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On the application of two Gauss–Legendre quadrature rules for composite numerical integration over a tetrahedral region

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Abstract

In this paper we first present a Gauss–Legendre quadrature rule for the evaluation of $I = \int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, where $f(x,y,z)$ is an analytic function in x, y, z and T is the standard tetrahedral region: $\{(x,y,z)|0 \leqslant x,y,z \leqslant 1,\ x+y+z \leqslant 1\}$ in three space (x,y,z). We then use a transformation $x = x(\xi,\eta,\zeta)$, $y = y(\xi,\eta,\zeta)$ and $z = z(\xi,\eta,\zeta)$ to change the integral into an equivalent integral $I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta,\zeta),y(\xi,\eta,\zeta),z(\xi,\eta,\zeta))\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$ over the standard 2-cube in (ξ,η,ζ) space: $\{(\xi,\eta,\zeta)|-1 \leqslant \xi,\eta,\zeta \leqslant 1\}$. We then apply the one-dimensional Gauss–Legendre quadrature rules in ξ, η and ζ variables to arrive at an efficient quadrature rule with new weight coefficients and new sampling points. Then a second Gauss–Legendre quadrature rule of composite type is obtained. This rule is derived by discretising the tetrahedral region T into four new tetrahedra T_i^c $(i = 1,2,3,4)$ of equal size which are obtained by joining the centroid of T, $c = (1/4,1/4,1/4)$ to the four vertices of T. By use of the affine transformations defined over each T_i^c and the linearity property of integrals leads to the result:

$$I = \sum_{i=1}^{4}\int\int_{T_i^c}\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{4}\int\int_T\int G(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z,$$

where

$$G(X,Y,Z) = \frac{1}{p^3}\sum_{k=1}^{4} f(x^{(k)}(X,Y,Z),y^{(k)}(X,Y,Z),z^{(k)}(X,Y,Z)),$$

$$x^{(k)} = x^{(k)}(X,Y,Z),\quad y^{(k)} = y^{(k)}(X,Y,Z)\quad \text{and}\quad z^{(k)} = z^{(k)}(X,Y,Z)$$

refer to an affine transformations which map each T_i^c into the standard tetrahedral region T.

We then write

$$I = \int\int_T\int G(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = \int_0^1\int_0^{1-\xi}\int_0^{1-\xi-\eta} G(X(\xi,\eta,\zeta),Y(\xi,\eta,\zeta),Z(\xi,\eta,\zeta))\left|\frac{\partial(X,Y,Z)}{\partial(\xi,\eta,\zeta)}\right|\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$

and a composite rule of integration is thus obtained. We next propose the discretisation of the standard tetrahedral region T into p^3 tetrahedra T_i $(i = 1(1)p^3)$ each of which has volume equal to $1/(6p^3)$ units. We have again shown that the use of affine transformations over each T_i and the use of linearity property of integrals leads to the result:

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$$\int\int_T\int f(x,y,z)\,dx\,dy\,dz=\sum_{i=1}^{p^3}\int\int_{T_i^c}\int f(x,y,z)\,dx\,dy\,dz=\sum_{\alpha=1}^{p^3}\int\int_{T_\alpha^{(p)}}\int f(x^{(\alpha,p)},y^{(\alpha,p)},z^{(\alpha,p)})\,dx^{(\alpha,p)}\,dy^{(\alpha,p)}\,dz^{(\alpha,p)}$$
$$=\frac{1}{p^3}\int\int_T\int H(X,Y,Z)\,dX\,dY\,dZ,$$

where

$$H(X,Y,Z)=\sum_{\alpha=1}^{P^3}f(x^{(\alpha,P)}(X,Y,Z),y^{(\alpha,P)}(X,Y,Z),z^{(\alpha,P)}(X,Y,Z)),$$
$$x^{(\alpha,p)}=x^{(\alpha,p)}(X,Y,Z),\quad y^{(\alpha,p)}=y^{(\alpha,p)}(X,Y,Z)\quad\text{and}\quad z^{(\alpha,p)}=z^{(\alpha,p)}(X,Y,Z)$$

refer to the affine transformations which map each T_i in $(x^{(\alpha,p)},y^{(\alpha,p)},z^{(\alpha,p)})$ space into a standard tetrahedron T in the (X,Y,Z) space. We can now apply the two rules earlier derived to the integral $\int\int_T\int H(X,Y,Z)\,dX\,dY\,dZ$, this amounts to the application of composite numerical integration of T into p^3 and $4p^3$ tetrahedra of equal sizes. We have demonstrated this aspect by applying the above composite integration method to some typical triple integrals.

Keywords: Finite element method; Composite numerical integration; Tetrahedral regions; Gauss–Legendre quadrature rules; Triangular prisms; Standard 2-cube; Standard tetrahedron

1. Introduction

In recent years, we have been witnessing finite element method (FEM) gaining importance due to the most obvious reason that it can provide solutions to many complicated problems that would be intractable by other numerical techniques [1,2]. In FEM it may be possible to perform some of the integration analytically, particularly if constant or linear elements are used to discretise the surface or boundary curve of the given region. However, with higher order elements or for more complex distorted elements the integrals become too complicated for analytical integration and the numerical integration is essential, among various integration schemes, Gauss–Legendre quadrature which can evaluate exactly the $(2n-1)$th order polynomial with n-Gaussian points is most commonly used in view of the accuracy and efficiency of calculations [3]. The triangular and tetrahedral elements are very widely used in finite element analysis. The versatility of these elements can be further enhanced by improved numerical integration schemes.

Mathematically, the problem can be defined as the evaluation of the following integrals:

$$\mathrm{II}=\int_0^1\int_0^{1-L_1}F(L_1,L_2,L_3)\,dL_2\,dL_1, \tag{1}$$

where L_1, L_2, L_3 are the well known area co-ordinates and

$$\mathrm{III}=\int_0^1\int_0^{1-L_1}\int_0^{1-L_1-L_2}G(L_1,L_2,L_3,L_4)\,dL_3\,dL_2\,dL_1, \tag{2}$$

where L_1, L_2, L_3, L_4 are the well known volume co-ordinates.

The basic problem of integrating an arbitrary function of two variables over the surface of the triangle were first given by Hammer et al. [4], and Hammer and Stroud [5,6]. Cowper [7] provided a table of Gaussian quadrature formulae with symmetrically placed integration points. Lyness and Jespersen [8] made an elaborate study of symmetric quadrature rules by formulating the problem in polar co-ordinates. Lannoy [9] discussed the symmetric 4-point integration formula, which is presented in [7]. Laurie [10] derived a 7-point integration rule and discussed the numerical error in integrating some functions. Laursen and Gellert [11] gave a table of symmetric integration formulae up to a precision of degree ten. Dunavant [12] presented some extensions to the integration formulae given by Lyness and Jespersen [8] and also gave tables of integration formulae with precisions of degree from eleven to twenty. Sylvester [13] derived some numerical integration formulae for triangles as product of one-dimensional Newton Cotes rules of closed type as well as open type. The precision of these integration formulae is limited to a degree ten atmost for various reasons. Lether [14] and Hillion [15] derived the formulae for triangles as product of one-dimensional Gauss–Legendre and Gauss Jacobi quadrature rules. The precision of these formulae is again up to degree seven. This is because the zeros and weight coefficients of Gauss Jacobi orthogonal polynomials with weight functions x, x^2, x^3 were available for polynomials of degree up to six only. Even today the zeros and weights for the integral $\int_0^1 x^r f(x)\,dx$, $r = 1, 2, 3$ are not available beyond a formula of order-eight as documented in Abramowicz and Stegun [16]. Reddy [17] and Reddy and Shippy [18] derived the 3-point, 4-point, 6-point and 7-point rules of precision 3, 4, 6 and 7 respectively which gave improved accuracy. Since the precision of all the formulae derived by the authors [4–18] is limited to a precision of degree ten and it is not likely that the techniques can be extended much further to give a greater accuracy which may be demanded in future, Lague and Baldur [19] proposed the product formulae based only on the sampling points and weight coefficients of Gauss–Legendre quadrature rules. By the proposed method of [19] this restriction is removed and one can now obtain numerical integration rules of very high degree of precision as the derivation now rely on standard Gauss–Legendre Quadarature rules. However, the Lague and Baldur [19] have not worked out explicit weight coefficients and sampling points for application to integrals over a triangular surface. Rathod et al [20–22] provided this information in a systematic manner in their recent work. For tetrahedral regions, four volume coordinates L_1, L_2, L_3, L_4 are involved and we have to compute numerically the integral III stated in Eq. (2). Numerical integration formulae for III with a degree of precision $d = 1, 2, 3$ are listed in Zienkiewicz [1] and these are based on reference [4]. Numerical integration formulae of precision higher than cubic are not available in the current literature and hence we propose here the derivation of higher order formulae for tetrahedral regions.

Integration formulae resulting from interval subdivision and repeated application of a low order formula are called composite numerical integration formulae [23–26]. One of the ways to reduce the error associated with low order integration formula in one-dimension is to subdivide the interval of integration, say, $[a, b]$ into smaller intervals and then to use the formula separately on each subinterval. We adopt a strategy similar to the above, which is normally used for the treatment of line integrals over arbitrary shaped curves for evaluation of triple integrals also. We segment the given region into sub-regions and effect a transformation over each sub-region into a standard region. The success of this strategy follows from the linearity property of triple integrals. Repeated application of low order formula is usually preferred to the single application of a high order formula, partly because of the simplicity of lower order formulae and partly because of computational difficulties; one such difficulty is due to the errors introduced because of only a fixed, usually small number of digits can be retained after each computer operation. In addition, there exist many functions for which the magnitude of the derivative increases without bound as the order of differentiation increases. Therefore a higher order formula may produce a larger error than a lower order one. It is in view of this fact that the numerical integration formulae employing more than eight points (for Newton Cotes rules) are almost never used. We feel that these important details cannot be simply ignored, and they need to be addressed in great rigor. Hence the derivation of algorithms for composite numerical integration formulae over dimensions higher than one is important for practical applications and it should be used wherever necessary. One of the purposes of this paper is to evolve a practical and workable algorithm for composite numerical integration over tetrahedral regions by using the well known Gauss–Legendre quadrature rules.

2. Formulation of integrals over a tetrahedron

The finite element method for three-dimensional problems with tetrahedron element requires the numerical integration of expressions containing product of shape functions and their global derivatives over a standard tetrahedron T with coordinates $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in the natural coordinate space (x, y, z) say. Since either an affine or an isoparametric coordinate transformation makes it possible to transform any tetrahedron (either a linear or curved) into global coordinate system, say (X, Y, Z). We thus have to consider the numerical integration over a standard tetrahedron T. The numerical integration of an arbitrary function f, over the tetrahedron T is given by

$$\mathrm{I} = \int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z = \int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}x\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z. \tag{3}$$

It is now required to find the value of the integral by a quadrature formula:

$$\mathrm{I} = \sum_{m=1}^{N} c_m f(x_m, y_m, z_m), \tag{4}$$

where c_m are the weights associated with the sampling points (x_m, y_m, z_m) and N is the number of pivotal points related to the required precision.

The integral I of Eq. (3) can be transformed into an integral over the cube: $\{(u,v,w)|0 \leqslant u,v,w \leqslant 1\}$ by the substitution

$$x = u, \quad y = (1-u)v, \quad z = (1-u)(1-v)w. \tag{5}$$

Then the determinant of the Jacobian and the differential volume are

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = (1-u)^2(1-v) \quad \text{and}$$

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\partial(x,y,z)}{\partial(u,v,w)}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = (1-u)^2(1-v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \tag{6}$$

Then on using Eqs. (5) and (6) in Eq. (3), we have

$$\begin{aligned}\mathrm{I} &= \int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1\int_0^1\int_0^1 f(u,(1-u)v,(1-u)(1-v)w) \times (1-u)^2(1-v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u. \end{aligned}\tag{7}$$

The integral I of Eq. (7) can be further transformed into an integral over the standard 2-cube: $\{(\xi,\eta,\zeta)|-1 \leqslant \xi,\eta,\zeta \leqslant 1\}$ by the substitution

$$u = \frac{(1+\xi)}{2}, \quad v = \frac{(1+\eta)}{2}, \quad w = \frac{(1+\zeta)}{2}. \tag{8}$$

Then clearly the determinant of the Jacobian and the differential volume are

$$\frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)} = \frac{1}{8} \quad \text{and} \quad \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w = \frac{\partial(u,v,w)}{\partial(\xi,\eta,\zeta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta = \frac{1}{8}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{9}$$

Now on using Eqs. (8) and (9) in Eq. (7), we have

$$\begin{aligned} I &= \int_0^1\left(\int_0^{1-x}\left(\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-1}^1\int_{-1}^1\int_{-1}^1 f\left(\frac{(1+\xi)}{2},\frac{(1-\xi)(1+\eta)}{4},\frac{(1-\xi)(1-\eta)(1+\zeta)}{8}\right) \times \frac{(1-\xi)^2(1-\eta)}{64}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \end{aligned}\tag{10}$$

Eq. (10) represents an integral over the standard 2-cube $\{(\xi,\eta,\zeta)|-1 \leqslant \xi,\eta,\zeta \leqslant 1\}$.

Efficient quadrature coefficients are readily available in the literature so that any desired accuracy can be obtained [16].

From Eqs. (4) and (10), we find that

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f\left(\frac{(1+\xi)}{2},\frac{(1-\xi)(1+\eta)}{4},\frac{(1-\xi)(1-\eta)(1+\zeta)}{8}\right)\times\frac{(1-\xi)^2(1-\eta)}{64}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta \\
&= \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}\sum_{k=1}^{\gamma}\frac{(1-\xi_i^{(\alpha)})^2(1-\eta_j^{(\beta)})}{64}w_i^{(\alpha)}w_j^{(\beta)}w_k^{(\gamma)} \\
&\quad\times f\left(\frac{(1+\xi_i^{(\alpha)})}{2},\frac{(1-\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})}{4},\frac{(1-\xi_i^{(\alpha)})(1-\eta_j^{(\beta)})(1+\zeta_k^{(\gamma)})}{8}\right) \\
&= \sum_{m=1}^{N=(\alpha\times\beta\times\gamma)} c_m f(x_m,y_m,z_m), \qquad (11)
\end{aligned}
$$

where, it is obvious that

$$
\begin{aligned}
c_m &= \frac{(1-\xi_i^{(\alpha)})^2(1-\eta_j^{(\beta)})}{64}w_i^{(\alpha)}w_j^{(\beta)}w_k^{(\gamma)}, \quad x_m=\frac{(1+\xi_i^{(\alpha)})}{2}, \quad y_m=\frac{(1-\xi_i^{(\alpha)})(1+\eta_j^{(\beta)})}{4}, \\
z_m &= \frac{(1-\xi_i^{(\alpha)})(1-\eta_j^{(\beta)})(1+\zeta_k^{(\gamma)})}{8} \qquad (12)
\end{aligned}
$$

in which $\xi_i^{(\alpha)}$, $\eta_j^{(\beta)}$, $\zeta_k^{(\gamma)}$ are the sampling points and $w_i^{(\alpha)}$, $w_j^{(\beta)}$, $w_k^{(\gamma)}$ are the corresponding weight coefficients of Gauss–Legendre quadrature rules of order α, β and γ respectively. Though quadrature rules of orders i.e., $\alpha\neq\beta\neq\gamma$ can be used, for convenience we derive the formulae with $\alpha=\beta=\gamma=s$ (say). The weight coefficients c_m and corresponding sampling points (x_m, y_m, z_m) of various orders i.e., $s=2,3,4$ etc. can be now easily computed by the formulae of Eq. (12) and the approximation to the integral I can be then computed by Eq. (11). We have listed here a C-Program which generates c_m, (x_m,y_m,z_m) and then computes the integral $\mathrm{I}=\int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ for some sample functions $f(x,y,z)$. We have also given here a sample output of the C-Program for $n=2$ and 3.

