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A new quadrature rule based on a generalized mixed interpolation formula of exponential type

Hamsapriye

Department of Mathematics, Central College Campus, Bangalore University, Bangalore – 560001, India

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Abstract

A new method of approximating a function $f(x)$ uniquely by a function $f_n(x)$ of the form $f_n(x) = \mathrm{e}^{lx}(aU_1(kx) + bU_2(kx) + \sum_{i=0}^{n-2} c_i x^i)$, so that $f_n(x)$ interpolates $f(x)$ at $(n+1)$ equidistant points $x_0, x_0+h, \ldots, x_0+nh$, with $h > 0$, is derived in a closed-form. Various equivalent forms of the interpolation formula are also derived. A closed-form expression is derived for the error involved in such an approximation. With the aid of the newly derived interpolation formula, a set of Newton Cotes quadrature rules of the closed type are also derived. The total truncation error involved in these quadrature rules are analysed and closed-form expressions for error terms are proposed as conjectures in the two cases when n is odd and when n is even, separately. A more general exponential-type interpolation formula and quadrature rules based upon the generalized mixed interpolation formula are also explained and discussed. A few numerical examples are worked out as illustrations and the results are compared with the results of some of the earlier methods. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently (see [7]), new quadrature formulae, based on an "exponential-type interpolation formula", of the form

$$f_n(x) = \mathrm{e}^{lx}\left(\sum_{i=0}^{n} c_i x^i\right) \tag{1.1}$$

have been derived, which extends (more precisely, generalizes) the classical Newton Cotes closed-type quadrature rules and which have been proved to be advantageous in cases wherein the integrands show exponential behaviour. Further, various quadrature formulae based on the idea of mixed trigonometric interpolation formula have been derived (see [8]), which are again an extension of the Newton

E-mail address: preethi@bgl.vsnl.net.in (Hamsapriye)

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Cotes quadrature rules. These rules have been proved to be advantageous in cases wherein the integrands show a periodic or quasi-periodic behaviour (see [2,3,8,9,12]). The mixed trigonometric interpolation consists of the two basic trigonometric functions $\cos(kx)$ and $\sin(kx)$ and a polynomial of certain degree. Furthermore (see [4]), the idea of mixed interpolation has been generalized to what is called as "generalized mixed interpolation formula" and several quadrature formulae have been developed, of the Newton Cotes closed type, as well as Newton Gregory type (see [5,6]). These quadrature rules have been proved to be advantageous for the class of integrands having oscillatory nature.

In the generalized mixed interpolation theory (see [4]), any function $f(x)$ is thought of being approximated by a function $\hat{f}_n(x)$, considered in the form

$$\hat{f}_n(x) = aU_1(kx) + bU_2(kx) + \sum_{i=0}^{n-2} c_i x^i, \quad n \geqslant 1, \tag{1.2}$$

where k is a free parameter. Here $U_1(kx)$ and $U_2(kx)$ are so chosen that their wronskian is nonzero (linearly independent) and that they satisfy a linear second-order ordinary differential equation (ODE), of the type

$$y''(x) + kp(kx)y'(x) + k^2 q(kx)y(x) = 0, \tag{1.3}$$

where $p(kx)$ and $q(kx)$ are given functions. The mode of approximation is to interpolate $f(x)$ at certain $(n+1)$ equidistant points of an interval $[\alpha, \beta]$. We remark here that the two functions $U_1(kx)$ and $U_2(kx)$ are chosen based on the well-established oscillation theory of ODEs (see [11]).

In the present paper, new quadrature rules of Newton Cotes closed type have been derived based on a "generalized exponential type of interpolation function", which is of the form

$$f_n(x) = e^{lx}\left[aU_1(kx) + bU_2(kx) + \sum_{i=0}^{n-2} c_i x^i\right], \quad n \geqslant 1, \tag{1.4}$$

where l and k are two parameters, freely chosen. The procedure adopted here is that the integrand $f(x)$ in

$$\int_\alpha^\beta f(x)\,dx \tag{1.5}$$

is approximated by $f_n(x)$ and the mode of approximation is again based on the idea of interpolating $f(x)$ at the $(n+1)$ equidistant points, chosen to be $x_0, x_0+h, x_0+2h, \ldots, x_0+nh,\ h > 0$.

The following is the plan of the paper: In Section 2, generalized exponential-type interpolation formula has been derived in a closed form. Various equivalent forms of the formula have also been derived. The error involved in the approximation of $f(x)$ by $f_n(x)$, as given by relation (1.4), has been discussed and a closed-form expression for the error term has been derived for the class of functions $f(x) \in C^{n+1}([\alpha, \beta])$. In Section 3, numerical examples have been considered using the newly derived interpolation formula for various choices of $U_1(kx)$ and $U_2(kx)$. Also the results have been compared with the exponential-type interpolation formulae of De Meyer et al. [7]. In Section 4, Newton Cotes closed-type quadrature rules have been derived, by way of approximating the integrand by the interpolation function of form (1.4). The errors involved in such quadrature rules have been proposed as conjectures in cases when n is odd and when n is even. Here n refers to the order of the generalized mixed interpolation function. Section 5 has been reserved for the study of certain examples, which show the applicability of the formulae derived in Section 4. The choices of the parameters l and k have been explained, which also involve the utility of the error terms that have been proposed as conjectures. The numerical results have been compared with those obtained by the aid of the other known methods. The idea of generalized exponential-type interpolation has been further generalized and extended to develop Newton Cotes closed-type quadrature rules. This has been dealt with in Section 6. Also the corresponding errors involved in such approximation have been analysed and an attempt has been made to derive a closed-form expression for these error terms.

2. Derivation of a generalized exponential-type interpolation formula

We consider the problem of approximating any function $f(x)$, by a generalized mixed interpolation function $f_n(x)$ as given by relation (1.4). That is,

$$f_n(x) = \mathrm{e}^{lx}\left[aU_1(kx) + bU_2(kx) + \sum_{i=0}^{n-2} c_i x^i\right], \quad n \geqslant 1, \tag{2.1}$$

where l and k are two parameters and $n(\in \mathbb{N}$, the set of natural numbers) is called the order of the interpolation function $f_n(x)$. Further, the error term is defined as

$$E_n(f;x) = f(x) - f_n(x). \tag{2.2}$$

As in any other problem of interpolation we require that

$$f(x_j) = f_n(x_j), \quad x_j = x_0 + jh, \quad j = 0,1,\ldots,n \tag{2.3}$$

which when used in relation (2.2) gives rise to the set of equations

$$E_n(f;x_j) = 0, \quad j = 0,1,\ldots,n. \tag{2.4}$$

Eq. (2.3) constitutes a system of linear equations in the $(n+1)$ unknowns a, b, c_i $(i = 0,1,\ldots,n-2)$, whose coefficient determinant is different from zero, with some restrictions on the parameter k (to be explained later). As in [7], relation (2.1) can be viewed as an interpolation problem of approximating the function $\mathrm{e}^{-lx}f(x)$, by a generalized mixed interpolation function of type (1.1), at the same set of data points $x_0 + jh$ $(j = 0,1,\ldots,n)$. Thus, $f_n(x)$ can be explicitly written in terms of the functional values $\mathrm{e}^{-l(x_0+jh)}\, f(x_0+jh)$ $(j = 0,1,\ldots,n)$. From the theory of generalized mixed interpolation (see [4]), we can immediately write down that

$$\begin{aligned}\mathrm{e}^{-lx} f_n(x) = {} & \sum_{j=0}^{n} \binom{s}{j} \nabla_h^j[\mathrm{e}^{-lx} f(x)]\Big|_{x=x_0+jh} - k^2 \tilde{\phi}_n(x) \nabla_h^{n-1}[\mathrm{e}^{-l(x_0+\overline{n-1}h)} f(x_0+\overline{n-1}h)] \\ & - k^2 \left(\tilde{\phi}_{n+1}(x) - \tilde{\phi}_n(x)\frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\right) \nabla_h^n[\mathrm{e}^{-l(x_0+nh)} f(x_0+nh)],\end{aligned} \tag{2.5}$$

where $s = (x-x_0)/h$, $\theta := kh$ and for any function $g(x)$, the forward difference $\nabla_h^j g(x_0+jh)$ is defined as

$$\nabla_h^j g(x)\Big|_{x=x_0+jh} = \sum_{p=0}^{j} \binom{j}{p} (-1)^{j-p} g(x_0+ph). \tag{2.6}$$

Also we have defined

$$\tilde{\phi}_n(x) = \frac{1}{k^2 \tilde{D}_n(\theta)} \left[\left(\sum_{p=0}^{n-1} \binom{s}{p} \nabla_\theta^p U_1(kx_0 + p\theta) - U_1(kx) \right) \nabla_\theta^n U_2(kx_0 + n\theta) - \left(\sum_{p=0}^{n-1} \binom{s}{p} \nabla_\theta^p U_2(kx_0 + p\theta) - U_2(kx) \right) \nabla_\theta^n U_1(kx_0 + n\theta) \right], \tag{2.7}$$

$$\tilde{D}_n^{1,1}(\theta) = \nabla_\theta^{n+1} U_2(kx_0 + \overline{n+1}\theta) \nabla_\theta^{n-1} U_1(kx_0 + \overline{n-1}\theta) - \nabla_\theta^{n+1} U_1(kx_0 + \overline{n+1}\theta) \nabla_\theta^{n-1} U_2(kx_0 + \overline{n-1}\theta), \tag{2.8}$$

$$\tilde{D}_n(\theta) = \nabla_\theta^n U_2(kx_0 + n\theta) \nabla_\theta^{n-1} U_1(kx_0 + \overline{n-1}\theta) - \nabla_\theta^n U_1(kx_0 + n\theta) \nabla_\theta^{n-1} U_2(kx_0 + \overline{n-1}\theta). \tag{2.9}$$

Thus, for those values of k, for which $\tilde{D}_n(\theta) \neq 0$, the interpolation function (2.5) can be uniquely determined (see [4]).

