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Numerical integration of some functions over an arbitrary linear tetrahedra in Euclidean three-dimensional space

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Abstract

In this paper it is proposed to compute the volume integral of certain functions whose antiderivates with respect to one of the variates (say either x or y or z) is available. Then by use of the well known Gauss Divergence theorem, it can be shown that the volume integral of such a function is expressible as sum of four integrals over the unit triangle. The present method can also evaluate the triple integrals of trivariate polynomials over an arbitrary tetrahedron as a special case. It is also demonstrated that certain integrals which are nonpolynomial functions of trivariates x, y, z can be computed by the proposed method. We have applied Gauss Legendre Quadrature rules which were recently derived by Rathod et al. [H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, N.L. Ramesh, Gauss Legendre Quadrature over a Triangle, J. Indian Inst. Sci. 84 (2004) 183–188] to evaluate the typical integrals governed by the proposed method.

Keywords: Finite element method; Numerical integration; Gauss Legendre quadrature; Tetrahedron; Triangular elements; Extended numerical integration

1. Introduction

Volume, center of mass, moment of inertia and other geometric properties of rigid homogeneous solids frequently arise in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modelling as well as in robotics. Integration formulas for multiple integrals have always been of great interest in computer applications, a good overview of available methods for evaluating volume integrals is given by Lee and Requicha [2]. Timmer and Stern [3] discussed a theoretical approach to the evaluation of volume integrals by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajiya [4] presented an outline of a closed form formula for volume integration by decomposing the solid into a set of solid tetrahedra. Cattani and paoluzzi [5] gave a symbolic solution to both the surface and volume integration of polynomials by using a triangulation of the solid boundary. In a recent paper, Bernardini [6] has presented explicit formulas and algorithms for computing integrals of polynomials over

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n-dimensional polyhedra by using the decomposition representation and the boundary representation of the polyhedron. Rathod and Govind Rao [7] derived explicit integration formulas for computing volume integrals of trivariate polynomials over an arbitrary tetrahedron in Euclidean three-dimensional space. They proposed two different approaches; the first method evaluates this volume integral by mapping the tetrahedron into orthogonal unit tetrahedron and the second method computes the same integral as a sum of four integrals over the unit triangle. The present work enhances the second method by considering the evaluation of some functions by use of the well known Gauss Divergence theorem.

2. Problem statement for present work

Most computational studies of triple integrals deal with problems in which the domain of integration is very simple, like a cube or a sphere, but the integrand is complicated. However, in real applications, we confront the inverse problem: the integrating function $f(x,y,z)$ is usually simple; but the domain is very complicated. Hence in this paper and in other previous works [4,6] an attempt is made to obtain practical formulas for the exact evaluation of integrals.

$$\int\int_P\int f(x,y,z)\mathrm{d}V,$$

where P is a three-polyhedron in R^3 and $\mathrm{d}V$ is the differential volume. The integrating-function is a trivariate monomial

$$f(x,y,z) = x^\alpha y^\beta z^\gamma,$$

where α, β, γ are nonnegative integers

or

$$f(x,y,z) = \frac{\partial F}{\partial x}, \quad \text{or} \quad f(x,y,z) = \frac{\partial F}{\partial y}, \quad \text{or} \quad f(x,y,z) = \frac{\partial F}{\partial z}.$$

However the paper is focused on the calculation of the following integral:

$$III_V = \int\int_V\int f(x,y,z)\mathrm{d}V,$$

where $f(x,y,z) = \frac{\partial F}{\partial x}$, or $f(x,y,z) = \frac{\partial F}{\partial y}$, or $f(x,y,z) = \frac{\partial F}{\partial z}$, for some suitable function F and V is an arbitrary tetrahedron with four vertices (x_i, y_i, z_i), $(i = 1,2,3,4)$. Two different approaches are possible. The first approach is direct and it transforms an arbitrary tetrahedron into an orthogonal tetrahedron by means of a mapping. The second approach is based on the fact that certain triple integrals can be reduced to surface integrals by use of the well known Gauss's divergence theorem. This paper is concerned with the second approach.

3. Volume integration over an arbitrary tetrahedron

In this section, we first obtain the volume integral of a scalar function

$$f(p) = f(x,y,z)$$

(α, β, γ are positive integers) over an arbitrary tetrahedron by transforming it to an orthogonal unit tetrahedron. That is we are actually interested in evaluating

$$III_V = \int\int_V\int f(x,y,z)\mathrm{d}V, \tag{1}$$

where V is an arbitrary tetrahedron in the x,y,z Cartesian coordinate system.

