$

The critical wave number a_c satisfies the equation

$$\left[(1+L)\left(a_c^2\right)^2 + 2a_c^2 + 1\right]\left[2a_c^2 - 1\right] + La_c^2\left[a_c^2 - 2\right] = 0. \tag{22}$$

Eq. (22) clearly shows that a_c depends on the electric number L. When we take $L = 0$, we obtain the results of the classical Rayleigh–Bénard instability with $a_c^2 = 0.5$ and $R_{Tc}^s = 657.5$ [16]. Rearranging Eq. (21), we may write

$$R_T^s = \frac{\eta_1^6}{(\pi a)^2} - \frac{(\pi a)^2}{\eta_1^2}R_{ET},$$

which coincides exactly with the expression of the stationary Rayleigh number of Roberts [19].

2.2. Marginal oscillatory state

Taking $\omega_1 = i\omega$ (ω being the frequency of oscillations) in Eq. (5) and separating the real and imaginary parts, we obtain the oscillatory thermal Rayleigh number

$$R_T^o = \frac{(1+a^2)\left[\eta_1^6 - \frac{\omega^2\eta_1^2}{Pr}\right]}{\pi^2 a^2[1+(1+L)a^2]} + i\omega N, \tag{23}$$

where

$$N = \frac{\eta_1^4(1+\frac{1}{Pr})(1+a^2)}{\pi^2 a^2[1+(1+L)a^2]}.$$

Since R_o^T is a real quantity, the imaginary part of Eq. (23) has to vanish. This gives us two possibilities:

(i) $\omega \neq 0$, $N = 0$ (oscillatory instability),
(ii) $\omega = 0$, $N \neq 0$ (stationary instability).

Taking $N = 0$, we get $\eta_1^4(1+\frac{1}{Pr})(1+a^2) = 0$, which is independent of ω. In problems wherein oscillatory convection is preferred to stationary, the condition $N = 0$ leads to an expression for ω^2 that is in turn substituted in the real part of the expression for R_T^o, thereby yielding the oscillatory thermal Rayleigh number. In view of the fact that N is independent of ω, we infer that oscillatory convection is not possible in the present problem. This essentially means that the *PES* holds good for the problem at hand.

In the next section we explore the possibility of cross-interaction of several modes of steady electroconvection by considering a spectral representation of the stream function ψ and perturbation temperature T as an infinite series of orthogonal space functions. To proceed with this approach we first assume steady state and then eliminate Φ in the linear term between Eqs. (12) and (14) to obtain

$$\nabla^6\psi + L\frac{\partial^3 T}{\partial x\partial z^2} - (1+L)\frac{\partial}{\partial x}(\nabla^2 T) + L\nabla^2\left(\frac{\partial\left(T, \frac{\partial\Phi}{\partial z}\right)}{\partial(x,z)}\right) + \frac{1}{Pr}\nabla^2\left(\frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial(x,z)}\right) = 0. \tag{24}$$

Eq. (13) in the steady state is

$$\nabla^2 T - R_T\frac{\partial\psi}{\partial x} + \frac{\partial(\psi, T)}{\partial(x,z)} = 0. \tag{25}$$

We now proceed with the spectral analysis of electroconvection by writing $R_{T\gamma}$ for R_T suggesting the use of a Rayleigh number that helps in taking into account the different modes that compete in influencing convection.

2.3. Spectral analysis of electroconvection

The linearized non-dimensional vorticity and heat transport equations (24) and (25) are transformed into the spectral domain by representing ψ and T as an infinite series of orthogonal space functions given by

$$\psi = -\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\psi_\gamma \exp i(lax + nz), \tag{26}$$

$$T = -i\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\theta_\gamma \exp i(lax + nz), \tag{27}$$

where $\gamma = (l,n)$, l and n are integer multiples of π and 'a' is the horizontal wave number.