From relation (2.5) we obtain that

$$f_n(x) = e^{lx} \left[\sum_{j=0}^{n} \nabla_h^j (e^{-l(x_0+jh)} f(x_0 + jh)) - k^2 \tilde{\phi}_n(x) \nabla_h^{n-1} (e^{-l(x_0+\overline{n-1}h)} f(x_0 + \overline{n-1}h)) - k^2 \left(\tilde{\phi}_{n+1}(x) - \tilde{\phi}_n(x) \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)} \right) \nabla_h^n (e^{-l(x_0+nh)} f(x_0 + nh)) \right]. \tag{2.10}$$

The expression on the right-hand side of Eq. (2.10) can be expressed in three different but equivalent forms, as explained below.

First form: Using relation (2.6) for $g(x) = e^{-lx} f(x)$, relation (2.10) can be written in the form

$$f_n(x) = e^{lx} \left[\sum_{j=0}^{n} \binom{s}{j} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} e^{-l(x_0+ph)} f(x_0 + ph) - k^2 \tilde{\phi}_n(x) \sum_{p=0}^{n-1} (-1)^{n-1-p} \binom{n-1}{p} e^{-l(x_0+ph)} f(x_0 + ph) - k^2 \left(\tilde{\phi}_{n+1}(x) - \tilde{\phi}_n(x) \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)} \right) \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} e^{-l(x_0+ph)} f(x_0 + ph) \right]. \tag{2.11}$$

Using the identity

$$\binom{s}{j} \binom{j}{p} = \binom{s}{p} \binom{s-p}{j-p} \tag{2.12}$$

and interchanging the summations in (2.11), the first term of (2.11) gets simplified to (see [7])

$$\sum_{p=0}^{n} \binom{s}{p} e^{-l(x_0+ph)} f(x_0 + ph) \sum_{j=0}^{n-p} (-1)^j \binom{s-p}{j}. \tag{2.13}$$

After using the result

$$\sum_{j=0}^{n-p}(-1)^j\binom{s-p}{j}=(-1)^{n-p}\binom{s-p-1}{n-p}, \tag{2.14}$$

relation (2.11) takes the form

$$f_n(x)=\mathrm{e}^{s\theta'}\left[\sum_{p=0}^{n}\left\{(-1)^{n-p}\binom{s}{p}\binom{s-p-1}{n-p}-k^2\tilde{\phi}_n(x)(-1)^{n-j-1}\binom{n-1}{j}\right.\right.$$
$$\left.\left.-k^2\left(\tilde{\phi}_{n+1}(x)-\tilde{\phi}_n(x)\frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\right)(-1)^{n-j}\binom{n}{j}\right\}\mathrm{e}^{-p\theta'}f(x_0+ph)\right], \tag{2.15}$$

where we have denoted $\theta' := lh$.

Second form: The inverse formula of relation (2.7) is given to be (see [1])

$$g(x_0+ph)=\sum_{j=0}^{p}\binom{p}{j}\nabla_h^j g(x_0+ph). \tag{2.16}$$

Using relation (2.16) for the function $f(x)$ and substituting for $f(x_0+ph)$ in relation (2.11), we obtain

$$f_n(x)=\mathrm{e}^{s\theta'}\left[\sum_{j=0}^{n}\binom{s}{j}\sum_{p=0}^{j}\binom{j}{p}(-1)^{j-p}\mathrm{e}^{-p\theta'}\sum_{i=0}^{p}\binom{p}{i}\nabla_h^i f(x_0+ih)-k^2\tilde{\phi}_n(x)\right.$$
$$\times\sum_{p=0}^{n-1}(-1)^{n-1-p}\binom{n-1}{p}\mathrm{e}^{-p\theta'}\sum_{i=0}^{p}\binom{p}{i}\nabla_h^i f(x_0+ih)$$
$$\left.-k^2\left(\tilde{\phi}_{n+1}(x)-\tilde{\phi}_n(x)\frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\right)\sum_{p=0}^{n}(-1)^{n-p}\binom{n}{p}\mathrm{e}^{-p\theta'}\sum_{i=0}^{p}\binom{p}{i}\nabla_h^i f(x_0+ih)\right]. \tag{2.17}$$

Using identity (2.12) and interchanging the orders of the summations twice, we finally arrive at (see [7])

$$f_n(x)=\mathrm{e}^{s\theta'}\left[\sum_{i=0}^{n}\left\{\binom{s}{i}\sum_{j=0}^{n-i}\binom{s-i}{j}(-1)^j(1-\mathrm{e}^{-\theta'})^j-k^2\tilde{\phi}_n(x)\right.\right.$$
$$\times(-1)^{n-1-i}\binom{n-1}{i}(1-\mathrm{e}^{-\theta'})^{n-1-i}-k^2\left(\tilde{\phi}_{n+1}(x)-\frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\tilde{\phi}_n(x)\right)$$
$$\left.\left.\times\ (-1)^{n-i}\binom{n}{i}(1-\mathrm{e}^{-\theta'})^{n-i}\right\}\mathrm{e}^{-i\theta'}\nabla_h^i f(x_0+ih)\right]. \tag{2.18}$$

The form of the interpolation function as given by relation (2.18) is suitable, whenever a difference table is given.

Third form: Introducing a new forward difference idea (see [7]) defined by

$$\nabla_h^j(\mathrm{e}^{-lx}f(x)) = \mathrm{e}^{-lx+j\theta'}U_h^j f(x), \quad \forall f(x),\ j \geqslant 0, \tag{2.19}$$

we can rewrite relation (2.10) in terms of U_h^j's. Also the new forward difference $U_h^j f(x)$ can be recursively defined by

$$\begin{aligned} U_h^j f(x) &= U_h^{j-1}[U_h f(x)], \quad j = 1,2,\ldots, \\ U_h f(x) &= \mathrm{e}^{-\theta'}(f(x) - \mathrm{e}^{\theta'}f(x-h)), \\ U_h^0 f(x) &= f(x). \end{aligned} \tag{2.20}$$

The following important identities can be derived using relations (2.6), (2.16), (2.19) and (2.20) (see also [7]):

$$\nabla_h^j[\mathrm{e}^{-(lx_0+j\theta')}f(x_0+jh)] = \mathrm{e}^{-lx_0}U_h^j[f(x_0+jh)], \tag{R1}$$

$$\nabla_h^j[f(x_0+jh)] = \mathrm{e}^{-lx_0}U_h^j[\mathrm{e}^{lx_0+j\theta'}f(x_0+jh)]. \tag{R2}$$

More generally, relation (R2) can be written in the form as

$$\nabla_h^j[f(x)] = \mathrm{e}^{-lx+j\theta'}U_h^j[\mathrm{e}^{lx}f(x)]. \tag{R3}$$

Further,

$$U_h^j g(x_0+jh) = \sum_{p=0}^{j}\binom{j}{p}(-1)^{j-p}\mathrm{e}^{-p\theta'}g(x_0+ph), \tag{R4}$$

$$g(x_0+jh) = \mathrm{e}^{-lx_0}\sum_{p=0}^{j}\binom{j}{p}U_h^p[\mathrm{e}^{l(x_0+ph)}f(x_0+ph)]. \tag{R5}$$

Thus, we can express the interpolation formula (2.10), in the form as

$$\begin{aligned} f_n(x) = \mathrm{e}^{s\theta'}\Bigg[&\sum_{j=0}^{n}\binom{s}{j}U_h^j f(x_0+jh) - k^2\tilde{\phi}_n(x)U_h^{n-1}f(x_0+\overline{n-1}h) \\ &- k^2\left(\tilde{\phi}_{n+1}(x) - \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\tilde{\phi}_n(x)\right)U_h^n f(x_0+nh)\Bigg], \end{aligned} \tag{2.21}$$

wherein identity (R1) has been made use of.

Limiting Cases.

(1) It can be proved that as $l \to 0$, relations (2.10), (2.15), (2.18) and (2.21) tend to the generalized mixed interpolation formula of Chakrabarti and Hamsapriye [4].

(2) As the parameter $k \to 0$, it has been proved that the generalized exponential-type interpolation formulae (2.10),(2.15), (2.18) and (2.21) tend to the exponential-type interpolation formula of De Meyer et al. [7], provided the functions $p(kx)$ and $q(kx)$ of relation (1.2) are such that $p(0) \neq 0$, $q(0) \neq 0$ and in which case we see that

$$\tilde{\phi}_n(x) \to h^2\binom{s}{n+1}\frac{\bar{U}_{n+1}(0)}{\bar{U}_n(0)} \quad \text{as } k \to 0. \tag{2.22}$$

(3) Finally as $k \to 0$ and $l \to 0$, formulae (2.10), (2.15), (2.18) and (2.21) tend to the classical Newton's interpolation formula.