2.1. C-program for generating sampling points (x_m, y_m, z_m) and weight coefficients (c_m)

```
#include⟨stdio.h⟩
#include⟨conio.h⟩
#include⟨math.h⟩
main()
{
int i, j, k, n;
double xm, ym, zm, cm, a[20], w[20];
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of alphas (a's)");
for(i=1; i<=n; i++)
scanf("%lf", &a[i]);
printf("Enter the values of weights (w's)");
for(i=1; i<=n; i++)
scanf("%lf", &w[i]);
printf("    xm     ym     zm     cm\n");
for(i=1; i<=n; i++)
{for(j=1; j<=n; j++)
{for(k=1; k<=n; k++)
{
xm=(1+a[i])/2;
```

```
ym=(1−a[i])*(1+a[j])/4;
zm=(1−a[i])*(1−a[j])*(1+a[k])/8;
cm=(1−a[i])*(1−a[i])*(1−a[j])*w[i]*w[j]*w[k]/64;
printf(" %0.15lf %0.15lf %0.15lf %0.15lf\n", xm, ym, zm, cm);
}}}
getch();
}
```

2.2. Sample output for $n = 2$ and 3

x_m	y_m	z_m	c_m
$n = 2$			
0.211324865405187	0.166666666666667	0.131445855765802	0.061320326520293
0.211324865405187	0.166666666666667	0.490562612162344	0.061320326520293
0.211324865405187	0.622008467928146	0.035220810900864	0.016430731970725
0.211324865405187	0.622008467928146	0.131445855765802	0.016430731970725
0.788675134594813	0.044658198738520	0.035220810900864	0.004402601362608
0.788675134594813	0.044658198738520	0.131445855765802	0.004402601362608
0.788675134594813	0.166666666666667	0.009437387837656	0.001179673479707
0.788675134594813	0.166666666666667	0.035220810900864	0.001179673479707
$n = 3$			
0.112701665379259	0.100000000000000	0.088729833462074	0.014972747367084
0.112701665379259	0.100000000000000	0.698568501158667	0.014972747367084
0.112701665379259	0.100000000000000	0.393649167310371	0.023956395787334
0.112701665379259	0.787298334620741	0.011270166537926	0.001901788268649
0.112701665379259	0.787298334620741	0.088729833462074	0.001901788268649
0.112701665379259	0.787298334620741	0.050000000000000	0.003042861229838
0.112701665379259	0.443649167310371	0.050000000000000	0.013499628508586
0.112701665379259	0.443649167310371	0.393649167310371	0.013499628508586
0.112701665379259	0.443649167310371	0.221824583655185	0.021599405613738
0.887298334620741	0.012701665379258	0.011270166537926	0.000241558782106
0.887298334620741	0.012701665379258	0.088729833462074	0.000241558782106
0.887298334620741	0.012701665379258	0.050000000000000	0.000386494051369
0.887298334620741	0.100000000000000	0.001431498841332	0.000030681988197
0.887298334620741	0.100000000000000	0.011270166537926	0.000030681988197
0.887298334620741	0.100000000000000	0.006350832689629	0.000049091181116
0.887298334620741	0.056350832689629	0.006350832689629	0.000217792616242
0.887298334620741	0.056350832689629	0.050000000000000	0.000217792616242
0.887298334620741	0.056350832689629	0.028175416344815	0.000348468185988
0.500000000000000	0.056350832689629	0.050000000000000	0.007607153074595
0.500000000000000	0.056350832689629	0.393649167310371	0.007607153074595
0.500000000000000	0.056350832689629	0.221824583655185	0.012171444919352
0.500000000000000	0.443649167310371	0.006350832689629	0.000966235128423
0.500000000000000	0.443649167310371	0.050000000000000	0.000966235128423
0.500000000000000	0.443649167310371	0.028175416344815	0.001545976205477
0.500000000000000	0.250000000000000	0.028175416344815	0.006858710562414
0.500000000000000	0.250000000000000	0.221824583655185	0.006858710562414
0.500000000000000	0.250000000000000	0.125000000000000	0.010973936899863

2.3. C-program for evaluation of triple integrals of Examples 1–4

```
#include⟨stdio.h⟩
#include⟨conio.h⟩
#include⟨math.h⟩
main()
{
int i, j, k, n;
double x, y, z, c, a[20],w[20], X, Y, Z, I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11,
        S1=0, S2=0, S3=0, S4=0, S5=0, S6=0, S7=0, S8=0, S9=0, S10=0, S11=0;
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of sampling points (a's)");
for(i=1; i<=n; i++)
scanf("%lf", &a[i]);
printf("Enter the values of weight coefficients (w's)");
for(i=1; i<=n; i++)
scanf("%lf", &w[i]);
for(i=1; i<=n; i++)
{   for(j=1; j<=n; j++)
{   for(k=1; k<=n; k++)
{
x=(1+a[i])/2;
y=(1-a[i])*(1+a[j])/4;
z=(1-a[i])*(1-a[j])*(1+a[k])/8;
c=(1-a[i])*(1-a[i])*(1-a[j])*w[i]*w[j]*w[k]/64;
I1=c*sqrt(x+y+z);
S1=S1+I1;
I2=c*1/sqrt(x+y+z);
S2=S2+I2;
I3=c*1/sqrt(pow(1-x-y,2)+pow(z,2));
S3=S3+I3;
I4=c*sin(x+2*y+4*z);
S4=S4+I4;
I5=c*pow(1+x+y+z,-4);
S5=S5+I5;
X=10-5*x-2*z; Y=5+5*y+2*z; Z=8*z;
I6=200*c*pow(X,2)*Y;
S6=S6+I6;
I7=200*c*pow(X,2)*pow(Y,2);
S7=S7+I7;
I8=200*c*pow(X,4)*pow(Y,4);
S8=S8+I8;
I9=200*c*(pow(X,2)*Y/sqrt(X+Y+Z));
S9=S9+I9;
I10=200*c*(pow(X,2)*pow(Y,2)/sqrt(X+Y+Z));
S10=S10+I10;
I11=200*c*(pow(X,4)*pow(Y,4)/sqrt(X+Y+Z));
S11=S11+I11;
}}}
```

```
printf("I1=%0.15lf\n", S1);
printf("I2=%0.15lf\n", S2);
printf("I3=%0.15lf\n", S3);
printf("I4=%0.15lf\n", S4);
printf("I5=%0.15lf\n", S5);
printf("I6=%0.15lf\n", S6);
printf("I7=%0.15lf\n", S7);
printf("I8=%0.15lf\n", S8);
printf("I9=%0.15lf\n", S9);
printf("I10=%0.15lf\n", S10);
printf("I11=%0.15lf\n", S11);
getch();
}
```

3. Composite integration rule over a standard tetrahedron T

We now derive a new composite integration rule over the standard tetrahedron T with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. We can evaluate the integrals over T by adopting a strategy similar to that used for the treatment of line integrals over arbitrarily shaped curves. We discretise the tetrahedron T into four new tetrahedra of equal volume $= 1/24$ units by joining the centroidal point $c = (1/4, 1/4, 1/4)$ to four vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, so that we can write $T = T_1^c + T_2^c + T_3^c + T_4^c$. This is depicted in Fig. 1.

We have, on using the linearity property of integrals:

$$I = \int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \stackrel{\text{def}}{=} \int_0^1\int_0^{1-x}\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$
$$= \sum_i^4 \int\int_{T_i^c}\int f(x^{(i)},y^{(i)},z^{(i)})\,\mathrm{d}x^{(i)}\,\mathrm{d}y^{(i)}\,\mathrm{d}z^{(i)}. \tag{13}$$

We can transform each T_i^c $(i = 1,2,3,4)$ into a standard tetrahedron T by use of the well known affine transformations:

We have from the above figure (viz Fig. 1)

T_1^c is spanned by vertices 2, 3, 4 and c;
T_2^c is spanned by vertices 3, 1, 4 and c;
T_3^c is spanned by vertices 1, 2, 4 and c; and
T_4^c is spanned by vertices 1, 2, 3 and c.

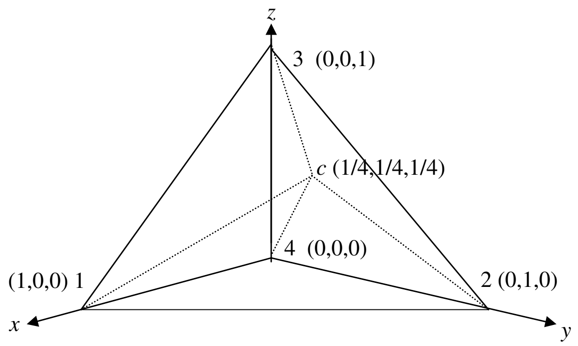

Fig. 1. Discretisation of tetrahedron T into four tetrahedra T_i^c $(i = 1,2,3,4)$, each of equal volume 1/24 units, c = centroid of T.

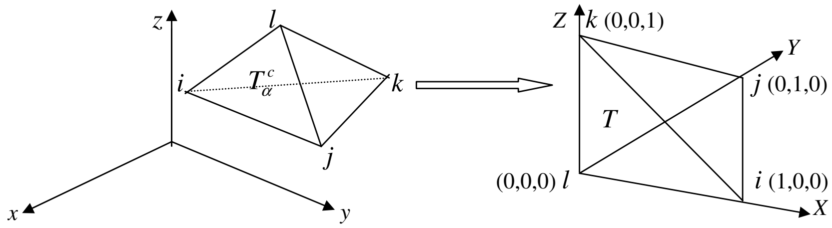

Fig. 2. Discretisation an arbitrary linear tetrahedron T_α in (x, y, z) space into a standard tetrahedron in (X, Y, Z) space.

Hence we can now use the affine transformation:

$$\begin{aligned} x^{(\alpha)} &= x_l + (x_i - x_l)X + (x_j - x_l)Y + (x_k - x_l)Z, \\ y^{(\alpha)} &= y_l + (y_i - y_l)X + (y_j - y_l)Y + (y_k - y_l)Z, \\ z^{(\alpha)} &= z_l + (z_i - z_l)X + (z_j - z_l)Y + (z_k - z_l)Z, \end{aligned} \tag{14}$$

which $(\alpha = 1, 2, 3, 4)$ transforms an arbitrary linear tetrahedron T_α in (x, y, z) space into a standard tetrahedron in (X, Y, Z) space as shown in Fig. 2.

Thus, on using the above affine transformation of Eq. (13), we obtain

$$\int\int_{T_k^c}\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{4}\int\int\int_T f(x^{(k)}(X,Y,Z), y^{(k)}(X,Y,Z), z^{(k)}(X,Y,Z))\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \quad (k = 1,2,3,4), \tag{15}$$

where

$$\begin{aligned} x^{(1)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y - \frac{1}{4}Z, \\ y^{(1)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X + \frac{3}{4}Y - \frac{1}{4}Z, \\ z^{(1)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y + \frac{3}{4}Z, \end{aligned} \tag{16}$$

$$\begin{aligned} x^{(2)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y + \frac{3}{4}Z, \\ y^{(2)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y - \frac{1}{4}Z, \\ z^{(2)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X + \frac{3}{4}Y - \frac{1}{4}Z, \end{aligned} \tag{17}$$

$$\begin{aligned} x^{(3)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X + \frac{3}{4}Y - \frac{1}{4}Z, \\ y^{(3)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y + \frac{3}{4}Z, \\ z^{(3)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y - \frac{1}{4}Z, \end{aligned} \tag{18}$$

$$\begin{aligned} x^{(4)}(X,Y,Z) &= \frac{1}{4} + \frac{3}{4}X - \frac{1}{4}Y - \frac{1}{4}Z, \\ y^{(4)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X + \frac{3}{4}Y - \frac{1}{4}Z, \\ z^{(4)}(X,Y,Z) &= \frac{1}{4} - \frac{1}{4}X - \frac{1}{4}Y + \frac{3}{4}Z. \end{aligned} \tag{19}$$

Hence on using the results of Eqs. (16)–(19) in Eq. (15) , we obtain

$$\int\int_{T_1^c}\int f(x^{(1)},y^{(1)},z^{(1)})\,\mathrm{d}x^{(1)}\,\mathrm{d}y^{(1)}\,\mathrm{d}z^{(1)}=\frac{1}{4}\int\int_T\int f(S(X,Y,Z),Q(X,Y,Z),R(X,Y,Z))\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{20}$$

$$\int\int_{T_2^c}\int f(x^{(2)},y^{(2)},z^{(2)})\,\mathrm{d}x^{(2)}\,\mathrm{d}y^{(2)}\,\mathrm{d}z^{(2)}=\frac{1}{4}\int\int_T\int f(R(X,Y,Z),S(X,Y,Z),Q(X,Y,Z))\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{21}$$

$$\int\int_{T_3^c}\int f(x^{(3)},y^{(3)},z^{(3)})\,\mathrm{d}x^{(3)}\,\mathrm{d}y^{(3)}\,\mathrm{d}z^{(3)}=\frac{1}{4}\int\int_T\int f(Q(X,Y,Z),R(X,Y,Z),S(X,Y,Z))\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{22}$$

$$\int\int_{T_4^c}\int f(x^{(4)},y^{(4)},z^{(4)})\,\mathrm{d}x^{(4)}\,\mathrm{d}y^{(4)}\,\mathrm{d}z^{(4)}=\frac{1}{4}\int\int_T\int f(P(X,Y,Z),Q(X,Y,Z),R(X,Y,Z))\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z. \tag{23}$$

Now, we can substitute the results of Eqs. (20)–(23) into Eq. (15) and obtain the following:

$$\int\int_T\int f(x,y,z)\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{4}\int\int_T\int G(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{24}$$

where

$$G(X,Y,Z)=\begin{bmatrix} f(S(X,Y,Z),Q(X,Y,Z),R(X,Y,Z)) \\ f(R(X,Y,Z),S(X,Y,Z),Q(X,Y,Z)) \\ f(Q(X,Y,Z),R(X,Y,Z),S(X,Y,Z)) \\ f(P(X,Y,Z),Q(X,Y,Z),R(X,Y,Z)) \end{bmatrix} \tag{25}$$

and

$$\begin{aligned} P(X,Y,Z)&=\frac{1}{4}+\frac{3}{4}X-\frac{1}{4}Y-\frac{1}{4}Z,\\ Q(X,Y,Z)&=\frac{1}{4}-\frac{1}{4}X+\frac{3}{4}Y-\frac{1}{4}Z,\\ R(X,Y,Z)&=\frac{1}{4}-\frac{1}{4}X-\frac{1}{4}Y+\frac{3}{4}Z,\\ S(X,Y,Z)&=\frac{1}{4}-\frac{1}{4}X-\frac{1}{4}Y-\frac{1}{4}Z. \end{aligned} \tag{26}$$

Now, we can apply the quadrature rule of Eqs. (11) and (12) to the Eq. (24) and thus obtain

$$\int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{4}\sum_{k=1}^{s\times s\times s}c_kG(x_k,y_k,z_k), \tag{27}$$

where

$$G(x_k,y_k,z_k)=f(S_k,Q_k,R_k)+f(R_k,S_k,Q_k)+f(Q_k,R_k,S_k)+f(P_k,Q_k,R_k),$$
$$P_k=P(x_k,y_k,z_k),\quad Q_k=Q(x_k,y_k,z_k),\quad R_k=R(x_k,y_k,z_k),\quad S_k=S(x_k,y_k,z_k),$$

and 's' refers to the order of the Gauss–Legendre quadrature rule and $(c_k,(x_k,y_k,z_k),\ k=1,2,3,\ldots,s^3)$ are the weight coefficients and sampling points. We have listed here a C- Program which generates P_k, Q_k, R_k and S_k. (c_k already listed in Section 2). We have also given a sample output of the C-Program for $n=2$ and 3.