We have, over the unit orthogonal tetrahedron $\overline{V} = \langle(0,0,0),(1,0,0),(0,1,0),(0,1,0)\rangle$ in the ξ,η,ζ Cartesian coordinate system. The volume (natural) coordinates are related to Cartesian coordinates by the well-known relations [7]

$$\begin{aligned} x &= L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4, \\ y &= L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4, \\ z &= L_1z_1 + L_2z_2 + L_3z_3 + L_4z_4 \end{aligned} \tag{2}$$

and

$$L_1 + L_2 + L_3 + L_4 = 1,$$

where (x_i, y_i, z_i) refer to the Cartesian coordinates of vertex i of the tetrahedron. Letting $L_1 = \xi, L_2 = \eta, L_3 = \zeta$, we can rewrite the relations (2) as

$$\begin{aligned} x(\xi,\eta,\zeta) &= x_4 + \xi x_{14} + \eta x_{24} + \gamma x_{34}, \\ y(\xi,\eta,\zeta) &= y_4 + \xi y_{14} + \eta y_{24} + \gamma y_{34}, \\ z(\xi,\eta,\zeta) &= z_4 + \xi z_{14} + \eta z_{24} + \gamma z_{34}, \end{aligned} \tag{3}$$

with

$$x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_{j,} \quad z_{ij} = z_i - z_j.$$

If we consider the mapping (see Fig. 1) between the three dimensional space (X, Y, Z) and the three dimensional space (ξ,η,ζ) by the parametric Eq. (3), we have for the volume element

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |\det J|\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{4}$$

where

$$\det J = \begin{vmatrix} (x_1 - x_4) & (x_2 - x_4) & (x_3 - x_4) \\ (y_1 - y_4) & (y_2 - y_4) & (y_3 - y_4) \\ (z_1 - z_4) & (z_2 - z_4) & (z_3 - z_4) \end{vmatrix},$$

$$|\det J| = 6 \times \text{volume of tetrahedron} = \text{absolute value of } \det J. \tag{5}$$

So, if we change the coordinates according to Eq. (3) and express consistently the volume element, we obtain

$$III_V = |\det J| \int\int_{\overline{V}}\int f(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{6}$$

where $\overline{V}$ is the unit orthogonal tetrahedron $\langle(0,0,0),(1,0,0),(0,1,0),(0,0,1)\rangle$.

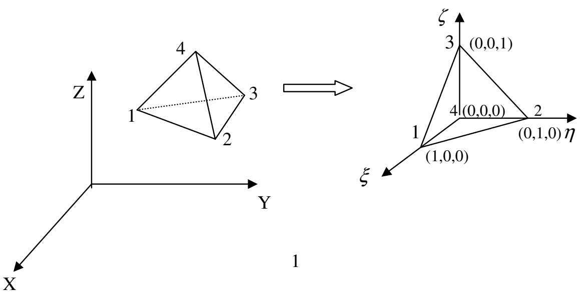

Fig. 1. Three-dimensional mapping of an arbitrary tetrahedron in (X, Y, Z) space into an unit orthogonal tetrahedron in (ξ,η,ζ) space.

4. Surface integration over an arbitrary tetrahedron

Theorem 1. *Let V be a three-dimensional linear tetrahedron bounded by a tetrahedral surface S. Then the structure product over a liner three-tetrahedron (linear arbitrary tetrahedron in three-dimensional space) is given by the equation*

$$III_V = \int\int_{\overline{V}}\int f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{|\det J|}{\det J}\int\int_{\overline{\tau}}\{A(u,v)-B(u,v)-C(u,v)-D(u,v)\}\,\mathrm{d}u\,\mathrm{d}v, \tag{7}$$

where $\overline{\tau}$ is the unit triangle $\langle(0,0),(1,0),(0,1)\rangle$ in the uv-plane and $A(u,v), B(u,v), C(u,v)$ and $D(u,v)$ are explained in the body of the following proof of this theorem.

Proof. Let us define

$$III_V = \int\int_{V}\int f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where V is an arbitrary linear tetrahedron with vertices at $((x_k,y_k,z_k), k=1,2,3,4)$ and $f(x,y,z)=\frac{\partial F}{\partial x}$, or $f(x,y,z)=\frac{\partial F}{\partial y}$, or $f(x,y,z)=\frac{\partial F}{\partial z}$, for some suitable function F. Then it follows that from the transformation:

$$III_V = |\det J|\int\int_{\overline{V}}\int f(x(\xi,\eta,\zeta),y(\xi,\eta,\zeta),z(\xi,\eta,\zeta))\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \tag{8}$$

where $\overline{V}$ is the orthogonal unit tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

We have from the chain rule of partial differentiation

$$\frac{\partial F}{\partial x} = \frac{1}{|J|}\left[\frac{\partial(y,z)}{\partial(\eta,\zeta)}\frac{\partial F}{\partial \xi} - \frac{\partial(y,z)}{\partial(\xi,\zeta)}\frac{\partial F}{\partial \eta} - \frac{\partial(y,z)}{\partial(\xi,\eta)}\frac{\partial F}{\partial \zeta}\right], \tag{9}$$

$$\frac{\partial F}{\partial y} = \frac{1}{|J|}\left[-\frac{\partial(x,z)}{\partial(\eta,\zeta)}\frac{\partial F}{\partial \xi} + \frac{\partial(x,z)}{\partial(\xi,\zeta)}\frac{\partial F}{\partial \eta} - \frac{\partial(x,z)}{\partial(\xi,\eta)}\frac{\partial F}{\partial \zeta}\right], \tag{10}$$

