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Constraints on the uncertainties of entangled symmetric qubits

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Abstract

We derive necessary and sufficient inseparability conditions imposed on the variance matrix of symmetric qubits. These constraints are identified by examining a structural parallelism between continuous variable states and two-qubit states. Pairwise entangled symmetric multiqubit states are shown here to obey these constraints. We also bring out an elegant local invariant structure exhibited by our constraints.

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Entanglement is a central property of multipartite quantum systems as it forms the corner stone of all aspects of quantum information, communication, and computation [1]. The first task is to find a criterion if a given state is entangled or not. Peres–Horodecki inseparability criterion [2] viz., *positivity under partial transpose* (PPT) has been extremely fruitful in addressing this question and provides necessary and sufficient conditions for (2×2)- and (2×3)-dimensional systems. It is found that the PPT criterion is significant in the case of infinite-dimensional bipartite continuous variable (CV) states too. An important advance came about through an identification of how Peres–Horodecki criterion gets translated elegantly into the properties of the second moments (uncertainties) of CV states [3]. This results in restrictions [3,4] on the covariance matrix of an entangled bipartite CV state. In the special case of two-mode Gaussian states, where the basic entanglement properties are imbibed in the structure of its covariance matrix, the restrictions on the covariance matrix are found [3,4] to be necessary and sufficient for inseparability. Investigations on the structure of variance matrix have proved to be crucial in understanding the issue of entanglement in CV states [5–7] and a great deal of interest has been catching up in experimentally accessible, simple conditions of inseparability involving higher order moments [8–10].

In a parallel direction, growing importance is being evinced towards quantum correlated macroscopic atomic ensembles [11–15]. In the last few years experimental generation of entangled multiqubit states in trapped-ion systems [16–19], where individual particles can be manipulated, has been accomplished, giving new hopes for scalable quantum information processing.

In the present Letter, we explore a structural parallelism in CV states and two-qubit systems by constructing covariance matrix of the latter. We show that pairwise entanglement between any two-qubits of a symmetric N-qubit state is completely characterized by the off-diagonal block of the two-qubit covariance matrix. We establish the inseparability constraints satisfied by the covariance matrix and these are identified to be equivalent to the generalized spin squeezing inequalities [20] for two-qubit entanglement. The interplay between two basic principles viz., the uncertainty principle and the nonseparability gets highlighted through the restriction on the covariance matrix of a quantum correlated state.

We first recapitulate succinctly the approach employed by Simon [3] for bipartite CV states: the basic variables of bipartite CV states are the conjugate quadratures of two field modes, $\hat{\xi} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)$, which satisfy the canonical com-

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mutation relations

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = i\Omega_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4;$$
$$\Omega = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

The $4 \otimes 4$ real symmetric covariance matrix of a bipartite CV state is defined through its elements $V_{\alpha\beta} = \frac{1}{2}\langle\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\}\rangle$, where $\Delta\hat{\xi} = \hat{\xi}_\alpha - \langle\hat{\xi}_\alpha\rangle$ and $\{\Delta\hat{\xi}_\alpha, \Delta\hat{\xi}_\beta\} = \Delta\hat{\xi}_\alpha\Delta\hat{\xi}_\beta + \Delta\hat{\xi}_\beta\Delta\hat{\xi}_\alpha$. Under canonical transformations, the variables of the two-mode system transform as $\hat{\xi} \to \hat{\xi}' = S\hat{\xi}$, where $S \in Sp(4, R)$ corresponds to a real symplectic 4×4 matrix. Under such transformations, the covariance matrix goes as $V \to V' = SVS^T$. The entanglement properties hidden in the covariance matrix V remain unaltered under a local $Sp(2, R) \otimes Sp(2, R)$ transformation. It is convenient to caste the covariance matrix in a 2×2 block form:

