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Boundary integration of polynomials over an arbitrary linear tetrahedron in Euclidean three-dimensional space

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Abstract

This paper is concerned with explicit integration formulas and algorithms for computing integrals of trivariate polynomials over an arbitrary linear tetrahedron in Euclidean three-dimensional space. This basic three-dimensional integral governing the problem is transformed to surface integrals by use of the divergence theorem. The resulting two-dimensional integrals are then transformed into convenient and computationally efficient line integrals. These algorithms and explicit finite integration formulas are followed by an application—example for which we have explained the detailed computational scheme. The numerical result thus found is in complete agreement with previous works. Further, it is shown that the present algorithms are much simpler and more economical as well, in terms of arithmetic operations. The symbolic finite integration formulas presented in this paper may lead to an easy incorporation of geometric properties of solid objects, for example, the centre of mass, moment of inertia, etc. required in the engineering design process as well as several applications of numerical analysis where integration is required, for example in the finite element and boundary integral equation methods.

0. Nomenclature

$$II^{\alpha\beta\gamma}_{\pi_{xy}} = \int\int_{\pi_{xy}} x^\alpha y^\beta (h + lx + my)^{\gamma+1}\, \mathrm{d}x\, \mathrm{d}y$$

= surface integration over a plane polygon in the XY-plane

$II^{\alpha\beta\gamma}_{\pi_{yz}}$, $II^{\alpha\beta\gamma}_{\pi_{zx}}$ have a similar meaning

h, l, m
h', l', m' arbitrary constants
h'', l'', m''
α, β, γ Positive integers (including zero)

$$II^{\alpha\beta\gamma}_{T^{xy}_{ijk}} = \int\int_{T^{xy}_{ijk}} x^\alpha y^\beta (h + lx + my)^{\gamma+1}\, \mathrm{d}x\, \mathrm{d}y$$

T^{xy}_{ijk} = a triangle in the xy-plane with vertices at (x_i, y_i), (x_j, y_j) and (x_k, y_k)

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$II^{\alpha\beta\gamma}_{T^{yz}_{ijk}}$, $II^{\alpha\beta\gamma}_{T^{zx}_{ijk}}$ have similar meanings

Δ^{xy}_{ijk} = area of the triangle with vertices (x_i, y_i), (x_j, y_j) and (x_k, y_k)

Δ^{yz}_{ijk}, Δ^{zx}_{ijk} have similar meanings

$$III^{\alpha\beta\gamma}_{T_{ijk}} = \int\int_{T_{ijk}} x^{\alpha}y^{\beta}z^{\gamma+1}\hat{k}\cdot\hat{n}\ \mathrm{d}s$$

= surface integral over T_{ijk}

T_{ijk} = a linear triangle in Euclidean three-dimensional space with vertices (x_i, y_i, z_i), (x_j, y_j, z_j) and (x_k, y_k, z_k)

$\hat{k}$ = unit normal vector along z-axis

$(n_i, i = 1, 2, 3, 4)$—outward unit normal to triangles T_{312}, T_{432}, T_{413} and T_{421}, respectively.
Ω_{ijk} is either $1, 0, -1$ and it depends on the normal of linear T_{ijk}

$$III^{\alpha\beta\gamma}_{v} = \int\int\int x^{\alpha}y^{\beta}z^{\gamma}\ \mathrm{d}V$$

= volume integral of trivariate monomial $x^{\alpha}v^{\beta}z^{\gamma}$ over a linear three polyhedron in Euclidean three-dimensional space

$$\int\int_{S} x^{\alpha}y^{\beta}z^{\gamma+1}\hat{k}\cdot\hat{n}\ \mathrm{d}s = \text{surface integral over the projected area } s \text{ in the } xy\text{-plane},$$

$$= \sum_{T_{ijk}\in T} III^{\alpha\beta\gamma}_{T_{ijk}}$$

S = is a surface of R^3 decomposable in a set T of triangles such that any pair of triangles T_{ijk} and $T_{i'j'k'}$ do not intersect.

1. Introduction

Volume, centre of mass, moment of inertia and other geometrical properties of rigid homogenous solids frequently arise in a large number of engineering applications, in CAD/CAE/CAM applications in geometric modelling as well as in robotics. Integration formulas for multiple integrals have always been of great interest in computer applications [1]. Computation of mass properties of both plane and space objects is discussed by Wesley [2] and Mortenson [3]. A good description of integration methods in solid modelling is given by Lee and Requicha [4] in their survey article. Lee and Reqquicha [4] observe that most computational studies in multiple integration deal with problems where the integration domain is a very simple solid, such as a cube or a sphere and the integrating function is very complicated, conversely, in most engineering applications the opposite is the usual problem. In such problems the integration domain may have a nonconvex shape and the function inside the integral sign is a trivariate polynomial. Timmer and Stern [5] discussed a theoretical approach to the evaluation of volume integrals by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajaya [6] presented an outline of a closed form formula for volume integration for a linear tetrahedron and suggested that volume integration over a linear polyhedron can be obtained by simple means of disjoint decomposition technique. Cattani and Paoluzzi [7,8] have obtained finite integration over plane polygons and space polyhedra via surface and volume integration methods based on Green's and Gauss's Divergence theorems. In a recent paper, Bernardini [9] has presented explicit formulas and algorithms over a n-dimensional solid by using decomposition representation and boundary representation. In recent works, Rathod and Govinda Rao [10,11] addressed these problems, with an aim of giving more efficient and explicit algorithms than the previous works of Cattani and Paoluzzi [7,8] which made reference to combined use of well-known Taylor series expansion and Leibniz's theorem on differentation to obtain finite integration formulas for the integration of monomials over plane polygons and space polyhedra. Integration of a triple product, viz.

$x^m y^n (h + lx + ky)^{p+1}$, ($m, n, p$-positive integers and zero, h, l, k arbitrary constants), an expression in bivariates x, y plays a very important role [10,11] in the computation of volume integrals of a trivariate monomial viz. $x^m y^n z^p$ over the domain of a linear polyhedron. The integral evaluation of this bivariate expression over a plane polygon is computed by application of Green's theorem which reduces the area integral to a line integral. Because of the presence of $(h + lx + ky)^{p+1}$ the integration of the expression $x^m y^n (h + lx + ky)^{p+1}$ has to be expressed as a sum of $p(p+1)/2$ line integrals, for each line segment of the plane-polygon. In this paper we have found a means of overcoming this complication and now the same computation can be done only once for each line segment of the plane polygon. We have further applied this technique (which is discussed in Lemmas 1, 2 and 3 of this paper) to evaluate the volume integral of monomials over a linear tetrahedron (which is discussed in Theorems 1, 2 and 3). With help of an application example, we have shown that the present computational scheme is superior to earlier works [10,11]. We have further proposed three more theorems (Theorems 4, 5 and 6) which express the volume integral over a linear tetrahedron in terms of six line integrals over the boundary edges. In Lemma 4, we have proposed an efficient means of computing each of these line integrals which is again an improvement over the earlier works [10,11].

2. Surface integration

In this section we first establish a preliminary result giving closed analytical formulas for surface integration over a plane polygon either in the xy-plane, yz-plane or zx-plane. Then, we wish to use these formulas to derive a closed formula for surface integration over a linear tetrahedral surface in R^3.

2.1. Let π_{xy} be a simple polygon in the xy-plane: we want to evaluate the following structure product:

$$II_{\Pi_{xy}}^{\alpha,\beta,\gamma+1} \overset{\text{def}}{=} \int\int_{\pi_{xy}} x^\alpha y^\beta (h + lx + my)^{\gamma+1} \, dx \, dy \tag{1}$$

where l, m, h are arbitrary constants and α, β, γ are positive integers; including zero.

LEMMA 1. The structure product $II_{\Pi_{xy}}^{\alpha,\beta,\gamma}$ over a simple polygon with N-oriented edges $l_{i,k}$ $(i = 1, 2, 3, \ldots, N)$ each with end points at (x_i, y_i) and (x_k, y_k), $(k = i + 1)$ and $(x_N, x_N) = (x_1, y_1)$ *in the xy-plane is expressible as*

$$II_{\Pi_{xy}}^{\alpha,\beta,\gamma+1} = \sum_{i=1}^{N} \left[A_{iok}^{xy} \sum_{n=0}^{\alpha+\beta+\gamma+1} \sum_{n_1+n_2+n_3=n} F(\alpha - n_1, n_1) G(\beta - n_2, n_2) H(\gamma + 1 - n_3, n_3) \right] \tag{1a}$$

where

$$F(\alpha - n_1, n_1) = \binom{\alpha}{n_1} x_i^{\alpha - n_1} x_{k_i}^{n_1}$$

$$G(\beta - n_2, n_2) = \binom{\beta}{n_2} y_i^{\beta - n_2} y_{k_i}^{n_2}$$

$$H(\gamma + 1 - n_3, n_3) = \begin{cases} \binom{\gamma+1}{n_3} z_i^{\gamma+1-n_3} z_{k_i}^{n_3}, & z_0 = h = 0 \\ \left[\sum_{p=n_3}^{\gamma+1} \dfrac{\binom{\gamma+1}{p} z_0^{\gamma+1}}{(\alpha + \beta + p + 2)} \binom{p}{n_3} Z_{io}^{p-n_3} \right] z_{ki}^{n_3} \end{cases} \tag{1b}$$

$$(0 \leq \alpha \leq n_1, 0 \leq \beta \leq n_2, 0 \leq \gamma + 1 \leq n_3)$$

$$A_{iok}^{xy} = \begin{cases} \dfrac{2\Delta_{iok}^{xy}}{(\alpha + \beta + \gamma + 3)}, & \text{if } h = z_0 = 0 \\ 2\Delta_{iok}^{xy} & \text{if } h = z_0 \neq 0 \end{cases}$$

$$2\Delta_{iok}^{xy} = x_k y_i - x_i y_k$$

PROOF. Let us consider the integral of Eq. (1):

$$
\begin{aligned}
II^{\alpha,\beta,\gamma+1}_{\pi_{xy}} &\overset{\text{def}}{=} \int\int_{\pi_{xy}} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1}\, \mathrm{d}x\, \mathrm{d}y \\
&= \int\int_{\pi_{xy}} \frac{\partial \Phi(x, y)}{\partial x}\, \mathrm{d}x\, \mathrm{d}y
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(x, y) &= \int x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1}\, \mathrm{d}x \\
&= \int_{\partial \pi_{xy}} \Phi(x, y)\, \mathrm{d}y\,, \quad \text{on using Green's theorem.} \\
&= \sum_{i=1}^{N} \int_{l_{i,k}} \Phi(x, y)\, \mathrm{d}y
\end{aligned}
\tag{2}
$$

(where $\partial \pi_{xy}$ refers to boundary of π_{xy}).

We shall now show that

$$
II^{\alpha,\beta,\gamma+1}_{\pi_{xy}} = \sum_{i=1}^{N} \int\int_{T^{xy}_{iok}} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1}\, \mathrm{d}x\, \mathrm{d}y \tag{3}
$$

where T^{xy}_{iok} refers to a triangle in the xy plane with vertices at (x_i, y_i), $(0, 0)$ and (x_k, y_k).

We can think of π_{xy} as a region in R^2 decomposable in a set T of triangles such that any pair of members T^{xy}_{ijk} (a triangle in the xy plane with vertices at (xi, yi), (xj, yj)), and $T^{xy}_{i',j',k'}$ do not intersect.

