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The effects of variable fluid properties on the hydro-magnetic flow and heat transfer over a non-linearly stretching sheet

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ABSTRACT

The influence of temperature-dependent fluid properties on the hydro-magnetic flow and heat transfer over a stretching surface is studied. The stretching velocity and the transverse magnetic field are assumed to vary as a power of the distance from the origin. It is assumed that the fluid viscosity and the thermal conductivity vary as an inverse function and linear function of temperature, respectively. Using the similarity transformation, the governing coupled non-linear partial differential equations are transformed into coupled non-linear ordinary differential equations and are solved numerically by the Keller–Box method. The governing equations of the problem show that the flow and heat transfer characteristics depend on five parameters, namely the stretching parameter, viscosity parameter, magnetic parameter, variable thermal conductivity parameter, and the Prandtl number. The numerical values obtained for the velocity, temperature, skin friction, and the Nusselt number are presented through graphs and tables for several sets of values of the parameters. The effects of the parameters on the flow and heat transfer characteristics are discussed.

1. Introduction

The problem of flow and heat transfer in the boundary layer adjacent to a continuous moving surface has attracted many researchers because of its numerous applications in engineering/manufacturing processes, namely continuous casting, glass fiber production, metal extrusion, hot rolling of paper and textiles, and wire drawing. The physical situation was recognized as a backward boundary layer problem by Sakiadis [1,2]. He was the first, among others, to investigate the flow behavior for this class of boundary layer problems. In his pioneering papers, solutions were obtained to the boundary layer flows on continuous moving surfaces which are substantially different from those of boundary layer flows on stationary surfaces. The thermal behavior of the problem was studied by Erickson et al. [3] using finite difference and integral methods, and experimentally verified by Tsou et al. [4]. Thereafter various aspects of the above boundary layer problem on continuous moving surface were considered by many researchers (Crane [5], Grubka and Bobba [6], Vleggaar [7], Soundalgekar and Ramana Murthy [8], Gupta and Gupta [9], Chen and Char [10], Chen and Strobel [11]).

All the above investigators restricted their analyses to flow and heat transfer in the absence of magnetic field. But in recent years, we find several applications in polymer industry (where one deals with stretching of plastic sheets) and metallurgy where hydro-magnetic techniques are being used. To be more specific, it may be pointed out that many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid and that in the process of drawing, these strips are sometimes stretched. Mention may be made of drawing, annealing, and thinning of copper wires. In all these cases, the properties of final product depend to a great extent on the rate of cooling by drawing such strips in an electrically conducting fluid subject to a magnetic field and the characteristics desired in the final product. In view of these applications Pavlov [12] investigated the flow of an electrically conducting fluid caused solely by the stretching of an elastic sheet in the presence of a uniform magnetic field. Chakrabarti and Gupta [13] considered the flow and heat transfer of an electrically conducting fluid past a porous stretching sheet and presented the analytical solution for the flow and the numerical solution for the heat transfer problem. Andersson [14] extended the work of Chakrabarti and Gupta [13] to MHD flow of a non-Newtonian viscoelastic fluid over an impermeable stretching sheet and

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found that the magnetic parameter has the same effect as the viscoelastic parameter. Andersson [15] obtained an analytical solution of the magneto-hydro-magnetic flow using a similarity transformation for the velocity and temperature fields. Chiam [16] investigated the boundary layer flow of Newtonian fluid: The flow is caused by sheet stretching according to a power law velocity in the presence of a transverse magnetic field. Chamkha [17] and Abo-Eldlahab [18] considered the problems related to hydro-magnetic three-dimensional flow on a stretching surface. Further, Ishak et al. [19] studied the effect of a uniform transverse magnetic field on the stagnation point flow toward a vertical stretching sheet. Ali [20] extended the work of Chiam [16] to heat transfer characteristics by assuming the non-linear magnetic field strength and obtained similarity solutions for different thermal boundary conditions.

In all the above mentioned papers, the thermophysical properties of the ambient fluid were assumed to be constant. However, it is well known that (Herwig and Wickern [21], Lai and Kulacki [22], Takhar et al. [23], Pop et al. [24], Hassanien [25], Subhas Abel et al. [26], Seedbeek [27], Ali [28], Andersson and Aarseth [29] Prasad et al. [30]) these physical properties may change with temperature, especially for fluid viscosity and thermal conductivity. For lubricating fluids, heat generated by internal friction and the corresponding rise in the temperature affects the physical properties of the fluid, and the properties of the fluid are no longer assumed to be constant. The increase in temperature leads to increase in the transport phenomena by reducing the physical properties across the thermal boundary layer, and so the heat transfer at the wall is also affected. Therefore to predict the flow and heat transfer rates, it is necessary to take into account the variable fluid properties. In view of this, the problem studied here extends the work of Vajravelu [31] by considering the temperature-dependent variable fluid properties. Thus in the present paper, we study the effects of variable viscosity and variable thermal conductivity on the hydromagnetic flow and heat transfer over a non-linear stretching sheet. The coupled non-linear partial differential equations governing the problem are reduced to a system of coupled non-linear ordinary differential equations by applying a suitable similarity transformation. These non-linear coupled differential equations are solved numerically by the Keller–Box method for different values of the parameters.