$$\frac{\partial F}{\partial z} = \frac{1}{|J|}\left[\frac{\partial(x,y)}{\partial(\eta,\zeta)}\frac{\partial F}{\partial \xi} - \frac{\partial(x,y)}{\partial(\xi,\zeta)}\frac{\partial F}{\partial \eta} + \frac{\partial(x,y)}{\partial(\xi,\eta)}\frac{\partial F}{\partial \zeta}\right] \tag{11}$$

clearly from Eqs. (9)–(11), we can find:

$$\frac{\partial(y,z)}{\partial(\eta,\zeta)} = \begin{vmatrix} y_{24} & y_{34} \\ z_{24} & z_{34}\end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(\xi,\zeta)} = \begin{vmatrix} y_{34} & y_{14} \\ z_{34} & z_{14}\end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(\xi,\eta)} = \begin{vmatrix} y_{14} & y_{24} \\ z_{14} & z_{24}\end{vmatrix}, \tag{12a}$$

$$\frac{\partial(x,z)}{\partial(\eta,\zeta)} = \begin{vmatrix} x_{34} & x_{24} \\ z_{34} & z_{24}\end{vmatrix}, \quad \frac{\partial(x,z)}{\partial(\xi,\zeta)} = \begin{vmatrix} x_{14} & x_{34} \\ z_{14} & z_{34}\end{vmatrix}, \quad \frac{\partial(x,z)}{\partial(\xi,\zeta)} = \begin{vmatrix} x_{14} & x_{34} \\ z_{14} & z_{34}\end{vmatrix}, \tag{12b}$$

$$\frac{\partial(x,y)}{\partial(\eta,\zeta)} = \begin{vmatrix} x_{24} & x_{34} \\ y_{24} & y_{34}\end{vmatrix}, \quad -\frac{\partial(x,y)}{\partial(\xi,\zeta)} = \begin{vmatrix} x_{34} & x_{14} \\ y_{34} & y_{14}\end{vmatrix}, \quad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} x_{14} & x_{24} \\ y_{14} & y_{24}\end{vmatrix} \tag{12c}$$

and these quantities (12a)–(12c) are constants.

Using chain rule of partial differentiation described above in Eq. (8), now we can write.

$$\begin{aligned} III_V &= \frac{|\det J|}{(\det J)}\int\int_{\overline{V}}\int\left\{\frac{\partial F_1}{\partial \xi} + \frac{\partial F_2}{\partial \eta} + \frac{\partial F_3}{\partial \zeta}\right\}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \\ III_V &= \frac{|\det J|}{(\det J)}\int\int_{\overline{V}}\int \hat{\nabla}\bullet\hat{F}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta, \end{aligned} \tag{13}$$

where

(1)

$$F_1 = F\frac{\partial(y,z)}{\partial(\eta,\zeta)},\quad F_2 = -F\frac{\partial(y,z)}{\partial(\xi,\zeta)},\quad F_3 = F\frac{\partial(y,z)}{\partial(\xi,\eta)}$$

when $f = \frac{\partial F}{\partial x}$, for some arbitrary function F and as a particular case $F = \frac{x^{\alpha+1}y^{\beta}z^{\gamma}}{\alpha+1}$ if $f = x^{\alpha}y^{\beta}z^{\gamma}$

(2)

$$F_1 = -F\frac{\partial(x,z)}{\partial(\eta,\zeta)},\quad F_2 = F\frac{\partial(x,z)}{\partial(\xi,\zeta)},\quad F_3 = -F\frac{\partial(x,z)}{\partial(\xi,\eta)}$$

when $f = \frac{\partial F}{\partial y}$ and as a particular case $F = \frac{x^{\alpha}y^{\beta+1}z^{\gamma}}{\beta+1}$ if $f = x^{\alpha}y^{\beta}z^{\gamma}$

(3)

$$F_1 = F\frac{\partial(x,y)}{\partial(\eta,\zeta)},\quad F_2 = -F\frac{\partial(x,y)}{\partial(\xi,\zeta)},\quad F_3 = F\frac{\partial(x,y)}{\partial(\xi,\eta)}$$

when $f = \frac{\partial F}{\partial z}$ and as a particular case $F = \frac{x^{\alpha}y^{\beta}z^{\gamma+1}}{\gamma+1}$ if $f = x^{\alpha}y^{\beta}z^{\gamma}$

where

$$\begin{aligned} x(\xi,\eta,\zeta) &= x_4 + \xi x_{14} + \eta x_{24} + \gamma x_{34},\\ y(\xi,\eta,\zeta) &= y_4 + \xi y_{14} + \eta y_{24} + \gamma y_{34},\\ z(\xi,\eta,\zeta) &= z_4 + \xi z_{14} + \eta z_{24} + \gamma z_{34}, \end{aligned}$$

with $t_{ij} = t_i - t_j, t = x, y, z;\ i,j = 1,2,3,4$ is the affine transformation which maps a linear arbitrary tetrahedron in (x,y,z) coordinate system into an orthogonal tetrahedron in the local coordinate system (ξ,η,ζ).