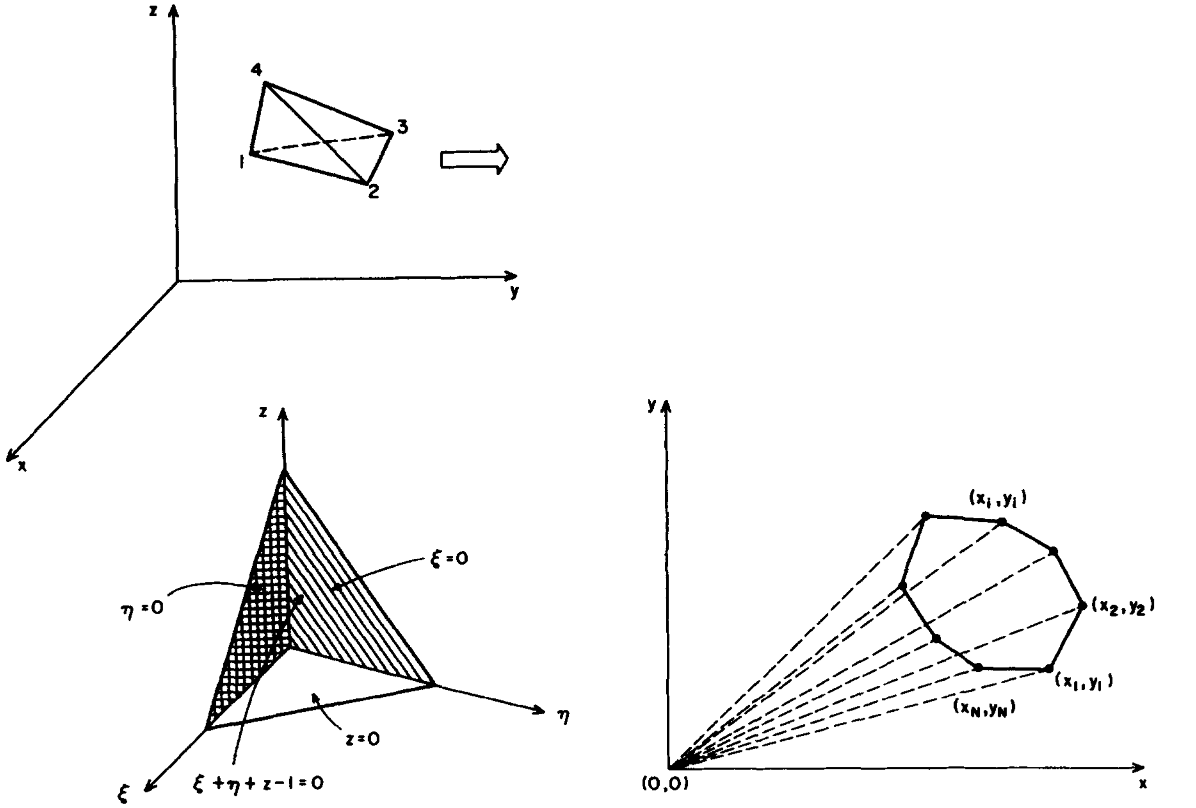

Fig. 1. (a) Three-dimensional mapping of an arbitrary linear tetrahedron in xyz-space into a unit orthogonal tetrahedron in $\xi\eta z$-space; (b) a simple polygon Π_{xy} in the xy-plane with N-oriented edges which expands into N-triangles with respect to the origin.

Thus, we may also write

$$II^{\alpha,\beta,\gamma+1}_{\pi_{xy}} = \iint_{\pi_{xy}} x^\alpha y^\beta (h + lx + my)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y = \sum_{T^{xy}_{ijk} \in T} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}} \tag{4}$$

Let us now prove the result of Eq. (3) when $\pi_{xy} = T^{xy}_{ijk}$ clearly the edges are l_{ij}, l_{jk}, l_{ik} (i.e. l_{pq} refers to the edges of triangle joining (x_p, y_p and x_q, y_q). We have (see Fig. 2)

$$\begin{aligned}
&\left(\iint_{T^{xy}_{joi}} + \iint_{T^{xy}_{koj}} + \iint_{T^{xy}_{iok}}\right) x^\alpha y^\beta (h + lx + my)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y \\
&= \left(\int_{\partial T^{xy}_{joi}} + \int_{\partial T^{xy}_{koj}} + \int_{\partial T^{xy}_{iok}}\right) \Phi(x, y)\,\mathrm{d}y \\
&= \left(\int_{l_{jo}} + \int_{l_{oi}} + \int_{l_{ij}}\right) \Phi(x, y)\,\mathrm{d}y \\
&\quad + \left(\int_{l_{ko}} + \int_{l_{oj}} + \int_{l_{jk}}\right) \Phi(x, y)\,\mathrm{d}y + \left(\int_{l_{io}} + \int_{l_{ok}} + \int_{l_{ki}}\right) \Phi(x, y)\,\mathrm{d}y \\
&= \int_{\partial T^{xy}_{ijk}} \Phi(x, y)\,\mathrm{d}y = \iint_{T^{xy}_{ijk}} x^\alpha y^\beta (h + lx + my)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y \\
&= II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}}
\end{aligned} \tag{5}$$

This proves Eq. (3) when $\pi_{xy} = T^{xy}_{ijk}$.

The general result of Eq. (3) can be readily proved on similar lines. This completes the proof of Eq. (3).

Now, let us consider the integral,

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}} = \iint_{T^{xy}_{ijk}} x^\alpha y^\beta (h + lx + my)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

The parametric equation of the oriented triangle T^{xy}_{ijk} in the xy plane with vertices at (x_i, y_i), (x_j, y_j) and (x_k, y_k) are

$$\begin{aligned} x &= x_i + x_{ji}u + x_{ki}v \\ y &= y_i + y_{ji}u + y_{ki}v \end{aligned} \tag{7}$$

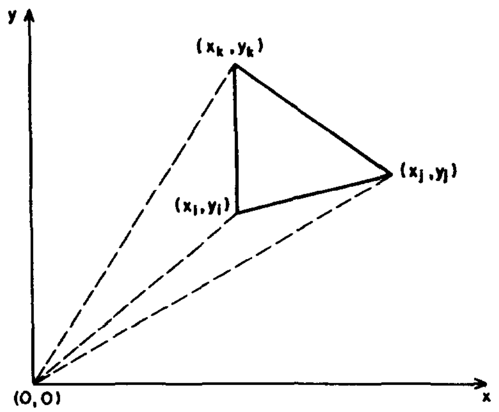

Fig. 2. A linear triangle T^{xy}_{ijk} with vertices at (x_i, y_i), (x_j, y_j) and (x_k, y_k) in xy-plane which expands into three new triangles with respect to the origin.

where

$$0 \leqslant u, v \leqslant 1\,, \qquad u + v \leqslant 1$$
$$x_{ji} = x_j - x_i\,, \qquad x_{ki} = x_k - x_i$$

Using Eq. (7), we can map an oriented triangle T^{xy}_{ijk} in the xy-plane to a unit triangle in the uv plane (see Fig. 1).

$$\begin{aligned} \mathrm{d}x\,\mathrm{d}y = \frac{\partial(x, y)}{\partial(u, v)}\,\mathrm{d}u\,\mathrm{d}v &= (x_{ji}y_{ki} - x_{ki}y_{ji})\,\mathrm{d}u\,\mathrm{d}v \\ &= 2\Delta^{xy}_{ijk}\,\mathrm{d}u\,\mathrm{d}v \\ &= (2 \times \text{area of triangle } T^{xy}_{ijk})\,\mathrm{d}u\,\mathrm{d}v \end{aligned} \tag{8}$$

and we define

$$2\Delta^{xy}_{ijk} = x_{ji}y_{ki} - x_{ki}y_{ji} \tag{9}$$

Use of Eqs. (7), (8) and (9) into Eq. (6) gives us

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}} = (2\Delta^{xy}_{ijk}) \int_0^1 \int_0^{1-u} [x_i + x_{ji}u + x_{ki}v]^{\alpha}[y_i + y_{ji}u + y_{ki}v]^{\beta} \times [z_i + z_{ji}u + z_{ki}v]^{\gamma+1}\,\mathrm{d}u\,\mathrm{d}v \tag{10a}$$

where

$$z = z(x, y) = h + 1x + my \tag{10b}$$

Let us further use the transformation in Eq. (10)

$$u = 1 - r\,, \qquad v = rs \tag{11}$$

Use of Eq. (11) into Eq. (10) gives us

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}} = 2\Delta^{xy}_{ijk} \int_0^1 \int_0^1 [x_i + x_{ij} \cdot \gamma + x_{ki}rs]^{\alpha}[y_i + y_{ij}r + y_{ki}rs]^{\beta} \times [z_i + z_{ij}r + z_{ki}rs]^{\gamma+1} r\,\mathrm{d}r\,\mathrm{d}s \tag{12}$$

We have defined $z(x, y) = h + lx + my$.

Clearly, $z(0, 0) = z_0$ (say) $= h$ and z_0 can be either zero or nonzero. Choosing $x_j = 0$, $y_j = 0$, $z_j = z_0$ and recalling:

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{iok}} \overset{\text{def}}{=} \int\!\!\int_{T^{xy}_{iok}} x^{\alpha}y^{\beta}(lx + my + h)^{\gamma+1} \tag{13}$$

where T^{xy}_{iok}—a triangle in the xy-plane with vertices at (x_i, y_i), $(0, 0)$ and (x_k, y_k). We have now from Eq. (12) and the above explanations:

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{iok}} = (2\Delta^{xy}_{iok}) \int_0^1 \int_0^1 r^{\alpha+\beta+1}[x_i + x_{ki}r]^{\alpha}[y_i + y_{ki}s]^{\beta} \times [z_0 + r(z_{io} + z_{kis})]^{\gamma+1}\,\mathrm{d}r\,\mathrm{d}s \tag{14}$$

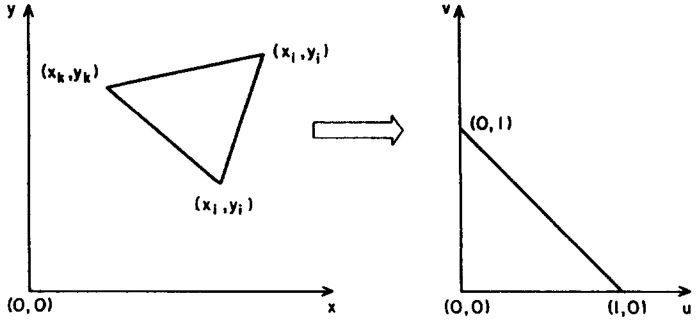

Fig. 3. The mapping between an oriented triangle in the xy-plane and the unit triangle in the uv-plane.

If $z_0 = 0$, Eq. (14) reduces to

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{iok}} = \frac{2\Delta^{xy}_{iok}}{(\alpha+\beta+\gamma+3)} \int_0^1 (x_i + x_{ki}s)^\alpha (y_i + y_{ki}s)^\beta (z_i + z_{ki}s)^{\gamma+1}\, ds \tag{15}$$

If $z_0 \neq 0$, Eq. (14) reduces to

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{iok}} = (2\Delta^{xy}_{iok}) \int_0^1 (x_i + x_{ki}s)^\alpha (y_i + y_{ki}s)^\beta \left\{ \sum_{p=0}^{\gamma+1} \frac{p^{\gamma+1} z_0^{\gamma+1-p}}{(\alpha+\beta+p+2)} (z_{io} + z_{ki}s)^\alpha \right\} ds \tag{16}$$

Let us define

$$\begin{aligned}
X(s) &= (x_i + x_{ki}s)^\alpha \\
Y(s) &= (y_i + y_{ki}s)^\beta \\
Z(s) &= \begin{cases} (z_{io} + z_{ki}s)^{\gamma+1}\,, & \text{if } z_o = 0 \\ \displaystyle\sum_{p=0}^{\gamma+1} \frac{(p^{\gamma+1}) z_0^{\gamma+1-p}}{(\alpha+\beta+p+2)} (z_{io} + z_{ki}s)^p\,, & \text{if } z_0 \neq 0 \end{cases} \\
f(s) &= X(s)Y(s)Z(s)
\end{aligned} \tag{17}$$

Using Eq. (17), we can write Eq. (15), and Eq. (16) as

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{iok}} = \begin{cases} \dfrac{2\Delta^{xy}_{iok}}{(\alpha+\beta+\gamma+3)} \displaystyle\int_0^1 f(s)\, ds\,, & z_0 = 0 \\ 2\Delta^{xy}_{iok} \displaystyle\int_0^1 f(s)\, ds\,, & z_0 \neq 0 \end{cases} \tag{18}$$

where we have to note that $f(s)$ is defined as in Eq. (17) for $zo = 0$ and $z_0 \neq 0$. Using Taylor's series expansion of a function of a single variable and from Eq. (17),

$$f(s) = \sum_{n=0}^{\alpha+\beta+\gamma+1} \left\{ \frac{f(0)^{(n)}}{\underline{|n}} \right\} s^n \tag{19}$$

where

$$f^{(n)}(0) = \left[\frac{d^n}{ds^n} \{X(s) \quad Y(s) \quad Z(s)\} \right] s = 0 \tag{20}$$

We have now on using the generalised form of the Leibnitz's theorem of differential calculus:

$$f^{(n)}(0) = \sum_{n_1+n_2+n_3=n} \frac{\underline{|n}}{\underline{|n_1}\,\underline{|n_2}\,\underline{|n_3}} \left\{ \frac{d_x^{n_1}(s)}{ds^{n_1}} \right\} s = 0\,, \quad \left\{ \frac{dy^{n_2}(s)}{ds^{n_2}} \right\} s = 0\,, \quad \left\{ \frac{dz^{n_3}(s)}{ds^{n_3}} \right\} s = 0 \tag{21}$$