The basic coupled partial differential equations (24) and (25) are transformed into the spectral domain using Eqs. (26) and (27) resulting in the following spectral equations:

$$\alpha_\gamma^6\psi_\gamma - al\left[n^2L - (1+L)\alpha_\gamma^2\right]\theta_\gamma = 0, \tag{28}$$

$$\alpha_\gamma^2\theta_\gamma + laR_{T\gamma}\psi_\gamma = 0, \tag{29}$$

where $\alpha_\gamma^2 = a^2l^2 + n^2$. For a non-trivial solution of the above homogeneous system we require

$$\begin{vmatrix} \alpha_\gamma^6 & -(n^2L-(1+L)\alpha_\gamma^2)al \\ R_{T\gamma}al & \alpha_\gamma^2 \end{vmatrix} = 0.$$

This yields us the expression for the modal Rayleigh number $R_{T\gamma}$ in the form

$$R_{T\gamma} = \frac{\alpha_\gamma^8}{a^2 l^2 \left[(1+L)\alpha_\gamma^2 - n^2 L\right]}. \tag{30}$$

We see from Eq. (15) that $R_{T\gamma}$ is a continuous function of a^2. For a given mode $\gamma = (l, n)$, the critical modal Rayleigh number $R_{T\gamma c}$ and the corresponding critical wave number a_c for various values of the parameter L have to be determined in each case separately. The mode $l = \pi$, $n = \pi$ gives us the expression for the stationary thermal Rayleigh number (Eq. (6)) discussed earlier.

Our main objective in the next section is to decipher analytically the effect of the electric number, L, on the monotonicity of the thermal Rayleigh number R_T using the concept of self-adjoint operator. In view of the fact that the *PES* is valid, we consider only steady motions.

2.4. Parametric perturbation method

We assume the steady solution to the linear version of Eqs. (12)–(14) in the form

$$\begin{bmatrix} \psi \\ T \\ \Phi \end{bmatrix} = \begin{bmatrix} \psi(z)\sin\pi a x \\ T(z)\cos\pi a x \\ \Phi(z)\cos\pi a x \end{bmatrix}. \tag{31}$$

Substituting Eq. (16) into the linear form of Eqs. (12)–(14), we obtain

$$(D^2 - \pi^2 a^2)^2\psi + \pi a(1+L)T - L\pi a D\Phi = 0, \tag{32}$$

$$R_T \pi a\psi - (D^2 - \pi^2 a^2)T = 0, \tag{33}$$

$$DT - (D^2 - \pi^2 a^2)\Phi = 0, \tag{34}$$

where $D = \frac{d}{dz}$. Eliminating Φ between Eqs. (32) and (34), we obtain

$$(D^2 - \pi^2 a^2)^3\psi + \pi a\{D^2 - (1+L)\pi^2 a^2\}T = 0. \tag{35}$$

Eq. (33) can be rewritten as

$$R_T \pi a[D^2 - (1+L)\pi^2 a^2]\psi - (D^2 - \pi^2 a^2)[D^2 - (1+L)\pi^2 a^2]T = 0. \tag{36}$$

We now define a symmetric operator L_1 as follows:

$$L_1 = \begin{bmatrix} R_T(D^2 - \pi^2 a^2)^3 R_T \pi a\{D^2 - (1+L)\pi^2 a^2\} \\ R_T \pi a\{D^2 - (1+L)\pi^2 a^2\} - (D^2 - \pi^2 a^2)\{D^2 - (1+L)\pi^2 a^2\} \end{bmatrix}. \tag{37}$$

We next define a vector $\vec{V}$ such that $\vec{V} = \begin{bmatrix} \psi \\ T \end{bmatrix}$. Eqs. (35) and (36) can now be written as

$$L_1 \vec{V} = \vec{0}. \tag{38}$$

We define the inner product between two vectors $\vec{a}$ and $\vec{b}$ such that

$$\langle \vec{a}, \vec{b} \rangle = \int_V \vec{a}^{*Tr} \cdot \vec{b}\, dV, \tag{39}$$

where V represents the domain of the integral operator in which $\vec{a}$ and $\vec{b}$ are defined, the asterisk represents the complex conjugate and Tr represents the transpose. As the operator L_1 and the boundary conditions on ψ and T in Eq. (15) are symmetric, one may easily prove that L_1 is self-adjoint and so are the boundary conditions on ψ and T in Eq. (15).

