Linear and non-linear analyses of convection in a micropolar fluid occupying a porous medium

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Abstract

Linear and weakly non-linear analyses of convection in a micropolar fluid occupying a high-porosity medium are performed. The Brinkman–Eringen momentum equation is considered. The linear and non-linear analyses are, respectively, based on the normal mode technique and truncated representation of Fourier series. The linear theory for a two-phase system reiterates that the preferred mode of convection is stationary as in the case of a single-phase system. An autonomous system of differential equations representing cellular convection arising in the study is considered to analyse the critical points. The Nusselt number is obtained as a function of micropolar and porous medium parameters.
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1. Introduction

It is now common knowledge that fluids with suspended particles are a working medium in many an industrial application. The fluids most often do not subscribe to a Newtonian description and the advent of microcontinuum field theories in such a situation threw open new fertile fields of application (see [1–4]). The common comment in respect of micropolar fluids was that there are no experiments whatsoever in which any of the material moduli could be measured. The work of Migoun [5] demonstrated an experimental method of determining parameters characterizing the microstructure of such fluids and seems to have laid to rest many doubts on the theory.

The theory of thermomicropolar convection began with Datta and Sastry [6] and interestingly continued by Ahmadi [7], Lebon and Perez Garcia [8], Bhattacharya and Jena [9], and Payne and Straughan [10]. The above works give a good understanding of Rayleigh–Benard convection in micropolar fluids. The study of natural convection in micropolar fluids in the presence of external constraints has been also investigated. Qin and Kaloni [11] have discussed the thermal instability in rotating micropolar fluids and established that the non-inertial acceleration is a possible means of controlling convection. Recently Siddheshwar and Pranesh

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[12–17] and Siddheshwar and Dessaux [18] have shown that the magnetic field, non-uniform basic temperature/salinity and thermal/gravity modulation are also effective mechanisms of controlling convection. All the above investigations mentioned earlier have been addressed for a purely viscous case, i.e., no porous matrix.

Investigations related to convection in micropolar fluids occupying a porous medium are sparse. Sharma and Gupta [19] have studied the thermal convection in micropolar fluids in a porous medium. The effect of rotation on thermal convection in micropolar fluids occupying a porous medium has been studied by Sharma and Kumar [20]. El-Hakiem [21] has investigated the effect of transverse magnetic field on natural convection in a boundary layer flow field as well as in the porous medium.

The aforementioned works concern linear theory and a good understanding of non-linear convection is warranted in this type of practically important fluids to assess the amount of heat transfer, finite amplitude cellular motion and other related aspects. A perusal of available literature reveals that Payne and Straughan [22] have studied non-linear convection in micropolar fluids by employing the energy method and existence of sub-critical instabilities was not inferred from their work. Franchi and Straughan [23] have investigated non-linear stability for thermal convection in a micropolar fluid with temperature dependent viscosity. They infer that due to complicated non-linearities which arise, it is necessary to use a generalised energy analysis which requires an intricate use of embedding inequalities. The second-order non-linear systems arising in convective flow of a micropolar fluid has also been studied by Vajravelu et al. [24] in a porous medium.

The analog of problems in micropolar fluids reported above, does not seem to have been addressed in a porous medium. The non-linear realm of convection in micropolar fluids in a porous medium is also presently unaddressed. In view of this, the non-linear convection problem in micropolar fluids occupying a porous medium is investigated in this paper.

2. Mathematical formulation

Consider a horizontal porous layer of infinite extent occupied by a Boussinesquian micropolar fluid (fluid with suspended particles). We note here that the Oberbeck–Boussinesq approximation even for the classical linearly viscous fluid is a rather subtle matter. At this point of time there seems to be no reported work that addresses this issue in micropolar fluids. We take the view that a method similar to that carried out by Rajagopal et al. [25] needs to be followed in justifying the Oberbeck–Boussinesq approximation for micropolar fluids. As there does not seem to be any major factor in micropolar fluids that could contradict the Oberbeck–Boussinesq approximation, we assume the same in the present paper.

We assume that the fluid layer has thickness 'd' and ΔT to be the adverse temperature gradient. The fluid is assumed to be non-Newtonian and describable by Eringen [3] constitutive equation. The present investigation in a porous medium demands an appropriate theory to study two-phase systems. There are rigorous theories in place to study diffusion of fluid through a porous medium (see [26–28]). For Newtonian fluids flowing through a porous medium, equations for momentum and heat transfer have been rigorously derived and are now common knowledge (see [27]). In the case of present problem involving a fluid with microstructure flowing through a porous medium we believe that the microstructures modify the Darcy resistance just as they would the viscous shear. With this understanding and firm belief we introduce a microstructure modified Darcy friction, in addition to the viscous shear. A similar reasoning can be put forth in respect of the assumed single-phase heat transfer equation arrived at by deftly combining the laws of micropolar heat conduction and porous media heat transfer.

In the present work, we use Dupit–Forchheimer relation to give corrections to momentum and angular momentum equations to account for changes in velocity and spin and assume that the porous medium is in thermal equilibrium (i.e. temperature of fluid and temperature of solid particles in porous medium is constant). This is a reasonable assumption of porous medium to consider the general balance laws of mixtures. For

mathematical tractability we confine ourselves to two-dimensional rolls so that all the quantities except the spin are independent of y, a horizontal co-ordinate. Further the boundaries are assumed to be free-free, spin-free with isothermal conditions. Thus the governing equations for a Rayleigh–Benard situation in a Boussinesquian micropolar fluid-saturated porous medium are:

$$q_{i,i} = 0, \tag{1}$$

$$\left\{ \rho_0 \left[\frac{1}{\Phi} \frac{\partial q_i}{\partial t} + \frac{1}{\Phi^2} q_j q_{i,j} \right] + p_{,i} + \rho g \delta_{i3} + \left(\frac{\zeta + \mu_f}{K} \right) q_i \right\} = \left(\frac{2\zeta}{\Phi} + \mu' \right) q_{i,jj} + \frac{\zeta}{\Phi} \varepsilon_{ijk} \tilde{\omega}_{k,j}, \tag{2}$$

$$\rho_0 I \left[\frac{\partial \tilde{\omega}_i}{\partial t} + \frac{1}{\Phi} q_j \tilde{\omega}_{i,j} \right] = (\lambda' + \eta') \tilde{\omega}_{i,ij} + \eta' \tilde{\omega}_{i,kk} + \zeta(\varepsilon_{ijk} q_{k,j} - 2\tilde{\omega}_i), \tag{3}$$

$$M \frac{\partial T}{\partial t} + q_j T_{,j} = \frac{\beta}{\rho_0 C_v \Phi} \varepsilon_{ijk} \tilde{\omega}_{k,j} T_{,k} + \chi_e T_{,kk}, \tag{4}$$

$$\rho = \rho_0 [1 - \alpha_t (T - T_0)], \tag{5}$$

where q_i is the component of velocity vector, M is the heat capacity ratio given by

$$M = \frac{(1 - \Phi)(\rho C_p)_{\text{solid}} + \Phi(\rho C_p)_{\text{fluid}}}{(\rho C_p)_{\text{fluid}}}.$$

The effective thermal diffusivity χ_e is given by

$$\chi_e = \frac{\Phi \chi_{\text{fluid}} + (1 - \Phi) \chi_{\text{solid}}}{(\rho C_p)_{\text{fluid}}},$$

where Φ is the porosity, μ_f is the viscosity of the fluid, μ' is the effective viscosity of the porous medium, ζ is the coupling viscosity, η' is the bulk viscosity co-efficient, λ' is the shear spin viscosity co-efficient, β is the micropolar heat conduction co-efficient, $\tilde{\omega}$ is the spin and C_p is the specific heat at constant pressure. In Eq. (2) K is the permeability and is related to Φ by Karman–Kozeny relation:

$$K = \frac{d_p^2 \Phi^2}{150(1 - \Phi)^2},$$

where d_p is the diameter of spherical particles making up the porous medium. Other quantities in Eqs. (1)–(5) have conventional meaning (see [29]).