Now, referring Eq. (17) on the definition of $X(s)$, $Y(s)$, $Z(s)$, we obtain

$$\frac{\left[\dfrac{d^{n_1} x(s)}{ds^{n_1}} \right]_{s=0}}{\underline{|n_1}} = \binom{\alpha}{n_1} x_i^{\alpha-n_1} x_{ki}^{n_i} \overset{\text{def}}{=} F(\alpha - n_1, n_1)$$

$$\frac{\left[\dfrac{d^{n_2} y(s)}{ds^{n_2}} \right]_{s=0}}{\underline{|n_2}} = \binom{\beta}{n_2} y_i^{\beta-n_2} y_{ki}^{n_2} \overset{\text{def}}{=} G(\beta - n_2, n_2)$$

and

$$\frac{\left[\frac{\mathrm{d}^{n_3} z(s)}{\mathrm{d}s^{n_3}}\right]_{s=0}}{\underline{|n_3}} = \begin{cases} \binom{\gamma+1}{n_3} z_i^{\gamma+1-n_3} z_{ki}^{n_3} & \text{for } z_0 = 0 \\ \left\{ \sum_{p=n_3}^{\gamma+1} \frac{\binom{\gamma+1}{p} z_0^{\gamma+1-p}}{(\alpha+\beta+p+2)} \binom{p}{n_3} z_{io}^{p-n_3} \right\} z_{ki}^{n_3} & \text{for } z_o \neq 0 \end{cases}$$

$$\stackrel{\text{def}}{=} H(\gamma+1-n_3, n_3) \tag{22}$$

from Eqs. (21) and (22), we have

$$\frac{f^{(n)}(0)}{\underline{|n}} = \sum_{n_1+n_2+n_3=n} F(\alpha-n_1, n_1) \quad G(\beta-n_2, n_2) \quad H(\gamma+1-n_3, n_3) \tag{23}$$

Thus, from Eqs. (19) and (23), we obtain

$$\int_0^1 f(s)\,\mathrm{d}s = \sum_{n=0}^{\alpha+\beta+\gamma+1} \frac{1}{n+1} \left\{ \sum_{n_1+n_2+n_3=n} F(\alpha-n_1, n_1) G(\beta-n_2, n_2) \quad H(\gamma+1-n_3, n_3) \right\} \tag{24}$$

Combining Eqs. (18) and (23) and substituting in Eq. (3) we obtain the desired result claimed in Eq. (1a,b). This completes the proof of Lemma 1. □

Let π_{yz} be a simple polygon in the yz-plane and we define the structure product:

$$II_{\pi_{yz}} \stackrel{\text{def}}{=} \int\int_{\pi_{yz}} (h' + l'y + m'z)^{\alpha+1} y^{\beta} z^{\gamma} \,\mathrm{d}y\,\mathrm{d}z$$

where l', m', h' are arbitrary constants and α, β, γ are positive integers including zero.

LEMMA 2. The structure product $II_{\pi_{yz}}^{\alpha+1,p,\gamma}$ over a simple polygon with N-oriented edges l_{ik} $(i = 1, 2, \ldots, N)$ each with end points at (y_i, z_i) and (y_k, z_k) $(k = i+1)$ and $(y_N, z_N) = (y_1, z_1)$ in the yz-plane is expressible as

$$II_{\pi_{yz}}^{\alpha+1,p,\gamma} = \sum_{i=1}^{N} \left[A_{iok}^{yz} \sum_{n=0}^{\alpha+\beta+\gamma+1} \sum_{n_1+n_2+n_3=n} F(\alpha+1-n_1, n_1) G(\beta-n_2, n_2) H(\gamma-n_3, n_3) \right] \tag{25a}$$

where

$$F(\alpha+1-n_1, n_1) = \binom{\alpha+1}{n_1} x_i^{\alpha+1-n_1} x_{k_i}^{n_1}, \qquad x_0 = h' = 0$$

$$\left[\sum_{p=n_1}^{\alpha+1} \frac{\binom{\alpha+1}{p} x_0^{\alpha+1-p}}{(\beta+\alpha+p+2)} {p \atop n_1} x_{io}^{p-n_1} \right] x_{k_i}^{n_1}, \qquad x_0 = h' \neq 0$$

$$G(\beta-n_2, n_2) = \binom{\beta}{n_2} y_i^{\beta-n_2} y_{k_i}^{n_2}$$

$$H(\gamma-n_3, n_3) = \binom{\gamma}{n_3} z_i^{\gamma-n_3} z_{k_i}^{n_3}$$

$$(0 \leqslant n_1 \leqslant \alpha+1, 0 \leqslant n_2 \leqslant \beta, 0 \leqslant n_3 \leqslant \gamma)$$

$$A_{iok}^{yz} = \begin{cases} \frac{2\Delta_{iok}^{yz}}{(\alpha+\beta+\gamma+3)}, & \text{if } h' = x_0 = 0 \\ 2\Delta_{iok}^{yz}, & \text{if } h' = x_0 \neq 0 \end{cases}$$

$$2\Delta_{iok}^{yz} = y_k z_i - y_i z_k \tag{25b}$$

PROOF. Proof can follow on similar lines as in Lemma 1. □

Let us now define π_{zx} be a simple polygon in the zx-plane and we define the structure product:

$$II_{\pi_{zx}}^{\alpha,\beta+1\gamma} \stackrel{\text{def}}{=} \int\int_{\pi_{zx}} x^{\alpha}(h''+l''y+m''z)^{\beta+1}z^{\gamma}\,dz\,dx$$

where l'', m'', h'' are arbitrary constants and α, β, γ are positive integers including zero.

LEMMA 3. The structure product $II_{\pi_{zx}}^{\alpha,\beta+1,\gamma}$ over a simple polygon with N-oriented edges l_{ik} $(i=1,2,\ldots,N)$ each with end points at (z_i,x_i) and (z_k,x_k), $(k=i+1)$ and $(z_N,x_N)=(y_1,z_1)$ in the zx-plane is expressible as

$$II_{\pi_{zx}}^{\alpha,\beta+1,\gamma} = \sum_{i=1}^{N}\left[A_{iok}^{zx}\sum_{n=0}^{\alpha+\beta+\gamma+1}\ \sum_{n_1+n_2+n_3=n} F(\alpha-n_1,n_1)G(\beta+1-n_2,n_2)H(\gamma-n_3,n_3)\right] \tag{26a}$$

where

$$\begin{aligned}
&F(\alpha-n_1,n_1)=\binom{\alpha}{n_1}x_i^{\alpha-n_1}x_{ki}^{n_1}\,,\\
&G(\beta+1-n_2,n_2)=\binom{\beta+1}{n_2}y_i^{\beta+1-n_2}y_{k_i}^{n_2}\,, \qquad y_0=h''=0\\
&\qquad\left[\sum_{p=n_2}^{\beta+1}\frac{\binom{\beta+1}{p}y_0^{\beta+1-p}}{(\beta+\alpha+p+2)}\binom{p}{n_2}y_{io}^{p-n_2}\right]y_{k_i}^{n_2}\,, \qquad y_0=h''\neq 0\\
&H(\gamma-n_3,n_3)=\binom{\gamma}{n_3}z_i^{\gamma-n_3}Z_{k_i}^{n_3}\\
&\qquad(0\leq n_1\leq\alpha,\ 0\leq n_2\leq\beta+1,\ 0\leq n_3\leq\gamma)\\
&A_{iok}^{zx}=\begin{cases}\dfrac{2\Delta_{iok}^{zx}}{(\alpha+\beta+\gamma+3)}\,, & \text{if } h''=y_o=0\\ 2\Delta_{iok}^{zx}\,, & \text{if } h''=y_o\neq 0\end{cases}\\
&2\Delta_{iok}^{zx}=z_kx_i-z_ix_k
\end{aligned} \tag{26b}$$

PROOF. Proof can be developed on similar lines as in Lemma 1. □

3. Volume integration over an arbitrary tetrahedron

Referring to Rathod and Govinda Rao [10,11] we can obtain the volume integral of a monomial $x^{\alpha}\,y^{\beta}\,z^{\gamma}(\alpha,\beta,\gamma)$ positive integers and zero) over an arbitrary tetrahedron in Euclidean three-dimensional space. Let us define structure product over an arbitrary tetrahedron T as

$$III_T^{\alpha,\beta,\gamma} \stackrel{\text{def}}{=} \int\int\int_T x^{\alpha}y^{\beta}z^{\gamma}\,dx\,dy\,dz \tag{27}$$

THEOREM 1. Let $T_{1,2,3,4}$ be an arbitrary linear tetrahedron in three-dimensional space with vertices at (x_p,y_p,z_p) $(p=1,2,3,4)$ bounded by the tetrahedron surface consisting of four arbitrary $T_{i,j,k}$ with vertices at (x_i,y_i,z_i), (x_j,y_j,z_j) and (x_k,y_k,z_k) with $(i,j,k)\in\{(1,2,3),(4,3,2),(4,1,2),(4,2,1)\}$, then the structure product $III_{T_{1,2,3,4}}^{\alpha,\beta,\gamma}$ over a linear tetrahedron $T_{1,2,3,4}$ is pressible as

$$III_{T_{1,2,3,4}}^{\alpha,\beta,\gamma}=\frac{\Omega(1,2,3,4)}{(\gamma+1)}\left[II_{T_{1,2,3}^{xy}}^{\alpha,\beta,\gamma+1}+II_{T_{4,3,2}^{xy}}^{\alpha,\beta,\gamma+1}+II_{T_{4,1,3}^{xy}}^{\alpha,\beta,\gamma+1}+II_{T_{4,2,1}^{xy}}^{\alpha,\beta,\gamma+1}\right] \tag{27a}$$

where

$$\begin{aligned} &\Omega(1,2,3,4) = \frac{|\det J|}{(\det J)} \\ &\det J = \begin{vmatrix} x_{14} & x_{24} & x_{34} \\ y_{14} & y_{24} & y_{34} \\ z_{14} & z_{24} & z_{34} \end{vmatrix} \\ &x_{pq} = x_p - x_q\,, \qquad y_{pq} = y_p - y_q\,, \qquad z_{pq} = z_p - z_q \\ &(p, q) \in \{(1,4), (2,4), (3,4)\} \end{aligned} \tag{27b}$$

PROOF. For the three-dimensional tetrahedron $T_{1,2,3,4}$ bounded by tetrahedral surface $\partial T_{1,2,3,4}$ consisting of arbitrary triangles $T_{1,2,3}$, $T_{4,3,2}$, $T_{4,1,3}$ and $T_{4,2,1}$ in three space, let

$$\hat{\psi}(x, y, z) = \left(0, 0, \frac{x^\alpha y^\beta z^{\gamma+1}}{\gamma + 1}\right)$$

be a vector field. Then, clearly $x^\alpha y^\beta z^\gamma = \hat{\nabla}\hat{\psi}$ and if we now further assume the regularity of the integration domain and continuity of the integration function we can then write from the Gauss's divergence theorem:

$$\begin{aligned} \iiint_{T_{1,2,3,4}} x^\alpha y^\beta z^\gamma \, dx\, dy\, dz &= \iiint_{T_{1,2,3,4}} \hat{\nabla} \cdot \hat{\psi} \, dx\, dy\, dz \\ &= \iint_{\partial T_{1,2,3,4}} \hat{\psi} \cdot \hat{n} \, ds \end{aligned}$$

(where $\hat{n}$ is the outward unit normal vector to $\partial T_{1,2,3,4}$)

$$\begin{aligned} = \frac{1}{(\gamma + 1)} \Bigg[&\iint_{T_{1,2,3}} x^\alpha y^\beta z^{\gamma+1} \hat{k} \cdot \hat{n}_1 \, ds + \iint_{T_{4,3,2}} x^\alpha y \beta z^{\gamma+1} \hat{k} \cdot \hat{n}_2 \, ds \\ &+ \iint_{T_{4,1,3}} x^\alpha y^\beta z^{\gamma+1} \hat{k} \cdot \hat{n}_3 \, ds + \iint_{T_{4,2,1}} x^\alpha \gamma^\beta z^{\gamma+1} \hat{k} \cdot \hat{n}_4 \, ds \Bigg] \end{aligned} \tag{28a}$$

where

$$\begin{aligned} &\hat{k} \cdot n_i \, ds = \begin{cases} \dfrac{\hat{k} \cdot \hat{n}_i}{|\hat{k} \cdot \hat{n}_i|} \hat{k} \cdot n_i| \, ds\,, & \text{if } |\hat{k} \cdot \hat{n}_i| \neq 0 \\ 0\,, & \text{if } \hat{k} \cdot \hat{n}_i = 0 \end{cases} \\ &|\hat{k} \cdot \hat{n}_i| \, ds = \cos\gamma \, ds = dx\, dy \end{aligned} \tag{28b}$$

γ = angle between the normal $\hat{n}_i$ and the positive z-axis

$$\frac{\hat{k} \cdot \hat{n}_i}{|\hat{k} \cdot \hat{n}_i|} = +1 \quad \text{or } -1 \quad \square$$

Now, we shall give an alternative proof which can help us in determining $\hat{k} \cdot \hat{n}_1$, $\hat{k} \cdot \hat{n}_2$, $\hat{k} \cdot \hat{n}_3$, $\hat{k} \cdot \hat{n}_4$.