3.1. C-Program for generating P_k, Q_k, R_k, S_k

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
```

```
int i, j, k, n;
double xk, yk, zk, ck, a[10], w[10], Pk, Qk, Rk, Sk;
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of alphas(a's) in order");
for(i=1; i<=n; i++)
scanf("%lf", &a[i]);
printf("Enter the values of weights(w's) in order");
for(i=1; i<=n; i++)
scanf("%lf", &w[i]);
printf("Pk      Qk      Rk      Sk\n");
for(i=1; i<=n; i++)
{for(j=1; j<=n; j++)
{for(k=1; k<=n; k++) {
xk=(1+a[i])/2; yk=(1-a[i])*(1+a[j])/4;
zk=(1-a[i])*(1-a[j])*(1+a[k])/8;
ck=(1-a[i])*(1-a[i])*(1-a[j])*w[i]*w[j]*w[k]/64;
Pk=(1+3*x-y-z)/4; Qk=(1-x+3*y-z)/4;
Rk=(1-x-y+3*z)/4; Sk=(1-x-y-z)/4;
printf("%0.15lf   %0.15lf   %0.15lf   %0.15lf   %0.15lf\n", Pk, Qk, Rk, Sk);
}}}
getch();
}
```

3.2. Sample output for $n = 2$ and 3

P_k	Q_k	R_k	S_k
$n = 2$			
0.333965518445773	0.289307319707253	0.254086508806388	0.122640653040586
0.244186329346638	0.199528130608117	0.523424076103795	0.032861463941451
0.244186329346638	0.654869931869597	0.068082274842315	0.032861463941451
0.220130068130403	0.630813670653362	0.140251058491018	0.008805202725216
0.821536598536263	0.077519662679971	0.068082274842315	0.032861463941451
0.797480337320029	0.053463401463737	0.140251058491018	0.008805202725216
0.797480337320029	0.175471869391883	0.018242590562872	0.008805202725216
0.791034481554227	0.169026013626081	0.037580157860278	0.002359346959414
$n = 3$			
0.287343790668925	0.274642125289667	0.263371958751741	0.174642125289667
0.134884123744777	0.122182458365519	0.720750959524186	0.022182458365519
0.211113957206851	0.198412291827593	0.492061459137963	0.098412291827593
0.134884123744777	0.809480792986260	0.033452624903444	0.022182458365519
0.115519207013740	0.790115876255223	0.091547375096556	0.002817541634481
0.125201665379259	0.799798334620741	0.062500000000000	0.012500000000000
0.211113957206851	0.542061459137963	0.148412291827593	0.098412291827593
0.125201665379259	0.456149167310371	0.406149167310371	0.012500000000000
0.168157811293055	0.499105313224167	0.277280729568982	0.055456145913796
0.909480792986260	0.034884123744777	0.033452624903444	0.022182458365519

(continued on next page)

Table (*continued*)

P_k	Q_k	R_k	S_k
0.890115876255223	0.015519207013740	0.091547375096556	0.002817541634481
0.899798334620742	0.025201665379258	0.062500000000000	0.012500000000000
0.890115876255223	0.102817541634482	0.004249040475814	0.002817541634481
0.887656209331075	0.100357874710333	0.011628041248259	0.000357874710333
0.888886042793149	0.101587708172407	0.007938540862036	0.001587708172407
0.899798334620742	0.068850832689629	0.018850832689629	0.012500000000000
0.888886042793149	0.057938540862037	0.051587708172407	0.001587708172407
0.894342188706945	0.063394686775833	0.035219270431018	0.007043854086204
0.598412291827593	0.154763124517222	0.148412291827593	0.098412291827593
0.512500000000000	0.068850832689629	0.406149167310371	0.012500000000000
0.555456145913796	0.111806978603426	0.277280729568982	0.055456145913796
0.512500000000000	0.456149167310371	0.018850832689629	0.012500000000000
0.501587708172407	0.445236875482778	0.051587708172407	0.001587708172407
0.507043854086204	0.450693021396574	0.035219270431018	0.007043854086204
0.555456145913796	0.305456145913796	0.083631562258611	0.055456145913796
0.507043854086204	0.257043854086204	0.228868437741389	0.007043854086204
0.531250000000000	0.281250000000000	0.156250000000000	0.031250000000000

4. Composite integration rule over the standard tetrahedron T, by a discretisation of T into p^3 tetrahedra

We can discretise the standard tetrahedron $T:\{(x,y,z)|0\leqslant x,y,z\leqslant 1, x+y+z\leqslant 1\}$ in (x,y,z) space into p^3 orthogonal tetrahedra each of volume $1/6\times(1/p\times 1/p\times 1/p)$. For example, by choosing $p=2$, we can discretise T into $2^3=8$ tetrahedra each of volume $1/6\times(1/2\times 1/2\times 1/2)$; and choosing $p=3$, we can discretise T into $3^3=27$ tetrahedra each of volume $1/6\times(1/3\times 1/3\times 1/3)$. We have developed here a discretisation procedure which works for composite integration rule with 8, 27, 64, 125, 216, 343 and 512 tetrahedra, i.e., we have described here a procedure in terms of parameter p, and by choosing $p=2,3,\ldots,8$ the discretisation of T into

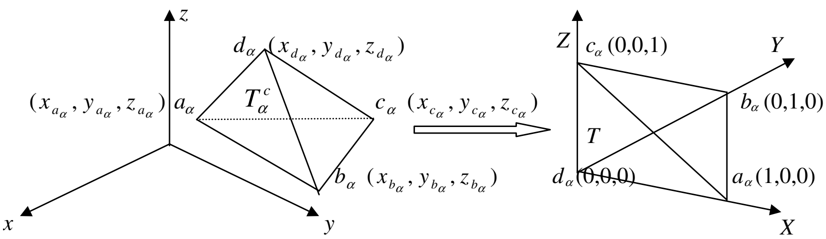

Fig. 3. Affine transformation which transforms $\widehat{T}_\alpha^{(p)}$ into a standard tetrahedron T.

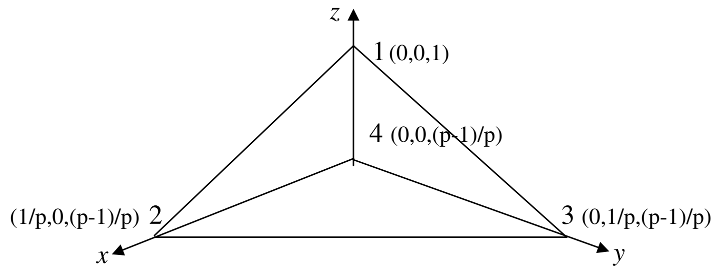

Fig. 4. Orthogonal tetrahedron $\widehat{T}_{1,p}$ of volume $1/6\times(1/p\times 1/p\times 1/p)$.

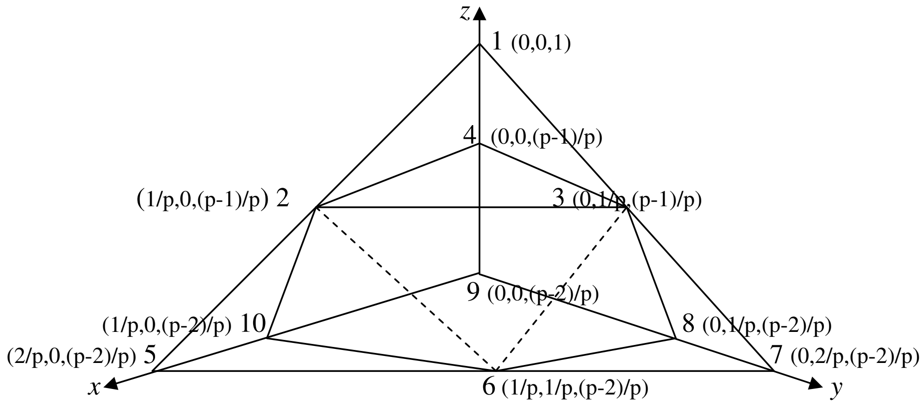

Fig. 5. Orthogonal tetrahedron $\widehat{T}_{2,p}$ of volume $2/6 \times (2/p \times 2/p \times 2/p)$.

smaller tetrahedra of equal size upto 512 is generated. We consider here the discretisation of $\widehat{T}_{k,p} : \{(x, y, z) | 0 \leqslant x, y, z \leqslant \frac{k}{p}, x + y + z \leqslant 1\}$, for $k = 1, 2, 3, \ldots, 8$. We have now for $k = 1$, $\widehat{T}_{1,p}$, a tetrahedron of volume $1/6 \times (1/p \times 1/p \times 1/p)$ which is shown in Fig. 4. We have for $k = 2$, $\widehat{T}_{2,p}$, a tetrahedron of volume $1/6 \times (2/p \times 2/p \times 2/p)$ which can be further discretised into $2^3 = 8$ tetrahedra of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$ and this is depicted in Fig. 5. We have for $k = 3$, $\widehat{T}_{3,p}$, a tetrahedron of volume $1/6 \times (3/p \times 3/p \times 3/p)$ which can be further discretised into $3^3 = 27$ tetrahedra of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$ and this is depicted in Fig. 6. We observe that the depiction of $\widehat{T}_{k,p}$, for $k = 4, 5, \ldots, 8$ is really complicated. It is interesting to note that $\widehat{T}_{\alpha,p} \subset \widehat{T}_{\beta,p}$ for $\alpha < \beta$, and α, β as integers. This implies that $\widehat{T}_{1,p} \subset \widehat{T}_{2,p} \subset \widehat{T}_{3,p} \subset \widehat{T}_{4,p} \subset \cdots \widehat{T}_{8,p}$. Further, we note that $\widehat{T}_{k,p} = T$, for $k = p$. These properties can be used to our advantage. We also see that depicting $\widehat{T}_{k,p}$ for $k > 3$ becomes complicated with each increasing k value. We have $\widehat{T}_{p,p} = T$, and it can be discretised into p^3 tetrahedra each of equal volume $1/6 \times (1/p \times 1/p \times 1/p)$. Let us denote $T^{(p)}_{\alpha}$, a tetrahedron with index of α having volume $1/6 \times (1/p \times 1/p \times 1/p)$. Clearly, we have $T = \widehat{T}_{p,p} = \sum_{\alpha=1}^{p^3} T^{(p)}_{\alpha}$ We can transform each of these tetrahedra $T^{(p)}_{\alpha}$, into a unit orthogonal tetrahedron T by use of the well known affine transformations:

$$
\begin{aligned}
x^{(\alpha,p)}(X,Y,Z) &= x_{d_\alpha} + (x_{a_\alpha} - x_{d_\alpha})X + (x_{b_\alpha} - x_{d_\alpha})Y + (x_{c_\alpha} - x_{d_\alpha})Z, \\
y^{(\alpha,p)}(X,Y,Z) &= y_{d_\alpha} + (y_{a_\alpha} - y_{d_\alpha})X + (y_{b_\alpha} - y_{d_\alpha})Y + (y_{c_\alpha} - y_{d_\alpha})Z, \\
z^{(\alpha,p)}(X,Y,Z) &= z_{d_\alpha} + (z_{a_\alpha} - z_{d_\alpha})X + (z_{b_\alpha} - z_{d_\alpha})Y + (z_{c_\alpha} - z_{d_\alpha})Z \quad (\alpha = 1, 2, \ldots, p^3),
\end{aligned}
\tag{28}
$$

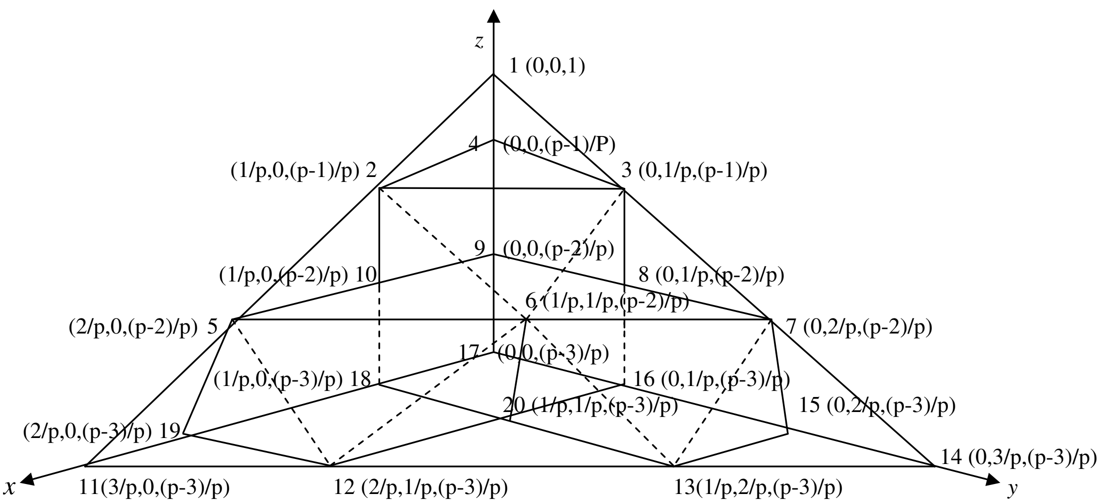

Fig. 6. Orthogonal tetrahedron $\widehat{T}_{3,p}$ of volume $3/6 \times (3/p \times 3/p \times 3/p)$.

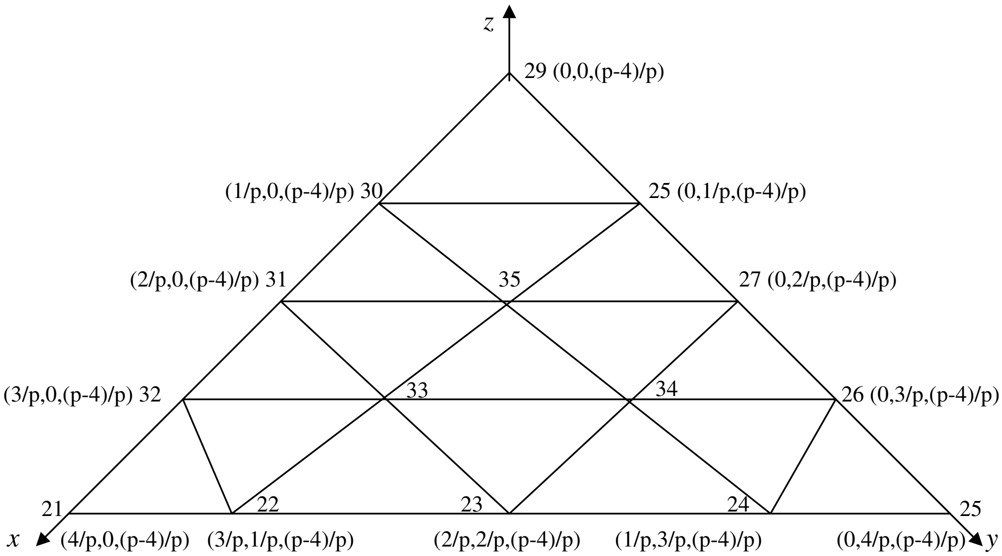

Fig. 7. Base triangle on $z = (p-4)/p$ for an orthogonal tetrahedron $\widehat{T}_{4,p}$ of volume $1/6 \times (4/p \times 4/p \times 4/p)$.

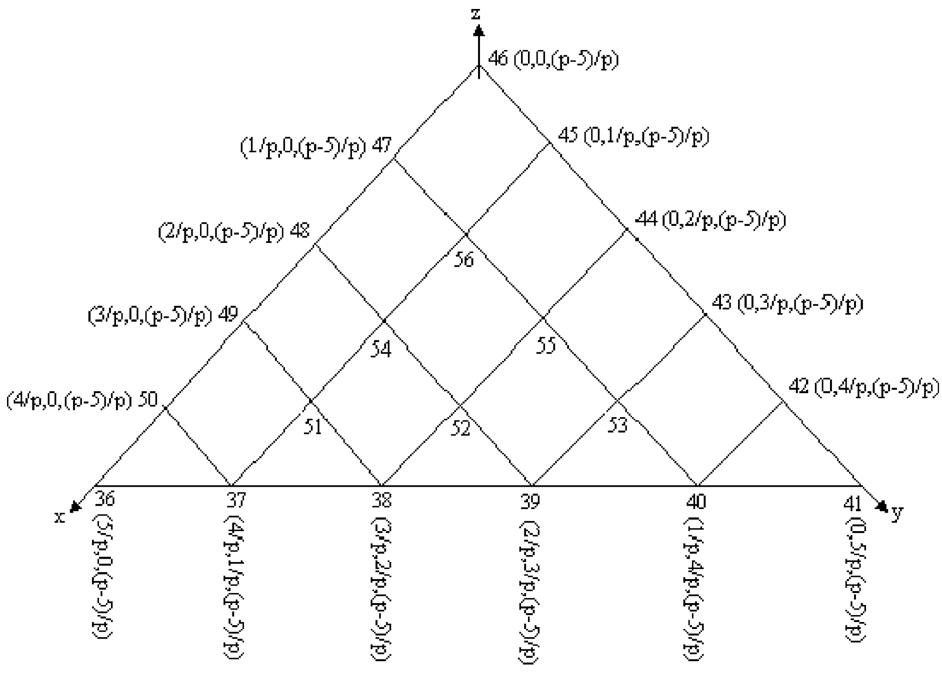

Fig. 8. Base triangle on $z = (p-5)/p$ for an orthogonal tetrahedron $\widehat{T}_{5,p}$ of volume $1/6 \times (5/p \times 5/p \times 5/p)$.

where $(a_\alpha, b_\alpha, c_\alpha, d_\alpha)$ are the nodes spanning four vertices of the $T_\alpha^{(p)}$, this information is listed for $T_\alpha^{(p)}$, $(\alpha = 1, 2, \ldots, 512)$, $p = 2, 3, \ldots, 8$ and this information is depicted in Fig. 3

The discretisation of $\widehat{T}_{k,p}$, $(k = 2, 3, \ldots, 8)$ consists of cubes, triangular prisms and orthogonal tetrahedra. Hence, one has further discretise the triangular prisms and cubes into orthogonal tetrahedra and each of these are to be of volume $1/6 \times (1/p \times 1/p \times 1/p)$. The procedure adopted to subdivide the triangular prisms and cubes can be found in Zienkiewicz [1], Chandrupatla and Belegundu [27]. This is explained here:

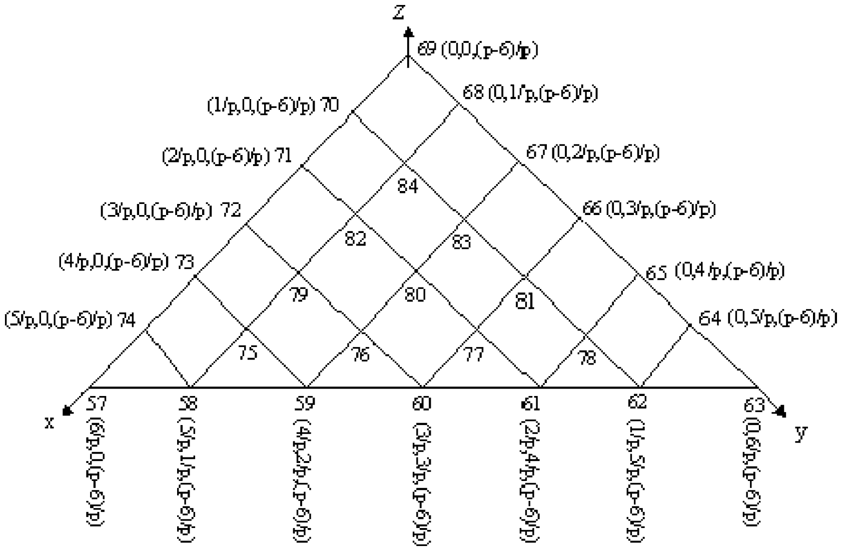

Fig. 9. Base triangle on $z = (p - 6)/p$ for an orthogonal tetrahedron $\widehat{T}_{6,p}$ of volume $1/6 \times (6/p \times 6/p \times 6/p)$.

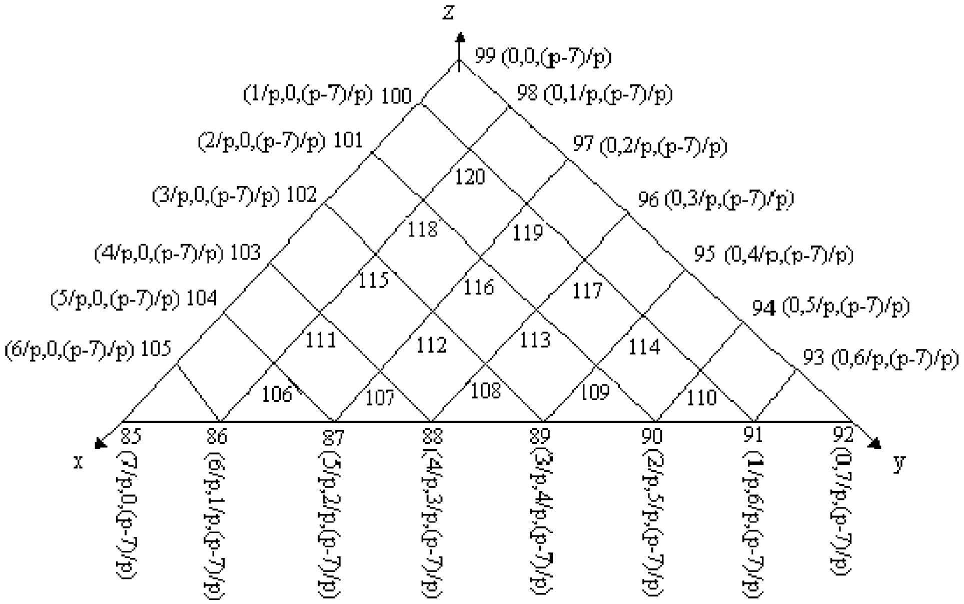

Fig. 10. Base triangle on $z = (p - 7)/p$ for an orthogonal tetrahedron $\widehat{T}_{7,p}$ of volume $1/6 \times (7/p \times 7/p \times 7/p)$.

5. Division of a cube into two triangular prisms

We consider here a cube spanned by nodes $\langle i,j,k,l,m,n,o,p \rangle$. Fig. 12 is self explanatory:

6. Division of a triangular prism into three tetrahedra

We consider here a triangular prism spanned by vertices: $\langle i,j,k,l,m,n \rangle$. Fig. 13 is self explanatory:

From the above two figures: Figs. 12 and 13, it is clear that a cube can be subdivided into six tetrahedra of equal size. Let the cube of Fig. 12 be denoted by C and the resulting tetrahedra be denoted by T_i, then $C = \sum_{i=1}^{6} T_i$. These tetrahedra are spanned by four vertices. Table 1 describes this spanning.

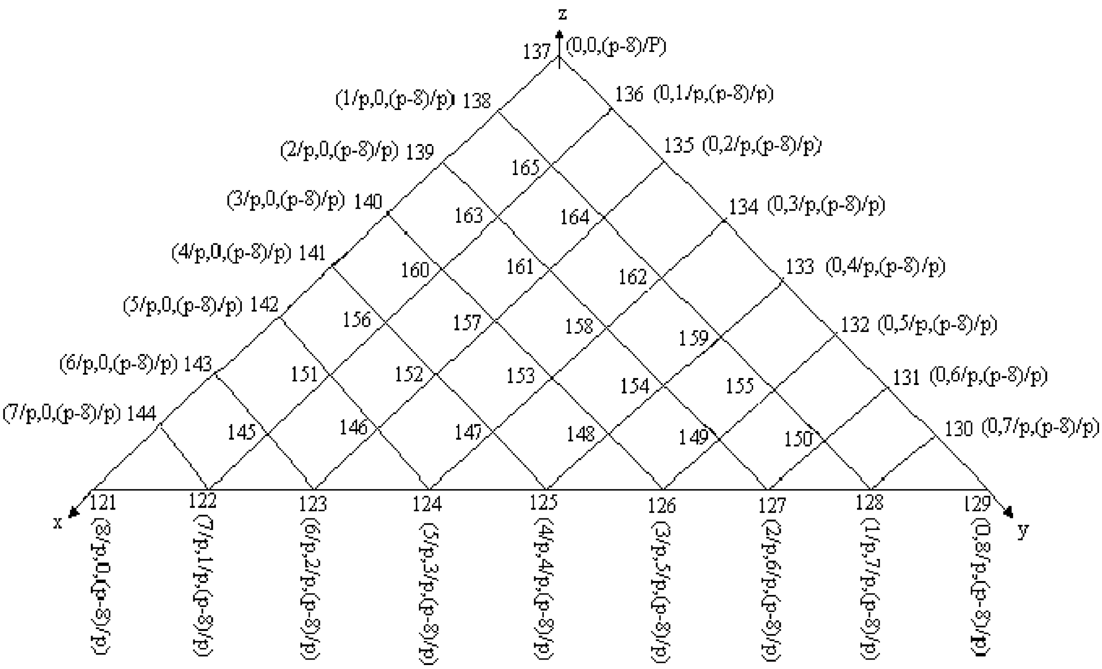

Fig. 11. Base triangle on $z = (p-8)/p$ for an orthogonal tetrahedron $\widehat{T}_{8,p}$ of volume $1/6 \times (8/p \times 8/p \times 8/p)$.

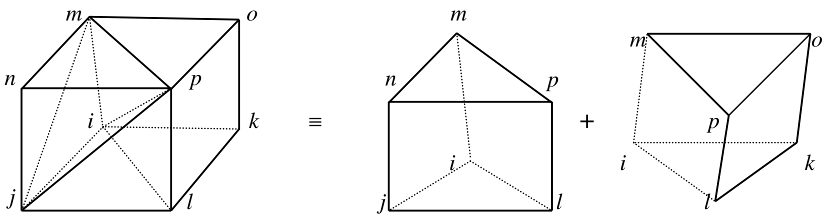

Fig. 12. Subdivision of a cube into two triangular prisms.

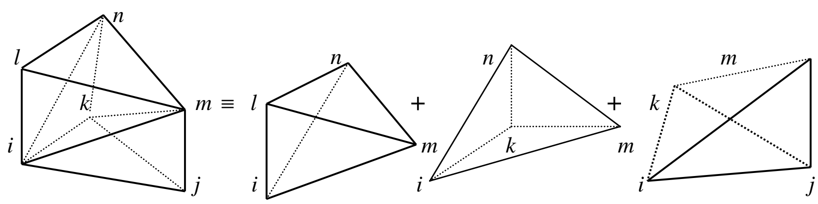

Fig. 13. Subdivision of a triangular prism into three tetrahedra.

Table 1
Division of a cube spanned by vertices $\langle i,j,k,l,m,n,o,p\rangle$ into six tetrahedra

Tetrahedra (T_i)	Local nodes spanning the tetrahedron			
	1	2	3	4
T_1	i	j	l	p
T_2	i	j	p	m
T_3	j	p	m	n
T_4	i	k	l	o
T_5	i	o	p	m
T_6	i	p	l	o

On using the above discretisation procedure explained in Figs. 4–11 and the method of subdivision of triangular prisms and cubes as explained in Figs. 12 and 13, the affine transformations of Eq. (28) and the linearity property of integrals, we obtain:

$$\begin{aligned}\mathrm{I} &= \int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1\int_0^{1-x}\int_0^{1-x-y} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \int\int_{T=\sum_{\alpha=1}^{p^3}T_\alpha^{(p)}}\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_{\alpha=1}^{p^3}\int\int_{T_\alpha^p}\int f(x^{(\alpha,p)},y^{(\alpha,p)},z^{(\alpha,p)})\,\mathrm{d}x^{(\alpha,p)}\,\mathrm{d}y^{(\alpha,p)}\,\mathrm{d}z^{(\alpha,p)}\\ &= \sum_{\alpha=1}^{p^3}\int\int_T\int f(x^{(\alpha,p)}(X,Y,Z),y^{(\alpha,p)}(X,Y,Z),z^{(\alpha,p)}(X,Y,Z))\left|\frac{\partial(x^{(\alpha,p)},y^{(\alpha,p)},z^{(\alpha,p)})}{\partial(X,Y,Z)}\right|\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z. \end{aligned} \tag{29}$$

We have tabulated the expressions for nodal vertices spanning $T_\alpha{}^{(p)}\ \langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle, \alpha=1,2,\ldots,8^3$ in Table 2, which are valid for $p=2,3,4,5,6,7$ and 8.

Computation of $(x^{(\alpha,p)}(X,Y,Z),\ y^{(\alpha,p)}(X,Y,Z),\ z^{(\alpha,p)}(X,Y,Z))$:

We shall illustrate the above computation.

We have from Table 2, the first two entries are noted as $T_1^{(p)}\langle 2,3,1,4\rangle$ and $T_2^{(p)}\angle 5,6,2,10\rangle$, from this we find for $\alpha=1$, $a_1=2$, $b_1=3$, $c_1=1$, $d_1=4$ and for $\alpha=2$, $a_2=5$, $b_2=6$, $c_2=2$, $d_2=10$.

We have from Eq. (28), for $\alpha=1$ and $\alpha=2$

$$\begin{aligned}&x^{(1,p)}(X,Y,Z)=x_4+(x_2-x_4)X+(x_3-x_4)Y+(x_1-x_4)Z,\\&y^{(1,p)}(X,Y,Z)=y_4+(y_2-y_4)X+(y_3-y_4)Y+(y_1-y_4)Z,\\&z^{(1,p)}(X,Y,Z)=z_4+(z_2-z_4)X+(z_3-z_4)Y+(z_1-z_4)Z,\end{aligned}\tag{30a}$$

$$\begin{aligned}&x^{(2,p)}(X,Y,Z)=x_{10}+(x_5-x_{10})X+(x_6-x_{10})Y+(x_2-x_{10})Z,\\&y^{(2,p)}(X,Y,Z)=y_{10}+(y_5-y_{10})X+(y_6-y_{10})Y+(y_2-y_{10})Z,\\&z^{(2,p)}(X,Y,Z)=z_{10}+(z_5-z_{10})X+(z_6-z_{10})Y+(z_2-z_{10})Z.\end{aligned}\tag{30b}$$

We have from Figs. 3 and 11, the nodal coordinates are given by

$$\begin{aligned}&x_1=0,\quad y_1=0,\quad z_1=1,\quad x_2=1/p,\quad y_2=0,\quad z_2=(p-1)/p,\quad x_3=0,\quad y_3=1/p,\\&z_3=(p-1)/p,\quad x_4=0,\quad y_4=0,\quad z_4=(p-1)/p,\quad x_5=2/p,\quad y_5=0,\quad z_5=(p-2)/p,\\&x_6=1/p,\quad y_6=1/p,\quad z_6=(p-2)/p,\quad x_{10}=1/p,\quad y_{10}=0,\quad z_{10}=(p-2)/p.\end{aligned}\tag{31}$$

