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General complete Lagrange interpolations with applications to three-dimensional finite element analysis

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Abstract

In this paper, we have first derived the interpolation polynomials for the general serendipity solid elements of rectangular shape which allow arbitrarily placed nodes along the edges. We have then presented a method to determine the interpolation functions for the general complete Lagrange elements which allow arbitrarily placed nodes. Explicit expressions for interpolation functions of the serendipity and complete Lagrange families with uniform spacing of nodes over the element domain are derived for elements of orders 4–10. We have also modified the shape functions of complete Lagrange family to correctly interpolate the complete polynomials in the global space for angular distortions of quadrilaterals over the six planar facets of linear solid hexahedron elements.

1. Introduction

In finite element analysis, interpolation theories based on Lagrangian and Hermitian interpolation polynomials have produced a variety of useful finite elements [1]. For rectangular finite elements in two and three dimensions, Argyris et al. [2,3] have presented the regular Lagrange family for which the nodes are uniformly placed everywhere on the grid. Lagrange rectangular elements are conceptually simple but are of limited use because of the large number of internal nodes. Nevertheless, they provide a natural introduction to interpolation functions for serendipity rectangular elements which have no internal nodes. Zalmal [4] presented a general formula for the derivation of interpolation functions for serendipity elements with uniformly spaced nodes. Ball [5] derived an explicit expression for interpolation functions in the form of matrix triple product for the most general serendipity rectangular element. Zienkiewicz [6] intended to define serendipity family so that polynomial completeness is realised with necessary minimum nodes and presented a few lower-order elements (linear, quadratic, and cubic) with equal number of nodes along each side which are uniformly spaced. It is obvious that the basis functions for serendipity elements with nodes placed only along the edges cannot generate complete polynomials beyond cubic.

For this reason, Zienkiewicz [6] suggested a central node for the quartic member of rectangular serendipity family and remarks that progression to yet higher members is difficult and requires some ingenuity. Taylor [1] suggested serendipity elements composed of vertices and side nodes placed at regular intervals with some internal nodeless variables. The interpolation bases are given mainly by geometrical consideration of shape functions and do not result in an explicit formula but provide insight into the form of interpolation functions and is applicable to all serendipity rectangular elements. Okabe et al. [8] rephrase

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Taylor's serendipity definition in the conventional nodal concept according to which an mth serendipity 2-cube ($m > 4$) should contain apparently the $(m-4)$th complete triangle composed of $(m-3)(m-2)/2$ internal nodes. However, Okabe et al. [8] noticed that Taylor's serendipity element is lacking in nodal symmetry with respect to coordinate axes. In some application areas, nodal symmetry may be important. From this viewpoint, serendipity family is classified into two independent families:

- the complete Lagrange family and
- the mixed complete Lagrange family.

In complete Lagrange family, the polynomial completeness is realised with necessary minimum nodes without destroying the nodal symmetry, whereas in mixed complete Lagrange family, the nodal symmetry is not maintained. Okabe [9] revealed the interpolation bases for the complete Lagrange family which realise the polynomial completeness with necessary minimum nodes without destroying the nodal symmetry. The vigorous research efforts so far are confined only to the derivation of rectangular elements [1–14]. Rathod and Sridevi [15] derived interpolation functions for the general serendipity and general complete Lagrange elements that allow arbitrarily placed nodes over the domain of element geometry in two dimensions. This paper concentrates on the derivation of interpolation functions for the general serendipity and general complete Lagrange elements that allow arbitrarily placed nodes over the domain of element geometry in three dimensions. We have obtained explicit formulae for the interpolation functions of both the families viz the general serendipity and general complete Lagrange. We have then illustrated these formulae by obtaining explicit interpolation functions when the nodes are uniformly spaced over the element domains from quartic to tenth-order members.

The inferences drawn for two-dimensional quadrilateral elements [15–20] are valid for three-dimensional linear solid hexahedron elements with all planar facets in arbitrary geometry, i.e., x is trilinear in ξ, η, ζ,

$$x = a_{000} + a_{100}\xi + a_{010}\eta + a_{001}\zeta + a_{110}\xi\eta + a_{011}\eta\zeta + a_{101}\zeta\xi + a_{111}\xi\eta\zeta.$$

Then x^p will include local monomial term $\xi^p\eta^p\zeta^p$ which is in the basis of a pth-order Lagrange element but not either in the pth-order serendipity or complete Lagrange rectangular prism element. This, then, is the undisputed advantage of the pth-order Lagrange rectangular prism element. The monomial basis set for pth-order regular Lagrange rectangular prism element is given by

$$(((\xi^m\eta^n\zeta^l, \quad m = 0, 1, \ldots, p),\ n = 0, 1, \ldots, p),\ l = 0, 1, \ldots, p)$$

which clearly has $(p+1)^3$ terms. Following Rathod and Sridevi [15], we have developed a method for the modification of shape functions for the complete Lagrange rectangular prism elements.

2. A fundamental lemma

Let $f(x) = \sum_{i=0}^{p} f_i x^i$ be a polynomial function of degree p in the variable x defined over the interval $x_0 < x_1 < x_2 < \cdots < x_l < \cdots < x_p$ as

$$f(x_k) = \begin{cases} 1 & k = l, \\ 0 & k \neq l, \end{cases} \quad k = 0(1)p. \tag{1}$$

Then the sequence of unknowns $\{f_i\},\ i = 0(1)p$ is a solution of the set of linear equations

$$-1 = \sum_{i=1}^{p} \left(x_k^i - x_l^i\right) f_i \quad (k \neq l) \tag{2}$$

and the function $f(x)$ can be uniquely determined by the relation

$$f(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{l-1})(x-x_{l+1})\cdots(x-x_p)}{(x_l-x_0)(x_l-x_1)\cdots(x_l-x_{l-1})(x_l-x_{l+1})\cdots(x_l-x_p)}. \tag{3}$$

Proof. Follows from Lagrangian interpolation theory.

3. General serendipity elements over a 3-cube

Serendipity elements have no internal nodes. The three simplest elements of serendipity family were found by inspection [6]. (See Fig. 1.)

We consider the serendipity rectangular prism element that has same number of nodes along each edge as shown in Fig. 2.

Nodal coordinates of node k for the general serendipity rectangular prism element of Fig. 2 are assumed as $(\xi_k, \eta_k, \varsigma_k),\ k = 1(1)12n-4$. We understand that the serendipity rectangular prism element in its most general form that has different number of nodes along each edge can be worked out on similar lines. Monomial bases for $(m+1)$th-order serendipity rectangular prism element are shown in Fig. 3.

3.1. Determination of interpolation functions for three-dimensional general serendipity elements

3.1.1. Vertex interpolation functions

Consider the nth-order general serendipity rectangular prism element shown in Fig. 2. We assume the following expression for interpolation function at the corner node 1:

$$N_1(\xi,\eta,\varsigma) = \frac{(\xi-\xi_7)(\eta-\eta_7)(\varsigma-\varsigma_7)}{(\xi_1-\xi_7)(\eta_1-\eta_7)(\varsigma_1-\varsigma_7)}\left[a_{000} + \sum_{k=1}^{n-1}\left\{a_{k00}\xi^k + a_{0k0}\eta^k + a_{00k}\varsigma^k\right\}\right]. \quad (4)$$

By the properties of shape functions, we have

$$N_1(\xi_1,\eta_1,\varsigma_1) = 1 \quad (5)$$

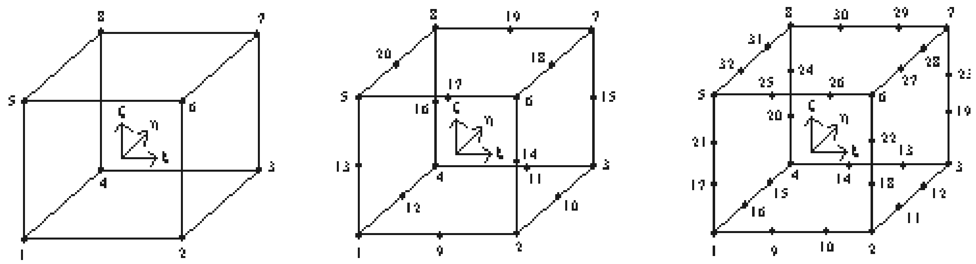


Fig. 1. Serendipity rectangular prism elements over a 3-cube: (a) linear (8 nodes); (b) quadratic (20 nodes); (c) cubic (32 nodes).

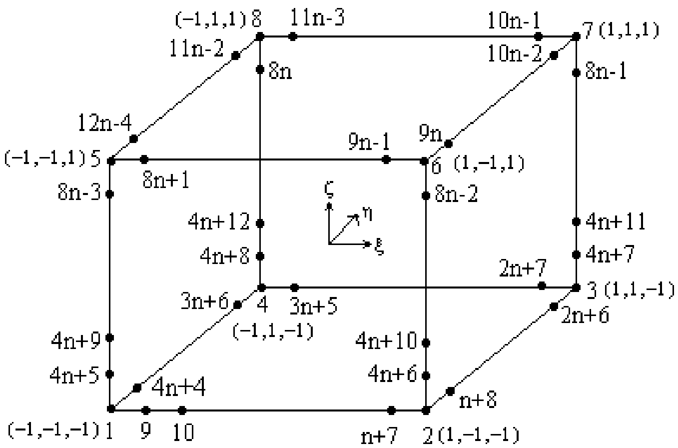


Fig. 2. nth-order serendipity rectangular prism element ($12n-4$ nodes) with corner nodal coordinates $(\mp 1, \mp 1, \mp 1)$.

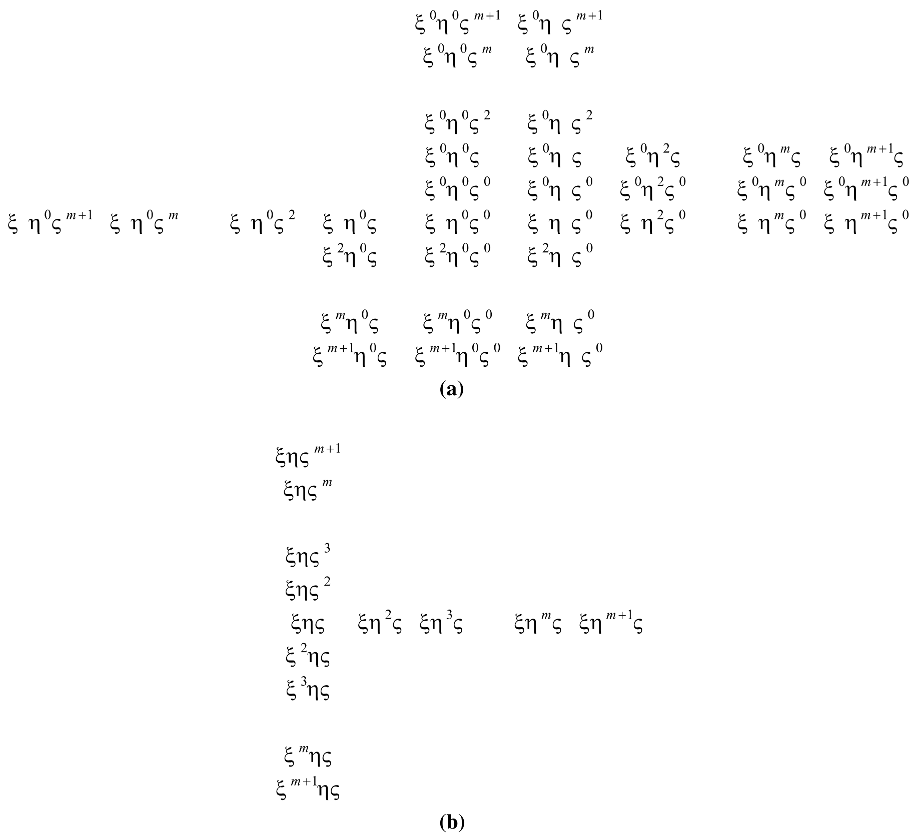


Fig. 3. (a) and (b) Monomial basis for $(m+1)$th-order serendipity rectangular prism element.

and

$$N(\xi_i, \eta_i, \varsigma_i) = 0, \quad i \neq 1. \tag{6}$$

Using Eqs. (5) and (6), Eq. (4) becomes

$$1 = N_1(\xi_1, \eta_1, \varsigma_1) = a_{000} + \sum_{k=1}^{n-1} \left(a_{k00}\xi_1^k + a_{0k0}\eta_1^k + a_{00k}\varsigma_1^k \right), \tag{7}$$

$$0 = N_1(\xi_{8+i}, \eta_1, \varsigma_1) = a_{000} + \sum_{k=1}^{n-1} \left(a_{k00}\xi_{8+i}^k + a_{0k0}\eta_1^k + a_{00k}\varsigma_1^k \right), \quad i = 1, 2, \ldots, n-1. \tag{8}$$

Subtracting Eq. (8) from Eq. (7), we obtain

$$\sum_{k=1}^{n-1} a_{k00}\left(\xi_{8+i}^k - \xi_1^k \right) = -1, \quad i = 1, 2, \ldots, n-1. \tag{9}$$

Comparing Eqs. (4) and (9) with Eqs. (1) and (2), we infer that the solution to the set of equations (7) and (8) can be immediately written as

$$\left(a_{000} + \sum_{k=1}^{n-1} a_{0k0}\eta_1^k + \sum_{k=1}^{n-1} a_{00k}\varsigma_1^k \right) + \sum_{k=1}^{n-1} a_{k00}\xi^k = \frac{(\xi - \xi_9)(\xi - \xi_{10}) \cdots (\xi - \xi_{n+7})}{(\xi_1 - \xi_9)(\xi_1 - \xi_{10}) \cdots (\xi_1 - \xi_{n+7})}. \tag{10}$$

We also have along the edges 1–4 of the 3-cube $\xi = \xi_1$, and $\varsigma = \varsigma_1$, so that we can write

$$N_1\left(\xi_1, \eta_{3n+5+i}, \varsigma_1 \right) = 0, \quad i = 1, 2, \ldots, n-1. \tag{11}$$

Eq. (11) is further equivalent to

$$a_{000} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi_1^k + a_{00k}\varsigma_1^k \right\} + \sum_{k=1}^{n-1} a_{0k0}\eta_{3n+5+i}^k = 0, \quad i = 1, 2, \ldots, n-1. \tag{12}$$

Referring the fundamental lemma, the solution to the set of equations (7) and (12) is given by

$$a_{000} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi_1^k + a_{00k}\varsigma_1^k \right\} + \sum_{k=1}^{n-1} a_{0k0}\eta^k = \frac{(\eta - \eta_{3n+6})(\eta - \eta_{3n+7})\cdots(\eta - \eta_{4n+4})}{(\eta_1 - \eta_{3n+6})(\eta_1 - \eta_{3n+7})\cdots(\eta_1 - \eta_{4n+4})}. \tag{13}$$

We also have along the edges 1–5 of the 3-cube $\xi = \xi_1$, and $\eta = \eta_1$, so that we can write $N_1(\xi_1, \eta_1, \varsigma_{4(n+i)+1}) = 0,\ i = 1, 2, \ldots, n-1$,

$$a_{000} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi_1^k + a_{0k0}\eta_1^k \right\} + \sum_{k=1}^{n-1} a_{00k}\varsigma_{4(n+i)+1}^k = 0, \quad i = 1, 2, \ldots, n-1. \tag{14}$$

Again, referring the fundamental lemma, the solution to the set of equations (7) and (14) is

$$a_{000} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi_1^k + a_{0k0}\eta_1^k \right\} + \sum_{k=1}^{n-1} a_{00k}\varsigma^k = \frac{(\varsigma - \varsigma_{4n+5})(\varsigma - \varsigma_{4n+9})\cdots(\varsigma - \varsigma_{8n-3})}{(\varsigma_1 - \varsigma_{4n+5})(\varsigma_1 - \varsigma_{4n+9})\cdots(\varsigma_1 - \varsigma_{8n-3})}. \tag{15}$$

Adding Eqs. (10), (13) and (15), we get

$$\begin{aligned} &3a_{000} + 2\sum_{k=1}^{n-1} \left\{ a_{k00}\xi_1^k + a_{0k0}\eta_1^k + a_{00k}\varsigma_1^k \right\} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi^k + a_{0k0}\eta^k + a_{00k}\varsigma^k \right\} \\ &\quad = \frac{(\xi - \xi_9)(\xi - \xi_{10})\cdots(\xi - \xi_{n+7})}{(\xi_1 - \xi_9)(\xi_1 - \xi_{10})\cdots(\xi_1 - \xi_{n+7})} + \frac{(\eta - \eta_{3n+6})(\eta - \eta_{3n+7})\cdots(\eta - \eta_{4n+4})}{(\eta_1 - \eta_{3n+6})(\eta_1 - \eta_{3n+7})\cdots(\eta_1 - \eta_{4n+4})} \\ &\qquad + \frac{(\varsigma - \varsigma_{4n+5})(\varsigma - \varsigma_{4n+9})\cdots(\varsigma - \varsigma_{8n-3})}{(\varsigma_1 - \varsigma_{4n+5})(\varsigma_1 - \varsigma_{4n+9})\cdots(\varsigma_1 - \varsigma_{8n-3})}. \end{aligned} \tag{16}$$

Substituting Eq. (7) in Eq. (16), we obtain

$$\begin{aligned} a_{000} + \sum_{k=1}^{n-1} \left\{ a_{k00}\xi^k + a_{0k0}\eta^k + a_{00k}\varsigma^k \right\} &= -2 + \frac{(\xi - \xi_9)(\xi - \xi_{10})\cdots(\xi - \xi_{n+7})}{(\xi_1 - \xi_9)(\xi_1 - \xi_{10})\cdots(\xi_1 - \xi_{n+7})} \\ &\quad + \frac{(\eta - \eta_{3n+6})(\eta - \eta_{3n+7})\cdots(\eta - \eta_{4n+4})}{(\eta_1 - \eta_{3n+6})(\eta_1 - \eta_{3n+7})\cdots(\eta_1 - \eta_{4n+4})} \\ &\quad + \frac{(\varsigma - \varsigma_{4n+5})(\varsigma - \varsigma_{4n+9})\cdots(\varsigma - \varsigma_{8n-3})}{(\varsigma_1 - \varsigma_{4n+5})(\varsigma_1 - \varsigma_{4n+9})\cdots(\varsigma_1 - \varsigma_{8n-3})}. \end{aligned} \tag{17}$$

From Eqs. (4) and (17), it follows that

$$\begin{aligned} N_1(\xi, \eta, \zeta) = {} & \frac{(\xi - \xi_2)(\eta - \eta_4)(\varsigma - \varsigma_5)}{(\xi_1 - \xi_7)(\eta_1 - \eta_7)(\varsigma_1 - \varsigma_7)} \\ & \times \left[-2 + \frac{(\xi - \xi_9)(\xi - \xi_{10})\cdots(\xi - \xi_{n+7})}{(\xi_1 - \xi_9)(\xi_1 - \xi_{10})\cdots(\xi_1 - \xi_{n+7})} + \frac{(\eta - \eta_{3n+6})(\eta - \eta_{3n+7})\cdots(\eta - \eta_{4n+4})}{(\eta_1 - \eta_{3n+6})(\eta_1 - \eta_{3n+7})\cdots(\eta_1 - \eta_{4n+4})} \right. \\ & \left. + \frac{(\varsigma - \varsigma_{4n+5})(\varsigma - \varsigma_{4n+9})\cdots(\varsigma - \varsigma_{8n-3})}{(\varsigma_1 - \varsigma_{4n+5})(\varsigma_1 - \varsigma_{4n+9})\cdots(\varsigma_1 - \varsigma_{8n-3})} \right]. \end{aligned} \tag{18}$$

Similarly, we can obtain $N_i(\xi, \eta, \zeta),\ i = 2(1)8$.

3.1.2. Edge interpolation functions

Let us assume the coordinates of corner nodes as

$$\{(\xi_i,\eta_i,\varsigma_i),\ i=1(1)8\}=\{(-\theta,-\theta,-\theta),(\theta,-\theta,-\theta),(\theta,\theta,-\theta),(-\theta,\theta,-\theta),(-\theta,-\theta,\theta),(\theta,-\theta,\theta),(\theta,\theta,\theta),(-\theta,\theta,\theta)\}. \tag{19}$$

For midside nodes along edges 1–2,

$$N_k(\xi,\eta,\varsigma)=\frac{(\eta-\theta)(\varsigma-\theta)}{4\theta^2}\frac{(\xi^2-\theta^2)}{(\xi_k^2-\theta^2)}\frac{(\xi-\xi_9)(\xi-\xi_{10})\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_{n+7})}{(\xi_k-\xi_9)(\xi_k-\xi_{10})\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_{n+7})},\quad k=9(1)n+7. \tag{20}$$

Similarly, we can obtain other edge interpolation functions.

3.2. Determination of interpolation functions for serendipity rectangular prism elements of conventional type

Consider the general serendipity rectangular prism element over the 3-cube $-\theta\leqslant\xi,\eta,\varsigma\leqslant\theta$ having equal number of nodes along each side as shown in Fig. 4.

Let $(\xi_k^{(n)},\eta_k^{(n)},\varsigma_k^{(n)}),\ k=1(1)12n-4$, refer to coordinates of node k for nth-order serendipity rectangular prism element. We can define the conventional serendipity rectangular prism element of nth-order by choosing uniform spacing over the 2θ-cube $-\theta\leqslant\xi,\eta,\varsigma\leqslant\theta$ as

$$\begin{aligned}
&\xi_{8+i}^{(n)}=\eta_{n+7+i}^{(n)}=\varsigma_{4(n+i)+1}^{(n)}=-\theta+(2i\theta/n),\\
&\xi_{8+i}^{(n)}=\xi_{3n+6-i}^{(n)}=\xi_{8n+i}^{(n)}=\xi_{11n-2-i}^{(n)},\\
&\eta_{n+7+i}^{(n)}=\eta_{12n-3-i}^{(n)}=\eta_{4n+5-i}^{(n)}=\eta_{9n-1+i}^{(n)},\\
&\varsigma_{4(n+i)+1}^{(n)}=\zeta_{4(n+i)+2}^{(n)}=\varsigma_{4(n+i)+3}^{(n)}=\zeta_{4(n+i)+1}^{(n)},\quad i=1,2,\ldots,n-1
\end{aligned} \tag{21}$$

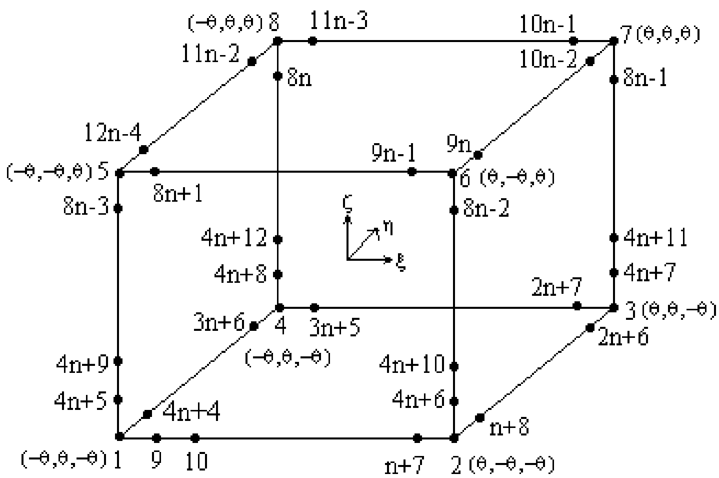


Fig. 4. nth-order serendipity rectangular prism element ($12n-4$ nodes) with corner nodal coordinates ($\mp\theta,\ \mp\theta,\ \mp\theta$).