PROOF. We can write

$$\begin{aligned} III^{\alpha\beta\gamma} &\overset{\text{def}}{=} \iiint_{T_{1,2,3,4}} x^\alpha y^\beta z^\gamma \, dx\, dy\, dz \\ &= \iiint_{T_{1,2,3,4}} \frac{\partial}{\partial z} \left\{ \frac{x^\alpha y^\beta z^{\gamma+1}}{\gamma + 1} \right\} dx\, dy\, dz \end{aligned} \tag{29}$$

We shall now make use of the isoparametric co-ordinates transformation in Eq. (29)

$$\begin{aligned} x &= x(\xi, \eta, \zeta) = (\xi x_1 + \eta x_2 + \zeta x_3) + (1 - \xi - \eta - \zeta)x_4 \\ y &= y(\xi, \eta, \zeta) = (\xi y_1 + \eta y_2 + \zeta y_3) + (1 - \xi - \eta - \zeta)y_4 \\ z &= z(\xi, \eta, \zeta) = (\xi z_1 + \eta z_2 + \zeta z_3) + (1 - \xi - \eta - \zeta)z_4 \end{aligned} \tag{30}$$

which maps an arbitrary tetrahedron $T_{1,2,3,4}$ into a unit orthogonal tetrahedron $\bar{T}$ with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).

We should note that $\langle(x_4, y_4, z_4), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\rangle$ are, respectively, mapped to the points (see Fig. 1a)

$$\begin{aligned} &\langle(\xi_4 = 0, \eta_4 = 0, \zeta_4 = 0), (\xi_1 = 1, \eta_1 = 0, \zeta_1 = 0) \\ &(\xi_2 = 0, \eta_2 = 1, \zeta_2 = 0), (\xi_3 = 0, \eta_3 = 0, \zeta_3 = 1)\rangle \end{aligned}$$

Now, using the chain rule on partial differentiation and the Gauss's divergence theorem, we can express Eq. (29) as

$$\begin{aligned} III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} &= \frac{|\det(J_{1,2,3,4})|}{(\gamma+1)\det(J_{1,2,3,4})} \int\int\int_{\bar{T}_{1,2,3,4}} \left[\frac{\partial}{\partial\xi}\left\{x^\alpha y^\beta z^{\gamma+1}\frac{\partial(x, y)}{\partial(\eta, \zeta)}\right\}\right. \\ &\quad \left. + \frac{\partial}{\partial\eta}\left\{-x^\alpha y^\beta z^{\gamma+1}\frac{\partial(x, y)}{\partial(\eta, \zeta)}\right\} + \frac{\partial}{\partial\zeta}\left\{x^\alpha y^\beta z^{\gamma+1}\frac{\partial(x, y)}{\partial(\eta, \zeta)}\right\}\right] \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \\ &= \frac{1}{(\gamma+1)}\Omega(1, 2, 3, 4)\int\int_{\bar{T}_{1,2,3,4}}\int \hat{\nabla}\hat{F}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \\ &= \frac{1}{(\gamma+1)}\Omega(1, 2, 3, 4)\int\int_{\bar{S}} \hat{F}\cdot\hat{n}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \end{aligned} \tag{31}$$

where $\bar{S}$ is the surface of the unit orthogonal tetrahedron spanned by the vertices 1, 2, 3, 4 at $\langle(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\rangle$ and $\hat{n}$ is the unit normal vector pointing outward to $\bar{T}_{1,2,3,4}$

$$\begin{aligned} \hat{F} &= (\hat{F}_1, \hat{F}_2, \hat{F}_3) \\ F_1 &= x^\alpha y^\beta z^{\gamma+1}\frac{\partial(x, y)}{\partial(\eta, \zeta)} \\ F_2 &= -x^\alpha y^\beta z^{\gamma+1}\frac{\partial(x, y)}{\partial(\xi, \zeta)} \\ F_3 &= x^\alpha y^\beta z^{\gamma+1}\frac{\partial'(x, y)}{\partial(\xi, \eta)} \end{aligned} \tag{32}$$

Clearly, from Eq. (30) we find

$$\begin{aligned} \frac{\partial(x, y)}{\partial(\eta, \zeta)} &= 2\Delta^{xy}_{4,2,3} = \begin{vmatrix} x_{2,4} & x_{3,4} \\ y_{2,4} & y_{3,4} \end{vmatrix} \\ \frac{\partial(x, y)}{\partial(\xi, \zeta)} &= 2\Delta^{xy}_{4,1,3} = \begin{vmatrix} x_{1,4} & x_{3,4} \\ y_{1,4} & y_{3,4} \end{vmatrix} \\ \frac{\partial(x, y)}{\partial(\xi, \eta)} &= 2\Delta^{xy}_{4,1,2} = \begin{vmatrix} x_{1,4} & x_{2,4} \\ y_{1,4} & y_{2,4} \end{vmatrix} \end{aligned} \tag{33}$$

In order to obtain a working relationship of Eq. (3), let us examine the surface integral $\int\int_{\bar{S}} \hat{F}\cdot\hat{n}\, \mathrm{d}\bar{s}$, now clearly from Fig. 1, $\bar{S}$ consists of triangular surface $\bar{S}_m$ $(m = 1, 2, 3, 4)$.

We define surfaces $\bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4$ as the triangular surfaces of $\bar{T}_{1,2,3,4}$ spanned by vertices 1, 2, 3; 2, 3, 4; 1, 3, 4; and 1, 2, 4, respectively.

Thus, we may now write

$$III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1, 2, 3, 4)}{(\gamma+1)}\sum_{m=1}^{4}\int\int_{\bar{S}_m} \hat{F}\cdot\hat{n}_m\, \mathrm{d}\bar{s}_m \tag{34}$$

where $\hat{n}_m$ $(m = 1, 2, 3, 4)$ are the outward pointing unit normals to $\bar{S}_m$ $(m = 1, 2, 3, 4)$, respectively.

By considering the projection of $\bar{S}_1$ on ξ, η-plane and noting the equation of the surface $\bar{S}$: $\xi + \eta + \zeta - 1 = 0$, we obtain

$$\iint_{\bar{S}_1} \hat{F} \cdot \hat{n}_1 \, d\bar{s}_1 = \int_0^1 \int_0^{1-\xi} \left(\sum_{m=1}^{3} F_m(\xi, \eta, 1 - \xi - \eta) \right) d\xi \, d\eta \tag{35}$$

$$\iint_{\bar{S}_2} \hat{F} \cdot \hat{n}_2 \, d\bar{s}_2 = - \int_0^1 \int_0^{1-\eta} F_1(0, \eta, \zeta) \, d\eta \, d\zeta \tag{36}$$

$$\iint_{\bar{S}_3} \hat{F} \cdot \hat{n}_3 \, d\bar{s}_3 = - \int_0^1 \int_0^{1-\xi} F_2(\xi, 0, \eta) \, d\xi \, d\zeta \tag{37}$$

$$\iint_{\bar{S}_4} \hat{F} \cdot \hat{n}_4 \, d\bar{s}_4 = - \int_0^1 \int_0^{1-\zeta} F_2(\xi, \eta, 0) \, d\xi \, d\eta \tag{37a}$$

Substituting from Eqs. (35)–(37a) in Eq. (34) we obtain

$$\begin{aligned} III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1,2,3,4)}{(\gamma+1)} \Bigg[& \int_0^1 \int_0^{1-\xi} \left(\sum_{m=1}^{3} F_m(\xi, \eta, 1 - \xi - \eta) \right) d\xi \, d\eta \\ & - \int_0^1 \int_0^{1-\eta} F_1(0, \eta, \zeta) \, d\eta \, d\zeta - \int_0^1 \int_0^{1-\xi} F_2(\xi, 0, \zeta) \, d\xi \, d\zeta \\ & - \int_0^1 \int_0^{1-\xi} F_3(\xi, \eta, 0) \, d\xi \, d\eta \Bigg] \end{aligned} \tag{38}$$

Simplification of Eq. (38) leads us to the results:

$$\begin{aligned} III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1,2,3,4)}{(\gamma+1)} & [(2\Delta^{xy}_{312}) II^{\alpha,\beta,\gamma+1}_{\bar{T}_{312}} \\ & - (2\Delta^{xy}_{423}) II^{\alpha,\beta,\gamma+1}_{\bar{T}_{423}} \\ & + (2\Delta^{xy}_{413}) II^{\alpha,\beta,\gamma+1}_{\bar{T}_{413}} - (2\Delta^{xy}_{412}) II^{\alpha,\beta,\gamma+1}_{\bar{T}_{412}}] \end{aligned} \tag{39}$$

where

$$\begin{aligned} & II^{\alpha,\beta,\gamma+1}_{\bar{T}_{pqr}} = \int_0^1 \int_0^{1-u} x^\alpha(u,v) y^\beta(u,v) z^{\gamma+1}(u,v) \, du \, dv \\ & x = x(u,v) = u x_p + v x_q + (1 - u - v) x_r \\ & y = y(u,v) = u y_p + v y_q + (1 - u - v) y_r \\ & z = z(u,v) = u z_p + v z_q + (1 - u - v) z_r \\ & (p, q, r) \in \{(3,1,2), (4,2,3), (4,1,3), (4,1,2)\} \end{aligned} \tag{40}$$

and $\bar{T}pqr$ refers to the unit triangle in the uv plane with vertices $(p(0,0), q(1,0)$ and $r(0,1))$.

Using the property of integrals, it can be shown that integrals

$$\begin{aligned} II^{\alpha,\beta,\gamma+1}_{\bar{T}_{pqr}} & = II^{\alpha,\beta,\gamma+1}_{\bar{T}_{qrp}} = II^{\alpha,\beta,\gamma+1}_{\bar{T}_{rpq}} = II^{\alpha,\beta,\gamma+1}_{\bar{T}_{qpr}} \\ & = II^{\alpha,\beta,\gamma+1}_{\bar{T}_{rqp}} = II^{\alpha,\beta,\gamma+1}_{\bar{T}_{prq}} \end{aligned} \tag{41}$$

We can also show that under the transformation (40)

$$z(u,v) = l x(u,v) + m y(u,v) + h \tag{42}$$

where l, m, h are arbitrary constants.

Let us now consider,

$$II^{\alpha,\beta,\gamma+1}_{T_{pqr}} \overset{\text{def}}{=} \int\int_{T^{xy}_{pqr}} x^{\alpha} y^{\beta} (lx + my + h)^{\gamma+1} \, \mathrm{d}x \, \mathrm{d}y \tag{43}$$

where T^{xy}_{pqr} is an arbitrary triangle with vertices (x_p, y_p), (x_q, y_q) and (x_r, y_r) in counter-clockwise direction.

Using Eqs. (40), (42) and (43) we can show that

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{pqr}} = \frac{\partial(x, y)}{\partial(u, v)} = II^{\alpha,\beta,\gamma+1}_{T_{pqr}} = 2\Delta^{xy}_{pqr} II^{\alpha,\beta,\gamma+1}_{T_{pqr}} \tag{44}$$

and

$$2\Delta^{xy}_{pqr} = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_{pr} & x_{qr} \\ y_{pr} & y_{qr} \end{vmatrix} > 0 \tag{45}$$

From the property of determinants we can show that

$$\begin{aligned} 2\Delta^{xy}_{pqr} &= 2\Delta^{xy}_{qrp} = 2\Delta^{xy}_{rpq} \\ &= -2\Delta^{xy}_{qpr} = -2\Delta^{xy}_{rpq} = -2\Delta^{xy}_{prq} \end{aligned} \tag{46}$$

Thus, we have from Eqs. (41), (44), (45) and (46)

$$\begin{aligned} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{pqr}} &= II^{\alpha,\beta,\gamma+1}_{T^{xy}_{qrp}} = II^{\alpha,\beta,\gamma+1}_{T^{xy}_{rpq}} \\ &= -II^{\alpha,\beta,\gamma+1}_{T^{xy}_{qpr}} = -II^{\alpha,\beta,\gamma+1}_{T^{xy}_{rpq}} = -II^{\alpha,\beta,\gamma+1}_{T^{xy}_{prq}} \end{aligned} \tag{47}$$

and

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{pqr}} = |2\Delta^{xy}_{pqr}| II^{\alpha,\beta,\gamma+1}_{T^{xy}_{pqr}} \tag{48}$$

Whenever p, q, r are in counter clockwise orientation using Eqs. (39) and (48) we have

$$III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1,2,3,4)}{(\gamma+1)} [\delta^{xy}_{312} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{3,1,2}} + \delta^{xy}_{432} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{4,3,2}} + \delta^{xy}_{413} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{4,1,3}} + \delta^{xy}_{421} II^{\alpha,\beta,\gamma+1}_{T^{xy}_{4,2,1}}] \tag{49}$$

where

$$\delta^{xy}_{pqr} = \begin{cases} \dfrac{2\Delta^{xy}_{pqr}}{|2\Delta^{xy}_{pqr}|}, & \text{if } 2\Delta^{xy}_{pqr} \neq 0 \\ 0, & \text{if } 2\Delta^{xy}_{pqr} = 0 \end{cases} \tag{50}$$

$$(p, q, r) \in \{(3,1,2), (4,3,2), (4,1,3), (4,2,1)\}$$

On comparing Eqs. (28a,b) and (49) we have

$$\begin{aligned} \Omega(1,2,3,4)\delta^{xy}_{312} &= \frac{\hat{k} \cdot \hat{n}_1}{|\hat{k} \cdot \hat{n}_1|} \\ \Omega(1,2,3,4)\delta^{xy}_{432} &= \frac{\hat{k} \cdot \hat{n}_2}{|\hat{k} \cdot \hat{n}_2|} \\ \Omega(1,2,3,4)\delta^{xy}_{413} &= \frac{\hat{k} \cdot \hat{n}_3}{|\hat{k} \cdot \hat{n}_3|} \\ \Omega(1,2,3,4)\delta^{xy}_{421} &= \frac{\hat{k} \cdot \hat{n}_4}{|\hat{k} \cdot \hat{n}_4|} \end{aligned} \tag{51}$$

The above theorem was clearly proved by considering projection of the arbitrary tetrahedron on the xy-plane. □

Similar results follow if we project the arbitrary tetrahedron on the yz and xz planes. This is further stated here in the form of Theorems 2 and 3.