2. Mathematical formulation

Consider a steady, two-dimensional boundary layer flow of an incompressible electrically conducting fluid, in the presence of a transverse magnetic field $B(x)$ with variable fluid properties, past an impermeable stretching sheet coinciding with the plane $y = 0$. The origin is located at the slit, through which the sheet is drawn through the fluid medium. The x-axis is taken in the direction of the main flow along the sheet, and the y-axis is normal to it. Two equal and opposite forces are applied along the x-axis so that the wall is stretched, keeping the origin fixed. The continuous stretching surface is assumed to have a power law velocity $u = u_w = bx^m$, where b is a constant and m is an exponent. Here, we assume that the induced magnetic field produced by the motion of an electrically conducting fluid is negligible. This assumption is valid for small magnetic Reynolds number. Further, since there is no external electric field, the electric field due to polarization of charges is negligible. The viscous dissipation and the ohmic heating terms are not included in the energy equation since they are, generally small. Under the foregoing assumptions and invoking the usual boundary layer approximation, the governing equations of mass, momentum and energy for the problem under consideration, in the presence of variable fluid properties (i.e. fluid viscosity and thermal conductivity), can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\sigma B^2(x)}{\rho_\infty}u, \tag{2}$$

$$\rho_\infty c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k(T)\frac{\partial T}{\partial y}\right), \tag{3}$$

where, u and v are the velocity components in the streamwise x and cross-stream y directions, respectively. Here ρ_∞ is the constant fluid density and μ is the coefficient of viscosity, and μ is considered to vary as an inverse function of temperature (Lai and Kulacki [22]) as

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}[1+\delta(T-T_\infty)], \tag{4}$$

This can be rewritten as

$$\frac{1}{\mu} = a(T-T_r),$$

$$\text{where,}\quad a = \frac{\delta}{\mu_\infty}\ \text{and}\ T_r = T_\infty-\frac{1}{\delta}. \tag{5}$$

Here, both a and T_r are constants, and their values depend on the reference state and the small parameter δ, reflecting a thermal property of the fluid. In general, $a > 0$ corresponds to liquids and $a < 0$ to gases when the temperature at the sheet (T_w) is larger than that of the temperature at far away from the sheet (T_∞). The correlations between the viscosity and the temperature for air and water are given below:

$$\text{For air}:\ \frac{1}{\mu} = -123.2(T-742.6),\quad \text{based on}\quad T_\infty = 293\ \text{K}(20^\circ\text{C}),$$

$$\text{and for water}:\ \frac{1}{\mu} = -29.83(T-258.6),\quad \text{based on}\quad T_\infty = 288\ \text{K}(15^\circ\text{C}).$$

The reference temperatures selected here for the correlations are practically meaningful. The viscosity of a liquid usually decreases with increase in temperature while it increases for gases, when $(T_w - T_\infty)$ is positive.

In the above equations, σ is the electrical conductivity, and $B^2(x)$ is the strength of the magnetic field. The special form of the magnetic field $B(x) = B_0x^{(m-1)/2}$ is chosen to obtain a similarity solution. This form of $B^2(x)$ has also been considered by Chiam [16] in MHD flow past a moving flat plate. Here, C_p is the specific heat at constant pressure and $k(T)$ is the temperature-dependent thermal conductivity. We consider the temperature-dependent thermal conductivity relationship in the form (Chiam [32])

$$k(T) = k_\infty\left(1+\frac{\varepsilon}{\Delta T}(T-T_\infty)\right) \tag{6}$$

where $\Delta T = T_w - T_\infty$, $\varepsilon = k_w{-}k_\infty/k_\infty$ is assumed to be small in magnitude and, k_w and k_∞ are respectively the thermal conductivities of the fluid at the sheet and far away from the sheet. Substituting equations (4)–(6) in Equations (2) and (3), we obtain:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\frac{\mu_\infty}{1+\delta(T-T_\infty)}\frac{\partial u}{\partial y}\right)-\frac{\sigma B^2(x)}{\rho_\infty}u, \tag{7}$$

$$\rho c_p u \frac{\partial T}{\partial x} + \left(\rho c_p v - \frac{k_\infty \varepsilon}{\Delta T}\frac{\partial T}{\partial y}\right)\frac{\partial T}{\partial y} = \left(k_\infty \left(1 + \frac{\varepsilon}{\Delta T}(T - T_\infty)\right)\right)\frac{\partial^2 T}{\partial y^2}. \quad (8)$$

The appropriate boundary conditions on the velocity and the temperature fields are

$$u = u_w = bx^m, \quad v = 0, \; T = T_w \text{ at } y = 0,$$
$$u \to 0, \quad T \to T_\infty \text{ as } y \to \infty, \quad (9)$$

where b is a stretching rate [(1/s) for $m = 1$]. It should be noted that the positive or negative m indicates respectively that the surface is accelerated or decelerated from the extruded slit.

Now we transform the system of equations (1)–(3) into a dimensionless form. To this end, let the dimensionless similarity variable be

$$\eta = \frac{y}{x}\sqrt{\frac{m+1}{2}}\sqrt{Re_x}, \text{ where } Re_x = \frac{u_w(x)}{\gamma_\infty}x, \quad (10)$$

and the dimensionless stream function $f(\eta)$ and dimensionless temperature $\theta(\eta)$ are

$$f(\eta) = \psi(x,y)/\left[u_w x (Re_x)^{\frac{-1}{2}}\right], \quad (11)$$

$$\theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \quad (12)$$

where the dimensionless stream function $\psi(x,y)$ identically satisfies the continuity equation (1) with