3. Basic state

The basic state of the fluid is quiescent and is described by

$$q_{ib} = (0, 0, 0), \quad \tilde{\omega}_{ib} = (0, 0, 0),$$

$$T = T_b(z), \quad \rho = \rho_b(z), \quad p = p_b(z), \tag{6}$$

where the subscript b denotes quantities in the basic state. The basic state pressure p_b, temperature T_b and density ρ_b satisfy the equations

$$\frac{d p_b}{d z} + \rho_b g = 0, \tag{7}$$

$$\frac{d^2 T_b}{d z^2} = 0 \tag{8}$$

and

$$\rho_b = \rho_0[1 - \alpha_t(T_b - T_0)]. \tag{9}$$

Eqs. (1) and (3) are also satisfied by the basic state. The basic state of the fluid is perturbed and hence we assume

$$q_i = q_{ib} + q'_i, \quad \tilde{\omega}_i = \tilde{\omega}_{ib} + \tilde{\omega}'_i,$$

$$T = T_b + T', \quad \rho = \rho_b + \rho', \quad p = p_b + p',$$

thereby leading to the following perturbed equations:

$$\left\{ \rho_0 \left[\frac{1}{\Phi} \frac{\partial q_i}{\partial t} + \frac{1}{\Phi^2} q'_j q'_{i,j} \right] + p'_{,i} + \rho' g \delta_{i3} + \left(\frac{\zeta + \mu_f}{K} \right) q'_i \right\} = \left(\frac{2\zeta}{\Phi} + \mu' \right) q'_{i,jj} + \frac{\zeta}{\Phi} \varepsilon_{ijk} \tilde{\omega}'_{k,j}, \tag{10}$$

$$\rho_0 I \left[\frac{\partial \tilde{\omega}'_i}{\partial t} + \frac{1}{\Phi} q'_j \tilde{\omega}'_{i,j} \right] = (\lambda' + \eta') \tilde{\omega}'_{i,ij} + \eta' \tilde{\omega}'_{i,jj} + \zeta[\varepsilon_{ijk} q'_{k,j} - 2\tilde{\omega}'_i], \tag{11}$$

$$M \frac{\partial T'}{\partial t} + q'_j T'_{,j} = \frac{\beta}{\rho_0 C_v \Phi} \varepsilon_{ijk} \tilde{\omega}'_{k,j} + \chi_e T'_{,jj} \tag{12}$$

$$\rho' = -\alpha_t \rho_0 T', \tag{13}$$

where the prime for q, ω, T, p and ρ denotes quantities of the perturbed state.

We now non-dimensionalise the above equations using the following definitions:

$$(x^*, y^*, z^*,) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d} \right), \quad t^* = \left(\frac{t}{Md^2/\chi_e} \right),$$

$$q_i^* = \frac{q'_i}{(\chi_e/d)}, \quad \Omega = \frac{(\varepsilon_{ijk} \tilde{\omega}'_{k,j})_z}{\chi_e/d^3},$$

$$T^* = \frac{T'}{\Delta T}, \quad \tilde{\omega}_i^* = \frac{\tilde{\omega}'_i}{\chi_e/d^2} \quad \text{and}$$

$$p^* = \frac{p'}{\chi_e (\frac{\zeta + \mu_f}{d^2})}. \tag{14}$$

Using (14) in Eqs. (10)–(12), we get

$$\frac{\lambda}{Pr} \frac{\partial q_i}{\partial t} + \frac{1}{Pr} q_j q_{i,j} + p_{,i} + Da q_i = -R_m T \delta_{i3} + (N_1 + N'_1) q_{i,jj} + N_1 \varepsilon_{ijk} \tilde{\omega}_{k,j}, \tag{15}$$

$$\frac{\lambda N_2}{Pr} \frac{\partial \tilde{\omega}_i}{\partial t} + \frac{N_2}{Pr} q_j \tilde{\omega}_{i,j} = N_4 \tilde{\omega}_{i,ij} + N_3 \tilde{\omega}_{i,jj} + N_1[\varepsilon_{ijk} q_{k,j} - 2\tilde{\omega}_i], \tag{16}$$

$$\frac{\partial T}{\partial t} + q_i - N_5 \varepsilon_{ijk} \tilde{\omega}_{k,j} = T_{,kk}, \tag{17}$$

where the asterisks have been dropped for simplicity and use has been made of Eq. (13) in Eq. (10).

The non-dimensional quantities appearing in Eqs. (15)–(17) are

$$\lambda = \frac{\Phi}{M},$$

$$N_1 = \frac{\zeta}{\Phi(\mu_f + \zeta)} \quad \text{(Coupling parameter)},$$

$$N_2 = \frac{I}{d^2} \quad \text{(Microinertia parameter)},$$

$$N_3 = \frac{\eta'}{d^2(\mu_f + \zeta)\Phi} \quad \text{(Couple stress parameter)},$$

$$N_5 = \frac{\beta}{\Phi \rho_0 C_v d^2} \quad \text{(Micropolar heat conduction parameter)},$$

$$Pr = \frac{(\zeta + \mu_f)\Phi^2}{\rho_0 \chi_e} \quad \text{(Effective Prandtl number)},$$

$$N_1' = \frac{\mu' + \zeta/\Phi}{(\mu_f)} \quad \text{(Modified viscosity ratio)},$$

$$Da = \frac{d^2}{k} \quad \text{(Inverse Darcy number)},$$

$$R_m = \frac{\rho_0 \alpha g \Delta T d^3}{(\mu_f + \zeta)\chi_e} \quad \text{(Effective micropolar Rayleigh number)}.$$

We note here that for $\zeta = 0$, we get the classical non-dimensional parameters associated with Newtonian fluids occupying a porous medium.

Operating curl on the linear momentum transport Eq. (15) and on the angular momentum Eq. (16) we get the following non-dimensional form of vorticity, microrotation and the heat transport:

$$\left(\frac{\lambda}{Pr}\frac{\partial}{\partial t} + Da\right)\nabla^2\psi = -R_m\frac{\partial T}{\partial x} + (N_1 + N_1')\nabla^4\psi - N_1\nabla^2\tilde{\omega}_y + \frac{\lambda}{Pr}J(\psi, \nabla^2\psi), \tag{18}$$

$$\frac{\lambda N_2}{Pr}\frac{\partial \tilde{\omega}_y}{\partial t} = N_3\nabla^2\tilde{\omega}_y = N_1\nabla^2\psi - 2N_1\tilde{\omega}_y + \frac{N_2}{Pr}J(\psi, \tilde{\omega}_y), \tag{19}$$

$$\frac{\partial T}{\partial t} = \frac{\partial \psi}{\partial x} + \nabla^2 T - N_5\frac{\partial \tilde{\omega}_y}{\partial x} + N_5 J(\tilde{\omega}_y, T) + J(\psi, T), \tag{20}$$

where ψ is the stream function defined as

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}. \tag{21}$$

Eqs. (18)–(20) are solved for stress-free, isothermal, spin-free boundaries and hence the boundary conditions are

$$\psi = \frac{\partial^2\psi}{\partial z^2} = T = \tilde{\omega}_y = 0 \quad \text{at } z = 0, 1. \tag{22}$$

In what follows we now discuss the linear theory.

