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Boundary integration of polynomials over an arbitrary linear hexahedron in Euclidean three-dimensional space

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Abstract

This paper is concerned with explicit integration formulas and algorithms for computing volume integrals of trivariate polynomials over an arbitrary linear hexahedron in Euclidean three-dimensional space. Three different approaches are discussed. The first algorithm is obtained by transforming a volume integral into a sum of surface integrals and then into convenient and computationally efficient line integrals. The second algorithm is obtained by transforming a volume integral into a sum of surface integrals over the boundary quadrilaterals. The third algorithm is obtained by transforming a volume integral into a sum of surface integrals over the triangulation of boundary. These algorithms and finite integration formulas are then followed by an application example, for which we have explained the detailed computational scheme. The symbolic finite integration formulas presented in this paper may lead to efficient and easy incorporation of integral properties of arbitrary linear polyhedra required in the engineering design process. © 1998 Elsevier Science S.A. All rights reserved.

0. Nomenclature

$$II_{\pi_{xy}}^{\alpha\beta\gamma} = \int \int_{\pi_{xy}} x^\alpha y^\beta (h + lx + my)^{\gamma+1} dx dy$$

= surface integration over a plane polygon in the XY-plane

$II_{\pi_{yz}}^{\alpha\beta\gamma}$, $II_{\pi_{zx}}^{\alpha\beta\gamma}$ have a similar meaning

h, l, m

h', l', m' arbitrary constants

h'', l'', m''

α, β, γ Positive integers (including zero)

$$II_{T_{ijk}^{xy}}^{\alpha\beta\gamma} = \int \int_{T_{ijk}^{xy}} x^\alpha y^\beta (h + lx + my)^{\gamma+1} dx dy$$

T_{ijk}^{xy} = a triangle in the XY-plane with vertices (x_i, y_i) , (x_j, y_j) and (x_k, y_k)

$II_{T_{ijk}^{yz}}^{\alpha\beta\gamma}$, $II_{T_{ijk}^{zx}}^{\alpha\beta\gamma}$ have similar meanings

Δ_{ijk}^{xy} = area of the triangle with vertices (x_i, y_i) , (x_j, y_j) and (x_k, y_k)

Δ_{ijk}^{yz} , Δ_{ijk}^{zx} have similar meanings

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$$III_{T_{ijk}}^{\alpha\beta\gamma} = \int \int_{T_{ijk}} x^\alpha y^\beta z^{\gamma+1} \hat{k} \cdot \hat{n} \, d\gamma$$

= surface integral over T_{ijk}

T_{ijk} = a linear triangle in Euclidean three-dimensional space with vertices (x_i, y_i, z_i) , (x_j, y_j, z_j) and (x_k, y_k, z_k)

\hat{k} = unit normal vector along z -axis

$(n_i, i = 1, 2, 3, 4)$ —outward unit normal to triangles T_{312} , T_{432} , T_{413} and T_{421} , respectively.

Ω_{ijk} is either 1, 0, -1 and it depends on the normal of linear T_{ijk}

$$III_v^{\alpha\beta\gamma} = \int \int \int x^\alpha y^\beta z^\gamma \, dV$$

= volume integral of trivariate monomial $x^\alpha y^\beta z^\gamma$ over a linear three polyhedron in Euclidean three-dimensional space

$$\int \int_S x^\alpha y^\beta z^{\gamma+1} \hat{k} \cdot \hat{n} \, ds = \text{surface integral over the projected area } s \text{ in the } xy\text{-plane,}$$

$$= \sum_{T_{ijk} \in T} III_{T_{ijk}}^{\alpha\beta\gamma}$$

S = is a surface of R^3 decomposable in a set T of triangles such that any pair of triangles T_{ijk} and $T_{i'j'k'}$ do not intersect.

$$II_\pi^{\alpha,\beta,\gamma+1} \stackrel{\text{def}}{=} \int \int_\pi x^\alpha y^\beta (h + lx + my)^{\gamma+1} \, dx \, dy$$

= surface integration over a plane polygon in the xy plane.

$II_\pi^{\alpha,\beta+1,\gamma}$, $II_\pi^{\alpha+1,\beta,\gamma}$ have a similar meaning and h, l, m are arbitrary constants, α, β, γ are positive integers (including zero)

$Q_{i,j,k,l}^{xy}$ = linear quadrilateral spanned by point $((x_p, y_p, z_p), p = i, j, k, l)$ in the xy plane

$T_{i,l,j,k}^{xy}$ = a linear triangle in the xy plane with vertices at $((x_a, y_a), a = i, j, k)$

= linear triangle obtained from a linear quadrilateral $Q_{i,j,k,l}^{xy}$ by letting $(x_i, y_i) = (x_l, y_l)$

$$III_{H_{1,2,3,\dots,8}}^{\alpha,\beta,\gamma} = \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma \, dx \, dy \, dz$$

= volume integral of trivariate monomial $x^\alpha y^\beta z^\gamma$ over a linear hexahedron

$H_{1,2,3,\dots,8}$ = an arbitrary linear hexahedron with vertices at $(x_A, y_A, z_A) \quad A = 1, 2, 3, \dots, 8$

$$III_{T_e}^{\alpha,\beta,\gamma} = \int \int \int_{T_e} x^\alpha y^\beta z^\gamma \, dv$$

= volume integral of tetrahedron T_e

T_e = tetrahedron with vertices at $((x_p, y_p, z_p), p = i, j, k, l)$

1. Introduction

Volume centre of mass, moment of inertia and other geometric properties of rigid homogeneous solids arise very frequently in a large number of engineering applications such as CAD/CAE/CAM, geometric modelling, as well as in a variety of scientific disciplines and robotics. Integration formulas for multiple integrals have always been of great interest in computer applications [1]. Computation of mass properties of both plane and

space objects is discussed by Wesley [2] and Mortenson [3]. A good overview of various methods for evaluating volume (triple) integrals in this context is given by Lee and Requicha [4]. Lee and Requicha [4] observe that most computational studies in multiple integration deal with problems where the integration domain is a very simple solid, such as a cube or sphere and the integrating function is very complicated, conversely, in most engineering application, the opposite is the usual problem. In such problems, the integration domain may have a non-convex shape and the function inside the integral sign is a trivariate polynomial. Timmer and Stem [5] discussed a theoretical approach to the evaluation of volume integrals by transforming the volume integral to a surface integral over the boundary of the integration domain. Lien and Kajiya [6] presented an outline of a closed form formula for volume integration for a linear tetrahedron and suggested that volume integration over a linear polyhedron can be obtained by disjoint decomposition of several tetrahedra. Cattani and Paoluzzi [7,8] have obtained finite integration formulas for integrals of monomials over plane polygons and space polyhedra via Gauss's Divergence theorems (in two dimension Green's Theorem). Bernardini [9] has further generalized these integration methods to integrals of polynomials over n -dimensional polyhedra. In recent works, Rathod and Govinda Rao [10,11], Rathod and Hiremath [15] addressed these problems, and derived explicit integration formulas which are more convenient and efficient than earlier studies [7,8]. Integration of a triple product [10,11], viz. $x^\alpha y^\beta (h + Lx + My)^{\gamma+1}$ (α, β, γ positive integers, zero h, l, m arbitrary constants), an expression in bivariates x, y , plays a very important role in the computation of volume integrals of the trivariate monomial $x^\alpha y^\beta z^\gamma$ over the domain of a linear polyhedron in Euclidean three-dimensional space. The integral of this bivariate expression $x^\alpha y^\beta (h + Lx + My)^{\gamma+1}$ over a linear polygon in the xy -plane is computed by use of Green's theorem (see [10]) which reduces the area integral to a sum of line integrals along the boundary line segments. Because of the presence of the term $(h + Lx + My)^{\gamma+1}$ the area integral of the bivariate expression $x^\alpha y^\beta (h + Lx + My)^{\gamma+1}$ generates a sum ($= (\gamma + 1) + \gamma + \dots + 2 + 1 = (\gamma + 2) \approx (\gamma + 1)/2$) $(\gamma + 2)(\gamma + 1)/2$) line integrals on application of the Green's theorem along each line segment of the linear plane polygon [10]. In this paper, we have found a means of overcoming this complication and in Lemma 1, it is shown that the same computation can be done only once for each line segment of the linear plane polygon.