THEOREM 2. Let $T_{1,2,3,4}$ be an arbitrary linear tetrahedron in three-dimensional space with vertices at $((x_p, y_p, z_p),\ p = 1, 2, 3, 4)$ bounded by the tetrahedral surface consisting of four arbitrary triangles T_{ijk} with vertices at $((x_t, y_t, z_t)t = i, j, k)$ and $(i, j, k) \in ((1, 2, 3)\ (4, 3, 2),\ (4, 1, 3),\ (4, 2, 1))$, then the structure product $III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}}$ over a linear tetrahedron $T_{1,2,3,4}$ is expressible as

$$III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1,2,3,4)}{(\alpha+1)}\left[II^{\alpha+1,\beta,\gamma}_{T^{yz}_{1,2,3}} + II^{\alpha+1,\beta,\gamma}_{T^{yz}_{4,3,2}} + II^{\alpha+1,\beta,\gamma}_{T^{yz}_{4,1,3}} + II^{\alpha+1,\beta,\gamma}_{T^{yz}_{4,2,1}}\right] \tag{52}$$

with Ω $(1, 2, 3, 4)$ as given in Eq. (27b).

THEOREM 3. Let $T_{1,2,3,4}$ be an arbitrary linear tetrahedron in three-dimensional space with vertices at $((x_p, y_p, z_p),\ p = 1, 2, 3, 4)$ bounded by the tetrahedral surface consisting of four arbitrary triangles T_{ijk} with vertices at $((x_t, y_t, z_t)t = i, j, k)$ and $(i, j, k) \in ((1, 2, 3)\ (4, 3, 2),\ (4, 1, 3),\ (4, 2, 1))$, then the structure product $III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}}$ over a linear tetrahedron $T_{1,2,3,4}$ is expressible as

$$III^{\alpha,\beta,\gamma}_{T_{1,2,3,4}} = \frac{\Omega(1,2,3,4)}{(\beta+1)}\left[II^{\alpha,\beta+1,\gamma}_{T^{zx}_{1,2,3}} + II^{\alpha,\beta+1,\gamma}_{T^{zx}_{4,3,2}} + II^{\alpha,\beta+1,\gamma}_{T^{zx}_{4,1,3}} + II^{\alpha,\beta+1,\gamma}_{T^{zx}_{4,2,1}}\right] \tag{53}$$

with $\Omega(1, 2, 3, 4)$ as given in Eq. (27b).

4. Application–example

We consider as an example, the evaluation of the volume integral:

$$\iiint_{T_{1,2,3,4}} x^2 y \, dx \, dy \, dz \tag{54}$$

where $T_{1,2,3,4}$ is the tetrahedron in R^3 bounded by tetrahedron surface $\partial T_{1,2,3,4}$ with vertices

$$\begin{aligned} v_1 &= (x_1, y_1, z_1) = (5, 5, 0) \\ v_2 &= (x_2, y_2, z_2) = (10, 10, 0) \\ v_3 &= (x_3, y_3, z_3) = (8, 7, 8) \end{aligned} \tag{55}$$

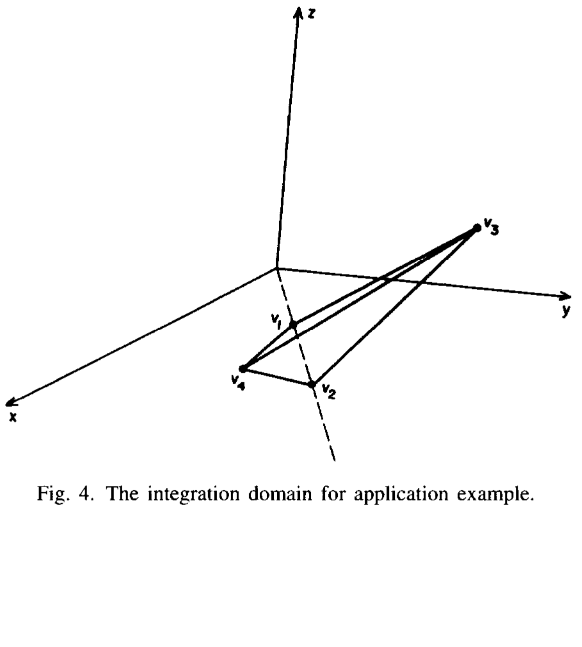

Fig. 4. The integration domain for application example.

$v_4 = (x_4, y_4, z_4) = (10, 5, 0)$

ALGORITHM 1. We can show that on using the concepts developed in Rathod and Govinda Rao [10,11] and Theorem 1. Eq. (54) is expressible as

$$\iiint_{T_{1,2,3,4}} x^2 y \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{T^{xy}_{2,3,4}} x^2 y(-4(x-10))\,\mathrm{d}x\,\mathrm{d}y - \iint_{T^{xy}_{1,2,3}} x^2 y\,(8(x-y))\,\mathrm{d}x\,\mathrm{d}y + \iint_{T^{xy}_{4,3,1}} x^2 y(4(y-5))\,\mathrm{d}x\,\mathrm{d}y \tag{56}$$

We can evaluate the integrals in the right-hand-side of Eq. (56) by using Lemma 1.

We shall now illustrate the evaluation of the integral of the type

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}} = \iint_{T^{xy}_{ijk}} x^{\alpha} y^{\beta} (lx + my + h)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y$$

For the application example, we have $\alpha = 2$, $\beta = 1$, $\gamma = 0$ now by using the formula of Eq. (5) we can write

$$II^{2,1,1}_{T^{xy}_{ijk}} = \iint_{T^{xy}_{joi}} x^2 y(lx + my + h)\,\mathrm{d}x\,\mathrm{d}y + \iint_{T^{xy}_{koj}} x^2 y(lx + my + h)\,\mathrm{d}x\,\mathrm{d}y + \iint_{T^{xy}_{iok}} x^2 y(lx + my + h)\,\mathrm{d}x\,\mathrm{d}y \tag{57}$$

note that $lx + my + h = z(x, y)$ is the equation of the plane spanning points $\langle i, j, k\rangle$ we shall illustrate the use of Lemma 1 to compute one of these integrals, say

$$II^{2,1,0}_{T^{xy}_{iok}} = \iint_{T^{xy}_{iok}} x^2 y(lx + my + h)\,\mathrm{d}x\,\mathrm{d}y \tag{58}$$

using Eqs. (1a) and (1b), we have for the integral of Eq. (58)

$$\begin{aligned} II^{2,1,1}_{T^{xy}_{iok}} = (A^{xy}_{iok})\Big[&\{F(2,0)G(1,0)H(1,0)\} \\ &+ \frac{1}{2}\{F(1,1)G(1,0)H(1,0) + F(2,0)G(0,1)H(1,0) + F(2,0)G(1,0)H(0,1)\} \\ &+ \frac{1}{3}\{F(1,1)G(1,0)H(0,1) + F(2,0)G(0,1)H(0,1) + F(0,2)G(1,0)H(1,0) \\ &+ F(1,1)G(0,1)H(1,0)\} \\ &+ \frac{1}{4}\{F(0,2)G(1,0)H(0,1 + F(1,1)G(0,1)H(0,1) + F(0,2)G(0,1)H(1,0)\} \\ &+ \frac{1}{5}\{F(0,2)G(0,1)H(0,1)\}\Big] \end{aligned} \tag{59}$$

where

$$\begin{aligned} F(2,0) &= x_i^2 \\ F(1,1) &= 2x_i x_{ki} \\ F(0,2) &= x_{ki}^2 \\ G(1,0) &= y_i \\ G(0,1) &= y_{ki} \\ H(1,0) &= \begin{cases} Z_i & \text{if } Z(0,0) = h = 0 \\ \dfrac{Z_o}{6} + \dfrac{Z_o}{5}, & \text{if } Z(0,0) = h = Z_o \neq 0 \end{cases} \end{aligned} \tag{60}$$

$$H(0,1)=\begin{cases} Z_{ki}, & \text{if } Z(0,0)=h=0 \\ \dfrac{Z_{ki}}{6}, & \text{if } Z(0,0)=h=Z_o\neq 0 \end{cases}$$

$$2\Delta^{xy}_{iok}=x_k y_i - x_i y_k$$

$$z(x,y)=h+lx+my$$

Hence, on using the above equations (57), (58), (59) and (60) we have

$$\begin{aligned} II^{2,1,0}_{T^{xy}_{234}} &= \int\int_{T^{xy}_{234}} x^2y(-4x+40)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{T^{xy}_{302}} + \int\int_{T^{xy}_{403}} + \int\int_{T^{xy}_{204}} x^2y(-4x+40)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{-120552}{9} - 28744 + 50000 \\ &= \frac{23584}{3} \end{aligned} \tag{61}$$

$$\begin{aligned} II^{2,1,1}_{T^{xy}_{123}} &= \int\int_{T^{xy}_{123}} x^2y(8x-8y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{T^{xy}_{201}} + \int\int_{T^{xy}_{302}} + \int\int_{T^{xy}_{103}} x^2y(8x-8y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 0+\left(\frac{-36512}{9}\right)+\left(\frac{9541}{9}\right) \\ &= -\frac{26971}{9} \end{aligned} \tag{62}$$

$$\begin{aligned} II^{2,1,0}_{T^{xy}_{431}} &= \int\int_{T^{xy}_{431}} x^2y(4y-20)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{T^{xy}_{304}} + \int\int_{T^{xy}_{103}} + \int\int_{T^{xy}_{401}} x^2y(4y-20)\,\mathrm{d}x\,\mathrm{d}y \\ &= -176+\left(\frac{1606}{9}\right)+\left(\frac{43750}{9}\right) \\ &= \frac{43772}{9} \end{aligned} \tag{63}$$

Thus, from Eqs. (56), (61), (62) and (63) we obtained

$$\int\int\int_{T_{1234}} x^2y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{23584}{3} + \frac{26971}{9} + \frac{43772}{9} = \frac{47165}{9} \tag{64}$$

ALGORITHM 2. We can also show on using the concepts developed in Rathod and Govinda Rao [10,11] and Theorem 2 that the integral of Eq. (54) is expressible as

$$\int\int\int_{T_{1,2,3,4}} x^2y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{1}{3}\int\int_{T^{yz}_{1,2,3}}\left(y+\frac{z}{8}\right)^3 y\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{3}\int\int_{T^{yz}_{2,3,4}}\left(-\frac{z}{4}+10\right)^3 y\,\mathrm{d}y\,\mathrm{d}z \tag{65}$$

We shall now illustrate the evaluation of the integrals of the type:

$$II^{\alpha+1,\beta,\gamma}_{T^{yz}_{ijk}} = \int\int_{T^{yz}_{ijk}} (l'y+m'z+h')^{\alpha+1} y^{\beta} z^{\gamma}\,\mathrm{d}y\,\mathrm{d}z$$

for the application example, we have $\alpha=2$, $\beta=1$, $\gamma=0$ and we can write (by using Green's theorem)

$$II^{3,1,0}_{T^{yz}_{ijk}} = \left(\int\int_{T^{yz}_{ioi}} + \int\int_{T^{yz}_{koj}} + \int\int_{T^{yz}_{iok}}\right) x^3(y,z) y \, dy \, dz \tag{66}$$

where $l'y + m'z + h' = x(y, z)$ is the equation of the plane spanning the points $\langle i, j, k\rangle$. We shall now illustrate the use of Lemma 2 to compute one of these integrals say

$$II^{3,1,0}_{T^{yz}_{iok}} = \int\int_{T^{yz}_{iok}} (l'y + m'z + h')^3 y \, dy \, dz \tag{67}$$