4. Linear stability analysis

To study linear theory we use the linearised version of (18)–(20) and seek the solutions to the linearised equations in the form (see [29])

$$
\begin{bmatrix} \psi \\ \tilde{\omega}_y \\ T \end{bmatrix} = e^{i\omega_1 t} \begin{bmatrix} A \sin \pi\alpha x \\ B \sin \pi\alpha x \\ C \cos \pi\alpha x \end{bmatrix} \sin \pi z. \tag{23}
$$

which satisfy (22). Substituting (23) into Eqs. (18)–(20), we get a system of three homogeneous equations for A, B and C. For a non-trivial solution to A, B and C, we require

$$
R_m = \frac{\left[\dfrac{N_2 \lambda i\omega_1}{Pr} + (2N_1 + N_3\delta^2)\right] \left\{\dfrac{\lambda i\omega_1}{Pr} + [Da + (N_1 + N_1')\delta^2]\delta^2 - N_1^2\delta^4\right\}(i\omega_1 + \delta^2)}{(\pi\alpha)^2 \left[\dfrac{N_2 \lambda i\omega_1}{Pr} + (2N_1 + N_3\delta^2) - N_1 N_5\delta^2\right]}. \tag{24}
$$

By separating the real and imaginary parts in Eq. (24), we get

$$
R_m = \frac{K_5\{K_{10}(K_6 - \omega_1^2 K_8) + \omega_1^2 K_3(K_7 - \omega_1^2 K_9)\}}{(K_{10}^2 + \omega_1^2 K_3^2)} + i\omega_1 N, \tag{25}
$$

where

$$
N = K_5 \left\{ \frac{K_{10}\{(K_7 - \omega_1^2 K_9) + K_3(K_6 - \omega_1^2 K_8)\}}{(K_{10}^2 + \omega_1^2 K_3^2)} \right\},
$$

$$
K_1 = \frac{\lambda}{Pr}, \quad K_2 = [Da\delta^2 + (N + N_1')\delta^4],
$$

$$
\delta^2 = \pi^2(1 + \alpha^2), \quad \alpha^2 = \frac{a^2}{\pi^2},
$$

$$
K_3 = N_2 K_1, \quad K_4 = 2N_1 + N_3\delta^2,
$$

$$
K_5 = \frac{\delta^2}{\pi^2\alpha^2}, \quad K_6 = (K_2 K_4 - N_1^2\delta^4),
$$

$$
K_7 = [(K_2 K_4 - N_1^2\delta^4) + \delta^4(K_1 K_4 + K_2 K_3)],
$$

$$
K_8 = [K_1 K_4 + \delta^2(K_1 K_4 + K_2 K_3)],
$$

$$
K_9 = \frac{\lambda}{Pr}(2N_1 + N_3\delta^2),
$$

and

$$
K_{10} = [(2N_1 + N_3\delta^2) - N_1 N_5\delta^2].
$$

In view of the observation on the 'Principle of Exchange of Stability' made by Lebon and Perez Garcia [8] and Siddheshwar and Pranesh [11,12] we consider only stationary convection, i.e. $\omega_1 = 0$. Thus the expression

Table 1
Comparison of critical Rayleigh number and critical wave number between Newtonian and Micropolar for different values of Da

Da	Newtonian		Micropolar	
	R_c	a_c^2	R_{mc}	a_c^2
1.0	642.60	5.10	1156.54	5.00
5.0	816.69	5.70	1340.67	5.40
10.0	1028.73	6.20	1566.41	5.80
15.0	1238.19	6.60	1788.72	6.10
20.0	1443.45	6.90	2008.57	6.30

Chosen parameter values: $N_1' = 1.5$, $N_1 = 0.10$, $N_3 = 2.0$, $N_5 = 1.0$. Remarks: (i) $R_c < R_{mc}$, $(a_c^2)_{\text{Micropolar}} < (a_c^2)_{\text{Newtonian}}$. (ii) The effect of Da on R_{mc} is qualitatively similar to that of the Newtonian problem in a porous medium.

Table 2
Comparison of the effect of viscosity ratio (Brinkman number) on the critical Rayleigh number and critical wave number for Newtonian and Micropolar fluids

Viscosity ratio	Newtonian		Micropolar	
(N_1')	R_c	a_c^2	R_{mc}	a_c^2
0.80	749.75	5.80	852.97	5.70
0.90	816.69	5.70	922.78	5.60
1.00	883.58	5.60	992.53	5.60
1.50	1217.22	5.40	1340.67	5.40
2.00	1550.25	5.30	1688.29	5.30
2.50	1883.00	5.30	2035.68	5.20

Chosen parameter values: $Da = 5.0$, $N_1 = 0.1$, $N_3 = 2.0$, $N_5 = 1.0$. Remarks: (i) The effect of N_1' on R_{mc} is qualitatively similar to that of the effect of ratio of viscosities, Λ, for Newtonian problem in a porous medium. (ii) $R_c < R_{mc}$, $(a_c^2)_{\text{Micropolar}} < (a_c^2)_{\text{Newtonian}}$.

Table 3
Comparison of critical Rayleigh number, R_{mc}, and critical wave number, a_c^2, for different values coupling parameter, N_1

N_1	Porous case		Purely viscous case [30]	
	R_{mc}	a_c^2	R_{mc}	a_c^2
0.00	1204.85	5.42	657.51	4.94
0.01	1217.85	5.42	667.42	4.93
0.05	1271.11	5.41	707.98	4.93
0.10	1340.67	5.39	760.83	4.93
0.20	1490.69	5.36	874.42	4.93
0.50	2051.47	5.27	1293.95	4.90
0.80	2862.55	5.16	1885.35	4.83
1.00	3636.41	5.05	2433.42	4.77

Chosen parameter values: (i) Viscous case $Da = 0$, $N_1' = 1.0$, $N_3 = 2.0$, $N_5 = 1.0$. (ii) Porous case $Da = 5.0$, $N_1' = 1.5$, $N_3 = 2.0$, $N_5 = 1.0$. Remarks: (i) $(R_{mc}, a_c^2)_{\text{viscous}} < (R_{mc}, a_c^2)_{\text{porous}}$. (ii) R_{mc} increases with N_1.

to find the stationary critical effective micropolar Rayleigh number, R_{mc} corresponding to the critical wave number, a_c^2 is given by

$$R_{mc} = \frac{\{\delta^2[Da + (N_1 + N_1')\delta^2](2N_1 + N_3\delta^2)N_1^2\delta^4\}\delta^2}{(\pi\alpha)^2\{(2N_1 + N_3\delta^2) - N_1N_5\delta^2\}}. \tag{26}$$