To seek information on the variation of R_T with respect to L, we differentiate Eq. (38) with respect to L and obtain

$$L_1 \vec{V}_d = \vec{u}_d, \tag{40}$$

where

$$\vec{u}_d = \begin{bmatrix} \pi^3 a^3 T \\ -\pi a R_{Td}\psi \end{bmatrix}$$

and the subscript 'd' represents the derivative with respect to L. Applying a Fredholm alternative condition to Eq. (40), we obtain

$$\pi^3 a^3 \int_V \psi^* T\, dV = -\pi a R_{Td} \int_V T^* \psi\, dV. \tag{41}$$

From the above equation it is clear that $R_{Td} < 0$. This means that R_T is a decreasing function of L and hence the effect of L is to destabilize the system.

The linear theory discussed in a previous section reveals that the stationary mode of instability is preferred to the oscillatory one. In deed, the linear theory predicts only the condition for the onset of convection and is silent about the heat transfer. We now embark on a weakly nonlinear analysis by means of a truncated representation of Fourier series for velocity, temperature and electric fields to find the effect of various parameters on finite amplitude steady convection and to know the amount of heat transfer. We note that the results obtained from such an analysis can serve as starting values while solving a more general nonlinear convection problem.

3. Local nonlinear stability analysis

The first effect of nonlinearity is to distort the temperature field through the interaction of ψ and T, and Φ and T. The distortion of temperature field will correspond to a change in the horizontal mean, *i.e.*, a component of the form $\sin(2\pi z)$ will be generated. Thus a minimal double Fourier series which describes the finite amplitude convection in a dielectric liquid is

$$\begin{bmatrix} \psi \\ T \\ \Phi \end{bmatrix} = \begin{bmatrix} A(t)\sin\pi ax\sin\pi z \\ B(t)\cos\pi ax\sin\pi z \\ \frac{1}{\pi}E(t)\cos\pi ax\cos\pi z \end{bmatrix} + \begin{bmatrix} 0 \\ C(t) \\ 0 \end{bmatrix}\sin(2\pi z), \tag{42}$$

where the amplitudes A, B, C and E are to be determined from the dynamics of the system. Substituting Eq. (42) into Eqs. (12)–(14), equating the coefficients of like terms, we obtain the following nonlinear autonomous system (*generalized Lorenz model*, Sparrow [17]) of differential equations

$$\dot{A} = -Pr\eta_1^2 A - \frac{\pi aPr(1+L)}{\eta_1^2}B - \frac{\pi aPrL}{\eta_1^2}E - \frac{L\pi^2 aPr}{\eta_1^2}CE, \tag{43}$$

$$\dot{B} = -R_T\pi aA - \eta_1^2 B - \pi^2 aAC, \tag{44}$$

$$\dot{C} = \frac{\pi^2 a}{2}AB - 4\pi^2 C, \tag{45}$$

$$0 = B + (1+a^2)E, \tag{46}$$

where the over dot denotes time derivative. It is advantageous to eliminate the variable E between Eqs. (43) and (46) noting that Eq. (46) does not have a time derivative term on the left side. This process reduces the system of Eqs. (43)–(46) to

$$\dot{A} = -Pr\eta_1^2 A - \frac{\pi aPr[1+(1+L)a^2]}{(1+a^2)\eta_1^2}B + \frac{L\pi^2 aPr}{(1+a^2)\eta_1^2}BC, \tag{47}$$

$$\dot{B} = -R\pi aA - \eta_1^2 B - \pi^2 aAC, \tag{48}$$

$$\dot{C} = \frac{\pi^2 a}{2}AB - 4\pi^2 C. \tag{49}$$

The third order Lorenz system described by Eqs. (47)–(49) is uniformly bounded in time and possesses many properties of the full problem. Moreover, the phase-space volume contracts at a uniform rate

$$\frac{\partial\dot{A}}{\partial A} + \frac{\partial\dot{B}}{\partial B} + \frac{\partial\dot{C}}{\partial C} = -\left[(Pr+1)\eta_1^2 + 4\pi^2\right], \tag{50}$$

which is always negative and therefore the system is bounded and dissipative. As a result, the trajectories are attracted to a set of measure zero in the phase-space; in particular, they may be attracted to a fixed point, a limit cycle or perhaps, a strange attractor. Before solving the nonlinear system of equations, we consider the linear autonomous system and analyze the critical points. The nature of the critical points obtained from the linear system discloses information about the trajectories in the phase plane. The nature of these trajectories is retained by the nonlinear system but with distortions dictated by the nonlinear terms.