Table 1
Quartic serendipity element shape functions

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(4)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(4)}$	$(1/24)(1-\xi)(1-\eta)(1-\zeta)[-4\xi^3+\eta^3+\zeta^3)+(\xi+\eta+\zeta)-6]$
2	$(1,-1,-1)$	$N_2^{(4)}$	$(1/24)(1+\xi)(1-\eta)(1-\zeta)[4(\xi^3-\eta^3-\zeta^3)-(\xi-\eta-\zeta)-6]$
3	$(1,1,-1)$	$N_3^{(4)}$	$(1/24)(1+\xi)(1+\eta)(1-\zeta)[4(\xi^3+\eta^3-\zeta^3)-(\xi+\eta-\zeta)-6]$
4	$(-1,1,-1)$	$N_4^{(4)}$	$(1/24)(1-\xi)(1+\eta)(1-\zeta)[-4(\xi^3-\eta^3+\zeta^3)+(\xi-\eta+\zeta)-6]$
5	$(-1,-1,1)$	$N_5^{(4)}$	$(1/24)(1-\xi)(1-\eta)(1+\zeta)[-4(\xi^3+\eta^3-\zeta^3)+(\xi+\eta-\zeta)-6]$
6	$(1,-1,1)$	$N_6^{(4)}$	$(1/24)(1+\xi)(1-\eta)(1+\zeta)[4(\xi^3-\eta^3+\zeta^3)-(\xi-\eta+\zeta)-6]$
7	$(1,1,1)$	$N_7^{(4)}$	$(1/24)(1+\xi)(1+\eta)(1+\zeta)[4(\xi^3+\eta^3+\zeta^3)-(\xi+\eta+\zeta)-6]$
8	$(-1,1,1)$	$N_8^{(4)}$	$(1/24)(1-\xi)(1+\eta)(1+\zeta)[-4(\xi^3-\eta^3-\zeta^3)+(\xi-\eta-\zeta)-6]$
9	$(-1/2,-1,-1)$	$N_9^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1-\zeta)\xi(2\xi-1)$
10	$(0,-1,-1)$	$N_{10}^{(4)}$	$-(1/4)(1-\xi^2)(1-\eta)(1-\zeta)(4\xi^2-1)$
11	$(1/2,-1,-1)$	$N_{11}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1-\zeta)\xi(2\xi+1)$
12	$(1,-1/2,-1)$	$N_{12}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1-\zeta)\eta(2\eta-1)$
13	$(1,0,-1)$	$N_{13}^{(4)}$	$-(1/4)(1+\xi)(1-\eta^2)(1-\zeta)(4\eta^2-1)$
14	$(1,1/2,-1)$	$N_{14}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1-\zeta)\eta(2\eta+1)$
15	$(1/2,1,-1)$	$N_{15}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1-\zeta)\xi(2\xi+1)$
16	$(0,1,-1)$	$N_{16}^{(4)}$	$-(1/4)(1-\xi^2)(1+\eta)(1-\zeta)(4\xi^2-1)$
17	$(-1/2,1,-1)$	$N_{17}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1-\zeta)\xi(2\xi-1)$
18	$(-1,1/2,-1)$	$N_{18}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1-\zeta)\eta(2\eta+1)$
19	$(-1,0,-1)$	$N_{19}^{(4)}$	$-(1/4)(1-\xi)(1-\eta^2)(1-\zeta)(4\eta^2-1)$
20	$(-1,-1/2,-1)$	$N_{20}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1-\zeta)\eta(2\eta-1)$
21	$(-1,-1,-1/2)$	$N_{21}^{(4)}$	$(1/3)(1-\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta-1)$
22	$(1,-1,-1/2)$	$N_{22}^{(4)}$	$(1/3)(1+\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta-1)$
23	$(1,1,-1/2)$	$N_{23}^{(4)}$	$(1/3)(1+\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta-1)$
24	$(-1,1,-1/2)$	$N_{24}^{(4)}$	$(1/3)(1-\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta-1)$
25	$(-1,-1,0)$	$N_{25}^{(4)}$	$-(1/4)(1-\xi)(1-\eta)(1-\zeta^2)(4\zeta^2-1)$
26	$(1,-1,0)$	$N_{26}^{(4)}$	$-(1/4)(1+\xi)(1-\eta)(1-\zeta^2)(4\zeta^2-1)$
27	$(1,1,0)$	$N_{27}^{(4)}$	$-(1/4)(1+\xi)(1+\eta)(1-\zeta^2)(4\zeta^2-1)$
28	$(-1,1,0)$	$N_{28}^{(4)}$	$-(1/4)(1-\xi)(1+\eta)(1-\zeta^2)(4\zeta^2-1)$
29	$(-1,-1,1/2)$	$N_{29}^{(4)}$	$(1/3)(1-\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta+1)$
30	$(1,-1,1/2)$	$N_{30}^{(4)}$	$(1/3)(1+\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta+1)$
31	$(1,1,1/2)$	$N_{31}^{(4)}$	$(1/3)(1+\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta+1)$
32	$(-1,1,1/2)$	$N_{32}^{(4)}$	$(1/3)(1-\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta+1)$
33	$(1/2,-1,1)$	$N_{33}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1+\zeta)\xi(2\xi+1)$
34	$(0,-1,1)$	$N_{34}^{(4)}$	$-(1/4)(1-\xi^2)(1-\eta)(1+\zeta)(4\xi^2-1)$
35	$(-1/2,-1,1)$	$N_{35}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1+\zeta)\xi(2\xi-1)$
36	$(1,-1/2,1)$	$N_{36}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1+\zeta)\eta(2\eta-1)$

(*continued overleaf*)

Table 1 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(4)}$	Shape function
37	$(1, 0, 1)$	$N_{37}^{(4)}$	$-(1/4)(1+\xi)(1-\eta^2)(1+\zeta)(4\eta^2-1)$
38	$(1, 1/2, 1)$	$N_{38}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1+\zeta)\eta(2\eta+1)$
39	$(1/2, 1, 1)$	$N_{39}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1+\zeta)\xi(2\xi+1)$
40	$(0, 1, 1)$	$N_{40}^{(4)}$	$-(1/4)(1-\xi^2)(1+\eta)(1+\zeta)(4\xi^2-1)$
41	$(-1/2, 1, 1)$	$N_{41}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1+\zeta)\xi(2\xi-1)$
42	$(-1, 1/2, 1)$	$N_{42}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1+\zeta)\eta(2\eta+1)$
43	$(-1, 0, 1)$	$N_{43}^{(4)}$	$-(1/4)(1-\xi)(1-\eta^2)(1+\zeta)(4\eta^2-1)$
44	$(-1, 1/2, 1)$	$N_{44}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1+\zeta)\eta(2\eta-1)$

and the nodal coordinates along the edges of the element are

$$\begin{aligned}
&\eta_i = \varsigma_i = -\theta, \quad i = 1, 2, 9, 10, \ldots, n+7,\\
&\eta_i = -\varsigma_i = -\theta, \quad i = 5, 6, 8n+1, 8n+2, \ldots, 9n-1,\\
&\eta_i = -\varsigma_i = \theta, \quad i = 1, 3, 2n+7, 2n+8, \ldots, 3n+5,\\
&\eta_i = \varsigma_i = \theta, \quad i = 7, 8, 10n-1, 10n, \ldots, 11n-3,\\
&\xi_i = \varsigma_i = -\theta, \quad i = 1, 4, 3n+6, 3n+7, \ldots, 4n+4,\\
&\xi_i = -\varsigma_i = -\theta, \quad i = 5, 8, 11n-2, 11n-1, \ldots, 12n-4,\\
&\xi_i = -\varsigma_i = \theta, \quad i = 2, 3, n+8, n+9, \ldots, 2n+6,\\
&\xi_i = \varsigma_i = \theta, \quad i = 6, 7, 9n, 9n+1, \ldots, 10n-2,\\
&\xi_i = \eta_i = -\theta, \quad i = 1, 5, 4n+5, 4n+9, \ldots, 8n-3,\\
&\xi_i = \eta_i = \theta, \quad i = 3, 7, 4n+7, 4n+11, \ldots, 8n-1,\\
&\xi_i = -\eta_i = \theta, \quad i = 2, 6, 4n+6, 4n+10, \ldots, 8n-2,\\
&\xi_i = -\eta_i = -\theta, \quad i = 4, 8, 4n+8, 4n+12, \ldots, 8n.
\end{aligned} \tag{22}$$

Using Eqs. (18), (20)–(22), the element shape functions can be immediately written. For $\theta = 1$ and $n = 1, 2, 3$, we obtain the conventional shape functions of linear, quadratic, and cubic elements reported in [6].

We have determined the interpolation functions for higher order serendipity rectangular prism elements – quartic-, quintic-, sextic-, septic-, octic-, ninth- and tenth-orders. These are listed in Tables 1–7, respectively. Node numbering sequence for these elements is described in Fig. 5.

4. General complete Lagrange elements over a 3-cube

Zienkiewicz [6] defined Lagrange family for the 2-cube and 3-cube so that polynomial completeness is realised with necessary minimum nodes and presented a few lower-order elements (linear, quadratic, and cubic) of serendipity family which depend on nodal coordinate values uniformly spaced on the element boundary over the domains $-1 \leqslant \xi, \eta \leqslant 1$ and $-1 \leqslant \xi, \eta, \varsigma \leqslant 1$. However, Zienkiewicz [6] observed that interior nodes are necessary for elements of orders higher than the cubic serendipity element to realise polynomial completeness. Okabe [8,9] proposed the complete Lagrange family for the 2-cube which realised polynomial completeness. In this section, we wish to determine the element shape functions for the general complete Lagrange family over the 3-cube in explicit forms, as far as possible, for the quartic-, quintic-, sextic-, septic-, octic-, ninth-, and tenth-order rectangular prism elements. Following Okabe [8,9], the monomial bases for the general complete Lagrange rectangular prism elements and the respective element geometry are shown in Figs. 6–12. Consider the nth-order general complete Langrange element over the 3-cube which allows arbitrary placement of nodes in all its orbits.

Table 2
Quintic serendipity element shape functions

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(5)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(5)}$	$(1/3072)(1-\xi)(1-\eta)(1-\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
2	$(1,-1,-1)$	$N_2^{(5)}$	$(1/3072)(1+\xi)(1-\eta)(1-\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
3	$(1,1,-1)$	$N_3^{(5)}$	$(1/3072)(1+\xi)(1+\eta)(1-\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
4	$(-1,1,-1)$	$N_4^{(5)}$	$(1/3072)(1-\xi)(1+\eta)(1-\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
5	$(-1,-1,1)$	$N_5^{(5)}$	$(1/3072)(1-\xi)(1-\eta)(1+\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
6	$(1,-1,1)$	$N_6^{(5)}$	$(1/3072)(1+\xi)(1-\eta)(1+\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
7	$(1,1,1)$	$N_7^{(5)}$	$(1/3072)(1+\xi)(1+\eta)(1+\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
8	$(-1,1,1)$	$N_8^{(5)}$	$(1/3072)(1-\xi)(1+\eta)(1+\zeta)[625(\xi^4+\eta^4+\zeta^4)-250(\xi^2+\eta^2+\zeta^2)-741]$
9	$(-3/5,-1,-1)$	$N_9^{(5)}$	$-(25/3072)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-1)(5\xi-3)$
10	$(-1/5,-1,-1)$	$N_{10}^{(5)}$	$(25/1536)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-9)(5\xi-1)$
11	$(1/5,-1,-1)$	$N_{11}^{(5)}$	$-(25/1536)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-9)(5\xi+1)$
12	$(3/5,-1,-1)$	$N_{12}^{(5)}$	$(25/3072)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-1)(5\xi+3)$
13	$(1,-3/5,-1)$	$N_{13}^{(5)}$	$-(25/3072)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-1)(5\eta-3)$
14	$(1,-1/5,-1)$	$N_{14}^{(5)}$	$(25/1536)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(5\eta-1)$
15	$(1,1/5,-1)$	$N_{15}^{(5)}$	$-(25/1536)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(5\eta+1)$
16	$(1,3/5,-1)$	$N_{16}^{(5)}$	$(25/3072)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-1)(5\eta+3)$
17	$(3/5,1,-1)$	$N_{17}^{(5)}$	$-(25/3072)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-1)(5\xi+3)$
18	$(1/5,1,-1)$	$N_{18}^{(5)}$	$-(25/1536)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(5\xi+1)$
19	$(-1/5,1,-1)$	$N_{19}^{(5)}$	$(25/1536)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(5\xi-1)$
20	$(-3/5,1,-1)$	$N_{20}^{(5)}$	$-(25/3072)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-1)(5\xi-3)$
21	$(-1,3/5,-1)$	$N_{21}^{(5)}$	$(25/3072)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-1)(5\eta+3)$
22	$(-1,1/5,-1)$	$N_{22}^{(5)}$	$-(25/1536)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(5\eta+1)$
23	$(-1,-1/5,-1)$	$N_{23}^{(5)}$	$(25/1536)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(5\eta-1)$
24	$(-1,-3/5,-1)$	$N_{24}^{(5)}$	$-(25/3072)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-1)(5\eta-3)$
25	$(-1,-1,-3/5)$	$N_{25}^{(5)}$	$-(25/3072)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta-3)$
26	$(1,-1,-3/5)$	$N_{26}^{(5)}$	$-(25/3072)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta-3)$
27	$(1,1,-3/5)$	$N_{27}^{(5)}$	$-(25/3072)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta-3)$
28	$(-1,1,-3/5)$	$N_{28}^{(5)}$	$-(25/3072)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta-3)$
29	$(-1,-1,-1/5)$	$N_{29}^{(5)}$	$(25/1536)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta-1)$
30	$(1,-1,-1/5)$	$N_{30}^{(5)}$	$(25/1536)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta-1)$
31	$(1,1,-1/5)$	$N_{31}^{(5)}$	$(25/1536)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta-1)$
32	$(-1,1,-1/5)$	$N_{32}^{(5)}$	$(25/1536)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta-1)$
33	$(-1,-1,1/5)$	$N_{33}^{(5)}$	$-(25/1536)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta+1)$
34	$(1,-1,1/5)$	$N_{34}^{(5)}$	$-(25/1536)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta+1)$
35	$(1,1,1/5)$	$N_{35}^{(5)}$	$-(25/1536)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta+1)$
36	$(-1,1,1/5)$	$N_{36}^{(5)}$	$-(25/1536)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(5\zeta+1)$

(*continued overleaf*)

Table 2 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(5)}$	Shape function
37	$(-1, -1, 3/5)$	$N_{37}^{(5)}$	$(25/3072)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta+3)$
38	$(1, -1, 3/5)$	$N_{38}^{(5)}$	$(25/3072)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta+3)$
39	$(1, 1, 3/5)$	$N_{39}^{(5)}$	$(25/3072)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta+3)$
40	$(-1, 1, 3/5)$	$N_{40}^{(5)}$	$(25/3072)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-1)(5\zeta+3)$
41	$(3/5, -1, 1)$	$N_{41}^{(5)}$	$(25/3072)(1-\xi^2)(1-\eta)(1+\zeta)(25\xi^2-1)(5\xi+3)$
42	$(1/5, -1, 1)$	$N_{42}^{(5)}$	$-(25/1536)(1-\xi^2)(1-\eta)(1+\zeta)(25\xi^2-9)(5\xi+1)$
43	$(-1/5, -1, 1)$	$N_{43}^{(5)}$	$(25/1536)(1-\xi^2)(1-\eta)(1+\zeta)(25\xi^2-9)(5\xi-1)$
44	$(-3/5, -1, 1)$	$N_{44}^{(5)}$	$-(25/3072)(1-\xi^2)(1-\eta)(1+\zeta)(25\xi^2-1)(5\xi-3)$
45	$(1, -3/5, 1)$	$N_{45}^{(5)}$	$-(25/3072)(1+\xi)(1-\eta^2)(1+\zeta)(25\eta^2-1)(5\eta-3)$
46	$(1, -1/5, 1)$	$N_{46}^{(5)}$	$(25/1536)(1+\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(5\eta-1)$
47	$(1, 1/5, 1)$	$N_{47}^{(5)}$	$-(25/1536)(1+\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(5\eta+1)$
48	$(1, 3/5, 1)$	$N_{48}^{(5)}$	$(25/3072)(1+\xi)(1-\eta^2)(1+\zeta)(25\eta^2-1)(5\eta+3)$
49	$(3/5, 1, 1)$	$N_{49}^{(5)}$	$(25/3072)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-1)(5\xi+3)$
50	$(1/5, 1, 1)$	$N_{50}^{(5)}$	$-(25/1536)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-9)(5\xi+1)$
51	$(-1/5, 1, 1)$	$N_{51}^{(5)}$	$(25/1536)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-9)(5\xi-1)$
52	$(-3/5, 1, 1)$	$N_{52}^{(5)}$	$-(25/3072)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-1)(5\xi-3)$
53	$(-1, 3/5, 1)$	$N_{53}^{(5)}$	$(25/3072)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-1)(5\eta+3)$
54	$(-1, 1/5, 1)$	$N_{54}^{(5)}$	$-(25/1536)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(5\eta+1)$
55	$(-1, -1/5, 1)$	$N_{55}^{(5)}$	$(25/1536)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(5\eta-1)$
56	$(-1, -3/5, 1)$	$N_{56}^{(5)}$	$-(25/3072)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-1)(5\eta-3)$

Let $N_i^{(n)}(\xi, \eta, \varsigma),\ i = 1, 2, \ldots, 12n-4$, refer to shape functions of nth-order general serendipity rectangular prism element. Following Okabe [9], the shape functions $\hat{N}_i^{(n)}(\xi, \eta, \varsigma),\ i = 1, 2, \ldots, 12n-4+r_0$ on the zeroth orbit of the nth-order general complete Lagrange rectangular prism element can be determined by the formulae

$$\hat{N}_i^{(n)}(\xi, \eta, \varsigma) = N_i^{(n)}(\xi, \eta, \varsigma) - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right)\hat{N}_i^{(n)}(\xi, \eta, \varsigma), \quad i = 1(1)12n-4, \ \ n \geqslant 4, \tag{23a}$$

$$\hat{N}_i(\xi, \eta, \varsigma) = \hat{N}_i^{(n)}(\xi, \eta, \varsigma) - \sum_{k=12n-3+r_0}^{12n-4+i_n} \hat{N}_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right)\hat{\hat{N}}_k^{(n)}(\xi, \eta, \varsigma), \quad i = 1(1)12n-4+r_0, \ \ n \geqslant 4, \tag{23b}$$

where i_n refers to number of interior nodes required for the nth-order general complete Lagrange rectangular prism element,

$$\hat{\hat{N}}_k^{(n)}(\xi, \eta, \varsigma), \quad k = r_0 + 12n - 3, r_0 + 12n - 2, \ldots, 12n - 4 + i_n$$

refer to shape functions of interior nodes and hence they belong to higher orbits, say 1, 2, 3, etc, $(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)})$ refer to coordinates of node k for the nth-order general serendipity rectangular prism element, r_0 refers to nodes of general complete Lagrange rectangular prism element on the zeroth orbit.

Table 3
Sextic serendipity element shape functions

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(6)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(6)}$	$(1/320)(1-\xi)(1-\eta)(1-\zeta)[-81(\xi^5+\eta^5+\zeta^5)+45(\xi^3+\eta^3+\zeta^3)$ $-4(\xi+\eta+\zeta)-80]$
2	$(1,-1,-1)$	$N_2^{(6)}$	$(1/320)(1+\xi)(1-\eta)(1-\zeta)[81(\xi^5-\eta^5-\zeta^5)-45(\xi^3-\eta^3-\zeta^3)$ $+4(\xi-\eta-\zeta)-80]$
3	$(1,1,-1)$	$N_3^{(6)}$	$(1/320)(1+\xi)(1+\eta)(1-\zeta)[81(\xi^5+\eta^5-\zeta^5)-45(\xi^3+\eta^3-\zeta^3)$ $+4(\xi+\eta-\zeta)-80]$
4	$(-1,1,-1)$	$N_4^{(6)}$	$(1/320)(1-\xi)(1+\eta)(1-\zeta)[-81(\xi^5-\eta^5+\zeta^5)+45(\xi^3-\eta^3+\zeta^3)$ $-4(\xi-\eta+\zeta)-80]$
5	$(-1,-1,1)$	$N_5^{(6)}$	$(1/320)(1-\xi)(1-\eta)(1+\zeta)[-81(\xi^5+\eta^5-\zeta^5)+45(\xi^3+\eta^3-\zeta^3)$ $-4(\xi+\eta-\zeta)-80]$
6	$(1,-1,1)$	$N_6^{(6)}$	$(1/320)(1+\xi)(1-\eta)(1+\zeta)[81(\xi^5-\eta^5+\zeta^5)-45(\xi^3-\eta^3+\zeta^3)$ $+4(\xi-\eta+\zeta)-80]$
7	$(1,1,1)$	$N_7^{(6)}$	$(1/320)(1+\xi)(1+\eta)(1+\zeta)[81(\xi^5+\eta^5+\zeta^5)-45(\xi^3+\eta^3+\zeta^3)$ $+4(\xi+\eta+\zeta)-80]$
8	$(-1,1,1)$	$N_8^{(6)}$	$(1/320)(1-\xi)(1+\eta)(1+\zeta)[-81(\xi^5-\eta^5-\zeta^5)+45(\xi^3-\eta^3-\zeta^3)$ $-4(\xi-\eta-\zeta)-80]$
9	$(-2/3,-1,-1)$	$N_9^{(6)}$	$(9/160)(1-\xi^2)(1-\eta)(1-\zeta)(9\xi^2-1)\xi(3\xi-2)$
10	$(-1/3,-1,-1)$	$N_{10}^{(6)}$	$-(9/64)(1-\xi^2)(1-\eta)(1-\zeta)(9\xi^2-4)\xi(3\xi-1)$
11	$(0,-1,-1)$	$N_{11}^{(6)}$	$(1/16)(1-\xi^2)(1-\eta)(1-\zeta)(9\xi^2-4)(9\xi^2-1)$
12	$(1/3,-1,-1)$	$N_{12}^{(6)}$	$-(9/64)(1-\xi^2)(1-\eta)(1-\zeta)(9\xi^2-4)\xi(3\xi+1)$
13	$(2/3,-1,-1)$	$N_{13}^{(6)}$	$(9/160)(1-\xi^2)(1-\eta)(1-\zeta)(9\xi^2-1)\xi(3\xi+2)$
14	$(1,-2/3,-1)$	$N_{14}^{(6)}$	$(9/160)(1+\xi)(1-\eta^2)(1-\zeta)(9\eta^2-1)\eta(3\eta-2)$
15	$(1,-1/3,-1)$	$N_{15}^{(6)}$	$-(9/64)(1+\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)\eta(3\eta-1)$
16	$(1,0,-1)$	$N_{16}^{(6)}$	$(1/16)(1+\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)(9\eta^2-1)$
17	$(1,1/3,-1)$	$N_{17}^{(6)}$	$-(9/64)(1+\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)\eta(3\eta+1)$
18	$(1,2/3,-1)$	$N_{18}^{(6)}$	$(9/160)(1+\xi)(1-\eta^2)(1-\zeta)(9\eta^2-1)\eta(3\eta+2)$
19	$(2/3,1,-1)$	$N_{19}^{(6)}$	$(9/160)(1-\xi^2)(1+\eta)(1-\zeta)(9\xi^2-1)\xi(3\xi+2)$
20	$(1/3,1,-1)$	$N_{20}^{(6)}$	$-(9/64)(1-\xi^2)(1+\eta)(1-\zeta)(9\xi^2-4)\xi(3\xi+1)$
21	$(0,1,-1)$	$N_{21}^{(6)}$	$(1/16)(1-\xi^2)(1+\eta)(1-\zeta)(9\xi^2-4)(9\xi^2-1)$
22	$(-1/3,1,-1)$	$N_{22}^{(6)}$	$-(9/64)(1-\xi^2)(1+\eta)(1-\zeta)(9\xi^2-4)\xi(3\xi-1)$
23	$(-2/3,1,-1)$	$N_{23}^{(6)}$	$(9/160)(1-\xi^2)(1+\eta)(1-\zeta)(9\xi^2-1)\xi(3\xi-2)$
24	$(-1,2/3,-1)$	$N_{24}^{(6)}$	$(9/160)(1-\xi)(1-\eta^2)(1-\zeta)(9\eta^2-1)\eta(3\eta+2)$
25	$(-1,1/3,-1)$	$N_{25}^{(6)}$	$-(9/64)(1-\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)\eta(3\eta+1)$
26	$(-1,0,-1)$	$N_{26}^{(6)}$	$(1/16)(1-\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)(9\eta^2-1)$
27	$(-1,-1/3,-1)$	$N_{27}^{(6)}$	$-(9/64)(1-\xi)(1-\eta^2)(1-\zeta)(9\eta^2-4)\eta(3\eta-1)$
28	$(-1,-2/3,-1)$	$N_{28}^{(6)}$	$(9/160)(1-\xi)(1-\eta^2)(1-\zeta)(9\eta^2-1)\eta(3\eta-2)$
29	$(-1,-1,-2/3)$	$N_{29}^{(6)}$	$(9/160)(1-\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta-2)$
30	$(1,-1,-2/3)$	$N_{30}^{(6)}$	$(9/160)(1+\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta-2)$
31	$(1,1,-2/3)$	$N_{31}^{(6)}$	$(9/160)(1+\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta-2)$