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{2}$$

A local operation $S_1 \oplus S_2 \in Sp(2, R) \otimes Sp(2, R)$ transforms the blocks A, B, C of the variance matrix as

$$A \to A' = S_1 A S_1^T, \qquad B \to B' = S_2 B S_2^T,$$
$$C \to C' = S_1 C S_2^T. \tag{3}$$

There are four local invariants associated with V: $I_1 = \det A$, $I_2 = \det B$, $I_3 = \det C$, $I_4 = \mathrm{Tr}(AJCBC^TJ)$ and the Peres–Horodecki criterion imposes the restriction [3]

$$I_1 I_2 + \left(\frac{1}{4} - |I_3|\right)^2 - I_4 \geqslant \frac{1}{4}(I_1 + I_2) \tag{4}$$

on the second moments of every separable CV state. The signature of the invariant $I_3 = \det C$ has an important consequence: *Gaussian states with $I_3 \geqslant 0$ are necessarily separable, whereas those with $I_3 < 0$ and violating (4) are entangled*. In other words, for Gaussian states violation of the condition (4) is both necessary and sufficient for entanglement.

Let us now turn our attention on qubits. An arbitrary two-qubit density operator belonging to the Hilbert–Schmidt space $\mathcal{H} = \mathcal{C}^2 \otimes \mathcal{C}^2$ is given by

$$\rho = \frac{1}{4}\left[I \otimes I + \sum_{i=x,y,z}(\sigma_{1i}s_{1i} + \sigma_{2i}s_{2i}) + \sum_{i,j=x,y,z}\sigma_{1i}\sigma_{2j}t_{ij}\right], \tag{5}$$

where I denotes the 2×2 unit matrix, $\sigma_{1i} = \sigma_i \otimes I$, and $\sigma_{2i} = I \otimes \sigma_i$ (σ_i are the standard Pauli spin matrices); $s_{\alpha i} = \mathrm{Tr}(\rho\sigma_{\alpha i})$, $\alpha = 1, 2$, denote average spin components of the αth qubit and $t_{ij} = \mathrm{Tr}(\rho\sigma_{1i}\sigma_{2j})$ are elements of the real 3×3 matrix T corresponding to two-qubit correlations. The set of 15 state parameters $\{s_{1i},\ s_{2i},\ t_{ij}\}$ transform [21] under local unitary operations $U_1 \otimes U_2$ on the qubits as,

$$s'_{1i} = \sum_{j=x,y,z} O^{(1)}_{ij} s_{1j}, \qquad s'_{2i} = \sum_{j=x,y,z} O^{(2)}_{ij} s_{2j},$$
$$t'_{ij} = \sum_{k,l=x,y,z} O^{(1)}_{ik} O^{(2)}_{jl} t_{kl} \quad \text{or} \quad T' = O^{(1)} T O^{(2)T}, \tag{6}$$

where $O^{(\alpha)} \in SO(3)$ denote the 3×3 rotation matrices, corresponding uniquely to the 2×2 unitary matrices $U_\alpha \in SU(2)$.

Our interest here is on two-qubit states, which are completely symmetric under interchange. The symmetric two-qubit states are confined to the three-dimensional subspace $H_s = \mathrm{Sym}(\mathcal{C}^2 \otimes \mathcal{C}^2)$ ('Sym' denotes symmetrization), spanned by the eigenstates $\{|J_{\max} = 1, M\rangle;\ M = \pm 1, 0\}$ of the total angular momentum of the qubits. The state parameters of a symmetric two-qubit system obey the following constraints due to exchange symmetry[1]:

$$s_{1i} = s_{2i} \equiv s_i, \qquad t_{ij} = t_{ji}, \qquad \mathrm{Tr}(T) = 1, \tag{7}$$

and thus 8 real state parameters viz., s_i and the elements t_{ij} of the real symmetric correlation matrix T, which has unit trace, characterize a symmetric two-qubit system.