We have also proposed two more lemmas (Lemmas 1 and 2) which are useful in evaluating integrals of the above-mentioned bivariate expression over a linear quadrilateral and a linear triangle. These lemmas use the well-known isoparametric coordinate transformations used in the finite element method [12,13]. We have then proposed three different algorithms based on these lemmas (Lemmas 1, 2 and 3) for computing volume integrals of trivariate monomial $x^\alpha y^\beta z^\gamma$ (α, β, γ positive integer including zero) over an arbitrary linear hexahedron in three-dimensional space. Integration over a simple trivariate polynomial function

$$f(x, y, z) = \sum_{\alpha=0}^I \sum_{\beta=0}^J \sum_{\gamma=0}^K a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$$

(where α, β, γ positive integer including zero) can be obtained by linearity property of integrals. We have also proposed theorems (Theorems 1–5) which gradually develop the numerical scheme to compute volume integrals in terms of line integrals and surface integrals (over quadrilaterals and triangles) by use of Lemmas 1–3. In Theorem 6, we express the volume integral over a linear arbitrary hexahedron as a sum of twelve line integrals along the boundary edges. Lemma 4 develops a computational scheme to evaluate each of these line integrals which is again an improvement over earlier studies [10,11].

2. Surface integration

In this section we first establish three preliminary results giving closed form analytical integration formulas for surface integration over a plane polygon in the xy -plane. Then, we wish to use these closed form finite integration formulas in computing volume integral of polynomials over a linear arbitrary hexahedron.

Let π be a simple polygon in the xy -plane. We want to evaluate the following structure product

$$H_\pi^{\alpha, \beta, \gamma+1} \stackrel{\text{def}}{=} \int \int_{\pi} x^\alpha y^\beta (h + lx + my)^{\gamma+1} dx dy \quad (1)$$

where l, m, h are arbitrary constants and α, β, γ are positive integers (including zeros).

LEMMA 1. The structure product $H_{\pi}^{\alpha, \beta, \gamma+1}$ over a simple polygon with N -oriented edges $l_{i,k}$ ($k = i + 1$), ($i = 1, 2, 3, \dots, N$) each with end points (x_i, y_i) and (x_k, y_k) in the xy -plane is expressible as

$$H_{\pi}^{\alpha, \beta, \gamma+1} = \sum_{i=1}^N \left[A_{koi}^{xy} \sum_{n=0}^{\alpha+\beta+\gamma+1} \sum_{n_1+n_2+n_3=n} \{F_{koi}(\alpha - n_1, n_1)G_{koi}(\beta - n_2, n_2)H_{koi}(\gamma + 1 - n_3, n_3)\} \right] \quad (2a)$$

where

$$\begin{aligned} F_{koi}(\alpha - n_1, n_1) &= \binom{\alpha}{n_1} x_k^{\alpha-n_1} x_{ik}^{n_1}, \\ G_{koi}(\beta - n_2, n_2) &= \binom{\beta}{n_2} y_k^{\beta-n_2} y_{ik}^{n_2}, \\ H_{koi}(\gamma + 1 - n_3, n_3) &= \begin{cases} \binom{\gamma+1}{n_3} z_k^{\gamma+1-n_3} z_{ik}^{n_3}, & \text{if } z_0 = h = 0 \\ \left[\sum_{p=n_3}^{r+1} \frac{\binom{\gamma+1}{p} z_0^{r+1-p}}{(\alpha + \beta + p + 2)} \binom{p}{h_3} z_{k_0}^{p-n_3} \right] z_{ik}^{n_3}, & \text{if } z_0 = h \neq 0 \end{cases} \end{aligned} \quad (2b)$$

$$x_{ik} = x_i - x_k$$

$$y_{ik} = y_i - y_k$$

$$z_{ik} = z_i - z_k$$

$$z_{k0} = z_k - z_0$$

$$\begin{aligned} A_{koi}^{xy} &= \frac{2\Delta_{koi}^{xy}}{(\alpha + \beta + p + 3)}, \quad \text{if } z_0 = h = 0 \\ &= 2\Delta_{koi}^{xy}, \quad \text{if } z_0 = h \neq 0 \end{aligned}$$

$$2\Delta_{koi}^{xy} = x_i y_k - x_k y_i$$

PROOF. We have from Eq. (1) (see Fig. 1):

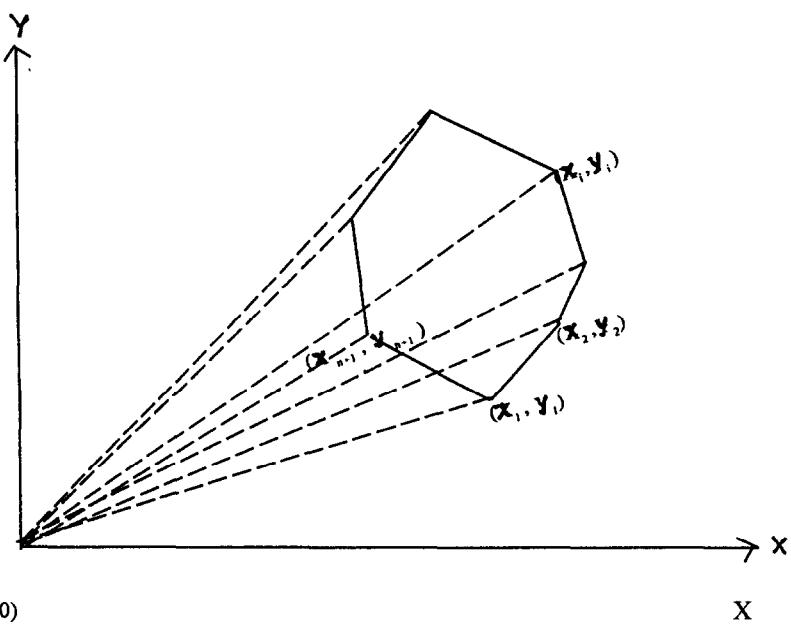


Fig. 1. A simple polygon in the xy -plane with N -oriented edges which expands into N -triangles with respect to the origin.

$$\begin{aligned}
H_{\pi}^{\alpha, \beta, \gamma+1} &\stackrel{\text{def}}{=} \int \int_{\pi} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1} dx dy \\
&= \int \int_{\pi} \frac{\partial \Phi(x, y)}{\partial x} dx dy, \quad \text{where } \Phi(x, y) = \int x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1} dx \\
&= \int_{\partial \pi} \Phi(x, y) dy, \quad \text{on using Green's theorem with } \partial \pi = \text{boundary of } \pi \\
&= \sum_{i=1}^N \int_{l_{ik}} \Phi(x, y) dy
\end{aligned} \tag{3}$$

We shall now show that

$$H_{\pi}^{\alpha, \beta, \gamma+1} = \sum_{i=1}^N \int \int_{T_{koi}^{xy}} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1} dx dy \tag{4}$$

