DISCUSSION

Comments on the articles "Hyperbolic thermoelasticity: A review of recent literature" (Chandrasekharaiah DS, 1998, *Appl Mech Rev* 51(12), 705–729) and "Thermoelasticity with second sound: A review" (Chandrasekharaiah DS, 1986, *Appl Mech Rev* 39(3), 355–376)

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In the articles *Thermoelasticity with second sound: A review* [1] and *Hyperbolic thermoelasticity: A review of recent literature* [2], Chandrasekharaiah has presented an in-depth look at nonconventional (a.k.a. generalized or non-Fourier) theories of thermoelasticity. The motivation driving the formulation of these theories is the desire to overcome the infinite propagation speed of thermal signals predicted by conventional thermoelasticity (CTE), the so-called "paradox of heat conduction."

In [1], two of these nonconventional theories are examined in the context of the Danilovskaya problem (DP). (In the DP, the homogeneous and isotropic thermoelastic half-space $x>0$, under a stress free boundary condition (BC) at $x=0$, is subjected to a Heaviside, or step, temperature BC at time $t=0^+$.) The first he refers to as extended thermoelasticity (ETE) and the second as temperature-rate dependent thermoelasticity (TRDTE). In both ETE and TRDTE, the parabolic diffusion equation of CTE is replaced with a hyperbolic heat transport equation. As a result, both theories predict thermal waves (*ie*, second sound) propagating with finite speeds.

In ETE, a single relaxation time $\tau>0$ appears and second sound propagates with speed $v_T=\sqrt{\kappa/\tau}$, where κ is used here to denote the thermal diffusivity. It is noted that ETE reduces to CTE in the limit $\tau\to 0$. TRDTE was presented in 1972 by Green and Lindsay [3]. This theory involves the two relaxation times α_0 and α, where $\alpha\geqslant\alpha_0>0$ and, in the case of homogeneous and isotropic materials, reduces to CTE in the limit $\alpha\to 0$. (While it has been postulated that α_0 is actually non-negative, it must be noted that TRDTE admits second sound only when $\alpha_0>0$ [1].) An important aspect of TRDTE is that Fourier's heat law is not violated in materials that have a center of symmetry at each point [2,3].

Although it was not pointed out in [1], Chandrasekharaiah did note in [2] several physically unrealistic results associated with TRDTE, in particular the fact that the displacement suffers jump discontinuities in the presence of a step temperature BC. A natural question that arises is why was this problem with TRDTE not reported in [1], especially since the author of that paper derived parts of the small-time solution to the DP for a TRDTE medium? (See [2] and the references therein for a discussion of the problems with TRDTE.)

The intent of the present Letter is the following: *i*) Show that the small-time expression given in [1] for the normal stress corresponding to the DP for a TRDTE medium is incorrect; *ii*) Show how this erroneous expression could have lead to the aforementioned shortcoming of TRDTE being missed in [1]; and *iii*) Give for the record the correct small-time expressions for the normal stress, displacement, and strain corresponding to the DP for a TRDTE medium. Lastly, all quantities below are dimensionless, unless stated otherwise the same notation employed in [1] is used here, and the reader is referred to [1] for the definition of all undefined symbols.

In the Laplace transform domain, the normal stress is given by (see Eq. (5.53) of [1])

$$\bar{\sigma}(x,s)=\frac{T_0 s(1+\alpha s)\{e^{-n_1x}-e^{-n_2x}\}}{n_1^2-n_2^2}, \tag{1}$$

where s is the transform parameter. For large s, it can be shown that

$$n_{1,2}\approx s/V_{1,2}^*+r_{1,2},$$
$$\frac{1}{n_1^2-n_2^2}\approx\frac{1}{s^2\sqrt{M_0}}+\frac{N_0}{s^3M_0^{3/2}}-\frac{1}{2s^4M_0^{3/2}}\left((1+\epsilon)^2-\frac{3N_0^2}{M_0}\right), \tag{2}$$

where $V_{1,2}^*$ ($V_2^*>V_1^*$) and M_0 are positive constants and V_2^* denotes the speed of the second sound (*ie*, thermal) wave. Using Eqs. (2), the large-s expression for $\bar{\sigma}(x,s)$ is found to be

$$\bar{\sigma}(x,s)\approx\frac{T_0(1+\alpha s)}{\sqrt{M_0}}\left[\frac{1}{s}+\frac{N_0}{s^2M_0}-\frac{1}{2s^3M_0}\times\left((1+\epsilon)^2-\frac{3N_0^2}{M_0}\right)\right]\{\exp[-(r_1+s/V_1^*)x]-\exp[-(r_2+s/V_2^*)x]\}. \tag{3}$$

Expanding and rearranging Eq. (3) into increasing powers of $1/s$, and then truncating all terms after $1/s^2$ so as to match [1], gives

$$\bar{\sigma}(x,s)\approx\frac{T_0}{\sqrt{M_0}}\left[\alpha+\frac{1}{s}(1+\alpha R_0)+\frac{S_0}{s^2}\right]\{\exp[-(r_1+s/V_1^*)x]-\exp[-(r_2+s/V_2^*)x]\}, \tag{4}$$