The basic variables of a two-qubit system are $\hat{\zeta} = (\sigma_{1i}, \sigma_{2j})$ and the covariance matrix $\mathcal{V}$ of a two-qubit system may be defined through

$$\mathcal{V}_{\alpha i;\beta j} = \frac{1}{2}\big\langle\{\Delta\hat{\zeta}_{\alpha i}, \Delta\hat{\zeta}_{\beta j}\}\big\rangle, \quad \alpha, \beta = 1, 2;\ i, j = x, y, z, \tag{8}$$

which can be written in the 3×3 block form as

$$\mathcal{V} = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}, \tag{9}$$

where

$$\mathcal{A}_{ij} = \frac{1}{2}\big[\big\langle\{\sigma_{1i}, \sigma_{1j}\}\big\rangle - \langle\sigma_{1i}\rangle\langle\sigma_{1j}\rangle\big]$$
$$= \delta_{ij} - \langle\sigma_{1i}\rangle\langle\sigma_{1j}\rangle = \delta_{ij} - s_{1i}s_{1j},$$
$$\mathcal{B}_{ij} = \frac{1}{2}\big[\big\langle\{\sigma_{2i}, \sigma_{2j}\}\big\rangle - \langle\sigma_{2i}\rangle\langle\sigma_{2j}\rangle\big]$$
$$= \delta_{ij} - \langle\sigma_{2i}\rangle\langle\sigma_{2j}\rangle = \delta_{ij} - s_{2i}s_{2j},$$
$$\mathcal{C}_{ij} = \frac{1}{2}\big[\langle\sigma_{1i}\sigma_{2j}\rangle - \langle\sigma_{1i}\rangle\langle\sigma_{2j}\rangle\big] = t_{ij} - s_{1i}s_{2j},$$
$$\text{or} \quad \mathcal{A} = \mathcal{I} - s_1 s_1^T, \qquad \mathcal{B} = \mathcal{I} - s_2 s_2^T, \qquad \mathcal{C} = T - s_1 s_2^T. \tag{10}$$

Here $\mathcal{I}$ denotes a 3×3 identity matrix.

In the case of symmetric states considerable simplicity ensues as a result of (7) and we obtain $\mathcal{A} = \mathcal{B} = 1 - ss^T$, $\mathcal{C} = T - ss^T$. We now establish an important property exhibited by the off-diagonal block $\mathcal{C}$ of the covariance matrix $\mathcal{V}$ of a symmetric two-qubit state.

Lemma 1. *For every separable symmetric state, $\mathcal{C} = T - ss^T$ is a positive definite matrix.*

Proof. A two-qubit separable symmetric state is given by

$$\varrho = \sum_w p_w \rho_w \otimes \rho_w, \quad \sum_w p_w = 1;\ 0 \leqslant p_w \leqslant 1, \tag{11}$$

[1] Permutation symmetry demands that $\langle\sigma_{1i}\rangle = \langle\sigma_{2i}\rangle$ and $\langle\sigma_{1i}\sigma_{2j}\rangle = \langle\sigma_{1j}\sigma_{2i}\rangle$. Moreover, the squared total angular momentum operator of symmetric two-qubits satisfies the condition $\langle(J_x^2 + J_y^2 + J_z^2)\rangle = J_{\max}(J_{\max} + 1) = \frac{N}{2}(\frac{N}{2} + 1) = 2$ i.e., $\frac{1}{4}\langle(\vec{\sigma}_1 + \vec{\sigma}_2)^2\rangle = 2$. This leads, in turn, to the unit trace condition $\mathrm{Tr}(T) = 1$ on the correlation matrix of the two-qubit symmetric states.

where $\rho_w = \frac{1}{2}(1+\sum_{i=x,y,z}\sigma_i\, s_{iw})$, denotes an arbitrary single qubit density matrix. The state variables s_i and t_{ij} of the two-qubit separable symmetric system are given by

$$s_i = \langle\sigma_{\alpha i}\rangle = \mathrm{Tr}(\varrho\sigma_{\alpha i}) = \sum_w p_w s_{iw},$$
$$t_{ij} = \langle\sigma_{1i}\sigma_{2j}\rangle = \mathrm{Tr}(\varrho\sigma_{1i}\sigma_{2j}) = \sum_w p_w s_{iw}s_{jw}. \tag{12}$$