Using Eqs. (25a) and (25b) we have for the integral of Eq. (67):

$$\begin{aligned} II^{3,1,0}_{T^{yz}_{iok}} = (A^{yz}_{iok}) \Big[& \{F(3,0)G(1,0)\} \\ & + \frac{1}{2}\{F(3,0)G(0,1) + F(2,1)G(1,0)\} \\ & + \frac{1}{3}\{F(2,1)G(0,1) + F(1,2)G(1,0)\} \\ & + \frac{1}{4}\{F(1,2)G(0,1) + F(0,3)G(1,0)\} \\ & + \frac{1}{5}\{F(0,3)G(0,1)\}\Big] \end{aligned} \tag{68}$$

where

$$\begin{aligned} F(3,0) &= \begin{cases} \dfrac{x_o^3}{3} + \dfrac{3x_o^2 x_{io}}{4} + \dfrac{3x_o x_{io}^2}{5} + \dfrac{x_{io}^3}{6}, & \text{for } x_o = x(0,0) = h' \neq 0 \\ x_i^3, & \text{for } x_o = x(0,0) = h' = 0 \end{cases} \\ F(2,1) &= \begin{cases} \left(\dfrac{3x_o^2}{4} + \dfrac{6x_o^2 x_{io}}{5} + \dfrac{3x_{io}^2}{6}\right) x_{ki}, & \text{for } x_o = x(0,0) = h' \neq 0 \\ 3x_i^2 x_{ki}, & \text{for } x_o = x(0,0) = h' = 0 \end{cases} \\ F(1,2) &= \begin{cases} \left(\dfrac{3x_o}{5} + \dfrac{3x_{io}}{6}\right) x_{ki}^2, & \text{for } x_o = x(0,0) = h' \neq 0 \\ 3x_i x_{ki}^2, & \text{for } x_o = x(0,0) = h' = 0 \end{cases} \\ F(0,3) &= \begin{cases} \left(\dfrac{1}{6}\right) x_{ki}^3, & \text{for } x_o = x(0,0) = h' \neq 0 \\ x_{ki}^3, & \text{for } x_o = x(0,0) = h' = 0 \end{cases} \\ G(1,0) &= y_i \\ G(0,1) &= y_{ki} \\ 2A^{yz}_{iok} &= y_k z_i - y_i z_k \\ x(y,z) &= h' + l'y + m'z \end{aligned} \tag{69}$$

Hence, on using the above equations (66), (67), (68) and (69) we obtain

$$\begin{aligned} II^{3,1,0}_{T^{yz}_{123}} &= \int\int_{T^{yz}_{123}} \left(y + \frac{z}{8}\right)^3 dy \, dz \\ &= \left(\int\int_{T^{yz}_{201}} + \int\int_{T^{yz}_{302}} + \int\int_{T^{yz}_{103}}\right)\left(y + \frac{z}{8}\right)^3 y \, dy \, dz \\ &= 0 + 85264 - 11997 \\ &= 73267 \end{aligned} \tag{70}$$

$$
\begin{aligned}
II^{3,1,0}_{T^{yz}_{234}} &= \int\int_{T^{yz}_{234}} \left(-\frac{z}{4}+10\right)^3 y \, dy \, dz \\
&= \left(\int\int_{T^{yz}_{302}} + \int\int_{T^{yz}_{403}} + \int\int_{T^{yz}_{204}}\right)\left(-\frac{z}{4}+10\right)^3 y \, dy \, dz \\
&= (183424) - 62992 \\
&= 120432
\end{aligned}
\tag{71}
$$

Substituting from Eqs. (70) and (71) into Eq. (65) we obtain

$$
\begin{aligned}
\int\int\int_{T_{1,2,3,4}} x^2 y \, dx \, dy \, dz &= -\frac{1}{3}\int\int_{T^{yz}_{123}} \left(y+\frac{z}{8}\right)^3 y \, dy \, dz + \frac{1}{3}\int\int_{T^{yz}_{234}} \left(-\frac{z}{4}+10\right)^3 y \, dy \, dz \\
&= \frac{-73267+120432}{3} \\
&= \frac{47165}{3}
\end{aligned}
\tag{72}
$$

ALGORITHM 3. We can also show on using the concepts developed in Rathod and Govinda Rao [10,11] and Theorem 3 that the integral of Eq. (54) is expressible as

$$
\begin{aligned}
\int\int\int_{T_{1,2,3,4}} x^2 y \, dx \, dy \, dz = &-\frac{1}{2}\int\int_{T^{zx}_{4,1,3}} x^2\left(5+\frac{z}{4}\right)^2 dz \, dx \\
&-\frac{1}{2}\int\int_{T^{zx}_{4,1,2}} x^2\left(x-\frac{z}{8}\right)^2 dz \, dx
\end{aligned}
\tag{73}
$$

We shall now illustrate the evaluation of the integrals of the type:

$$
II^{\alpha,\beta+1,\gamma}_{T^{zx}_{ijk}} = \int\int_{T^{zx}_{ijk}} x^\alpha (l''z + m''x + h'')^{\beta+1} z^\gamma \, dz \, dx
$$

For the application example, we have $\alpha = 2$, $\beta = 1$, $\gamma = 0$ and we can write on using Green's theorem:

$$
II^{2,2,0}_{T^{zx}_{ijk}} = \left(\int\int_{T^{zx}_{joi}} + \int\int_{T^{zx}_{koi}} + \int\int_{T^{zx}_{iok}}\right) x^2 y^2(z,x) \, dz \, dx
$$

where $(l''z + m''x + h'' = y(z,x))$ is the equation of the plane spanning the points $\langle i, j, k\rangle$).

We shall now illustrate the use of Lemma 3 to compute one of these integrals, say

$$
II^{3,1,0}_{T^{zx}_{iok}} = \int\int_{T^{zx}_{iok}} x^2 (l''z + m''x + h'')^2 \, dz \, dx
\tag{74}
$$

where $(l''z + m''x + h'' = y(z,x))$ is the equation of the plane spanning the points $\langle i, j, k\rangle$).

Using Eqs. (26a) and (26b), we have for the integral of Eq. (74):

$$
\begin{aligned}
II^{2,2,0}_{T^{zx}_{iok}} = (A^{zx}_{iok})\Big[&\{F(2,0)G(2,0)\} \\
&+\frac{1}{2}\{F(1,1)G(2,0) + F(2,0)G(1,1)\} \\
&+\frac{1}{3}\{F(2,0)G(0,2) + F(1,1)G(1,1) + F(0,2)G(2,0)\} \\
&+\frac{1}{4}\{F(1,1)G(0,2) + F(0,2)G(1,1)\} \\
&+\frac{1}{5}\{(0,2)G(0,2)\}\Big]
\end{aligned}
\tag{75}
$$

where

$$
\begin{aligned}
&F(2,0)=x_i^2,\ F(1,1)=2x_ix_{ki}, \qquad F(0,2)=x_{ki}^2\\
&G(2,0)=\begin{cases}\left[\frac{1}{4}y_o^2+\frac{1}{5}y_oy_{io}+\frac{1}{6}y_{io}^2\right], & \text{if } y(0,0)=y_o=h''\neq 0\\ y_i^2, & \text{if } y(0,0)=y_o=h''=0\end{cases}\\
&G(1,1)=\begin{cases}\left[\frac{y_o}{5}+\frac{2y_{io}}{6}\right]y_{ki}\,; & \text{if } y(0,0)=y_o=h''=0\\ 2y_iy_ki\,; & \text{if } y(0,0)=y_o=h''=0\end{cases}\\
&G(0,2)=\begin{cases}\frac{1}{6}y_{ki}^2, & \text{if } y(0,0)=y_o=h''\neq 0\\ y_{ki}^2, & \text{if } y(0,0)=y_o=h''=0\end{cases}\\
&2\Delta_{iok}^{zx}=z_kx_i-z_ix_k\\
&y(z,x)=h''+l''z+m''x
\end{aligned} \tag{76}
$$

Hence, on using the above equations (74) and (76) we obtain

$$
\begin{aligned}
II_{T_{4,1,3}^{zx}}^{2,2,0}&=\left(\int\int_{T_{104}^{zx}}+\int\int_{T_{301}^{zx}}+\int\int_{T_{403}^{zx}}\right)x^2\left(5+\frac{z}{4}\right)^2\,\mathrm{d}z\,\mathrm{d}x\\
&=0+\left(\frac{-136622}{9}\right)+\left(\frac{486064}{9}\right)\\
&=\frac{349442}{9}
\end{aligned} \tag{77}
$$

$$
\begin{aligned}
II_{T_{312}^{zx}}^{2,2,0}&=\left(\int\int_{T_{103}^{zx}}+\int\int_{T_{201}^{zx}}+\int\int_{T_{302}^{zx}}\right)x^2\left(x-\frac{z}{8}\right)^2\,\mathrm{d}z\,\mathrm{d}x\\
&=\frac{98432}{9}+0+\left(\frac{-730864}{9}\right)\\
&=-\frac{632432}{9}
\end{aligned} \tag{78}
$$

Substituting from Eqs. (77) and (78) into Eq. (73), we obtain

$$
\begin{aligned}
\int\int_{T_{1,2,3,4}}x^2y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z&=-\frac{1}{2}\int\int_{T_{4,1,3}^{zx}}x^2\left(5+\frac{z}{4}\right)\mathrm{d}z\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int\int_{T_{3,1,2}^{zx}}x^2\left(x-\frac{z}{8}\right)^2\mathrm{d}z\,\mathrm{d}x\\
&=\frac{-174721}{9}+\frac{316216}{9}\\
&=\frac{141495}{9}\\
&=\frac{47165}{3}
\end{aligned} \tag{79}
$$

The results obtained in Eqs. (64), (72) and (79) are again incomplete with that of Bernardini [9].

5. Line integration over arbitrary tetrahedron

5.1. We have shown in the previous section that the surface integrals of the type:

$$
II_{T_{iok}^{xy}}^{\alpha,\beta,\gamma+1},\quad II_{T_{iok}^{yz}}^{\alpha+1,\beta,\gamma},\quad II_{T_{iok}^{zx}}^{\alpha,\beta+1,\gamma}
$$

are in fact reducible to line integra over the interval $0 \leq t \leq 1$. Let us now define the equation of the plane spanning points $\langle i, j, k \rangle$ in three alternative forms which is dependent on the projection over which the volume integral $III^{\alpha,\beta,\gamma}_{T_{1234}}$ is considered. Letting the equation of the plane spanning $\langle (x_p, y_p, z_p), (x_q, y_q, z_q), (x_r, y_r, z_r) \rangle$ be denoted by three alternative forms: $z(x, y) = h + lx + my$, $x(y, z) = h' + l'y + m'z$ and $y(z, x) = h'' + l''z + m''x$, this may not be possible if the corresponding area integrals

$$II^{\alpha,\beta,\gamma+1}_{T^{xy}_{ijk}}, \quad II^{\alpha+1,\beta,\gamma}_{T^{yz}_{ijk}}, \quad II^{\alpha,\beta+1,\gamma}_{T^{zx}_{ijk}}$$

do not exit. Let us further denote

$$z(x, y) = h + lx + my = Z(x, y, p, q, r)$$
$$x(y, z) = h' + l'y + m'z = X(y, z, p, q, r)$$
$$y(z, x) = h'' + l''z + m''x = Y(z, x, p, q, r)$$

We should note that l, m, h; l', m', h'; and l'', m'', h'' depend only on (x_p, y_p, z_p), (x_q, y_q, z_q), (x_r, y_r, z_r).

