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Special Finsler Spaces Admitting Metric Like Tensor Field

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Abstract

In this work we modify the special Finsler spaces like C-reducible, semi-C-reducible, quasi-C-reducible are admitting the tensor field $X_{hk} = h_{hk} + X_{00}l_h l_k$, which satisfies the condition $C^h_{ij}X_{hk} = C_{ijk}$. Similarly, we have also worked out for S3-like, C^h-recurrent, P-reducible and T-conditions of Finsler spaces.

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1 Introduction

The terminology and notations are referred to [1], [4] and [6]. Let $F^n = (M^n, L)$ be a Finsler space on a differentiable manifold M endowed with a fundamental function $L(x,y)$. We use the following notations: [4][6]

$$a) \qquad g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \;\; \dot{\partial}_i = \frac{\partial}{\partial y^i},$$

$$b) \qquad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij},$$

$$\begin{array}{ll} c) & h_{ij} = g_{ij} - l_i l_j, \\ d) & C^h h_{hk} = C_k, C^h l_h = 0, \\ e) & h_k^m h_{mj} = h_{jk}, h_j^m l_m = 0, \\ f) & C_{hr}^m g_{mj} = C_{hjr}, \\ g) & l_m l^m = 0,\ p^m l_m = 0, \\ h) & h_j^m X_{mk} = X_{jk} - X_{k0} l_j, \\ i) & X_{i0} l_j = X_{j0} l_i,\ X_{i0} = X_{00} l_i. \end{array} \tag{1}$$

There are three kinds of torsion tensors in carton's theory of Finsler spaces.Two of them are $h(h\nu)$-torsion tensor C_{ijk} and $(\nu)h\nu$-torsion tensor P_{ijk}, which are symmetric in all their indices. The contravarient components of $(\nu)h\nu$-torsion tensor is given by $C_{ij}^h = g^{hk} C_{ijk}$, which may be treated as Christoffel symbols of second kind of each tangent Riemannian space of Finsler space F^n. Here, g^{hk} is the inverse of metric tensor g_{hk} of F^n. If l_i is the normalized element of support h_{ij} is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$, then

$$C_{ij}^h g_{hk} = C_{ijk} = C_{ij}^h h_{hk}. \tag{2}$$

If b_i are components of a concurrent vector field, then $b_{i/j} = -g_{ij}$ and $b_{i|j} = 0$, where $/j$ and $|\ j$ denote the h and ν-covarient derivatives with respect to cartons connection $C\Gamma$. From this it follows that b_i are functions of position only, and $C_{ij}^h b_h = 0$. Thus if we consider a tensor field is given by $B_{ij} = g_{ij} + \alpha l_i l_j + \beta b_i b_j$, where α and β are scalar functions, then

$$C_{ij}^h B_{hk} = C_{ijk}. \tag{3}$$

The purpose of the present paper is to study the existence of any symmetric covariant tensor X_{hk} which satisfies

$$X_{hk} = h_{hk} + X_{00} l_h l_k. \tag{4}$$

Throughout the paper we are conserned with non-Riemannian Finsler space having positive definite metric tensor g_{ij}. From (4) we have,

$$C^h X_{hk} = C_k \quad and \quad C_{ij}^h X_{h0} = 0, \tag{5}$$

where $C^h = C_{ij}^h g^{ij}$ and 0 denotes the contraction with l^i.

2 The Existence Of Covariant Tensor X_{hk} In C-Reducible Finsler Space :

In a C-reducible Finsler space the $(h)h\nu$-torsion tensor C_{ij}^h is given by,[2][5]

$$C_{ij}^h = (C^h h_{ij} + C_i h_j^h + C_j h_i^h)/(n+1). \tag{6}$$

Now contracting above equation by X_{hk}, then from equations (4) and (1)(d), we have