Table 4
Comparison of R_{mc} and a_c^2 for different couple stress parameter, N_3

N_3	Porous case		Purely viscous case [30]	
	R_{mc}	a_c^2	R_{mc}	a_c^2
0.50	1597.68	5.40	897.10	4.92
1.00	1417.64	5.40	801.96	4.93
1.50	1365.49	5.40	774.12	4.93
2.00	1340.67	5.40	760.83	4.93
2.50	1326.16	5.40	753.05	4.93
3.00	1316.64	5.40	747.94	4.93
5.00	1297.96	5.40	737.90	4.93
8.00	1287.65	5.40	732.35	4.93

Chosen parameter values: (i) Purely viscous case $Da = 0$, $N_1' = 1.0$, $N_1 = 0.1$, $N_5 = 1.0$. (ii) Porous case $Da = 5.0$, $N_1' = 1.5$, $N_1 = 0.1$, $N_5 = 1.0$. Remarks: (i) $(R_{mc}, a_c^2)_{viscous} < (R_{mc}, a_c^2)_{porous}$. (ii) R_{mc} decreases with increase in N_3.

Table 5
Comparison of R_{mc} and a_c^2 for different micropolar heat conduction parameter, N_5 and different phases

N_5	Porous case		Purely viscous case [30]	
	R_{mc}	a_c^2	R_{mc}	a_c^2
0.10	1275.90	5.40	726.66	4.93
0.30	1290.81	5.40	733.98	4.93
0.50	1304.68	5.40	741.46	4.93
0.80	1326.04	5.40	752.96	4.93
1.00	1340.67	5.40	760.83	4.93
2.00	1418.96	5.40	802.79	4.93

Chosen parameter values: (i) Purely viscous case $Da = 0$, $N_1' = 1.0$, $N_1 = 0.1$, $N_3 = 2.0$. (ii) Porous case $Da = 5.0$, $N_1' = 1.5$, $N_1 = 0.1$, $N_3 = 2.0$. Remarks: (i) $(R_{mc}, a_c^2)_{viscous} < (R_{mc}, a_c^2)_{porous}$. (ii) R_{mc} increases with N_5.

4.1. Limiting cases

The eigenvalue expression (26) is quite general. The results of viscous flow (i.e., non-porous) [30] and Darcy flow regimes can be obtained from it in the limit $Da \to 0$ (with $N_1' = 1$) and $Da \to \infty$, respectively. The effect of micropolar and porous medium parameters on R_m is shown in Tables 1–5. In the next section we discuss the non-linear theory.

5. Non-linear theory

The linear theory discussed in the previous section predicts only the condition for the onset of convection and is silent about the heat transfer. To know the amount of heat transfer we have to resort to the non-linear theory. We now follow Veronis [31] and undertake a weak non-linear analysis of the problem. Although the relationship between the solutions of the governing partial differential and the corresponding severely truncated ordinary differential system has not been established these low-order spectral modes may reproduce qualitatively the convective phenomenon observed in the full system. The result can be used as starting values while discussing the fully non-linear problem.

The first effect of non-linearity is to distort the temperature field through the interaction of ψ and T and $\tilde{\omega}_y$ and T. The distortion of temperature field will correspond to a change in the horizontal mean, i.e., a component of the form $\sin 2\pi z$ will be generated. Thus the minimal system which describes finite amplitude convection in a micropolar fluid-saturated porous layer is given by

$$
\begin{bmatrix} \psi \\ \tilde{\omega}_y \\ T \end{bmatrix} = \begin{bmatrix} A(t)\sin\pi\alpha x \\ B(t)\sin\pi\alpha x \\ C(t)\cos\pi\alpha x \end{bmatrix} \sin\pi z + \begin{bmatrix} 0 \\ 0 \\ D(t) \end{bmatrix} \sin 2\pi z, \tag{27}
$$

where the amplitudes A, B, C and D are to be determined from the dynamics of the system. The stream function and the spin do not contain an x-independent term because the spontaneous generation of large-scale flow has been discounted.

Substituting Eqs. (26) and first two relations of (27) into Eqs. (18)–(20), we get

$$
\begin{bmatrix} \dot{A} \\ \dot{B} \\ \dot{C} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} -\dfrac{Pr}{\lambda}(Da\delta^2 + (N_1' + N_1)\delta^4) & -\dfrac{N_1\lambda}{Pr} & -\dfrac{\pi\alpha R_m Pr}{\lambda} & 0 \\ -\dfrac{N_1}{N_2}\dfrac{\delta^2 Pr}{\lambda^2} & -(2N_1 + N_3\delta^2)\dfrac{Pr}{N_2\lambda} & 0 & 0 \\ -\pi\alpha & -N_5\pi\alpha & -\delta^2 & 0 \\ 0 & 0 & 0 & -4\pi^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}
$$

$$
-\pi^2\alpha D \begin{bmatrix} 0 \\ 0 \\ (A + N_5 B) \\ 0 \end{bmatrix} + \dfrac{\pi^2\alpha C}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ (A\pi + N_5\alpha B) \end{bmatrix}, \tag{28}
$$

where overdot indicates time derivatives.

In the above equations A is proportional to the intensity of convective motion, B is proportional to the intensity of microrotation and C is proportional to the temperature difference between ascending and descending currents. Sign of A and C denote that warm fluid is rising and cold fluid is descending. The variable D is proportional to the distortion of vertical temperature profile, a positive value indicating the strongest gradient occuring near boundaries.

The fourth-order Lorenz system (28) is uniformly bounded in time and possesses many properties of the full system. Also, the phase-space volume contracts at a uniform rate given by

$$
\frac{\partial \dot{A}}{\partial A} + \frac{\partial \dot{B}}{\partial B} + \frac{\partial \dot{C}}{\partial C} + \frac{\partial \dot{D}}{\partial D}
$$

$$
= -\left\{ \frac{Pr}{\lambda}[Da\delta^2 + (N_1' + N_1)\delta^4] + \frac{Pr}{N_2\lambda}(2N_1 + N_3\delta^2) + \delta^2 + 4\pi^2 \right\}, \tag{29}
$$

which is always negative and therefore the system is bounded and dissipative. As a result the trajectories are attracted to a fixed point, a limit cycle or perhaps, a strange attractor. An analytical solution to the full problem in Eq. (28) is prohibitive. However, in the steady state a closed form solution can be obtained. This is demonstrated below.

6. Steady-state solution

Assuming the amplitudes are steady, we set $\dot{A} = \dot{B} = \dot{C} = \dot{D} = 0$ and get the following set of equations as the deterministic set for the steady amplitudes:

$$A[Da\,\delta^2 + (N_1' + N_1)\delta^4] + N_1\delta_1^2 B + R_m C\pi = 0, \tag{30}$$

$$(2N_1 + N_3\delta^2)B + N_1\delta^2 A(t) = 0, \tag{31}$$

$$\pi\alpha(1 + \pi D)(A + N_5 B) + \delta^2 C = 0, \tag{32}$$

$$-4\pi^2 D + \frac{\pi^2\alpha C}{2}(A + N_5 B) = 0. \tag{33}$$

The steady solutions are very useful because they predict that a finite amplitude solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of R_m for which a steady solution is possible lies below the critical values for instability to either a marginal state or an overstable infinitesimal perturbation.