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}. \quad (13)$$

By using (10)–(12), the momentum equation (7) and energy equation (8) can be written as:

$$\beta f_\eta^2(\eta) - f(\eta) f_{\eta\eta}(\eta) = \frac{1}{1 - \frac{\theta(\eta)}{\theta_r}}\left[f_{\eta\eta\eta}(\eta) + \frac{f_{\eta\eta}(\eta)\theta_\eta(\eta)}{\theta_r - \theta(\eta)}\right] - M_n f_\eta(\eta), \quad (14)$$

$$[1 + \varepsilon\theta(\eta)]\theta_{\eta\eta}(\eta) = -\varepsilon\theta_\eta^2(\eta) - \Pr f(\eta)\theta_\eta(\eta), \quad (15)$$

and they are subjected to boundary conditions

$$f_\eta(\eta) = 1, \quad f(\eta) = 0, \; \theta(\eta) = 1 \text{ at } \eta = 0, \quad (16)$$

$$f_\eta(\eta) = 0, \quad \theta(\eta) = 0 \text{ as } \eta \to \infty. \quad (17)$$

Here, the subscript η denotes the differentiation with respect to η. The parameters β, θ_r, Mn, *and* Pr are respectively, the stretching parameter, fluid viscosity parameter, magnetic parameter, and the Prandtl number, which are defined as follows:

$$\beta = \frac{2m}{m+1}, \quad \theta_r = \frac{T_r - T_\infty}{T_w - T_\infty} = -\frac{1}{\delta(T_w - T_\infty)}\frac{2}{1+m},$$
$$Mn = \frac{2\sigma B_0^2}{\rho_\infty b(m+1)}, \quad Pr = \frac{\mu_\infty c_p}{k_\infty}. \quad (18)$$

The value of θ_r is determined by the viscosity of the fluid under consideration and the operating temperature difference. If θ_r is large, in other words, if $(T_\infty - T_w)$ is small, the effects of variable viscosity on the flow can be neglected. On other hand, for smaller values of θ_r, either the fluid viscosity changes markedly with temperature or the operating temperature difference is high. In either case, the effect of the variable fluid viscosity is expected to be very important. Also let us keep in mind that the liquid viscosity varies differently with temperature compared to the gas viscosity. Therefore it is important to note that θ_r is negative for liquids and positive for gases. It should be noted that the velocity $u = u_w(x)$ used to define the dimensionless stream function $f(\eta)$ in the equation (11) and the local Reynolds number in equation (10) is the velocity of the moving surface that drives the flow. This choice contrasts with conventional boundary layer analysis in which the free stream velocity is taken as the velocity scale. Although the transformation defined in (10) and (11) can be used for arbitrary variations of $u_w(x)$, the transformation results in a true similarity problem only if u_w varies as bx^m. Here m is an arbitrary constant, not necessarily an integer. Such surface velocity variations are therefore required for the ODE (14) to be valid. Non-similar stretching sheet problems, which require the solution of partial differential equations rather than ODEs, were considered by Jeng et al. [33] for Newtonian fluids.

It is worth mentioning here that $\theta_r \to \infty$ as $\delta \to 0$. In this situation for the constant magnetic field case, the equations (14)

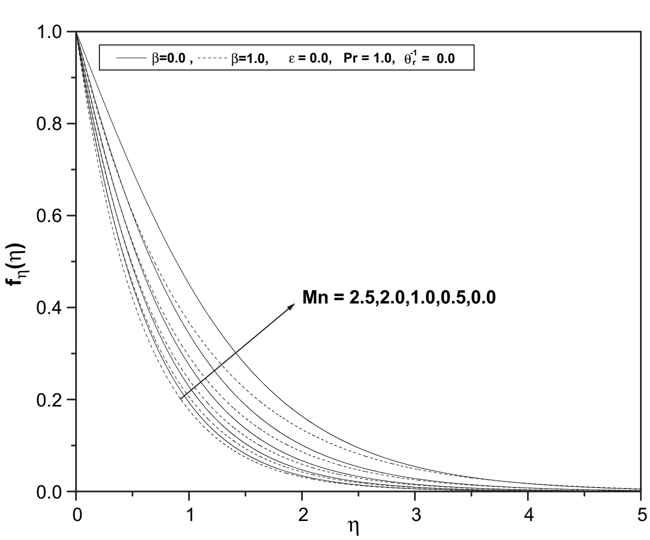

Fig. 1. Horizontal velocity profiles for different values of magnetic parameter.

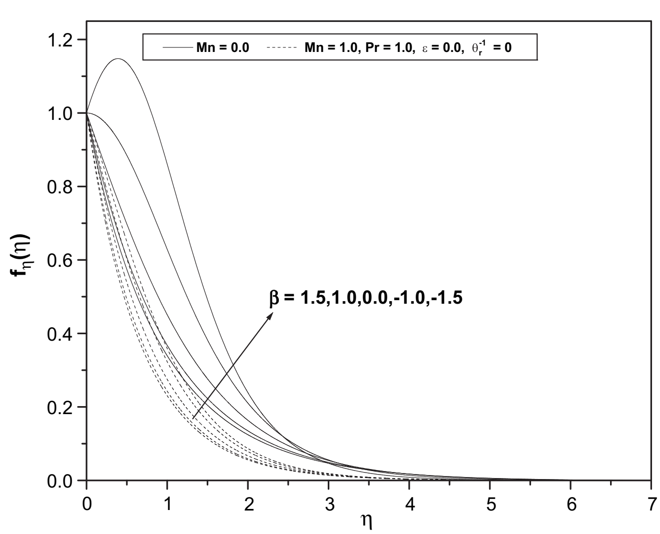

Fig. 2. Horizontal velocity profiles for different values of stretching parameter and magnetic parameter.