$$\begin{aligned}
C^h_{ij}X_{hk} &= X_{hk}(C^h h_{ij} + C_i h^h_j + C_j h^h_i)/(n+1),\\
C^h_{ij}X_{hk} &= 1/(n+1)[C^h h_{ij}(h_{hk} + X_{00}l_hl_k) + C_i h^h_j(h_{hk} + X_{00}l_hl_k) + C_j h^h_i(h_{hk} + X_{00}l_hl_k)]\\
C^h_{ij}X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i h^h_j X_{00}l_hl_k + C_j h_{ik} + C_j h^h_i X_{00}l_hl_k]\\
C^h_{ij}X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i X_{00}l_hl_k(\delta^h_j - l^h l_j) + C_j h_{ik} + C_j X_{00}l_hl_k(\delta^h_j - l^h l_j)]\\
C^h_{ij}X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_i X_{00}l_k(l_j - l_j) + C_j h_{ik} + C_j X_{00}l_k(l_i - l_i)]\\
C^h_{ij}X_{hk} &= 1/(n+1)[C_k h_{ij} + C_i h_{jk} + C_j h_{ik}]\\
C^h_{ij}X_{hk} &= C_{ijk}. \qquad (7)
\end{aligned}$$

Theorem 2.1 *In a C-reducible Finsler space the covarient tensor field X_{hk} satisfies (4) is of the form (7).*

Consider a Semi-C-reducible Finsler space C^h_{ij} is given by [3],

$$C^h_{ij} = (C^h h_{ij} + C_i h^h_j + C_j h^h_i)p/(n+1) + (C^h C_i C_j)q/C^2. \qquad (8)$$

Now contracting above equation by X_{hk} and using equations (4) and (1)(d), we have

$$\begin{aligned}
C^h_{ij}X_{hk} &= X_{hk}(C^h h_{ij} + C_i h^h_j + C_j h^h_i)p/(n+1) + X_{hk}(C^h C_i C_j)q/C^2,\\
C^h_{ij}X_{hk} &= (C^h h_{ij}(h_{hk} + X_{00}l_hl_k) + C_i h^h_j(h_{hk} + X_{00}l_hl_k) + C_j h^h_i(h_{hk} + X_{00}l_hl_k))\\
&\quad p/(n+1) + (C^h C_i C_j)(h_{hk} + X_{00}l_hl_k)q/C^2,\\
C^h_{ij}X_{hk} &= (C_k h_{ij} + C_i h_{jk} + C_j h_{ik})p/(n+1) + (C_k C_i C_j)q/C^2,\\
C^h_{ij}X_{hk} &= C_{ijk}. \qquad (9)
\end{aligned}$$

Theorem 2.2 *In a Semi-C-reducible Finsler space the tensor field X_{hk} satisfies (4) is of the form (9).*

Consider a Quasi-C-reducible Finsler space C^h_{ij} is given by [3],

$$C^h_{ij} = (C^h A_{ij} + C_i A^h_j + C_j A^h_i). \qquad (10)$$

Now contracting above equation by X_{hk} and using equations (4) and (1)(d), we have

$$\begin{aligned}
C^h_{ij}X_{hk} &= (C^h A_{ij} + C_i A^h_j + C_j A^h_i)X_{hk},\\
C^h_{ij}X_{hk} &= (C^h A_{ij}(h_{hk} + X_{00}l_hl_k) + C_i A^h_j(h_{hk} + X_{00}l_hl_k)\\
&\quad + C_j A^h_i(h_{hk} + X_{00}l_hl_k)),\\
C^h_{ij}X_{hk} &= (C_k A_{ij} + C_i A_{jk} + C_j A_{ik}),\\
C^h_{ij}X_{hk} &= C_{ijk}. \qquad (11)
\end{aligned}$$

Theorem 2.3 *In a Quasi-C-reducible Finsler space the tensor field X_{hk} satisfies (4) is of the form (11) if $A^h l_h = 0$.*

3 The Existence Of Covariant Tensor X_{hk} In S3-Like Finsler Space:

In a S3-like Finsler space, whose $\nu-$ curvature tensor of cartons connection $C\Gamma$ is given by [5],

$$L^2 S^m_{ihk} = S(h_{ih}h^m_k - h_{ik}h^m_h). \tag{12}$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned}
L^2 S^m_{ihk} X_{mj} &= S(h_{ih}h^m_k X_{mj} - h_{ik}h^m_h X_{mj}),\\
L^2 S^m_{ihk} X_{mj} &= S[h_{ih}h^m_k(h_{mj} - X_{00}l_m l_j) - h_{ik}h^m_h(h_{mj} - X_{00}l_m l_j)],\\
L^2 S^m_{ihk} X_{mj} &= S[h_{ih}h_{jk} - X_{00}h_{ih}h^m_k l_m l_j - h_{ik}h_{hj} + X_{00}h_{ik}h^m_h l_m l_j],\\
L^2 S^m_{ihk} X_{mj} &= S[h_{ih}h_{jk} - h_{ik}h_{hj}],\\
L^2 S^m_{ihk} X_{mj} &= S_{hijk}.
\end{aligned} \tag{13}$$