(*continued overleaf*)

Table 3 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(6)}$	Shape function
32	$(-1, 1, -2/3)$	$N_{32}^{(6)}$	$(9/160)(1-\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta-2)$
33	$(-1, -1, -1/3)$	$N_{33}^{(6)}$	$-(9/64)(1-\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta-1)$
34	$(1, -1, -1/3)$	$N_{34}^{(6)}$	$-(9/64)(1+\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta-1)$
35	$(1, 1, -1/3)$	$N_{35}^{(6)}$	$-(9/64)(1+\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta-1)$
36	$(-1, 1, -1/3)$	$N_{36}^{(6)}$	$-(9/64)(1-\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta-1)$
37	$(-1, -1, 0)$	$N_{37}^{(6)}$	$(1/16)(1-\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)(9\zeta^2-1)$
38	$(1, -1, 0)$	$N_{38}^{(6)}$	$(1/16)(1+\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)(9\zeta^2-1)$
39	$(1, 1, 0)$	$N_{39}^{(6)}$	$(1/16)(1+\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)(9\zeta^2-1)$
40	$(-1, 1, 0)$	$N_{40}^{(6)}$	$(1/16)(1-\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)(9\zeta^2-1)$
41	$(-1, -1, 1/3)$	$N_{41}^{(6)}$	$-(9/64)(1-\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta+1)$
42	$(1, -1, 1/3)$	$N_{42}^{(6)}$	$-(9/64)(1+\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta+1)$
43	$(1, 1, 1/3)$	$N_{43}^{(6)}$	$-(9/64)(1+\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta+1)$
44	$(-1, 1, 1/3)$	$N_{44}^{(6)}$	$-(9/64)(1-\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-4)\zeta(3\zeta+1)$
45	$(-1, -1, 2/3)$	$N_{45}^{(6)}$	$(9/160)(1-\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta+2)$
46	$(1, -1, 2/3)$	$N_{46}^{(6)}$	$(9/160)(1+\xi)(1-\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta+2)$
47	$(1, 1, 2/3)$	$N_{47}^{(6)}$	$(9/160)(1+\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta+2)$
48	$(-1, 1, 2/3)$	$N_{48}^{(6)}$	$(9/160)(1-\xi)(1+\eta)(1-\zeta^2)(9\zeta^2-1)\zeta(3\zeta+2)$
49	$(2/3, -1, 1)$	$N_{49}^{(6)}$	$(9/160)(1-\xi^2)(1-\eta)(1+\zeta)(9\xi^2-1)\xi(3\xi+2)$
50	$(1/3, -1, 1)$	$N_{50}^{(6)}$	$-(9/64)(1-\xi^2)(1-\eta)(1+\zeta)(9\xi^2-4)\xi(3\xi+1)$
51	$(0, -1, 1)$	$N_{51}^{(6)}$	$(1/16)(1-\xi^2)(1-\eta)(1+\zeta)(9\xi^2-4)(9\xi^2-1)$
52	$(-1/3, -1, 1)$	$N_{52}^{(6)}$	$-(9/64)(1-\xi^2)(1-\eta)(1+\zeta)(9\xi^2-4)\xi(3\xi-1)$
53	$(-2/3, -1, 1)$	$N_{53}^{(6)}$	$(9/160)(1-\xi^2)(1-\eta)(1+\zeta)(9\xi^2-1)\xi(3\xi-2)$
54	$(1, 2/3, 1)$	$N_{54}^{(6)}$	$(9/160)(1+\xi)(1-\eta^2)(1+\zeta)(9\eta^2-1)\eta(3\eta+2)$
55	$(1, 1/3, 1)$	$N_{55}^{(6)}$	$-(9/64)(1+\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)\eta(3\eta+1)$
56	$(1, 0, 1)$	$N_{56}^{(6)}$	$(1/16)(1+\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)(9\eta^2-1)$
57	$(1, -1/3, 1)$	$N_{57}^{(6)}$	$-(9/64)(1+\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)\eta(3\eta-1)$
58	$(1, -2/3, 1)$	$N_{58}^{(6)}$	$(9/160)(1+\xi)(1-\eta^2)(1+\zeta)(9\eta^2-1)\eta(3\eta-2)$
59	$(2/3, 1, 1)$	$N_{59}^{(6)}$	$(9/160)(1-\xi^2)(1+\eta)(1+\zeta)(9\xi^2-1)\xi(3\xi+2)$
60	$(1/3, 1, 1)$	$N_{60}^{(6)}$	$-(9/64)(1-\xi^2)(1+\eta)(1+\zeta)(9\xi^2-4)\xi(3\xi+1)$
61	$(0, 1, 1)$	$N_{61}^{(6)}$	$(1/16)(1-\xi^2)(1+\eta)(1+\zeta)(9\xi^2-4)(9\xi^2-1)$
62	$(-1/3, 1, 1)$	$N_{62}^{(6)}$	$-(9/64)(1-\xi^2)(1+\eta)(1+\zeta)(9\xi^2-4)\xi(3\xi-1)$
63	$(-2/3, 1, 1)$	$N_{63}^{(6)}$	$(9/160)(1-\xi^2)(1+\eta)(1+\zeta)(9\xi^2-1)\xi(3\xi-2)$
64	$(-1, 2/3, 1)$	$N_{64}^{(6)}$	$(9/160)(1-\xi)(1-\eta^2)(1+\zeta)(9\eta^2-1)\eta(3\eta+2)$
65	$(-1, 1/3, 1)$	$N_{65}^{(6)}$	$-(9/64)(1-\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)\eta(3\eta+1)$
66	$(-1, 0, 1)$	$N_{66}^{(6)}$	$(1/16)(1-\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)(9\eta^2-1)$
67	$(-1, -1/3, 1)$	$N_{67}^{(6)}$	$-(9/64)(1-\xi)(1-\eta^2)(1+\zeta)(9\eta^2-4)\eta(3\eta-1)$
68	$(-1, -2/3, 1)$	$N_{68}^{(6)}$	$(9/160)(1-\xi)(1-\eta^2)(1+\zeta)(9\eta^2-1)\eta(3\eta-2)$

Table 4
Septic serendipity element shape functions

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(7)}$	Shape function
1	$(-1, -1, -1)$	$N_1^{(7)}$	$(1/368640)(1-\xi)(1-\eta)(1-\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
2	$(1, -1, -1)$	$N_2^{(7)}$	$(1/368640)(1+\xi)(1-\eta)(1-\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
3	$(1, 1, -1)$	$N_3^{(7)}$	$(1/368640)(1+\xi)(1+\eta)(1-\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
4	$(-1, 1, -1)$	$N_4^{(7)}$	$(1/368640)(1-\xi)(1+\eta)(1-\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
5	$(-1, -1, 1)$	$N_5^{(7)}$	$(1/368640)(1-\xi)(1-\eta)(1+\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
6	$(1, -1, 1)$	$N_6^{(7)}$	$(1/368640)(1+\xi)(1-\eta)(1+\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
7	$(1, 1, 1)$	$N_7^{(7)}$	$(1/368640)(1+\xi)(1+\eta)(1+\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
8	$(-1, 1, 1)$	$N_8^{(7)}$	$(1/368640)(1-\xi)(1+\eta)(1+\zeta)[117649(\xi^6+\eta^6+\zeta^6) - 84035(\xi^4+\eta^4+\zeta^4) + 12691(\xi^2+\eta^2+\zeta^2) - 92835]$
9	$(-5/7, -1, -1)$	$N_9^{(7)}$	$-(49/368640)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
10	$(-3/7, -1, -1)$	$N_{10}^{(7)}$	$(49/122880)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
11	$(-1/7, -1, -1)$	$N_{11}^{(7)}$	$-(49/73728)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
12	$(-5/7, -1, -1)$	$N_{12}^{(7)}$	$(49/73728)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
13	$(3/7, -1, -1)$	$N_{13}^{(7)}$	$-(49/122880)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
14	$(5/7, -1, -1)$	$N_{14}^{(7)}$	$(49/368640)(1-\xi^2)(1-\eta)(1-\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
15	$(1, -5/7, -1)$	$N_{15}^{(7)}$	$-(49/368640)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
16	$(1, -3/7, -1)$	$N_{16}^{(7)}$	$(49/122880)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
17	$(1, -1/7, -1)$	$N_{17}^{(7)}$	$-(49/73728)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
18	$(1, 1/7, -1)$	$N_{18}^{(7)}$	$(49/73728)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
19	$(1, 3/7, -1)$	$N_{19}^{(7)}$	$-(49/122880)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
20	$(1, 5/7, -1)$	$N_{20}^{(7)}$	$(49/368640)(1+\xi)(1-\eta^2)(1-\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
21	$(5/7, 1, -1)$	$N_{21}^{(7)}$	$(49/368640)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
22	$(3/7, 1, -1)$	$N_{22}^{(7)}$	$-(49/122880)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
23	$(-5/7, 1, -1)$	$N_{23}^{(7)}$	$(49/73728)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
24	$(-1/7, 1, -1)$	$N_{24}^{(7)}$	$-(49/73728)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
25	$(-3/7, 1, -1)$	$N_{25}^{(7)}$	$(49/122880)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
26	$(-5/7, 1, -1)$	$N_{26}^{(7)}$	$-(49/368640)(1-\xi^2)(1+\eta)(1-\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
27	$(-1, 5/7, -1)$	$N_{27}^{(7)}$	$(49/368640)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
28	$(-1, 3/7, -1)$	$N_{28}^{(7)}$	$-(49/122880)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
29	$(-1, 1/7, -1)$	$N_{29}^{(7)}$	$(49/73728)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
30	$(-1, -1/7, -1)$	$N_{30}^{(7)}$	$-(49/73728)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
31	$(-1, -3/7, -1)$	$N_{31}^{(7)}$	$(49/122880)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$

(*continued overleaf*)

Table 4 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(7)}$	Shape function
32	$(-1, -5/7, -1)$	$N_{32}^{(7)}$	$-(49/368640)(1-\xi)(1-\eta^2)(1-\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
33	$(-1, -1, -5/7)$	$N_{33}^{(7)}$	$-(49/368640)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta-5)$
34	$(1, -1, -5/7)$	$N_{34}^{(7)}$	$-(49/368640)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta-5)$
35	$(1, 1, -5/7)$	$N_{35}^{(7)}$	$-(49/368640)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta-5)$
36	$(-1, 1, -5/7)$	$N_{36}^{(7)}$	$-(49/368640)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta-5)$
37	$(-1, -1, -3/7)$	$N_{37}^{(7)}$	$(49/122880)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta-3)$
38	$(1, -1, -3/7)$	$N_{38}^{(7)}$	$(49/122880)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta-3)$
39	$(1, 1, -3/7)$	$N_{39}^{(7)}$	$(49/122880)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta-3)$
40	$(-1, 1, -3/7)$	$N_{40}^{(7)}$	$(49/122880)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta-3)$
41	$(-1, -1, -1/7)$	$N_{41}^{(7)}$	$-(49/73728)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta-1)$
42	$(1, -1, -5/7)$	$N_{42}^{(7)}$	$-(49/73728)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta-1)$
43	$(1, 1, -1/7)$	$N_{43}^{(7)}$	$-(49/73728)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta-1)$
44	$(-1, 1, -1/7)$	$N_{44}^{(7)}$	$-(49/73728)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta-1)$
45	$(-1, -1, 1/7)$	$N_{45}^{(7)}$	$(49/73728)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta+1)$
46	$(1, -1, 1/7)$	$N_{46}^{(7)}$	$(49/73728)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta+1)$
47	$(1, 1, 1/7)$	$N_{47}^{(7)}$	$(49/73728)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta+1)$
48	$(-1, 1, 1/7)$	$N_{48}^{(7)}$	$(49/73728)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-9)(7\zeta+1)$
49	$(-1, -1, 3/7)$	$N_{49}^{(7)}$	$-(49/12280)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta+3)$
50	$(1, -1, 3/7)$	$N_{50}^{(7)}$	$-(49/12280)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta+3)$
51	$(1, 1, 3/7)$	$N_{51}^{(7)}$	$-(49/12280)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta+3)$
52	$(-1, 1, 3/7)$	$N_{52}^{(7)}$	$-(49/12280)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-25)(49\zeta^2-1)(7\zeta+3)$
53	$(-1, -1, 5/7)$	$N_{53}^{(7)}$	$(49/12280)(1-\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta+5)$
54	$(1, -1, 5/7)$	$N_{54}^{(7)}$	$(49/12280)(1+\xi)(1-\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta+5)$
55	$(1, 1, 5/7)$	$N_{55}^{(7)}$	$(49/12280)(1+\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta+5)$
56	$(-1, 1, 5/7)$	$N_{56}^{(7)}$	$(49/12280)(1-\xi)(1+\eta)(1-\zeta^2)(49\zeta^2-9)(49\zeta^2-1)(7\zeta+5)$
57	$(5/7, -1, 1)$	$N_{57}^{(7)}$	$(49/368640)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
58	$(3/7, -1, 1)$	$N_{58}^{(7)}$	$-(49/122880)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
59	$(1/7, -1, 1)$	$N_{59}^{(7)}$	$(49/73728)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
60	$(-1/7, -1, 1)$	$N_{60}^{(7)}$	$-(49/73728)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
61	$(-3/7, -1, 1)$	$N_{61}^{(7)}$	$(49/122880)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
62	$(-5/7, -1, 1)$	$N_{62}^{(7)}$	$-(49/368640)(1-\xi^2)(1-\eta)(1+\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
63	$(1, -5/7, 1)$	$N_{63}^{(7)}$	$-(49/368640)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$
64	$(1, -3/7, 1)$	$N_{64}^{(7)}$	$(49/122880)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
65	$(1, -1/7, 1)$	$N_{65}^{(7)}$	$-(49/73728)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
66	$(1, 1/7, 1)$	$N_{66}^{(7)}$	$(49/73728)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
67	$(1, 3/7, 1)$	$N_{67}^{(7)}$	$-(49/122880)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
68	$(1, 5/7, 1)$	$N_{68}^{(7)}$	$(49/368640)(1+\xi)(1-\eta^2)(1+\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$

Table 4 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(7)}$	Shape function
69	$(5/7, 1, 1)$	$N_{69}^{(7)}$	$(49/368640)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi+5)$
70	$(3/7, 1, 1)$	$N_{70}^{(7)}$	$-(49/122880)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi+3)$
71	$(-5/7, 1, 1)$	$N_{71}^{(7)}$	$(49/73728)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi+1)$
72	$(-1/7, 1, 1)$	$N_{72}^{(7)}$	$-(49/73728)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-9)(7\xi-1)$
73	$(-3/7, 1, 1)$	$N_{73}^{(7)}$	$(49/122880)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-25)(49\xi^2-1)(7\xi-3)$
74	$(-5/7, 1, 1)$	$N_{74}^{(7)}$	$-(49/368640)(1-\xi^2)(1+\eta)(1+\zeta)(49\xi^2-9)(49\xi^2-1)(7\xi-5)$
75	$(-1, 5/7, 1)$	$N_{75}^{(7)}$	$(49/368640)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta+5)$
76	$(-1, 3/7, 1)$	$N_{76}^{(7)}$	$-(49/122880)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta+3)$
77	$(-1, 1/7, 1)$	$N_{77}^{(7)}$	$(49/73728)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta+1)$
78	$(-1, -1/7, 1)$	$N_{78}^{(7)}$	$-(49/73728)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-9)(7\eta-1)$
79	$(-1, -3/7, 1)$	$N_{79}^{(7)}$	$(49/122880)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-25)(49\eta^2-1)(7\eta-3)$
80	$(-1, -5/7, 1)$	$N_{80}^{(7)}$	$-(49/368640)(1-\xi)(1-\eta^2)(1+\zeta)(49\eta^2-9)(49\eta^2-1)(7\eta-5)$

Thus on the zeroth orbit of the element, we have all nodes of the general serendipity rectangular prism element as well as some of the general complete Lagrange rectangular prism element for quintic and higher orders. Without loss of generality, we can omit the double-rank formalism – nodal rank and orbital rank introduced by Okabe [8,9] to determine the explicit expressions of shape functions of interior nodes.

Referring Okabe [8,9], the shape functions of interior nodes can be obtained by multiplying the shape functions of general serendipity rectangular prism element over the 3-cubes $-\theta \leqslant \xi, \eta, \varsigma \leqslant \theta$, and $-\varepsilon \leqslant \xi, \eta, \varsigma \leqslant \varepsilon, 0 < \varepsilon < \theta$ by a multiplication factor called *orbital modifier*. On using this procedure, we can assume that the explicit expressions for shape functions $\hat{\hat{N}}_k^{(n)}(\xi, \eta, \zeta),\ k = r_0 + 12n - 3, r_0 + 12n - 2, \ldots, 12n - 4 + i_n,\ n \geqslant 4$ are known. The shape functions over the edges and faces of zeroth orbit, i.e., on the boundary of $-1 \leqslant \xi, \eta, \varsigma \leqslant 1$ are obtained by using Eqs. (23a) and (23b).