Using the values of $((x_i,y_i,z_i),i=1,2,3,4,5,6,10)$ from the above Eq. (31) into the Eq. (30), we find

$$\begin{aligned}&(x^{(1,p)}(X,Y,Z),y^{(1,p)}(X,Y,Z),z^{(1,p)}(X,Y,Z))=(X/p,Y/p,(p-1)/p+Z/p),\\&(x^{(2,p)}(X,Y,Z),y^{(2,p)}(X,Y,Z),z^{(2,p)}(X,Y,Z))=(1/p+X/p,Y/p,(p-2)/p+Z/p).\end{aligned}$$

We can compute the remaining expressions for $(x^{(\alpha,p)}(X,Y,Z),\ y^{(\alpha,p)}(X,Y,Z),\ z^{(\alpha,p)}(X,Y,Z))$ from the values $T_\alpha^{(p)}\langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle$ of Table 2.

We can further write the Eq. (29) as

$$I=\int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\frac{1}{p^3}\int\int_T\int H(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z,\tag{32}$$

where

$$H(X,Y,Z)=\sum_{\alpha=1}^{p^3}f(x^{(\alpha,p)}(X,Y,Z),y^{(\alpha,p)}(X,Y,Z),z^{(\alpha,p)}(X,Y,Z)).\tag{33}$$

We can now apply Gauss–Legendre quadrature rules on the integral of Eq. (32) in a manner similar to the procedure which we have already developed for the integral $\mathrm{I}=\int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Following the method already developed in Section 2, we have now on using the transformations

Table 2
Nodal vertices spanning $T_\alpha^{(p)}\langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle$, $\alpha=1,2,\ldots,8^3$

$T_\alpha^{(p)}\langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle$	$T_\alpha^{(p)}\langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle$	$T_\alpha^{(p)}\langle a_\alpha,b_\alpha,c_\alpha,d_\alpha\rangle$
$T_{1}^{(p)}\langle 2,3,1,4\rangle$	$T_{2}^{(p)}\langle 5,6,2,10\rangle$	$T_{3}^{(p)}\langle 6,7,3,8\rangle$
$T_{4}^{(p)}\langle 10,6,2,3\rangle$	$T_{5}^{(p)}\langle 10,6,3,8\rangle$	$T_{6}^{(p)}\langle 3,4,10,2\rangle$
$T_{7}^{(p)}\langle 10,3,4,9\rangle$	$T_{8}^{(p)}\langle 9,10,3,8\rangle$	$T_{9}^{(p)}\langle 11,12,5,19\rangle$
$T_{10}^{(p)}\langle 19,12,5,6\rangle$	$T_{11}^{(p)}\langle 19,12,6,20\rangle$	$T_{12}^{(p)}\langle 12,13,6,20\rangle$
$T_{13}^{(p)}\langle 20,13,6,7\rangle$	$T_{14}^{(p)}\langle 20,13,7,15\rangle$	$T_{15}^{(p)}\langle 13,14,7,15\rangle$
$T_{16}^{(p)}\langle 6,10,19,5\rangle$	$T_{17}^{(p)}\langle 19,6,10,18\rangle$	$T_{18}^{(p)}\langle 18,19,6,20\rangle$
$T_{19}^{(p)}\langle 7,8,20,6\rangle$	$T_{20}^{(p)}\langle 20,7,8,16\rangle$	$T_{21}^{(p)}\langle 16,20,7,15\rangle$
$T_{22}^{(p)}\langle 17,18,20,6\rangle$	$T_{23}^{(p)}\langle 17,18,6,9\rangle$	$T_{24}^{(p)}\langle 18,6,9,10\rangle$
$T_{25}^{(p)}\langle 17,16,20,8\rangle$	$T_{26}^{(p)}\langle 17,8,6,9\rangle$	$T_{27}^{(p)}\langle 17,6,20,8\rangle$
$T_{28}^{(p)}\langle 21,22,11,32\rangle$	$T_{29}^{(p)}\langle 22,23,12,33\rangle$	$T_{30}^{(p)}\langle 23,24,13,34\rangle$
$T_{31}^{(p)}\langle 24,25,14,26\rangle$	$T_{32}^{(p)}\langle 32,22,11,12\rangle$	$T_{33}^{(p)}\langle 32,22,12,33\rangle$
$T_{34}^{(p)}\langle 33,23,12,13\rangle$	$T_{35}^{(p)}\langle 33,23,13,34\rangle$	$T_{36}^{(p)}\langle 34,24,13,14\rangle$
$T_{37}^{(p)}\langle 34,24,14,26\rangle$	$T_{38}^{(p)}\langle 12,19,32,11\rangle$	$T_{39}^{(p)}\langle 32,12,19,31\rangle$
$T_{40}^{(p)}\langle 31,32,12,33\rangle$	$T_{41}^{(p)}\langle 13,20,33,12\rangle$	$T_{42}^{(p)}\langle 33,13,20,35\rangle$
$T_{43}^{(p)}\langle 35,33,13,34\rangle$	$T_{44}^{(p)}\langle 14,15,34,13\rangle$	$T_{45}^{(p)}\langle 34,14,15,27\rangle$
$T_{46}^{(p)}\langle 27,34,14,26\rangle$	$T_{47}^{(p)}\langle 30,31,33,12\rangle$	$T_{48}^{(p)}\langle 30,31,12,18\rangle$
$T_{49}^{(p)}\langle 31,12,18,19\rangle$	$T_{50}^{(p)}\langle 30,35,33,20\rangle$	$T_{51}^{(p)}\langle 30,20,12,18\rangle$
$T_{52}^{(p)}\langle 30,12,33,20\rangle$	$T_{53}^{(p)}\langle 29,30,35,20\rangle$	$T_{54}^{(p)}\langle 29,30,20,17\rangle$
$T_{55}^{(p)}\langle 30,20,17,18\rangle$	$T_{56}^{(p)}\langle 29,28,35,16\rangle$	$T_{57}^{(p)}\langle 29,16,20,17\rangle$
$T_{58}^{(p)}\langle 29,20,35,16\rangle$	$T_{59}^{(p)}\langle 28,35,34,13\rangle$	$T_{60}^{(p)}\langle 28,35,13,16\rangle$
$T_{61}^{(p)}\langle 35,13,16,20\rangle$	$T_{62}^{(p)}\langle 28,27,34,15\rangle$	$T_{63}^{(p)}\langle 28,15,13,16\rangle$
$T_{64}^{(p)}\langle 28,13,34,15\rangle$	$T_{65}^{(p)}\langle 36,37,21,50\rangle$	$T_{66}^{(p)}\langle 37,38,22,51\rangle$
$T_{67}^{(p)}\langle 38,39,23,52\rangle$	$T_{68}^{(p)}\langle 39,40,24,53\rangle$	$T_{69}^{(p)}\langle 40,41,25,42\rangle$
$T_{70}^{(p)}\langle 50,37,21,22\rangle$	$T_{71}^{(p)}\langle 50,37,22,51\rangle$	$T_{72}^{(p)}\langle 51,38,22,23\rangle$
$T_{73}^{(p)}\langle 51,38,23,52\rangle$	$T_{74}^{(p)}\langle 52,39,23,24\rangle$	$T_{75}^{(p)}\langle 52,39,24,53\rangle$
$T_{76}^{(p)}\langle 53,40,24,25\rangle$	$T_{77}^{(p)}\langle 53,40,25,42\rangle$	$T_{78}^{(p)}\langle 22,32,50,21\rangle$
$T_{79}^{(p)}\langle 50,22,32,49\rangle$	$T_{80}^{(p)}\langle 49,50,22,51\rangle$	$T_{81}^{(p)}\langle 23,33,51,22\rangle$
$T_{82}^{(p)}\langle 51,23,33,54\rangle$	$T_{83}^{(p)}\langle 54,51,23,52\rangle$	$T_{84}^{(p)}\langle 24,34,52,23\rangle$
$T_{85}^{(p)}\langle 52,24,34,55\rangle$	$T_{86}^{(p)}\langle 55,52,24,53\rangle$	$T_{87}^{(p)}\langle 25,26,53,24\rangle$
$T_{88}^{(p)}\langle 53,25,26,43\rangle$	$T_{89}^{(p)}\langle 43,53,25,42\rangle$	$T_{90}^{(p)}\langle 48,49,51,22\rangle$
$T_{91}^{(p)}\langle 48,49,22,31\rangle$	$T_{92}^{(p)}\langle 49,22,31,32\rangle$	$T_{93}^{(p)}\langle 48,54,51,33\rangle$
$T_{94}^{(p)}\langle 48,33,22,31\rangle$	$T_{95}^{(p)}\langle 48,22,51,33\rangle$	$T_{96}^{(p)}\langle 47,48,54,33\rangle$
$T_{97}^{(p)}\langle 47,48,33,30\rangle$	$T_{98}^{(p)}\langle 48,33,30,31\rangle$	$T_{99}^{(p)}\langle 47,56,54,35\rangle$
$T_{100}^{(p)}\langle 47,35,33,30\rangle$	$T_{101}^{(p)}\langle 47,33,54,35\rangle$	$T_{102}^{(p)}\langle 46,47,56,35\rangle$
$T_{103}^{(p)}\langle 46,47,35,29\rangle$	$T_{104}^{(p)}\langle 47,35,29,30\rangle$	$T_{105}^{(p)}\langle 46,45,56,28\rangle$
$T_{106}^{(p)}\langle 46,28,35,29\rangle$	$T_{107}^{(p)}\langle 46,35,56,28\rangle$	$T_{108}^{(p)}\langle 45,56,55,34\rangle$
$T_{109}^{(p)}\langle 45,56,34,28\rangle$	$T_{110}^{(p)}\langle 56,34,28,35\rangle$	$T_{111}^{(p)}\langle 45,44,55,27\rangle$
$T_{112}^{(p)}\langle 45,27,34,28\rangle$	$T_{113}^{(p)}\langle 45,34,55,27\rangle$	$T_{114}^{(p)}\langle 44,55,53,24\rangle$
$T_{115}^{(p)}\langle 44,55,24,27\rangle$	$T_{116}^{(p)}\langle 55,24,27,34\rangle$	$T_{117}^{(p)}\langle 44,43,53,26\rangle$
$T_{118}^{(p)}\langle 44,26,24,27\rangle$	$T_{119}^{(p)}\langle 44,24,53,26\rangle$	$T_{120}^{(p)}\langle 56,54,52,23\rangle$
$T_{121}^{(p)}\langle 56,54,23,35\rangle$	$T_{122}^{(p)}\langle 54,23,35,33\rangle$	$T_{123}^{(p)}\langle 56,55,52,34\rangle$
$T_{124}^{(p)}\langle 56,34,23,35\rangle$	$T_{125}^{(p)}\langle 56,23,52,34\rangle$	$T_{126}^{(p)}\langle 57,58,36,74\rangle$
$T_{127}^{(p)}\langle 74,58,36,37\rangle$	$T_{128}^{(p)}\langle 74,58,37,75\rangle$	$T_{129}^{(p)}\langle 58,59,37,75\rangle$

Table 2 (*continued*)

$T_\alpha^{(p)}\langle a_\alpha, b_\alpha, c_\alpha, d_\alpha\rangle$	$T_\alpha^{(p)}\langle a_\alpha, b_\alpha, c_\alpha, d_\alpha\rangle$	$T_\alpha^{(p)}\langle a_\alpha, b_\alpha, c_\alpha, d_\alpha\rangle$
$T_{130}^{(p)}\langle 75,59,37,38\rangle$	$T_{131}^{(p)}\langle 75,59,38,76\rangle$	$T_{132}^{(p)}\langle 59,60,38,76\rangle$
$T_{133}^{(p)}\langle 76,60,38,39\rangle$	$T_{134}^{(p)}\langle 76,60,39,77\rangle$	$T_{135}^{(p)}\langle 60,61,39,77\rangle$
$T_{136}^{(p)}\langle 77,61,39,40\rangle$	$T_{137}^{(p)}\langle 77,61,40,78\rangle$	$T_{138}^{(p)}\langle 61,62,40,78\rangle$
$T_{139}^{(p)}\langle 78,62,40,41\rangle$	$T_{140}^{(p)}\langle 78,62,41,64\rangle$	$T_{141}^{(p)}\langle 62,63,41,64\rangle$
$T_{142}^{(p)}\langle 37,50,74,36\rangle$	$T_{143}^{(p)}\langle 74,37,50,73\rangle$	$T_{144}^{(p)}\langle 73,74,37,75\rangle$
$T_{145}^{(p)}\langle 38,51,,75,37\rangle$	$T_{146}^{(p)}\langle 75,38,51,79\rangle$	$T_{147}^{(p)}\langle 79,75,38,76\rangle$
$T_{148}^{(p)}\langle 39,52,76,38\rangle$	$T_{149}^{(p)}\langle 76,39,52,80\rangle$	$T_{150}^{(p)}\langle 80,76,39,77\rangle$
$T_{151}^{(p)}\langle 40,53,77,39\rangle$	$T_{152}^{(p)}\langle 77,40,53,81\rangle$	$T_{153}^{(p)}\langle 81,77,40,78\rangle$
$T_{154}^{(p)}\langle 41,42,78,40\rangle$	$T_{155}^{(p)}\langle 78,41,42,65\rangle$	$T_{156}^{(p)}\langle 65,78,41,64\rangle$
$T_{157}^{(p)}\langle 72,73,75,37\rangle$	$T_{158}^{(p)}\langle 72,73,37,49\rangle$	$T_{159}^{(p)}\langle 73,37,49,50\rangle$
$T_{160}^{(p)}\langle 72,79,75,51\rangle$	$T_{161}^{(p)}\langle 72,51,37,49\rangle$	$T_{162}^{(p)}\langle 72,37,75,51\rangle$
$T_{163}^{(p)}\langle 71,72,79,51\rangle$	$T_{164}^{(p)}\langle 71,72,51,48\rangle$	$T_{165}^{(p)}\langle 72,51,48,49\rangle$
$T_{166}^{(p)}\langle 71,82,79,54\rangle$	$T_{167}^{(p)}\langle 71,54,51,48\rangle$	$T_{168}^{(p)}\langle 71,51,79,54\rangle$
$T_{169}^{(p)}\langle 70,71,82,54\rangle$	$T_{170}^{(p)}\langle 70,71,54,47\rangle$	$T_{171}^{(p)}\langle 71,54,47,48\rangle$
$T_{172}^{(p)}\langle 70,84,82,56\rangle$	$T_{173}^{(p)}\langle 70,56,54,47\rangle$	$T_{174}^{(p)}\langle 70,54,82,56\rangle$
$T_{175}^{(p)}\langle 69,70,84,56\rangle$	$T_{176}^{(p)}\langle 69,70,56,46\rangle$	$T_{177}^{(p)}\langle 70,56,46,47\rangle$
$T_{178}^{(p)}\langle 69,68,84,45\rangle$	$T_{179}^{(p)}\langle 69,45,56,46\rangle$	$T_{180}^{(p)}\langle 69,56,84,45\rangle$
$T_{181}^{(p)}\langle 82,79,76,38\rangle$	$T_{182}^{(p)}\langle 82,79,38,54\rangle$	$T_{183}^{(p)}\langle 79,38,54,51\rangle$
$T_{184}^{(p)}\langle 82,80,76,52\rangle$	$T_{185}^{(p)}\langle 82,52,38,54\rangle$	$T_{186}^{(p)}\langle 82,38,76,52\rangle$
$T_{187}^{(p)}\langle 84,82,80,52\rangle$	$T_{188}^{(p)}\langle 84,82,52,56\rangle$	$T_{189}^{(p)}\langle 82,52,56,54\rangle$
$T_{190}^{(p)}\langle 84,83,80,55\rangle$	$T_{191}^{(p)}\langle 84,55,52,56\rangle$	$T_{192}^{(p)}\langle 84,52,80,55\rangle$
$T_{193}^{(p)}\langle 68,84,83,55\rangle$	$T_{194}^{(p)}\langle 68,84,55,45\rangle$	$T_{195}^{(p)}\langle 84,55,45,56\rangle$
$T_{196}^{(p)}\langle 68,67,83,44\rangle$	$T_{197}^{(p)}\langle 68,44,55,45\rangle$	$T_{198}^{(p)}\langle 68,55,83,44\rangle$
$T_{199}^{(p)}\langle 83,80,77,39\rangle$	$T_{200}^{(p)}\langle 83,80,39,55\rangle$	$T_{201}^{(p)}\langle 80,39,55,52\rangle$
$T_{202}^{(p)}\langle 83,81,77,53\rangle$	$T_{203}^{(p)}\langle 83,53,39,55\rangle$	$T_{204}^{(p)}\langle 83,39,77,53\rangle$
$T_{205}^{(p)}\langle 67,83,81,53\rangle$	$T_{206}^{(p)}\langle 67,83,53,44\rangle$	$T_{207}^{(p)}\langle 83,53,44,55\rangle$
$T_{208}^{(p)}\langle 67,66,81,43\rangle$	$T_{209}^{(p)}\langle 67,43,53,44\rangle$	$T_{210}^{(p)}\langle 67,53,81,43\rangle$
$T_{211}^{(p)}\langle 66,81,78,40\rangle$	$T_{212}^{(p)}\langle 66,81,40,43\rangle$	$T_{213}^{(p)}\langle 81,40,43,53\rangle$
$T_{214}^{(p)}\langle 66,65,78,42\rangle$	$T_{215}^{(p)}\langle 66,42,40,43\rangle$	$T_{216}^{(p)}\langle 66,40,78,42\rangle$
$T_{217}^{(p)}\langle 85,86,57,105\rangle$	$T_{218}^{(p)}\langle 105,86,57,58\rangle$	$T_{219}^{(p)}\langle 105,86,58,106\rangle$
$T_{220}^{(p)}\langle 86,87,58,106\rangle$	$T_{221}^{(p)}\langle 106,87,58,59\rangle$	$T_{222}^{(p)}\langle 106,87,59,107\rangle$
$T_{223}^{(p)}\langle 87,88,59,107\rangle$	$T_{224}^{(p)}\langle 107,88,59,60\rangle$	$T_{225}^{(p)}\langle 107,88,60,108\rangle$
$T_{226}^{(p)}\langle 88,89,60,108\rangle$	$T_{227}^{(p)}\langle 108,89,60,61\rangle$	$T_{228}^{(p)}\langle 108,89,61,109\rangle$
$T_{229}^{(p)}\langle 89,90,61,109\rangle$	$T_{230}^{(p)}\langle 109,90,61,62\rangle$	$T_{231}^{(p)}\langle 109,90,62,110\rangle$
$T_{232}^{(p)}\langle 90,91,62,110\rangle$	$T_{233}^{(p)}\langle 110,91,62,63\rangle$	$T_{234}^{(p)}\langle 110,91,63,93\rangle$
$T_{235}^{(p)}\langle 91,92,63,93\rangle$	$T_{236}^{(p)}\langle 58,74,105,57\rangle$	$T_{237}^{(p)}\langle 105,58,74,104\rangle$
$T_{238}^{(p)}\langle 104,105,58,106\rangle$	$T_{239}^{(p)}\langle 59,75,106,58\rangle$	$T_{240}^{(p)}\langle 106,59,75,111\rangle$
$T_{241}^{(p)}\langle 111,106,59,107\rangle$	$T_{242}^{(p)}\langle 60,76,107,59\rangle$	$T_{243}^{(p)}\langle 107,60,76,112\rangle$
$T_{244}^{(p)}\langle 112,107,60,108\rangle$	$T_{245}^{(p)}\langle 61,77,108,60\rangle$	$T_{246}^{(p)}\langle 108,61,77,113\rangle$
$T_{247}^{(p)}\langle 113,108,61,109\rangle$	$T_{248}^{(p)}\langle 62,78,109,61\rangle$	$T_{249}^{(p)}\langle 109,62,78,114\rangle$
$T_{250}^{(p)}\langle 114,109,62,110\rangle$	$T_{251}^{(p)}\langle 63,64,110,62\rangle$	$T_{252}^{(p)}\langle 110,63,64,94\rangle$
$T_{253}^{(p)}\langle 94,110,63,93\rangle$	$T_{254}^{(p)}\langle 103,104,106,58\rangle$	$T_{255}^{(p)}\langle 103,104,58,73\rangle$
$T_{256}^{(p)}\langle 104,58,73,74\rangle$	$T_{257}^{(p)}\langle 103,111,106,75\rangle$	$T_{258}^{(p)}\langle 103,75,58,73\rangle$
$T_{259}^{(p)}\langle 103,58,106,75\rangle$	$T_{260}^{(p)}\langle 102,103,111,75\rangle$	$T_{261}^{(p)}\langle 102,103,75,72\rangle$