Theorem 3.1 *In a S3-like Finsler space, the covariant tensor field X_{mj} satisfies (4) is of the form (13).*

Next we consider S4-like Finsler space, whose ν-curvature tensor of cartons connection $C\Gamma$ is given by [7],

$$L^2 S^m_{ihk} = h^m_h M_{ik} + h_{ik}M^m_h - h_{hk}M^m_i - h^m_i M_{hk}. \tag{14}$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned}
L^2 S^m_{ihk} X_{mj} &= h^m_h M_{ik}X_{mj} + h_{ik}M^m_h X_{mj} - h_{hk}M^m_i X_{mj} - h^m_i M_{hk}X_{mj},\\
L^2 S^m_{ihk} X_{mj} &= h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk},\\
L^2 S^m_{ihk} X_{mj} &= L^2 S_{hijk}.
\end{aligned} \tag{15}$$

Theorem 3.2 *In a S4-like Finsler space, the covariant tensor field X_{ij} satisfies (4) is of the form (15).*

Next we consider a ν-curvature tensor of Cartons connection $C\Gamma$ is given by [2],

$$S^m_{ihk} = C^m_{hr}C^r_{ik} - C^m_{kr}C^r_{ih}. \tag{16}$$

Contacting above equation by X_{mj} and using equations (4) and (1)(d), we have

$$\begin{aligned}
S^m_{ihk} X_{mj} &= C^m_{hr}C^r_{ik}X_{mj} - C^m_{kr}C^r_{ih}X_{mj},\\
S^m_{ihk} X_{mj} &= C^m_{hr}C^r_{ik}(h_{mj} + X_{00}l_m l_j) - C^m_{kr}C^r_{ih}(h_{mj} + X_{00}l_m l_j),\\
S^m_{ihk} X_{mj} &= C_{hjr}C^r_{ik} - C_{krj}C^r_{ih},\\
S^m_{ihk} X_{mj} &= S_{hijk}.
\end{aligned} \tag{17}$$

Theorem 3.3 *In a ν-curvature tensor, the covariant tensor field X_{mj} satisfies (4) is of the form (17).*

Now we concerned with a space of scalar curvature in Berwald's sence. It is characterized by the equation is [2]

$$R^i_{jk} = h^i_k k_j - h^i_j k_k. \tag{18}$$

where h_{ik} is the angular metric tensor and the scalar curvature K is a function scalar field.
Contacting above equation by X_{il} and using equations (4) and (1)(d), we get

$$\begin{aligned}
R^i_{jk}X_{il} &= h^i_k K_j X_{il} - h^i_j K_k X_{il}, \\
R^i_{jk}X_{il} &= h^i_k K_j (h_{il} + X_{00} l_i l_l) - h^i_j K_k X_{il}(h_{il} + X_{00} l_i l_l), \\
R^i_{jk}X_{il} &= K_j h_{ki} - K_k h_{jl}, \\
R^i_{jk}X_{il} &= R_{ijk}.
\end{aligned} \tag{19}$$

Theorem 3.4 *In a space of scalar curvature tensor, the covariant tensor field X_{il} satisfies (4) is of the form (19).*

4 The Existence Of Covariant Tensor X_{hk} In P-Reducible Finsler Space :

The P-reducible Finsler space is given as [5],

$$P^m_{jk} = (h^m_j P_k + h_{jk} P^m + h^m_k P_j)/(n+1), \tag{20}$$

Contacting above equation by X_{mi} and using equations (4) and (1)(d), we have

$$\begin{aligned}
P^m_{jk}X_{mi} &= (h^m_j P_k X_{mi} + h_{jk} P^m X_{mi} + h^m_k P_j X_{mi})/(n+1), \\
P^m_{jk}X_{mi} &= (h^m_j P_k (h_{mi} + X_{00} l_m l_i) + h_{jk} P^m (h_{mi} + X_{00} l_m l_i) + \\
&\quad h^m_k P_j (h_{mi} + X_{00} l_m l_i))/(n+1), \\
P^m_{jk}X_{mi} &= (h_{ij} P_k + h_{jk} P_i + h_{ki} P_j)/(n+1), \\
P^m_{jk}X_{mi} &= P_{ijk}.
\end{aligned} \tag{21}$$