3.1. Linear autonomous system

The linearized autonomous system is

$$\dot{A} = -Pr\eta_1^2 A - \frac{\pi aPr[1+(1+L)a^2]}{(1+a^2)\eta_1^2}B, \tag{51}$$

$$\dot{B} = -R\pi aA - \eta_1^2 B, \tag{52}$$

$$\dot{C} = -4\pi^2 C. \tag{53}$$

To explore the critical points of the above linear autonomous system of equations, we follow Simmons [18] and write the auxiliary equation

$$\begin{vmatrix} -Pr\eta_1^2-\xi & \frac{-\pi a Pr[1+(1+L)a^2]}{(1+a^2)\eta_1^2} & 0 \\ -R_T\pi a & -\eta_1^2-\xi & 0 \\ 0 & 0 & -4\pi^2-\xi \end{vmatrix} = 0.$$

On expansion, we obtain

$$\xi^2+(Pr+1)\eta_1^2\xi+\left[Pr\eta_1^4-\frac{R_T\pi^2a^2Pr[1+(1+L)a^2]}{(1+a^2)\eta_1^2}\right]=0. \tag{54}$$

Let ξ_1 and ξ_2 be the roots of Eq. (54). We now discuss three cases according to the nature of these roots.

Case (i) ξ_1 and ξ_2 are real and equal.
In this case, we have

$$(Pr+1)^2\eta_1^4=4\left[Pr\eta_1^4-\frac{R_T\pi^2a^2Pr[1+(1+L)a^2]}{(1+a^2)\eta_1^2}\right].$$

On simplification, the above yields an expression for R_T

$$R_T=\frac{[4Pr-(Pr+1)^2](1+a^2)\eta_1^6}{4\pi^2a^2Pr[1+(1+L)a^2]}. \tag{55}$$

For the above value of R_T, the critical point is a *node*. In this case the system becomes stable as the paths approach towards the critical point.

Case (ii) ξ_1 and ξ_2 are real and distinct.
In this case, we have the condition

$$R_T>\frac{[4Pr-(Pr+1)^2](1+a^2)\eta_1^6}{4\pi^2a^2Pr[1+(1+L)a^2]}. \tag{56}$$

For this range of values of R_T, the critical point is a *saddle point* and the system is unstable as paths never approach the critical points.

Case (iii) ξ_1 and ξ_2 are imaginary.
The requirement in this case takes the form

$$R_T<\frac{[4Pr-(Pr+1)^2](1+a^2)\eta_1^6}{4\pi^2a^2Pr[1+(1+L)a^2]}. \tag{57}$$

For this range of values of R_T, the critical point is a *spiral* and the system is asymptotically stable if paths approach the critical point as $t\to-\infty$ and the system becomes unstable as $t\to\infty$ if the paths spiral out.

Having made a qualitative analysis of the linear autonomous system, we note that the nonlinear system of autonomous differential equations (47)–(49) is not amenable to analytical treatment for the general time-dependent variables and we need to solve it by means of a numerical method. However, in the case of steady motions, these equations can be solved in closed form. Such solutions prove very useful because they may show that a finite amplitude steady solution to the system is possible for sub-critical values of the thermal Rayleigh number and that the minimum value of R_T for which finite amplitude steady solution is possible lies below the critical values corresponding to a steady infinitesimal disturbance or an overstable one. In the case of steady motions, Eqs. (47)–(49) take the form

$$(1+a^2)\eta_1^4A+\pi a[1+(1+L)a^2]B-L\pi^2aBC=0, \tag{58}$$

$$R_T\pi aA+\eta_1^2B+\pi^2aAC=0, \tag{59}$$

$$8C-aAB=0. \tag{60}$$

Writing B and C in terms of A using Eqs. (59) and (60) and substituting the resulting expressions into Eq. (58), we obtain

$$A\left\{\begin{array}{l} a^4\pi^4(1+a^2)\eta_1^4\left(\frac{A^2}{8}\right)^2+\left[a^2\pi^2\left\{\begin{array}{c} 2\eta_1^6(1+a^2) \\ -R_Ta^2\pi^2[1+(1+L)a^2] \\ -R_T^2a^2\pi^2L \end{array}\right\}\right]\left(\frac{A^2}{8}\right) \\ +\ \eta_1^2\left[\eta_1^6(1+a^2)-R_Ta^2\pi^2[1+(1+L)a^2]\right] \end{array}\right\}=0. \tag{61}$$