Error analysis. We use the fact that $e^{-lx} f_n(x)$ interpolates the function $e^{-lx} f(x)$, at the points $x_0 + jh$ $(j = 0, 1, 2, \ldots, n)$. Thus, for $f(x) \in C^{(n+1)}([\alpha, \beta])$ (here $\alpha = x_0$, $\beta = x_0 + nh$), we have (see [4])

$$e^{-lx} E_n(f, x) = h^{n-1} \tilde{\phi}_n(x) \tilde{L}_n(e^{-lx} f(x))||_{x=\xi}, \quad x_0 < \xi(x) < x_0 + nh, \tag{2.23}$$

where $\tilde{L}_n$ is the operator as defined by

$$\tilde{L}_n \equiv \left(\frac{\bar{U}_n(kx)}{\bar{U}_{n+1}(kx)} \frac{d^2}{dx^2} - k \frac{\bar{U}'_n(kx)}{\bar{U}_{n+1}(kx)} \frac{d}{dx} + k^2 \right) \frac{d^{n-1}}{dx^{n-1}}, \tag{2.24}$$

$$\bar{U}_n(kx) = U_2^{(n)}(kx) U_1^{(n-1)}(kx) - U_2^{(n-1)}(kx) U_1^{(n)}(kx). \tag{2.25}$$

Using the result proved in [7] that

$$D_x^n[e^{-lx} f(x)] = e^{-lx}(D_x - l)^n f(x), \quad \forall n \tag{2.26}$$

we can write $\tilde{L}_n[e^{-lx} f(x)]$ as

$$\tilde{L}_n[e^{-lx} f(x)] = e^{-lx} \tilde{\tilde{L}}_n[f(x)], \tag{2.27}$$

where $\tilde{\tilde{L}}_n$ stands for the operator

$$\tilde{\tilde{L}}_n \equiv \left(\frac{\bar{U}_n(kx)}{\bar{U}_{n+1}(kx)} (D_x - l)^2 - k \frac{\bar{U}'_n(kx)}{\bar{U}_{n+1}(kx)} (D_x - l) + k^2 \right) (D_x - l)^{n-1}. \tag{2.28}$$

Using (2.27) in (2.23) we arrive at the relation

$$E_n(f; x) = e^{l(x-\xi)} h^{n-1} \tilde{\phi}_n(x) \tilde{\tilde{L}}_n[f(x)], \quad \xi \in (x_0, x_0 + nh). \tag{2.29}$$

It is to be noted that the operator $\tilde{\tilde{L}}_n$ annihilates the function as given by relation (1.4). That is

$$\tilde{\tilde{L}}_n f_n(x) = 0 \tag{2.30}$$

and therefore we have from relation (2.2)

$$\tilde{\tilde{L}}_n E_n(f; x) = \tilde{\tilde{L}}_n f(x) \tag{2.31}$$

which satisfies the boundary conditions (2.4).

It is possible to express $E_n(f; x)$ without the factor $e^{-l\xi}$, as explained below.

It is proved that (see [7])

$$(D_x - l)^{n-1} \rho_{n-2}(x) = h^{1-n}, \tag{2.32}$$

where $\rho_n(x)$ is defined to be

$$\rho_n(x) = \frac{1}{\theta'^{n+1}} \left\{ \left[\sum_{p=0}^{n} (-1)^{n-p} \binom{s}{j} (1 - e^{-\theta'})^p \right] e^{\theta' s} - (-1)^n \right\}. \tag{2.33}$$

Thus, the function chosen to be

$$\tilde{\rho}_n(x) = e^{lx} \left(a U_1(kx) + b U_2(kx) + \sum_{i=0}^{n-2} c_i x^i \right) + \frac{1}{k^2} \rho_{n-2}(x) \tag{2.34}$$

is such that it satisfies the differential equation

$$\tilde{\tilde{L}}_n\tilde{\rho}_n(x) = h^{1-n}, \tag{2.35}$$

where a, b, c_i are $(n+1)$ constants, which are determined by imposing the conditions that

$$\tilde{\rho}_n(x_0 + jh) = 0, \quad j = 0, 1, \ldots, n. \tag{2.36}$$

By virtue of the conditions in (2.36), relation (2.34) gives rise to the problem of interpolating the functional values $-(1/k^2)\rho_{n-2}(x_0 + jh)$ $(j = 0, 1, \ldots, n)$, by an interpolation function of form (2.1). Using the interpolation formula as given by relation (2.21), we derive $\tilde{\rho}_n(x)$ in the form

$$\begin{aligned}\tilde{\rho}_n(x) = e^{s\theta'}\Bigg[&-\frac{1}{k^2}\sum_{p=0}^{n}\binom{s}{p}U_h^p\rho_{n-2}(x_0 + ph) + \tilde{\phi}_n(x)U_h^{n-1}\rho_{n-2}(x_0 + \overline{n-1}h)\\ &+\left(\tilde{\phi}_{n+1}(x) - \tilde{\phi}_n(x)\frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\right)U_h^n\rho_{n-2}(x_0 + nh)\Bigg] + \frac{1}{k^2}\rho_{n-2}(x).\end{aligned} \tag{2.37}$$

Thus the function $\tilde{\rho}_n(x)$, as in relation (2.37) satisfies Eq. (2.35) and conditions (2.36).

Now we define a new function $G_t(f;x)$, for $t \notin \{x_0, \ldots, x_0 + nh\}$ by

$$G_t(f;x) = E_n(f;x)\tilde{\rho}_n(t) - E_n(f;t)\tilde{\rho}_n(x). \tag{2.38}$$

Clearly, the function $G_t(f;x)$ vanishes at the $(n+2)$ points $(x_0 + ih)$, for $i = 0, 1, \ldots, n$ and t. A slightly modified version of the Rolle's theorem states that if $f(x) \in C^{(1)}([\alpha, \beta])$ and $f(\alpha) = 0 = f(\beta)$, then for all $l \in \mathbb{R}$, there exists a $\xi \in (\alpha, \beta)$, such that $D_x f(\xi) = l f(\xi)$. This theorem can be extended to the case of $f(x)$ vanishing at any $(n+2)$ points and we may conclude that $\forall l \in \mathbb{R}$, $\exists \xi \in (\alpha, \beta) \ni$: $(D_x - l)^{n+1} f(\xi) = 0$. Using this generalization as well as by following the steps of the proof as explained in [9], we conclude that for a function $g(x)$ vanishing at any $(n+2)$ distinct points, there exists a $\xi \in (\alpha, \beta)$, such that $\tilde{\tilde{L}}_n g(\xi) = 0$. Applying this theorem to the function $G_t(f;x)$, we derive the expression

$$\tilde{\tilde{L}}_n G_t(f;\xi') = 0 \quad \text{for } \xi' \in (\alpha, \beta). \tag{2.39}$$

This further leads to the result that

$$E_n(f;t) = h^{n-1}\tilde{\rho}_n(t)\tilde{\tilde{L}}_n f(\xi') \quad \text{for } \xi' \in (x_0, x_0 + nh), \tag{2.40}$$

wherein relations (2.31) and (2.32) are used. Since t is any arbitrary point, we finally obtain that

$$E_n(f;x) = h^{n-1}\tilde{\rho}_n(x)\tilde{\tilde{L}}_n f(\xi) \quad \text{for } \xi \in (x_0, x_0 + nh), \tag{2.41}$$

where ξ depends on x.

3. Numerical examples

In this section, two examples are worked out which show the applicability of the various formulae as derived in Section 2.

Table 1
Comparison of maximum errors: $U_1(kx)=\cos(kx), U_2(kx)=\sin(kx)$ $\alpha=0, \beta=2,$ $n=8$

$f(x)$	k	l	Exponential interpolation	Generalized interpolation
e^{-x^2}	2.2	2.2	3.619 E-03	1.523 E-03
	4.0	0.3	1.974 E-04	3.689 E-05
	2.8	0.3	1.974 E-04	9.043 E-05
	2.89	2.2	3.619 E-03	8.382 E-05
$x^2e^{-x^2}$	2.9	2.4	1.163 E-02	3.378 E-04
	4.4	2.0	1.146 E-02	3.153 E-04
	4.0	0.3	6.170 E-04	1.459 E-04
	4.8	1.9	1.034 E-02	3.216 E-04

For numerical experiments we have worked with the choice of pairs of functions $U_1(kx)$ and $U_2(kx)$ as listed below:

$$U_1(kx)=\cos(kx) \quad \text{and} \quad U_2(kx)=\sin(kx), \tag{3.1}$$

$$U_1(kx)=e^{kx}\cos(kx) \quad \text{and} \quad U_2(kx)=e^{kx}\sin(kx), \tag{3.2}$$

$$U_1(kx)=\mathrm{Ai}(-kx-1) \quad \text{and} \quad U_2(kx)=\mathrm{Bi}(-kx-1), \tag{3.3}$$

where $\mathrm{Ai}(-kx)$ and $\mathrm{Bi}(-kx)$ are the standard Airy functions and this choice enforces the limiting case 2.

We have considered the following two functions, as examples, for numerical purposes. The formula is more suitable to the class of functions with damped oscillations and we have considered such examples under the applications of the quadrature formulae derived in Section 4:

$$f(x)=e^{-x^2} \quad \text{for } x\in[0,2],$$
$$f(x)=x^2e^{-x^2} \quad \text{for } x\in[0,2].$$

For various choices of k and l, we have tabulated the maximum errors in the exponential-type interpolation formula, as well the generalized exponential-type interpolation formula, by considering 50 nonnodal points in the interval $[0,2]$. On comparison, we observe that the newly derived generalized exponential-type interpolation formula gives better results.

Tables 1–3 compare the maximum errors involved in the use of the newly derived interpolation formula with that of the exponential interpolation formula as derived in [7]. The interval $[\alpha,\beta]$ in all the examples is fixed to be $[0,2]$ and the functions $f(x)$ are specified in the tables. The order of the interpolation function is fixed at $n=8$, for all the examples and the nonnodal points are chosen to be $y_i=\frac{2}{50}i$, for $i=1,50$. The values of l and k are fixed arbitrarily.