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where $R_0=N_0/M_0$ and $S_0=(2R_0-\alpha M_0^{-1}(1+\epsilon)^2+3\alpha R_0^2)/2$. (The quantity S_0 does not appear in [1], it is introduced here for convenience.) Inverting Eq. (4), the small-time expression for the normal stress is found to be

$$\sigma(x,t)\approx\frac{T_0}{\sqrt{M_0}}\sum_{j=1}^{2}(-1)^{j+1}\{\alpha\delta(t-x/V_j^*)+H(t-x/V_j^*)\times[(1+\alpha R_0)+(t-x/V_j^*)S_0]\}e^{-r_jx}, \tag{5}$$

where $H(\cdot)$ is the Heaviside unit step function and $\delta(\cdot)$ denotes the Dirac delta function. (The notation used here for $H(\cdot)$ is slightly different than that of [1].) Equation (5) is the correct form of Eq. (5.58) in [1]. Comparing the former with the latter it is clear why the latter is incorrect; the contribution of the term αs, which is part of the quantity $(1+\alpha s)$ in the numerator of Eq. (5.53) of [1], is missing in the inverse. Indeed, no term with coefficient α that appears in Eq. (5) is present in Eq. (5.58) of [1]. In particular, Eq. (5.58) of [1] does not contain the two delta function terms that it should. A second consequence of these missing terms is that the expressions given in Eq. (5.61) of [1] for $\sigma_{1,2}^*$, where $T_0\sigma_{1,2}^*\equiv[\sigma^+-\sigma^-]_{x=x_{1,2}^*}\equiv\sigma(x_{1,2}^*,t+0)-\sigma(x_{1,2}^*,t-0)$ denote the amplitudes of the jumps in σ across the wavefronts $x=x_{1,2}^*$, are also incorrect. Specifically, since the (correct) expression for $\sigma(x,t)$ exhibits two propagating delta functions, $|\sigma_{1,2}^*|=\infty$ in the sense of [4].

From the Laplace transforms of Eqs. (5.49) and (5.51) in [1], it can be shown that $\bar{u}=s^{-2}(\partial\bar{\sigma}/\partial x)$, where $\bar{u}$ is the image of the x-component of the displacement vector in the Laplace transform domain. Consequently, using Eq. (1), it follows that

$$\bar{u}(x,s)=-\frac{T_0(1+\alpha s)\{n_1e^{-n_1x}-n_2e^{-n_2x}\}}{s(n_1^2-n_2^2)}. \tag{6}$$

Again using the approximations given in Eqs. (2), the large-s expansion of $\bar{u}$ turns out to be

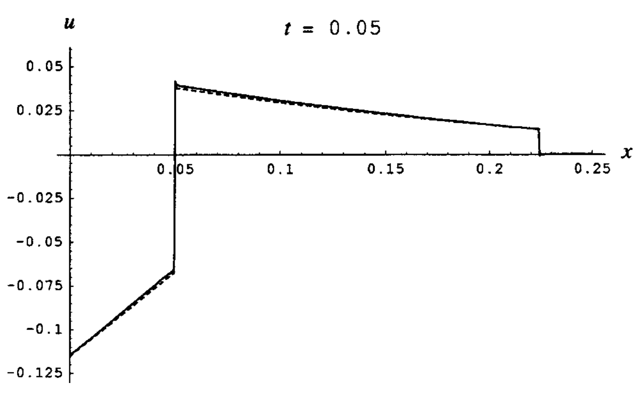

Fig. 1 u vs. x for $\alpha_0=0.05$, $\alpha=0.1$, $\epsilon=0.0356$, and $T_0=1$. Solid: Inverse of Eq. (6) computed using TRSIA with 1000 terms. Broken: Small-time solution given in Eq. (8).

$$\bar{u}(x,s)\approx\frac{T_0}{\sqrt{M_0}}\sum_{j=1}^{2}(-1)^j\left[\frac{1}{s}\left(\frac{\alpha}{V_j^*}\right)+\frac{1}{s^2}\left(\alpha r_j+\frac{\alpha R_0+1}{V_j^*}\right)\right]\exp[-(r_j+s/V_j^*)x]. \tag{7}$$

On inverting Eq. (7), the small-time solution for u is found to be

$$u(x,t)\approx\frac{T_0}{\sqrt{M_0}}\sum_{j=1}^{2}(-1)^j\left[\frac{\alpha}{V_j^*}+(t-x/V_j^*)\left(\alpha r_j+\frac{\alpha R_0+1}{V_j^*}\right)\right]e^{-r_jx}H(t-x/V_j^*). \tag{8}$$

From Eq. (8), it is clear that u always admits two propagating jump discontinuities, the amplitudes of which are given by

$$u_{1,2}^*=\mp\left(\frac{\alpha}{V_{1,2}^*\sqrt{M_0}}\right)e^{-r_{1,2}x_{1,2}^*}. \tag{9}$$

In Eq. (9), $T_0u_{1,2}^*\equiv[u^+-u^-]_{x=x_{1,2}^*}$ denote the amplitudes of the jumps in u across $x=x_{1,2}^*$. (The quantities $u_{1,2}^*$ are introduced here in a manner consistent with the notation convention of [1].)