4.1. Theorem

Theorem 1. *Let* $N_i^{(n)}(\xi, \eta, \varsigma),\ i = 1, 2, \ldots, 12n - 4$ *refer to shape functions of* nth-*order general serendipity rectangular prism element, and the shape functions*

$$\hat{N}_k(\xi, \eta, \varsigma), \quad k = 12n - 3, 12n - 2, \ldots, 12n - 4 + r_0$$

satisfy the properties

$$\begin{aligned} &\hat{N}_k^{(n)}\left(\xi_j, \eta_j, \varsigma_j\right) = \delta_{kj}, \quad k, j = 12n - 3, 12 - 2, \ldots, 12n - 4 + r_0, \\ &\hat{N}_j^{(n)}(\xi_l, \eta_l, \varsigma_l) = 0, \quad j = 12n - 3, 12 - 2, \ldots, 12n - 4 + r_0, \ \ l = 1, 2, \ldots, 12n - 4, \\ &\hat{N}_i^{(n)}(\xi, \eta, \varsigma) = N_i^{(n)}(\xi, \eta, \varsigma) - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right) \hat{N}_k^{(n)}(\xi, \eta, \varsigma), \quad i = 1, 2, \ldots, 12n - 4. \end{aligned} \tag{24a}$$

Then

$$\hat{N}_i^{(n)}(\xi_l, \eta_l, \varsigma_l) = \delta_{il}, \quad i, l = 1, 2, \ldots, 12n - 4, 12n - 3, \ldots, 12n - 4 + r_0, \qquad \sum_{i=1}^{12n-4+r_0} \hat{N}_i^{(n)}(\xi, \eta, \varsigma) = 1. \tag{24b}$$

Table 5
Octic serendipity element shape functions

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(8)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(8)}$	$(1/10080)(1-\xi)(1-\eta)(1-\zeta)[-4096(\xi^7+\eta^7+\zeta^7)+3584(\xi^5+\eta^5+\zeta^5)$ $-784(\xi^3+\eta^3+\zeta^3)+36(\xi+\eta+\zeta)-2520]$
2	$(1,-1,-1)$	$N_2^{(8)}$	$(1/10080)(1+\xi)(1-\eta)(1-\zeta)[4096(\xi^7-\eta^7-\zeta^7)-3584(\xi^5-\eta^5-\zeta^5)$ $+784(\xi^3-\eta^3-\zeta^3)-36(\xi-\eta-\zeta)-2520]$
3	$(1,1,-1)$	$N_3^{(8)}$	$(1/10080)(1+\xi)(1+\eta)(1-\zeta)[4096(\xi^7+\eta^7-\zeta^7)-3584(\xi^5+\eta^5-\zeta^5)$ $+784(\xi^3+\eta^3-\zeta^3)-36(\xi+\eta-\zeta)-2520]$
4	$(-1,1,-1)$	$N_4^{(8)}$	$(1/10080)(1-\xi)(1+\eta)(1-\zeta)[-4096(\xi^7-\eta^7+\zeta^7)+3584(\xi^5-\eta^5+\zeta^5)$ $-784(\xi^3-\eta^3+\zeta^3)+36(\xi-\eta+\zeta)-2520]$
5	$(-1,-1,1)$	$N_5^{(8)}$	$(1/10080)(1-\xi)(1-\eta)(1+\zeta)[-4096(\xi^7+\eta^7-\zeta^7)+3584(\xi^5+\eta^5-\zeta^5)$ $-784(\xi^3+\eta^3-\zeta^3)+36(\xi+\eta-\zeta)-2520]$
6	$(1,-1,1)$	$N_6^{(8)}$	$(1/10080)(1+\xi)(1-\eta)(1+\zeta)[4096(\xi^7-\eta^7+\zeta^7)-3584(\xi^5-\eta^5+\zeta^5)$ $+784(\xi^3-\eta^3+\zeta^3)-36(\xi-\eta+\zeta)-2520]$
7	$(1,1,1)$	$N_7^{(8)}$	$(1/10080)(1+\xi)(1+\eta)(1+\zeta)[4096(\xi^7+\eta^7+\zeta^7)-3584(\xi^5+\eta^5+\zeta^5)$ $+784(\xi^3+\eta^3+\zeta^3)-36(\xi+\eta+\zeta)-2520]$
8	$(-1,1,1)$	$N_8^{(8)}$	$(1/10080)(1-\xi)(1+\eta)(1+\zeta)[-4096(\xi^7-\eta^7-\zeta^7)+3584(\xi^5-\eta^5-\zeta^5)$ $-784(\xi^3-\eta^3-\zeta^3)+36(\xi-\eta-\zeta)-2520]$
9	$(-3/4,-1,-1)$	$N_9^{(8)}$	$(1/315)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
10	$(-2/4,-1,-1)$	$N_{10}^{(8)}$	$-(1/90)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
11	$(-1/4,-1,-1)$	$N_{11}^{(8)}$	$(1/45)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
12	$(0,-1,-1)$	$N_{12}^{(8)}$	$-(1/144)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
13	$(1/4,-1,-1)$	$N_{13}^{(8)}$	$(1/45)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
14	$(2/4,-1,-1)$	$N_{14}^{(8)}$	$-(1/90)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
15	$(3/4,-1,-1)$	$N_{15}^{(8)}$	$(1/315)(1-\xi^2)(1-\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
16	$(1,-3/4,-1)$	$N_{16}^{(8)}$	$(1/315)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$
17	$(1,-2/4,-1)$	$N_{17}^{(8)}$	$(-(1/90)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
18	$(1,-1/4,-1)$	$N_{18}^{(8)}$	$(1/45)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
19	$(1,0,-1)$	$N_{19}^{(8)}$	$-(1/144)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
20	$(1,1/4,-1)$	$N_{20}^{(8)}$	$(1/45)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
21	$(1,2/4,-1)$	$N_{21}^{(8)}$	$-(1/90)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
22	$(1,3/4,-1)$	$N_{22}^{(8)}$	$(1/315)(1+\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
23	$(3/4,1,-1)$	$N_{23}^{(8)}$	$(1/315)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
24	$(2/4,1,-1)$	$N_{24}^{(8)}$	$-(1/90)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
25	$(1/4,1,-1)$	$N_{25}^{(8)}$	$(1/45)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
26	$(0,1,-1)$	$N_{26}^{(8)}$	$(-(1/144)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
27	$(-1/4,1,-1)$	$N_{27}^{(8)}$	$(1/45)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
28	$(-2/4,1,-1)$	$N_{28}^{(8)}$	$-(1/90)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
29	$(-3/4,1,-1)$	$N_{29}^{(8)}$	$(1/315)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
30	$(-1,3/4,-1)$	$N_{30}^{(8)}$	$(1/315)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
31	$(-1,2/4,-1)$	$N_{31}^{(8)}$	$-(1/90)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
32	$(-1,1/4,-1)$	$N_{32}^{(8)}$	$(1/45)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$

Table 5 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(8)}$	Shape function
33	$(-1, 0, -1)$	$N_{33}^{(8)}$	$-(1/144)(1-\xi)(1-\eta)(1-\zeta)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
34	$(-1, -1/4, -1)$	$N_{34}^{(8)}$	$(1/45)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
35	$(-1, -2/4, -1)$	$N_{35}^{(8)}$	$-(1/90)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
36	$(-1, -3/4, -1)$	$N_{36}^{(8)}$	$(1/315)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$
37	$(-1, -1, -3/4)$	$N_{37}^{(8)}$	$(1/315)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta-3)$
38	$(1, -1, -3/4)$	$N_{38}^{(8)}$	$(1/315)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta-3)$
39	$(1, 1, -3/4)$	$N_{39}^{(8)}$	$(1/315)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta-3)$
40	$(-1, 1, -3/4)$	$N_{40}^{(8)}$	$(1/315)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta-3)$
41	$(-1, -1, -2/4)$	$N_{41}^{(8)}$	$-(1/90)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta-2)$
42	$(1, -1, -2/4)$	$N_{42}^{(8)}$	$-(1/90)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta-2)$
43	$(1, 1, -2/4)$	$N_{43}^{(8)}$	$-(1/90)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta-2)$
44	$(-1, 1, -2/4)$	$N_{44}^{(8)}$	$-(1/90)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta-2)$
45	$(-1, -1, -1/4)$	$N_{45}^{(8)}$	$(1/45)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta-1)$
46	$(1, -1, -1/4)$	$N_{46}^{(8)}$	$(1/45)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta-1)$
47	$(1, 1, -1/4)$	$N_{47}^{(8)}$	$(1/45)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta-1)$
48	$(-1, 1, -1/4)$	$N_{48}^{(8)}$	$(1/45)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta-1)$
49	$(-1, -1, 0)$	$N_{49}^{(8)}$	$-(1/144)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)(16\zeta^2-1)$
50	$(1, -1, 0)$	$N_{50}^{(8)}$	$-(1/144)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)(16\zeta^2-1)$
51	$(1, 1, 0)$	$N_{51}^{(8)}$	$-(1/144)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)(16\zeta^2-1)$
52	$(-1, 1, 0)$	$N_{52}^{(8)}$	$-(1/144)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)(16\zeta^2-1)$
53	$(-1, -1, 1/4)$	$N_{53}^{(8)}$	$(1/45)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta+1)$
54	$(1, -1, 1/4)$	$N_{54}^{(8)}$	$(1/45)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta+1)$
55	$(1, 1, 1/4)$	$N_{55}^{(8)}$	$(1/45)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta+1)$
56	$(-1, 1, 1/4)$	$N_{56}^{(8)}$	$(1/45)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-4)\zeta(4\zeta+1)$
57	$(-1, -1, 2/4)$	$N_{57}^{(8)}$	$-(1/90)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta+2)$
58	$(1, -1, 2/4)$	$N_{58}^{(8)}$	$-(1/90)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta+2)$
59	$(1, 1, 2/4)$	$N_{59}^{(8)}$	$-(1/90)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta+2))$
60	$(-1, 1, 2/4)$	$N_{60}^{(8)}$	$-(1/90)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-9)(16\zeta^2-1)\zeta(4\zeta+2))$
61	$(-1, -1, 3/4)$	$N_{61}^{(8)}$	$(1/315)(1-\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta+3)$
62	$(1, -1, 3/4)$	$N_{62}^{(8)}$	$(1/315)(1+\xi)(1-\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta+3)$
63	$(1, 1, 3/4)$	$N_{63}^{(8)}$	$(1/315)(1+\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta+3)$
64	$(-1, 1, 3/4)$	$N_{64}^{(8)}$	$(1/315)(1-\xi)(1+\eta)(1-\zeta^2)(16\zeta^2-4)(16\zeta^2-1)\zeta(4\zeta+3)$
65	$(3/4, -1, 1)$	$N_{65}^{(8)}$	$(1/315)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
66	$(2/4, -1, 1)$	$N_{66}^{(8)}$	$-(1/90)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
67	$(1/4, -1, 1)$	$N_{67}^{(8)}$	$(1/45)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
68	$(0, -1, 1)$	$N_{68}^{(8)}$	$-(1/144)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$

(*continued overleaf*)

Table 5 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(8)}$	Shape function
69	$(-1/4, -1, 1)$	$N_{69}^{(8)}$	$(1/45)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
70	$(-2/4, -1, 1)$	$N_{70}^{(8)}$	$-(1/90)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
71	$(-3/4, -1, 1)$	$N_{71}^{(8)}$	$(1/315)(1-\xi^2)(1-\eta)(1+\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
72	$(1, 3/4, 1)$	$N_{72}^{(8)}$	$(1/315)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
73	$(1, 2/4, 1)$	$N_{73}^{(8)}$	$-(1/90)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
74	$(1, 1/4, 1)$	$N_{74}^{(8)}$	$(1/45)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
75	$(1, 0, 1)$	$N_{75}^{(8)}$	$-(1/144)(1+\xi)(1-\eta)(1+\zeta)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
76	$(1, -1/4, 1)$	$N_{76}^{(8)}$	$(1/45)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
77	$(1, -2/4, 1)$	$N_{77}^{(8)}$	$-(1/90)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
78	$(1, -3/4, 1)$	$N_{78}^{(8)}$	$(1/315)(1+\xi)(1-\eta^2)(1+\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$
79	$(3/4, 1, -1)$	$N_{79}^{(8)}$	$(1/315)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi+3)$
80	$(2/4, 1, -1)$	$N_{80}^{(8)}$	$-(1/90)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi+2)$
81	$(1/4, 1, -1)$	$N_{81}^{(8)}$	$(1/45)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi+1)$
82	$(0, 1, -1)$	$N_{82}^{(8)}$	$-(1/144)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)(16\xi^2-1)$
83	$(-1/4, 1, -1)$	$N_{83}^{(8)}$	$(1/45)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-4)\xi(4\xi-1)$
84	$(-2/4, 1, -1)$	$N_{84}^{(8)}$	$-(1/90)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-9)(16\xi^2-1)\xi(4\xi-2)$
85	$(-3/4, 1, -1)$	$N_{85}^{(8)}$	$(1/315)(1-\xi^2)(1+\eta)(1-\zeta)(16\xi^2-4)(16\xi^2-1)\xi(4\xi-3)$
86	$(-1, 3/4, -1)$	$N_{86}^{(8)}$	$(1/315)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta+3)$
87	$(-1, 2/4, -1)$	$N_{87}^{(8)}$	$-(1/90)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta+2)$
88	$(-1, 1/4, -1)$	$N_{88}^{(8)}$	$(1/45)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta+1)$
89	$(-1, 0, -1)$	$N_{89}^{(8)}$	$-(1/144)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)(16\eta^2-1)$
90	$(-1, -1/4, -1)$	$N_{90}^{(8)}$	$(1/45)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-4)\eta(4\eta-1)$
91	$(-1, -2/4, -1)$	$N_{91}^{(8)}$	$-(1/90)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-9)(16\eta^2-1)\eta(4\eta-2)$
92	$(-1, -3/4, -1)$	$N_{92}^{(8)}$	$(1/315)(1-\xi)(1-\eta^2)(1-\zeta)(16\eta^2-4)(16\eta^2-1)\eta(4\eta-3)$

Proof. By Eq. (23a) and property of serendipity element shape functions,

$$\sum_{i=1}^{12n-4} \hat{N}_i^{(n)}(\xi,\eta,\varsigma) = \sum_{i=1}^{12n-4} N_i^{(n)}(\xi,\eta,\varsigma) - \sum_{k=12n-3}^{12n-4+r_0} \left\{ \sum_{i=1}^{12n-4} N_i^{(n)}\left(\xi_k^{(n)},\eta_k^{(n)},\varsigma_k^{(n)}\right) \right\} \hat{N}_k^{(n)}(\xi,\eta,\varsigma)$$

$$= 1 - \sum_{k=12-3}^{12n-4+r_0} \hat{N}_k^{(n)}(\xi,\eta,\varsigma)$$

from which we obtain

$$\sum_{i=1}^{12n-4+r_0} \hat{N}_i^{(n)}(\xi,\eta,\varsigma) = 1.$$

Table 6
Ninth-order serendipity element shape functions

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(9)}$	Shape function
1	$(-1, -1, -1)$	$N_1^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
2	$(1, -1, -1)$	$N_2^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
3	$(1, 1, -1)$	$N_3^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
4	$(-1, 1, -1)$	$N_4^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
5	$(-1, -1, 1)$	$N_5^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
6	$(1, -1, 1)$	$N_6^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
7	$(1, 1, 1)$	$N_7^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
8	$(-1, 1, 1)$	$N_8^{(9)}$	$(1/82575360)(1-\xi)(1-\eta)(1-\zeta)[43046721(\xi^8+\eta^8+\zeta^8)$ $-44641044(\xi^6+\eta^6+\zeta^6)+12951414(\xi^4+\eta^4+\zeta^4)$ $-1046196(\xi^2+\eta^2+\zeta^2)-20610765]$
9	$(-7/9, -1, -1)$	$N_9^{(9)}$	$-(81/82575360)(1-\xi^2)(1-\eta)(1-\zeta)(81\xi^2-25)$ $(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
10	$(-5/9, -1, -1)$	$N_{10}^{(9)}$	$(81/20643840)(1-\xi^2)(1-\eta)(1-\zeta)(81\xi^2-49)$ $(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
11	$(-3/9, -1, -1)$	$N_{11}^{(9)}$	$-(81/8847360)(1-\xi^2)(1-\eta)(1-\zeta)(81\xi^2-49)$ $(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
12	$(-1/9, -1, -1)$	$N_{12}^{(9)}$	$(81/5898240)(1-\xi^2)(1-\eta)(1-\zeta)$ $(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
13	$(1/9, -1, -1)$	$N_{13}^{(9)}$	$-(81/5898240)(1-\xi^2)(1-\eta)(1-\zeta)(81\xi^2-49)$ $(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
14	$(3/9, -1, -1)$	$N_{14}^{(9)}$	$(81/8847360)(1-\xi^2)(1-\eta)(1-\zeta)$ $(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
15	$(5/9, -1, -1)$	$N_{15}^{(9)}$	$(81/2063840)(1-\xi^2)(1-\eta)(1-\zeta)$ $(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
16	$(7/9, -1, -1)$	$N_{16}^{(9)}$	$-(81/82575360)(1-\xi^2)(1-\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)$ $(81\xi^2-1)(9\xi+7)$
17	$(1, -7/9, -1)$	$N_{17}^{(9)}$	$-(81/82575360)(1+\xi)(1-\eta^2)(1-\zeta)$ $(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$
18	$(1, -5/9, -1)$	$N_{18}^{(9)}$	$(81/20643840)(1+\xi)(1-\eta^2)(1-\zeta)$ $(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$

(continued overleaf)

Table 6 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(9)}$	Shape function
19	$(1, -3/9, -1)$	$N_{19}^{(9)}$	$-(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)$ $(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
20	$(1, -1/9, -1)$	$N_{20}^{(9)}$	$(81/5898240)(1+\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta-1)$
21	$(1, 1/9, -1)$	$N_{21}^{(9)}$	$-(81/5898240)(1+\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta+1)$
22	$(1, 3/9, -1)$	$N_{22}^{(9)}$	$(81/8847360)(1+\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-1)(9\eta+3)$
23	$(1, 5/9, -1)$	$N_{23}^{(9)}$	$(81/2063840)(1+\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+5)$
24	$(1, 7/9, -1)$	$N_{24}^{(9)}$	$-(81/82575360)(1+\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+7)$
25	$(7/9, 1, -1)$	$N_{25}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)$ $(81\xi^2-1)(9\xi+7)$
26	$(5/9, 1, -1)$	$N_{26}^{(9)}$	$(81/2063840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)$ $(81\xi^2-1)(9\xi+5)$
27	$(3/9, 1, -1)$	$N_{27}^{(9)}$	$(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-1)(9\xi+3)$
28	$(1/9, 1, -1)$	$N_{28}^{(9)}$	$-(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-9)(9\xi+1)$
29	$(-1/9, 1, -1)$	$N_{29}^{(9)}$	$(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-9)(9\xi-1)$
30	$(-3/9, 1, -1)$	$N_{30}^{(9)}$	$-(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-1)(9\xi-3)$
31	$(-5/9, 1, -1)$	$N_{31}^{(9)}$	$(81/20643840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)$ $(81\xi^2-1)(9\xi-5)$
32	$(-7/9, 1, -1)$	$N_{32}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)$ $(81\xi^2-1)(9\xi-7)$
33	$(-1, 7/9, -1)$	$N_{33}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+7)$
34	$(-1, 5/9, -1)$	$N_{34}^{(9)}$	$(81/2063840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+5)$
35	$(-1, 3/9, -1)$	$N_{35}^{(9)}$	$(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-1)(9\eta+3)$
36	$(-1, 1/9, -1)$	$N_{36}^{(9)}$	$-(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta+1)$
37	$(-1, -1/9, -1)$	$N_{37}^{(9)}$	$(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta-1)$
38	$(-1, -3/9, -1)$	$N_{38}^{(9)}$	$-(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-1)(9\eta-3)$
39	$(-1, -5/9, -1)$	$N_{39}^{(9)}$	$(81/20643840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta-5)$
40	$(-1, -7/9, -1)$	$N_{40}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta-7)$
41	$(-1, -1, -7/9)$	$N_{41}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)$ $(81\zeta^2-1)(9\zeta-7)$

Table 6 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(9)}$	Shape function
42	$(1, -1, -7/9)$	$N_{42}^{(9)}$	$-(81/82575360)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-7)$
43	$(1, 1, -7/9)$	$N_{43}^{(9)}$	$-(81/82575360)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-7)$
44	$(-1, 1, -7/9)$	$N_{44}^{(9)}$	$-(81/82575360)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-7)$
45	$(-1, -1, -5/9)$	$N_{45}^{(9)}$	$(81/20643840)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-5)$
46	$(1, -1, -5/9)$	$N_{46}^{(9)}$	$(81/20643840)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-5)$
47	$(1, 1, -5/9)$	$N_{47}^{(9)}$	$(81/20643840)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-5)$
48	$(-1, 1, -5/9)$	$N_{48}^{(9)}$	$(81/20643840)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta-5)$
49	$(-1, -1, -3/9)$	$N_{49}^{(9)}$	$-(81/8847360)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta-3)$
50	$(1, -1, -3/9)$	$N_{50}^{(9)}$	$-(81/8847360)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta-3)$
51	$(1, 1, -3/9)$	$N_{51}^{(9)}$	$-(81/8847360)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta-3)$
52	$(-1, 1, -3/9)$	$N_{52}^{(9)}$	$-(81/8847360)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta-3)$
53	$(-1, -1, -1/9)$	$N_{53}^{(9)}$	$(81/5898240)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta-1)$
54	$(1, -1, -1/9)$	$N_{54}^{(9)}$	$(81/5898240)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta-1)$
55	$(1, 1, -1/9)$	$N_{55}^{(9)}$	$(81/5898240)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta-1)$
56	$(-1, 1, -1/9)$	$N_{56}^{(9)}$	$(81/5898240)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta-1)$
57	$(-1, -1, 1/9)$	$N_{57}^{(9)}$	$-(81/5898240)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta+1)$
58	$(1, -1, 1/9)$	$N_{58}^{(9)}$	$-(81/5898240)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta+1)$
59	$(1, 1, 1/9)$	$N_{59}^{(9)}$	$-(81/5898240)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta+1)$
60	$(-1, 1, 1/9)$	$N_{60}^{(9)}$	$-(81/5898240)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-9)(9\zeta+1)$
61	$(-1, -1, 3/9)$	$N_{61}^{(9)}$	$(81/8847360)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta+3)$
62	$(1, -1, 3/9)$	$N_{62}^{(9)}$	$(81/8847360)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta+3)$
63	$(1, 1, 3/9)$	$N_{63}^{(9)}$	$(81/8847360)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta+3)$

(*continued overleaf*)

Table 6 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(9)}$	Shape function
64	$(-1, 1, 3/9)$	$N_{64}^{(9)}$	$(81/8847360)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-25)(81\zeta^2-1)(9\zeta+3)$
65	$(-1, -1, 5/9)$	$N_{65}^{(9)}$	$(81/2063840)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+5)$
66	$(1, -1, 5/9)$	$N_{66}^{(9)}$	$(81/2063840)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+5)$
67	$(1, 1, 5/9)$	$N_{67}^{(9)}$	$(81/2063840)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+5)$
68	$(-1, 1, 5/9)$	$N_{68}^{(9)}$	$(81/2063840)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-49)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+5)$
69	$(-1, -1, 7/9)$	$N_{69}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+7)$
70	$(1, -1, 7/9)$	$N_{70}^{(9)}$	$-(81/82575360)(1+\xi)(1-\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+7)$
71	$(1, 1, 7/9)$	$N_{71}^{(9)}$	$-(81/82575360)(1+\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+7)$
72	$(-1, 1, 7/9)$	$N_{72}^{(9)}$	$-(81/82575360)(1-\xi)(1+\eta)(1-\zeta^2)(81\zeta^2-25)(81\zeta^2-9)(81\zeta^2-1)(9\zeta+7)$
73	$(-5/7, -1, -1)$	$N_{73}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi+7)$
74	$(-5/7, -1, -1)$	$N_{74}^{(9)}$	$(81/2063840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi+5)$
75	$(-5/7, -1, -1)$	$N_{75}^{(9)}$	$(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi+3)$
76	$(-5/7, -1, -1)$	$N_{76}^{(9)}$	$-(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi+1)$
77	$(-5/7, -1, -1)$	$N_{77}^{(9)}$	$(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-9)(9\xi-1)$
78	$(-5/7, -1, -1)$	$N_{78}^{(9)}$	$-(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)(81\xi^2-1)(9\xi-3)$
79	$(-5/7, -1, -1)$	$N_{79}^{(9)}$	$(81/20643840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)(81\xi^2-1)(9\xi-5)$
80	$(-5/7, -1, -1)$	$N_{80}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)(81\xi^2-1)(9\xi-7)$
81	$(-5/7, -1, -1)$	$N_{81}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)(81\eta^2-1)(9\eta-7)$
82	$(-5/7, -1, -1)$	$N_{82}^{(9)}$	$(81/20643840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)(81\eta^2-1)(9\eta-5)$
83	$(-5/7, -1, -1)$	$N_{83}^{(9)}$	$-(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta-3)$
84	$(-5/7, -1, -1)$	$N_{84}^{(9)}$	$(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta-1)$
85	$(-5/7, -1, -1)$	$N_{85}^{(9)}$	$-(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)(81\eta^2-9)(9\eta+1)$
86	$(-5/7, -1, -1)$	$N_{86}^{(9)}$	$(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)(81\eta^2-1)(9\eta+3)$

Table 6 (*continued*)