So let us consider the RHS of Eq. (4)

$$\begin{aligned}
\text{RHS} &= \sum_{i=1}^N \int \int_{T_{koi}^{xy}} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1} dx dy \\
&= \sum_{i=1}^N \int \int_{T_{koi}^{xy}} \frac{\partial \Phi(x, y)}{\partial x} dx dy, \quad \text{where } \Phi(x, y) = \int x^{\alpha} y^{\beta} (h + lx + my)^{\gamma+1} dx \\
&= \sum_{i=1}^N \left(\int_{l_{ko}} + \int_{l_{oi}} + \int_{l_{ik}} \right) \Phi(x, y) dy \\
&= \left(\int_{l_{2o}} + \int_{l_{o1}} + \int_{l_{12}} \right) \Phi(x, y) dy \\
&\quad + \left(\int_{l_{3o}} + \int_{l_{o2}} + \int_{l_{23}} \right) \Phi(x, y) dy + \left(\int_{l_{4o}} + \int_{l_{o3}} + \int_{l_{34}} \right) \Phi(x, y) dy \\
&\quad + \cdots \\
&\quad + \left(\int_{l_{N,o}} + \int_{l_{o,N-1}} + \int_{l_{N-1,N}} \right) \Phi(x, y) dy + \left(\int_{l_{N+1,o}} + \int_{l_{oN}} + \int_{l_{N,N+1}} \right) \Phi(x, y) dy \\
&= \sum_{i=1}^N \int_{l_{ik}} \Phi(x, y) dy.
\end{aligned} \tag{5}$$

Eq. (5) follows from the fact that

$$(x_{N+1}, y_{N+1}) = (x_1, y_1)$$

so that we have

$$\int_{l_{oi}} \Phi(x, y) dy = \int_{l_{o,N+1}} \Phi(x, y) dy$$

and

$$\int_{l_{pq}} \Phi(x, y) dy = - \int_{l_{qp}} \Phi(x, y) dy \quad (p, q = 0, 1, 2, \dots, N+1) \tag{6}$$

From Eqs. (5) and (3), we conclude that Eq. (4) is true.

Now, we wish to obtain a simple finite integration formula for the integral

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = \int \int_{T_{koi}^{xy}}^{x^\alpha y^\beta} (h + lx + my)^{\gamma+1} dy dx \quad (7)$$

Let us now consider the integral over oriented triangle T_{kji}^{xy}

$$II_{T_{kji}^{xy}}^{\alpha, \beta, \gamma+1} = \int \int_{T_{kji}^{xy}}^{x^\alpha y^\beta} (h + lx + my)^{\gamma+1} dx dy \quad (8)$$

The parametric equations of the oriented triangle T_{kji}^{xy} in the xy -plane with vertices at (x_k, y_k) , (x_j, y_j) and (x_i, y_i) are

$$\begin{aligned} x &= x_k + x_{jk}u + x_{ik}v \\ y &= y_k + y_{jk}u + y_{ik}v \\ (0 \leq u, v \leq 1, 0 \leq u+v \leq 1, x_{jk} &= x_j - x_k, x_{ik} = x_i - x_k, y_{jk} = y_j - y_k, y_{ik} = y_i - y_k) \end{aligned} \quad (9)$$

Using Eq. (9), we can map an oriented triangle T_{kji}^{xy} in the xy -plane to an oriented unit triangle \bar{T}_{kji} in the uv -plane (see Fig. 2, we have for the area element

$$\begin{aligned} dx dy &= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du dv \\ &= (x_{jk}y_{ik} - x_{ik}y_{jk}) du dv \\ &= 2\Delta_{kji}^{xy} du dv \\ &= (2 \times \text{area of triangle } T_{kji}^{xy}) du dv \end{aligned} \quad (10)$$

where we have defined

$$2\Delta_{kji}^{xy} = (x_{jk}y_{ik} - x_{ik}y_{jk}) \quad (11)$$

we have from Eqs. (8)–(11)

$$II_{T_{kji}^{xy}}^{\alpha, \beta, \gamma+1} = (2\Delta_{kji}^{xy}) \int_0^1 \int_0^{1-u} (x_k + x_{jk}u + x_{ik}v)^\alpha (y_k + y_{jk}u + y_{ik}v)^\beta (z_k + z_{jk}u + z_{ik}v)^\gamma du dv \quad (12)$$

where

$$\begin{aligned} Z_p &= h + lx_p + my_p, \quad p = k, j, i \\ Z_{jk} &= Z_j - Z_k, \quad Z_{ik} = Z_i - Z_k \end{aligned} \quad (13)$$

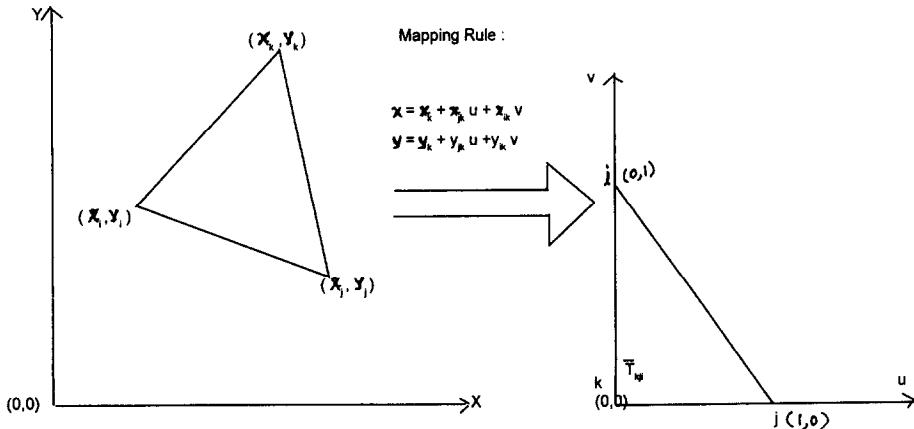


Fig. 2. The mapping between an oriented triangle in the xy -plane and the unit triangle in the uv -plane.

Let us further use the transformation in Eq. (12) which maps the unit triangle in uv plane to a unit square in the rs -plane as

$$u = 1 - r, \quad v = rs \quad (14)$$

Use of Eq. (14) in Eq. (12) leads us to

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = (2\Delta_{kji}^{xy}) \int_0^1 \int_0^1 (x_j + x_{kj}r + x_{ik}rs)^{\alpha} (y_j + y_{kj}r + y_{ik}rs)^{\beta} (z_j + z_{kj}r + z_{ik}rs)^{\gamma+1} r dr ds \quad (15)$$

we have defined $z(x, y) = h + lx + my$, and hence clearly $z(0, 0) = h = Z_0$ (say), may be either zero or non-zero. Choosing $x_j = 0$ $y_j = 0$, we have $z(0, 0) = z_j = h = z_0$, so that we have from Eq. (15)

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = (2\Delta_{koi}^{xy}) \int_0^1 \int_0^1 r^{\alpha+\beta+1} (x_k + x_{ik}s)^{\alpha} (y_k + y_{ik}s)^{\beta} [z_0 + r(z_{k0} + z_{ik}s)]^{\gamma+1} dr ds \quad (16)$$

If $Z_0 = h = 0$, then Eq. (16) reduces to

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = \frac{2\Delta_{koi}^{xy}}{(\alpha + \beta + \gamma + 3)} \int_0^1 (x_k + x_{ik}s)^{\alpha} (y_k + y_{ik}s)^{\beta} (z_k + z_{ik}s)^{\gamma+1} ds \quad (17)$$

If $Z_0 - h \neq 0$, then Eq. (16) reduces to

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = (2\Delta_{koi}^{xy}) \int_0^1 (x_k + x_{ik}s)^{\alpha} (y_k + y_{ik}s)^{\beta} \left\{ \sum_{p=0}^{r+1} \frac{\binom{r+1}{p} z_0^{\gamma+1-p}}{\alpha + \beta + p + 2} (z_{k0} + z_{ik}s)^p \right\} ds \quad (18)$$