$$\det J = \begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\eta} & \frac{\partial z}{\partial\zeta} \end{vmatrix} = \begin{vmatrix} x_{14} & x_{24} & x_{34} \\ y_{14} & y_{24} & y_{34} \\ z_{14} & z_{24} & z_{34} \end{vmatrix}.$$

In order to obtain a working relationship of Eq. (12), let us examine the surface integral $\int\int_{\overline{S}} \vec{F}\cdot\hat{n}\,\mathrm{d}\overline{S}$
Now clearly from Fig. 2 $\overline{S}$ consists of four triangular surfaces

$$\overline{S_1} = \Delta_{123},\quad \overline{S_2} = \Delta_{423},\quad \overline{S_3} = \Delta_{413},\quad \overline{S_4} = \Delta_{412},$$

where Δ_{ijk} means the triangular surface formed by vertices i,j,k. Thus we can write

$$\int\int_{\overline{S}} F\cdot n\,\mathrm{d}\overline{S} = \sum_{i=1}^{4}\int\int_{\overline{S_i}} F\cdot n_i\,\mathrm{d}\overline{S_i}, \tag{14}$$

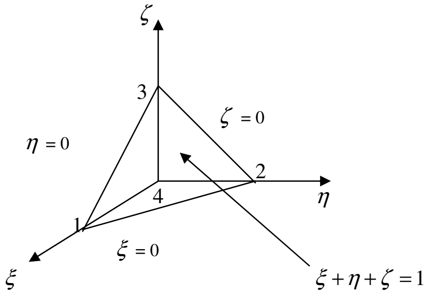

Fig. 2. The unit orthogonal tetrahedron in (ξ,η,ζ) space.

where n_1, n_2, n_3 and n_4 are outward pointing unit normal vectors to $\overline{S_1}, \overline{S}_2, \overline{S_3}$ and $\overline{S_4}$, respectively. By considering the projection of $\overline{S_1}$ on $\xi\eta$ plane, and the equation of surface

$$\overline{S_1} : \xi + \eta + \zeta - 1 = 0,$$

we find

$$\int\int_{\overline{S_1}} F \cdot n_1 d\overline{S_1} = \int_0^1 \int_0^{1-\xi} \sum_{i=1}^{3} F_i(\xi, \eta, 1 - \xi - \eta) \mathrm{d}\xi \, \mathrm{d}\eta. \tag{15}$$

Similarly, we can show that

$$\begin{aligned}
\int\int_{\overline{S_2}} F \cdot n_2 \, \mathrm{d}\overline{S_2} &= -\int_0^1 \int_0^{1-\eta} F_1(0, \eta, \zeta) \, \mathrm{d}\eta \, \mathrm{d}\zeta, \\
\int\int_{\overline{S_3}} F \cdot n_3 \, \mathrm{d}\overline{S_3} &= -\int_0^1 \int_0^{1-\xi} F_2(\xi, 0, \zeta) \, \mathrm{d}\xi \, \mathrm{d}\zeta, \\
\int\int_{\overline{S_4}} F \cdot n_4 \, \mathrm{d}\overline{S_4} &= -\int_0^1 \int_0^{1-\xi} F_3(\xi, \eta, 0) \, \mathrm{d}\xi \, \mathrm{d}\eta.
\end{aligned} \tag{16}$$

Substituting Eqs. (15) and (16) into (14) we obtain

$$\begin{aligned}
\int\int_{\overline{S}} F \cdot n \, \mathrm{d}\overline{S} = &\int_0^1 \int_0^{1-\xi} \sum_{i=1}^{3} F_i(\xi, \eta, 1 - \xi - \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta - \int_0^1 \int_0^{1-\eta} F_1(0, \eta, \zeta) \mathrm{d}\eta \, \mathrm{d}\zeta \\
&- \int_0^1 \int_0^{1-\xi} F_2(\xi, 0, \zeta) \mathrm{d}\xi \, \mathrm{d}\zeta - \int_0^1 \int_0^{1-\xi} F_3(\xi, \eta, 0) \mathrm{d}\xi \, \mathrm{d}\eta.
\end{aligned} \tag{17}$$

From Eqs. (11–13) and (17) we can show that

$$\int\int_V \int f \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \frac{|\det J|}{\det J} \int\int_{\overline{\tau}} \{A(u, v) - B(u, v) - C(u, v) - D(u, v)\} \, \mathrm{d}u \, \mathrm{d}v, \tag{18}$$

where $\overline{\tau}$ is the unit triangle $\langle (0, 0), (1, 0), (0, 1) \rangle$ in the uv-plane.

$$\begin{aligned}
A(u, v) &= \sum_{i=1}^{3} F_i(u, v, 1 - u - v), \\
B(u, v) &= F_1(0, u, v), \\
C(u, v) &= F_2(u, 0, v), \\
D(u, v) &= F_3(u, v, 0).
\end{aligned}$$