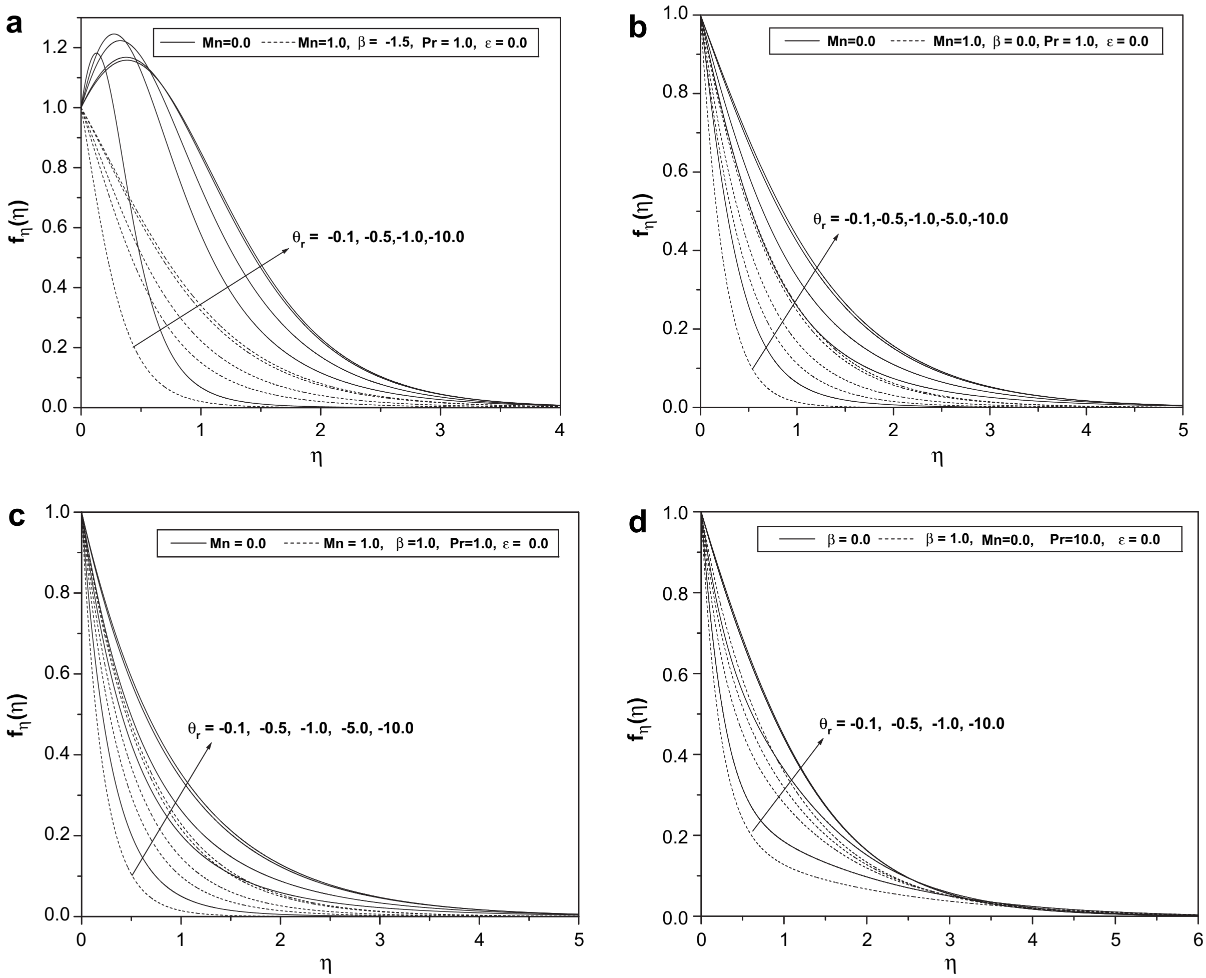

Fig. 3. (a) Horizontal velocity profiles for different values of fluid viscosity parameter. (b) Horizontal velocity profiles for different values of fluid viscosity parameter. (c) Horizontal velocity profiles for different values of fluid viscosity parameter. (d) Horizontal velocity profiles for different values of fluid viscosity parameter.

and (15) reduce to those of Chakrabarti and Gupta [13], and for $m = 0$ those of Vajravelu [31]. In the presence of a variable magnetic field and when there is no heat transfer, equation (14) reduces to that of Chiam [16]. Further, when the variable thermal conductivity parameter and the magnetic parameter are absent, equations (14) and (15) are similar to the ones studied by Crane [5] and Grubka and Bobba [6].

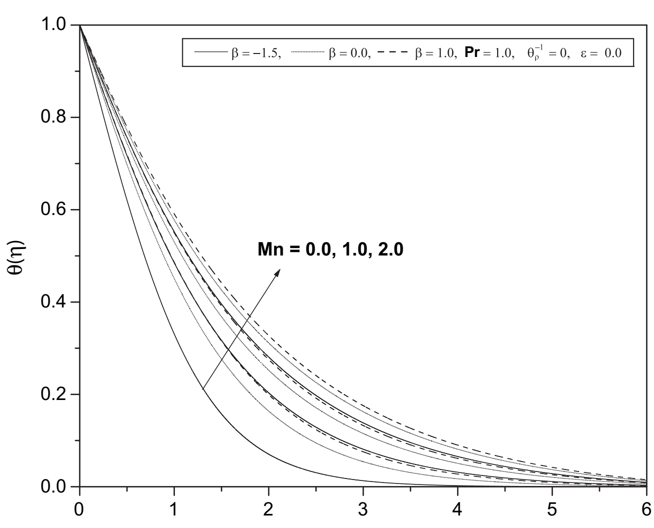

Fig. 4. Temperature profiles for different values of magnetic parameter and stretching parameter.