4. The quadrature

We derive in this section, the Newton Cotes closed-type quadrature rules, which are based on the newly derived interpolation formula, presented in Section 2. We subdivide the interval $[\alpha,\beta]$

Table 2
Comparison of maximum errors: $U_1(kx) = e^{kx}\cos(kx)$, $U_2(kx) = e^{kx}\sin(kx)$ $\alpha = 0, \beta = 2, n = 8$

$f(x)$	k	l	Exponential interpolation	Generalized interpolation
e^{-x^2}	2.953	0.3	1.973 E-04	1.025 E-04
	1.51	2.2	3.619 E-03	1.249 E-04
	2.70	2.0	3.518 E-03	2.827 E-04
	1.00	0.7	1.348 E-04	8.235 E-05
$x^2e^{-x^2}$	2.28	2.2	1.281 E-02	5.277 E-04
	3.61	2.0	1.146 E-02	6.791 E-04
	1.00	2.4	1.163 E-02	4.035 E-03
	1.30	2.4	1.163 E-02	6.459 E-04

Table 3
Comparison of maximum errors: $U_1(kx) = \mathrm{Ai}(-kx-1)$, $U_2(kx) = \mathrm{Bi}(-kx-1)$ $\alpha = 0, \beta = 2$

n	$f(x)$	k	l	Classical interpolation	Exponential interpolation	Generalized interpolation
4	e^{-x^2}	−0.1	0.2	1.38 E-02	9.99 E-03	9.42 E-03
6		0.3	−0.3	9.37 E-04	8.50 E-04	5.65 E-04
8		−0.421	−0.32	3.15 E-05	6.09 E-05	4.10 E-05
10		−0.3	−0.33	1.60 E-06	6.06 E-06	3.86 E-06
4	$x^2e^{-x^2}$	−0.4	−0.1	2.07 E-02	2.04 E-02	1.90 E-02
6		$-\frac{5}{9}$	−0.1	1.02 E-03	3.93 E-03	1.12 E-03
8		−0.85	−0.2	1.22 E-04	4.84 E-04	7.72 E-05
10		−0.54	−0.55	2.83 E-05	3.06 E-05	2.51 E-05
10		$-\frac{8}{9}$	$-\frac{8}{9}$	2.83 E-05	3.87 E-05	1.61 E-05

into n equal sub-intervals. Thus, we set $\alpha = x_0$, $\beta = x_0 + nh$ and $x_i = x_0 + ih$ $(i = 0, 1, \ldots, n)$, with $h = (b-a)/n$. We approximate the integral $\int_\alpha^\beta f(x)\,dx$ by integrating the function $f_n(x)$. With the aid of the $(n+1)$ equidistant points $x_0 + jh$, $j = 0, 1, \ldots,$ we obtain $f_n(x)$ and we write

$$f(x) = f_n(x) + E_n(f;x), \tag{4.1}$$

where $E_n(f;x)$ represents the error involved in such approximation. Thus,

$$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f_n(x) + \int_\alpha^\beta E_n(f;x)\,dx \tag{4.2}$$

and we take the approximate value of the integral to be

$$\int_\alpha^\beta f(x)\,dx \approx \int_\alpha^\beta f_n(x). \tag{4.3}$$

Using expressions (2.21) and (2.29) and setting $\alpha = x_0,\ \beta = x_0 + nh$ we get

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \left[\sum_{j=0}^{n} \binom{s}{j} U_h^j f(x_0 + jh) - k^2 \tilde{\phi}_n(x) U_h^{n-1} f(x_0 + \overline{n-1}h) \right.$$
$$\left. - k^2 \left(\tilde{\phi}_{n+1}(x) - \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)} \tilde{\phi}_n(x) \right) U_h^n f(x_0 + nh) \right] \mathrm{d}x$$
$$+ h^{n-1} \int_{x_0}^{x_0+nh} \mathrm{e}^{l(x-\xi)} \tilde{\phi}_n(x) \tilde{\tilde{L}}_n f(\xi)\,\mathrm{d}x \quad \text{for } x_0 < \xi(x) < x_0 + nh. \tag{4.4}$$

We define $A_j^{(n)}$ as in [7]. That is, by defining (see [7])

$$A_j^{(n)} := \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \binom{s}{j} \mathrm{d}x = nh \int_0^n \mathrm{e}^{s\theta'} \binom{s}{j} \mathrm{d}s, \tag{4.5}$$

one can prove that

$$\sum_{j=0}^{n} A_j^{(n)} U_h^j f(x_0 + jh) = nh \sum_{j=0}^{n} u_j^{(n)}(\theta') f(x_0 + jh). \tag{4.6}$$

Using the result in (R2), we can write immediately that

$$\sum_{j=0}^{n} A_j^{(n)} \nabla_h^j f(x_0 + jh) = nh \sum_{j=0}^{n} u_j^{(n)}(\theta')\, \mathrm{e}^{j\theta'} f(x_0 + jh). \tag{4.7}$$

Thus the terms $\int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \tilde{\phi}_n(x)\,\mathrm{d}x$ and $\int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \tilde{\phi}_{n+1}(x)\,\mathrm{d}x$ can be simplified to the forms which are proved to be more suitable for computational purposes:

$$\tilde{\phi}_1^{(n)} := \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \tilde{\phi}_n(x)\,\mathrm{d}x = \frac{1}{k^2 \tilde{D}_n(\theta)} \left[\left(nh \sum_{j=0}^{n} u_j^{(n)}(\theta')\, \mathrm{e}^{j\theta'} U_1(kx_0 + j\theta') \right. \right.$$
$$\left. - \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} U_1(kx)\,\mathrm{d}x \right) \nabla_\theta^n U_2(kx_0 + n\theta) - \left(nh \sum_{j=0}^{n} u_j^{(n)}(\theta')\, \mathrm{e}^{j\theta'} U_2(kx_0 + j\theta') \right.$$
$$\left. \left. - \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} U_2(kx)\,\mathrm{d}x \right) \nabla_\theta^n U_1(kx_0 + n\theta) \right] \tag{4.8}$$

and

$$\tilde{\phi}_2^{(n)} := \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} \tilde{\phi}_{n+1}(x)\,\mathrm{d}x = \frac{1}{k^2 \tilde{D}_{n+1}(\theta)} \left[\left(nh \sum_{j=0}^{n} u_j^{(n)}(\theta') e^{j\theta'} U_1(kx_0 + j\theta') \right. \right.$$
$$\left. - \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} U_1(kx)\,\mathrm{d}x \right) \nabla_\theta^{n+1} U_2(kx_0 + \overline{n+1}\theta) - \left(nh \sum_{j=0}^{n} u_j^{(n)}(\theta') e^{j\theta'} U_2(kx_0 + j\theta') \right.$$
$$\left. \left. - \int_{x_0}^{x_0+nh} \mathrm{e}^{s\theta'} U_2(kx)\,\mathrm{d}x \right) \nabla_\theta^{n+1} U_1(kx_0 + \overline{n+1}\theta) \right]. \tag{4.9}$$

Using relations (4.8) and (4.9), we can write the extended Newton Cotes quadrature rules (4.4) in the form (which is more suitable for the computational purposes)

$$\int_{x_0}^{x_0+nh} f_n(x)\,\mathrm{d}x \approx nh\sum_{j=0}^{n} u_j^{(n)}(\theta')f(x_0+jh) - k^2\tilde{\phi}_1^{(n)}U_h^{n-1}f(x_0+\overline{n-1}h)$$
$$-k^2\left(\tilde{\phi}_2^{(n)} - \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)}\tilde{\phi}_1^{(n)}\right)U_h^n f(x_0+nh)$$
$$+h^{n-1}\int_{x_0}^{x_0+nh} \mathrm{e}^{l(x-\xi)}\tilde{\phi}_n(x)\tilde{\tilde{L}}_n f(\xi)\,\mathrm{d}x, \quad x_0<\xi(x)<x_0+nh. \tag{4.10}$$

By putting $n=1,2,3,\ldots$, we obtain the modified trapezium rule, modified Simpson's $\frac{1}{3}$rd rule, modified Simpson's $\frac{3}{8}$th rule, etc.

Below, we have given the expressions for $u_j^{(n)}(\theta')$, for $j=0,1,\ldots,n$ and for $n=1,2,3$, which are helpful for computational purposes (also see [7]):

$$u_0^{(1)}(\theta') = \left(\frac{\mathrm{e}^{\theta'}}{\theta'^2} - \frac{1}{\theta'} - \frac{1}{\theta'^2}\right),$$

$$u_1^{(1)}(\theta') = \left(\frac{\mathrm{e}^{-\theta'}}{\theta'^2} + \frac{1}{\theta'} - \frac{1}{\theta'^2}\right),$$

$$u_0^{(2)}(\theta') = \left(-\frac{1}{4\theta'^2} + \frac{1}{2\theta'^3}\right)\mathrm{e}^{2\theta'} - \frac{1}{2\theta'} - \frac{3}{4\theta'^2} - \frac{1}{2\theta'^3},$$

$$u_1^{(2)}(\theta') = \left(\frac{1}{\theta'^2} - \frac{1}{\theta'^3}\right)\mathrm{e}^{\theta'} + \left(\frac{1}{\theta'^2} + \frac{1}{\theta'^3}\right)\mathrm{e}^{-\theta'},$$

$$u_2^{(2)}(\theta') = \left(-\frac{1}{4\theta'^2} - \frac{1}{2\theta'^3}\right)\mathrm{e}^{-2\theta'} + \frac{1}{2\theta'} - \frac{3}{4\theta'^2} + \frac{1}{2\theta'^3}.$$

$$u_0^{(3)}(\theta') = \left(\frac{1}{9\theta'^2} + \frac{1}{3\theta'^4} - \frac{1}{3\theta'^3}\right)\mathrm{e}^{3\theta'} - \frac{1}{3\theta'} - \frac{11}{18\theta'^2} - \frac{2}{3\theta'^3} - \frac{1}{3\theta'^4},$$

$$u_1^{(3)}(\theta') = \left(-\frac{1}{2\theta'^2} - \frac{4}{3\theta'^3} - \frac{1}{\theta'^4}\right)\mathrm{e}^{-2\theta'} + \left(\frac{1}{\theta'^2} - \frac{5}{3\theta'^3} + \frac{1}{\theta'^4}\right)\mathrm{e}^{\theta'},$$

$$u_2^{(3)}(\theta') = \left(-\frac{1}{2\theta'^2} + \frac{4}{3\theta'^3} - \frac{1}{\theta'^4}\right)\mathrm{e}^{2\theta'} + \left(\frac{1}{\theta'^2} + \frac{5}{3\theta'^3} + \frac{1}{\theta'^4}\right)\mathrm{e}^{-\theta'},$$

$$u_3^{(3)}(\theta') = \left(\frac{1}{9\theta'^2} + \frac{1}{3\theta'^4} + \frac{1}{3\theta'^3}\right)\mathrm{e}^{-3\theta'} + \frac{1}{3\theta'} - \frac{11}{18\theta'^2} + \frac{2}{3\theta'^3} - \frac{1}{3\theta'^4}.$$

It is clear that $u_j^{(n)}(\theta') = u_{n-j}^{(n)}(-\theta')$, see also [7].