Node i	Coordinates (ξ_i, η_i, ζ_i)	$N_i^{(9)}$	Shape function
87	$(-5/7, -1, -1)$	$N_{87}^{(9)}$	$(81/2063840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+5)$
88	$(-5/7, -1, -1)$	$N_{88}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+7)$
89	$(-5/7, -1, -1)$	$N_{89}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)$ $(81\xi^2-1)(9\xi+7)$
90	$(-5/7, -1, -1)$	$N_{90}^{(9)}$	$(81/2063840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)$ $(81\xi^2-1)(9\xi+5)$
91	$(-5/7, -1, -1)$	$N_{91}^{(9)}$	$(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-1)(9\xi+3)$
92	$(-5/7, -1, -1)$	$N_{92}^{(9)}$	$-(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-9)(9\xi+1)$
93	$(-5/7, -1, -1)$	$N_{93}^{(9)}$	$(81/5898240)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-9)(9\xi-1)$
94	$(-5/7, -1, -1)$	$N_{94}^{(9)}$	$-(81/8847360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-25)$ $(81\xi^2-1)(9\xi-3)$
95	$(-5/7, -1, -1)$	$N_{95}^{(9)}$	$(81/20643840)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-49)(81\xi^2-9)$ $(81\xi^2-1)(9\xi-5)$
96	$(-5/7, -1, -1)$	$N_{96}^{(9)}$	$-(81/82575360)(1-\xi^2)(1+\eta)(1-\zeta)(81\xi^2-25)(81\xi^2-9)$ $(81\xi^2-1)(9\xi-7)$
97	$(-5/7, -1, -1)$	$N_{97}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+7)$
98	$(-5/7, -1, -1)$	$N_{98}^{(9)}$	$(81/2063840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta+5)$
99	$(-5/7, -1, -1)$	$N_{99}^{(9)}$	$(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-1)(9\eta+3)$
100	$(-5/7, -1, -1)$	$N_{100}^{(9)}$	$-(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta+1)$
101	$(-5/7, -1, -1)$	$N_{101}^{(9)}$	$(81/5898240)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-9)(9\eta-1)$
102	$(-5/7, -1, -1)$	$N_{102}^{(9)}$	$-(81/8847360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-25)$ $(81\eta^2-1)(9\eta-3)$
103	$(-5/7, -1, -1)$	$N_{103}^{(9)}$	$(81/20643840)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-49)(81\eta^2-9)$ $(81\eta^2-1)(9\eta-5)$
104	$(-5/7, -1, -1)$	$N_{104}^{(9)}$	$-(81/82575360)(1-\xi)(1-\eta^2)(1-\zeta)(81\eta^2-25)(81\eta^2-9)$ $(81\eta^2-1)(9\eta-7)$

By the choice of $\hat{N}_k(\xi,\eta,\varsigma),\ k = 12n-3, 12n-2, \dots, 12n-4+r_0$, we have

(i) For $l, i = 1, 2, \dots, 12n-4$,

$$\hat{N}_i(\xi_l,\eta_l,\varsigma_l) = N_i^{(n)}(\xi_l,\eta_l,\varsigma_l) - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)},\eta_k^{(n)},\varsigma_k^{(n)}\right)\hat{N}_k^{(n)}(\xi_l,\eta_l,\varsigma_l)$$

$$= \delta_{il} - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)},\eta_k^{(n)},\varsigma_k^{(n)}\right)0 = \delta_{il}.$$

Table 7
Tenth-order serendipity element shape functions

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(10)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(10)}$	$(1/580608)(1-\xi)(1-\eta)(1-\zeta)[-390625(\xi^9+\eta^9+\zeta^9)$ $+468750(\xi^7+\eta^7+\zeta^7)-170625(\xi^5+\eta^5+\zeta^5)+20500(\xi^3+\eta^3+\zeta^3)$ $-576(\xi+\eta+\zeta)-145152]$
2	$(1,-1,-1)$	$N_2^{(10)}$	$(1/580608)(1+\xi)(1-\eta)(1-\zeta)[390625(\xi^9-\eta^9-\zeta^9)$ $-468750(\xi^7-\eta^7-\zeta^7)+170625(\xi^5-\eta^5-\zeta^5)-20500(\xi^3-\eta^3-\zeta^3)$ $+576(\xi-\eta-\zeta)-145152]$
3	$(1,1,-1)$	$N_3^{(10)}$	$(1/580608)(1+\xi)(1+\eta)(1-\zeta)[390625(\xi^9+\eta^9-\zeta^9)$ $-468750(\xi^7+\eta^7-\zeta^7)+170625(\xi^5+\eta^5-\zeta^5)-20500(\xi^3+\eta^3-\zeta^3)$ $+576(\xi+\eta-\zeta)-145152]$
4	$(-1,1,-1)$	$N_4^{(10)}$	$(1/580608)(1-\xi)(1+\eta)(1-\zeta)[-390625(\xi^9-\eta^9+\zeta^9)$ $+468750(\xi^7-\eta^7+\zeta^7)-170625(\xi^5-\eta^5+\zeta^5)+20500(\xi^3-\eta^3+\zeta^3)$ $-576(\xi-\eta+\zeta)-145152]$
5	$(-1,-1,1)$	$N_5^{(10)}$	$(1/580608)(1-\xi)(1-\eta)(1+\zeta)[-390625(\xi^9+\eta^9-\zeta^9)$ $+468750(\xi^7+\eta^7-\zeta^7)-170625(\xi^5+\eta^5-\zeta^5)+20500(\xi^3+\eta^3-\zeta^3)$ $-576(\xi+\eta-\zeta)-145152]$
6	$(1,-1,1)$	$N_6^{(10)}$	$(1/580608)(1+\xi)(1-\eta)(1+\zeta)[390625(\xi^9-\eta^9+\zeta^9)$ $-468750(\xi^7-\eta^7+\zeta^7)+170625(\xi^5-\eta^5+\zeta^5)-20500(\xi^3-\eta^3+\zeta^3)$ $+576(\xi-\eta+\zeta)-145152]$
7	$(1,1,1)$	$N_7^{(10)}$	$(1/580608)(1+\xi)(1+\eta)(1+\zeta)[390625(\xi^9+\eta^9+\zeta^9)$ $-468750(\xi^7+\eta^7+\zeta^7)+170625(\xi^5+\eta^5+\zeta^5)-20500(\xi^3+\eta^3+\zeta^3)$ $+576(\xi+\eta+\zeta)-145152]$
8	$(-1,1,1)$	$N_8^{(10)}$	$(1/580608)(1-\xi)(1+\eta)(1+\zeta)[-390625(\xi^9-\eta^9-\zeta^9)$ $+468750(\xi^7-\eta^7-\zeta^7)-170625(\xi^5-\eta^5-\zeta^5)+20500(\xi^3-\eta^3-\zeta^3)$ $-576(\xi-\eta-\zeta)-145152]$
9	$(-4/5,-1,-1)$	$N_9^{(10)}$	$(25/290304)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
10	$(-3/5,-1,-1)$	$N_{10}^{(10)}$	$-(25/64512)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
11	$(-2/5,-1,-1)$	$N_{11}^{(10)}$	$(25/24192)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
12	$(-1/5,-1,-1)$	$N_{12}^{(10)}$	$-(25/13824)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
13	$(0,-1,-1)$	$N_{13}^{(10)}$	$(1/2304)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
14	$(1/5,-1,-1)$	$N_{14}^{(10)}$	$-(25/13824)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
15	$(2/5,-1,-1)$	$N_{15}^{(10)}$	$(25/24192)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
16	$(3/5,-1,-1)$	$N_{16}^{(10)}$	$-(25/64512)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
17	$(4/5,-1,-1)$	$N_{17}^{(10)}$	$(25/290304)(1-\xi^2)(1-\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
18	$(1,-4/5,-1)$	$N_{18}^{(10)}$	$(25/290304)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
19	$(1,-3/5,-1)$	$N_{19}^{(10)}$	$-(25/64512)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
20	$(1,-2/5,-1)$	$N_{20}^{(10)}$	$(25/24192)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
21	$(1,-1/5,-1)$	$N_{21}^{(10)}$	$(25/290304)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
22	$(1,0,-1)$	$N_{22}^{(10)}$	$(1/2304)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
23	$(1,1/5,-1)$	$N_{23}^{(10)}$	$(25/290304)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
24	$(1,2/5,-1)$	$N_{24}^{(10)}$	$(25/24192)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
25	$(1,3/5,-1)$	$N_{25}^{(10)}$	$-(25/64512)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
26	$(1,4/5,-1)$	$N_{26}^{(10)}$	$(25/290304)(1+\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
27	$(4/5,1,-1)$	$N_{27}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$

Table 7 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(10)}$	Shape function
28	$(3/5, 1, -1)$	$N_{28}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
29	$(2/5, 1, -1)$	$N_{29}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
30	$(1/5, 1, -1)$	$N_{30}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
31	$(0, 1, -1)$	$N_{31}^{(10)}$	$(1/2304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
32	$(-1/5, 1, -1)$	$N_{32}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
33	$(-2/5, 1, -1)$	$N_{33}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
34	$(-3/5, 1, -1)$	$N_{34}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
35	$(-4/5, 1, -1)$	$N_{35}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
36	$(-1, 4/5, -1)$	$N_{36}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
37	$(-1, 3/5, -1)$	$N_{37}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
38	$(-1, 2/5, -1)$	$N_{38}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
39	$(-1, 1/5, -1)$	$N_{39}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
40	$(-1, 0, -1)$	$N_{40}^{(10)}$	$(1/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
41	$(-1, -1/5, -1)$	$N_{41}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
42	$(-1, -2/5, -1)$	$N_{42}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
43	$(-1, -3/5, -1)$	$N_{43}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
44	$(-1, -4/5, -1)$	$N_{44}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
45	$(-1, -1, -4/5)$	$N_{45}^{(10)}$	$(25/290304)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-4)$
46	$(1, -1, -4/5)$	$N_{46}^{(10)}$	$(25/290304)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-4)$
47	$(1, 1, -4/5)$	$N_{47}^{(10)}$	$(25/290304)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-4)$
48	$(-1, 1, -4/5)$	$N_{48}^{(10)}$	$(25/290304)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-4)$
49	$(-1, -1, -3/5)$	$N_{49}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-3)$
50	$(1, -1, -3/5)$	$N_{50}^{(10)}$	$-(25/64512)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-3)$
51	$(1, 1, -3/5)$	$N_{51}^{(10)}$	$-(25/64512)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-3)$
52	$(-1, 1, -3/5)$	$N_{52}^{(10)}$	$-(25/64512)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta-3)$
53	$(-1, -1, -2/5)$	$N_{53}^{(10)}$	$(25/24192)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta-2)$
54	$(1, -1, -2/5)$	$N_{54}^{(10)}$	$(25/24192)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta-2)$
55	$(1, 1, -2/5)$	$N_{55}^{(10)}$	$(25/24192)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta-2)$
56	$(-1, 1, -2/5)$	$N_{56}^{(10)}$	$(25/24192)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta-2)$
57	$(-1, -1, -1/5)$	$N_{57}^{(10)}$	$-(25/13824)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta-1)$
58	$(1, -1, -1/5)$	$N_{58}^{(10)}$	$-(25/13824)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta-1)$
59	$(1, 1, -1/5)$	$N_{59}^{(10)}$	$-(25/13824)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta-1)$
60	$(-1, 1, -1/5)$	$N_{60}^{(10)}$	$-(25/13824)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta-1)$
61	$(-1, -1, 0)$	$N_{61}^{(10)}$	$(1/2304)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)$
62	$(1, -1, 0)$	$N_{62}^{(10)}$	$(1/2304)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)$
63	$(1, 1, 0)$	$N_{63}^{(10)}$	$(1/2304)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)$

(*continued overleaf*)

Table 7 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(10)}$	Shape function
64	$(-1, 1, 0)$	$N_{64}^{(10)}$	$(1/2304)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)$
65	$(-1, -1, 1/5)$	$N_{65}^{(10)}$	$-(25/13824)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta+1)$
66	$(1, -1, 1/5)$	$N_{66}^{(10)}$	$-(25/13824)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta+1)$
67	$(1, 1, 1/5)$	$N_{67}^{(10)}$	$-(25/13824)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta+1)$
68	$(-1, 1, 1/5)$	$N_{68}^{(10)}$	$-(25/13824)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-4)\zeta(5\zeta+1)$
69	$(-1, -1, 2/5)$	$N_{69}^{(10)}$	$(25/24192)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta+2)$
70	$(1, -1, 2/5)$	$N_{70}^{(10)}$	$(25/24192)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta+2)$
71	$(1, 1, 2/5)$	$N_{71}^{(10)}$	$(25/24192)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta+2)$
72	$(-1, 1, 2/5)$	$N_{72}^{(10)}$	$(25/24192)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-9)(25\zeta^2-1)\zeta(5\zeta+2)$
73	$(-1, -1, 3/5)$	$N_{73}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+3)$
74	$(1, -1, 3/5)$	$N_{74}^{(10)}$	$-(25/64512)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+3)$
75	$(1, 1, 3/5)$	$N_{75}^{(10)}$	$-(25/64512)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+3)$
76	$(-1, 1, 3/5)$	$N_{76}^{(10)}$	$-(25/64512)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-16)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+3)$
77	$(-1, -1, 4/5)$	$N_{77}^{(10)}$	$(25/290304)(1-\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+4)$
78	$(1, -1, 4/5)$	$N_{78}^{(10)}$	$(25/290304)(1+\xi)(1-\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+4)$
79	$(1, 1, 4/5)$	$N_{79}^{(10)}$	$(25/290304)(1+\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+4)$
80	$(-1, 1, 4/5)$	$N_{80}^{(10)}$	$(25/290304)(1-\xi)(1+\eta)(1-\zeta^2)(25\zeta^2-9)(25\zeta^2-4)(25\zeta^2-1)\zeta(5\zeta+4)$
81	$(4/5, -1, 1)$	$N_{81}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
82	$(3/5, -1, 1)$	$N_{82}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$
83	$(2/5, -1, 1)$	$N_{83}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
84	$(1/5, -1, 1)$	$N_{84}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
85	$(0, -1, 1)$	$N_{85}^{(10)}$	$(1/2304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
86	$(-1/5, -1, 1)$	$N_{86}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
87	$(-2/5, -1, 1)$	$N_{87}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
88	$(-3/5, -1, 1)$	$N_{88}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
89	$(-4/5, -1, 1)$	$N_{89}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1-\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
90	$(4/5, -1, 1)$	$N_{90}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
91	$(3/5, -1, 1)$	$N_{91}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
92	$(2/5, -1, 1)$	$N_{92}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
93	$(1/5, -1, 1)$	$N_{93}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
94	$(0, -1, 1)$	$N_{94}^{(10)}$	$(1/2304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
95	$(-1/5, -1, 1)$	$N_{95}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
96	$(-2/5, -1, 1)$	$N_{96}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
97	$(-3/5, -1, 1)$	$N_{97}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
98	$(-4/5, -1, 1)$	$N_{98}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$
99	$(4/5, 1, 1)$	$N_{99}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+4)$
100	$(3/5, 1, 1)$	$N_{100}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi+3)$

Table 7 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(10)}$	Shape function
101	$(2/5, 1, 1)$	$N_{101}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi+2)$
102	$(1/5, 1, 1)$	$N_{102}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi+1)$
103	$(0, 1, 1)$	$N_{103}^{(10)}$	$(1/2304)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)$
104	$(-1/5, 1, 1)$	$N_{104}^{(10)}$	$-(25/13824)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-4)\xi(5\xi-1)$
105	$(-2/5, 1, 1)$	$N_{105}^{(10)}$	$(25/24192)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-9)(25\xi^2-1)\xi(5\xi-2)$
106	$(-3/5, 1, 1)$	$N_{106}^{(10)}$	$-(25/64512)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-16)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-3)$
107	$(-4/5, 1, 1)$	$N_{107}^{(10)}$	$(25/290304)(1-\xi^2)(1+\eta)(1+\zeta)(25\xi^2-9)(25\xi^2-4)(25\xi^2-1)\xi(5\xi-4)$
108	$(4/5, 1, 1)$	$N_{108}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+4)$
109	$(3/5, 1, 1)$	$N_{109}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta+3)$
110	$(2/5, 1, 1)$	$N_{110}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta+2)$
111	$(1/5, 1, 1)$	$N_{111}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta+1)$
112	$(0, 1, 1)$	$N_{112}^{(10)}$	$(1/2304)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)$
113	$(-1/5, 1, 1)$	$N_{113}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-4)\eta(5\eta-1)$
114	$(-2/5, 1, 1)$	$N_{114}^{(10)}$	$(25/24192)(1-\xi)(1-\eta^2)(1-\zeta)(25\eta^2-16)(25\eta^2-9)(25\eta^2-1)\eta(5\eta-2)$
115	$(-3/5, 1, 1)$	$N_{115}^{(10)}$	$-(25/64512)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-16)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-3)$
116	$(-4/5, 1, 1)$	$N_{116}^{(10)}$	$(25/290304)(1-\xi)(1-\eta^2)(1+\zeta)(25\eta^2-9)(25\eta^2-4)(25\eta^2-1)\eta(5\eta-4)$

(ii) For $j = 12n-3, 12n-2, \ldots, 12n-4+r_0,\ i = 1, 2, \ldots, 12n-4$,

$$\begin{aligned}\hat{N}_i(\xi_j, \eta_j, \varsigma_j) &= N_i^{(n)}(\xi_j, \eta_j, \varsigma_j) - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right)\hat{N}_k^{(n)}(\xi_j, \eta_j, \varsigma_j) \\ &= N_i^{(n)}(\xi_j, \eta_j, \varsigma_j) - \sum_{k=12n-3}^{12n-4+r_0} N_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right)\delta_{kj} = 0.\end{aligned}$$

(iii) By definition and choice stated in the theorem,

$$\hat{N}_k^{(n)}(\xi_i, \eta_i, \varsigma_i) = 0, \quad k = 12n-3, 12n-2, \ldots, 12n-4+r_0,\ i = 1, 2, \ldots, 12n-4$$

from which we obtain

$$\hat{N}_i^{(n)}(\xi_l, \eta_l, \zeta_l) = \delta_{il}, \quad i, l = 1, 2, \ldots, 12n-4, 12n-3, \ldots, 12n-4+r_0.$$

4.2. Theorem

Theorem 2. *Let* $\hat{N}_i^{(n)}(\xi, \eta, \varsigma),\ i = 1, 2, \ldots, 12n-4+r_0$ *refer to zeroth orbit* (*nodes along edges and over the six faces of the* 3*-cube* $-1 \leqslant \xi, \eta, \zeta \leqslant 1$) *shape functions of* nth*-order general complete Lagrange rectangular prism element, and the shape functions of interior nodes of the* 3*-cube* $-1 \leqslant \xi, \eta, \varsigma \leqslant 1$ *denoted by*

$$\hat{\hat{N}}_k(\xi, \eta, \varsigma), \quad k = 12n-3+r_0, 12n-2+r_0, \ldots, 12n-4+i_n$$

(*where* i_n *refers to the total number of interior nodes contained in the* 3*-cube*) *satisfy the properties*

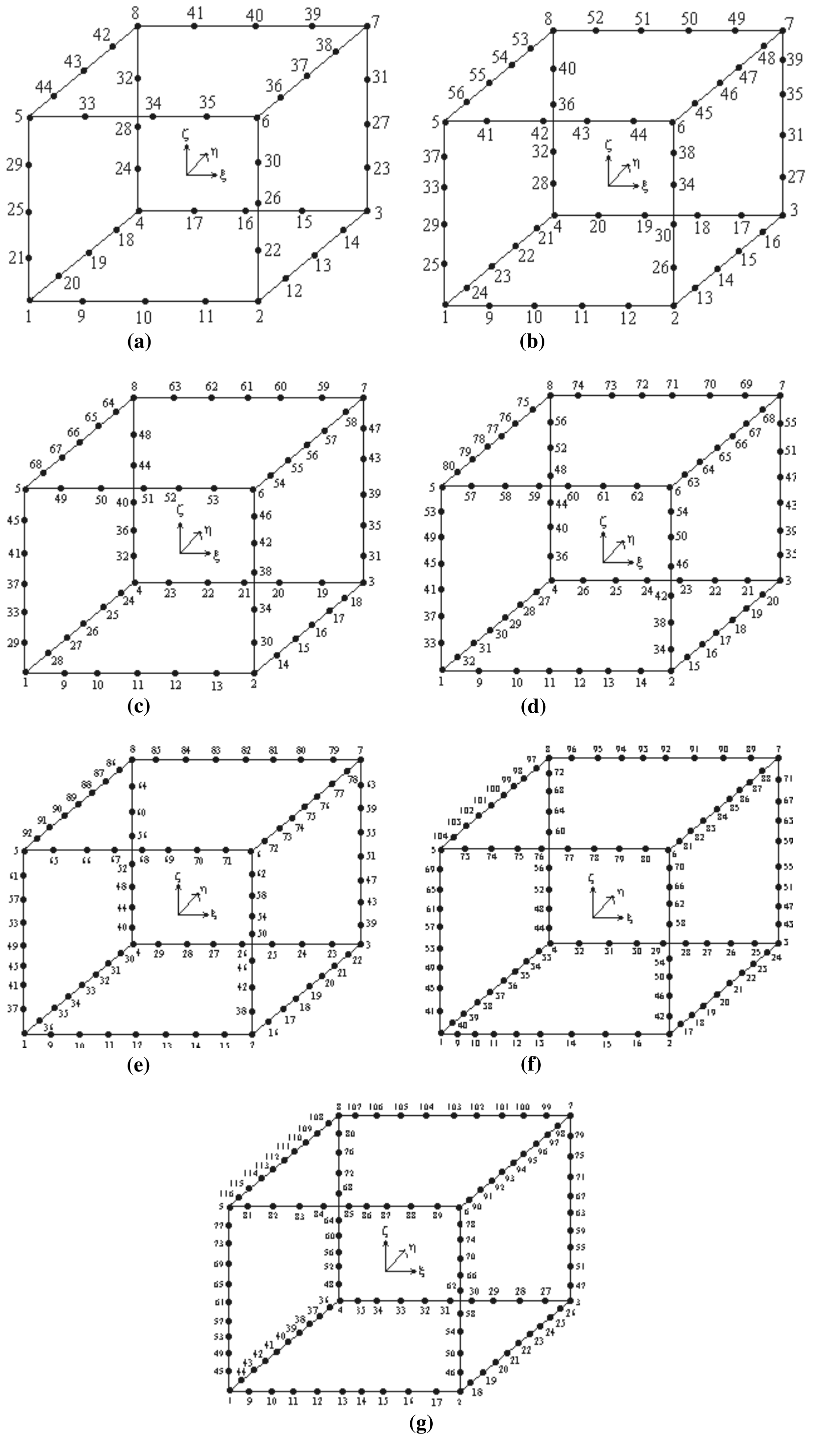
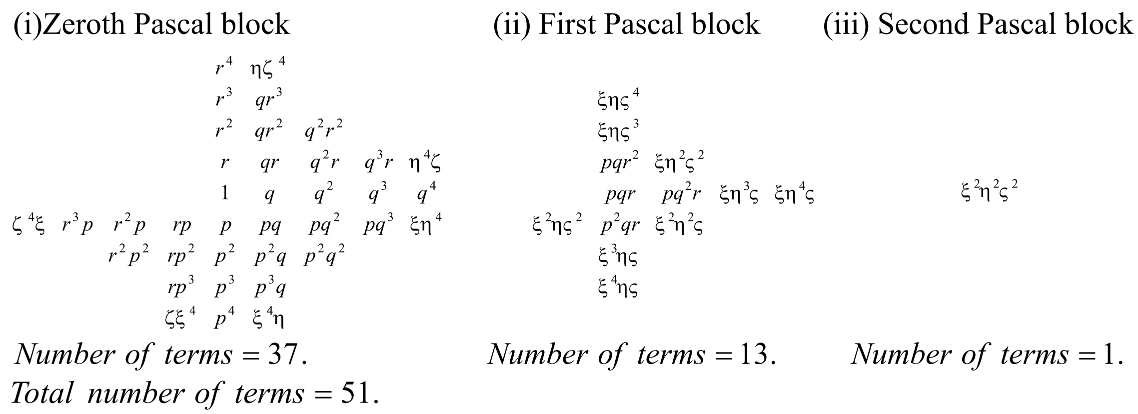
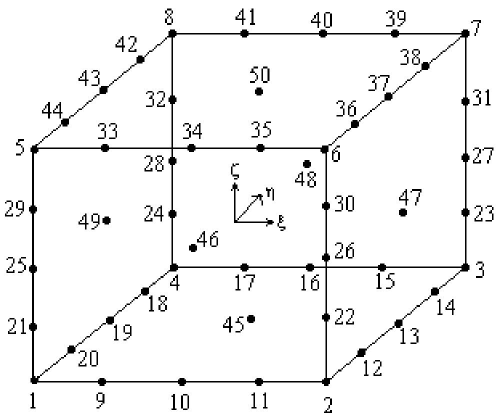


Fig. 5. General serendipity rectangular prism elements: (a) quartic (44 nodes); (b) quintic (56 nodes); (c) sextic (68 nodes); (d) septic (80 nodes); (e) octic (92 nodes); (f) ninth-order (104 nodes); (g) tenth-order (116 nodes).