We note that if we take $F = \frac{x^{\alpha+1} y^\beta z^\gamma}{\alpha+1}$ then $f = \frac{\partial F}{\partial x} = x^\alpha y^\beta z^\gamma$ and clearly f is a monomial term which is already considered in Rathod and Govinda Rao [7], and in this case

$$\begin{aligned}
F(u, v, 1 - u - v) &= x^{\alpha+1}(u, v, 1 - u - v) y^\beta(u, v, 1 - u - v) z^\gamma(u, v, 1 - u - v)/(\alpha + 1), \\
F(0, u, v) &= x^{\alpha+1}(0, u, v) y^\beta(0, u, v) z^\gamma(0, u, v)/(\alpha + 1), \\
F(u, 0, v) &= x^{\alpha+1}(u, 0, v) y^\beta(u, 0, v) z^\gamma(u, 0, v)/(\alpha + 1), \\
F(u, v, 0) &= x^{\alpha+1}(u, v, 0) y^\beta(u, v, 0) z^\gamma(u, v, 0)/(\alpha + 1),
\end{aligned} \tag{19}$$

$$x(u,v,1-u-v) = x_3 + x_{13}u + x_{23}v,$$

$$y(u,v,1-u-v) = y_3 + y_{13}u + y_{23}v,$$

$$z(u,v,1-u-v) = z_3 + z_{13}u + z_{23}v, \tag{20}$$

$$x(0,u,v) = x_4 + x_{24}u + x_{34}v,$$

$$y(0,u,v) = y_4 + y_{24}u + y_{34}v,$$

$$z(0,u,v) = z_4 + z_{24}u + z_{34}v, \tag{21}$$

$$x(u,0,v) = x_4 + x_{14}u + x_{34}v,$$

$$y(u,0,v) = y_4 + y_{14}u + y_{34}v,$$

$$z(u,0,v) = z_4 + z_{14}u + z_{34}v, \tag{22}$$

$$x(u,v,0) = x_4 + x_{14}u + x_{24}v,$$

$$y(u,v,0) = y_4 + y_{14}u + y_{24}v,$$

$$z(u,v,0) = z_4 + z_{14}u + z_{24}v. \qquad \square \tag{23}$$

5. Application example

We consider some typical integrals with known exact values.

5.1. Evaluate

$$I_1 = \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \sqrt{\xi+\eta+\zeta}\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi = \int_0^1 \int_0^{1-\xi} \frac{2}{3}[1-(\xi+\eta)^{\frac{3}{2}}]\mathrm{d}\eta\,\mathrm{d}\xi = 0.142857142857,$$

$$I_2 = \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \frac{\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta}{\sqrt{\xi+\eta+\zeta}} = \int_0^1 \int_0^{1-\xi} 2[1-\sqrt{\xi+\eta}]\mathrm{d}\eta\,\mathrm{d}\xi = 0.200000000000,$$

$$I_3 = \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} \sin(\xi+2\eta+4\zeta)\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi$$

$$= \int_0^1 \int_0^{1-\xi} \frac{1}{4}[\cos(\xi+2\eta) - \cos(4-3\xi-2\eta)]\mathrm{d}\eta\,\mathrm{d}\xi = 0.13190232688,$$

$$I_4 = \int_0^1 \int_0^{1-\xi} \int_0^{1-\xi-\eta} (1+\xi+\eta+\zeta)^{-4}\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi$$

$$= \int_0^1 \int_0^{1-\xi} \left[\frac{(1+\xi+\eta)^{-3}}{3} - \frac{1}{24}\right]\mathrm{d}\eta\,\mathrm{d}\xi = 0.0208333333333333.$$

5.2. Evaluate

$$I_{2,1} = \int\int_V\int \frac{x^2y}{\sqrt{x+y+z}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where V is the tetrahedron spanning the vertices $\langle(5,5,0),(10,10,0),(8,7,8),(10,8,0)\rangle$