It is to be remarked that the Newton Cotes quadrature rules of open type can be derived along similar lines, as also derived in [7].

Error analysis. From relation (4.3), we have

$$E_n^Q = h^{n-1}\int_{x_0}^{x_0+nh} \mathrm{e}^{l(x-\xi)}\tilde{\phi}_n(x)\tilde{\tilde{L}}_n f(\xi)\,\mathrm{d}x \tag{4.11}$$

which on using relation (2.27), gives rise to the relation

$$E_n^Q = h^{n-1}\int_{x_0}^{x_0+nh} \mathrm{e}^{lx}\tilde{\phi}_n(x)\tilde{L}_n[\mathrm{e}^{-l\xi}f(\xi)]\,\mathrm{d}x. \tag{4.12}$$

Applying the proposed conjecture of [4], we arrive at the relations

$$E_n^Q = h^{n-1}\tilde{L}_n[\mathrm{e}^{-l\eta}f(\eta)]\int_{x_0}^{x_0+nh} \tilde{\phi}_n(x)\,\mathrm{e}^{lx}\,\mathrm{d}x, \quad \text{for } n\text{-odd} \tag{4.13}$$

$$= \frac{h^{n-1}}{n+2}\frac{\mathrm{d}}{\mathrm{d}\eta}[\tilde{L}_n(\mathrm{e}^{-l\eta}f(\eta)]\int_{x_0}^{x_0+nh}\left(x-x_0+\frac{nh}{2}\right)\tilde{\phi}_n(x)\,\mathrm{e}^{lx}\,\mathrm{d}x, \quad \text{for } n\text{-even} \tag{4.14}$$

for some $\eta \in (x_0, x_0+nh)$.

We can rewrite relations (4.13) and (4.14), back in terms of $\tilde{\tilde{L}}_n$ as

$$E_n^Q = h^{n-1}\mathrm{e}^{-l\eta}\tilde{\tilde{L}}_n f(\eta)\int_{x_0}^{x_0+nh} \mathrm{e}^{lx}\tilde{\phi}_n(x)\,\mathrm{d}x, \quad \text{for } n\text{-odd} \tag{4.15}$$

$$= \frac{h^{n-1}}{n+2}\,\mathrm{e}^{-l\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}[\tilde{\tilde{L}}_n f(\eta)]\int_{x_0}^{x_0+nh}\left(x-x_0+\frac{nh}{2}\right)\tilde{\phi}_n(x)\mathrm{e}^{lx}\,\mathrm{d}x \quad \text{for } n\text{-even} \tag{4.16}$$

for some $\eta \in (x_0, x_0+nh)$.

In fact, the numerical results show that the above conjectures proposed in relations (4.15) and (4.16) are valid for the choice of the pairs of functions $U_1(kx)$ and $U_2(kx)$ made in Section 3.

Also, it is clearly seen that relations (4.15) and (4.16) tend to the corresponding error terms of the generalized mixed interpolation formula, as $l \to 0$. Further as $k \to 0$, we retrieve the error terms of the exponential-type quadrature rules, if the stipulated condition stated earlier holds good. Finally, as $l \to 0$ and $k \to 0$, we obtain the error expressions of the classical Newton Cotes quadrature rules, under the same stipulated condition.

The above-proposed conjectures are valid for those choices of $U_1(kx)$ and $U_2(kx)$, which form the fundamental set for a linear second-order, ordinary differential equations, with constant coefficients (see [10]).

5. Numerical experiments

In this section, we discuss the utility of the error expressions, given by relations (4.15) and (4.16), in fixing numerical values of l and k. For computational purposes few examples are studied and the tables show the applicability of the quadrature rule (4.10). These tables compare the absolute errors between the present and the earlier methods.

Firstly, we explain briefly the choice of k and l, which are free parameters, independent of each other. If we look back at the derivation of the generalized mixed interpolation formula $f_n(x)$ of Section 2 (see [4]), based on which the present quadrature rules are derived, we find that there is a possibility for the parameter k to have a complex value, as long as the quantity $\tilde{D}_n(\theta)$, given by relation (2.9) does not vanish.

With this in the background, the choice of k and l can be made as explained below. For n odd, we can choose k and l, such that the function

$$x \to \tilde{\tilde{L}}_n f(x) \tag{5.1}$$

vanishes at certain intermediate point, of the interval of interest. For n even, we can choose k and l, such that the function

$$x \to (D_x - l)\tilde{\tilde{L}}_n f(x) \tag{5.2}$$

vanishes at certain intermediate point, of the interval of interest.

Now since (5.1) or (5.2) gives a single equation for a fixed x, to be solved for the two parameters k and l, there are two possibilities. One is that by fixing l (or k), we can solve the resulting equation for k (or l). The second possibility is to set $l = k$ and then solve the resulting equation for one parameter k. From the numerical results we infer that the second possibility is more fruitful. Also, for all numerical purposes, we have fixed the intermediate point x to be the middle point of the interval concerned (see [7]). It has been verified numerically, that choices of the intermediate points other than the mid-point, do not change the error significantly.

Secondly, we have worked with the pair of functions given by relations (3.1)–(3.3), which also meet the condition stated in relation (2.22). The pair of functions in (3.1)–(3.3) are the linearly independent solutions of the following ODEs:

$$y''(x) + k^2 y(x) = 0, \tag{5.3}$$

$$y''(x) - 2ky'(x) + 2k^2 y(x) = 0, \tag{5.4}$$

$$y''(x) + k^2(kx + 1)y(x) = 0, \tag{5.5}$$

respectively. Further, the operators $\tilde{\tilde{L}}_n$ for the pair of functions (3.1) and (3.2) are given to be

$$\tilde{\tilde{L}}_n \equiv [(D_x - l)^2 + k^2](D_x - l)^{n-1}, \quad n \geqslant 1, \tag{5.6}$$

$$\tilde{\tilde{L}}_n \equiv [(D_x - l)^2 - 2k(D_x - l) + 2k^2](D_x - l)^{n-1}, \quad n \geqslant 1. \tag{5.7}$$

For the pair of functions in (3.3), $\tilde{\tilde{L}}_n$ for $n = 1$ is derived to be

$$\tilde{\tilde{L}}_1 \equiv (D_x - l)^2 + k^2(kx + 1). \tag{5.8}$$

It is to be noted that, for the choice of functions $U_1(kx)$ and $U_2(kx)$ as given by relations (3.1) and (3.2), for a fixed x and l, we always obtain a quadratic equation in k, for any n. But, by setting $l = k$ and for a fixed x, we obtain a polynomial equation, whose degree varies with n.

Thirdly, the computation of the numerical values of k, via relations (5.1) and (5.2) is easy, when the derivatives of the function $f(x)$ can be obtained in a closed analytic form. Otherwise, as stated in [7], a computational scheme has to be adopted to approximate these derivatives numerically and it is preferred that the scheme involves the given functional values only. This is an additional requirement for solving linear integral equations.

Lastly, for all practical purposes, we divide the interval of integration into N sub-intervals of equal lengths. On each of these sub-intervals the same quadrature rule of form (4.10) is applied for some fixed n. Also at each stage, the lower limit has to be carefully relocated. In all the examples discussed, we have fixed $n=1,2$ and 3, for the choice of pairs of functions (3.1) and (3.2) and for the pair of functions as in (3.3), we have worked with $n=1$, for simplicity. Further, we have worked with both the possibilities of choosing the values of l and k. The details are given below.

(I) *For the choice of* $U_1(kx)=\cos(kx)$ *and* $U_2(kx)=\sin(kx)$. For a fixed l, we obtain the following values of k, for $n=1,2,3$:

$$\tilde{\tilde{L}}_1 f(\eta)=0 \Rightarrow k=\left(-\frac{f''(\eta)-2lf'(\eta)+l^2f(\eta)}{f(\eta)}\right)^{1/2}, \tag{5.9}$$

$$(D_x-l)\tilde{\tilde{L}}_2 f(\eta)=0 \Rightarrow k=\left(-\frac{f^{\text{iv}}(\eta)-4lf'''(\eta)+6l^2f''(\eta)-4l^3f'(\eta)+l^4f(\eta)}{f''(\eta)-2lf'(\eta)+l^2f(\eta)}\right)^{1/2} \tag{5.10}$$

$$\tilde{\tilde{L}}_3 f(\eta)=0 \Rightarrow k=\left(-\frac{f^{\text{iv}}(\eta)-4lf'''(\eta)+6l^2f''(\eta)-4l^3f'(\eta)+l^4f(\eta)}{f''(\eta)-2lf'(\eta)+l^2f(\eta)}\right)^{1/2}. \tag{5.11}$$

For $l=k$, we obtain the following relations for $n=1,2$ and 3:

$$\tilde{\tilde{L}}_1 f(\eta)=0 \Rightarrow f''(\eta)-2kf'(\eta)+2k^2f(\eta)=0, \tag{5.12}$$

$$(D_x-k)\tilde{\tilde{L}}_2 f(\eta)=0 \Rightarrow f^{\text{iv}}(\eta)-4kf'''(\eta)+7k^2f''(\eta)-6k^3f'(\eta)+2k^4f(\eta)=0, \tag{5.13}$$

$$\tilde{\tilde{L}}_3 f(\eta)=0 \Rightarrow f^{\text{iv}}(\eta)-4kf'''(\eta)+7k^2f''(\eta)-6k^3f'(\eta)+2k^4f(\eta)=0. \tag{5.14}$$