(*continued on next page*)

Table 2 (*continued*)

$T_{\alpha}^{(p)}\langle a_{\alpha},b_{\alpha},c_{\alpha},d_{\alpha}\rangle$	$T_{\alpha}^{(p)}\langle a_{\alpha},b_{\alpha},c_{\alpha},d_{\alpha}\rangle$	$T_{\alpha}^{(p)}\langle a_{\alpha},b_{\alpha},c_{\alpha},d_{\alpha}\rangle$
$T_{262}^{(p)}\langle 103,75,72,73\rangle$	$T_{263}^{(p)}\langle 102,115,111,79\rangle$	$T_{264}^{(p)}\langle 102,79,75,72\rangle$
$T_{265}^{(p)}\langle 102,75,111,79\rangle$	$T_{266}^{(p)}\langle 101,102,115,79\rangle$	$T_{267}^{(p)}\langle 101,102,79,71\rangle$
$T_{268}^{(p)}\langle 102,79,71,72\rangle$	$T_{269}^{(p)}\langle 101,118,115,82\rangle$	$T_{270}^{(p)}\langle 101,82,79,71\rangle$
$T_{271}^{(p)}\langle 101,79,115,82\rangle$	$T_{272}^{(p)}\langle 100,101,118,82\rangle$	$T_{273}^{(p)}\langle 100,101,82,70\rangle$
$T_{274}^{(p)}\langle 101,82,70,71\rangle$	$T_{275}^{(p)}\langle 100,120,118,84\rangle$	$T_{276}^{(p)}\langle 100,84,82,70\rangle$
$T_{277}^{(p)}\langle 100,82,118,84\rangle$	$T_{278}^{(p)}\langle 99,100,120,84\rangle$	$T_{279}^{(p)}\langle 99,100,84,69\rangle$
$T_{280}^{(p)}\langle 100,84,69,70\rangle$	$T_{281}^{(p)}\langle 99,98,120,68\rangle$	$T_{282}^{(p)}\langle 99,68,84,69\rangle$
$T_{283}^{(p)}\langle 99,84,120,68\rangle$	$T_{284}^{(p)}\langle 115,111,107,59\rangle$	$T_{285}^{(p)}\langle 115,111,59,79\rangle$
$T_{286}^{(p)}\langle 111,59,79,75\rangle$	$T_{287}^{(p)}\langle 115,112,107,76\rangle$	$T_{288}^{(p)}\langle 115,76,59,79\rangle$
$T_{289}^{(p)}\langle 115,59,107,76\rangle$	$T_{290}^{(p)}\langle 118,115,112,76\rangle$	$T_{291}^{(p)}\langle 118,115,76,82\rangle$
$T_{292}^{(p)}\langle 115,76,82,79\rangle$	$T_{293}^{(p)}\langle 118,116,112,80\rangle$	$T_{294}^{(p)}\langle 118,80,76,82\rangle$
$T_{295}^{(p)}\langle 118,76,112,80\rangle$	$T_{296}^{(p)}\langle 120,118,116,80\rangle$	$T_{297}^{(p)}\langle 120,118,80,84\rangle$
$T_{298}^{(p)}\langle 118,80,84,82\rangle$	$T_{299}^{(p)}\langle 120,119,116,83\rangle$	$T_{300}^{(p)}\langle 120,83,80,84\rangle$
$T_{301}^{(p)}\langle 120,80,116,83\rangle$	$T_{302}^{(p)}\langle 98,120,119,83\rangle$	$T_{303}^{(p)}\langle 98,120,83,68\rangle$
$T_{304}^{(p)}\langle 120,83,68,84\rangle$	$T_{305}^{(p)}\langle 98,97,119,67\rangle$	$T_{306}^{(p)}\langle 98,67,83,68\rangle$
$T_{307}^{(p)}\langle 98,83,119,67\rangle$	$T_{308}^{(p)}\langle 116,112,108,60\rangle$	$T_{309}^{(p)}\langle 116,112,60,80\rangle$
$T_{310}^{(p)}\langle 112,60,80,76\rangle$	$T_{311}^{(p)}\langle 116,113,108,77\rangle$	$T_{312}^{(p)}\langle 116,77,60,80\rangle$
$T_{313}^{(p)}\langle 116,60,108,77\rangle$	$T_{314}^{(p)}\langle 119,116,113,77\rangle$	$T_{315}^{(p)}\langle 119,116,77,83\rangle$
$T_{316}^{(p)}\langle 116,77,83,80\rangle$	$T_{317}^{(p)}\langle 119,117,113,81\rangle$	$T_{318}^{(p)}\langle 119,81,77,83\rangle$
$T_{319}^{(p)}\langle 119,77,113,81\rangle$	$T_{320}^{(p)}\langle 97,119,117,81\rangle$	$T_{321}^{(p)}\langle 97,119,81,67\rangle$
$T_{322}^{(p)}\langle 119,81,67,83\rangle$	$T_{323}^{(p)}\langle 97,96,117,66\rangle$	$T_{324}^{(p)}\langle 97,66,81,67\rangle$
$T_{325}^{(p)}\langle 97,81,117,66\rangle$	$T_{326}^{(p)}\langle 117,113,109,61\rangle$	$T_{327}^{(p)}\langle 117,113,61,81\rangle$
$T_{328}^{(p)}\langle 113,61,81,77\rangle$	$T_{329}^{(p)}\langle 117,114,109,78\rangle$	$T_{330}^{(p)}\langle 117,78,61,81\rangle$
$T_{331}^{(p)}\langle 117,61,109,78\rangle$	$T_{332}^{(p)}\langle 96,117,114,78\rangle$	$T_{333}^{(p)}\langle 96,117,78,66\rangle$
$T_{334}^{(p)}\langle 117,78,66,81\rangle$	$T_{335}^{(p)}\langle 96,95,114,65\rangle$	$T_{336}^{(p)}\langle 96,65,78,66\rangle$
$T_{337}^{(p)}\langle 96,78,114,65\rangle$	$T_{338}^{(p)}\langle 95,114,110,62\rangle$	$T_{339}^{(p)}\langle 95,114,62,65\rangle$
$T_{340}^{(p)}\langle 114,62,65,78\rangle$	$T_{341}^{(p)}\langle 95,94,110,64\rangle$	$T_{342}^{(p)}\langle 95,64,62,65\rangle$
$T_{343}^{(p)}\langle 95,62,110,64\rangle$	$T_{344}^{(p)}\langle 121,122,85,144\rangle$	$T_{345}^{(p)}\langle 144,122,85,86\rangle$
$T_{346}^{(p)}\langle 144,122,86,145\rangle$	$T_{347}^{(p)}\langle 122,123,86,145\rangle$	$T_{348}^{(p)}\langle 145,123,86,87\rangle$
$T_{349}^{(p)}\langle 145,123,87,146\rangle$	$T_{350}^{(p)}\langle 123,124,87,146\rangle$	$T_{351}^{(p)}\langle 146,124,87,88\rangle$
$T_{352}^{(p)}\langle 146,124,88,147\rangle$	$T_{353}^{(p)}\langle 124,125,88,147\rangle$	$T_{354}^{(p)}\langle 147,125,88,89\rangle$
$T_{355}^{(p)}\langle 147,125,89,148\rangle$	$T_{356}^{(p)}\langle 125,126,89,148\rangle$	$T_{357}^{(p)}\langle 148,126,89,90\rangle$
$T_{358}^{(p)}\langle 148,126,90,149\rangle$	$T_{359}^{(p)}\langle 126,127,90,149\rangle$	$T_{360}^{(p)}\langle 149,127,90,91\rangle$
$T_{361}^{(p)}\langle 149,127,91,150\rangle$	$T_{362}^{(p)}\langle 127,128,91,150\rangle$	$T_{363}^{(p)}\langle 150,128,91,92\rangle$
$T_{364}^{(p)}\langle 150,128,92,130\rangle$	$T_{365}^{(p)}\langle 128,129,92,130\rangle$	$T_{366}^{(p)}\langle 86,105,144,85\rangle$
$T_{367}^{(p)}\langle 144,86,105,143\rangle$	$T_{368}^{(p)}\langle 143,144,86,145\rangle$	$T_{369}^{(p)}\langle 87,106,145,86\rangle$
$T_{370}^{(p)}\langle 145,87,106,151\rangle$	$T_{371}^{(p)}\langle 151,145,87,146\rangle$	$T_{372}^{(p)}\langle 88,107,146,87\rangle$
$T_{373}^{(p)}\langle 146,88,107,152\rangle$	$T_{374}^{(p)}\langle 152,146,88,147\rangle$	$T_{375}^{(p)}\langle 89,108,147,88\rangle$
$T_{376}^{(p)}\langle 147,89,108,153\rangle$	$T_{377}^{(p)}\langle 153,147,89,148\rangle$	$T_{378}^{(p)}\langle 90,109,148,89\rangle$
$T_{379}^{(p)}\langle 148,90,109,154\rangle$	$T_{380}^{(p)}\langle 154,148,90,149\rangle$	$T_{381}^{(p)}\langle 91,110,149,90\rangle$
$T_{382}^{(p)}\langle 149,91,110,155\rangle$	$T_{383}^{(p)}\langle 155,149,91,150\rangle$	$T_{384}^{(p)}\langle 92,93,150,91\rangle$
$T_{385}^{(p)}\langle 150,92,93,131\rangle$	$T_{386}^{(p)}\langle 131,150,92,130\rangle$	$T_{387}^{(p)}\langle 142,143,145,86\rangle$
$T_{388}^{(p)}\langle 142,143,86,104\rangle$	$T_{389}^{(p)}\langle 143,86,104,105\rangle$	$T_{390}^{(p)}\langle 142,151,145,106\rangle$

Table 2 (*continued*)

$T_{\alpha}^{(p)}\langle a_{\alpha}, b_{\alpha}, c_{\alpha}, d_{\alpha}\rangle$	$T_{\alpha}^{(p)}\langle a_{\alpha}, b_{\alpha}, c_{\alpha}, d_{\alpha}\rangle$	$T_{\alpha}^{(p)}\langle a_{\alpha}, b_{\alpha}, c_{\alpha}, d_{\alpha}\rangle$
$T_{391}^{(p)}\langle 142, 106, 86, 104\rangle$	$T_{392}^{(p)}\langle 142, 86, 145, 106\rangle$	$T_{393}^{(p)}\langle 141, 142, 151, 106\rangle$
$T_{394}^{(p)}\langle 141, 142, 106, 103\rangle$	$T_{395}^{(p)}\langle 142, 106, 103, 104\rangle$	$T_{396}^{(p)}\langle 141, 156, 151, 111\rangle$
$T_{397}^{(p)}\langle 141, 111, 106, 103\rangle$	$T_{398}^{(p)}\langle 141, 106, 151, 111\rangle$	$T_{399}^{(p)}\langle 140, 141, 156, 111\rangle$
$T_{400}^{(p)}\langle 140, 141, 111, 102\rangle$	$T_{401}^{(p)}\langle 141, 111, 102, 103\rangle$	$T_{402}^{(p)}\langle 140, 160, 156, 115\rangle$
$T_{403}^{(p)}\langle 140, 115, 111, 102\rangle$	$T_{404}^{(p)}\langle 140, 111, 156, 115\rangle$	$T_{405}^{(p)}\langle 139, 140, 160, 115\rangle$
$T_{406}^{(p)}\langle 139, 140, 115, 101\rangle$	$T_{407}^{(p)}\langle 140, 115, 101, 102\rangle$	$T_{408}^{(p)}\langle 139, 163, 160, 118\rangle$
$T_{409}^{(p)}\langle 139, 118, 115, 101\rangle$	$T_{410}^{(p)}\langle 139, 115, 160, 118\rangle$	$T_{411}^{(p)}\langle 138, 139, 163, 118\rangle$
$T_{412}^{(p)}\langle 138, 139, 118, 100\rangle$	$T_{413}^{(p)}\langle 139, 118, 100, 101\rangle$	$T_{414}^{(p)}\langle 138, 165, 163, 120\rangle$
$T_{415}^{(p)}\langle 138, 120, 118, 100\rangle$	$T_{416}^{(p)}\langle 138, 118, 163, 120\rangle$	$T_{417}^{(p)}\langle 137, 138, 165, 120\rangle$
$T_{418}^{(p)}\langle 137, 138, 120, 99\rangle$	$T_{419}^{(p)}\langle 138, 120, 99, 100\rangle$	$T_{420}^{(p)}\langle 137, 136, 165, 98\rangle$
$T_{421}^{(p)}\langle 137, 98, 120, 99\rangle$	$T_{422}^{(p)}\langle 137, 120, 165, 98\rangle$	$T_{423}^{(p)}\langle 156, 151, 146, 87\rangle$
$T_{424}^{(p)}\langle 156, 151, 87, 111\rangle$	$T_{425}^{(p)}\langle 151, 87, 111, 106\rangle$	$T_{426}^{(p)}\langle 156, 152, 146, 107\rangle$
$T_{427}^{(p)}\langle 156, 107, 87, 111\rangle$	$T_{428}^{(p)}\langle 156, 87, 146, 107\rangle$	$T_{429}^{(p)}\langle 160, 156, 152, 107\rangle$
$T_{430}^{(p)}\langle 160, 156, 107, 115\rangle$	$T_{431}^{(p)}\langle 156, 107, 115, 111\rangle$	$T_{432}^{(p)}\langle 160, 157, 152, 112\rangle$
$T_{433}^{(p)}\langle 160, 112, 107, 115\rangle$	$T_{434}^{(p)}\langle 160, 107, 152, 112\rangle$	$T_{435}^{(p)}\langle 163, 160, 157, 112\rangle$
$T_{436}^{(p)}\langle 163, 160, 112, 118\rangle$	$T_{437}^{(p)}\langle 160, 112, 118, 115\rangle$	$T_{438}^{(p)}\langle 163, 161, 157, 116\rangle$
$T_{439}^{(p)}\langle 163, 116, 112, 118\rangle$	$T_{440}^{(p)}\langle 163, 112, 157, 116\rangle$	$T_{441}^{(p)}\langle 165, 163, 161, 116\rangle$
$T_{442}^{(p)}\langle 165, 163, 116, 120\rangle$	$T_{443}^{(p)}\langle 163, 116, 120, 118\rangle$	$T_{444}^{(p)}\langle 165, 164, 161, 119\rangle$
$T_{445}^{(p)}\langle 165, 119, 116, 120\rangle$	$T_{446}^{(p)}\langle 165, 116, 161, 119\rangle$	$T_{447}^{(p)}\langle 136, 165, 164, 119\rangle$
$T_{448}^{(p)}\langle 136, 165, 119, 98\rangle$	$T_{449}^{(p)}\langle 165, 119, 98, 120\rangle$	$T_{450}^{(p)}\langle 136, 135, 164, 97\rangle$
$T_{451}^{(p)}\langle 136, 97, 119, 98\rangle$	$T_{452}^{(p)}\langle 136, 119, 164, 97\rangle$	$T_{453}^{(p)}\langle 157, 152, 147, 88\rangle$
$T_{454}^{(p)}\langle 157, 152, 88, 112\rangle$	$T_{455}^{(p)}\langle 152, 88, 112, 107\rangle$	$T_{456}^{(p)}\langle 157, 153, 147, 108\rangle$
$T_{457}^{(p)}\langle 157, 108, 88, 112\rangle$	$T_{458}^{(p)}\langle 157, 88, 147, 108\rangle$	$T_{459}^{(p)}\langle 161, 157, 153, 108\rangle$
$T_{460}^{(p)}\langle 161, 157, 108, 116\rangle$	$T_{461}^{(p)}\langle 157, 108, 116, 112\rangle$	$T_{462}^{(p)}\langle 161, 158, 153, 113\rangle$
$T_{463}^{(p)}\langle 161, 113, 108, 116\rangle$	$T_{464}^{(p)}\langle 161, 108, 153, 113\rangle$	$T_{465}^{(p)}\langle 164, 161, 158, 113\rangle$
$T_{466}^{(p)}\langle 164, 161, 113, 119\rangle$	$T_{467}^{(p)}\langle 161, 113, 119, 116\rangle$	$T_{468}^{(p)}\langle 164, 162, 158, 117\rangle$
$T_{469}^{(p)}\langle 164, 117, 113, 119\rangle$	$T_{470}^{(p)}\langle 164, 113, 158, 117\rangle$	$T_{471}^{(p)}\langle 135, 164, 162, 117\rangle$
$T_{472}^{(p)}\langle 135, 164, 117, 97\rangle$	$T_{473}^{(p)}\langle 164, 117, 97, 119\rangle$	$T_{474}^{(p)}\langle 135, 134, 162, 96\rangle$
$T_{475}^{(p)}\langle 135, 96, 117, 97\rangle$	$T_{476}^{(p)}\langle 135, 117, 162, 76\rangle$	$T_{477}^{(p)}\langle 158, 153, 148, 89\rangle$
$T_{478}^{(p)}\langle 158, 153, 89, 113\rangle$	$T_{479}^{(p)}\langle 153, 89, 113, 108\rangle$	$T_{480}^{(p)}\langle 158, 154, 148, 109\rangle$
$T_{481}^{(p)}\langle 158, 109, 89, 113\rangle$	$T_{482}^{(p)}\langle 158, 89, 148, 109\rangle$	$T_{483}^{(p)}\langle 162, 158, 154, 109\rangle$
$T_{484}^{(p)}\langle 162, 158, 109, 117\rangle$	$T_{485}^{(p)}\langle 158, 109, 117, 113\rangle$	$T_{486}^{(p)}\langle 162, 159, 154, 114\rangle$
$T_{487}^{(p)}\langle 162, 114, 109, 117\rangle$	$T_{488}^{(p)}\langle 162, 109, 154, 114\rangle$	$T_{489}^{(p)}\langle 134, 162, 159, 114\rangle$
$T_{490}^{(p)}\langle 134, 162, 114, 96\rangle$	$T_{491}^{(p)}\langle 162, 114, 96, 117\rangle$	$T_{492}^{(p)}\langle 134, 133, 159, 95\rangle$
$T_{493}^{(p)}\langle 134, 95, 114, 96\rangle$	$T_{494}^{(p)}\langle 134, 114, 159, 95\rangle$	$T_{495}^{(p)}\langle 159, 154, 149, 90\rangle$
$T_{496}^{(p)}\langle 159, 154, 90, 114\rangle$	$T_{497}^{(p)}\langle 154, 90, 114, 109\rangle$	$T_{498}^{(p)}\langle 159, 155, 149, 110\rangle$
$T_{499}^{(p)}\langle 159, 110, 90, 114\rangle$	$T_{500}^{(p)}\langle 159, 90, 149, 110\rangle$	$T_{501}^{(p)}\langle 133, 159, 155, 110\rangle$
$T_{502}^{(p)}\langle 133, 159, 110, 95\rangle$	$T_{503}^{(p)}\langle 159, 110, 95, 114\rangle$	$T_{504}^{(p)}\langle 133, 132, 155, 94\rangle$
$T_{505}^{(p)}\langle 133, 94, 110, 95\rangle$	$T_{506}^{(p)}\langle 133, 110, 155, 94\rangle$	$T_{507}^{(p)}\langle 132, 155, 150, 91\rangle$
$T_{508}^{(p)}\langle 132, 155, 91, 94\rangle$	$T_{509}^{(p)}\langle 155, 91, 94, 110\rangle$	$T_{510}^{(p)}\langle 132, 131, 150, 93\rangle$
$T_{511}^{(p)}\langle 132, 93, 91, 94\rangle$	$T_{512}^{(p)}\langle 132, 91, 150, 93\rangle$	

Table 3
Numerical results for triple integrals of Example 1 by p^3 tetrahedra (s = order of the Gauss–Legendre quadrature rule)

p^3	$s = 2$	$s = 3$	...	$s = 10$
(a) Numerical results of the integral $I_1 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{(x+y+z)}\,dz\,dy\,dx = 0.142857142857143$				
1^3	0.143127410953799	0.142875312759851	...	0.142857148844769
2^3	0.142968549423603	0.142862491460237		0.142857145041424
3^3	0.142943290907788	0.142860315425514		0.142857143741210