The solution $A = 0$ corresponds to pure conduction and the rest of the solutions are given by

$$\frac{A^2}{8} = \left[\frac{1}{2\pi^2 a^2(1+a^2)\eta_1^4}\right] \times \left[\begin{array}{l} R_T\pi^2 a^2[1+(1+L)a^2] + LR_T^2\pi^2 a^2 - 2(1+a^2)\eta_1^6 \\ \pm\ \pi a R_T\sqrt{\begin{array}{l}\pi^2 a^2(1+a^2)^2 + L^2\pi^2 a^2(R_T+a^2)^2 \\ +2L(1+a^2)\{\pi^2 a^2(R_T+a^2) - 2\eta_1^6\}\end{array}} \end{array}\right]. \tag{62}$$

We take the positive sign in front of the radical in Eq. (62) on the ground that the amplitude of the stream function is real. The finite amplitude Rayleigh number R_{Tf} can be obtained from Eq. (62) by equating the discriminant to zero that gives us the following expression for R_{Tf}

$$a^4\pi^4L^2R_{Tf}^4 + 2a^4\pi^4L(1+(1+L)a^2)R_{Tf}^3 + a^2\pi^2(a^2\pi^2[1+(1+L)a^2]^2 - 4\eta_1^6(1+a^2)L)R_{Tf}^2 + 4a^2\pi^2(1+a^2)\eta_1^4[1+(1+L)a^2](1-\eta_1^2)R_{Tf} + 4(1+a^2)\eta_1^{10}((1+a^2)\eta_1^2 - 1) = 0. \tag{63}$$

Computation reveals that $R_{Tfc} > R_{Tc}^s$, thus ruling out sub-critical instability.

4. Heat transport

In the study of convection in dielectric liquids the quantification of heat transport across the layer plays a crucial role. This is because the onset of convection, as the thermal Rayleigh number R_T is increased, is more readily detected by its effect on the heat transfer.

The Nusselt number Nu is defined as

$$Nu(t) = \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by (conduction)}} = \frac{\left[\frac{\pi a_c}{2\pi}\int_0^{2\pi/\pi a_c}(1-z+T)_{,z}dx\right]_{z=0}}{\left[\frac{\pi a_c}{2\pi}\int_0^{2\pi/\pi a_c}(1-z)_{,z}dx\right]_{z=0}}. \tag{64}$$

Substituting T from Eq. (42) into Eq. (62) and then writing C in terms of A using Eqs. (58)–(60), we obtain

$$Nu = 1 + \frac{2\pi^2 a^2 R_T\left(\frac{A^2}{8}\right)}{\eta_1^2 + a^2\pi^2\left(\frac{A^2}{8}\right)}. \tag{65}$$

The second term on the right side of Eq. (65) characterizes the convective contribution to the heat transport.

5. Results and discussion

In the paper, AC electroconvection of infinitesimal and finite amplitude disturbances is considered using normal modes and truncated representation of Fourier series respectively. The principle of exchange of stabilities is shown to be valid in the case of the linear theory. Using the concept of self-adjoint operator qualitative effect of the effect of electric number (L) on the onset of convection is studied. Using a spectral representation relative domination of different modes of convection is also considered. The nonlinear theory helps in deciding whether subcritical motion is possible in the case of AC electroconvection. It also helps in quantifying heat transfer and in understanding the transition from periodic oscillations to a behavior that is apparently chaotic. Table 1 documents the fact that the effect of increasing electric number is to decrease the critical value of the thermal Rayleigh number $\left(R_{Tc}^S\right)$ and increase in the value of critical wave number (a_c), and there by the critical wave length (λ_c). The above result on R_{Tc}^S was also shown by using the concept of self-adjoint operator. Further, from Table 1 we see that the results of Roberts [19] can be recovered from the present study.