$$f=\frac{\partial}{\partial z}\{x^2y\times 2\sqrt{x+y+z}\}=\frac{\partial F}{\partial z},$$
$$F=2x^2y\sqrt{x+y+z},$$
$$F_1=F\frac{\partial(x,y)}{\partial(\eta,\zeta)},$$
$$F_2=-F\frac{\partial(x,y)}{\partial(\xi,\zeta)},$$
$$F_3=F\frac{\partial(x,y)}{\partial(\xi,\eta,)},$$
$$F_1=F\frac{\partial(x,y)}{\partial(\eta,\zeta)}=F\begin{vmatrix}x_{24} & x_{34}\\ y_{24} & y_{34}\end{vmatrix}=F\begin{vmatrix}0 & -2\\ 5 & 2\end{vmatrix}=10F,$$
$$F_2=-F\frac{\partial(x,y)}{\partial(\xi,\zeta)}=F\begin{vmatrix}x_{34} & x_{14}\\ y_{34} & y_{14}\end{vmatrix}=F\begin{vmatrix}-2 & -5\\ 2 & 0\end{vmatrix}=10F,$$
$$F_3=F\frac{\partial(x,y)}{\partial(\xi,\eta,)}F\begin{vmatrix}x_{14} & x_{24}\\ y_{14} & y_{24}\end{vmatrix}=F\begin{vmatrix}-5 & 0\\ 0 & 5\end{vmatrix}=-25F,$$
$$x(\xi,\eta,\zeta)=x_4+\xi x_{14}+\eta x_{24}+\gamma x_{34}=10-5\xi-2\zeta,$$
$$y(\xi,\eta,\zeta)=y_4+\xi y_{14}+\eta y_{24}+\gamma y_{34}=5+5\eta+2\zeta,$$
$$z(\xi,\eta,\zeta)=z_4+\xi z_{14}+\eta z_{24}+\gamma z_{34}=8\zeta$$
$$F=2x^2y\sqrt{x+y+z}=2(10-5\xi-2\zeta)^2(5+5\eta+2\zeta)\sqrt{15-5\xi+5\eta+8\zeta},$$
$$\sum_{i=1}^{3}F_i(\xi,\eta,1-\xi-\eta)=10F(\xi,\eta,1-\xi-\eta)+10F(\xi,\eta,1-\xi-\eta)-25F(\xi,\eta,1-\xi-\eta)$$
$$=-5F(\xi,\eta,1-\xi-\eta),$$
$$\sum F_i=-5F(\xi,\eta,1-\xi-\eta)=-1(8-3\xi+2\eta)^2(7-2\xi+3\eta)\times\sqrt{(23-13\xi-3\eta)},$$
$$F_1(0,\eta,\zeta)=20(10-2\zeta)^2(5+5\xi+2\zeta)\times\sqrt{(15+5\eta+8\zeta)},$$
$$F_2(\xi,0,\zeta)=20(10-5\xi-2\zeta)^2(5+2\zeta)\times\sqrt{(15-5\eta+8\zeta)},$$
$$F_3(\xi,\eta,0)=-50(10-5\xi)^2(5+5\eta)\times\sqrt{(15-5\xi+5\eta)}.$$

Using Eq. (17), we get

$$I_{2,1}=\int\int_V\int\frac{x^2y}{\sqrt{x+y+z}}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\int_0^{1-x}\left\{\begin{array}{l}10(8-3x+2y)^2(7-2x+3y)\sqrt{(23-13x-3y)}\\+20(10-2y)^2(5+5x+2y)\sqrt{(15+5x+8y)}\\+20(10-5x-2y)^2(5+2y)\sqrt{(15-5x+8y)}\\-50(10-5x)^2(5+5y)\sqrt{(15-5x+5y)}\end{array}\right\}\mathrm{d}x\mathrm{d}y. \tag{24}$$

In general, we can show that

$$I_{\alpha,\beta}=\int\int_V\int\frac{x^\alpha y^\beta}{\sqrt{x+y+z}}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_0^1\int_0^{1-x}\left\{\begin{array}{l}10(8-3x+2y)^\alpha(7-2x+3y)^\beta\sqrt{(23-13x-3y)}\\+20(10-2y)^\alpha(5+5x+2y)^\beta\sqrt{(15+5x+8y)}\\+20(10-5x-2y)^\alpha(5+2y)^\beta\sqrt{(15-5x+8y)}\\-50(10-5x)^\alpha(5+5y)^\beta\sqrt{(15-5x+5y)}\end{array}\right\}\mathrm{d}x\mathrm{d}y. \tag{25}$$

6. Formulation of integrals over a triangular area

The finite element method for two dimensional problems with triangular elements requires the numerical integration of shape functions on a triangle. Since an affine transformation makes it possible to transform any triangle into the two dimensional standard triangle T with coordinates $(0,0),(0,1),(1,0)$ in Cartesian frame, we have just to consider numerical integration on T. The integral of an arbitrary function, f, over the surface of a triangle T is given by

$$I = \int\int_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} f(x,y)\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_0^{1-y} f(x,y)\,\mathrm{d}x. \tag{26}$$

It is now required to find the value of the integral by a quadrature formula:

$$I = \sum_{m=1}^{N} c_m f(x_{m,} y_m), \tag{27}$$

where c_m are the weights associated with specific points (x_m, y_m) and N is the number of pivotal points related to the required precision. One such accurate method known to the present authors is based on 13 integration points [8] and it does not appear likely that this technique will be extended much further to give a greater accuracy which may be demanded in future. The other method is the approximation of I by product formulae [9] which is of the type (27) based on the roots and weights of Gauss Legendre and Gauss Jacobi polynomials. The precision of these formulae is limited to polynomials of degree seven; this is because the weights and roots of Jacobi polynomials are not tabulated in standard texts for sufficiently higher degree polynomials. The product formulae proposed in this paper and in the recent works [1,10–13] are based only on the roots and weights of Gauss Legendre polynomials.