(II) *For the choice of* $U_1(kx)=e^{kx}\cos(kx)$ *and* $U_2(kx)=e^{kx}\sin(kx)$. For a fixed l, we obtain the following relations, for $n=1,2$ and 3:

$$\tilde{\tilde{L}}_1 f(\eta)=0 \Rightarrow (f''(\eta)-2lf'(\eta)+2l^2f(\eta))-2k(f'(\eta)-lf(\eta))+2k^2f(\eta)=0, \tag{5.15}$$

$$\begin{aligned}(D_x-l)\tilde{\tilde{L}}_2 f(\eta)=0 \Rightarrow\; &(f^{\text{iv}}(\eta)-4lf'''(\eta)+6l^2f''(\eta)-4l^3f'(\eta)+l^4f(\eta))\\ &-2k(f'''(\eta)-3lf''(\eta)+3l^2f'(\eta)-l^3f(\eta))+2k^2(f''(\eta)\\ &-2lf'(\eta)+l^2f(\eta))=0,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\tilde{\tilde{L}}_3 f(\eta)=0 \Rightarrow\; &(f^{\text{iv}}(\eta)-4lf'''(\eta)+6l^2f''(\eta)-4l^3f'(\eta)+l^4f(\eta))\\ &-2k(f'''(\eta)-3lf''(\eta)+3l^2f'(\eta)-l^3f(\eta))\\ &+2k^2(f''(\eta)-2lf'(\eta)+l^2f(\eta))=0.\end{aligned} \tag{5.17}$$

Table 4
Trapezium rule: $U_1(kx)=\cos(kx)$, $U_2(kx)=\sin(kx)$, k as chosen by relation (5.9), for the values of l as displayed in the table, for varying values of a and $h=0.01$

a	l	n	Classical trapezium rule	Exponential trapezium rule	Generalized exponential trapezium
0.25	0.9	25	3.91 E−06	1.84 E−11	2.87 E−13
0.50	0.8	50	6.49 E−06	4.32 E−11	2.90 E−11
0.75	0.8	75	7.12 E−06	1.14 E−10	1.03 E−10
1.00	0.7	100	6.13 E−06	1.37 E−10	1.65 E−10
1.25	0.2	125	4.36 E−06	1.53 E−10	9.65 E−11
1.50	0.01	150	2.63 E−06	1.64 E−10	9.25 E−11
1.75	−0.5	175	1.36 E−06	1.69 E−10	1.03 E−11
2.00	−0.5	200	6.11 E−07	1.72 E−10	2.26 E−11
2.25	−0.02	225	2.37 E−07	1.73 E−10	3.63 E−11
2.50	−1.4	250	8.04 E−08	1.73 E−10	6.59 E−11
2.75	−1.5	275	2.38 E−08	1.74 E−10	2.04 E−11
3.00	−2.1	300	6.17 E−09	1.74 E−10	2.09 E−11

For $l=k$, we obtain the following three relations for $n=1,2$ and 3, respectively:

$$\tilde{\tilde{L}}_1 f(\eta)=0 \Rightarrow f''(\eta)-4kf'(\eta)+5k^2 f(\eta)=0, \tag{5.18}$$

$$(D_x-k)\tilde{\tilde{L}}_2 f(\eta)=0 \Rightarrow f^{\text{iv}}(\eta)-6kf'''(\eta)+14k^2 f''(\eta)-14k^3 f'(\eta)+5k^4 f(\eta)=0, \tag{5.19}$$

$$\tilde{\tilde{L}}_3 f(\eta)=0 \Rightarrow f^{\text{iv}}(\eta)-6kf'''(\eta)+14k^2 f''(\eta)-14k^3 f'(\eta)+5k^4 f(\eta)=0. \tag{5.20}$$

(III) *For the choice of* $U_1(kx)=Ai(-kx-1)$ *and* $U_2(kx)=Bi(-kx-1)$. For $l=k$, we obtain, when $n=1$ the following relation:

$$\tilde{\tilde{L}}_1 f(\eta)=0 \Rightarrow f''(\eta)-2kf'(\eta)+2k^2 f(\eta)+k^3\eta f(\eta)=0. \tag{5.21}$$

Example 1. As a first example we have considered the example, as also has been considered in [7]. That is,

$$\int_0^a e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}\,\text{Er } f(a) \tag{5.22}$$

Tables 4–6 compare the maximum errors involved in the use of the classical Newton-Cotes quadrature rules, the extended quadrature rules of De Meyer et al. [7] and the modified quadrature rules as presented in Section 4. The parameter l in the exponential-type interpolation formula of De Meyer et al. [7], is chosen by solving strategy (4.1) of De Meyer et al. [7]. This yields a polynomial equation of degree 1, 2, 3, respectively, in the case of extended trapezium, Simpson's $\frac{1}{3}$rd, Simpson's $\frac{3}{8}$th rule.

It is remarked here that while evaluating the definite integrals, the free parameter l and k appearing in the formula presently derived, are chosen by solving polynomial equations of certain degree, depending upon n. We have chosen those values of l and k, which are neither too very small nor too very large. Depending upon the nature of $f(x)$, we can fix the values of l and k. For instance, if

Table 5
Simpson's $\frac{1}{3}$rd rule: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$, k as chosen by relation (5.10), for the values of l as displayed in the table, for varying values of a and $h = 0.01$

a	l	n	Exponential rule	Generalized exponential rule
0.50	0.9	50	1.273 E−07	9.06 E−11
1.00	0.5	100	1.275 E−07	4.37 E−10
1.50	−0.1	150	1.276 E−07	1.63 E−10
2.00	0.8	200	1.277 E−07	3.26 E−09
2.50	0.9	250	1.277 E−07	1.32 E−11
3.00	−1.2	300	1.277 E−07	2.77 E−09
3.50	−1.2	350	1.277 E−07	2.47 E−08
4.00	−10.0	400	1.277 E−07	7.16 E−08

Table 6
Simpson's $\frac{3}{8}$th rule: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$, k as chosen by relation (5.11), for the values of l as displayed in the table, for varying values of a and $h = 0.01$

a	l	n	Classical rule	Exponential rule	Generalized rule
0.3	4.1	30	3.86 E−10	4.85 E−12	9.82 E−15
0.6	5.1	60	4.77 E−10	5.78 E−12	5.31 E−13
0.9	5.1	90	2.76 E−10	8.65 E−11	2.91 E−13
1.2	4.1	120	1.70 E−11	4.11 E−10	8.22 E−12
1.5	−6.0	150	1.18 E−10	6.29 E−09	1.00 E−11
1.8	−6.0	180	1.22 E−10	1.07 E−04	9.63 E−12
2.1	−10.1	210	7.42 E−11	1.07 E−04	6.17 E−12
2.4	−7.0	240	3.22 E−11	1.07 E−04	9.45 E−12
2.7	−7.0	270	1.06 E−11	1.07 E−04	9.46 E−12
3.0	−10.1	300	2.77 E−12	1.07 E−04	8.14 E−12

$f(x)$ is nonoscillatory and we wish to apply the exponential-type quadrature rule, obviously l cannot be chosen to be a very large negative or positive number. Accordingly, the choice of l has been made, and in Tables 7 and 8, we have displayed a column, which gives the range of l so chosen. In other words, it shows the interval in which the parameter l assumes its values, by solving Eq. (4.1) of De Meyer et al. [7]. Similarly, for oscillatory integrands, if we wish to use the generalized exponential quadrature rule, then the parameter $l=k$, has to be chosen so that the mixed interpolation function itself is an oscillatory one. Again if $f(x)$ is nonoscillatory, then the parameter $l = k$ has to be chosen in such a way that the mixed interpolation function shows a nonoscillatory behaviour. Accordingly, we have made the choice of k. For more details see [5].

Further, in Tables 7–12 we have set a column specifying the range of the chosen k, which is obtained by solving the polynomial equations mentioned earlier and which is expected to minimize the error involved in the use of that particular quadrature rule. In other words, the column shows that which particular root of the polynomial equation is used in the quadrature rule applied in all the sub-intervals and this has been specified by specifying their ranges, for various choices of $U_1(kx)$

Table 7
Trapezium, Simpson's $\frac{1}{3}$rd and $\frac{3}{8}$th rules: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$, $l = k$ as chosen by relation (5.12)–(5.14), for varying values of a and $h = 0.01$

n	a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
1	0.3	4.57 E−06	2.29 E−11	$-2.00 < l < -1.42$	1.21 E−12	$0.66 < k < 0.99$
2		1.71 E−10	3.14 E−09	$2.72 < l < 3.28$	2.65 E−10	$-1.49 < k < -1.01$
3		3.86 E−10	5.48 E−11	$2.73 < l < 3.27$	2.28 E−11	$-2.79 < k < -2.46$
1	0.6	6.97 E−06	5.32 E−11	$-2.60 < l < -1.42$	2.25 E−11	$0.20 < k < 0.99$
2		2.12 E−10	5.26 E−09	$2.12 < l < 3.28$	4.44 E−10	$-1.94 < k < -1.01$
3		4.77 E−10	7.70 E−11	$2.13 < l < 3.27$	2.92 E−11	$-3.16 < k < -2.46$
1	0.9	6.67 E−06	1.27 E−10	$-3.20 < l < -1.42$	5.29 E−11	$-0.44 < k < 0.99$
2		1.22 E−10	6.38 E−09	$-1.52 < l < 3.28$	5.01 E−10	$-2.27 < k < -1.01$
3		2.76 E−10	8.97 E−11	$1.53 < l < 3.27$	3.35 E−11	$-3.58 < k < -2.46$
1	1.2	4.73 E−06	1.49 E−10	$-3.80 < l <$-1.42	6.32 E−11	$-1.19 < \mathrm{Re}(k) < 0.99$
2		7.57 E−12	6.83 E−09	$0.92 < l < 3.28$	5.06 E−10	$-2.44 < k < -1.01$
3		1.70 E−11	3.66 E−10	$0.93 < l < 3.27$	3.47 E−11	$-4.06 < k < -2.46$
1	1.5	2.63 E−06	1.62 E−10	$-4.40 < l < -1.42$	7.02 E−11	$-1.49 < \mathrm{Re}(k) < 0.99$
2		5.27 E−11	6.95 E−09	$0.32, < l < 3.28$	5.04 E−10	$-2.45 < \mathrm{Re}(k) < -1.01$
3		1.18 E−10	1.87 E−09	$0.33 < l < 3.27$	3.45 E−11	$-4.57 < k < -2.46$
1	1.8	1.17 E−06	1.69 E−10	$-5.50 < l < -1.42$	7.51 E−11	$-1.79 < \mathrm{Re}(k) < 0.99$
2		5.45 E−11	8.21 E−05	$-0.27, < l < 3.28$	8.34 E−10	$-2.59 < \mathrm{Re}(k) < -0.20$
3		1.22 E−10	7.58 E−05	$-0.26 < l < 3.27$	3.44 E−11	$-5.12 < k < -2.46$