The physical quantities of interest here are the skin friction coefficient C_f and the Nusselt number Nu; and they are defined by

$$C_{fx} = \frac{2\tau_w(x)}{\rho u_w^2}, \quad Nu_x = \frac{q_w x}{k_\infty (T_w - T_\infty)}, \tag{19}$$

where $\tau_w(x) = -\mu_w(\partial u/\partial y)_{\text{at } y=0}$ and $q_w(x) = -k_\infty(\partial T/\partial y)_{\text{at } y=0}$.

Using equations (4), (5), (10)–(12) and (18), the skin friction and the Nusselt number can be written as

$$C_f\sqrt{Re_x} = -\frac{\sqrt{2(m+1)}\theta_r}{(\theta_r - 1)} f_{\eta\eta}(0, \theta_r),$$

$$\frac{Nu}{\sqrt{Re_x}} = -\sqrt{\frac{m+1}{2}}\,\theta_\eta(0, \theta_r).$$

3. Numerical procedure

By applying similarity transformation to the governing equations and the boundary conditions, the governing equations are reduced to a system of coupled, non-linear differential equations with appropriate boundary conditions. Finally the system of similarity equations with the boundary conditions is solved numerically

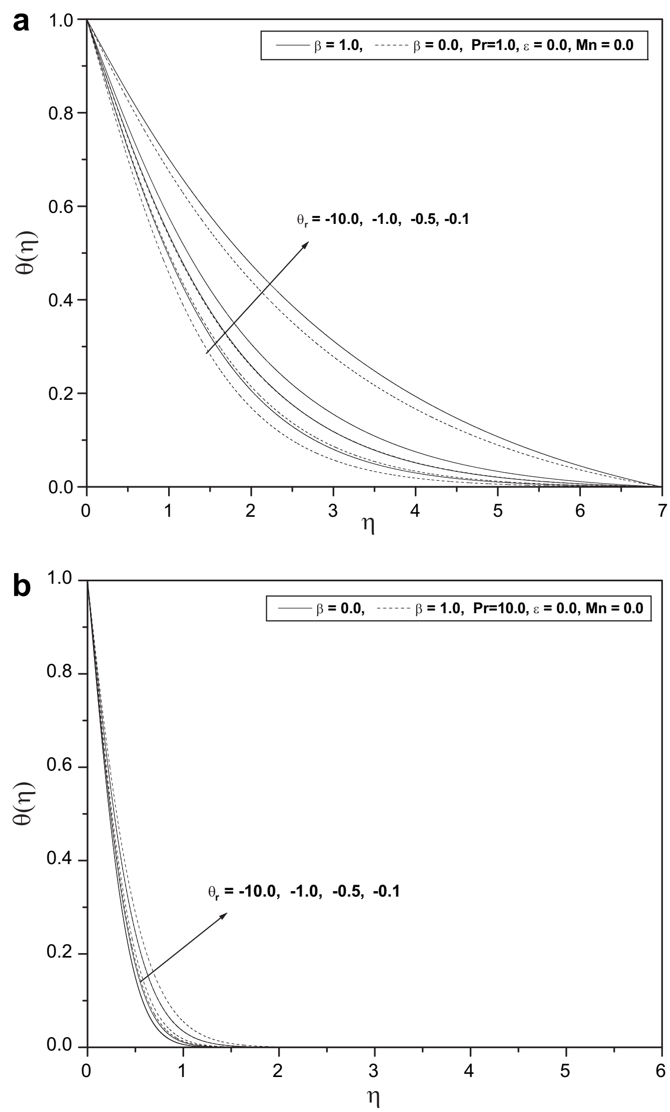

Fig. 5. (a) Temperature profiles for different values of fluid viscosity parameter. (b) Temperature profiles for different values of fluid viscosity parameter.

by the Keller–Box method (Cebeci and Bradshaw [34], Prasad et al. [30] Datti and Prasad [35]). This method is unconditionally stable and has a second order accuracy with arbitrary spacing. First, we write the transformed differential equations and the boundary conditions in terms of first order system, which is then converted to a set of finite difference equations using central differences. Then the non-linear algebraic equations are linearized by Newton's method and the resulting linear system of equations is then solved by block tri-diagonal elimination technique. For the sake of brevity, the details of the numerical solution procedure are not presented here. It is worth mentioning that a uniform grid of $\Delta\eta = 0.01$ is satisfactory in obtaining sufficient accuracy with an error tolerance less than 10^{-6}. To validate the present results, a comparison is made with the known results of Crane [5] and Soundalgekar and Murthy [8].