Writing all amplitudes in terms of A, we get

$$B = -\frac{N_1\delta^2 A}{(N_3\delta^2 + 2N_1)}, \tag{34}$$

$$C = \frac{A\{N_1^2\delta^4 - [Da\,\delta^2 + (N_1' + N_1)\delta^4](N_3\delta^2 + 2N_1)\}}{R_m\pi\alpha(N_3\delta^2 + 2N_1)} \tag{35}$$

and

$$D = \frac{A^2}{8}\{N_1^2\delta^4 - [Da\delta^2 + (N_1' + N_1)\delta^4](N_3\delta^2 + 2N_1)\}\{(N_3\delta^2 + 2N_1) - N_1 N_5\delta^2\}. \tag{36}$$

Substituting Eqs. (34)–(36) into (30), we get

$$\frac{A^2}{8} = \frac{\{R_m\pi^2\alpha^2(P_1) - \delta^2(P_2 - P_3)(N_3\delta^2 + 2N_1)^2\}}{\pi^2\alpha^2(P_2 - P_3)P_4^2}, \tag{37}$$

where

$$P_1 = N_1 N_5\delta^2 - (N_3\delta^2 + 2N_1),$$

$$P_2 = N_1^2\delta^4,$$

$$P_3 = (N_3\delta^2 + 2N_1)[Da\,\delta^2 + (N_1 + N_1')\delta^4],$$

$$P_4 = (N_3\delta^2 + 2N_1) - N_1 N_5\delta^2.$$

The solution $A = 0$ corresponds to pure conduction, which we know to be a possible solution though it is unstable when R is sufficiently large.

7. Heat transport

In the study of convection, the onset of convection as the Rayleigh number is increased is more readily detected by its effect on the heat transfer. In the quiescent state, the heat transfer is due to conduction alone. Hence, if H is the rate of heat transfer per unit area, then we have

$$H = -\chi_e \left[\left\langle \frac{\partial T_{\text{Total}}}{\partial z} \right\rangle \right]_{z=0}, \tag{38}$$

where the angular brackets $\langle \rangle$ correspond to a horizontal average and

$$T_{\text{Total}} = \left(T_0 - \Delta T \frac{z}{d} \right) + T(x, z, t). \tag{39}$$

The first term on the RHS of Eq. (39) is the temperature distribution of conduction state prevalent before convection sets in. Then, by using the last relation of (27) in Eq. (39) and using the resultant equation in Eq. (38), we get

$$\frac{H}{\frac{\chi_e \Delta T}{d}} = 1 - 2\pi D. \tag{40}$$

The second term on the RHS of Eq. (40) represents the heat which enters the bottom by conduction and carried to the top by both conduction and convection and hence heat transfer increases above that by conduction alone. This process can be explained physically by a relationship between the driving temperature difference ΔT and the heat transport. Expression (40) is nothing but the non-dimensional heat transport quantity namely the Nusselt number, Nu, which is a measure of the ratio of the heat transported across any layer to the heat which could be transported by conduction alone. Thus

$$Nu = 1 - 2\pi D(t). \tag{41}$$

The effects of micropolar parameters and porous parameter on heat transport/streamlines are studied and the results are discussed in the next section.

8. Results and discussions

In the paper we study both linear and non-linear convection in a micropolar fluid occupying a porous medium. The analyses are restricted to steady state as it is the preferred mode of convection for micropolar fluids. To study the linear theory the eigenvalue, R_{mc} is obtained and in the non-linear theory the heat transfer quantity, the Nusselt number, Nu, is obtained as a function of micropolar and porous medium parameters. The linear theory results are reported in Tables 1–5 and non-linear theory results are reported in the form of graphs in Figs. 1–5. Further the stream lines are also drawn to depict cell pattern.

Before we delve into a discussion of the results depicted by figures and tables we first make some comments on the parameters N_1, N_3 and N_5 arising due to the suspended particles. Assuming the Clausius–Duhem inequality, Eringen [3] presented certain thermodynamic restrictions which lead to non-negativeness of N_1, N_3 and N_5. For $G = 0$, i.e., $N_1 = 0$ it is clear that Eq. (10) becomes independent of G, i.e. it is uncoupled. As $G \to \infty$, we see that $N_1 \to 1$ and $N_3 \to 0$. This is the Stokesian description of suspension. Thus it is obvious that couple stress comes into play at small values of N_3. This supports the contention that $N_1 \in [0, 1]$ and that N_3 is a small positive real number. The parameter N_5 has to be finite because the increasing of concentration has to practically stop somewhere and hence it has to be a positive, finite real number.
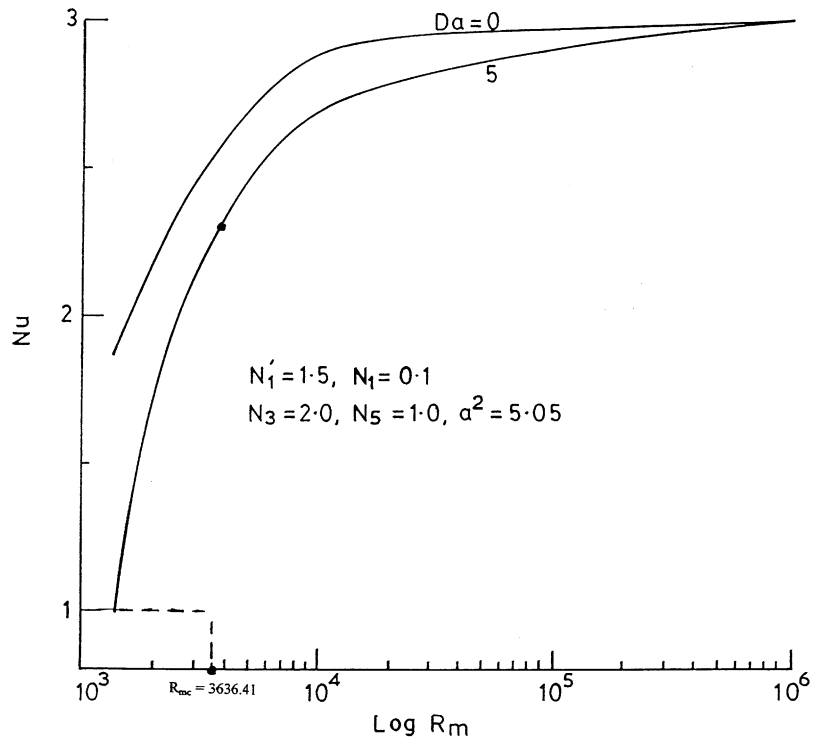
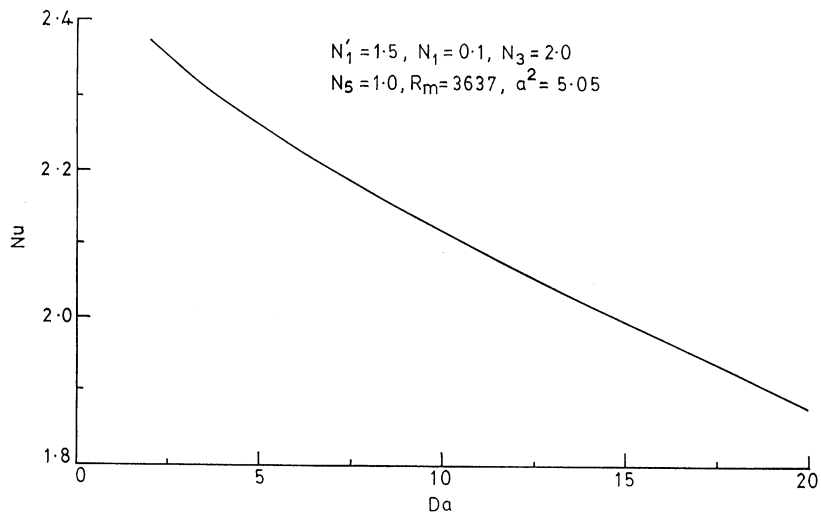Fig. 1. Plot of Nusselt number Nu vs. Rayleigh number R_m.
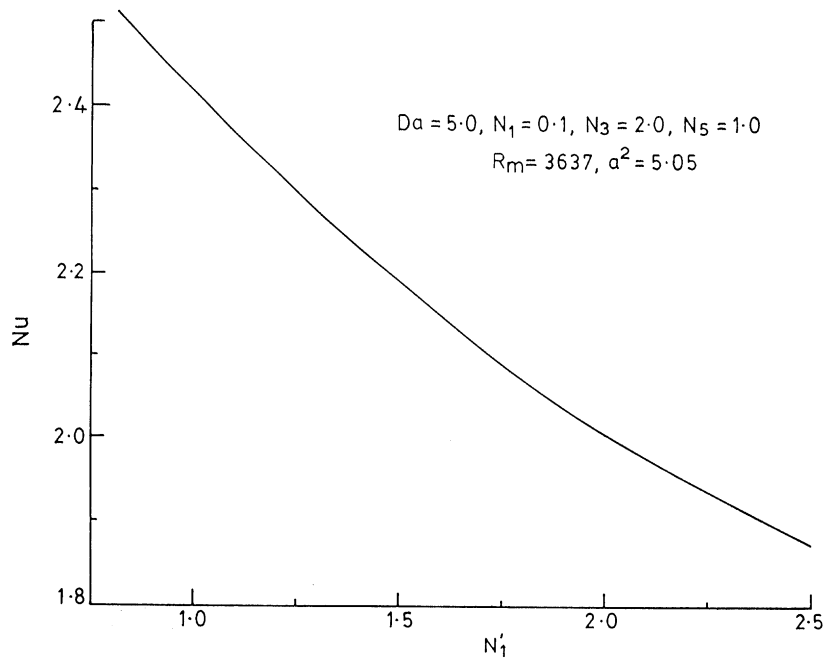