4^3	0.142925754472053	0.142858857605517		0.142857143180142
5^3	0.142910067322625	0.142858089259868		0.142857143005059
6^3	0.142898386146655	0.142857702544920		0.142857142935285
7^3	0.142889906246789	0.142857495304780		0.142857142902701
8^3	0.142883683469227	0.142857376596384		0.142857142885692
(b) Numerical results of the integral $I_2 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{\sqrt{(x+y+z)}}\,dz\,dy\,dx = 0.200000000000000$				
1^3	0.197660776240555	0.199583323221218	...	0.199998238575602
2^3	0.199289384231933	0.199888647749929		0.199999495304986
3^3	0.199727686899939	0.199975314543690		0.199999922512874
4^3	0.199733470424987	0.199982038699062		0.199999962252894
5^3	0.199769517063701	0.199988149357536		0.199999978392296
6^3	0.199806850518181	0.199991942165847		0.199999986302052
7^3	0.199838399462151	0.199994290724439		0.199999990682733
8^3	0.199863788284065	0.199995802131114		0.199999993327192
(c) Numerical results of the integral $I_3 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{\sqrt{(1-x-y)^2+z^2}}\,dz\,dy\,dx = 0.440686793509772$				
1^3	0.440894903222272	0.440665600968959	...	0.440686793509776
2^3	0.440667396711449	0.440934510261613		0.440692764491590
3^3	0.440322668703070	0.440904065895054		0.440692101049158
4^3	0.440213336639126	0.440869061004978		0.440691271746126
5^3	0.438172134196788	0.438811957547661		0.438660514270247
6^3	0.439527291801191	0.440154962076774		0.440024154598790
7^3	0.439675902035183	0.440264362428419		0.440148984820863
8^3	0.439553012064345	0.440105201387252		0.440002234985244

$$\begin{aligned} X(\xi,\eta,\zeta) &= \frac{(1+\xi)}{2}, \\ Y(\xi,\eta,\zeta) &= \frac{(1-\xi)(1+\eta)}{4}, \\ Z(\xi,\eta,\zeta) &= \frac{(1-\xi)(1-\eta)(1+\zeta)}{8}, \end{aligned} \tag{34}$$

the integral in Eq. (32) can be written as

$$\begin{aligned} \mathrm{I} &= \int\int_T\int f(x,y,z)\,dx\,dy\,dz = \frac{1}{p^3}\int\int_T\int H(X,Y,Z)\,dX\,dY\,dZ \\ &= \frac{1}{p^3}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-\xi)^2(1-\eta)}{64}H(X(\xi,\eta,\zeta),Y(\xi,\eta,\zeta),Z(\xi,\eta,\zeta))\,d\xi\,d\eta\,d\zeta \\ &= \frac{1}{p^3}\sum_{i=1}^{\lambda}\sum_{j=1}^{\mu}\sum_{k=1}^{v}\frac{(1-\xi_i^{(\lambda)})^2(1-\eta_j^{(\mu)})}{64}w_i^{(\lambda)}w_j^{(\mu)}w_k^{(v)} \\ &\quad\times H(X(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(v)}),Y(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(v)}),Z(\xi_i^{(\lambda)},\eta_j^{(\mu)},\zeta_k^{(v)})) \\ &= \frac{1}{p^3}\sum_{m=1}^{N=\lambda\mu v}c_m H(x_m,y_m,z_m), \end{aligned} \tag{35}$$

where, it is obvious that

$$x_m = \frac{(1+\xi_i^{(\lambda)})}{2}, \quad y_m = \frac{(1-\xi_i^{(\lambda)})(1+\eta_j^{(\mu)})}{4}, \quad z_m = \frac{(1-\xi_i^{(\lambda)})(1-\eta_j^{(\mu)})(1+\zeta_k^{(v)})}{8},$$
$$c_m = \frac{(1-\xi_i^{(\lambda)})^2(1-\eta_j^{(\mu)})}{64} w_i^{(\lambda)} w_j^{(\mu)} w_k^{(v)} \tag{36}$$

in which $\xi_i^{(\lambda)}$, $\eta_j^{(\mu)}$ and $\eta_k^{(v)}$ are the sampling points and $w_i^{(\lambda)}$, $w_j^{(\mu)}$ and $w_k^{(v)}$ are the corresponding weight coefficients of Gauss–Legendre quadrature rules of order λ, μ and v respectively.

7. Composite integration over the standard tetrahedron *T*, by a discretisation of *T* into 4p3 tetrahedra

We can discretise the standard tetrahedron T in (x,y,z) space into p^3 tetrahedra each of which has a volume $1/6\times p^3$ ($p = 2,3,4,5,6,7,8$). This is explained in the previous Section 3. We have seen that the use of this discretisation transforms the integral $I = \int\int_T \int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ into an equivalent integral as in Eq. (32)

$$\mathrm{I} = \int\int_T \int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{p^3}\int\int_T\int H(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{37}$$

where

$$H(X,Y,Z) = \sum_{\alpha=1}^{p^3} f(x^{(\alpha,p)}(X,Y,Z), y^{(\alpha,p)}(X,Y,Z), z^{(\alpha,p)}(X,Y,Z)).$$

Now we shall use the composite integration rule over the tetrahedron T with a subdivision of 4-tetrahedra $T_k^{(c)}$ ($k = 1,2,3,4$) which is derived in the previous Section 2 to compute the integral $I = \int\int_T \int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in (x,y,z) space. We shall apply this rule to the integral $\int\int_T \int H(X,Y,Z)$. Hence on application of the above stated composite integration rule, we have

$$I = \int\int_T\int f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{4p^3}\int\int_T\int H(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$$
$$= \frac{1}{4p^3}\sum_{k=1}^{s\times s\times s} c_k\{H(S_k,Q_k,R_k) + H(R_k,S_k,Q_k) + H(Q_k,R_k,S_k) + H(P_k,Q_k,R_k), \tag{38}$$

where

$$P_k = P(x_k,y_k,z_k), \quad Q_k = Q(x_k,y_k,z_k), \quad R_k = R(x_k,y_k,z_k), \quad S_k = S(x_k,y_k,z_k),$$

's' refers to the order of the Gauss–Legendre quadrature rule and (c_k, (x_k,y_k,z_k), $k = 1,2,3,\ldots,s^3$) are the weight coefficients and sampling points.

8. Some numerical results

We consider some typical integrals with known exact values.

Example 1. Let us consider the following multiple integrals which are generalised to three-dimensions from Reddy and Shippy [18]:

$$I_1 = \int_0^1\int_0^{1-x}\int_0^{1-x-y}\sqrt{(x+y+z)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0.142857142857143,$$
$$I_2 = \int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}{\sqrt{(x+y+z)}} = 0.200000000000000,$$
$$I_3 = \int_0^1\int_0^{1-x}\int_0^{1-x-y}[(1-x-y)^2+z^2]^{\frac{-1}{2}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0.440686793509772.$$

Table 4
Numerical results for triple integrals of Example 2 by p^3 tetrahedra ($s =$ order of the Gauss–Legendre quadrature rule)

p^3	$s = 2$	$s = 3$	…	$s = 10$
(a) Numerical results of the integral $I_4 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(x + 2y + 4z)\,dz\,dy\,dx = 0.131902326890181$				
1^3	0.131949528497795	0.131902664864686	…	0.131902326890181
2^3	0.133776332733418	0.131869224232551		0.131902326890182
3^3	0.133303270931640	0.131902424456643		0.131902326890182
4^3	0.132849768669041	0.131895368575070		0.131902326890182
5^3	0.132570359963437	0.131899004023074		0.131902326890182
6^3	0.132395175857975	0.131900568045736		0.131902326890182
7^3	0.132279820967538	0.131901316223466		0.131902326890181
8^3	0.132200304619986	0.131901707231827		0.131902326890182
(b) Numerical results of the integral $I_5 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 + x + y + z)^{-4}\,dz\,dy\,dx = 0.020833333333333$				
1^3	0.020103982733156	0.020798626362386	…	0.020833333333331
2^3	0.022488542242974	0.023296347515733		0.023371913673058
3^3	0.020646606627416	0.020822390285579		0.020833333333333
4^3	0.020684768683484	0.020828401131514		0.020833333333333
5^3	0.020719279915376	0.020830963682934		0.020833333333333
6^3	0.020744984572742	0.020832083337238		0.020833333333333
7^3	0.020763555926555	0.020832619115166		0.020833333333333
8^3	0.020777102696685	0.020832898005761		0.020833333333333

Table 5
Numerical results for triple integrals of Example 3 by p^3 tetrahedra ($s =$ order of the Gauss–Legendre quadrature rule)

p^3	$s = 2$	$s = 3$	…	$s = 10$
(a) Numerical results of the integral $I_6 = \int \int_T \int X^2 Y\,dX\,dY\,dZ = 15,721.6666666667$				
1^3	15,550.9773662551	15,721.6666666667	…	15,721.6666666667
2^3	15,691.3470735275	15,721.6666666667		15,721.6666666667
3^3	15,714.4670898321	15,721.6666666667		15,721.6666666667
4^3	15,719.7689888408	15,721.6666666667		15,721.6666666667
5^3	15,721.3511069958	15,721.6666666667		15,721.6666666667
6^3	15,721.8836258968	15,721.6666666667		15,721.6666666667
7^3	15,722.0618384713	15,721.6666666667		15,721.6666666667
8^3	15,722.1074285821	15,721.6666666667		15,721.6666666667
(b) Numerical results of the integral $I_7 = \int \int_T \int X^2 Y^2\,dX\,dY\,dZ = 109,662.063492063$				
1^3	107,484.179240969	109,657.491666667	…	109,662.063492064
2^3	109,140.344628995	109,661.741471354		109,662.063492064
3^3	109,503.862769064	109,661.970766652		109,662.063492064
4^3	109,600.765715007	109,662.027961222		109,662.063492064
5^3	109,634.847672300	109,662.047209067		109,662.063492064
6^3	109,649.117888292	109,662.055041718		109,662.063492064
7^3	109,655.816231740	109,662.058689245		109,662.063492064
8^3	109,659.212233043	109,662.060567013		109,662.063492064
(c) Numerical results of the integral $I_8 = \int \int_T \int X^4 Y^4\,dX\,dY\,dZ = 426,917,356.623377$				
1^3	387,905,448.629903	425,756,672.276489	…	426,917,356.623379
2^3	416,931,072.317321	426,833,124.356736		426,917,356.623378
3^3	423,448,083.898012	426,903,935.676678		426,917,356.623378
4^3	425,362,442.735248	426,913,828.379060		426,917,356.623378
5^3	426,099,107.479272	426,916,113.497101		426,917,356.623379
6^3	426,437,883.629226	426,916,826.984686		426,917,356.623378
7^3	426,614,058.055740	426,917,098.931230		426,917,356.623378
8^3	426,714,236.228519	426,917,218.358190		426,917,356.623378

Example 2. We now consider the following multiple integrals from Stroud [6]:

$$I_4 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(x+2y+4z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0.131902326890181,$$
$$I_5 = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-4}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0.020833333333333.$$

Example 3. Let us consider the following multiple integrals of the type from Rathod and Govinda Rao [20,21]:

$$\mathrm{III}_v^{\alpha,\beta,\gamma} = \int\int\int_v X^\alpha Y^\beta Z^\gamma\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{39}$$

where v is the tetrahedron in (X, Y, Z) space with vertices spanning the points $\langle(5,5,0),(10,10,0),(8,7,8)(10,5,0)\rangle$.

On using the following transformations:

$$X(x,y,z) = 10 - 5x - 2z, \quad Y(x,y,z) = 5 + 5y + 2z \quad \text{and} \quad Z(x,y,z) = 8z \tag{40}$$

we obtain,

$$\begin{aligned}\mathrm{III}_v^{\alpha,\beta,\gamma} &= \int\int\int_v X^\alpha Y^\beta Z^\gamma\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \\ &= 200\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (10-5x-2z)^\alpha \times (5+5y+2z)^\beta \times (8z)^\gamma\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{41}$$

Table 6
Numerical results for triple integrals of Example 4 by p^3 tetrahedra (s = order of the Gauss–Legendre quadrature rule)

p^3	$s=2$	$s=3$	...	$s=10$
(a) Numerical results of the integral $I_9 = \int\int_T\int \frac{X^2 Y}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 3784.40065050825$				
1^3	3760.92683460206	3784.40536151659	...	3784.40065050824
2^3	3780.80254809052	3784.40189622375		3784.40065050824
3^3	3783.80659878752	3784.40138719039		3784.40065050825
4^3	3784.40541345972	3784.40098499626		3784.40065050824
5^3	3784.40554843421	3784.40081526866		3784.40065050824
6^3	3784.40644048431	3784.40065050824		3784.40065050824
7^3	3784.40065050821	3784.40065050825		3784.40065050824
8^3	3784.40065050824	3784.40065050824		3784.40065050824
(b) Numerical results of the integral $I_{10} = \int\int_T\int \frac{X^2 Y^2}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 26,253.2913203869$				
1^3	25,902.3306380734	26,253.0118553023	...	26,253.2913203870
2^3	26,172.8987492499	26,253.2636819994		26,253.2913203870
3^3	26,231.8828502171	26,253.2756830985		26,253.2913203870
4^3	26,246.5372451585	26,253.2842805417		26,253.2913203877
5^3	26,251.2466876748	26,253.2878642826		26,253.2913203878
6^3	26,252.7268502173	26,253.2882826517		26,253.2913203877
7^3	26,252.8252451253	26,253.2913102526		26,253.2913203877
8^3	26,253.2462547696	26,253.2813102826		26,253.2913203877
(c) Numerical results of the integral $I_{11} = \int\int_T\int \frac{X^4 Y^4}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 100,719,764.240877$				
1^3	92,941,103.3625746	100,525,995.136937	...	100,719,764.240876
2^3	98,758,723.9087926	100,706,321.080616		100,719,764.240877
3^3	100,055,885.079954	100,717,584.232255		100,719,764.240877
4^3	100,428,865.188526	100,719,176.567812		100,719,764.240877
5^3	100,569,987.026842	100,719,551.874469		100,719,764.240877
6^3	100,595,885.209865	100,719,648.415262		100,719,764.240877
7^3	100,598,724.233851	100,719,763.348926		100,719,764.240877
8^3	100,693,527.210564	100,719,764.132965		100,719,764.240878

Table 7
Numerical results for triple integrals of Example 1 by $4p^3$ tetrahedra (s = order of the Gauss–Legendre quadrature rule)

$4\times p^3$	$s=2$	$s=3$	...	$s=10$
(a) Numerical results of the integral $I_1=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\sqrt{(x+y+z)}\,dz\,dy\,dx=0.142857142857143$				
4×1^3	0.143126164899338	0.142857594494647	...	0.142857148844769
4×2^3	0.142952716371517	0.142860128324534		0.142857143757019
4×3^3	0.142930447115489	0.142859843178828		0.142857143716475