4. Results and discussion

In order to have an insight in to the effects of the parameters on the MHD flow and heat transfer characteristics, we present the numerical results graphically in Figs. 1–8 and in Table 1 for several

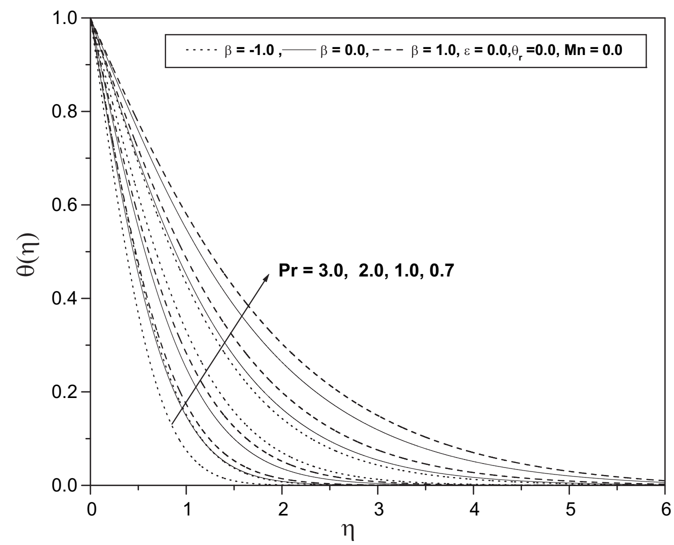

Fig. 6. Temperature profiles for different values of Prandtl number (Pr).

sets of values of the temperature-dependent fluid property parameters. We consider only the case of a liquid for which $\theta_r < 0$. We can have a glimpse of the physical layout of the boundary layer structure which develops near the slit by observing the horizontal profiles in Figs. 1–3.

Fig. 1 illustrates the effect of the magnetic parameter Mn on the horizontal velocity $f_\eta(\eta)$ in the presence/absence of stretching parameter. From Fig. 1 we see that $f_\eta(\eta)$ is considerably reduced with an increase in the magnetic parameter. It clearly indicates that the transverse magnetic field opposes the transport phenomena. This is due to the fact that, the transverse magnetic field has a tendency to create a drag force, known as the Lorentz force, and hence an increase in the absolute value of the velocity gradient at the surface. That is, the thickness of the boundary layer is reduced for higher values of the magnetic parameter Mn. This behavior is clearly noticeable when the surface is accelerated ($\beta > 0$) from the extruded slit.

The effect of the stretching parameter β on the horizontal velocity $f_\eta(\eta)$ in the presence/absence of the magnetic parameter Mn is depicted in Fig. 2. It is observed that an increase in the

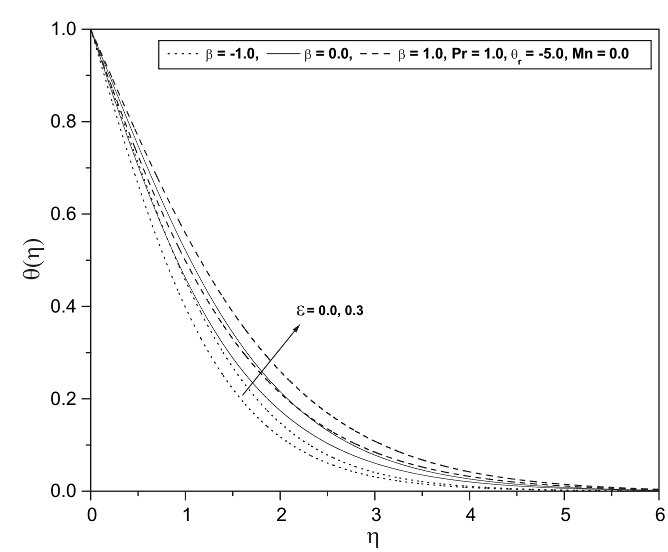

Fig. 7. Temperature profiles for different values of variable thermal conductivity parameter (ε).

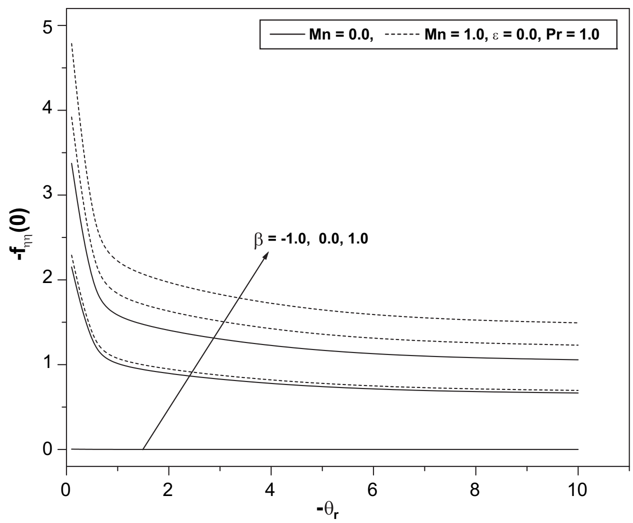

Fig. 8. Influence of Variable viscosity parameter, Stretching parameter and magnetic parameter on the dimensionless skin friction coefficient.

stretching parameter β reduces the momentum boundary layer thickness, which tends to zero as the variable η increases from the boundary. Physically, $\beta < 0$ implies the surface decelerating case, $\beta = 0$ implies the continuous movement of a flat surface, and $\beta > 0$ implies the surface accelerating case. A decrease in $f_\eta(\eta)$ discloses the fact that the effect of β is to decelerate the velocity and hence reduce the momentum boundary layer thickness.