Fig. 2. Plot of Nu vs. inverse Darcy number Da.

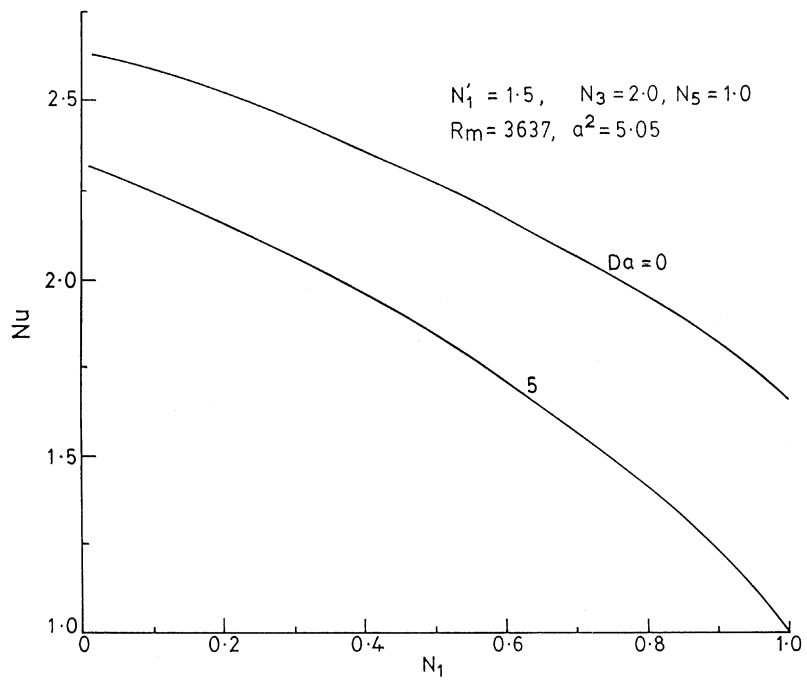Fig. 3. Plot of Nu vs. modified coupling parameter N_1'.

Fig. 4. Plot of Nu vs. coupling parameter N_1.

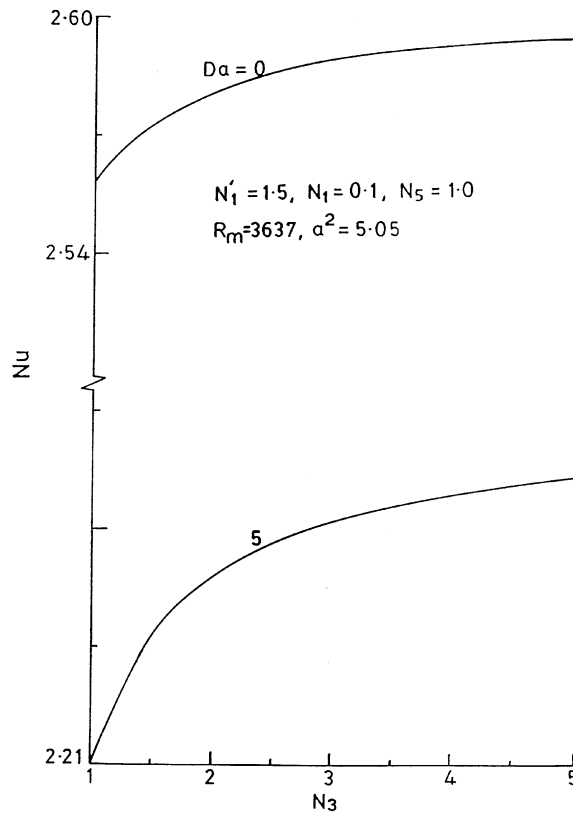Fig. 5. Plot of Nu vs. couple stress parameter N_3.

From Table 1 we observe that the qualitative effect of inverse Darcy number, Da, on the onset of convection is the same for the both Newtonian and micropolar fluids. The effect of increase in porous packing is to delay the onset of convection.

The effect of modified coupling parameter, N_1', on the onset of convection is illustrated in Table 2. Increase in the effective viscosity of the porous medium is to increase the critical effective micropolar Rayleigh number, R_{mc}.

Tables 1 and 2 indicate that the effect of parameters of porous medium is independent of type of fluid. However difference in Rayleigh number between viscous and porous case is more (significant) for micropolar fluids compared to classical fluids. Hence the Darcy friction proves to be an efficient measure of controlling convection in micropolar fluids.

Computation of a_c^2 for different values of Da and N_1' reveals that the effect of increase in the porous packing is to decrease the horizontal scale of cells.

Results indicated in Tables 3–5 reveal that the porous medium does not alter the effect of various micropolar parameters on R_{mc}. From these tables we find R_{mc} and a_c^2 increase with

(i) increase in N_1, N_1' and N_5 and
(ii) decrease in N_3.