THEOREM 4. Let $T_{1,2,3,4}$ be an arbitrary linear tetrahedron in the three-dimensional space with vertices at $((x_a, y_a, z_a), a = 1, 2, 3, 4)$ bounded by the tetrahedral surface consisting of four arbitrary triangles Tijk with vertices $((x_a, y_a, z_a), a = i, j, k)$ with $(i, j, k) \in \{(1, 2, 3), (4,3,2), (4, 1, 3), (4, 2, 1)\}$ then the structure product $III^{\alpha,\beta,\gamma}_{T_{1234}}$ is expressible as

$$\begin{aligned} III^{\alpha,\beta,\gamma}_{T_{1234}} = \frac{\Omega(1,2,3,4)}{(\gamma+1)} \Bigg[& \iint_{T^{xy}_{201}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 1, 2, 3, \ ) - Z^{\gamma+1}(x, y, 4, 2, 1, \ )\} \, dx\, dy \\ & + \iint_{T^{xy}_{302}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 1, 2, 3, \ ) - Z^{\gamma+1}(x, y, 4, 3, 2 \ )\} \, dx\, dy \\ & + \iint_{T^{xy}_{103}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 1, 2, 3, \ ) - Z^{\gamma+1}(x, y, 4, 1, 3, \ )\} \, dx\, dy \\ & + \iint_{T^{xy}_{304}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 4, 3, 2, \ ) - Z^{\gamma+1}(x, y, 4, 1, 3, \ )\} \, dx\, dy \\ & + \iint_{T^{xy}_{104}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 4, 1, 3, \ ) - Z^{\gamma+1}(x, y, 4, 2, 1, \ )\} \, dx\, dy \\ & + \iint_{T^{xy}_{402}} x^\alpha y^\beta \{Z^{\gamma+1}(x, y, 4, 3, 2, \ ) - Z^{\gamma+1}(x, y, 4, 2, 1, \ )\} \, dx\, dy \Bigg] \end{aligned}$$

PROOF. Follows from Theorem 1 and Eq. (5). □

THEOREM 5. Let $T_{1,2,3,4}$ be an arbitrary linear tetrahedron in three-dimensional space with vertices at $((x_a, y_a, z_a), a = 1, 2, 3, 4)$ bounded by the tetrahedral surface consisting of four arbitrary triangles Tijk with vertices $((x_a, y_a, z_a), a = i, j, k)$ with $(i, j, k) \in \{(1, 2, 3), (4, 3, 2), (4, 1, 3), (4, 2, 1)\}$ then the structure product $III^{\alpha,\beta,\gamma}_{T_{1234}}$ is expressible as

$$\begin{aligned} III^{\alpha,\beta,\gamma}_{T_{1234}} = \frac{\Omega(1,2,3,4)}{(\gamma+1)} \Bigg[& \iint_{T^{yz}_{201}} y^\beta z^\gamma \{X^{\alpha+1}(y, z, 1, 2, 3, \ ) - X^{\alpha+1}(y, z, 4, 2, 1, \ )\} \, dy\, dz \\ & + \iint_{T^{yz}_{302}} y^\beta z^\gamma \{X^{\alpha+1}(y, z, 1, 2, 3, \ ) - X^{\alpha+1}(y, z, 4, 3, 2, \ )\} \, dy\, dz \\ & + \iint_{T^{yz}_{103}} y^\beta z^\gamma \{X^{\alpha+1}(y, z, 1, 2, 3, \ ) - X^{\alpha+1}(y, z, 4, 1, 3, \ )\} \, dy\, dz \\ & + \iint_{T^{yz}_{304}} y^\beta z^\gamma \{X^{\alpha+1}(y, z, 4, 3, 2, \ ) - X^{\alpha+1}(y, z, 4, 1, 3, \ )\} \, dy\, dz \end{aligned}$$

$$+\int\int_{T^{yz}_{104}} y^{\beta}z^{\gamma+1}\{X^{\alpha+1}(y,z,4,1,3,\ )-X^{\alpha+1}(y,z,4,2,1,\ )\}\,dy\,dz$$
$$+\int\int_{T^{yz}_{402}} y^{\beta}z^{\gamma}\{X^{\alpha+1}(y,z,4,3,2,\ )-X^{\alpha+1}(y,z,4,2,1,\ )\}\,dy\,dz \tag{81}$$

PROOF. Follows from Theorem 2 and Eq. (5) with x, y replaced by y, z. □

THEOREM 6. Let $T1,2,3,4$ be an arbitrary linear tetrahedron in three-dimensional space with vertices at $((x_a, y_a, z_a),\ a=1,2,3,4)$ bounded by the tetrahedral surface consisting of four arbitrary triangles Tijk with vertices $((x_a, y_a, z_a),\ a=i,j,k)$ with $(i,j,k)\in\{(1,2,3),(4,3,2),(4,1,3),(4,2,1)\}$ then the structure product $III^{\alpha,\beta,\gamma}_{T_{1234}}$ is expressible as

$$III^{\alpha,\beta,\gamma}_{T_{1234}}=\frac{\Omega(1,2,3,4)}{(\beta+1)}\left[\int\int_{T^{zx}_{201}} z^{\gamma}x^{\alpha}\{Y^{\beta+1}(z,x,1,2,3,\ )-Y^{\beta+1}(z,x,4,2,1,\ )\}\,dz\,dx\right.$$
$$+\int\int_{T^{zx}_{302}} z^{\gamma}zx^{\alpha}\{Y^{\beta+1}(z,x,1,2,3,\ )-Y^{\beta+1}(z,zx\,4,3,2,\ )\}\,dz\,dx$$
$$+\int\int_{T^{zx}_{103}} z^{\gamma}x^{\alpha}\{Y^{\beta+1}(z,x,1,2,3,\ )-Y^{\beta+1}(z,x,4,1,3,\ )\}\,dz\,dx$$
$$+\int\int_{T^{zx}_{304}} z^{\gamma}x^{\alpha}\{Y^{\beta+1}(z,x,4,3,2,\ )-Y^{\beta+1}(z,x,4,1,3,\ )\}\,dz\,dx$$
$$+\int\int_{T^{zx}_{104}} z^{\gamma}x^{\alpha}\{Y^{\beta+1}(z,x,4,1,3,\ )-Y^{\beta+1}(z,x,4,2,1,\ )\}\,dz\,dx$$
$$\left.+\int\int_{T^{yx}_{402}} z^{\gamma}x^{\alpha}\{Y^{\beta+1}(z,x,4,3,2,\ )-Y^{\beta+1}(z,x,4,2,1,\ )\}\,dz\,dx\right] \tag{82}$$

PROOF. Follows from Theorem 3 and Eq. (5) with x, y replaced by z, x. □

Use of Eqs. (80), (81) and (82) as stated in Theorems 4, 5 and 6 will further reduce the computational effort by 50% and suitable modification in Lemmas 1, 2 and 3 will help us in proposing better alternatives to Algorithms 1, 2 and 3 discussed in previous sections. The speciality of Eqs. (80), (81) and (82) is that they can be in reality viewed as six line integrals along the edges of an arbitrary linear tetrahedron $T_{1,2,3,4}$.

5.2. Let $T^{uv}_{i\delta k}$ $(\delta=p,q)$ refers to arbitrary triangles in the uv-plane with vertices at (u_i, v_i), (u_δ, v_δ) and (u_k, v_k). We want to evaluate the following structure product:

$$JJ^{m,n,r}_{T^{uv}_{i\delta k}}\overset{\text{def}}{=}\left\{II^{m,n,r}_{T^{uv}_{i\delta k}}(ipk)-II^{m,n,r}_{T^{uv}_{i\delta k}}(iqk)\right\} \tag{83}$$

where

$$II^{m,n,r}_{T^{uv}_{i\delta k}}(i,\delta,k)\overset{\text{def}}{=}\int\int_{T^{uv}_{i\delta k}} u^m v^n w^r(u,v,i,\delta,k)\,du\,dv \tag{84}$$

$(u,v)\in(x,y),(y,z),(z,x)$ and $W(u,v,i,\delta,k)$, $(\delta=p,q)$ are the equation of the planes in u,v,w space spanned by points (u_i, v_i, w_i), $(u_\delta, v_\delta, w_\delta)$ and (u_k, v_k, w_k) we can write

$$W(u,v,i,p,k)=H+Lu+Mv$$
$$W(u,v,i,q,k)=H'+L'u+M'v \tag{85}$$

With the property

$$W(u_a, v_a, i, \delta, k) = Wa(a = i, k, \delta = p, q)$$
$$W(o, o, i, p, k) = H = W_o^{ipk} \text{ (say)} \tag{86}$$
$$W(o, o, i, q, k) = H' = W_o^{iqk} \text{ (say)},$$

It should be noted that H, L, M are determined in terms of $((u_a, v_a, w_a),\ a = i, p, k)$ and H', L', M' are determined in terms of $((u_b, v_b, w_b),\ b = i, q, k)$.

LEMMA 4. Let T^{uv}_{iok} be a triangle in the uv-plane with vertices at (u_i, v_i), (o, o) and (u_k, v_k) then the structure product

$$JJ^{m,n,r}_{T^{uv}_{iok}} \overset{\text{def}}{=} \{II^{m,n,r}_{T^{uv}_{iok}}(ipk) - II^{m,n,r}_{T^{uv}_{iok}}(iqk)\}$$
$$= (A^{uv}_{iok}) \sum_{\mu=0}^{m+n+r} \frac{1}{(\mu+1)} \sum_{\mu_1+\mu_2+\mu_3=\mu} P(m-\mu_1, \mu_1)$$
$$\times Q(n-\mu_2, \mu_2) R_{pq}(r-\mu_3, \mu_3) \tag{87}$$

where

$$P(m-\mu_1, \mu_1) = \binom{m}{\mu_1} u_i^{m-\mu_1} u_{ki}^{\mu_1}$$
$$Q(n-\mu_2, \mu_2) = \binom{n}{\mu_2} v_i^{n-\mu_2} v_{ki}^{\mu_2}$$
$$R_{pq}(r-\mu_3, \mu_3) = R_p(s-\mu_3 \mu_3) - R_q(r-\mu_3, \mu_3)$$
$$R_\delta(r-\mu_3, \mu_3) = \begin{cases} \binom{s}{\mu_3} w_i^{s-\mu_3} w_{ki}^{\mu_3}, & \text{if } w_0^{i\delta k} = 0 \\ \displaystyle\sum_{\lambda=\mu_3}^{r} \frac{\binom{s}{\mu_3}(w_o^{i\delta k})^{s-\lambda}}{(m+n+\lambda+2)} \binom{\lambda}{\mu_3} (w_i - w_o^{i\delta k})^{\lambda-s} & \text{if } w_o^{i\delta k} = 0 \end{cases}$$
$$\delta = (p, q)$$
$$A^{uv}_{iok} = \begin{cases} \dfrac{2\Delta^{uv}_{iok}}{(m+n+s+2)}, & \text{if } w_o^{i\delta k} = 0 \\ 2\Delta^{uv}_{iok}, & \text{if } w_o^{i\delta k} \neq 0 \end{cases} \tag{88}$$

PROOF. Let us consider the integral,

$$II^{m,n,r}_{T^{uv}_{i\delta k}} \overset{\text{def}}{=} \int\int_{T^{uv}_{i\delta k}} u^m v^n W^s(u, v, i, \delta, k)\, du\, dv \tag{89a}$$
$$(\delta = p, q)$$

The parametric equations of the oriented triangles $T^{uv}_{i\delta k}$ in the uv-plane with vertices at $((u_a, v_a),\ a = i, s, k)$, $s = p, q$ are

$$u = u_i + u_{\delta i}\xi + u_{ki}\eta$$
$$v = v_i + v_{\delta i}\xi + v_{ki}\eta$$
$$W(u, v, i, \delta, k) = w_i + w_{\delta i}\xi + w_{ki}\eta$$

where

$$o \leqslant \xi, n \leqslant 1, \qquad \xi + n \leqslant 1$$
$$u_{\delta i} = u_\delta - u_i, \qquad u_{ki} = u_k - u_i \qquad \text{etc.} \tag{89b}$$

Using Eqs. (89a,b), we can map oriented triangles $T^{uv}_{i\delta k}$ in the uv-plane to unit triangle in the $\xi\eta$-plane (see Fig. 3). We have for the area element,

$$\begin{aligned}\mathrm{d}u\,\mathrm{d}v &= \frac{\partial(u,v)}{\partial(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta = (u_{\delta i}v_{ki} - u_{ki}v_{\delta i})\\ &= (2\Delta^{uv}_{i\delta k})\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &= (2\times \sim\text{area of triangle } T^{uv}_{i\delta k})\,\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned} \tag{90}$$

and

$$2\Delta^{uv}_{i\delta k} = u_{\delta i}v_{ki} - u_{ki}v_{\delta i} \tag{91}$$

Using Eqs. (89a,b), (90) and (91), we obtain:

$$\begin{aligned} JJ^{m,n,\gamma}_{T^{uv}_{i\sigma k}} &\stackrel{\text{def}}{=} \left\{II^{m,n,\gamma}_{T^{uv}_{i\sigma k}}(ipk) - II^{m,n,\gamma}_{T^{uv}_{i\sigma k}}(iqk)\right\}\\ &= \left\{(2\Delta^{uv}_{ipk})\int_0^1\int_0^{1-\xi}[u_i + u_{pi}\xi + u_{ki}\eta]^m[v_i + v_{pi}\xi + v_{ki}\eta]^n\right.\\ &\quad\times[w_i + w_pi\xi + w_{ki}\eta]^\gamma\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad-(2\Delta^{uv}_{iqk})\int_0^1\int_0^{1-\xi}[u_i + u_{qi}\xi + u_{ki}\eta]^m[v_i + v_{qi}\xi + v_{ki}\eta]^n\\ &\quad\left.\times[w_i + w_{qi}\xi + w_{ki}\eta]^\gamma\,\mathrm{d}\xi\,\mathrm{d}\eta\right\} \end{aligned} \tag{92}$$

We also note from Eq. (85) that

$$\begin{aligned} W(u,v,i,p,k) &= w_i + w_{pi}\xi + w_{ki}\eta = H + Lu + Mv\,,\\ W(u,v,i,q,k) &= w_i + w_{qi}\xi + w_{ki}\eta = H' + L'u + M'v \end{aligned} \tag{93}$$