Fig. 3(a)–(c) respectively, show the effects of a decelerating surface ($\beta < 0$), continuously moving surface ($\beta = 0$), and an accelerating surface ($\beta > 0$) from the slit, on the horizontal velocity $f_\eta(\eta)$ for various values of the fluid viscosity parameter θ_r with $Pr = 1.0$. From these figures it can be seen that $f_\eta(\eta)$ decreases asymptotically to zero as the variable η increases. However, in the decelerating surface case ($\beta < 0$), the velocity profile increases from its value one and then decays to zero. The effect of increasing values of the fluid viscosity parameter θ_r is to decrease the momentum boundary layer thickness. Also, as $\theta_r \to 0$, the boundary layer thickness decreases and the velocity distribution is asymptotically tends to zero [see Fig. 3(d)]. This is due to the fact that, for a given fluid (air or water), when δ is fixed, smaller θ_r implies higher temperature difference between the wall and the ambient fluid. The results presented in this paper demonstrate clearly that θ_r, the indicator of the variation of fluid viscosity with temperature, has a substantial effect on the horizontal velocity $f_\eta(\eta)$ and hence on the skin friction.

In Figs. 4–7 the numerical results for the temperature $\theta(\eta)$ for several sets of values of the governing parameters are presented. Fig. 4 illustrates the effect of the stretching parameter β and the magnetic parameter Mn on $\theta(\eta)$. The effect of increasing values of the stretching parameter β is to increase the temperature $\theta(\eta)$. This is true even in the presence of the magnetic field. The effect of increasing values of the magnetic parameter Mn is to increase the temperature $\theta(\eta)$. Of course, the effect of Mn on the thermal transport, if any, is only an indirect effect through the changes in $f(\eta)$ and $f_\eta(\eta)$. Fig. 5(a) and (b) exhibit the temperature distribution $\theta(\eta)$ for several sets of values of the fluid viscosity parameter θ_r, the stretching parameter, β and the Prandtl number Pr. From the

Table 1
Skin friction and wall temperature gradient for different values of the physical parameters.

ε	$1/\theta_r$	Pr	$Mn = 0.0$						$Mn = 1.0$					
			$\beta = -1.0$		$\beta = 0.0$		$\beta = 1.0$		$\beta = -1.0$		$\beta = 0.0$		$\beta = 1.0$	
			$f_{\eta\eta}(0)$	$\theta_\eta(0)$	$f_{\eta\eta}(0)$	$\theta_\eta(0)$	$f_{\eta\eta}(0)$	$\theta_\eta(0)$	$f_{\eta\eta}(0)$	$\theta_\eta(0)$	$f_{\eta\eta}(0)$	$\theta_\eta(0)$	$f_{\eta\eta}(0)$	$\theta_\eta(0)$
0	$\vec{0}$	0.7	−2.89E-04	−0.56514	−0.6278	−0.49630	−1.00017	−0.45828	−0.85112	−0.43954	−1.16333	−0.41274	−1.4142	−0.39357
		1		−0.70710		−0.62779		−0.58267		−0.56403		−0.53023		−0.50546
		2		−1.06000		−0.96618		−0.91126		−0.90008		−0.8563		−0.82311
		3		−1.32577		−1.22447		−1.16517		−1.15994		−1.11198		−1.07522
		4		−1.54675		−1.44128		−1.37933		−1.37824		−1.32779		−1.28886
		5		−1.74012		−1.63177		−1.56801		−1.56996		−1.517807		−1.47739
0	$\vec{0}$	1	−2.89E-04	−0.70711	−0.6278	−0.6278	−1.00017	−0.58267	−0.85112	−0.56403	−1.16333	−0.53023	−1.14142	−0.50546
0.1				−0.65991		−0.58461		−0.51833		−0.52357		−0.49167		−0.46836
0.2				−0.61986		−0.54793		−0.50716		−0.48919		−0.45893		−0.43688
0.3				−0.58522		−0.5163		−0.46628		−0.45956		−0.43074		−0.40979
0.3	$\vec{0}$	0.7	−2.89E-04	−0.46538	−0.6278	−0.40634	−1.00017	−0.374	−0.85112	−0.35694	−1.16333	−0.33471	−1.41421	−0.31897
		1		−0.58522		−0.5163		−0.47728		−0.45956		−0.43074		−0.40979
		2		−0.88383		−0.80082		−0.75261		−0.74118		−0.70291		−0.67399
		3		−1.10743		−1.018		−0.96572		−0.9597		−0.91752		−0.88526
		4		−1.29355		−1.20025		−1.14549		−1.14329		−1.09877		−1.06449
		5		−1.4563		−1.36032		−1.30387		−1.30449		−1.25838		−1.22271
0	−5	1	−3.85E-04	−0.69853	−0.703	−0.61419	−1.11251	−0.56726	−0.95323	−0.5446	−1.29339	−0.51059	−1.56695	−0.48582
0.1			−3.87E-04	−0.65141	−0.70227	−0.57145	−1.1116	−0.52704	−0.95216	−0.50494	−1.29229	−0.47293	−1.56586	−0.44971
0.2			−3.87E-04	−0.61137	−0.70161	−0.53515	−1.11078	−0.49289	−0.95119	−0.47127	−1.29131	−0.44099	−1.5649	−0.41908
0.3			−3.88E-04	−0.57682	−0.701	−0.50386	−1.11004	−0.46347	−0.95029	−0.44227	−1.29043	−0.4135	−1.56404	−0.39275
0	−10	1	−3.35E-04	−0.70282	−0.66654	−0.62086	−1.05804	−0.57477	−0.69664	0.57732	−1.23039	−0.5201	−1.49289	−0.49532
	−5		−3.84E-04	−0.69853	−0.703	−0.61419	−1.11251	−0.56726	−0.73755	−0.56798	−1.29339	−0.51059	−1.56695	−0.48585
	−1		−8.64E-04	−0.66465	−0.93777	−0.56828	−1.46636	−0.51781	−0.99445	−0.50708	−1.70094	−0.45123	−2.05077	−0.42782
	−0.5		−0.00145	−0.624	−1.15163	−0.5234	−1.79737	−0.47279	−1.21978	−0.45474	−2.08276	−0.40308	−2.52268	−0.44654
	−0.1		−0.00433	−0.48602	−2.15181	−0.43116	−3.37462	−0.32524	−2.29136	−0.46648	−3.91992	−0.27761	−4.7866	−0.43615
0.3	−5	0.7	−3.77E-04	−0.4562	−0.69821	−0.39484	−1.10694	−0.36204	−0.94641	−0.34273	−1.28691	−0.32111	−1.56079	−0.3059
		1	−3.87E-04	−0.57682	−0.701	−0.50386	−1.11004	−0.46347	−0.95029	−0.44227	−1.29043	−0.4135	−1.56404	−0.39275
		2	−3.82E-04	−0.87756	−0.7076	−0.78788	−1.11796	−0.73698	−0.96031	−0.72124	−1.30019	−0.68176	−1.57347	−0.65205
		3	−3.75E-04	−1.10227	−0.7119	−1.00486	−1.12351	−0.94925	−0.96698	−0.93922	−1.30725	−0.89528	−1.58063	−0.86181
		4	−3.69E-04	−1.28907	−0.71504	−1.18692	−1.12777	−1.12842	−0.9718	−1.12255	−1.31267	−1.07591	−1.5863	−1.04014
		5	−3.66E-04	−1.45229	−0.71748	−1.34684	−1.13121	−1.28632	−0.97549	−1.28359	−1.31703	−1.23507	−1.59098	−1.1977