Let us define

$$\begin{aligned} X(s) &= (x_k + x_{ik}s)^{\alpha} \\ Y(s) &= (y_k + y_{ik}s)^{\beta} \\ Z_0(s) &= (z_k + z_{ik}s)^{\gamma+1}, \quad \text{if } z_0 = 0 \\ Z_1(s) &= \left\{ \sum_{p=0}^{r+1} \frac{\binom{r+1}{p} z_0^{\gamma+1-p}}{\alpha + \beta + p + 2} (z_{k0} + z_{ik}s)^p \right\}, \quad \text{if } z_0 \neq 0 \\ f(s) &= f_0(s), \quad \text{if } z_0 = 0 \\ f(s) &= f_1(s), \quad \text{if } z_0 \neq 0 \end{aligned} \quad (19)$$

and

$$\begin{aligned} f_0(s) &\stackrel{\text{def}}{=} X(s)Y(s)Z_0(s) \\ f_1(s) &\stackrel{\text{def}}{=} X(s)Y(s)Z_1(s) \end{aligned} \quad (20)$$

using Eqs. (19) and (20) we can write Eqs. (17) and (18) as

$$II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1} = \begin{cases} \frac{2\Delta_{kji}^{xy}}{(\alpha + \beta + \gamma + 3)} \int_0^1 f_0(s) ds, & \text{if } z_0 = 0 \\ 2\Delta_{kji}^{xy} \int_0^1 f_1(s) ds, & \text{if } z_0 \neq 0. \end{cases} \quad (21)$$

using Taylor series expansion for a function of single variable, we have from Eq. (20)

$$f_i(s) = \sum_{n=0}^{\alpha+\beta+\gamma+1} \left\{ \frac{f_i^{(n)}(0)}{n} \right\} s^n, \quad i = 0, 1 \quad (22)$$

where

$$f_{pq}^{(n)}(0) = \left[\frac{d^n}{ds^n} \{X(s)Y(s)Z_i(s)\} \right]_{s=0}, \quad i = 0, 1 \quad (23)$$

Using Eq. (19) on the definition of $X(s)$, $Y(s)$ and $Z_i(s)$ ($i = 0, 1$), we obtain

$$\begin{aligned} \frac{\left\{ \frac{d^{n_1} X(s)}{ds^{n_1}} \right\}_{s=0}}{|n_1|} &= \binom{\alpha}{n_1}_{x_k}^{\alpha - n_1} x_{ik}^{n_1} \stackrel{\text{def}}{=} F_{koi}^{(\alpha - n_1, n_1)}, \\ \frac{\left\{ \frac{d^{n_2} Y(s)}{ds^{n_2}} \right\}_{s=0}}{|n_2|} &= \binom{\beta}{n_2}_{y_k}^{\beta - n_2} y_{ik}^{n_2} \stackrel{\text{def}}{=} G_{koi}(\beta - n_2, n_2), \\ \frac{\left\{ \frac{d^{n_3} Z_0(s)}{ds^{n_3}} \right\}_{s=0}}{|n_3|} &= \left(\frac{\gamma + 1}{n_3} \right) z_k^{\gamma + 1 - n_3} z_{ik}^{n_3} \stackrel{\text{def}}{=} H_{koi}^{(\gamma + 1 - n_3, n_3)}, \quad \text{if } Z_0 = 0 \\ \frac{\left\{ \frac{d^{n_3} Z_1(s)}{ds^{n_3}} \right\}_{s=0}}{|n_3|} &= \left[\sum_{p=n_3}^{r+1} \frac{\binom{r+1}{p} Z_o^{r+1-p}}{(\alpha + \beta + p + 2)} \binom{p}{n_3} z_{ko}^{p - n_3} \right] z_{ik}^{n_3} \\ &\stackrel{\text{def}}{=} H_{koi}(\gamma + 1 - n_3, n_3), \quad \text{if } z_0 \neq 0 \end{aligned} \quad (24)$$

From Eqs. (22) and (23) we can write

$$\left\{ \frac{f^{(n)}(0)}{n} \right\} = \sum_{n_1 + n_2 + n_3 = n} F_{koi}(\alpha - n_1, n_1) G_{koi}(\beta - n_2, n_2) H_{koi}(\gamma + 1 - n_3, n_3) \quad (25)$$

Thus, from Eqs. (19) and (22), we obtain

$$\int_0^1 f(s) ds = \sum_{n=0}^{\alpha + \beta + \gamma + 1} \frac{1}{(n+1)} \{F_{koi}(\alpha - n_1, n_1) G_{koi}(\beta - n_2, n_2) H_{koi}(\gamma + 1 - n_3, n_3)\} \quad (26)$$

From Eqs. (19)–(26) we obtain the desired result claimed in Eqs. (2a) and (2b). This completes the proof of Lemma 1. \square

Consider again the integral of Eq. (1) which is already discussed in Lemma 1, i.e.

$$I_{\pi}^{\alpha, \beta, \gamma + 1} = \int \int_{\pi} x^{\alpha} y^{\beta} (h + lx + my)^{\gamma + 1} dx dy$$

Where π is the simple polygon in the xy -plane, l, m, h are arbitrary constants and α, β, γ are positive integers (including zero). We can think of π as region in R^2 decomposable in a set T of triangles such that any pair of members of T_{ijk}^{xy} (a triangle in the xy -plane with vertices at (x_i, y_i) , (x_j, y_j) and (x_k, y_k)) and T_{ijkl}^{xy} do not intersect. We can also think of π as region in R^2 decomposable in a set Q of quadrilaterals only or a combination of quadrilaterals and triangles such that any pair of member of Q_{ijkl}^{xy} (a quadrilateral in the xy -plane with vertices at (x_i, y_i) , (x_j, y_j) , (x_k, y_k) and (x_l, y_l)) and Q_{ijkl}^{xy} do not intersect or any pair of members of Q_{ijkl}^{xy} and T_{ijkl}^{xy} do not intersect. Thus, we may write (see Fig. 2)

$$\begin{aligned} I_{\pi}^{\alpha, \beta, \gamma + 1} &= \\ &\left[\begin{array}{ll} \sum_{T_{ijk}^{xy} \in T} I_{T_{ijk}^{xy}}^{\alpha, \beta, \gamma + 1}, & \text{if } \pi \text{ is decomposed only in a set } T \text{ of triangle} \\ \sum_{Q_{ijkl}^{xy} \in Q} I_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma + 1}, & \text{if } \pi \text{ is decomposed only in a set } Q \text{ of quadrilaterals} \\ \sum_{Q_{ijkl}^{xy} \in Q} I_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma + 1} + I_{T_{ijkl}^{xy}}^{\alpha, \beta, \gamma + 1}, & \text{if } \pi \text{ is decomposable in a set } Q \text{ of quadrilaterals and a single triangle.} \end{array} \right] \end{aligned} \quad (27)$$

In our recent work [15], the structure product $H_{T_{ijk}}^{\alpha, \beta, \gamma+1}$ has been already considered. We shall now obtain a finite integration formula for the structure product $H_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma+1}$, as we wish to use this integral formula for computation of volume integrals of polynomials over an arbitrary linear hexahedron.