(a)



(b)

Fig. 6. (a) Monomial basis for quartic complete Lagrange rectangular prism element. (b) Quartic complete Lagrange rectangular prism element (51 nodes)*. *Node number 51 (origin) not depicted in figure.

$$\hat{\hat{N}}_k^{(n)}(\xi_j, \eta_j, \varsigma_j) = \delta_{kj}, \quad k, j = 12n - 3 + r_0, 12n - 2 + r_0, \ldots, 12n - 4 + i_n,$$

$$\hat{\hat{N}}_k^{(n)}(\xi_l, \eta_l, \varsigma_l) = 0, \quad k = 12n - 3 + r_0, 12n - 2 + r_0, \ldots, 12n - 4 + i_n, \quad l = 1, 2, \ldots, 12n - 4 + r_0, \quad (25a)$$

$$\hat{\hat{N}}_i^{(n)}(\xi, \eta, \varsigma) = \hat{N}_i^{(n)}(\xi, \eta, \varsigma) - \sum_{k=12n-3+r_0}^{12n-4+i_n} \hat{N}_i^{(n)}\left(\xi_k^{(n)}, \eta_k^{(n)}, \varsigma_k^{(n)}\right) \hat{N}_k^{(n)}(\xi, \eta, \varsigma),$$

$$i = 1, 2, \ldots, 12n - 4 + r_0.$$

Then

$$\hat{\hat{N}}_i^{(n)}(\xi_l, \eta_l, \varsigma_l) = \delta_{il}, \quad i, l = 1, 2, \ldots, 12n - 4 + r_0, 12n - 3 + r_0, \ldots, 12n - 4 + i_n,$$

$$\sum_{i=1}^{12n-4+i_n} \hat{\hat{N}}_i^{(n)}(\xi, \eta, \varsigma) = 1. \quad (25b)$$

Proof. Similar to the proof outlined for Theorem 1.

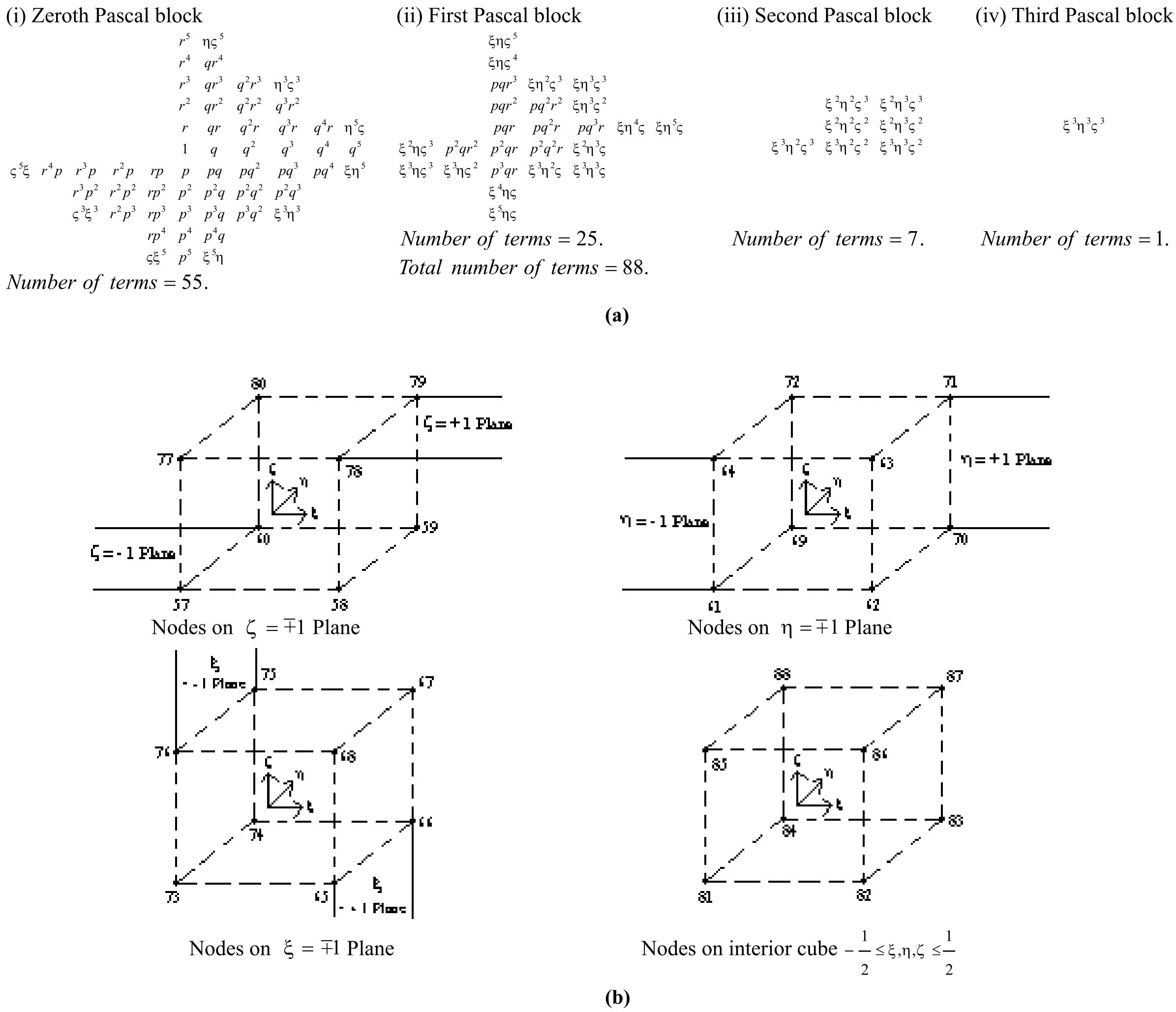


Fig. 7. (a) Monomial basis for quintic complete Lagrange rectangular prism element. (b) Quintic complete Lagrange rectangular prism element (88 nodes) (see Fig. 5(b) also).

Now, it is clear that the shape functions

$$\hat{\hat{N}}_i^{(n)}(\xi, \eta, \varsigma), \quad i = 1, 2, \ldots, 12n - 4 + r_0, 12n - 3 + r_0, \ldots, 12n - 4 + i_n$$

proposed in Eqs. (24a) and (24b) satisfied the fundamental properties necessary for their existence and uniqueness.

4.3. Node numbering scheme for higher order complete Lagrange elements

We have followed uniform node numbering scheme for elements of orders 4–10. On the six faces of the 3-cube $-1 \leqslant \xi, \eta, \varsigma \leqslant 1$, we have the interior nodes of respective 2-cube. Thus, the node numbering scheme is according to the usual order of two-dimensional elements. The node numbering on the faces of the 3-cube is as follows: (i) $\zeta = -1$ face; (ii) $\eta = -1$ face; (iii) $\xi = +1$ face; (iv) $\eta = +1$ face; (v) $\xi = -1$ face; (vi) $\zeta = +1$ face. Then follows the node numbering for interior network on the 3-cube $-1/2 \leqslant \xi, \eta, \zeta \leqslant 1/2$. If this is not sufficient, further we select the cube $-1/4 \leqslant \xi, \eta, \zeta \leqslant 1/4$. The node numbering scheme on each of these cubes is as mentioned above.

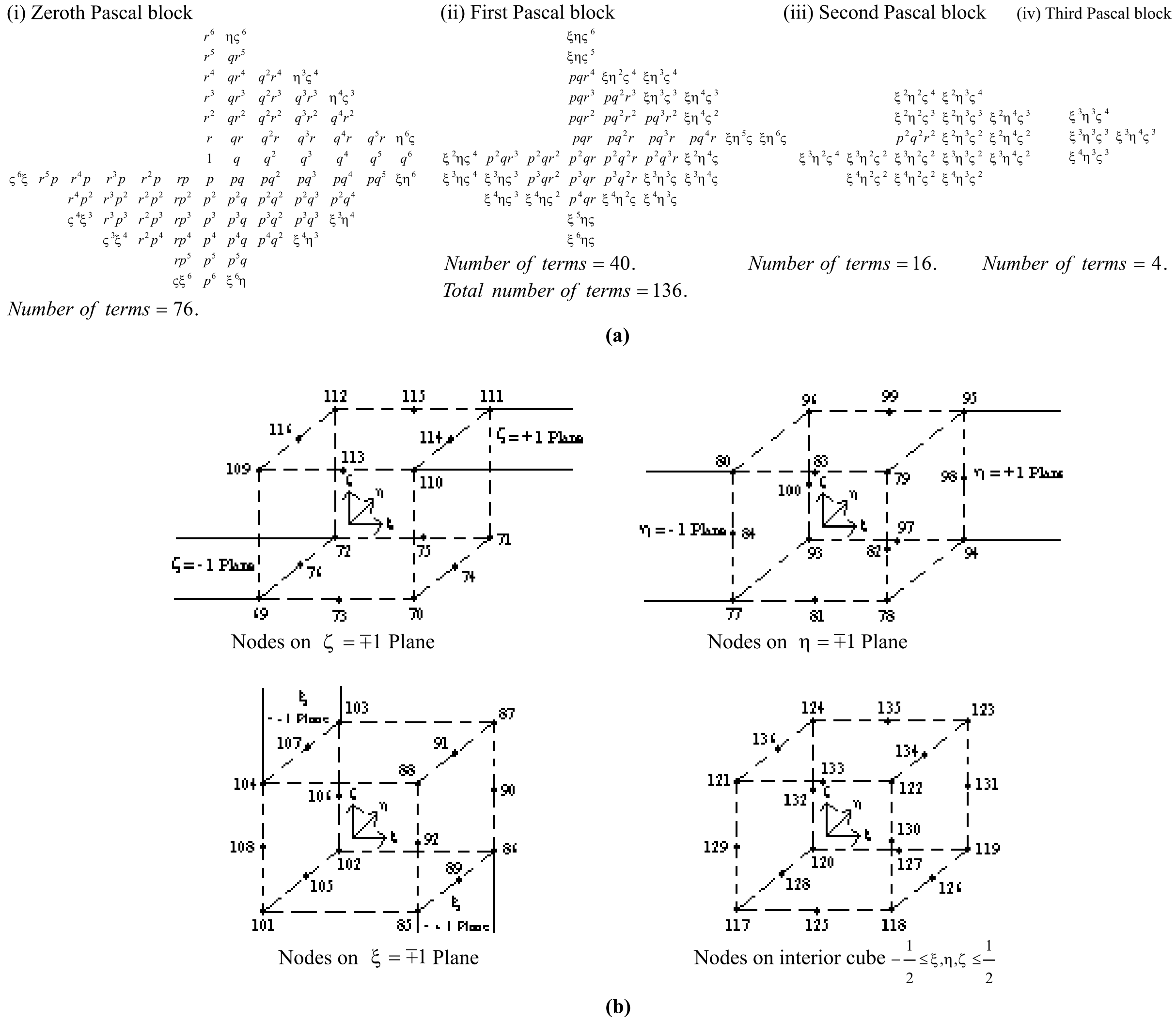


Fig. 8. (a) Monomial basis for sextic complete Lagrange rectangular prism element. (b) Sextic complete Lagrange rectangular prism element (136 nodes) (see Fig. 5(c) also).

4.4. Orbital modifiers for the six faces of the 3-cube $-1\leqslant\xi,\eta,\zeta\leqslant 1$

The shape functions $\widehat{N}_i(\xi,\eta,\zeta)$ on each of the faces should satisfy the property that $\hat{N}_i(\xi_k,\eta_k,\varsigma_k)=\delta_{ik}$, and this suggests the following orbital modifiers:

(i) Nodes on the faces $\varsigma=\mp 1$ but interior to the boundary of $-1\leqslant\xi,\eta\leqslant 1$ will have the orbital modifier $(1-\xi^2)(1-\eta^2)(1\mp\varsigma)$.

(ii) Nodes on the faces $\eta=\mp 1$ but interior to the boundary of $-1\leqslant\xi,\varsigma\leqslant 1$ will have the orbital modifier $(1-\xi^2)(1\mp\eta)(1-\varsigma^2)$.

(iii) Nodes on the faces $\xi=\mp 1$ but interior to the boundary of $-1\leqslant\eta,\varsigma\leqslant 1$ will have the orbital modifier $(1\mp\xi)(1-\eta^2)(1-\varsigma^2)$.

The shape functions $\hat{\hat{N}}_i(\xi,\eta,\zeta)$ interior to the 3-cube $-1\leqslant\xi,\eta\leqslant 1$ must be chosen so that $\hat{\hat{N}}_i(\xi_k,\eta_k,\varsigma_k)=\delta_{ik}$ which vanish on the edges $\xi=\pm 1$, $\eta=\pm 1$, $\varsigma=\pm 1$, and this suggests the orbital modifier $(1-\xi^2)(1-\eta^2)(1-\varsigma^2)$. We have illustrated the above procedure for quartic complete Lagrange rectangular prism element.

(i) Zeroth Pascal block

$$\begin{array}{ccccccccccccccc}
 & & & & & & & r^7 & \eta\varsigma^7 & & & & & & \\
 & & & & & & & r^6 & qr^6 & & & & & & \\
 & & & & & & & r^5 & qr^5 & q^2r^5 & \eta^3\varsigma^5 & & & & \\
 & & & & & & & r^4 & qr^4 & q^2r^4 & q^3r^4 & & & & \\
 & & & & & & & r^3 & qr^3 & q^2r^3 & q^3r^3 & q^4r^3 & \eta^5\varsigma^3 & & \\
 & & & & & & & r^2 & qr^2 & q^2r^2 & q^3r^2 & q^4r^2 & q^5r^2 & & \\
 & & & & & & & r & qr & q^2r & q^3r & q^4r & q^5r & q^6r & \eta^7\varsigma \\
 & & & & & & & 1 & q & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 \\
\varsigma^7\xi & r^6p & r^5p & r^4p & r^3p & r^2p & rp & p & pq & pq^2 & pq^3 & pq^4 & pq^5 & pq^6 & \xi\eta^7 \\
 & & r^5p^2 & r^4p^2 & r^3p^2 & r^2p^2 & rp^2 & p^2 & p^2q & p^2q^2 & p^2q^3 & p^2q^4 & p^2q^5 & & \\
 & & \varsigma^5\xi^3 & r^4p^3 & r^3p^3 & r^2p^3 & rp^3 & p^3 & p^3q & p^3q^2 & p^3q^3 & p^3q^4 & \xi^3\eta^5 & & \\
 & & & & r^3p^4 & r^2p^4 & rp^4 & p^4 & p^4q & p^4q^2 & p^4q^3 & & & & \\
 & & & & \varsigma^3\xi^5 & r^2p^5 & rp^5 & p^5 & p^5q & p^5q^2 & \xi^5\eta^3 & & & & \\
 & & & & & & rp^6 & p^6 & p^6q & & & & & & \\
 & & & & & & \varsigma\xi^7 & p^7 & \xi^7\eta & & & & & &
\end{array}$$

Number of terms = 97.

(ii) First Pascal block

$$\begin{array}{ccccccccccc}
 & & & & \xi\eta\varsigma^7 & & & & & & \\
 & & & & \xi\eta\varsigma^6 & & & & & & \\
 & & & & pqr^5 & \xi\eta^2\varsigma^5 & \xi\eta^3\varsigma^5 & & & & \\
 & & & & pqr^4 & pq^2r^4 & \xi\eta^3\varsigma^4 & & & & \\
 & & & & pqr^3 & pq^2r^3 & pq^3r^3 & \xi\eta^4\varsigma^3 & \xi\eta^5\varsigma^3 & & \\
 & & & & pqr^2 & pq^2r^2 & pq^3r^2 & pq^4r^2 & \xi\eta^5\varsigma^2 & & \\
 & & & & pqr & pq^2r & pq^3r & pq^4r & pq^5r & \xi\eta^6\varsigma & \xi\eta^7\varsigma \\
\xi^2\eta\varsigma^5 & p^2qr^4 & p^2qr^3 & p^2qr^2 & p^2qr & p^2q^2r & p^2q^3r & p^2q^4r & \xi^2\eta^5\varsigma & & \\
\xi^3\eta\varsigma^5 & \xi^3\eta\varsigma^4 & p^3qr^3 & p^3qr^2 & p^3qr & p^3q^2r & p^3q^3r & \xi^3\eta^4\varsigma & \xi^3\eta^5\varsigma & & \\
 & & \xi^4\eta\varsigma^3 & p^4qr^2 & p^4qr & p^4q^2r & \xi^4\eta^3\varsigma & & & & \\
 & & \xi^5\eta\varsigma^3 & \xi^5\eta\varsigma^2 & p^5qr & \xi^5\eta^2\varsigma & \xi^5\eta^3\varsigma & & & & \\
 & & & & \xi^6\eta\varsigma & & & & & & \\
 & & & & \xi^7\eta\varsigma & & & & & &
\end{array}$$

Number of terms = 55.

(iii) Second Pascal block

$$\begin{array}{ccccccc}
 & & & \xi^2\eta^2\varsigma^5 & \xi^2\eta^3\varsigma^5 & & \\
 & & & \xi^2\eta^2\varsigma^4 & \xi^2\eta^3\varsigma^4 & & \\
 & & & p^2q^2r^3 & \xi^2\eta^3\varsigma^3 & \xi^2\eta^4\varsigma^3 & \xi^2\eta^5\varsigma^3 \\
 & & & p^2q^2r^2 & p^2q^3r^2 & \xi^2\eta^4\varsigma^2 & \xi^2\eta^5\varsigma^2 \\
\xi^2\eta^3\varsigma^5 & \xi^2\eta^3\varsigma^4 & \xi^3\eta^3\varsigma^3 & p^3q^2r^2 & \xi^3\eta^3\varsigma^2 & \xi^3\eta^4\varsigma^2 & \xi^3\eta^5\varsigma^2 \\
 & & \xi^4\eta^3\varsigma^3 & \xi^4\eta^2\varsigma^2 & \xi^4\eta^3\varsigma^2 & & \\
 & & \xi^5\eta^3\varsigma^3 & \xi^5\eta^2\varsigma^2 & \xi^5\eta^3\varsigma^2 & &
\end{array}$$

Number of terms = 25.

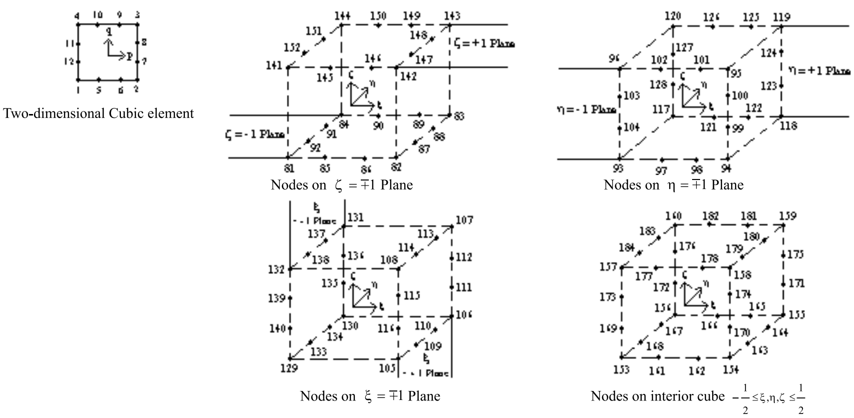
Total number of terms = 184.

(iv) Third Pascal block

$$\begin{array}{ccc}
\xi^3\eta^3\varsigma^5 & & \\
\xi^3\eta^3\varsigma^4 & & \\
\xi^3\eta^3\varsigma^3 & \xi^3\eta^4\varsigma^3 & \xi^3\eta^5\varsigma^3 \\
\xi^4\eta^3\varsigma^3 & & \\
\xi^5\eta^3\varsigma^3 & &
\end{array}$$

Number of terms = 7.

(a)



(b)

Fig. 9. Monomial basis for septic complete Lagrange rectangular prism element. Septic complete Lagrange rectangular prism element (184 nodes) (see Fig. 5(d) also).

(i) Zeroth Pascal block

$r^8 \quad \eta\varsigma^8$
$r^7 \quad qr^7$
$r^6 \quad qr^6 \quad q^2r^6 \quad \eta^3\varsigma^6$
$r^5 \quad qr^5 \quad q^2r^5 \quad q^3r^5$
$r^4 \quad qr^4 \quad q^2r^4 \quad q^3r^4 \quad q^4r^4$
$r^3 \quad qr^3 \quad q^2r^3 \quad q^3r^3 \quad q^4r^3 \quad q^5r^3 \quad \eta^6\varsigma^3$
$r^2 \quad qr^2 \quad q^2r^2 \quad q^3r^2 \quad q^4r^2 \quad q^5r^2 \quad q^6r^2$
$r \quad qr \quad q^2r \quad q^3r \quad q^4r \quad q^5r \quad q^6r \quad q^7r \quad \eta^8\varsigma$
$1 \quad q \quad q^2 \quad q^3 \quad q^4 \quad q^5 \quad q^6 \quad q^7 \quad q^8$
$\varsigma^8\xi \quad r^7p \quad r^6p \quad r^5p \quad r^4p \quad r^3p \quad r^2p \quad rp \quad p \quad pq \quad pq^2 \quad pq^3 \quad pq^4 \quad pq^5 \quad pq^6 \quad pq^7 \quad \xi\eta^8$
$r^6p^2 \quad r^5p^2 \quad r^4p^2 \quad r^3p^2 \quad r^2p^2 \quad rp^2 \quad p^2 \quad p^2q \quad p^2q^2 \quad p^2q^3 \quad p^2q^4 \quad p^2q^5 \quad p^2q^6$
$\varsigma^6\xi^3 \quad r^5p^3 \quad r^4p^3 \quad r^3p^3 \quad r^2p^3 \quad rp^3 \quad p^3 \quad p^3q \quad p^3q^2 \quad p^3q^3 \quad p^3q^4 \quad p^3q^5 \quad \xi^3\eta^6$
$r^4p^4 \quad r^3p^4 \quad r^2p^4 \quad rp^4 \quad p^4 \quad p^4q \quad p^4q^2 \quad p^4q^3 \quad p^4q^4$
$r^3p^5 \quad r^2p^5 \quad rp^5 \quad p^5 \quad p^5q \quad p^5q^2 \quad p^5q^3$
$\varsigma^3\xi^6 \quad r^2p^6 \quad rp^6 \quad p^6 \quad p^6q \quad p^6q^2 \quad \xi^6\eta^3$
$rp^7 \quad p^7 \quad p^7q$
$\varsigma\xi^8 \quad p^8 \quad \xi^8\eta$

Number of terms = 121.