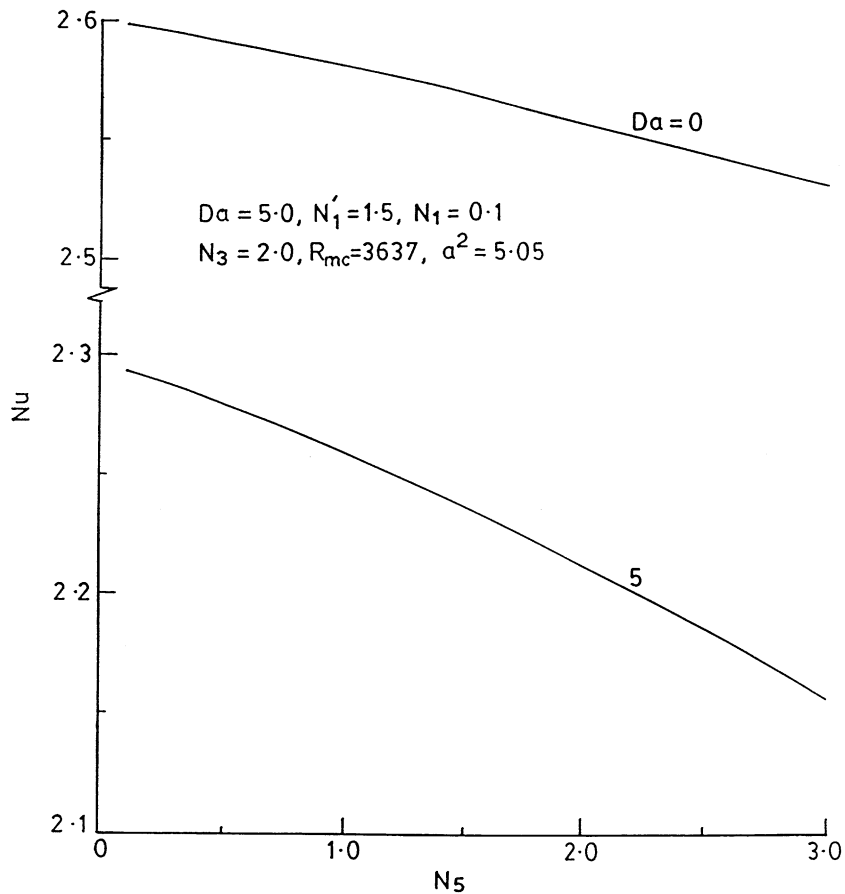These results are true for known physical reasons (see [11,12]).

Fig. 6. Plot of Nu vs. micropolar heat conduction parameter N_5.

Fig. 1 depicts the effect of effective micropolar Rayleigh number, R_m on Nu. As R_m increases, Nu increases exponentially and attains a maximum value of 3. From the Nu–R_m plane we observe that for any particular value of R_m

$$(Nu)_{\text{porous}} < (Nu)_{\text{viscous}}.$$

The most important result from the present non-linear study can be extracted from Fig. 1. For the range of parameters reported in Fig. 1, the critical Rayleigh number and wave number from linear theory are $R_{mc} = 3636.41$ and $a_c = 2.247$. We note from Fig. 1 that for values of Rayleigh number less than R_{mc} the Nusselt number has a super-linear value indicating 'subcritical' instability. In the absence of non-linear terms the Nu–R_m would go along the dotted line upto $R_m = R_{mc}$.

Effects of parameters arising due to porous medium, namely, the Inverse Darcy number, Da and the modified coupling parameter, N_1' on Nu are shown in Figs. 2 and 3, respectively. The linear theory results reveal that effect of Da and N_1' is to delay the onset of convection (see Tables 1 and 2). Hence the heat transfer across fluid layers is less, as the convective motion is restricted by these parameters and heat transfer is dominated by conduction at higher values of Da and N_1'.
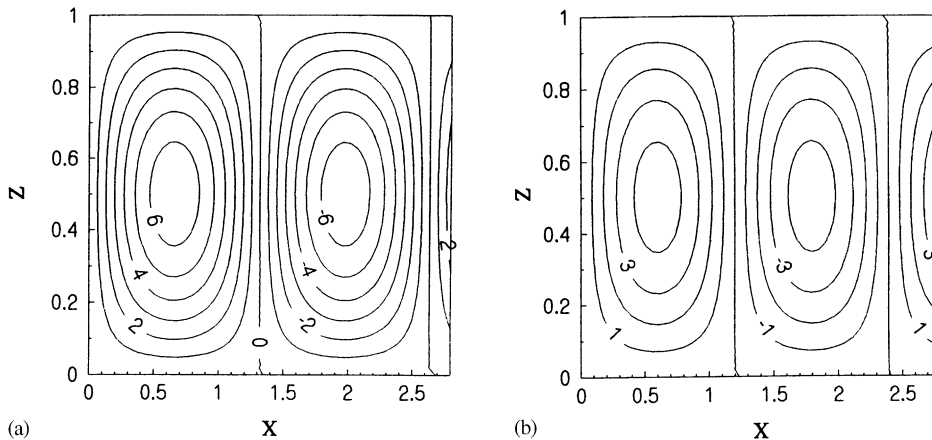
Fig. 7. Stream function profiles for different Da: (a) $Da=5.0$, (b) $Da=20.0$, and $N_1'=1.5, N_1=0.1, N_3=2.0, N_5=1.0$; $R_m=3637, a^2=5.05$.

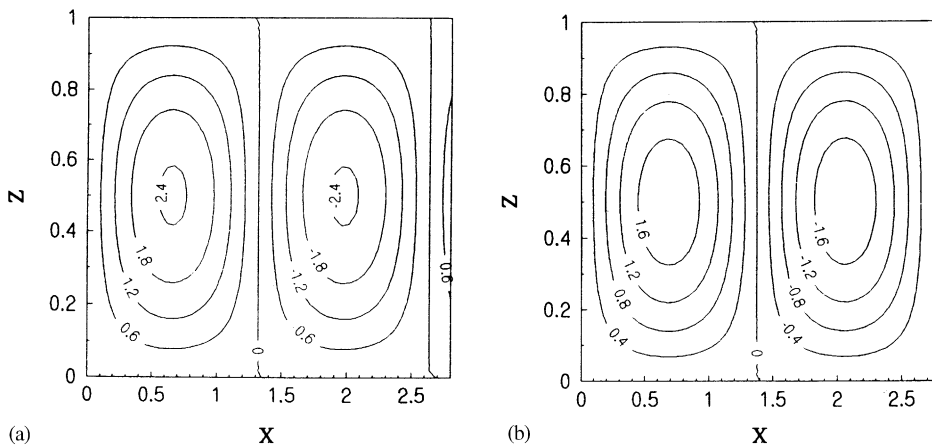Fig. 8. Stream function profiles for different N_1': (a) $N_1=1.5$, (b) $N_1=2.5$, and $Da=5.0, N_1'=1.5, N_1=0.1, N_3=2.0, N_5=1.0$; $R_m=3637, a^2=5.05$.