LEMMA 2. A structure product over the area of an oriented linear quadrilateral Q_{ijkl}^{xy} in the xy -plane with vertices at $((x_m, y_m), m = i, j, k, l)$:

$$H_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma+1} \stackrel{\text{def}}{=} \int \int_{Q_{ijkl}^{xy}} x^\alpha y^\beta z^\gamma (x, y) dx dy \quad (28)$$

where $Z(x, y) = Lx + My + h$, L, M, h , arbitrary constants can be expressed as

$$H_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma+1} = |\alpha| |\beta| |\gamma+1| \sum_{r=0}^n \sum_{s=0}^n \Phi(r, s) I(r, s) \quad (29)$$

where

$$I(r, s) = \frac{(r+2)(s+2)J_{00} + (r+1)(s+2)J_{10} + (r+2)(s+1)J_{01}}{(r+1)(r+2)(s+1)(s+2)}$$

$$\Phi(r, s) = \sum_{r_1+r_2+r_3=r} \sum_{s_1+s_2+s_3=s} E(r_1, s_1) F(r_2, s_2) G(r_3, s_3),$$

$$E(r_1, s_1) = \sum_{t_1=0}^{s_1} \frac{\alpha - r_1 - s_1 + t_1}{|\alpha - r_1 - s_1 + t_1|} \frac{r_1 - t_1}{|r_1 - t_1|} \frac{s_1 - t_1}{|s_1 - t_1|} \frac{t_1}{|t_1|},$$

$$F(r_2, s_2) = \sum_{t_2=0}^{s_2} \frac{\beta - r_2 - s_2 + t_2}{|\beta - r_2 - s_2 + t_2|} \frac{r_2 - t_2}{|r_2 - t_2|} \frac{s_2 - t_2}{|s_2 - t_2|} \frac{t_2}{|t_2|},$$

$$G(r_3, s_3) = \sum_{t_3=0}^{s_3} \frac{\gamma + 1 - r_3 - s_3 + t_3}{|\gamma + 1 - r_3 - s_3 + t_3|} \frac{r_3 - t_3}{|r_3 - t_3|} \frac{s_3 - t_3}{|s_3 - t_3|} \frac{t_3}{|t_3|},$$

$$0 \leq r_1 + s_1 \leq 2\alpha, \quad 0 \leq r_1, \quad s_1 \leq \alpha$$

$$0 \leq r_2 + s_2 \leq 2\beta, \quad 0 \leq r_2, \quad s_2 \leq \beta$$

$$0 \leq r_3 + s_3 \leq 2(\gamma + 1), \quad 0 \leq r_3, \quad s_3 \leq (\gamma + 1)$$

$$0 \leq r + s \leq 2(\alpha + \beta + \gamma + 1)$$

$$\alpha - (r_1 + s_1) + t_1 \geq 0, \quad r_1 \geq t_1, \quad s_1 \geq t_1$$

$$\beta - (r_2 + s_2) + t_2 \geq 0, \quad r_2 \geq t_2, \quad s_2 \geq t_2$$

$$(\gamma + 1) - (r_3 + s_3) + t_3 \geq 0, \quad r_3 \geq t_3, \quad s_3 \geq t_3$$

$$a_{00} = x_i, \quad a_{01} = x_{li}, \quad a_{10} = x_{ji}, \quad a_{11} = x_{ij} + x_{kl}$$

$$b_{00} = y_i, \quad b_{01} = y_{li}, \quad b_{10} = y_{ji}, \quad b_{11} = y_{ij} + y_{kl}$$

$$c_{00} = z_i, \quad c_{01} = z_{li}, \quad c_{10} = z_{ji}, \quad c_{11} = z_{ij} + z_{kl}$$

$$Z_m = Lx_m + My_m + h, \quad m = i, j, k, l$$

$$x_{mn} = x_m - x_n, \quad y_{mn} = y_m - y_n, \quad m = i, j, k, l$$

$$z_{mn} = z_m - z_n, \quad m = i, j, k, l \quad m \neq n, \quad n = i, j, k, l$$

$$J_{00} = x_{ji}y_{lj} - x_{lj}y_{ji} = 2A_{ijl}^{xy}$$

$= 2 \times \text{area of the triangle with vertices at } ((x_m, y_m), m = i, j, l)$

$$J_{10} = x_{kl}y_{ij} - x_{ij}y_{kl}$$

$$J_{01} = x_{li}y_{jk} - x_{jk}y_{li}$$

PROOF. From Zienkiewicz [12] and Rathod [13], we know that the arbitrary linear quadrilateral with vertices at (xm, ym) , $m = i, j, k, l$ in the xy plane can be mapped to a unit square $0 \leq u, v \leq 1$ in the uv -plane by use of the isoparametric coordinate transformation (see Fig. 3)

$$\begin{aligned} x &= \sum_{i=0}^1 \sum_{j=0}^1 a_{ij}u^i v^j = x(u, v) \\ y &= \sum_{i=0}^1 \sum_{j=0}^1 b_{ij}u^i v^j = y(u, v) \\ z &= L_x + M_y + h = \sum_{i=0}^1 \sum_{j=0}^1 c_{ij}u^i v^j = z(u, v) \end{aligned} \quad (31)$$

where $((a_{ij}, b_{ij}, c_{ij}, i = 0, 1), j = 0, 1)$ are as defined already in Eq. (30).

Using the transformations of Eq. (31), we obtain

$$\begin{aligned} H_{Q_{ijkl}}^{\alpha, \beta, \gamma+1} &\stackrel{\text{def}}{=} \int \int_{Q_{ijkl}} x^\alpha y^\beta (Lx + My + h)^{\gamma+1} dx dy \\ &= \int_0^1 \int_0^1 x^\alpha(u, v) y^\beta(u, v) z^{\gamma+1}(u, v) (J_{00} + J_{10}u + J_{01}v) du dv \end{aligned} \quad (32)$$

where J_{00}, J_{10}, J_{01} are defined in Eq. (30).

Let us now define

$$X(u, v) = x^\alpha(u, v), \quad Y(u, v) = y^\beta(u, v), \quad Z(u, v) = z^{\gamma+1}(u, v)$$

and

$$f(u, v) = X(u, v)Y(u, v)Z(u, v) \quad (33)$$

Using Eq. (33), we can write

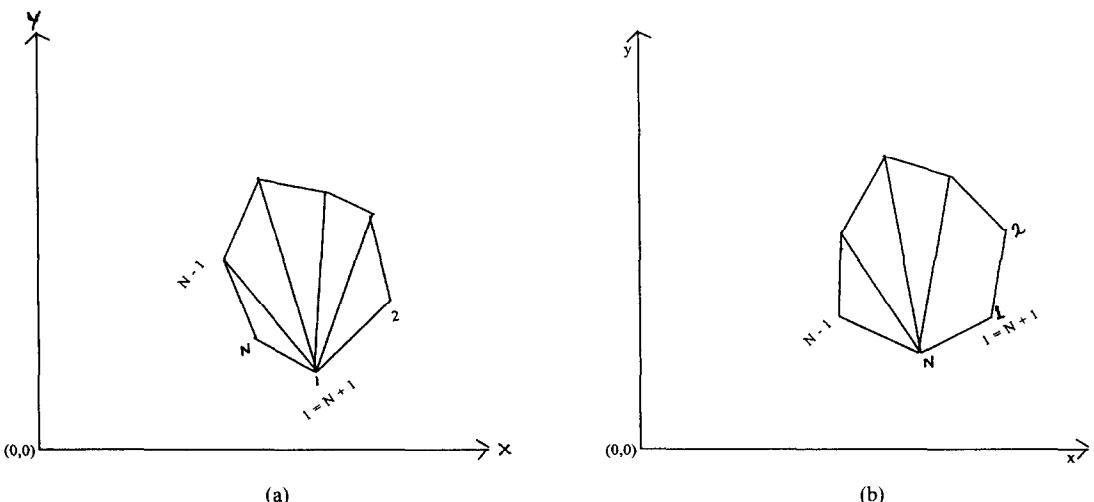


Fig. 3. (a) A simple polygon \bar{A} in the xy -plane which is decomposable in a set T of triangles; (b) a simple polygon \bar{A} in the xy -plane which is expressible (decomposable) in a set T of triangles and a set Q of quadrilaterals.