(ii) First Pascal block

$\xi\eta\varsigma^8$
$\xi\eta\varsigma^7$
$pqr^6 \quad \xi\eta^2\varsigma^6 \quad \xi\eta^3\varsigma^6$
$pqr^5 \quad pq^2r^5 \quad \xi\eta^3\varsigma^5$
$pqr^4 \quad pq^2r^4 \quad pq^3r^4 \quad \xi\eta^4\varsigma^4$
$pqr^3 \quad pq^2r^3 \quad pq^3r^3 \quad pq^4r^3 \quad \xi\eta^5\varsigma^3 \quad \xi\eta^6\varsigma^3$
$pqr^2 \quad pq^2r^2 \quad pq^3r^2 \quad pq^4r^2 \quad pq^5r^2 \quad \xi\eta^6\varsigma^2$
$pqr \quad pq^2r \quad pq^3r \quad pq^4r \quad pq^5r \quad pq^6r \quad \xi\eta^7\varsigma \quad \xi\eta^8\varsigma$
$\xi^2\eta\varsigma^6 \quad p^2qr^5 \quad p^2qr^4 \quad p^2qr^3 \quad p^2qr^2 \quad p^2qr \quad p^2q^2r \quad p^2q^3r \quad p^2q^4r \quad p^2q^5r \quad \xi^2\eta^6\varsigma$
$\xi^3\eta\varsigma^6 \quad \xi^3\eta\varsigma^5 \quad p^3qr^4 \quad p^3qr^3 \quad p^3qr^2 \quad p^3qr \quad p^3q^2r \quad p^3q^3r \quad p^3q^4r \quad \xi^3\eta^5\varsigma \quad \xi^3\eta^6\varsigma$
$\xi^4\eta\varsigma^4 \quad p^4qr^3 \quad p^4qr^2 \quad p^4qr \quad p^4q^2r \quad p^4q^3r \quad \xi^4\eta^4\varsigma$
$p^5qr^3 \quad p^5qr^2 \quad p^5qr \quad p^5q^2r \quad \xi^5\eta^3\varsigma$
$\xi^6\eta\varsigma^3 \quad \xi^6\eta\varsigma^2 \quad p^6qr \quad \xi^6\eta^2\varsigma \quad \xi^6\eta^3\varsigma$
$\xi^7\eta\varsigma$
$\xi^8\eta\varsigma$

Number of terms = 73.

(iii) Second Pascal block

$\xi^2\eta^2\varsigma^6 \quad \xi^2\eta^3\varsigma^6$
$\xi^2\eta^2\varsigma^5 \quad \xi^2\eta^3\varsigma^5$
$p^2q^2r^4 \quad \xi^2\eta^3\varsigma^4 \quad \xi^2\eta^4\varsigma^4$
$p^2q^2r^3 \quad p^2q^3r^3 \quad \xi^2\eta^4\varsigma^3 \quad \xi^2\eta^5\varsigma^3 \quad \xi^2\eta^6\varsigma^3$
$p^2q^2r^2 \quad p^2q^3r^2 \quad p^2q^4r^2 \quad \xi^2\eta^5\varsigma^2 \quad \xi^2\eta^6\varsigma^2$
$\xi^3\eta^2\varsigma^6 \quad \xi^3\eta^2\varsigma^5 \quad \xi^3\eta^2\varsigma^4 \quad p^3q^2r^3 \quad p^3q^2r^2 \quad p^3q^3r^2 \quad \xi^3\eta^4\varsigma^2 \quad \xi^3\eta^5\varsigma^2 \quad \xi^3\eta^6\varsigma^2$
$\xi^4\eta^2\varsigma^4 \quad \xi^4\eta^2\varsigma^3 \quad p^4q^2r^2 \quad \xi^4\eta^3\varsigma^2 \quad \xi^4\eta^4\varsigma^2$
$\xi^5\eta^2\varsigma^3 \quad \xi^5\eta^2\varsigma^2 \quad \xi^5\eta^3\varsigma^2$
$\xi^6\eta^2\varsigma^3 \quad \xi^6\eta^2\varsigma^2 \quad \xi^6\eta^3\varsigma^2$

Number of terms = 37.

(iv) Third Pascal block

$\xi^3\eta^3\varsigma^6$
$\xi^3\eta^3\varsigma^5$
$\xi^3\eta^3\varsigma^4 \quad \xi^3\eta^4\varsigma^4$
$\xi^3\eta^3\varsigma^3 \quad \xi^3\eta^4\varsigma^3 \quad \xi^3\eta^5\varsigma^3 \quad \xi^3\eta^6\varsigma^3$
$\xi^4\eta^3\varsigma^4 \quad \xi^4\eta^3\varsigma^3 \quad \xi^4\eta^4\varsigma^3$
$\xi^5\eta^3\varsigma^3$
$\xi^6\eta^3\varsigma^3$

Number of terms = 13.

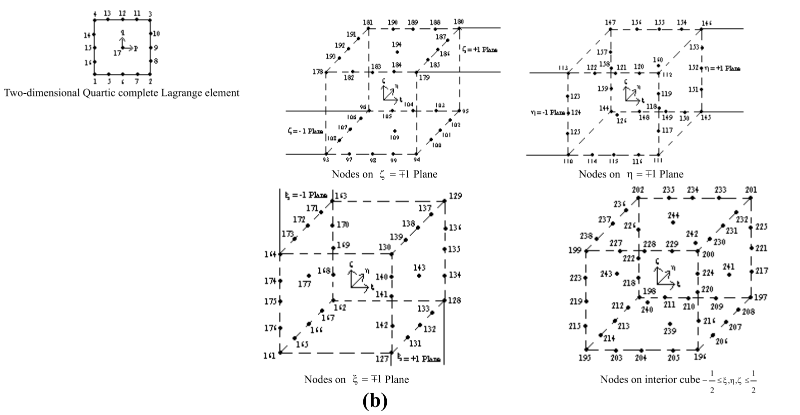
(v) Fourth Pascal hook

$\xi^4\eta^4\varsigma^4$

Number of terms = 1.

Total number of terms = 245.

(a)



(b)

Fig. 10. (a) Monomial basis for octic complete Lagrange rectangular prism element. (b) Octic complete Lagrange rectangular prism element (245 nodes) (see Fig. 5(e) also)*. *Node number 245 (Origin) not depicted in figure.

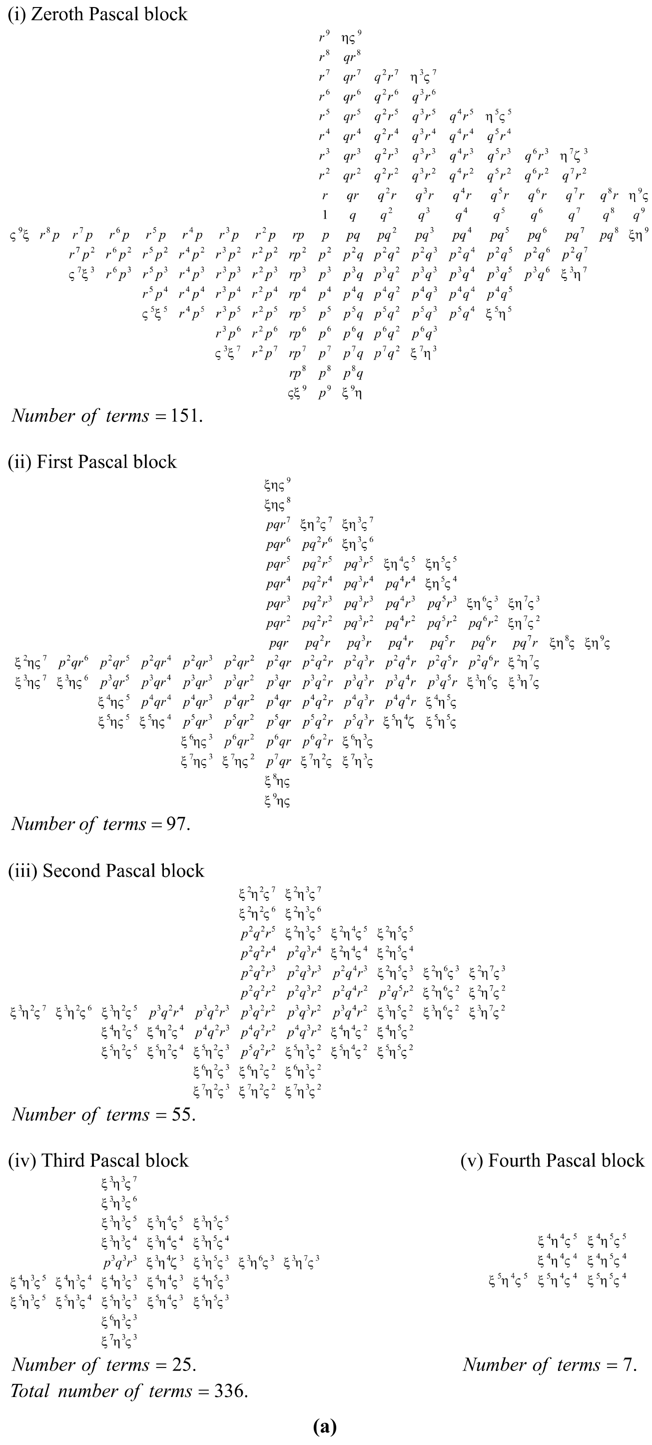
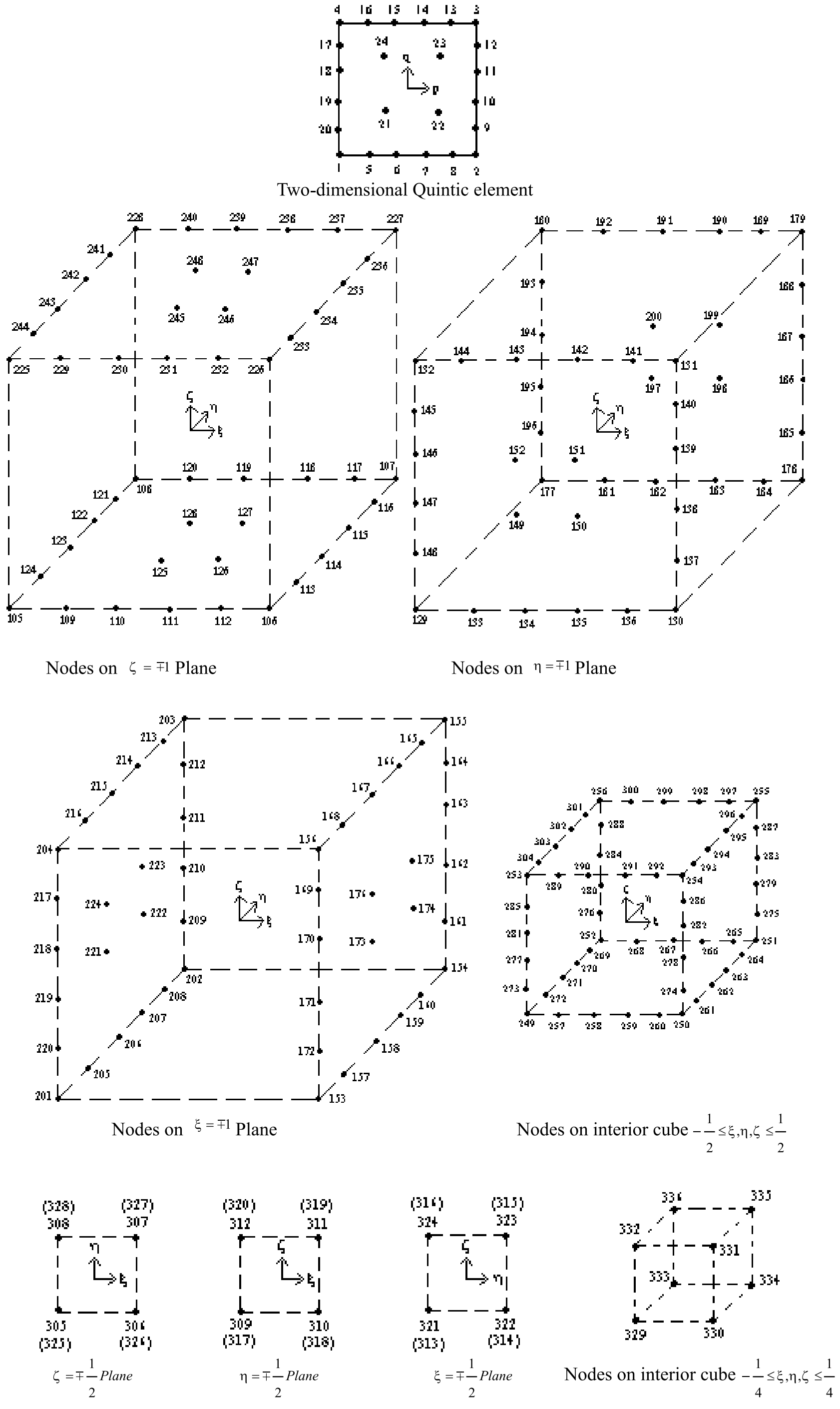


Fig. 11. (a) Monomial basis for ninth order complete Lagrange rectangular prism element. (b) Ninth-order complete Lagrange rectangular prism element (336 nodes) (see Fig. 5(f) also).

4.5. Quartic complete Lagrange rectangular prism element ($n = 4$, $i_4 = 7$, $r_0 = 6$)

See Fig. 6. Replacing p, q, r in Fig. 6(a) by ξ, η, ζ, respectively and using Eqs. (23a) and (23b), we can write



(b)

Fig. 11. (*continued*).

$$
\begin{aligned}
\hat{N}_i^{(4)}(\xi,\eta,\varsigma) &= N_i^{(n)}(\xi,\eta,\varsigma) - \sum_{k=45}^{50} \hat{N}_i^{(4)}\left(\xi_k^{(4)},\eta_k^{(4)},\varsigma_k^{(4)}\right)\hat{N}_k^{(4)}(\xi,\eta,\varsigma), \quad i = 1(1)44, \\
\hat{\hat{N}}_i^{(4)}(\xi,\eta,\varsigma) &= \hat{N}_i^{(4)}(\xi,\eta,\varsigma) - \hat{N}_i\left(\xi_{51}^{(4)},\eta_{51}^{(4)},\varsigma_{51}^{(4)}\right)\hat{\hat{N}}_{51}^{(4)}(\xi,\eta,\varsigma), \quad i = 1(1)50.
\end{aligned}
\tag{26}
$$

The interpolation functions are listed in Table 8.

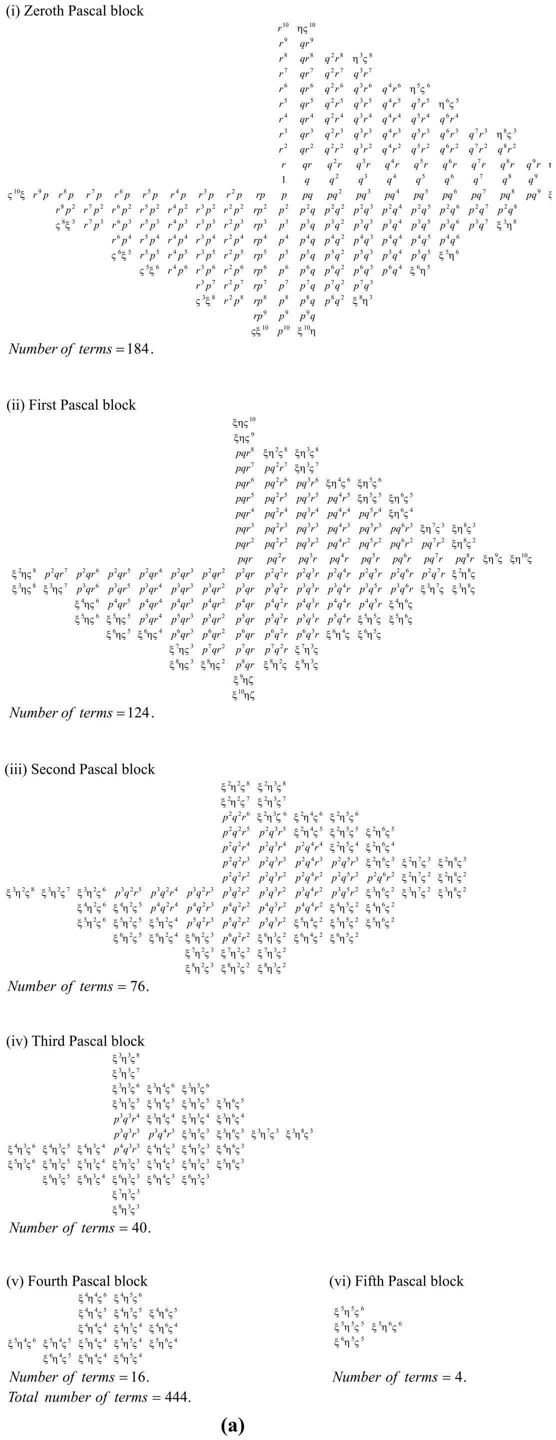
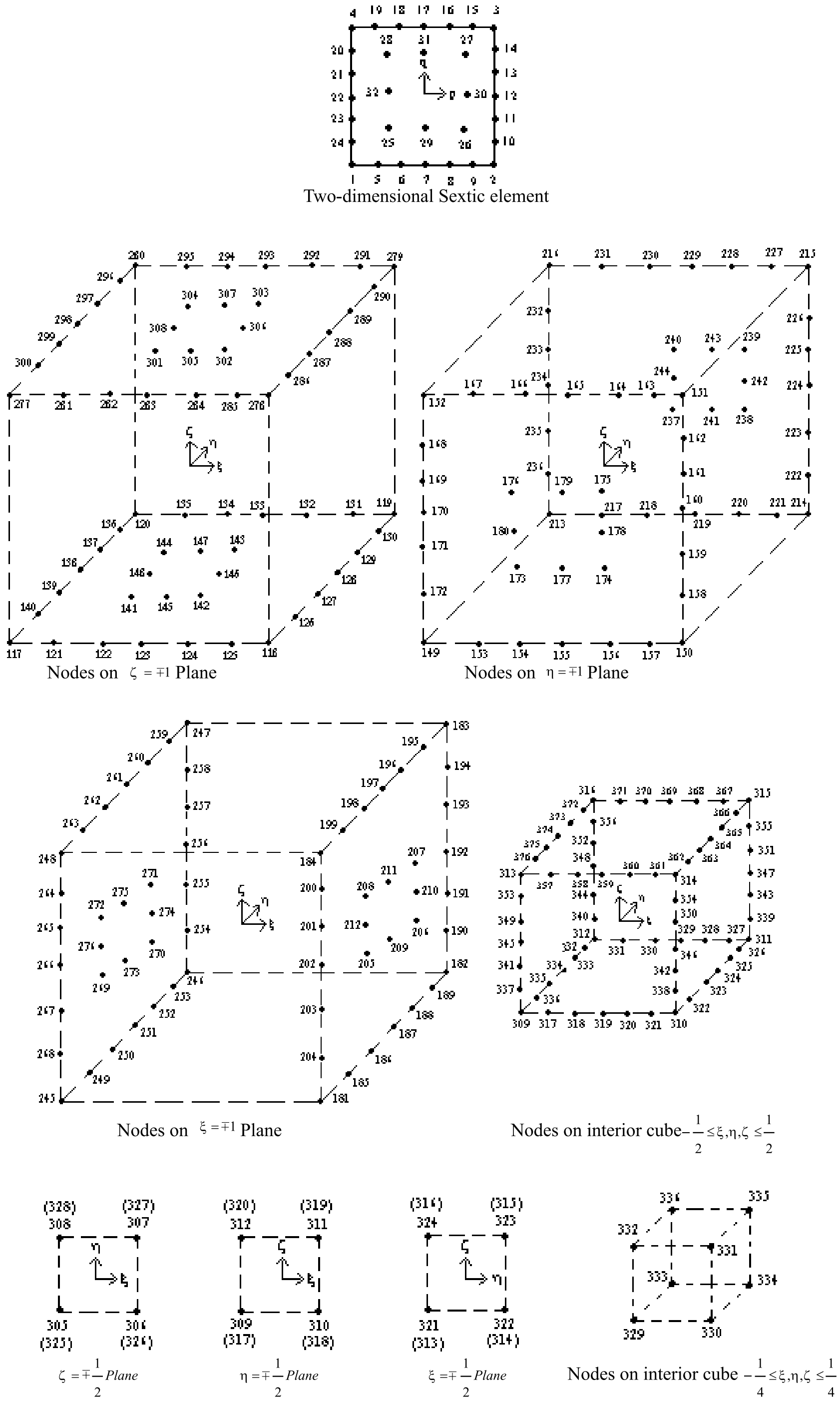


Fig. 12. (a) Monomial basis for tenth-order complete Lagrange rectangular prism element. (b) Tenth-order complete Lagrange rectangular prism element (444 nodes) (see Fig. 5(g) also).



(b)

Fig. 12. (*continued*).