4×4^3	0.142905899162925	0.142858377892635		0.142857143171104
4×5^3	0.142891158016211	0.142857774550862		0.142857143000921
4×6^3	0.142882042726974	0.142857500246332		0.142857142933098
4×7^3	0.142876097876429	0.142857361394577		0.142857142901427
4×8^3	0.142872030437808	0.142857284757020		0.142857142884894
(b) Numerical results of the integral $I_2=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{\sqrt{(x+y+z)}}\,dz\,dy\,dx=0.200000000000000$				
4×1^3	0.199511082837642	0.199993114634697	...	0.199999742021434
4×2^3	0.199571252828265	0.199957421513038		0.199999846912301
4×3^3	0.199507624151014	0.199937552898858		0.199999769239983
4×4^3	0.199664143049274	0.199966640958806		0.199999887587700
4×5^3	0.199761661597328	0.199980232656840		0.199999935651445
4×6^3	0.199823075867672	0.199987247066100		0.199999959206997
4×7^3	0.199863707947918	0.199991235098191		0.199999972252829
4×8^3	0.199891850790212	0.199993680302101		0.199999980128126
(c) Numerical results of the integral $I_3=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{\sqrt{(1-x-y)^2+z^2}}\,dz\,dy\,dx=0.440686793509772$				
4×1^3	0.391383844839317	0.431148430312508	...	0.438096875459463
4×2^3	0.413728561394546	0.428192921474296		0.439387641946574
4×3^3	0.422052167727767	0.432255245364635		0.439819760792517
4×4^3	0.426424657221525	0.434324623294889		0.440036169593660
4×5^3	0.427097658863712	0.433548448463196		0.438136026007479
4×6^3	0.430301401014733	0.435753643849637		0.439586855190488
4×7^3	0.431753684553353	0.436482185740039		0.439774018477423
4×8^3	0.432616686107687	0.436789594280571		0.439674048689021

Table 8
Numerical results for triple integrals of Example 2 by $4p^3$ tetrahedra (s = order of the Gauss–Legendre quadrature rule)

$4\times p^3$	$s=2$	$s=3$	...	$s=10$
(a) Numerical results of the integral $I_4=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\sin(x+2y+4z)\,dz\,dy\,dx=0.131902326890181$				
4×1^3	0.136541434613748	0.131732994797002	...	0.131902326890182
4×2^3	0.133596856910658	0.131876982020841		0.131902326890182
4×3^3	0.132803474840193	0.131902372752403		0.131902326890182
4×4^3	0.132475812030519	0.131898851544786		0.131902326890182
4×5^3	0.132298459270747	0.131900698681477		0.131902326890182
4×6^3	0.132191514372817	0.131901473550699		0.131902326890182
4×7^3	0.132122304404619	0.131901839629679		0.131902326890182
4×8^3	0.132075075943872	0.131902029484061		0.131902326890182
(b) Numerical results of the integral $I_5=\int_0^1\int_0^{1-x}\int_0^{1-x-y}(1+x+y+z)^{-4}\,dz\,dy\,dx=0.020833333333333$				
4×1^3	0.020435509596591	0.020770384268389	...	0.020833333333297
4×2^3	0.023006701619675	0.023350365575227		0.023371913610432
4×3^3	0.020671418355438	0.020827633749027		0.020833333333333
4×4^3	0.020726358272060	0.020831050865503		0.020833333333333
4×5^3	0.020759618900076	0.020832300138985		0.020833333333333
4×6^3	0.020779950465380	0.020832807554578		0.020833333333333
4×7^3	0.020793047063806	0.020833040041285		0.020833333333333
4×8^3	0.020801911735656	0.020833157606598		0.020833333333333

We have evaluated the above integrals for $\alpha = 2,\ \beta = 1,\ \gamma = 0,\ \alpha = 2,\ \beta = 2,\ \gamma = 0$ and $\alpha = 4,\ \beta = 4,\ \gamma = 0$.
That is

$$I_6 = \mathrm{III}_v^{2,1,0} = \int\int\int_v X^2 Y\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 15,721.6666666667,$$
$$I_7 = \mathrm{III}_v^{2,2,0} = \int\int\int_v X^2 Y^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 109,662.063492063,$$
$$I_8 = \mathrm{III}_v^{4,4,0} = \int\int\int_v X^4 Y^4\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 426,917,356.623377.$$

Again from Rathod and Govinda Rao [20,21], we know that $I_6 = 47165/3$, other integrals were computed in a similar way.

Example 4. We now consider the following multiple integrals of the type:

$$\mathrm{III}_v^{\alpha,\beta,\gamma} = \int\int\int_v \frac{X^\alpha Y^\beta Z^\gamma}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z, \tag{42}$$

where v is the tetrahedron in (X, Y, Z) space with vertices spanning the points $\langle(5,5,0),(10,10,0),(8,7,8),(10,5,0)\rangle$.

On using the following transformations:

$$X(x,y,z) = 10 - 5x - 2z, \quad Y(x,y,z) = 5 + 5y + 2z \quad \text{and} \quad Z(x,y,z) = 8z \tag{43}$$

Table 9
Numerical results for triple integrals of Example 3 by $4p^3$ tetrahedra (s = order of the Gauss–Legendre quadrature rule)

$4\times p^3$	$s=2$	$s=3$	…	$s=10$
(a) Numerical results of the integral $I_6 = \int\int_T\int X^2Y\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 15,721.6666666667$				
4×1^3	15,689.3048321759	15,721.6666666666	…	15,721.6666666667
4×2^3	15,707.6339988551	15,721.6666666667		15,721.6666666667
4×3^3	15,717.3458981982	15,721.6666666667		15,721.6666666667
4×4^3	15,720.0401043695	15,721.6666666667		15,721.6666666667
4×5^3	15,720.9861743827	15,721.6666666667		15,721.6666666667
4×6^3	15,721.3756048897	15,721.6666666667		15,721.6666666667
4×7^3	15,721.5532372850	15,721.6666666667		15,721.6666666667
4×8^3	15,721.6396157035	15,721.6666666667		15,721.6666666667
(b) Numerical results of the integral $I_7 = \int\int_T\int X^2Y^2\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 109,662.063492063$				
4×1^3	109,094.831466500	109,662.063492063	…	109,662.063492064
4×2^3	109,430.144438985	109,661.899263848		109,662.063492064
4×3^3	109,577.712362477	109,662.021597705		109,662.063492064
4×4^3	109,623.818419547	109,662.048417031		109,662.063492064
4×5^3	109,641.957676911	109,662.056827766		109,662.063492064
4×6^3	109,650.371072906	109,662.060112264		109,662.063492064
4×7^3	109,654.753824783	109,662.061601506		109,662.063492064
4×8^3	109,657.240087167	109,662.062354016		109,662.063492064
(c) Numerical results of the integral $I_8 = \int\int_T\int X^4Y^4\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 426917356.623377$				
4×1^3	412,498,387.482000	426,917,355.396991	…	426,917,356.623380
4×2^3	422,717,461.388815	426,882,920.114960		426,917,356.623378
4×3^3	425,191,223.205158	426,911,417.113909		426,917,356.623378
4×4^3	426,050,123.506266	426,915,733.124736		426,917,356.623377
4×5^3	426,417,820.340631	426,916,770.720206		426,917,356.623378
4×6^3	426,600,995.270582	426,917,102.796006		426,917,356.623378
4×7^3	426,702,757.876150	426,917,231.567149		426,917,356.623378
4×8^3	426,764,062.677285	426,917,288.855189		426,917,356.623378

we obtain,

$$\mathrm{III}_v^{\alpha,\beta,\gamma} = \int\int\int_v \frac{X^\alpha Y^\beta Z^\gamma}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 200\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{(10-5x-2z)^\alpha \times (5+5y+2z)^\beta \times (8z)^\gamma}{\sqrt{15-5x+5y+8z}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x. \quad (44)$$

We have evaluated the above integrals for $\alpha=2,\ \beta=1,\ \gamma=0$; $\alpha=2,\ \beta=2,\ \gamma=0$ and $\alpha=4,\ \beta=4,\ \gamma=0$.

$$I_9 = \mathrm{III}_v^{2,1,0} = \int\int\int_v \frac{X^2 Y}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 3784.40065050825,$$
$$I_{10} = \mathrm{III}_v^{2,2,0} = \int\int\int_v \frac{X^2 Y^2}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 26,253.2913203869,$$
$$I_{11} = \mathrm{III}_v^{4,4,0} = \int\int\int_v \frac{X^4 Y^4}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 100,719,764.240877.$$

We have tabulated the numerical values for I_1, I_2 and I_3 of Example 1, I_4 and I_5 of Example 2, I_6, I_7 and I_8 of Example 3 and I_9, I_{10} and I_{11} of Example 4 in Tables 3–6 using p^3 tetrahedra.

We have also tabulated the numerical values for I_1, I_2 and I_3 of Example 1, I_4 and I_5 of Example 2, I_6, I_7 and I_8 of Example 3 and I_9, I_{10} and I_{11} of Example 4 in Tables 7–10 using $4p^3$ tetrahedra.

Table 10
Numerical results for triple integrals of Example 4 by $4p^3$ tetrahedra (s = order of the Gauss–Legendre quadrature rule)

$4\times p^3$	$s=2$	$s=3$	…	$s=10$
(a) Numerical results of the integral $I_9 = \int\int_T\int \frac{X^2Y}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 3784.40065050825$				
4×1^3	3780.91040926571	3784.40495727689	…	3784.40065050824
4×2^3	3782.44831596703	3784.40131290504		3784.40065050824
4×3^3	3783.86876464398	3784.40096504194		3784.40065050825
4×4^3	3784.23932826683	3784.40078505936		3784.40065050824
4×5^3	3784.35863023875	3784.40071487029		3784.40065050824
4×6^3	3784.37865273835	3784.40065083515		3784.40065050824
4×7^3	3784.39495735488	3784.40065059333		3784.40065050824
4×8^3	3784.40131290504	3784.40065053512		3784.40065050825
(b) Numerical results of the integral $I_{10} = \int\int_T\int \frac{X^2Y^2}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 26,253.2913203869$				
4×1^3	26,181.4081206154	26,253.1032178572	…	26,253.2913203846
4×2^3	26,216.7666800829	26,253.2740377792		26,253.2913203867
4×3^3	26,241.0561103839	26,253.2844811557		26,253.2913203877
4×4^3	26,248.2848702891	26,253.2884739280		26,253.2913203877
4×5^3	26,250.9628816914	26,253.2899702724		26,253.2913203877
4×6^3	26,251.0561103839	26,253.2904745282		26,253.2913203877
4×7^3	26,252.1848703451	26,253.2910039282		26,253.2913203878
4×8^3	26,252.9628816914	26,253.2912702534		26,253.2913203877
(c) Numerical results of the integral $I_{11} = \int\int_T\int \frac{X^4Y^4}{\sqrt{X+Y+Z}}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = 100,719,764.240877$				
4×1^3	97,987,543.3040550	100,662,060.180032	…	100,719,764.240878
4×2^3	99,877,731.3724180	100,714,114.475863		100,719,764.240877
4×3^3	100,380,017.523110	100,718,780.724472		100,719,764.240876
4×4^3	100,551,947.150281	100,719,491.425302		100,719,764.240877
4×5^3	100,624,502.421325	100,719,664.271945		100,719,764.240877
4×6^3	100,680,017.523110	100,719,681.524658		100,719,764.240879
4×7^3	100,701,947.148642	100,719,701.422652		100,719,764.240877
4×8^3	100,718,552.102325	100,719,723.135945		100,719,764.240877

8.1. C-Program for evaluation of triple integrals of Examples 1–4 by a division of standard tetrahedron into 4-tetrahedra

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
main()
{
int i, j, k, n;
double x, y, z, c, P, Q, R, S, a[20], w[20], I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11, S1=0,
        S2=0, S3=0, S4=0, S5=0, S6=0, S7=0, S8=0, S9=0, S10=0, S11=0, XPQR, YPQR,
        ZPQR, XQRS, YQRS, ZQRS, XSQR, YSQR, ZSQR, XRSQ, YRSQ, ZRSQ;
clrscr();
printf("Enter the value of n= ");
scanf("%d", &n);
printf("Enter the values of a's in order");
for(i=1; i<=n; i++)
scanf("%lf", &a[i]);
printf("Enter the values of w's in order");
for(i=1; i<=n; i++)
scanf("%lf", &w[i]);
for(i=1; i<=n; i++)
{for(j=1; j<=n ;j++)
{for(k=1; k<=n; k++)
{
x=(1+a[i])/2; y=(1-a[i])*(1+a[j])/4;
z=(1-a[i])*(1-a[j])*(1+a[k])/8;
c=pow(1-a[i],2)*(1-a[j])*w[i]*w[j]*w[k]/64;
P=(1+3*x-y-z)/4;   Q=(1-x+3*y-z)/4;   R=(1-x-y+3*z)/4;   S=(1-x-y-z)/4;
XPQR=10-5*P-2*R;    YPQR=5+5*Q+2*R;    ZPQR=8*R;
XQRS=10-5*Q-2*S;    YQRS=5+5*R+2*S;    ZQRS=8*S;
XSQR=10-5*S-2*R;    YSQR=5+5*Q+2*R;    ZSQR=8*R;
XRSQ=10-5*R-2*Q;    YRSQ=5+5*S+2*Q;    ZRSQ=8*Q;
I1=c*(sqrt(P+Q+R)+sqrt(Q+R+S)+sqrt(S+Q+R)+sqrt(R+S+Q))/4;
S1=S1+I1;
I2=c*(1/sqrt(P+Q+R)+1/sqrt(Q+R+S)+1/sqrt(S+Q+R)+1/sqrt(R+S+Q))/4;
S2=S2+I2;
I3=c*(1/sqrt(pow(1-P-Q,2)+R*R)+1/sqrt(pow(1-Q-R,2)+S* S)
     +1/sqrt(pow(1-S-Q,2)+R*R)+1/sqrt(pow(1-R-S,2)+Q*Q))/4;
S3=S3+I3;
I4=c*(sin(P+2*Q+4*R)+sin(Q+2*R+4*S)+sin(S+2*Q+4*R)+sin(R+2*S+4*Q))/4;
S4=S4+I4;
I5=c*(pow(1+P+Q+R,-4)+pow(1+Q+R+S,-4)+pow(1+S+Q+R,-4)+pow(1+R+S+Q,-4))/4;
S5=S5+I5;
I6=200*c*(pow(XPQR,2)*YPQR+pow(XQRS,2)*YQRS
     +pow(XSQR,2)*YSQR+pow(XRSQ,2)*YRSQ)/4;
S6=S6+I6;
I7=200*c*(pow(XPQR,2)*pow(YPQR,2)+pow (XQRS,2)*pow(YQRS,2)
     +pow(XSQR,2)*pow(YSQR,2)+pow (XRSQ,2)*pow(YRSQ,2))/4;
S7=S7+I7;
I8=200*c*(pow(XPQR,4)*pow(YPQR,4)
     +pow (XQRS,4)*pow(YQRS,4)+pow(XSQR,4)*pow (YSQR,4)+pow(XRSQ,4)*pow(YRSQ,4))/4;
```

```
S8=S8+I8;
I9=200*c*(pow(XPQR,2)*YPQR/ sqrt(XPQR+YPQR+ZPQR)+pow(XQRS,2)*YQRS/
    sqrt(XQRS+YQRS+ZQRS)+pow(XSQR,2)*YSQR/ sqrt(XSQR+YSQR+ZSQR)
    +pow(XRSQ,2)*YRSQ/sqrt(XRSQ+YRSQ+ZRSQ))/4;
S9=S9+I9;
I10=200*c*(pow(XPQR,2)*pow(YPQR,2)/sqrt(XPQR+YPQR+ZPQR)+pow (XQRS,2)*
    pow(YQRS,2)/sqrt(XQRS+YQRS+ZQRS)+pow (XSQR,2)*pow(YSQR,2)/
    sqrt(XSQR+YSQR+ZSQR)+pow(XRSQ,2)*pow(YRSQ,2)/sqrt(XRSQ+YRSQ+ZRSQ))/4;
S10=S10+I10;
I11=200*c*(pow(XPQR,4)*pow (YPQR,4)/sqrt(XPQR+YPQR+ZPQR)+pow(XQRS,4)* pow(YQRS,4)/
sqrt(XQRS+YQRS+ZQRS)+pow(XSQR,4)*pow (YSQR,4)/sqrt(XSQR+YSQR+ZSQR)+pow (XRSQ,4)*
pow(YRSQ,4)/sqrt(XRSQ+YRSQ+ZRSQ))/4;
S11=S11+I11;
}}}
printf("I1=%0.15lf\n", S1);
printf("I2=%0.15lf\n", S2);
printf("I3=%0.15lf\n", S3);
printf("I4=%0.15lf\n", S4);
printf("I5=%0.15lf\n", S5);
printf("I6=%0.15lf\n", S6);
printf("I7=%0.15lf\n", S7);
printf("I8=%0.15lf\n", S8);
printf("I9=%0.15lf\n", S9);
printf("I10=%0.15lf\n", S10);
printf("I11=%0.15lf\n", S11);
getch();
}
```