$$II_{ijkl}^{\alpha, \beta, \gamma+1} = \int_0^1 \int_0^1 f(u, v)(J_{00} + J_{10}u + J_{01}v) du dv \quad (34)$$

We shall now use Taylor's theorem of two variables to expand $f(u, v)$ in powers of u, v and write

$$f(u, v) = \sum_{r=0}^n \sum_{s=0}^n \left[\frac{\partial^{r+s} f(u, v)}{\partial u^r \partial v^s} \right]_{(0,0)} \frac{u^r v^s}{|r|s}, \quad n = \alpha + \beta + \gamma + 1 \quad (35)$$

we shall now determine

$$\left[\frac{\partial^{r+s} f(u, v)}{\partial u^r \partial v^s} \right]_{(0,0)} / |r|s$$

by using generalised form of Leibnitz's theorem on differentiation:

We have on differentiating ' r ' times the function $f(u, v)$ partially w.r.t. ' u :

$$\frac{\partial^r}{\partial u^r} \{f(u, v)\} = \sum_{r_1+r_2+r_3=r} \frac{|r|}{|r_1|r_2|r_3|} \frac{\partial^{r_1} X(u, v)}{\partial u^{r_1}} \frac{\partial^{r_2} Y(u, v)}{\partial u^{r_2}} \frac{\partial^{r_3} Z(u, v)}{\partial u^{r_3}} \quad (36)$$

Again, differentiating Eq. (36) ' s ' times partially w.r.t. ' v ' we obtain

$$\begin{aligned} \frac{\partial^{r+s}}{\partial u^r \partial v^s} \{f(u, v)\} &= \sum_{r_1+r_2+r_3=r} \frac{|r|}{|r_1|r_2|r_3|} \left(\sum_{s_1+s_2+s_3=s} \frac{|s|}{|s_1|s_2|s_3|} \left[\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} \right] \left[\frac{\partial^{r_2+s_2} Y(u, v)}{\partial u^{r_2} \partial v^{s_2}} \right] \right. \\ &\quad \times \left. \left[\frac{\partial^{r_3+s_3} Z(u, v)}{\partial u^{r_3} \partial v^{s_3}} \right] \right) \\ 0 \leq r+s &\leq 2(\alpha + \beta + \gamma + 1) \\ 0 \leq r_1+s_1 &\leq 2\alpha, \quad 0 \leq r_1, \quad s_1 \leq \alpha, \\ 0 \leq r_2+s_2 &\leq 2\beta, \quad 0 \leq r_2, \quad s_2 \leq \beta, \\ 0 \leq r_3+s_3 &\leq 2(\gamma + 1), \quad 0 \leq r_3, \quad s_3 \leq (\gamma + 1) \end{aligned} \quad (37)$$

Thus, from Eq. (37) we obtain

$$\begin{aligned} \left[\frac{\partial^{r+s} f(u, v)}{\partial u^r \partial v^s} \right]_{(0,0)} / (|r|s) &\stackrel{\text{def}}{=} \Phi(r, s) \underline{\alpha} \underline{\beta} \underline{\gamma + 1} \quad (\text{say}) \\ &= \sum_{r_1+r_2+r_3=r} \frac{1}{|r_1|r_2|r_3|} \sum_{s_1+s_2+s_3=s} \frac{1}{|s_1|s_2|s_3|} \left[\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} \right]_{(0,0)} \left[\frac{\partial^{r_2+s_2} Y(u, v)}{\partial u^{r_2} \partial v^{s_2}} \right]_{(0,0)} \\ &\quad \times \left[\frac{\partial^{r_3+s_3} Z(u, v)}{\partial u^{r_3} \partial v^{s_3}} \right]_{(0,0)} \end{aligned} \quad (38)$$

Let us now illustrate the determination the quantities:

$$\left[\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} \right]_{(0,0)}, \left[\frac{\partial^{r_2+s_2} Y(u, v)}{\partial u^{r_2} \partial v^{s_2}} \right]_{(0,0)}, \left[\frac{\partial^{r_3+s_3} Z(u, v)}{\partial u^{r_3} \partial v^{s_3}} \right]_{(0,0)}$$

We have from Eqs. (31) and (33)

$$\begin{aligned} \frac{\partial^{r_1} X(u, v)}{\partial u^{r_1}} &= \frac{\partial^{r_1}}{\partial u^{r_1}} [\{(a_{00} + a_{01}v) + u(a_{10} + a_{11}v)\}^\alpha] \\ &\quad \times (a_{10} + a_{11}v)^{r_1} \alpha(\alpha - 1)(\alpha - 2) \dots (\alpha - r_1 + 1) x^{\alpha - r_1}(u, v) \end{aligned}$$

so that,

$$\begin{aligned}
\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} &= \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r_1+1) \frac{\partial^{s_1}}{\partial v^{s_1}} \{(a_{10} + a_{11}v)^{r_1} x^{\alpha-r_1}(u, v)\} \\
&= \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r_1+1) \sum_{t_1+t'_1=s_1} \frac{s_1}{|t_1| |t'_1|} \left\{ \frac{\partial^{t_1}}{\partial v^{t_1}} (a_{10} + a_{11}v)^{r_1} \right\} \\
&\quad \times \left\{ \frac{\partial^{t'_1}}{\partial v^{t'_1}} (a_{00} + a_{10}u) + v(a_{01} + a_{11}u)^{\alpha-r_1} \right\} \\
&= \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r_1+1) \sum_{t_1+t'_1=s_1} \frac{s_1}{|t_1| |t'_1|} \\
&\quad \times r_1(r_1-1)\dots(r_1-t_1+1) a_{11}^{t_1} (a_{10} + a_{11}v)^{r_1-t_1} \\
&\quad \times (\alpha-r_1)(\alpha-r_1-1)\dots(\alpha-r_1-1)(\alpha-r_1-t'_1+1) \\
&\quad \times (a_{01} + a_{11}u)^{t'_1} x^{\alpha-r_1-t'_1}(u, v)
\end{aligned} \tag{39}$$

$$\begin{aligned}
\therefore \left[\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} \right]_{(0,0)} &= \alpha(\alpha-1)(\alpha-2)\dots(\alpha-r_1+1) \sum_{t_1+t'_1=s_1} \frac{|s_1|}{|t_1| |t'_1|} \\
&\quad \times r_1(r_1-1)\dots(r_1-t_1+1) a_{11}^{t_1} a_{10}^{r_1-t_1} \\
&\quad \times (\alpha-r_1)(\alpha-r_1-1)(\alpha-r_1-2)\dots(\alpha-r_1-t'_1+1) a_{01}^{t'_1} a_{00}^{\alpha-r_1-t'_1} \\
&= \sum_{t_1+t'_1=s_1} |\underline{\alpha}| \frac{a_{00}^{\alpha-r_1-t'_1}}{|\alpha-r_1-t'_1|} \frac{a_{10}^{r_1-t_1}}{|r_1-t_1|} \frac{a_{01}^{t'_1}}{|t'_1|} \frac{a_{11}^{t_1}}{|t_1|} |\underline{r_1}| |\underline{s_1}| \\
&= \sum_{t_1=0}^{s_1} |\underline{\alpha}| |\underline{r_1}| |\underline{s_1}| \frac{a_{00}^{\alpha-r_1-s_1+t_1}}{|\alpha-r_1-s_1+t_1|} \frac{a_{10}^{r_1-t_1}}{|r_1-t_1|} \frac{a_{01}^{s_1-t_1}}{|s_1-t_1|} \frac{a_{11}^{t_1}}{|t_1|}
\end{aligned} \tag{40}$$