The integral I of Eq. (26) can be transformed into an integral over the surface of the square: $\{(u,v) \mid 0 \leqslant u, v \leqslant 1\}$ by the substitution:

$$x = u, \quad y = (1-u)v. \tag{28}$$

Then the determinant of the Jacobian and the differential area are

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (1)(1-u) - 0(-v) = 1-u \quad \text{and} \quad \mathrm{d}x\,\mathrm{d}y = \frac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}u\,\mathrm{d}v \\ = (1-u)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned} \tag{29}$$

Then on using Eqs. (28) and (29) in Eq. (26), we have

$$\int_0^1 \int_0^{1-x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^1 f(u,(1-u)v)(1-u)\,\mathrm{d}u\,\mathrm{d}v. \tag{30}$$

The integral I of Eq. (30) can be further transformed into an integral over the standard 2-square: $\{(\xi,\eta) | -1 \leqslant \xi, \eta \leqslant 1\}$ by the substitution

$$u = (1+\xi)/2, \quad v = (1+\eta)/2 \tag{31}$$

then clearly the determinant of the Jacobian and the differential area are

$$\begin{aligned}&\frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = (1/2)(1/2) - (0)(0) = 1/4, \\ &\mathrm{d}u\,\mathrm{d}v = \frac{\partial(u,v)}{\partial(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{4}\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned} \tag{32}$$

Now on using Eqs. (31) and (32) in Eq. (30), we have

$$I = \int_0^1 \int_0^{1-x} f(x,y)\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_0^1 f(u,(1-u)v)(1-u)\mathrm{d}u\,\mathrm{d}v$$
$$= \int_{-1}^1 \int_{-1}^1 f\left(\frac{1+\xi}{2}, \frac{(1-\xi)(1+\eta)}{4}\right)\left(\frac{1-\xi}{8}\right)\mathrm{d}\xi\,\mathrm{d}\eta. \quad (33)$$

Eq. (33) represents an integral over the surface of a standard 2-square:
$\{(\xi,\eta) \mid -1 \leqslant \xi,\eta \leqslant 1\}$. Efficient quadrature coefficients are readily available in the literature so that any desired accuracy can be readily obtained [14].

From Eq. (33), we can write

$$I = \int_{-1}^1 \int_{-1}^1 f(x(\xi,\eta), y(\xi,\eta))\left(\frac{1-\xi}{8}\right)\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$I = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{1-\xi_i}{8}\right) w_i w_j f(x(\xi_i,\eta_j), y(\xi_i,\eta_j)), \quad (34)$$

where ξ_i, η_j are Gaussian points in the ξ, η directions, respectively w_i and w_j are the corresponding weights.

We can rewrite Eq. (34) as:

$$I = \sum_{k=1}^{N=n\times n} c_k f(x_k, y_k), \quad (35)$$

where c_k, x_k and y_k can be obtained from the relations:

$$c_k = \frac{(1-\xi_i)}{8} w_i w_j, \quad x_k = \frac{(1+\xi_i)}{2}, \quad y_k = \frac{(1-\xi_i)(1+\eta_j)}{4} \quad (k = 1,2,\ldots,n), \ (i,j = 1,2,3,\ldots,n). \quad (36)$$

The weighting coefficients c_k and sampling points (x_k, y_k) of various order can be now easily computed by formulas of Eqs. (35) and (36). We have listed here a C-Program which generates c_k, x_k and y_k and then computes the integral $\int\int_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.We have also given the sample output of the C-Program for $n = 2, 3$.

6.1. C-Program for generating weight coefficients and sampling points

```
# include ⟨stdio.h⟩
# include ⟨conio.h⟩
# include ⟨math.h⟩
void main ()
{
   double c[10][10], x[10][10], y[10][10], p[10], q[10], w1[10], w2[10];
int k, i, j, n;
clrscr ();
printf (" input n\n");
scanf ("%d", &n);
printf ("enter %dp values\n",n);
for (i=0; i < n; ++i){
scanf ("% lf",& x[i] );
}
printf ("enter % dq values\n",n);
for (i=0; i < n; ++i){
```

```
scanf ("% lf",& y[i] );
}
printf ("enter % d w1 values\n",n);
for (i=0; i < n; ++i){
scanf ("% lf",& w1[i]);
}
printf ("enter % d w1 values\n",n);
for (i=0; i < n; ++i){
scanf ("% lf",& w2[i]);
}
for (i=0; i < n; ++i){
for (j=0; j < n; ++j){
c[i][j] = (( 1 - p[i] )/8.0)*(w1 [i]*w2[j]);
x[i][j] = ((1 + p[i] )*(1 + 0* q[j]))/4.0;
y[i][j] = ((1 - p[i])*(1 + q[j]))/4.0;
}
for (i = 0; i < n; ++i)
{
for (j = 0; j < n; ++j)
{
printf ("% 0.15lf \t % 0.15 lf \t % 0.15 lf \n", c[i][j], x[i][j], y[i][j]);
}
}
getch ( );
}
```

6.2. Sample output of weight coefficients and sampling points

x_k	y_k	c_k
$n = 2$		
0.211324865	0.166666667	0.197168783
0.211324865	0.622008467	0.197168783
0.788675134	0.044658198	0.052831216
0.788675134	0.166666667	0.052831216
$n = 3$		
0.112701665	0.100000000	0.068464377
0.112701665	0.443649167	0.109543004
0.112701665	0.787298334	0.068464377
0.500000000	0.056350832	0.061728395
0.500000000	0.250000000	0.098765432
0.500000000	0.443649167	0.061728395
0.887298334	0.012701665	0.008696116
0.887298334	0.056350832	0.013913785
0.887298334	0.100000000	0.008696116

We can now evaluate the triple integrals I_1, I_2, I_3, I_4, I_5 by application of the numerical integration rule of Eq. (27). We find excellent convergence to the exact value in each case. The results are summarized in Tables 1 and 2.