While not given in [1], the small-time relation for the strain will be given here for completeness. To this end, Eq. (6) is differentiated with respect to x, re-expressed using the identities $n_{1,2}^2=\{s(1+\epsilon)+s^2L_0\pm(n_1^2-n_2^2)\}/2$ and the approximations given in Eqs. (2), and then inverted to yield the small-time strain relation

$$\frac{\partial u}{\partial x}(x,t)\approx\frac{(1+\epsilon)}{2}\int_0^t\sigma(x,t')\mathrm{d}t'+\frac{L_0}{2}\sigma(x,t)+\frac{T_0}{2}\sum_{j=1}^{2}e^{-r_jx}\{H(t-x/V_j^*)+\alpha\delta(t-x/V_j^*)\}. \tag{10}$$

In Eq. (10), $\sigma(x,\cdot)$ denotes the right-hand side of Eq. (5).

Figure 1 shows a comparison of the inverse of Eq. (6) with the small-time solution given in Eq. (8). The inverse of Eq. (6) was computed numerically using Tzou's Riemann sum inversion algorithm (TRSIA) [5] and the values of the material parameters were obtained from Table II of [1]. As shown in Fig. 1, Eq. (8) is a very good/excellent approximation to u for $t\lesssim 0.05$. In addition, the two propagating jumps are clearly visible, with $|u_1^*|>|u_2^*|$, and it is noted that $x_{1,2}^*$ are the elastic (trailing) and thermal (leading) wavefronts, respectively.

It must be pointed out that the presence of propagating jumps in u violates the continuity of displacements requirement [[6], p. 142], and thus indicates that TRDTE is inconsistent with the continuum theory of matter under a step (actually any discontinuous) temperature BC (see [2] and the references therein). These jumps, which occur in both the coupled ($\epsilon > 0$) and uncoupled ($\epsilon = 0$) cases, vanish only in the limit $\alpha \rightarrow 0$. (For a treatment of the uncoupled, spherically symmetric case for a shell, see [7].)

Finally, it should be mentioned that an error similar to the one corrected here, in which all $\delta(\cdot)$ and $\delta'(\cdot)$ terms are missing from the Laplace inverse, occurs in the expression for the strain (*ie*, Eq. (48)) in [8]. (It is of interest to note that had the correct expression for the strain been obtained in [8], the drawbacks with TRDTE could have been uncovered in 1980.) However, while Eq. (5.58) of [1] is incorrect, and this error appears to have directly resulted in the primary physically objectionable feature of TRDTE being overlooked in [1] as well as to the mistaken claim ([1], p 371) that the TRDTE expression for $\sigma(x,t)$ reduces to its ETE counterpart (ie, Eq. (4.39) of [1]) when $\alpha = \alpha_0 = \tau$, Chandrasekharaiah's two articles [1,2] nevertheless provide an excellent review of the literature on nonconventional thermoelasticity and contain a wealth of information on the subject.

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REFERENCES

[1] Chandrasekharaiah DS (1986), Thermoelasticity with second sound: A review, *Appl. Mech. Rev.*, **39**(3), 355–376.
[2] Chandrasekharaiah DS (1998), Hyperbolic thermoelasticity: A review of recent literature, *Appl. Mech. Rev.*, **51**(12), 705–729.
[3] Green AE and Lindsay KA (1972), Thermoelasticity, *J. Elast.*, **2**, 1–7.
[4] Boley BA and Hetnarski RB (1968), Propagation of discontinuities in coupled thermo-elastic problems, *ASME J. Appl. Mech.*, **35**, 489–494.
[5] Tzou DY (1997), *Macro- to Microscale Heat Transfer: The Lagging Behavior*, Taylor and Francis, Washington DC.
[6] Achenbach JD (1973), *Wave Propagation in Elastic Solids*, North-Holland, Amsterdam.
[7] Jordan PM and Puri P (2001), Thermal stresses in a spherical shell under three thermo-elastic models, *J. Therm. Stresses*, **24**, 47–70.
[8] Chandrasekharaiah DS (1980), On generalised thermoelastic wave propagation, *Proc of Indian Acad Sci (Math Sci)* **89**, 43–52.

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The authors of the Letter to the Editor have reinvestigated the Danilovskaya's problem (DP) in the context of TRDTE. They have suggested some correction terms to the expressions (5.58) and (5.61) of [1]. They have also obtained small-time solutions for the displacement and strain fields, which were not reported in [1]. Their analysis brings out the fact that some physically unrealistic features of TRDTE (summarized in [2]) can be seen in DP as well.

But for the unnecessarily aggressive language, repetitive statements, and undue length, the letter makes a useful contribution to the literature on TRDTE. I am thankful to the authors of the letter for correcting a couple of mathematical expressions which I had derived some 20 years back.

I am moved by their overall opinion on my two Review Articles [1,2].