8.2. C-Program for evaluation of triple integrals of Examples 1–4 by a division of standard tetrahedron into $2^3 = 8$ tetrahedra

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{
int i, j, k, o, p, d;
double x, y, z, c, P, Q, R, S, a[20], w[20], I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11,
    S1=0, S2=0, S3=0, S4=0, S5=0, S6=0, S7=0, S8=0, S9=0, S10=0, S11=0,
    X[100], Y[100], Z[100], l[100], m[100], n[100];
clrscr();
printf("Enter the value of o= ");
scanf("%d",&o);
printf("Enter the value of p= ");
scanf("%d",&p);
printf("Enter the values of a's in order");
for(i=1; i<=o; i++)
scanf("%lf",&a[i]);
printf("Enter the values of w's in order");
for(i=1; i<=o; i++)
scanf("%lf",&w[i]);
```

```
for(i=1; i<=o; i++)
{for(j=1; j<=o; j++)
{for(k=1; k<=o; k++)
{
x=(1+a[i])/2;
y=(1-a[i])*(1+a[j])/4;
z=(1-a[i])*(1-a[j])*(1+a[k])/8;
c=(1-a[i])*(1-a[i])*(1-a[j])*w[i]*w[j]*w[k]/64;
   l[1]=x/p;   m[1]=y/p;   n[1]=(p-1+z)/p;
   l[2]=(1+x)/p;   m[2]=y/p;   n[2]=(p-2+z)/p;
   l[3]=x/p;   m[3]=(1+y)/p;   n[3]=(p-2+z)/p;
   l[4]=(x+y+z)/p;   m[4]=(1-x-z)/p;   n[4]=(p-1-x-y)/p;
   l[5]=(x+y)/p;   m[5]=(1-x)/p;   n[5]=(p-2+z)/p;
   l[6]=(1-x-y)/p;   m[6]=x/p;   n[6]=(p-1-z)/p;
   l[7]=x/p;   m[7]=y/p;   n[7]=(p-2+y+z)/p;
   l[8]=y/p;   m[8]=(1-x-y)/p;   n[8]=(p-2+z)/p;
for(d=1; d<=8; d++)
{
X[d]=10-5*l[d]-2*n[d]; Y[d]=5+5*m[d]+2*n[d]; Z[d]=8*n[d];
I1=c*sqrt(l[d]+m[d]+n[d])/8;
S1=S1+I1;
I2=c*1/sqrt(l[d]+m[d]+n[d])/8;
S2=S2+I2;
I3=c*1/sqrt(pow(1-l[d]-m[d],2)+n[d]*n[d])/8;
S3=S3+I3;
I4=c*sin(l[d]+2*m[d]+4*n[d])/8;
S4=S4+I4;
I5=c*pow(1+l[d]+m[d]+n[d],-4)/8;
S5=S5+I5;
I6=200*c*(pow(X[d],2)*Y[d])/8;
S6=S6+I6;
I7=200*c*(pow(X[d],2)*pow(Y[d],2))/8;
S7=S7+I7;
I8=200*c*(pow(X[d],4)*pow(Y[d],4))/8;
S8=S8+I8;
I9=200*c*(pow(X[d],2)*Y[d]/sqrt(X[d]+Y[d]+Z[d]))/8;
S9=S9+I9;
I10=200*c*(pow(X[d],2)*pow(Y[d],2)/sqrt(X[d]+Y[d]+Z[d]))/8;
S10=S10+I10;
I11=200*c*(pow(X[d],4)*pow(Y[d],4)/sqrt(X[d]+Y[d]+Z[d]))/8;
S11=S11+I11;
}}}
printf("I1=%0.15lf\n", S1);
printf("I2=%0.15lf\n", S2);
printf("I3=%0.15lf\n", S3);
printf("I4=%0.15lf\n", S4);
printf("I5=%0.15lf\n", S5);
printf("I6=%0.15lf\n", S6);
printf("I7=%0.15lf\n", S7);
printf("I8=%0.15lf\n", S8);
printf("I9=%0.15lf\n", S9);
printf("I10=%0.15lf\n", S10);
```

```
printf("I11=%0.15lf\n", S11);
getch();
}
```

Note: Similarly we can write a C-program for the evaluation of triple integrals by using $3^3 = 27$, $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$ and $8^3 = 512$ tetrahedra.

Note: Similarly we can write a C-program for the evaluation of triple integrals by using $4 \times 2^3 = 32$, $4 \times 3^3 = 108$, $4 \times 4^3 = 256$, $4 \times 5^3 = 500$, $4 \times 6^3 = 864$, $4 \times 7^3 = 1372$ and $4 \times 8^3 = 2048$ tetrahedra.

9. Conclusions

In this paper, we have presented the composite numerical integration formulae, which can be derived by decomposing the tetrahedron into four tetrahedra by joining the centroid to four vertices. We have further shown that the standard tetrahedron can be discretised into $2^3, 3^3, \ldots, 8^3$ tetrahedra of equal volumes. Over each of these the Gauss–Legendre quadrature rule developed in Section 2 is applicable. We have also applied the composite rule, which is derived in Section 3 by discretising the standard tetrahedron into four equal tetrahedra by joining the centroid to its four vertices. This integrates the standard tetrahedron by discretising into $4 \times 2^3, 4 \times 3^3, \ldots, 4 \times 8^3$ tetrahedra of equal volumes. These formulae are tested for the accuracy and efficiency by applying them to some non-polynomial and polynomial functions.

References

[1] O.C. Zienkiewicz, The Finite Element Method, third ed., Mc Graw-Hill, London, 1997.
[2] T.J.R. Hughes, The Finite Element Method, Static and Dynamic Analysis, Prentice Hall, Englewood Cliffs, NJ, 1987.
[3] K.J. Bathe, Finite Element Procedures, Prentice Hall, Englewood Cliffs, NJ, 1996.
[4] P.C. Hammer, O.J. Marlowe, A.H. Stroud, Numerical integration over simplexes and cones, Math. Tables Aids Comput. 10 (1956) 130–136.
[5] P.C. Hammer, A.H. Stroud, Numerical integration over simplexes, Math. Tables Aids Comput. 10 (1956) 137–139.
[6] P.C. Hammer, A.H. Stroud, Numerical evaluation of multiple integrals, Math. Tables Aids Comput. 12 (1958) 272–280.
[7] G.R. Cowper, Gaussian quadrature formulas for triangle, Int. J. Numer. Meth. Eng. 7 (1973) 405–408.
[8] J.N. Lyness, D. Jespersen, Moderate degree symmetric quadrature rules for the triangle, J. Inst. Math. Appl. 15 (1975) 19–32.
[9] F.G. Lannoy, Triangular finite elements and numerical integration, Comput. Struct. 7 (1977) 613.
[10] D.P. Laurie, Automatic numerical integration over a triangle, CSIR Spec. Rep. WISK, 273, National Research Institute of Mathematical Sciences, Pretoria, 1977.
[11] M.E. Laursen, M. Gellert, Some criteria for numerically integrated matrices and quadrature formaulas for triangles, Int. J. Numer. Meth. Eng. 12 (1978) 67–76.
[12] D.A. Dunavant, High degree efficient symmetrical Gaussian quadrature rules for the triangle, Int. J. Numer. Meth. Eng. 21 (1985) 1129–1148.
[13] P. Sylvester, Symmetric quadrature formulae for simplexes, Math. Comput. 24 (1970) 95–100.
[14] F.G. Lether, Computation of double integrals over a triangle, J. Comput. Appl. Math. 2 (1976) 219–224.
[15] P. Hillion, Numerical integration on a triangle, Int. J. Numer. Meth. Eng. 11 (1977) 797–815.
[16] M. Abramowicz, I.A. Stegun, Hand Book of Mathematical Functions, Dover Publications, Inc., New York, 1965.
[17] C.T. Reddy, Improved three point integration schemes for triangular finite elements, Int. J. Numer. Meth. Eng. 12 (1978) 1890–1896.
[18] C.T. Reddy, D.J. Shippy, Alternative integration formulae for triangular finite elements, Int. J. Numer. Meth. Eng. 17 (1981) 133–153.
[19] G. Lague, R. Baldur, Extended numerical integration method for triangular surfaces, Int. J. Numer. Meth. Eng. 11 (1977) 388–392.
[20] H.T. Rathod, H.S. Govinda Rao, Integration of polynomials over an arbitrary tetrahedron in Euclidean three dimensional space, Comput. Struct. 59 (1) (1996) 55–65.
[21] H.T. Rathod, H.S. Govinda Rao, Integration of trivariate polynomials over linear poly-hedra in Euclidean three dimensional space, J. Austral. Math. Soc. Ser. B 39 (1998) 355–385.
[22] H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, N.L. Ramesh, Gauss–Legendre Quadrature over a triangle, J. Indian Inst. Sci. 84 (2004) 183–188.
[23] Brice Carnahan, H.A. Luther, James O. Wilkes, Applied Numerical Methods, John Wiley and Sons, 1969.
[24] C.F. Gerald, P.O. Wheatley, Applied Numerical Analysis, sixth ed., Pearson Education, Asia, 1999, Lowprice edition.
[25] S.C. Chapra, Canale, Numerical Methods for Science and engineering, Tata McGraw-Hill, New Delhi, 2000.
[26] R.L. Burden, J.D. Faiers, Numerical Analysis, fourth ed., PWS-KENT Publishing Company, 1989.
[27] T.R. Chandrupatla, A.D. Belegundu, Introduction to Finite elements, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1991.