Table 8
Trapezium, Simpson's $\frac{1}{3}$rd and $\frac{3}{8}$th rules: $U_1(kx) = e^{kx}\cos(kx)$, $U_2(kx) = e^{kx}\sin(kx)$, $l = k$ as chosen by relation (5.18)–(5.20), for varying values of a and $h = 0.01$

n	a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
1	0.3	4.56 E−06	2.29 E−11	$-2.00 < l < -1.42$	7.63 E−12	$-0.85 < k < -0.63$
2		1.71 E−10	3.14 E−09	$2.72 < l < 3.28$	4.61 E−10	$-1.04 < k < -0.69$
3		3.86 E−10	5.48 E−11	$2.73 < l < 3.27$	4.36 E−11	$-2.71 < k < -2.28$
1	0.6	6.97 E−06	5.32 E−11	$-2.60 < l < -1.42$	3.14 E−11	$-1.06 < k < -0.63$
2		2.12 E−10	5.26 E−09	$2.12 < l < 3.28$	1.02 E−09	$-1.40 < k < -0.69$
3		4.77 E−10	7.70 E−11	$2.13 < l < 3.27$	3.80 E−11	$-3.21 < k < -2.28$
1	0.9	6.67 E−06	1.27 E−10	$-3.20 < l < -1.42$	5.85 E−11	$-1.23 < k < -0.63$
2		1.22 E−10	6.38 E−09	$-1.52 < l < 3.28$	1.60 E−09	$-1.76 < k < -0.69$
3		2.76 E−10	8.97 E−11	$1.53 < l < 3.27$	3.09 E−11	$-3.72 < k < -2.28$
1	1.2	4.73 E−06	1.49 E−10	$-3.80 < l < -1.42$	7.80 E−11	$-1.37 < k < -0.63$
2		7.57 E−12	6.83 E−09	$0.92 < l < 3.28$	2.11 E−09	$-2.12 < k < -0.69$
3		1.70 E−11	3.66 E−10	$0.93 < l < 3.27$	2.58 E−11	$-4.26 < k < -2.28$
1	1.5	2.63 E−06	1.62 E−10	$-4.40 < l < -1.42$	8.70 E−11	$-1.40 < k < -0.63$
2		5.27 E−11	6.95 E−09	$0.32, < l < 3.28$	2.51 E−09	$-2.52 < k < -0.69$
3		1.18 E−10	1.87 E−09	$0.33 < l < 3.27$	2.76 E−11	$-4.80 < k < -2.28$

Table 9
Trapezium rule: $U_1(kx) = \mathrm{Ai}(-kx-1)$, $U_2(kx) = \mathrm{Bi}(-kx-1)$, $l = k$ as chosen by relation (5.21) for varying values of a and $h = 0.01$

a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
0.1	1.65 E−06	6.31 E−12	$1.22 < l < 1.40$	1.59 E−12	$0.88 < k < 0.99$
0.2	3.20 E−06	1.40 E−11	$1.02 < l < 1.40$	6.07 E−13	$0.76 < k < 0.99$
0.3	4.56 E−06	2.29 E−11	$0.82 < l < 1.40$	3.05 E−12	$0.64 < k < 0.99$
0.4	5.68 E−06	3.27 E−11	$0.62 < l < 1.40$	9.23 E−12	$0.54 < k < 0.99$
0.5	6.49 E−06	4.32 E−11	$0.42 < l < 1.40$	1.74 E−11	$0.36 < k < 0.99$

Table 10
Trapezium rule: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$, $l = k$ as chosen by relation (5.12) for varying values of a and $h = 0.01$

a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
0.25	1.88 E−05	7.51 E−10	$1.08 < l < 3.26$	3.86 E−10	$0.97 < k < 2.31$
0.50	1.98 E−05	1.02 E−09	$-0.83 < l < 3.26$	6.40 E−10	$-0.84 < k < 2.31$
0.75	6.60 E−06	1.07 E−09	$-2.72 < l < 3.26$	7.38 E−10	$-2.21 < k < 2.31$
1.00	4.77 E−06	9.80 E−10	$-4.80 < l < 3.26$	8.73 E−10	$-3.00 < k < 2.31$
1.25	6.57 E−06	5.63 E−10	$-8.39 < l < 3.26$	9.95 E−10	$-3.82 < k < 2.31$

Table 11
Trapezium rule: Simpson's $\frac{1}{3}$rd and $\frac{3}{8}$th rules: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$, $l = k$ as chosen by relation (5.18) for varying values of a and $h = 0.01$

n	a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
1	0.3	2.07 E−05	8.29 E−10	$0.68 < l < 3.26$	6.93 E−10	$0.33 < k < 1.46$
2		2.27 E−09	3.23 E−09	$0.20 < l < 1.73$	1.76 E−08	$-2.78 < k < -1.70$
3		5.13 E−09	2.70 E−10	$-17.5 < l < -8.17$	2.85 E−09	$-2.21 < k < -1.22$
1	0.6	1.54 E−05	1.03 E−09	$-1.58 < l < 3.26$	9.15 E−10	$-0.78 < k < 1.46$
2		8.75 E−10	5.67 E−07	$-0.97 < l < 2.78$	4.95 E−08	$-3.74 < k < -1.08$
3		1.97 E−09	7.72 E−06	$-140.87 < l < -4.92$	1.45 E−07	$-3.10 < k < -0.69$
1	0.9	1.27 E−06	1.03 E−09	$-3.94 < l < 3.26$	9.56 E−10	$-1.74 < k < 1.46$
2		1.02 E−09	5.63 E−07	$-0.97 < l < 2.78$	4.67 E−08	$-3.74 < k < -1.08$
3		2.31 E−09	7.72 E−06	$-140.87 < l < -4.92$	1.38 E−07	$-3.10 < k < -0.69$
1	1.2	6.87 E−06	7.04 E−10	$-7.42 < l < 3.26$	8.98 E−10	$-2.94 < k < 1.46$
2		7.31 E−10	3.40 E−07	$-1.09 < l < 2.78$	4.08 E−08	$-3.74 < k < -1.08$
3		1.64 E−09	7.72 E−06	$-140.87 < l < -4.92$	1.38 E−07	$-3.10 < k < -0.69$

and $U_2(kx)$. To illustrate this, we take the example of the function $f(x) = e^{-x^2}$ and $N = 30$. The polynomial equation (5.12) gives a set of two values for k, say k_1 and k_2, by fixing η as the mid-point of each of the 30 sub-intervals. Suppose that we work with k_1 (or k_2), then this column gives the minimum and the maximum of these 30 values of k_1 (or k_2).

Table 12
Trapezium rule: $U_1(kx) = \mathrm{Ai}(-kx-1)$, $U_2(kx) = \mathrm{Bi}(-kx-1)$, $l = k$ as chosen by relation (5.21) for varying values of a and $h = 0.01$

a	Classical rule	Exponential rule	Range for l	Generalized rule	Range for k
0.1	8.89 E−06	3.99 E−10	$2.36 < l < 3.26$	2.15 E−10	$1.77 < k < 2.31$
0.2	1.62 E−05	6.57 E−10	$1.48 < l < 3.26$	3.86 E−10	$1.23 < k < 2.31$
0.3	2.07 E−05	8.29 E−10	$0.68 < l < 3.26$	5.32 E−10	$0.63 < k < 2.31$
0.4	2.19 E−05	9.47 E−10	$-0.07 < l < 3.26$	6.46 E−10	$-0.07 < k < 2.31$
0.5	1.98 E−05	1.02 E−09	$-0.83 < l < 3.26$	7.17 E−10	$-0.83 < k < 2.31$

It is to be mentioned here that whenever we have chosen a complex value of k, then we have taken the *real* part of the end result. Added to this, we have taken much care that, with the above choices of k, the function $\tilde{D}_n(\theta)$, remains nonzero.

Example 2. As a second example we have considered the integral

$$\int_0^a \mathrm{e}^{-x^2}\cos(3x)\,\mathrm{d}x. \tag{5.23}$$

See Tables 10–12.

Example 3. We consider the integral

$$\int_{0.1}^{3.1} \mathrm{e}^{2x}\sin(x)\,\mathrm{d}x = \mathrm{e}^{1/5}\left(\cos\left(\frac{1}{10}\right) - 2\sin\left(\frac{1}{10}\right) - \mathrm{e}^6\cos\left(\frac{31}{10}\right) + 2\mathrm{e}^6\sin\left(\frac{31}{10}\right)\right). \tag{5.24}$$

In this example, the two parameters $l = k$, appearing in the modified exponential quadrature formula, are chosen to be the smallest of the two roots obtained by solving relation (5.21), for various values of n. The parameter l of the exponential-type quadrature rule is chosen according to strategy (4.1) of De Meyer et al. [7] and again we have chosen the smallest of the two roots so obtained (see Table 13).

Example 4. We consider

$$\int_0^4 \mathrm{e}^{x^2}\,\mathrm{d}x = -\mathrm{i}\frac{\sqrt{\pi}}{2}\mathrm{Erf}(\mathrm{i}4). \tag{5.25}$$

In this example, we get two complex conjugate roots for the parameters and both the values give the same accuracy (see Table 14).