The variation of the modal Rayleigh number $R_{T\gamma}$ versus a^2 for some modes is shown in Fig. 2. Both even and odd parity modes are covered in the figure. The relative domination of different modes is brought out clearly in the figure. We note that (π, π) is the most fundamental mode and is not damped out by any higher mode. The effect of L clearly is to ensure that the mode (π, π) picks up instability earlier than the other modes. This essentially means that $L \neq 0$ favors the fundamental mode

Table 1
Critical thermal Rayleigh number (R_{Tc}^s), critical wave number (a_c) and critical wavelength (λ_c) as a function of the electric number L (or electric Rayleigh number R_E) for stationary instability in a constant viscosity dielectric liquid.

L present paper	R_E Roberts [19]	R_{Tc}^s	a_c	$\lambda_c = \frac{2\pi}{\pi a_c}$
0	0657.51	657.511	0.500	4.000
10	1293.90	129.390	0.956	2.092
100	1527.90	15.279	0.995	2.010
500	1551.50	3.105	0.999	2.002
1000	1555.00	1.555	0.999	2.002

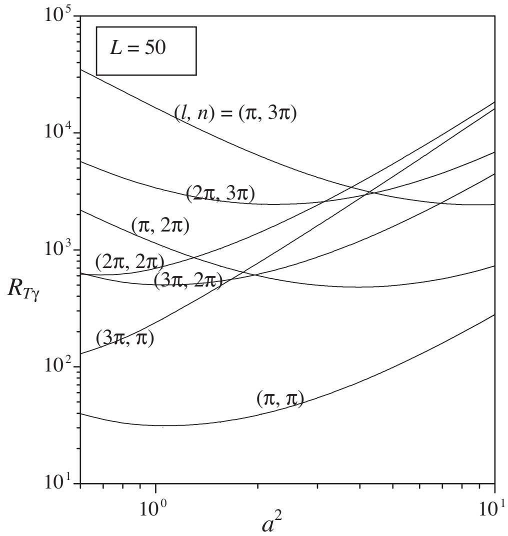


Fig. 2. Plot of $R_{T\gamma}$ vs. a^2 (for both even and odd parity modes).

(π, π) in effecting instability in the form of stationary convection. This justifies the choice of the normal mode solution used in the paper for studying linear stability.

Fig. 3 is a plot of the stream lines for different values of L and R_T, using Eq. (62) for obtaining A. These figures reiterate remarks on λ_c presented earlier in the context of Table 1.

We consider Nusselt number plots of both steady and unsteady finite amplitude electroconvection. The realm of steady nonlinear electroconvection warrants the quantification of heat transfer in the Nusselt–Rayleigh plane for different