Let us now evaluate the quadratic form $n^T(T - ss^T)n$ where n (n^T) denotes any arbitrary real three componential column (row), in a separable symmetric state:

$$\begin{aligned} n^T\big(T - ss^T\big)n &= \sum_{i,j}(t_{ij} - s_i s_j)n_i n_j \\ &= \sum_{i,j}\Big[\sum_w p_w s_{iw}s_{jw} - \sum_w p_w s_{iw}\sum_{w'} p_{w'}s_{jw'}\Big]n_i n_j \\ &= \sum_w p_w(\vec{s}\cdot\hat{n})^2 - \Big(\sum_w p_w(\vec{s}\cdot\hat{n})\Big)^2, \end{aligned} \tag{13}$$

which has the structure $\langle A^2\rangle - \langle A\rangle^2$ and is therefore a positive semi-definite quantity. □

The above Lemma 1 establishes the fact that the off diagonal block $\mathcal{C}$ of the covariance matrix is *necessarily* positive semi-definite for separable symmetric states. And therefore, $\mathcal{C} < 0$ serves as a sufficient condition for inseparability in two-qubit symmetric states.

We investigate pure entangled two-qubit states. A Schmidt decomposed pure entangled two-qubit state has the form,

$$|\Phi\rangle = \kappa_1|\uparrow_1\uparrow_2\rangle + \kappa_2|\downarrow_1\downarrow_2\rangle, \quad 0 < \kappa_2 \leqslant \kappa_1 < 1, \kappa_1^2 + \kappa_2^2 = 1. \tag{14}$$

(Here κ_1, κ_2 denote the Schmidt coefficients.) Obviously, every pure entangled two-qubit state is symmetric in the Schmidt basis. It is easy to see that in (14) the 3×3 correlation matrix T is diagonal,

$$T = \mathrm{diag}(2\kappa_1\kappa_2, -2\kappa_1\kappa_2, 1) \tag{15}$$

and $s_i = (0, 0, \kappa_1^2 - \kappa_2^2)$. The corresponding $\mathcal{C}$ matrix also has a diagonal form

$$\mathcal{C} = T - ss^T = \mathrm{diag}\big(2\kappa_1\kappa_2, -2\kappa_1\kappa_2, 4\kappa_1^2\kappa_2^2\big). \tag{16}$$

It is readily seen that $\mathcal{C}$ is *negative*, as its diagonal element $-2\kappa_1\kappa_2 < 0$, for an arbitrary entangled pure two-qubit state. In other words, the condition $\mathcal{C} < 0$ is both necessary and sufficient for pure entangled two-qubit states.

This discussion of the entangled two-qubit pure state leads naturally to state the following Theorem 1 for the corresponding symmetric mixed state.

Theorem 1. *The necessary condition for the inseparability of an arbitrary symmetric two-qubit mixed state is given by $\mathcal{C} < 0$.*

Proof. For the sake of brevity, we indicate here the steps leading to this condition and relegate the details to a separate communication [22]. An arbitrary two-qubit symmetric state, characterized by the density matrix (5), with the state parameters obeying the permutation symmetry requirements (7), gets transformed into a 3×3 block form,

$$\rho_S = \frac{1}{4}\begin{pmatrix} 1+2s_z+t_{zz} & a^*+b^* & t_{xx}-t_{yy}-2it_{xy} \\ a+b & 2(t_{xx}+t_{yy}) & a^*-b^* \\ t_{xx}-t_{yy}+2it_{xy} & a-b & 1-2s_z+t_{zz} \end{pmatrix}, \tag{17}$$