Theorem 4.1 *In a P-reducible Finsler space, the covariant tensor field X_{mi} satisfies (4) is of the form (21).*

5 The Existence Of Covariant Tensor X_{hk} In C^h-Recurrent Finsler Space:

Now we consider a C^h-recurrent Finsler space is given as [2],

$$C^m_{jk/h} = \alpha_h C^m_{jk}. \tag{22}$$

Contacting above equation by X_{mi} and using equations (4) and (1)(d), we can written as

$$\begin{aligned} C^m_{jk/h}X_{mi} &= \alpha_h C^m_{jk}X_{mi}, \\ C^m_{jk/h}X_{mi} &= \alpha_h C^m_{jk}(h_{mi} + X_{00}l_m l_i), \\ C^m_{jk/h}X_{mi} &= \alpha_h C_{ijk}, \\ C^m_{jk/h}X_{mi} &= C_{ijk/h}. \end{aligned} \tag{23}$$

Theorem 5.1 *In a C^h-recurrent Finsler space, the covariant tensor field X_{mi} satisfies (4) is of the form (23).*

6 The Existence Of Covariant Tensor X_{hk} In T-Condition :

Finsler space satisfying T-condition can be defined as,[6]

$$T^m_{ijk} = Lc^m_{ij/k} + l^m c_{ijk} + l_i c^m_{jk} + l_j c^m_{ik} + l_k c^m_{ij} = 0, \tag{24}$$

Contacting above equation by X_{hm} and using equations (4) and (1)(d), we have

$$\begin{aligned} T^m_{ijk}X_{hm} &= LC^m_{ij/k}X_{hm} + l^m C_{ijk}X_{hm} + l_i C^m_{jk}X_{hm} + l_j C^m_{ik}X_{hm} + l_k C^m_{ij}X_{hm} = 0, \\ T^m_{ijk}X_{hm} &= LC^m_{ij/k}X_{hm} + l^m C_{ijk}(h_{hm} + X_{00}l_h l_m) + l_i C^m_{jk}(h_{hm} + X_{00}l_h l_m) + \\ & \quad l_j C^m_{ik}(h_{hm} + X_{00}l_h l_m) + l_k C^m_{ij}(h_{hm} + X_{00}l_h l_m) = 0, \\ T^m_{ijk}X_{hm} &= LC_{hij/k} + l_h C_{ijk} + X_{00}l_h l_m l^m C_{ijk} + l_i C^{hjk} + l_j C^{hik} + l_k C^{hij} = 0, \\ T^m_{ijk}X_{hm} &= LC_{hij/k} + l_h C_{ijk} + l_i C^{hjk} + l_j C^{hik} + l_k C^{hij} + X_{00}l_h l_m l^m C_{ijk} = 0, \\ T^m_{ijk}X_{hm} &= T_{hijk} + X_{00}l_h c_{ijk} = 0, \\ T^m_{ijk}X_{hm} &= T_{hijk} = 0. \end{aligned} \tag{25}$$

Theorem 6.1 *If the Finsler space satisfying T-condition, then the covariant tensor field X_{hm} satisfies (4) is of the form (25) provided $X_{00}l_h C_{ijk} = 0$.*

Finsler space satisfying generalized T-condition can be defined as [5],

$$T^h_j = LC^h_{/j} + l^h C_j + l_j C^h = 0. \tag{26}$$

Contacting above equation by X_{ih} and using equations (4) and (1)(d), we obtain

$$\begin{aligned} T^h_j X_{ih} &= LC^h_{/j}X_{ih} + l^h C_j X_{ih} + l_j C^h X_{ih} = 0, \\ T^h_j X_{ih} &= LC^h_{/j}(h_{ih} + X_{00}l_i l_h) + l^h C_j(h_{ih} + X_{00}l_i l_h) + l_j C^h(h_{ih} + X_{00}l_i l_h) = 0, \\ T^h_j X_{ih} &= LC_{i/j} + l_i C_j + l_j C_i = 0, \\ T^h_j X_{ih} &= T_{ij}. \end{aligned} \tag{27}$$

Theorem 6.2 *If the Finsler space satisfying generalized T-Condition, then the covariant tensor field X_{ih} satisfies (4) is of the form (27).*

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