Fig. 4 illustrates the effect of coupling parameter, N_1 on Nu. Clearly Nu decreases with increase in N_1. Increase in concentration of microelements (suspended particles) leads to an increase in the critical value of R_m (see Table 3) and hence to a decrease in heat transfer across the fluid layers.

The effect of couple stress parameter, N_3, on Nu is demonstrated in Fig. 5. The figure indicates that as N_3 increases the heat transport increases and levels off to the Newtonian value, both in viscous and porous cases. We may infer from this figure that micropolar fluids characterised by small values of N_3 allow less heat transfer compared to Newtonian fluids.

Fig. 6 is the plot of micropolar heat conduction parameter, N_5, versus Nu. In the previous section we noted that effect of suspended particles is to deduct from the velocity (see Eq. (4)) and this results in reduced heat transfer by convection from the bottom to top. The figure essentially confirms the reduced heat transfer in fluids with suspended particles compared to that of fluids without suspended particles.

Figs. 7–11 show the contour plot of stream lines. The findings from the linear theory on the various parameters influence on a_c are reiterated by these figures. Increase in N_1 is to decrease a_c and thereby
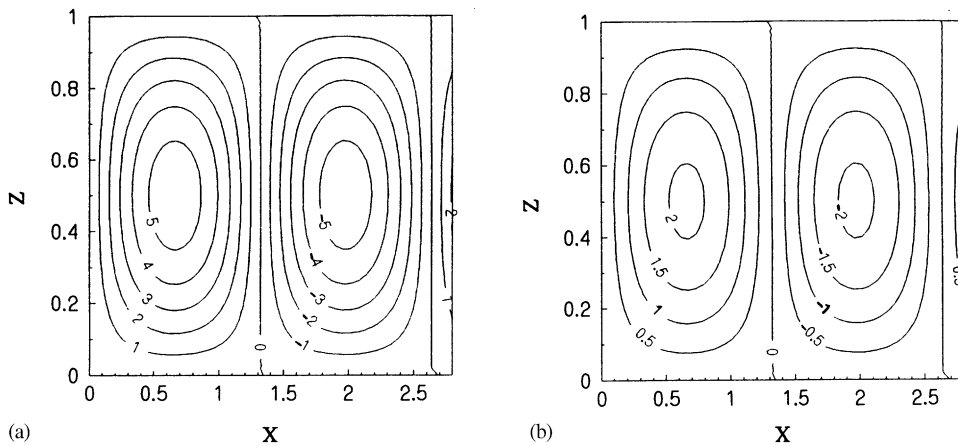
Fig. 9. Stream function profiles for different N_1: (a) $N_1 = 0.5$, (b) $N_1 = 1.0$, and $Da = 5.0, N_1' = 1.5, N_3 = 2.0, N_5 = 1.0$; $R_m = 3637, a^2 = 5.05$.
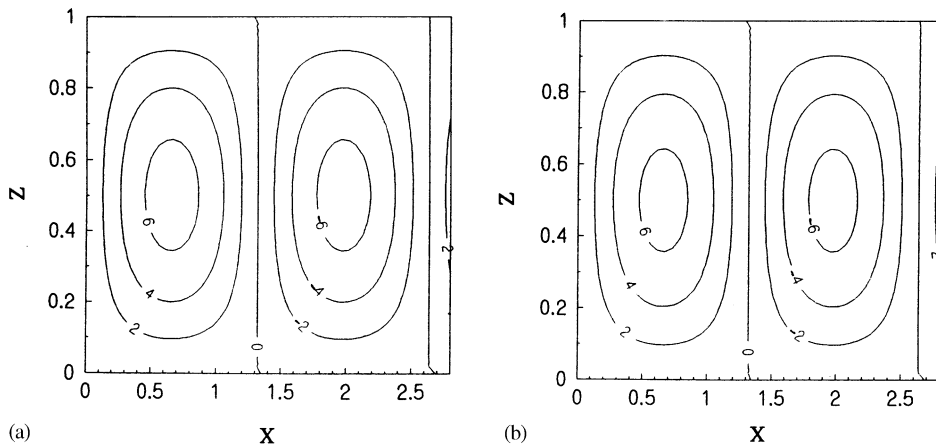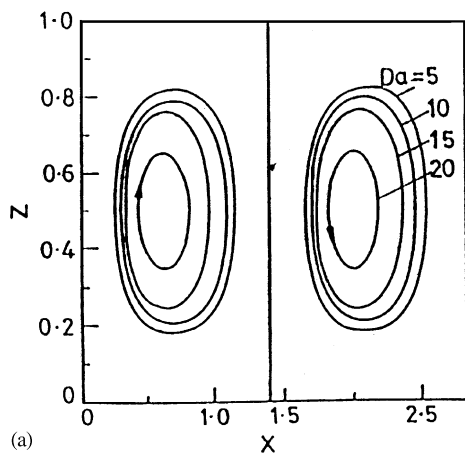

Fig. 10. Stream function profiles for different N_3/N_5: (a) $N_3 = 0.5$, $N_5 = 0.5$; (b) $N_3 = 3.0$, $N_5 = 5.0$, and $Da = 5.0$, $N_1' = 1.5$, $N_1 = 0.1$, $N_5 = 1.0/N_3 = 2.0$; $R_m = 3637$, $a^2 = 5.05$.
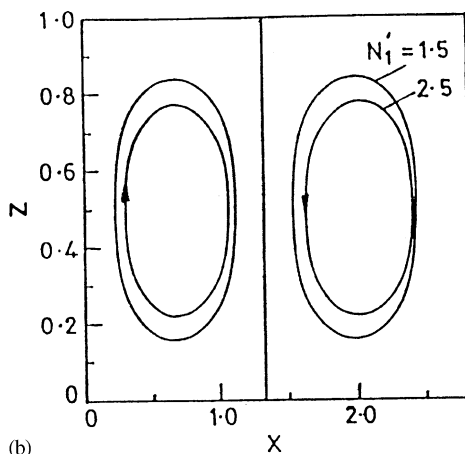
reduced cell size. The streamlines are virtually unaffected by N_3 and N_5 but are significantly affected by Da, N_1 and N_1'. Fig. 11 shows the decreasing cell-size with increase in Da and N_1'.
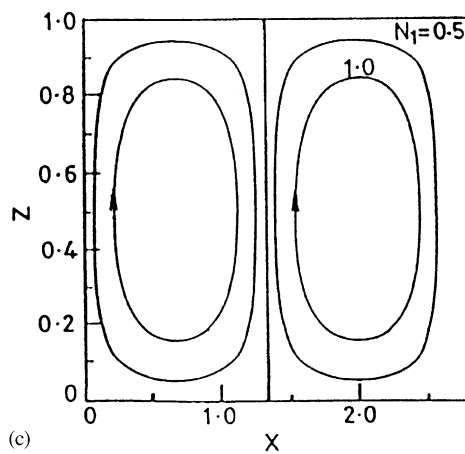
Acknowledgements

Fig. 11. Stream function profiles for (a) $\psi = 4$, $N_1' = 1.5$, $N_1 = 0.1$, $N_3 = 2.0$, $N_5 = 1.0$; (b) $\psi = 1.2$, $Da = 5$, $N_1 = 0$, $N_3 = 2.0$, $N_5 = 1.0$; (c) $\psi = 1.0$, $Da = 5$, $N_1' = 1.5$, $N_3 = 2.0$, $N_5 = 1.0$.

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