From Eq. (40), we thus obtain

$$\begin{aligned}
\left[\frac{\partial^{r_1+s_1} X(u, v)}{\partial u^{r_1} \partial v^{s_1}} \right]_{(0,0)} / (|\underline{r_1}| |\underline{s_1}|) &\stackrel{\text{def}}{=} |\underline{\alpha}| E(r_1, s_1) \quad (\text{say}) \\
&= |\underline{\alpha}| \sum_{t_1=0}^{s_1} \frac{a_{00}^{\alpha-r_1-s_1+t_1}}{|\alpha-r_1-s_1+t_1|} \frac{a_{10}^{r_1-t_1}}{|r_1-t_1|} \frac{a_{01}^{s_1-t_1}}{|s_1-t_1|} \frac{a_{11}^{t_1}}{|t_1|}
\end{aligned} \tag{41}$$

where

$$\alpha - r_1 - s_1 + t_1 \geq 0, \quad r_1 - t_1 \geq 0, \quad s_1 - t_1 \geq 0, \quad 0 \leq t_1 \leq s_1, \quad 0 \leq r_1, \quad s_1 \leq \alpha$$

In a similar manner, we can obtain,

$$\begin{aligned}
\left[\frac{\delta^{r_2+s_2} Y(u, v)}{\delta u^{r_2} \delta v^{s_2}} \right]_{(0,0)} / (|\underline{r_2}| |\underline{s_2}|) &\stackrel{\text{def}}{=} |\underline{\beta}| F(r_2, s_2) \quad (\text{say}) \\
&= |\underline{\beta}| \sum_{t_2=0}^{s_2} \frac{b_{00}^{\beta-r_2-s_2+t_2}}{|\beta-r_2-s_2+t_2|} \frac{b_{10}^{r_2-t_2}}{|r_2-t_2|} \frac{b_{01}^{s_2-t_2}}{|s_2-t_2|} \frac{b_{11}^{t_2}}{|t_2|}
\end{aligned} \tag{43}$$

where

$$\beta - r_2 - s_2 + t_2 \geq 0, \quad r_2 - t_2 \geq 0, \quad s_2 - t_2 \geq 0, \quad 0 \leq t_2 \leq s_2, \quad 0 \leq r_2, \quad s_2 \leq \beta \tag{44}$$

$$\begin{aligned}
\left[\frac{\delta^{r_3+s_3} Z(u, v)}{\delta u^{r_3} \delta v^{s_3}} \right]_{(0,0)} / (|\underline{r_3}| |\underline{s_3}|) &\stackrel{\text{def}}{=} |\underline{\gamma}+1| G(r_3, s_3) \quad (\text{say}) \\
&= |\underline{\gamma}+1| \sum_{t_3=0}^{s_3} \frac{c_{00}^{\gamma+1-r_3-s_3+t_3}}{|\gamma+1-r_3-s_3+t_3|} \frac{c_{10}^{r_3-t_3}}{|r_3-t_3|} \frac{c_{01}^{s_3-t_3}}{|s_3-t_3|} \frac{c_{11}^{t_3}}{|t_3|}
\end{aligned} \tag{45}$$

where

$$\gamma + 1 - r_3 + s_3 + t_3 \geq 0, \quad r_3 - t_3 \geq 0, \quad s_3 - t_3 \geq 0, \quad 0 \leq t_3 \leq s_3, \quad 0 \leq r_3, \quad s_3 \leq (\gamma + 1) \quad (46)$$

on using Eqs. (38)–(46), we can now rewrite Eq. (35) as

$$f(u, v) = \sum_{r=0}^n \sum_{s=0}^n \Phi(r, s) u^r v^s |\underline{\alpha}| |\underline{\beta}| |\underline{\gamma} + 1| \quad (47)$$

$$\Phi(r, s) = \sum_{r_1+r_2+r_3+r=s} \sum_{s_1+s_2+s_3+s=r} E(r_1, s_1) F(r_2, s_2) G(r_3, s_3) \quad (48)$$

with $E(r_1, s_1)$, $F(r_2, s_2)$ and $G(r_3, s_3)$ as defined in Eqs. (41)–(46) substituting from Eqs. (33), (34), (35) and (47) into Eq. (32), we obtain

$$\begin{aligned} II_{Q_{ijkl}^{xy}}^{\alpha, \beta, \gamma+1} &= \int_0^1 \int_0^1 f(u, v) (J_{00} + J_{10}u + J_{01}v) du dv \\ &= |\underline{\alpha}| |\underline{\beta}| |\underline{\gamma} + 1| \int_0^1 \int_0^1 \sum_{r=0}^n \sum_{s=0}^n \Phi(r, s) u^r v^s (J_{00} + J_{10}u + J_{01}v) du dv \\ &= |\underline{\alpha}| |\underline{\beta}| |\underline{\gamma} + 1| \sum_{r=0}^n \sum_{s=0}^n \Phi(r, s) I(r, s) \end{aligned} \quad (49)$$

where

$$\begin{aligned} I(r, s) &= \int_0^1 \int_0^1 u^r v^s (J_{00} + J_{10}u + J_{01}v) du dv \\ &= \frac{(r+2)(s+2)J_{00} + (r+1)(s+2)J_{10} + (r+2)(s+1)J_{01}}{(r+1)(r+2)(s+1)(s+2)} \end{aligned} \quad (50)$$

This completes the proof of Lemma 2. \square

Note: It should be noted that in order to avoid division, we should compute $|\underline{\alpha} E(r_1, s_1)$, $|\underline{\beta} F(r_2, s_2)$ and $|\underline{\gamma} + 1 G(r_3, s_3)$ rather than $E(r_1, s_1)$, $F(r_2, s_2)$ and $G(r_3, s_3)$ as we can show that

$$|\underline{\alpha} E(r_1, s_1) = \sum_{t_1=0}^{s_1} \binom{\alpha}{r_1 + s_1 - t_1} \binom{r_1 + s_1 - t_1}{s_1} \binom{s_1}{t_1} a_{00}^{\alpha - r_1 - s_1 + t_1} a_{10}^{r_1 - t_1} a_{01}^{s_1 - t_1} a_{11}^{t_1} \quad (50a)$$

$$|\underline{\beta} F(r_2, s_2) = \sum_{t_2=0}^{s_2} \binom{\beta}{r_2 + s_2 - t_2} \binom{r_2 + s_2 - t_2}{s_2} \binom{s_2}{t_2} b_{00}^{\beta - r_2 - s_2 + t_2} b_{10}^{r_2 - t_2} b_{01}^{s_2 - t_2} b_{11}^{t_2} \quad (50b)$$

$$|\underline{\gamma} + 1 G(r_3, s_3) = \sum_{t_3=0}^{s_3} \binom{\gamma + 1}{r_3 + s_3 - t_3} \binom{r_3 + s_3 - t_3}{s_3} \binom{s_3}{t_3} c_{00}^{\gamma + 1 - r_3 - s_3 + t_3} c_{10}^{r_3 - t_3} c_{01}^{s_3 - t_3} c_{11}^{t_3} \quad (50c)$$