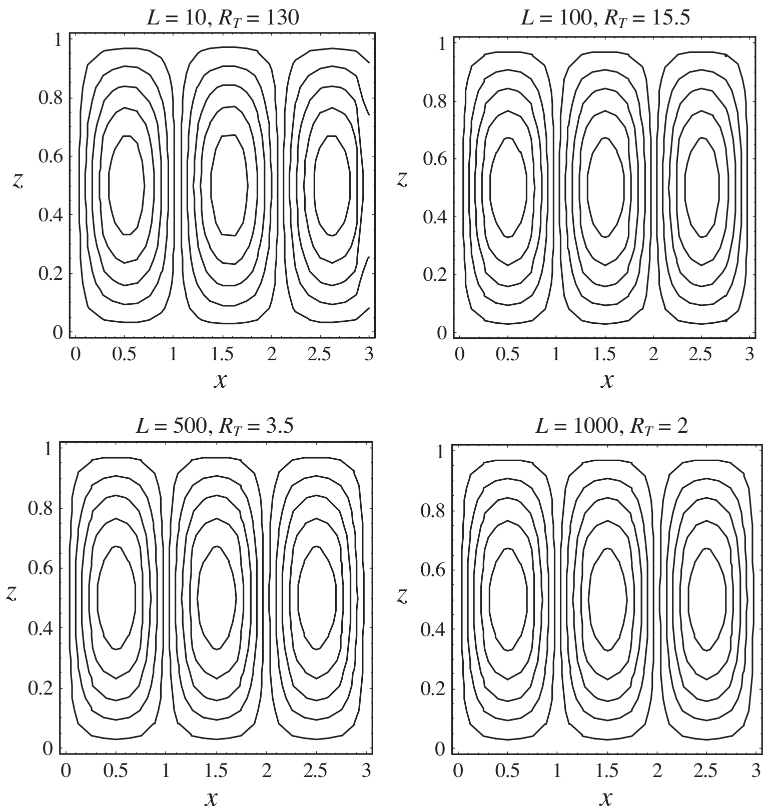


Fig. 3. Streamlines for different values of L in the case of steady finite amplitude convection.

values of L. We observe from Fig. 4 that the Nusselt number increases with increase in R_T for all values of L. Further the effect of increasing L is to increase the Nusselt number. We also notice that for large R_T, Nu approaches the asymptotic value 3. It can also be seen that Nu becomes independent of R_T for large values of L. Computations further reveal that steady finite amplitude subcritical instabilities can be ruled out in the case of AC electroconvection.

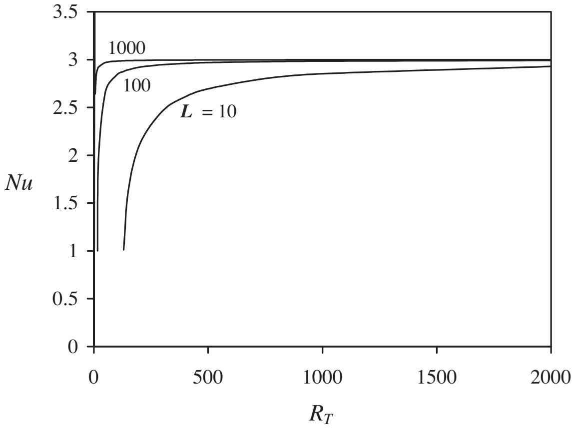


Fig. 4. Plot of the Nusselt number Nu vs. the thermal Rayleigh number R_T for different values of L in the case of steady finite amplitude convection.

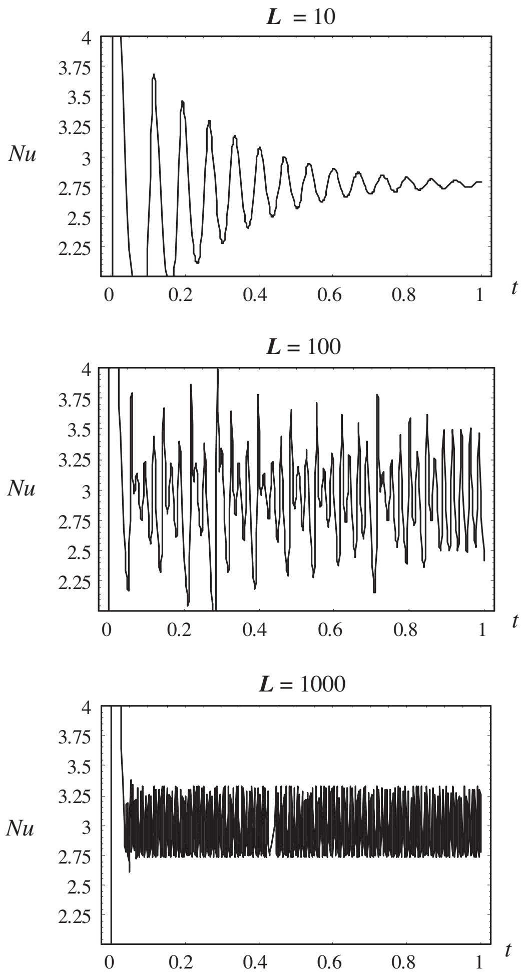
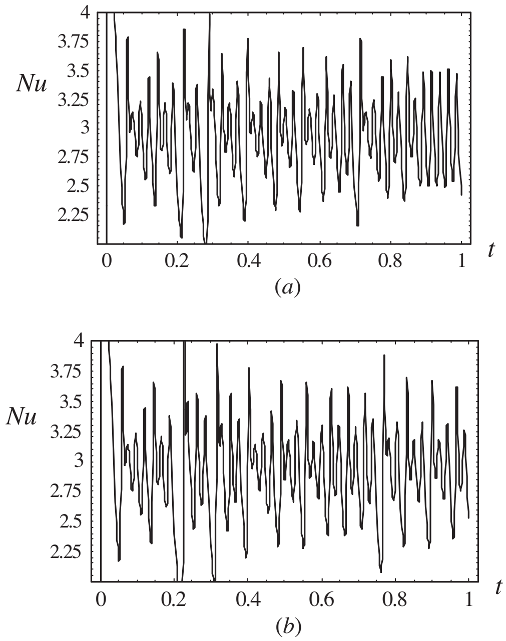