graphical representation, we observe that the effect of increasing values of the fluid viscosity parameter θ_r is to enhance the temperature. This is due to the fact that an increase in the fluid viscosity parameter θ_r results in an increase in the thermal boundary layer thickness. This is even true for the higher values of the Prandtl number *Pr* [see Fig. 5(b)]. The variations of $\theta(\eta)$ for different values of the Prandtl number *Pr* and the stretching parameter β are displayed in Fig. 6. The effect of increasing *Pr* is to decrease $\theta(\eta)$. That is, an increase in *Pr* means decrease in the thermal conductivity k_∞ and hence, there would be a decrease of thermal boundary layer thickness. The effect of the variable thermal conductivity parameter ε on $\theta(\eta)$ can be seen in Fig. 7. From this figure we observe that $\theta(\eta)$ increases with increasing ε.

Fig. 8 displays the variation of skin friction $-f_{\eta\eta}(0)$ against the fluid viscosity parameter for several sets of values of the stretching parameter and the magnetic parameter. Note that the skin friction decreases with an increase in the viscosity parameter or the stretching parameter. This observation is true even in the presence of the magnetic field. The impact of all the physical parameters on the skin friction $[-f_{\eta\eta}(0)]$ and the wall temperature gradient $[-\theta_\eta(0)]$ may be analyzed from Table 1. From Table 1 it can be seen that the effect of the magnetic parameter, stretching parameter, and the fluid viscosity parameter is to decrease the skin friction and to enhance the wall temperature gradient. This phenomenon is true even in the presence of the variable thermal conductivity. The effect of the Prandtl number is to decrease the wall temperature gradient even in the presence of the variable viscosity and the variable thermal conductivity.

5. Conclusions

(i) The effect of the magnetic field and the variable viscosity is to decrease the velocity and the skin friction. However we see opposite effects on the dimensionless temperature and the rate of heat transfer.
(ii) The velocity and the skin friction are reduced by the stretching parameter, while its effect is to increase the dimensionless temperature and the rate of heat transfer. This is true even in the presence of the temperature-dependent fluid properties.
(iii) The effect of the Prandtl number is to decrease the thermal boundary layer thickness and the wall temperature gradient in the presence of the other physical parameters of the model.

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Nomenclature

a	constant
b	stretching rate, positive constant
$B(x)$	magnetic field
C_f	skin friction
C_p	specific heat at constant pressure
f	dimensionless velocity variable
$h(x)$	heat transfer coefficient
$k(T)$	thermal conductivity
k_w	thermal conductivity at the sheet
k_∞	thermal conductivity far away from the sheet
m	index of power law velocity
M_n	magnetic parameter
Nu_x	Nusselt number
Pr	Prandtl number
q_w	local heat flux at the sheet
Re_x	local Reynolds number

T	temperature variable
T_r	transformed reference temperature
T_w	given temperature at the sheet
T_∞	constant temperature of the fluid far away from the sheet
x	horizontal distance
y	vertical distance
u	velocity in *x* direction
u_w	velocity of the sheet
v	velocity in *y*-direction

Greek symbols

ΔT	sheet temperature
ε	small parameter
η	similarity variable
γ	kinematic viscosity
β	stretching parameter
δ	thermal property of the fluid
μ	dynamic viscosity
μ_∞	constant value of dynamic viscosity
ψ	stream function
ρ	density
ρ_∞	constant fluid density
σ	electric conductivity
τ_{xy}	shear stress
θ	dimensionless temperature variable
θ_r	transformed dimensionless reference temperature.