in the symmetric subspace characterized by the maximal value of total angular momentum $J_{\max} = 1$ (with the ordering of the basis states given by $M = 1, 0, -1$). Here, $a = \sqrt{2}(s_x + is_y)$, and $b = \sqrt{2}(t_{xz} + it_{yz})$. The above 3×3 matrix form (17) for ρ_S is realized by a transformation from the two-qubit basis to the total angular momentum basis $|J, M\rangle$ with $J = 1, 0; -J \leqslant M \leqslant J$:

$$\begin{aligned} &|\uparrow_1\uparrow_2\rangle = |1,1\rangle, \qquad |\downarrow_1\downarrow_2\rangle = |1,-1\rangle, \\ &|\uparrow_1\downarrow_2\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle + |0,0\rangle\big), \\ &|\downarrow_1\uparrow_2\rangle = \frac{1}{\sqrt{2}}\big(|1,0\rangle - |0,0\rangle\big). \end{aligned} \tag{18}$$

But, the partial transpose (PT) of (5), with respect to the second qubit—identified as an operation leading to the complete sign reversal $\sigma_{2i} \to -\sigma_{2i}$—of a symmetric system does not get restricted to the symmetric subspace with $J_{\max} = 1$, after the basis change (18). However, following the transformation (18) with another basis change $|X\rangle = \frac{-1}{\sqrt{2}}(|1,1\rangle - |1,-1\rangle)$, $|Y\rangle = \frac{-i}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$, $|Z\rangle = |1,0\rangle$ leads to an elegant block structure for the PT symmetric density matrix:

$$\rho_S^{T_2} = \frac{1}{2}\begin{pmatrix} T & s \\ s^T & 1 \end{pmatrix} \tag{19}$$

with T being the 3×3 two-qubit real symmetric correlation matrix and s the 3×1 column of qubit averages. As a final step, we identify the congruence

$$\rho_S^{T_2} \sim \tilde{\rho}_S^{T_2} = L\rho_S^{T_2}L^\dagger = \frac{1}{2}\begin{pmatrix} \mathcal{C} & 0 \\ 0 & 1 \end{pmatrix}, \tag{20}$$

with $L = \begin{pmatrix} \mathcal{I} & -s \\ 0 & 1 \end{pmatrix}$, leading us to the result

$$\rho_S^{T_2} < 0 \quad \Leftrightarrow \quad \mathcal{C} < 0, \tag{21}$$

thus proving our theorem. □

We have therefore established that the $\mathcal{C}$ matrix approach provides a simpler equivalent procedure to verify the inseparability status of a symmetric two-qubit system.

Now, we proceed to explore how this basic structure $\mathcal{C} < 0$ of inseparability reflects itself via collective second moments of a symmetric N-qubit system. Collective observables of a N-qubit system are expressible in terms of the angular momentum operator

$$\vec{J} = \sum_{\alpha=1}^{N}\frac{1}{2}\vec{\sigma}_\alpha, \tag{22}$$

where $\vec{\sigma}_\alpha$ denote the Pauli spin operator of the αth qubit. Symmetric N-qubit states are confined to the $(N+1)$-dimensional subspace $\{|J_{\max}=N/2, M\rangle, -\frac{N}{2}\leqslant M\leqslant\frac{N}{2}\}$ of maximum angular momentum $J_{\max}=N/2$. A collective correlation matrix involving first and second moments of $\vec{J}$ may be defined by

$$V_{ij}^{(N)}=\frac{1}{2}\langle J_iJ_j+J_jJ_i\rangle-\langle J_i\rangle\langle J_j\rangle;\quad i,j=x,y,z. \tag{23}$$