Let us further use the transformation:

$$\xi = 1 - r\,, \qquad v = rt \tag{94}$$

Use of Eq. (94) into Eq. (92) gives

$$\begin{aligned} JJ^{m,n,\gamma}_{T^{uv}_{ipqk}} &= \left\{(2\Delta^{uv}_{ipk})\int_0^1\int_0^1[u_p + u_{ip}r + u_{ki}t]^m[v_p + v_{ip}r + v_{ki}t]^n\right.\\ &\quad\times[w_p + w_{ip}\xi + w_{ki}rt]^\gamma r\,\mathrm{d}r\,\mathrm{d}t\\ &\quad-(2\Delta^{uv}_{iqk})\int_0^1\int_0^1[u_q + u_{iq}r + u_{ki}t]^m[v_q + v_{iq}r + v_{ki}t]^n\\ &\quad\left.\times[w_q + w_{iq}r + w_{ki}rt]^\gamma r\,\mathrm{d}r\,\mathrm{d}t\right\} \end{aligned} \tag{95}$$

From Eqs. (85), (86) and (93) we have

$$\begin{aligned} W(o,o,i,p,k) &= H = w_o^{ipk}\\ W(o,o,i,q,k) &= H' = w_o^{iqk} \end{aligned} \tag{96}$$

choosing

$$\begin{aligned} &u_p = 0\,, \quad v_p = 0\,, \quad \text{we obtain } w_p = w_o^{ipk}\\ &u_q = 0\,, \quad v_q = 0\,, \quad \text{we obtain } w_q = w_o^{iqk} \end{aligned} \tag{97}$$

Let us recall from Eqs. (83) and (84) that

$$II^{m,n,r}_{T^{uv}_{iok}}(i,p,k)=\int\int_{T^{uv}_{iok}} u^m v^n W^r(u,v,i,p,k)\,\mathrm{d}u\,\mathrm{d}v$$

$$II^{m,n,r}_{T^{uv}_{iok}}(i,q,k)=\int\int_{T^{uv}_{iok}} u^m v^n W^r(u,v,i,q,k)\,\mathrm{d}u\,\mathrm{d}v$$

and

$$JJ^{m,n,r}_{T^{uv}_{iok}}=\{II^{m,n,r}_{T^{uv}_{iok}}(ipk)-II^{m,n,r}_{T^{uv}_{iok}}(iqk)\} \tag{98}$$

Using Eqs. (96) and (97) in (95), we obtain Eq. (98), viz.

$$\begin{aligned} JJ^{m,n,\gamma}_{T^{uv}_{iok}} &=\{II^{m,n,\gamma}_{T^{uv}_{iok}}(ipk)-II^{m,n,\gamma}_{T^{uv}_{iok}}(iqk)\} \\ &=(2\Delta^{uv}_{iok})\int_0^1\int_0^1 r^{m+n+1}(u_i+u_{ki}t)^m(v_i+v_{ki}t)^n \\ &\quad\times\{[w^{ipk}_o+r(w_i-w^{ipk}_o+w_{ki}t)]^\gamma \\ &\quad-[w^{iqk}_o+r(w_i-w^{iqk}_o+w_{ki}t)]^\gamma\}\,\mathrm{d}r\,\mathrm{d}t \end{aligned} \tag{99}$$

Let us now define

$$\begin{aligned} &U(t)=(u_i+u_{ki}t)^m \\ &V(t)=(v_i+v_{ki}t)^n \\ &W^{i\delta k}(t)=\begin{cases} (w_i+w_{ki}t)^r\,, & \text{if } w^{i\delta k}_o=0 \\ \sum_{\lambda=0}^{\gamma}\dfrac{\binom{\gamma}{\lambda}(W^{i\delta k}_o)^{\gamma-\lambda}}{(m+n+\lambda+2)}[(w_i+w^{i\delta k}_o)+w_{ki}t]^\lambda\,, & \text{if } w^{i\delta k}_o\neq 0 \end{cases} \\ &A^{uv}_{iok}=\begin{Bmatrix} \dfrac{2\Delta^{uv}_{iok}}{(m+n+\lambda+3)}\,, & \text{if } w^{i\delta k}_o=0 \\ 2\Delta^{uv}_{iok}\,, & \text{if } w^{i\delta k}_o\neq 0 \end{Bmatrix} \\ &\qquad(\delta=p,q) \end{aligned} \tag{100}$$

$$JJ^{m,n,r}_{T^{uv}_{iok}}=A^{uv}_{iok}\int_o^1 u(t)V(t)\{W^{ipk}(t)-W^{iqk}(t)\}\,\mathrm{d}t \tag{101}$$

Letting $f(t)=U(t)V(t)\{W^{ipk}(t)-W^{iqk}(t)\}$ and using Taylor series expansion, we can write

$$f(t)=\sum_{\mu=0}^{m+n+r}\frac{f^{\mu}(0)}{\underline{|\mu}}t^{\mu} \tag{102}$$

where

$$f^{\mu}(0)=\left(\frac{\mathrm{d}^\mu}{\mathrm{d}r^\mu}[u(t)V(t)W^{ipk}(t)-W^{iqk}(t)]\right)_{t=0} \tag{103}$$

we have, now on using the generalised form of Leibnitz's theorem for Eq. (103), we obtain the expression

$$\begin{aligned} f^{(\mu)}(0)=&\sum_{\mu=\mu_1+\mu_2+\mu_3}\frac{\underline{|\mu}}{\underline{|\mu_1}\,\underline{|\mu_2}\,\underline{|\mu_3}}\left\{\frac{\mathrm{d}^{\mu_1}u(t)}{\mathrm{d}r^{\mu_1}}\right\}_{t=0}\left\{\frac{\mathrm{d}^{\mu_2}v(t)}{\mathrm{d}r^{\mu_2}}\right\}_{t=0} \\ &\times\left[\left[\frac{\mathrm{d}^{\mu_3}W^{ipk}(t)}{\mathrm{d}t^{\mu_3}}\right]_{t=0}-\left[\frac{\mathrm{d}^{\mu_3}W^{iqk}(t)}{\mathrm{d}t^{\mu_3}}\right]_{t=0}\right] \end{aligned} \tag{104}$$

Now, from Eq. (100), we obtain

$$\frac{\left[\dfrac{\mathrm{d}^{\mu_1}u(t)}{\mathrm{d}r^{\mu^1}}\right]_{t=0}}{\underline{|\mu}} = \binom{m}{\mu_1} u_i^{m-\mu_1} u_{ki}^{\mu_1} \overset{\text{def}}{=} P(m-\mu_1, \mu_1)\,,$$

$$\frac{\left[\dfrac{\mathrm{d}^{\mu_2}v(t)}{\mathrm{d}r^{\mu^2}}\right]_{t=0}}{\underline{|\mu_2}} = \binom{\eta}{\mu_2} v_i^{n-\mu_2} v_{ki}^{\mu_2} \overset{\text{def}}{=} Q(n-\mu_2, \mu_2)\,,$$

$$\frac{\left[\dfrac{\mathrm{d}^{\mu_3}w^{i\delta k}(t)}{\mathrm{d}r^{\mu^3}}\right]_{t=0}}{\underline{|\mu_3}} = R_\delta(\gamma-\mu_3, \mu_3)\,; \quad (\delta = p, q)$$

$$= \begin{cases} \binom{\gamma}{\mu_3} w_i^{\gamma-\mu_3} w_{ki}^{\mu_3}\,, & \text{if } w_o^{i\delta k} = 0 \\ \left[\left[\displaystyle\sum_{\lambda=0}^{\gamma} \frac{\binom{\lambda}{\gamma}(W_o^{i\delta k})^{\gamma-\lambda}}{(m+n+\lambda+2)} \binom{\gamma}{\mu_3} [(w_i + w_o^{i\delta k}) + w_{ki}t]^{\gamma-\mu_3}\right]\right] w_{ki}^{\mu_3} & \text{if } w_o^{i\delta k} \neq 0 \end{cases}$$

and

$$R_{pq}(\gamma-\mu_3, \mu_3) \overset{\text{def}}{=} R_p(\gamma-\mu_3, \mu_3) - R_q(\gamma-\mu_3, \mu_3) \tag{105}$$

from Eqs. (104) and (105) we obtain

$$\frac{\delta^\mu(0)}{\underline{|\mu}} = \sum_{\mu_1+\mu_2+\mu_3=\mu} P(m-\mu_1, \mu_1) Q(m-\mu_2, \mu_2) R_{pq}(m-\mu_3, \mu_3) \tag{106}$$

Thus, from Eqs. (101), (102) and (106), we cbtain

$$JJ_{T_{iok}^{uv}}^{m,n,\gamma} = (A_{iok}^{uv}) \sum_{\mu=0}^{m+n+\gamma} \frac{1}{(\mu+1)} \left\{ \sum_{\mu_1+\mu_2+\mu_3=\mu} P(m-\mu_1, \mu_1) Q(n-\mu_2, \mu_2) R_{pq}(\gamma-\mu_3, \mu_3) \right\} \tag{107}$$

This completes the proof of Lemma 4. □

Application of Lemma 4 in conjunction with Theorems 4, 5 and 6 will further provide us with a better algorithm to compute the volume integral of monomials over a linear tetrahedron in Euclidean three-dimensional space. The new algorithm can be easily implemented for the application example illustrated in the previous section of this paper.

6. Conclusion

The theorems and lemmas we have presented on integration are interesting for various reasons. We have expressed the integral of spatial expression $x^\alpha y^\beta (lx + my + h)^{\gamma+1} (\alpha, \beta, \gamma) \geq 0$ and positive integers into line integrals not via the use of Green's theorem which was normally done in all previous works, but by means of a simple transformation which joins the line segments end points to the origin of the xy-plane. This transforms the area integral over a plane polygon to a sum of line integrals along its segments. The line integrals thus obtained have a product of three linear functions as their integrand viz.

$$(x_i + x_{ki}r)^\alpha (y_i + y_{ki}r)^\beta (z_i + z_{ki}r)^{\gamma+1}$$

or

$$(x_i + x_{ki}r)^\alpha (y_i + y_{ki}r)^\beta \sum_{p=0}^{\gamma+1} \frac{\binom{\gamma+1}{p} z_0^{\gamma+1-p}}{(\alpha+\beta+p+2)} (z_i - z_0 + z_{ki}r)^p$$

We have used the technique developed in our earlier works [10,11] which apply the Taylor series expansion, generalised form of Leibnitz's theorem and multinomial theorem. We have demonstrated these derivations by means of an application example which explains the detailed computational scheme to evaluate the structure product $III_{T_{1234}}^{\alpha,\beta,\gamma}$. The volume integral of the monomial with vertices $x^{\alpha}y^{\beta}z^{\gamma}$ over a linear arbitrary tetrahedron with vertices at $((x_p, y_p, z_p),\ p = 1, 2, 3, 4)$ we have further developed a finite integration formula for the structure product $III_{T_{1234}}^{\alpha,\beta,\gamma}$ fully expressed in terms of six-line integrals over the boundary edges of the linear tetrahedron $T_{1,2,3,4}$. The application example illustrated in the earlier sections of this paper can be similarly worked out by use of the latter development proposed in terms of six-line integrals. The proposed algorithms of this paper are much simpler and economical in terms of arithmetic operations.

References

[1] V.I. Krylov, Approximate Calculation of Integrals (Macmillan London, 1962).

[2] M.A. Wesley, Construction and uses of geometric modules, in: Computer Aided Design (Springer-Verlag, New York, 1980) 79–136.

[3] M.E. Mortenson, Geometric Modelling (John Wiley, New York, 1985).

[4] Y.T. Lee and A.A.G. Requicha, Algorithms for commuting the volume and other integral properties of solids I known methods and open issues communication, ACM 25 (1982) 635–641.

[5] H.G. Timmer and J.M. Stern, Computation of Global Geometric properties of solid objects, Comput. Aided Des. 12 (1980) 301–304.

[6] S. Lean and J.T. Kajiya, A symbolic method for computing the integral properties of arbitrary nonconvex polyhedra, IEEE Comput. Graph. Applic. 4 (1984) 35–41.

[7] C. Cattani and A. Poluzzi, Boundary integration over linear polyhedra, Comput. Aided Des. 22 (1990) 130–135.

[8] C. Cattani and A. Poluzzi, Symbolic analysis of linear polyhedra, Engrg. Comput. 6 (1990) 17–29.

[9] F. Bernardini, Integration of polynomials over n-dimensional polyhedra, Comput. Aided Des. 23 (1991) 51–58.

[10] H.T. Rathod and H.S. Govinda Rao, Integration of polynomials over Linear Polyhedra in Euclidean three dimension space, Comput. Method Appl. Mech. Engrg. 126 (1955) 373–392.

[11] H.T. Rathod and H.S. Govinda Rao, Integration of polynomials over an arbitrary tetrahedron in Euclidean three dimensional space, Comput. Struct. 59 (1996) 35–65.

[12] O.C. Zienkiewicz, The Finite Element Method (McGraw Hill, London, 1977).