Example 5. We consider (see Table 15)

$$\int_0^3 \mathrm{Ai}(-3x-1)\,\mathrm{d}x = 0.1000081398890482. \tag{5.26}$$

Table 13
Trapezium rule: $U_1(kx) = \cos(kx)$, $U_2(kx) = \sin(kx)$

n	Classical quadrature rule	Exponential quadrature rule	Generalized quadrature rule
50	1.35 E−01	3.00 E−02	1.07 E−02
100	3.39 E−02	2.26 E−03	7.72 E−04
150	1.51 E−02	4.68 E−04	1.58 E−04
200	8.49 E−03	1.51 E−04	5.06 E−05
250	5.43 E−03	6.23 E−05	2.08 E−05
300	3.77 E−03	3.02 E−05	1.01 E−05
350	2.77 E−03	1.64 E−05	5.46 E−06
400	2.12 E−03	9.61 E−06	3.20 E−06
450	1.68 E−03	6.06 E−06	2.00 E−06
500	1.36 E−03	3.96 E−06	1.31 E−06

Table 14
Trapezium rule: $U_1(kx) = e^{kx}\cos(kx)$, $U_2(kx) = e^{kx}\sin(kx)$

n	Exponential quadrature rule	Generalized quadrature rule
100	1.79 E−01	4.10 E−02
150	3.54 E−02	8.25 E−03
200	1.12 E−02	2.62 E−03
250	4.59 E−03	1.08 E−03
300	2.21 E−03	5.23 E−04
350	1.19 E−03	2.71 E−04
400	7.01 E−04	1.59 E−04
450	4.37 E−04	1.11 E−04
500	2.87 E−04	7.26 E−05

Table 15
Trapezium rule: $U_1(kx) = \mathrm{Ai}(-kx-1)$, $U_2(kx) = \mathrm{Bi}(-kx-1)$. $k = l = 1.9$

n	Exponential quadrature rule	Generalized quadrature rule
50	9.11 E−04	2.83 E−05
100	2.26 E−04	8.72 E−06
150	1.01 E−04	4.01 E−06
200	5.66 E−05	2.28 E−06
250	3.62 E−05	1.47 E−06
300	2.51 E−05	1.02 E−06
350	1.85 E−05	7.51 E−07
400	1.41 E−05	5.76 E−07

6. Further generalization of exponential-type interpolation and extension to quadrature rules

The idea of exponential-type interpolation can be further generalized by way of approximating any function $f(x)$ by $\tilde{f}_n(x)$ as given by

$$\tilde{f}_n(x) = e^{w(lx)} \left[aU_1(kx) + bU_2(kx) + \sum_{i=0}^{n-2} c_i x^i \right], \tag{6.1}$$

where $w(lx)$ is a known function, k and l are free parameters. By following the ideas and steps of Section 2, we can obtain a closed-form expression for (6.1), which interpolates the function $f(x)$ at the same $(n+1)$ equidistant points $x_0 + jh$, $j = 0, 1, \ldots, n$. Without repeating the details of the derivation we give below the interpolation formula, corresponding to relation (2.21):

$$\begin{aligned} \tilde{f}_n(x) = e^{w(lx)} \Bigg[&\sum_{j=0}^{n} \binom{s}{j} e^{-w(lx_0 + j\theta') + w(j\theta')} \tilde{U}_h^j f(x_0 + jh) \\ &- k^2 \tilde{\phi}_n(x) e^{-w(lx_0 + \overline{n-1}\theta') + w(\overline{n-1}\theta')} \tilde{U}_h^{n-1} f(x_0 + \overline{n-1}h) \\ &- \left(\tilde{\phi}_{n+1}(x) - \tilde{\phi}_n(x) \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)} \right) e^{-w(lx_0 + n\theta') + w(n\theta')} \tilde{U}_h^n f(x_0 + nh) \Bigg], \end{aligned} \tag{6.2}$$

where we have defined recursively that

$$\tilde{U}_f f(x) = e^{w(lx - \theta')} [e^{-w(lx)} f(x) - e^{-w(lx - \theta')} f(x - h)], \quad \forall f(x), \tag{6.3}$$

$$\tilde{U}_h^j f(x) e^{w(lx - j\theta')} = \nabla_h^j [e^{-w(lx)} f(x)],$$

$$\tilde{U}_h^{(0)} f(x) = f(x) \tag{6.4}$$

The counterpart of relation (R3) would be

$$\nabla_h^j [f(x)] = e^{-w(lx - j\theta')} \tilde{U}_h^j [e^{w(lx) f(x)}]. \tag{6.5}$$

It can be verified that

$$\tilde{U}_h^j e^{w(lx)} \equiv 0 \quad \text{for } j = 1, 2, \ldots . \tag{6.6}$$

Further, the error involved in such an approximation can be derived to be

$$\tilde{E}_n(f; x) = e^{w(lx)} h^{n-1} \tilde{\phi}_n(x) \tilde{L}_n (e^{-w(lx)} f(x))_{x=\xi}, \quad x_0 < \xi(x) < x_n, \tag{6.7}$$

where $\tilde{L}_n$ is as given by relation (2.24).

Now by setting $W(x) = e^{-w(x)}$, and using Leibnitz differentiation rule, we can write

$$D_x^n (W(x) f(x)) = (W(x) + f(x))^{(n)}. \tag{6.8}$$

Thus relation (6.7) can be expressed in the form

$$\tilde{E}_n(f; x) = h^{n-1} e^{w(lx)} \tilde{\phi}_n(x) \tilde{\tilde{\mathscr{L}}}_n f(\xi), \quad x_0 < \xi(x) < x_0 + nh, \tag{6.9}$$

where $\tilde{\bar{\mathscr{L}}}_n$ denotes the differential operator

$$\tilde{\bar{\mathscr{L}}}_n \equiv \frac{\bar{U}_n(kx)}{\bar{U}_{n+1}(kx)}(W(x)+f(x))^{(n+1)} - k\frac{\bar{U}'_n(kx)}{\bar{U}_{n+1}(kx)}(W(x)+f(x))^{(n)} + k^2(W(x)+f(x))^{(n-1)}. \tag{6.10}$$

The quadrature: As in Section 4, we can obtain the Newton Cotes closed-type quadrature rules in the form

$$\begin{aligned}\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \Bigg[&\sum_{j=0}^{n} \tilde{A}_j^{(n)} \mathrm{e}^{-w(lx_0+j\theta')+w(j\theta')} \tilde{U}_h^{j} f(x_0+jh) \\ &- k^2 \mathrm{e}^{-w(lx_0+\overline{n-1}\theta')+w(\overline{n-1}\theta')} \tilde{U}_h^{n-1} f(x_0+\overline{n-1}h) \int_{x_0}^{x_0+nh} \mathrm{e}^{w(lx)} \tilde{\phi}_n(x)\,\mathrm{d}x \\ &- k^2 \mathrm{e}^{-w(lx_0+n\theta')+w(n\theta')} \tilde{U}_h^{n} f(x_0+nh) \Bigg\{ \int_{x_0}^{x_0+nh} \mathrm{e}^{w(lx)} \tilde{\phi}_{n+1}(x)\,\mathrm{d}x \\ &- \frac{\tilde{D}_n^{1,1}(\theta)}{\tilde{D}_{n+1}(\theta)} \int_{x_0}^{x_0+nh} \mathrm{e}^{w(lx)} \tilde{\phi}_n(x)\,\mathrm{d}x \Bigg\} \Bigg],\end{aligned} \tag{6.11}$$

where $\tilde{A}_j^{(n)}$'s are defined by

$$\tilde{A}_j^{(n)} = \int_{x_0}^{x_0+nh} \mathrm{e}^{w(lx)} \binom{s}{j}\,\mathrm{d}x. \tag{6.12}$$

The error in the quadrature rule is then proposed as conjectures, as stated below:

$$\tilde{E}_n^{Q} = h^{n-1} \tilde{L}_n(\mathrm{e}^{-w(l\eta)} f(\eta)) \int_{x_0}^{x_0+nh} \mathrm{e}^{w(lx)} \tilde{\phi}_n(x)\,\mathrm{d}x \quad \text{for } n\text{-odd}, \tag{6.13}$$

$$\tilde{E}_n^{Q} = \frac{h^{n-1}}{n+2} \frac{\mathrm{d}}{\mathrm{d}\eta} \tilde{L}_n(\mathrm{e}^{-w(l\eta)} f(\eta)) \int_{x_0}^{x_0+nh} \left(x - \left(x_0 + \frac{nh}{2}\right)\right) \tilde{\phi}_n(x) \mathrm{e}^{w(lx)}\,\mathrm{d}x \quad \text{for } n\text{-even} \tag{6.14}$$

for some $x_0 < \eta < x_0 + nh$.

For the sake of completeness, we have included this idea of a 'more general' exponential-type interpolation formula and the quadrature rules based on it. This further generalization of the idea of interpolation and quadrature, are expected to give accurate results for a more general class of functions, than what has been derived and discussed in the previous sections (Sections 2 and 4). In fact, a deeper study is yet to be done on the choice of the function $w(lx)$, for any given function $f(x)$. It is intended to take up such studies in future, with the support of few numerical examples.

7. Conclusions

A new interpolation formula of exponential-type based on the generalized mixed interpolation formula has been derived. The newly derived 'generalized exponential-type interpolation' formula has been expressed in three different forms. The error involved in such generalized exponential-type interpolation formula has been discussed and a closed-form expression has been derived for the error term. Several numerical experiments have been performed, which show the applicability of the newly derived interpolation formula. In fact, a more general exponential-type interpolation formula has been derived in a closed-form and the error involved has also been discussed. An attempt has been made to derive a closed-form expression for the error term involved in approximating any function $f(x)$ by this further generalized exponential-type interpolation function $f_n(x)$.

The classical Newton Cotes quadrature rules, of the closed type, have been extended based on the newly derived generalized exponential-type interpolation formula, in a form that is more suitable for numerical computational purposes. The error involved in the 'generalized Newton Cotes quadrature rules' has been proposed as conjectures in the two cases when n is odd and when n is even. Several numerical examples have been taken up for study and the newly derived quadrature rules show the efficiency over the classical Newton Cotes quadrature rules. Also the idea of extending the classical Newton Cotes rules, via the exponential-type interpolation, has been further generalized as in Section 6 and the error involved has been proposed as conjectures, in the two cases when n is odd and when n is even, only to a certain extent.

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