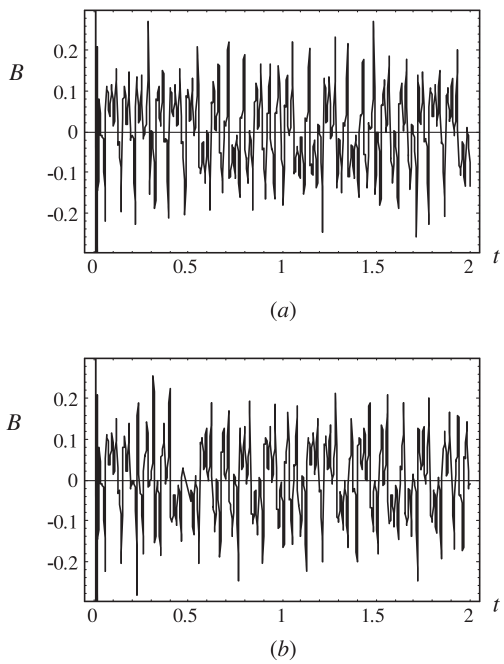
Fig. 5. Variations in the Nusselt number, Nu, with time for different values of L and for $R_T = 658$ and $Pr = 10$.

For unsteady finite amplitude convection we have the plots of Nu versus t for different values of L and these are depicted in Figs. 5–7. Before we discuss the results we note that the choice of $R_T = 658$ has been made with the intention of having time series plots in the convective regime of electroconvection for small and large L. In this regime Nu takes a value greater than 1 to signify that heat transport is by conduction and convection. When $R_T < R_{Tc}$ we have the conduction regime and onset at $R_T = R_{Tc}$. The value of the critical Rayleigh number for different values of L is recorded in Table 1. The transients in the



(a) $(A, B, C) = (0, 1, 0)$ and (b) $(A, B, C) = (0, 1.0001, 0)$.

Fig. 6. Variations in the Nusselt number, Nu, with time for different initial conditions and for $L = 100$, $R_T = 658$ and $Pr = 10$.



(a) $(A, B, C) = (0, 1, 0)$ and (b) $(A, B, C) = (0, 1.0001, 0)$.

Fig. 7. Variations in amplitude, $B(t)$ versus time for different initial conditions and for $L = 100$, $R_T = 658$ and Pr = 10.

Figs. 5–7 clearly demonstrate the approach of Nu towards the value 3 for large values of L as discussed earlier in the context of Fig. 4.

The sensitive dependence on the initial conditions is considered in Fig. 7 by studying the variation of Nu with time for fixed value of R_T and Pr, and different values of L. The chosen initial conditions are $A(0) = 0$, $B(0) = 1.0001$ and $C(0) = 0$, the departure from the earlier chosen initial conditions of $(A,B,C = 0,1,0)$. As the system is sensitive to the initial conditions, we may conclude that the time evolution eventually leads to chaotic motion. Due to the destabilizing nature of L on the onset of electroconvection (see Table 1), it is obvious that chaos is realized earlier in the case of $L \neq 0$ compared to that in the case of $L = 0$.

Some general results from the linear and non-linear stability analyses are:

(i) $[\pi a_c]_{L=0} < [\pi a_c]_{L\neq 0}$,
(ii) $\left[R^s_{Tc}\right]_{L=0} > \left[R^s_{Tc}\right]_{L\neq 0}$,
(iii) $[Nu]_{L=0} < [Nu]_{L\neq 0}$ and
(iv) Chaos manifests earlier in electroconvection compared to that in classical Rayleigh–Bénard convection.

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