Table 8
Quartic complete Lagrange element shape functions

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(4)}$	Shape function
1	$(-1,-1,-1)$	$N_1^{(4)}$	$(1/24)(1-\xi)(1-\eta)(1-\zeta)\{-4(\xi^3+\eta^3+\zeta^3)+4(\xi+\eta+\zeta)-3\xi\eta\zeta\}$
2	$(1,-1,-1)$	$N_2^{(4)}$	$(1/24)(1+\xi)(1-\eta)(1-\zeta)\{4(\xi^3-\eta^3-\zeta^3)-4(\xi-\eta-\zeta)+3\xi\eta\zeta\}$
3	$(1,1,-1)$	$N_3^{(4)}$	$(1/24)(1+\xi)(1+\eta)(1-\zeta)\{4(\xi^3+\eta^3-\zeta^3)-4(\xi+\eta-\zeta)-3\xi\eta\zeta\}$
4	$(-1,1,-1)$	$N_4^{(4)}$	$(1/24)(1-\xi)(1+\eta)(1-\zeta)\{-4(\xi^3-\eta^3+\zeta^3)+4(\xi-\eta+\zeta)+3\xi\eta\zeta\}$
5	$(-1,-1,1)$	$N_5^{(4)}$	$(1/24)(1-\xi)(1-\eta)(1+\zeta)\{-4(\xi^3+\eta^3-\zeta^3)+4(\xi+\eta-\zeta)+3\xi\eta\zeta\}$
6	$(1,-1,1)$	$N_6^{(4)}$	$(1/24)(1+\xi)(1-\eta)(1+\zeta)\{4(\xi^3-\eta^3+\zeta^3)-4(\xi-\eta+\zeta)-3\xi\eta\zeta\}$
7	$(1,1,1)$	$N_7^{(4)}$	$(1/24)(1+\xi)(1+\eta)(1+\zeta)\{4(\xi^3+\eta^3+\zeta^3)-4(\xi+\eta+\zeta)+3\xi\eta\zeta\}$
8	$(-1,1,1)$	$N_8^{(4)}$	$(1/24)(1-\xi)(1+\eta)(1+\zeta)\{-4(\xi^3-\eta^3-\zeta^3)+4(\xi-\eta-\zeta)-3\xi\eta\zeta\}$
9	$(-1/2,-1,-1)$	$N_9^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1-\zeta)\xi(2\xi-1)$
10	$(0,-1,-1)$	$N_{10}^{(4)}$	$-(1/4)(1-\xi^2)(1-\eta)(1-\zeta)(4\xi^2+\eta+\zeta+\eta\zeta)$
11	$(1/2,-1,-1)$	$N_{11}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1-\zeta)\xi(2\xi+1)$
12	$(1,-1/2,-1)$	$N_{12}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1-\zeta)\eta(2\eta-1)$
13	$(1,0,-1)$	$N_{13}^{(4)}$	$-(1/4)(1+\xi)(1-\eta^2)(1-\zeta)(4\eta^2+\zeta-\xi-\zeta\xi)$
14	$(1,1/2,-1)$	$N_{14}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1-\zeta)\eta(2\eta+1)$
15	$(1/2,1,-1)$	$N_{15}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1-\zeta)\xi(2\xi+1)$
16	$(0,1,-1)$	$N_{16}^{(4)}$	$-(1/4)(1-\xi^2)(1+\eta)(1-\zeta)(4\xi^2-\eta+\zeta-\eta\zeta)$
17	$(-1/2,1,-1)$	$N_{17}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1-\zeta)\xi(2\xi-1)$
18	$(-1,1/2,-1)$	$N_{18}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1-\zeta)\eta(2\eta+1)$
19	$(-1,0,-1)$	$N_{19}^{(4)}$	$-(1/4)(1-\xi)(1-\eta^2)(1-\zeta)(4\eta^2+\zeta+\xi+\zeta\xi)$
20	$(-1,-1/2,-1)$	$N_{20}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1-\zeta)\eta(2\eta-1)$
21	$(-1,-1,-1/2)$	$N_{21}^{(4)}$	$(1/3)(1-\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta-1)$
22	$(1,-1,-1/2)$	$N_{22}^{(4)}$	$(1/3)(1+\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta-1)$
23	$(1,1,-1/2)$	$N_{23}^{(4)}$	$(1/3)(1+\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta-1)$
24	$(-1,1,-1/2)$	$N_{24}^{(4)}$	$(1/3.)(1-\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta-1)$
25	$(-1,-1,0)$	$N_{25}^{(4)}$	$-(1/4)(1-\xi)(1-\eta)(1-\zeta^2)(4\zeta^2+\xi+\eta+\xi\eta)$
26	$(1,-1,0)$	$N_{26}^{(4)}$	$-(1/4)(1+\xi)(1-\eta)(1-\zeta^2)(4\zeta^2-\xi+\eta-\xi\eta)$
27	$(1,1,0)$	$N_{27}^{(4)}$	$-(1/4)(1+\xi)(1+\eta)(1-\zeta^2)(4\zeta^2-\xi-\eta+\xi\eta)$
28	$(-1,1,0)$	$N_{28}^{(4)}$	$-(1/4)(1-\xi)(1+\eta)(1-\zeta^2)(4\zeta^2+\xi-\eta-\xi\eta)$
29	$(-1,-1,1/2)$	$N_{29}^{(4)}$	$(1/3)(1-\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta+1)$
30	$(1,-1,1/2)$	$N_{30}^{(4)}$	$(1/3)(1+\xi)(1-\eta)(1-\zeta^2)\zeta(2\zeta+1)$
31	$(1,1,1/2)$	$N_{31}^{(4)}$	$(1/3)(1+\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta+1)$
32	$(-1,1,1/2)$	$N_{32}^{(4)}$	$(1/3)(1-\xi)(1+\eta)(1-\zeta^2)\zeta(2\zeta+1)$
33	$(1/2,-1,1)$	$N_{33}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1+\zeta)\xi(2\xi-1)$
34	$(0,-1,1)$	$N_{34}^{(4)}$	$-(1/4)(1-\xi^2)(1-\eta)(1+\zeta)(4\xi^2+\eta-\zeta-\eta\zeta)$
35	$(-1/2,-1,1)$	$N_{35}^{(4)}$	$(1/3)(1-\xi^2)(1-\eta)(1+\zeta)\xi(2\xi+1)$
36	$(1,-1/2,1)$	$N_{36}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1+\zeta)\eta(2\eta-1)$
37	$(1,0,1)$	$N_{37}^{(4)}$	$-(1/4)(1+\xi)(1-\eta^2)(1+\zeta)(4\eta^2-\zeta-\xi+\zeta\xi)$

Table 8 (*continued*)

Node i	Coordinates(ξ_i, η_i, ζ_i)	$N_i^{(4)}$	Shape function
38	$(1, 1/2, 1)$	$N_{38}^{(4)}$	$(1/3)(1+\xi)(1-\eta^2)(1+\zeta)\eta(2\eta+1)$
39	$(1/2, 1, 1)$	$N_{39}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1+\zeta)\xi(2\xi+1)$
40	$(0, 1, 1)$	$N_{40}^{(4)}$	$-(1/4)(1-\xi^2)(1+\eta)(1+\zeta)(4\xi^2-\eta-\zeta+\eta\zeta)$
41	$(-1/2, 1, 1)$	$N_{41}^{(4)}$	$(1/3)(1-\xi^2)(1+\eta)(1+\zeta)\xi(2\xi-1)$
42	$(-1, 1/2, 1)$	$N_{42}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1+\zeta)\eta(2\eta+1)$
43	$(-1, 0, 1)$	$N_{43}^{(4)}$	$-(1/4)(1-\xi)(1-\eta^2)(1+\zeta)(4\eta^2-\zeta+\xi-\zeta\xi)$
44	$(-1, 1/2, 1)$	$N_{44}^{(4)}$	$(1/3)(1-\xi)(1-\eta^2)(1+\zeta)\eta(2\eta-1)$
45	$(0, 0, -1)$	$N_{45}^{(4)}$	$(1/2)(1-\xi^2)(1-\eta^2)(1-\zeta)$
46	$(0, -1, 0)$	$N_{46}^{(4)}$	$(1/2)(1-\xi^2)(1+\eta)(1-\zeta^2)$
47	$(1, 0, 0)$	$N_{47}^{(4)}$	$(1/2)(1+\xi)(1-\eta^2)(1-\zeta^2)$
48	$(0, 1, 0)$	$N_{48}^{(4)}$	$(1/2)(1-\xi^2)(1-\eta)(1-\zeta^2)$
49	$(-1, 0, 0)$	$N_{49}^{(4)}$	$(1/2)(1-\xi)(1-\eta^2)(1-\zeta^2)$
50	$(0, 0, 1)$	$N_{50}^{(4)}$	$(1/2)(1-\xi^2)(1-\eta^2)(1+\zeta)$
51	$(0, 0, 0)$	$N_{51}^{(4)}$	$(1-\xi^2)(1-\eta^2)(1-\zeta^2)$

In Figs. 7(a)–12(a), replace p, q, r by ξ, η, ζ, respectively, to get monomial bases and in Figs. 7(b)–12(b) nodes on edges of the cube $-1 \leqslant \xi, \eta, \zeta \leqslant 1$ are given in Fig. 5(a)–(g).

4.6. *Quintic complete Lagrange rectangular prism element ($n = 5$, $i_5 = 32$, $r_0 = 24$)*

See Fig. 7.

4.7. *Sextic complete Lagrange rectangular prism element ($n = 6$, $i_6 = 68$, $r_0 = 48$)*

See Fig. 8.

4.8. *Septic complete Lagrange rectangular prism element ($n = 7$, $i_7 = 104$, $r_0 = 72$)*

See Fig. 9.

4.9. *Octic complete Lagrange rectangular prism element ($n = 8$, $i_8 = 153$, $r_0 = 102$)*

See Fig. 10.

4.10. *Ninth-order complete Lagrange rectangular prism element ($n = 9$, $i_9 = 232$, $r_0 = 144$)*

See Fig. 11.

4.11. *Tenth-order complete Lagrange rectangular prism element ($n = 10$, $i_{10} = 328$, $r_0 = 192$)*

See Fig. 12.

5. Modified interpolation functions for complete Lagrange elements

The shape functions of complete Lagrange element can be modified to correctly interpolate pth-order displacement states under the same conditions as a pth-order regular Lagrange element. We assume for the present that these modifications are applicable to arbitrary linear hexahedrons which have arbitrary straight-sided planar quadrilateral facets in the global (x, y, z) space. The hexahedron can be mapped to a 3-cube in the local space by the standard trilinear shape functions in the natural coordinates (ξ, η, ζ). This is achieved by the application of a suitably designed constraint which constrains the displacement at the nodes of a complete Lagrange element. Any component of the element's displacement field can then be expressed interms of the shape functions of a pth-order regular Lagrange element. The equation of the constraint is

$$u = \sum_{j=1}^{(p+1)^3} M_j^{(p)}(\xi, \eta, \zeta) u_j, \tag{27}$$

$$u_k = \sum_{i=1}^{c_p} T_i^{k,p} u_i, \quad k = c_p + 1, c_p + 2, \ldots, (p+1)^3, \ \ p = 2, 3, \ldots, 10, \tag{28}$$

where $M_j^{(p)}(\xi, \eta, \zeta)$ is the pth-order regular Lagrange element shape function, and $T_i^{k,p}$ is a coefficient of constraint for kth interior node of regular Lagrange element of order p, and c_p is the number of nodes of pth-order complete Lagrange element. The element is, practically speaking, a pth-order complete Lagrange element with modified shape functions. Thus

$$u = \sum_{i=1}^{c_p} \overline{M}_i^{(p)}(\xi, \eta, \zeta) u_i, \tag{29}$$

where

$$\overline{M}_i^{(p)}(\xi, \eta, \zeta) = M_j^{(p)}(\xi, \eta, \zeta) + \sum_{k=c_p+1}^{(p+1)^3} M_k^{(p)}(\xi, \eta, \zeta) T_i^{k,p}. \tag{30}$$

This result can also be expressed in terms of standard complete Lagrange element shape functions $\hat{N}_i^{(p)}(\xi, \eta, \zeta)$.

To accomplish this we put Eq. (29) in hierarchical form

$$u = \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi, \eta, \zeta) u_i + \sum_{k=c_p+1}^{(p+1)^3} M_k^{(p)}(\xi, \eta, \zeta) \Delta u_k, \tag{31}$$

assuming that the node k is located at the interior point (ξ_k, η_k, ζ_k) of the 3-cube $-1 \leqslant \xi, \eta, \zeta \leqslant 1$, so that we obtain

$$u_k = u(\xi_k, \eta_k, \zeta_k) = \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) u_i + \Delta u_k, \quad k = c_p + 1, c_p + 2, \ldots, (p+1)^3. \tag{32}$$

Since the shape functions $M_i^{(p)}(\xi, \eta, \zeta)$ satisfy the property

$$M_i^{(p)}(\xi_k, \eta_k, \zeta_k) = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$$

we have from Eq. (32),

$$\Delta u_k = u_k - \sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) u_i. \tag{33}$$

From Eqs. (33) and (28), we have

$$\begin{aligned}(\Delta u_k) M_k^{(p)} &= u_k M_k^{(p)} - M_k^{(p)} \left[\sum_{i=1}^{c_p} \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) u_i \right] \\ &= M_k^{(p)} \left[\sum_{i=1}^{c_p} T_i^{k,p} u_i \right] - M_k^{(p)} \left[\sum_{i=1}^{c_p} \hat{N}_k^{(p)}(\xi_k, \eta_k, \zeta_k) u_i \right] \\ &= M_k^{(p)} \sum_{i=1}^{c_p} \left[T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) \right] u_i. \end{aligned} \tag{34}$$

Substituting Eq. (34) in Eq. (31), we obtain

$$u = \sum_{i=1}^{c_p} \left[\hat{N}_i^{(p)}(\xi, \eta, \zeta) + \sum_{k=c_p+1}^{(p+1)^3} M_k^{(p)} \left\{ T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) \right\} \right] u_i. \tag{35}$$

Thus, from Eqs. (29) and (35), we get

$$\overline{M}_i^{(p)}(\xi, \eta, \zeta) = \hat{N}_i^{(p)}(\xi, \eta, \zeta) + \sum_{k=c_p+1}^{(p+1)^3} M_k^{(p)}(\xi, \eta, \zeta) \left[T_i^{k,p} - \hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k) \right]. \tag{36}$$

We see that the shape functions are revised from those of a complete Lagrange element of order p to the extent that $T_i^{k,p}$ differ from $\hat{N}_i^{(p)}(\xi_k, \eta_k, \zeta_k)$. The trick is to design $T_i^{k,p}$ so that it gives the correct value to u_k, $k = c_p + 1, c_p + 2, \ldots, (p+1)^3$ at the interior nodes for the pth-order displacement field. If that is done, then the pth-order ($p \geqslant 2$) regular Lagrange element shape functions in Eq. (27) and consequently the shape functions in Eq. (29) or (35) will correctly interpolate a pth-order displacement field for trilinear element shapes.

The constraint co-efficients $T_i^{k,p}$, $i = 1, 2, \ldots, c_p$ are constructed with the aid of a special interpolation formula

$$u = \sum_{i=1}^{c_p} N_i^{*(p)} u_i = \left[N_i^{*(p)} \right] \{u_i\}, \tag{37}$$

where

$$\lfloor N_i^{*(p)} \rfloor = \lfloor X_m^{(p)} \rfloor \lfloor A_{mi}^{(p)} \rfloor, \quad i = 2(1)10. \tag{38}$$

The selection of elements of basis vector $\lfloor X_m^{(k)} \rfloor$, $k = 2(1)10$, is as follows: Let $\lfloor X_{\mathrm{C}}^{(k)} \rfloor$ refer to the elements of basis vector for general complete Lagrange rectangular prism element of order k in natural coordinates ξ, η, ζ, $\lfloor X_{\mathrm{G}}^{(k)} \rfloor$ refer to those elements of $\lfloor X_{\mathrm{C}}^{(k)} \rfloor$ for which $\alpha + \beta + \gamma \leqslant k$, but the monomials $\xi^\alpha \eta^\beta \zeta^\gamma$ are now replaced by the global monomials $x^\alpha y^\beta z^\gamma$, and $\lfloor X_{\mathrm{L}}^{(k)} \rfloor$ refer to those elements of $\lfloor X_{\mathrm{C}}^{(k)} \rfloor$ which contain the local monomials $\xi^\alpha \eta^\beta \zeta^\gamma$ for which $\alpha + \beta + \gamma > k$. Clearly, then

$$\lfloor X_m^{(k)} \rfloor = \lfloor X_{\mathrm{G}}^{(k)} \rfloor \oplus \lfloor X_{\mathrm{L}}^{(k)} \rfloor. \tag{39}$$

(i) Zeroth Pascal block (Quadratic) clustered around $p^0q^0r^0$

$r^2 \quad \eta\zeta^2$
$r \quad qr \quad \eta^2\zeta$
$1 \quad q \quad q^2$
$\zeta^2\xi \quad rp \quad p \quad pq \quad \xi\eta^2$
$\zeta\xi^2 \quad p^2 \quad \xi^2\eta$

Number of terms = Local terms (6) + Global terms (9) = 16.

(ii) First Pascal block (Quadratic) clustered around $\xi\eta\zeta$

$\xi\eta\zeta^2$
$\xi\eta\zeta \quad \xi\eta^2\zeta$
$\xi^2\eta\zeta$

Number of terms = Local terms (4) + Global terms (0) = 4.

Total number of terms=16+4=20.

Fig. 13. Monomials of global variables (p, q, r) and local variables (ξ, η, ζ) for basis vector $[X_m^{(2)}]$.

(i)Zeroth Pascal block (Cubic) clustered around $p^0q^0r^0$

$r^3 \quad \eta\zeta^3$
$r^2 \quad qr^2$
$r \quad qr \quad q^2r \quad \eta^3\zeta$
$1 \quad q \quad q^2 \quad q^3$
$\zeta^3\xi \quad r^2p \quad rp \quad p \quad pq \quad pq^2 \quad \xi\eta^3$
$r^2p \quad p^2 \quad p^2q$
$\zeta\xi^3 \quad p^3 \quad \xi^3\eta$

Number of terms = Local terms (6) + Global terms (19) = 25.

(ii) First Pascal block (Cubic) clustered around $\xi\eta\zeta$

$\xi\eta\zeta^3$
$\xi\eta\zeta^2$
$pqr \quad \xi\eta^2\zeta \quad \xi\eta^2\zeta$
$\xi^2\eta\zeta$
$\xi^3\eta\zeta$

Number of terms = Local terms (6) + Global terms (1) = 7.

Total number of terms = 25+7 = 32.

Fig. 14. Monomials of global variables (p, q, r) and local variables (ξ, η, ζ) for basis vector $[X_m^{(3)}]$.

For example, in Fig. 13, we replace p, q, r by ξ, η, ζ, respectively, to get the elements of basis vector for quadratic complete Lagrange rectangular prism element as

$$\lfloor X_C^{(2)} \rfloor = \lfloor 1, \xi, \eta, \zeta, \xi\eta, \eta\zeta, \zeta\xi, \xi^2, \eta^2, \zeta^2, \xi^2\eta, \xi\eta^2, \eta^2\zeta, \eta\zeta^2, \zeta^2\xi, \zeta\xi^2, \xi\eta\zeta, \xi^2\eta\zeta, \xi\eta^2\zeta, \xi\eta\zeta^2 \rfloor \tag{40}$$

and we also have

$$\lfloor X_G^{(2)} \rfloor = \lfloor 1, x, y, z, xy, yz, zx, x^2, y^2, z^2 \rfloor, \tag{41a}$$

$$\lfloor X_L^{(2)} \rfloor = \lfloor \xi^2\eta, \xi\eta^2, \eta^2\zeta, \eta\zeta^2, \zeta^2\xi, \zeta\xi^2, \xi\eta\zeta, \xi^2\eta\zeta, \xi\eta^2\zeta, \xi\eta\zeta^2 \rfloor. \tag{41b}$$

From Eq. (39),

$$\lfloor X_m^{(2)} \rfloor = \lfloor 1, x, y, z, xy, yz, zx, x^2, y^2, z^2, \xi^2\eta, \xi\eta^2, \eta^2\zeta, \eta\zeta^2, \zeta^2\xi, \zeta\xi^2, \xi\eta\zeta, \xi^2\eta\zeta, \xi\eta^2\zeta, \xi\eta\zeta^2 \rfloor. \tag{42}$$

The basis vectors $\lfloor X_C^{(k)} \rfloor$, and $\lfloor X_m^{(k)} \rfloor$, $k = 3(1)10$ can be similarly obtained from Figs. 6(a)–12(a) (see Fig. 14). The elements of $\lfloor A_{mi}^{(p)} \rfloor$ are just constant co-efficients to be determined. The trivariate monomial terms of degree p in x, y, z specified at the exterior and interior nodes of complete Lagrange linear solid hexahedron element correctly interpolate any trivariate complete polynomial function. This would not be possible if these global (metric) terms were replaced by corresponding $\xi, \eta,$ and ζ. Some monomial terms like $\xi^\alpha\eta^\beta\zeta^\gamma$, $\alpha + \beta + \gamma = p + 1$ are somewhat arbitrary and are required to ensure the existence of $\lfloor A_{mi}^{(p)} \rfloor$. To find $\lfloor A_{mi}^{(p)} \rfloor$, note that Eqs. (37) and (38) give the same values of u at nodes

$$u_i = \lfloor X_{im}^{(p)} \rfloor \lfloor A_{mi}^{(p)} \rfloor \{u_i\}, \tag{43}$$

where $\lfloor X_{im}^{(p)} \rfloor$ is the value of $\lfloor X_m^{(p)} \rfloor$ at node i.

Consequently,

$$[A_{mi}^{(p)}] = [X_{im}^{(p)}]^{-1}, \tag{44}$$

we require the value of u from Eq. (37) at the interior points $k = c_p + 1(1)(p+1)^3$, and we thus have

$$
\begin{aligned}
u_k &= \lfloor X_{km}^{(p)} \rfloor \lfloor A_{mi}^{(p)} \rfloor \{u_i\} \\
&= \left[X_{k1}^{(p)}\, X_{k2}^{(p)}, \cdots, X_{km}^{(p)} \right] \begin{bmatrix} A_{11}^{(p)} & A_{12}^{(p)} & \cdots & A_{1i}^{(p)} & \cdots & A_{1m}^{(p)} \\ A_{21}^{(p)} & A_{22}^{(p)} & \cdots & A_{2i}^{(p)} & \cdots & A_{2m}^{(p)} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{m1}^{(p)} & A_{m2}^{(p)} & \cdots & A_{mi}^{(p)} & \cdots & A_{mm}^{(p)} \end{bmatrix} \{u_i\} \\
&= \left[\sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha 1}^{(p)}, \cdots, \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)}, \cdots, \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha m}^{(p)} \right] \{u_i\}. \\
&= \sum_{i=1}^{c_p} \left(\sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)} \right) u_i. \qquad (45)
\end{aligned}
$$

Comparing Eq. (45) with (28), we obtain

$$T_i^{k,p} = \sum_{\alpha=1}^{m} X_{k\alpha}^{(p)} A_{\alpha i}^{(p)} \qquad (46)$$

or

$$T_i^{k,p} = N_i^{*(p)}(x_k, y_k, z_k). \qquad (47)$$

6. Conclusions

A general formula is derived for the interpolation functions of general serendipity family for the 3-cube with arbitrarily placed nodes along each side. It is then shown that the interpolation functions for the general complete Lagrange family are derivable through the basis transformation without matrix inversion. Explicit expressions for interpolation functions of the serendipity family and the complete Lagrange family elements $-1 \leqslant \xi, \eta, \varsigma \leqslant 1$ which allow uniform spacing of nodes over the domain of the element from 4–10 orders are obtained for the first time. Recent successes of adaptive finite element procedures [10–13] justify the use of higher order approximations with harmonious combination of lower-order elements. The native Lagrange family [8] may produce varieties of higher order transitive elements combining different order finite elements conformably. However from the viewpoint of polynomial completeness, serendipity elements are undoubtedly preferable. The complete Lagrange family presented here realises polynomial completeness with necessary minimum nodes without destroying the nodal symmetry. We have not pursued the derivation of explicit interpolation functions for the mixed Lagrange family which neglects nodal symmetry of interior nodes as regularity of nodal placement may be important in practical finite element applications. Needless to say these reconstructed interpolation functions produce a wide range of possibilities with reference to improved isoparametric transformation for finite element analysis [14]. We have developed a method for the modification of interpolation functions of pth-order complete Lagrange elements in such a way that pth-order displacement states are correctly interpolated for trilinear element geometry which refers to angular distortions.

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