The collective observables (up to second order in $\vec{J}$) in a symmetric multiqubit state are expressible in terms of the constituent qubit variables as,

$$\frac{1}{2}\big\langle(J_iJ_j+J_jJ_i)\big\rangle=\frac{1}{8}\sum_{\alpha,\beta=1}^{N}\big\langle(\sigma_{\alpha i}\sigma_{\beta j}+\sigma_{\beta i}\sigma_{\alpha j})\big\rangle$$
$$=\frac{1}{4}\sum_{\alpha,\beta=1}^{N}\big\langle(\sigma_{\alpha i}\sigma_{\beta j})\big\rangle=\frac{N}{4}\big[\delta_{ij}+(N-1)t_{ij}\big],$$
$$\langle J_i\rangle=\frac{1}{2}\sum_{\alpha=1}^{N}\langle\sigma_{\alpha i}\rangle=S_i=\frac{N}{2}s_i, \tag{24}$$

where we have made use of the following fact: the bipartite reductions $\rho_{\alpha\beta}$ of a symmetric multi-qubit state are all identical and the average values of two-qubit correlations $\langle(\sigma_{\alpha i}\sigma_{\beta j})\rangle=t_{ij}$—irrespective of the qubit labels α and β—for any random pair of qubits drawn from a symmetric state. Moreover, $\langle\sigma_{\alpha i}\rangle=s_i$ for all qubits belonging to a symmetric N-qubit system.

So, the correlation matrix $V^{(N)}$ of (23) assumes the form

$$V^{(N)}=\frac{N}{4}\big(\mathcal{I}-ss^T+(N-1)\big(T-ss^T\big)\big) \tag{25}$$

with $\mathcal{I}$ being a 3×3 identity matrix; T and s are the state variables characterizing any two-qubit partition $\rho_{\alpha\beta}$ of a symmetric N-qubit system. It is convenient to express (25) as

$$V^{(N)}+\frac{1}{N}SS^T=\frac{N}{4}\big(\mathcal{I}+(N-1)\mathcal{C}\big), \tag{26}$$

by shifting the second term i.e., $\frac{N}{4}ss^T$ to the left-hand side and expressing $\frac{N}{2}s_i=\langle J_i\rangle=S_i$ (see (24)), in terms of the collective average angular momentum. As has been established by our theorem, $\mathcal{C}$ is positive semi-definite for all separable symmetric two-qubit states, implying that the condition

$$V^{(N)}+\frac{1}{N}SS^T<\frac{N}{4}\mathcal{I} \tag{27}$$

can only be satisfied by an entangled symmetric N-qubit state.

Under identical local unitary transformations $U\otimes U\otimes\cdots\otimes U$ on the qubits the variance matrix $V^{(N)}$ and the average spin S transform as

$$V^{(N)'}=OV^{(N)}O^T\quad\text{and}\quad S'=OS, \tag{28}$$

where O is a real orthogonal rotation matrix corresponding to the local unitary transformation U. Thus, the 3×3 real symmetric matrix $V^{(N)}+\frac{1}{N}SS^T$ can always be diagonalized by a suitable identical local unitary transformation on all the qubits. In other words, (27) is a local invariant condition and it essentially implies:

The symmetric N-qubit system is pairwise entangled iff the least eigenvalue of the real symmetric matrix $V^{(N)}+\frac{1}{N}SS^T$ is less than $N/4$,

Local invariant structure. Now, we explore how the negativity of the matrix $\mathcal{C}$ reflects itself on the structure of the local invariants associated with the two-qubit state [23,24].

We denote the eigenvalues[2] of the off-diagonal block $\mathcal{C}$ of the covariance matrix (9) by c_1, c_2, and c_3. Restricting ourselves to identical local unitary transformations [23,24], we define three local invariants, which completely determine the eigenvalues of $\mathcal{C}$:

$$\mathcal{I}_1=\det(\mathcal{C})=c_1c_2c_3,$$
$$\mathcal{I}_2=\mathrm{Tr}(\mathcal{C})=c_1+c_2+c_3,$$
$$\mathcal{I}_3=\mathrm{Tr}\big(\mathcal{C}^2\big)=c_1^2+c_2^2+c_3^2. \tag{29}$$