Table 1
Numerical results of double integration using Gauss Legendre quadrature rules ($n = 2, 3, 4, \ldots, 10$)

n	I_1	I_2	I_3	I_4
$n = 2$	0.143110598	0.197845653	0.132949941	0.020163481
$n = 3$	0.142872769	0.19964004	0.131878572	0.020802826
$n = 4$	0.142859506	0.199900863	0.131902583	0.020832132
$n = 5$	0.142857690	0.199964167	0.131902325	0.020833289
$n = 6$	0.142857309	0.199984574	0.131902328	0.020833332
$n = 7$	0.142857203	0.199992491	0.131902327	0.020833333
$n = 8$	0.142857167	0.199995994	0.131902327	0.020833333
$n = 9$	0.142857155	0.199997709	0.131902328	0.020833333
$n = 10$	0.142857147	0.199998611	0.131902326	0.020833333

Table 2
Numerical results of double integration using Gauss Legendre quadrature rules ($n = 2, 3, 4, \ldots, 10$)

n	$I_{2,1}$	$I_{2,2}$	$I_{4,4}$
$n = 2$	3885.026445	28352.804692	201298122.324282
$n = 3$	3784.373937	26254.724496	102607699.018003
$n = 4$	3784.400584	26253.291791	100727688.818777
$n = 5$	3784.400843	26253.293286	100719784.572472
$n = 6$	3784.400787	26253.292725	100719778.788107
$n = 7$	3784.400471	26253.289423	100719744.374018
$n = 8$	3784.400960	26253.294563	100719798.263909
$n = 9$	3784.400519	26253.289912	100719750.558736
$n = 10$	3784.400996	26253.294944	100719799.301592

7. Conclusions

This paper proposes to compute the volume integral of certain functions whose antiderivates with respect to one of the variates (say either x or y or z) is available. Then by use of the well known Gauss Divergence theorem, it can be shown that the volume integral of such a function is expressible as sum of four integrals over the unit triangle. The present method can also evaluate the triple integrals of trivariate polynomials over an arbitrary tetrahedron as a special case. It is also demonstrated that certain integrals which are nonpolynomial functions of trivariates x, y, z can be computed by the proposed method. We have applied Gauss Legendre Quadrature rules [1] to evaluate the typical integrals governed by the proposed method. The results obtained are in excellent agreement with the exact value.

References

[1] H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, N.L. Ramesh, Gauss Legendre Quadrature over a Triangle, J. Indian Inst. Sci. 84 (2004) 183–188.
[2] Y.T. Lee, A.A.G. Requicha, Algorithms for computing the volume and other integral properties of solids I: known methods and open issues, Commun. ACM 25 (1982) 635–641.
[3] H.G. Timmer, J.M. Stern, Computation of global geometric properties of solid objects, Comput. Aid. Des. 12 (1980) 301–304.
[4] S. Lien, J.T. Kajiya, A symbolic method for calculating the integral properties of arbitrary nonconvex polyhedra, IEEE Comput. Graph. Appl. 4 (1984) 35–41.
[5] C. Cattani, A. Paoluzzi, Boundary integration over linear polyhedra, Comput. Aid. Des. 22 (1990) 130–135.
[6] F. Bernardini, Integrating of polynomials over n-dimensional polyhedra, Comput. Aid. Des. 23 (1991) 51–58.
[7] H.T. Rathod, H.S. Govinda Rao, Integration of polynomials over an arbitrary tetrahedron in Euclidean three-dimensional space, Int. J. Comput. Struct. 59 (1) (1996) 55–65.
[8] G.R. Cowper, Gaussian quadrature formulas for triangles, Int. J. Num. Meth. Eng. 7 (1973) 405–408.
[9] P. Hillion, Numerical integration on a triangle, Int. J. Num. Meth. Eng. 11 (1977) 797–815.
[10] H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, Gauss Legendre Quadrature over a triangle, Acharya Nagarjuna Int. J. Math. Inform. Technol. 1 (1) (2004) 33–52.

[11] H.T. Rathod, K.V. Nagaraja, Symmetric Gauss Legendre quadrature over a triangle, J. Sci. Eng. Technol. 17 (1) (2005) 1–8.
[12] H.T. Rathod, K.V. Nagaraja, B. Venkatesudu, Symmetric Gauss Legendre quadrature formulas for composite numerical integration over a triangular surface, Appl. Math. Comput., in press.
[13] H.T. Rathod, B. Venkatesudu, K.V. Nagaraja, On the application of two Gauss- Legendre quadrature rules for composite numerical integration over a tetrahedral region, Appl. Math. Comput., in press.
[14] F.G. Lethor, Computation of double integrals over a triangle, J. Comp. Appl. Math. 2 (1976) 219–224.