The invariant $\mathcal{I}_2$ may be rewritten as $\mathcal{I}_2=\mathrm{Tr}(T-ss^T)=1-s_0^2$, since $\mathrm{Tr}(T)=1$ for a symmetric state[3] and we have denoted $\mathrm{Tr}(ss^T)=s_1^2+s_2^2+s_3^2=s_0^2$. Another useful invariant, which is a combination of the invariants defined through (29), may be constructed as

$$\mathcal{I}_4=\frac{\mathcal{I}_2^2-\mathcal{I}_3}{2}=c_1c_2+c_2c_3+c_1c_3. \tag{30}$$

Positivity of the single qubit reduced density operator demands $s_0^2\leqslant1$ and leads in turn to the observation that the invariant $\mathcal{I}_2$ is positive for all symmetric states. Thus, all the three eigenvalues c_1,c_2,c_3 of $\mathcal{C}$ can never assume negative values for symmetric qubits and at most two of them can be negative.

We consider three distinct cases encompassing all pairwise entangled symmetric states.

Case (i) Let one of the eigenvalues $c_1=0$ and of the remaining two, let $c_2<0$ and $c_3>0$.[4]
Clearly, the invariant $\mathcal{I}_1=0$ in this case. But we have

$$\mathcal{I}_4=c_2c_3<0, \tag{31}$$

which leads to a local invariant condition for two-qubit entanglement.

Case (ii) Suppose any two eigenvalues say, c_1,c_2, are negative and the third one c_3 is positive.
Obviously, $\mathcal{I}_1>0$ in this case. But the invariant $\mathcal{I}_4$ assumes negative value:

$$\mathcal{I}_4=c_1\mathcal{I}_2-c_1^2+c_2c_3<0 \tag{32}$$

as each term in the right-hand side is negative. In other words, $\mathcal{I}_4<0$ gives the criterion for bipartite entanglement in this case too.

[2] The off-diagonal block $\mathcal{C}$ of the covariance matrix can be diagonalized by an appropriate identical local unitary transformation on the qubits.

[3] See footnote 1.

[4] We must note that if one of the eigenvalues i.e., c_1, c_2, or c_3 is equal to zero and another negative, the remaining one should be positive in order to preserve the condition $\mathcal{I}_2>0$.

Case (iii) Let $c_1 < 0$; c_2 and c_3 be positive.
In this case we have

$$\mathcal{I}_1 < 0, \tag{33}$$

giving the inseparability criterion in terms of a local invariant.

In conclusion, we have investigated a structural similarity between continuous variable systems and symmetric two-qubit states by constructing two-qubit variance matrix. We have shown here that the off-diagonal block of the variance matrix $\mathcal{C}$ of a separable symmetric two-qubit state is a positive semidefinite quantity. Symmetric two-qubit states satisfying the condition $\mathcal{C} < 0$ are therefore identified as inseparable. An equivalence between the Peres–Horodecki criterion and the negativity of the covariance matrix $\mathcal{C}$ is established, showing that our condition is both necessary and sufficient for entanglement in symmetric two-qubit states. We have identified the constraints satisfied by the collective correlation matrix $V^{(N)}$ of pairwise entangled symmetric N-qubit states. Local invariant structure of our inseparability constraints is also investigated. In a recent publication [20], which appeared since completion of this work, Korbicz et al. have generalized the concept of spin squeezing connecting it to the theory of entanglement witnesses and proposed the inequality $\frac{4\langle \Delta J_n^2\rangle}{N} < 1 - \frac{4\langle J_n\rangle^2}{N^2}$ (where $J_n = \vec{J}\cdot\hat{n}$, $\hat{n}$ denoting an arbitrary unit vector, $\Delta J_n^2 = \langle J_n^2\rangle - \langle J_n\rangle^2$) as a necessary condition for two-qubit entanglement in symmetric states. After some simple algebra, we find that this inequality is equivalent to $\mathcal{C} < 0$, thus establishing a connection between the covariance matrix and the theory of entanglement witnesses. Further, the approach outlined in this Letter has been extended recently [25] to obtain constraints on higher order covariance matrices, which in turn lead to a family of inseparability conditions for various even partitions of symmetric N-qubit systems.

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