LEMMA 3. Let $T_{i,j,k}^{xy}$ be linear triangle in the xy -plane with vertices at $((x_a, y_a), a = i, j, k)$, that is linear triangle obtained from a linear quadrilateral $Q_{i,j,k,l}^{xy}$ by letting $(x_i, y_i) = (x_1, y_1)$. Then the structure product over the area of the triangle $T_{i,j,k,l}^{xy}$ defined by

$$II_{T_{i,j,k}^{xy}}^{\alpha, \beta, \gamma+1} \stackrel{\text{def}}{=} \int \int_{T_{i,j,k}^{xy}} x^\alpha y^\beta z^{\gamma+1} dx dy \quad (51)$$

with $z = z(x, y) = Lx + My + h$ as the equation of the plane spanning points (x_m, y_m, z_m) , $m = i, l, j, k$ in the three space is expressible as

$$I_{T_{i,j,k}^{xy}}^{\alpha, \beta, \gamma+1} = (2\Delta_{i,j,k}^{xy})^{\alpha + \beta + \gamma + 1} \sum_{m=0}^m \sum_{n=0}^m \frac{\Lambda(m, n)}{(m+2)(n+1)} \quad (52)$$

where

$$\begin{aligned}
 A(m, n) &= \sum_{p=0}^m \sum_{q=0}^{m-p} \sum_{\lambda=0}^n \sum_{\mu=0}^{n-\lambda} F(p, \lambda) G(q, \mu) H(m-p-q, n-\lambda-\mu) \\
 F(p, \lambda) &= \begin{cases} 0, & p < \lambda \\ \binom{\alpha}{p} a_{00}^{\alpha-p} a_{11}^p, & p = \lambda \\ \binom{\alpha}{p} \binom{p}{\lambda} a_{00}^{\alpha-p} a_{10}^{p-\lambda} a_{11}^\lambda, & p > \lambda \end{cases} \\
 G(q, \mu) &= \begin{cases} 0, & q < \mu \\ \binom{\beta}{q} b_{00}^{\beta-q} b_{11}^q, & q = \mu \\ \binom{\beta}{q} \binom{q}{\mu} b_{00}^{\beta-q} b_{10}^{q-\mu} b_{11}^\mu, & q > \mu \end{cases} \\
 H(m-p-q, n-\lambda-\mu) &= \begin{cases} 0, & \text{if } (m-p-q) < (n-\lambda-\mu) \\ \binom{\gamma+1}{m-p-q} c_{00}^{\gamma+1-(m-p-q)} c_{11}^{m-p-q}, & \text{if } (m-p-q) = (n-\lambda-\mu) \\ \binom{\gamma+1}{m-p-q} \binom{m-p-q}{n-\lambda-\mu} c_{00}^{(\gamma+1)-(m-p-q)} \\ \quad \times c_{10}^{(m-p-q)-(n-\lambda-\mu)} c_{11}^{(n-\lambda-\mu)}, & \text{if } (m-p-q) = (n-\lambda-\mu) \end{cases} \\
 a_{0,0} &= x_i, \quad a_{1,0} = x_{ji}, \quad a_{1,1} = x_{kj} \\
 b_{0,0} &= y_i, \quad b_{1,0} = y_{ji}, \quad b_{1,1} = y_{kj} \\
 c_{0,0} &= z_i, \quad c_{1,0} = z_{ji}, \quad c_{1,1} = z_{kj} \\
 x_{ab} &= x_a - x_b, \quad y_{ab} = y_a - y_b, \quad z_{ab} = z_a - z_b \\
 a &= i, j, k, \quad a \neq b, \quad b = i, j, k \\
 2\Delta_{ijk}^{xy} &= (x_j - x_i)(y_k - y_j) - (x_k - x_j)(y_j - y_i) \\
 &= x_{ji}y_{kj} - x_{kj}y_{ji} \tag{53}
 \end{aligned}$$

PROOF. From Zienkiewicz [12] and Rathod [13] we know that the arbitrary linear quadrilateral with vertices at $((x_a, y_a), a = i, j, k, l)$ can be mapped to a unit square $0 \leq u, v \leq 1$ in the uv -plane by use of the isoparametric coordinate transformation (see Fig. 3) is already given in Eq. (31). If we now further assume that the nodes i and l are tied together, so that we have $(x_i, y_i) = (x_l, y_l)$, then we obtain a new isoparametric coordinate transformation which maps an arbitrary linear triangle to a unit square $0 \leq u, v \leq 1$ (see Fig. 4)

$$\begin{aligned}
 x &= x(u, v) = a_{00} + a_{10}u + a_{11}uv, \\
 y &= y(u, v) = b_{00} + b_{10}u + b_{11}uv, \\
 z &= z(u, v) = Lx(u, v) + My(u, v) + h \\
 &\quad = c_{00} + c_{10}u + c_{11}uv, \tag{54}
 \end{aligned}$$

where the constants $a_{00}, a_{10}, a_{11}, b_{00}, b_{10}, b_{11}, c_{00}, c_{10}, c_{11}$ are already defined in Eq. (53).

Using the transformation of Eq. (54), we obtain

$$\begin{aligned}
 II_{T_{i,j,k}^{xy}}^{\alpha, \beta, \gamma+1} &\stackrel{\text{def}}{=} \int \int_{T_{i,j,k}^{xy}} x^\alpha y^\beta z^{\gamma+1} dx dy \\
 &= \int_0^1 \int_0^1 x^\alpha(u, v) y^\beta(u, v) z^{\gamma+1}_{z(u,v)} \frac{\partial(x, y)}{\partial(u, v)} du dv \\
 &= (2\Delta_{ijk}^{xy}) \int_0^1 \int_0^1 ux^\alpha(u, v) y^\beta(u, v) z^{\gamma+1}_{z(u,v)} du dv \tag{55}
 \end{aligned}$$

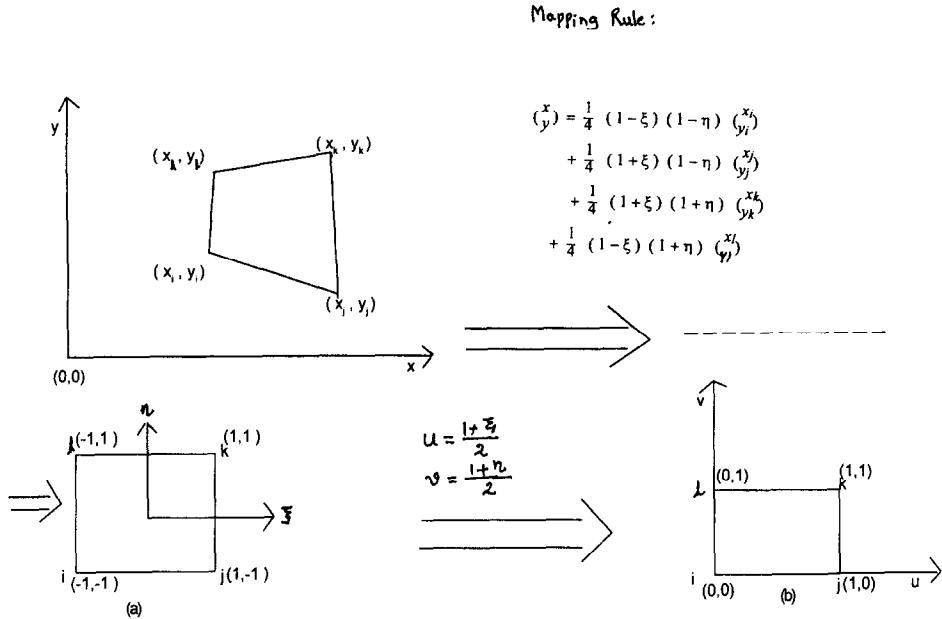


Fig. 4. Mapping of an arbitrary linear quadrilateral element into: (a) a square of 2-unit side length; (b) a square of 1-unit side length.

where

$$\begin{aligned} 2\Delta_{ijk}^{xy} &= (x_j - x_i)(y_k - y_j) - (x_k - x_j)(y_j - y_i) \\ &= 2 \times \text{area of triangle spanning vertices } (x_i, y_i), (x_j, y_j) \text{ and } (x_k, y_k) \\ &= \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} \end{aligned} \quad (56)$$

Let us now define

$$X(u, v) = x^\alpha(u, v), \quad Y(u, v) = y^\beta(u, v), \quad Z(u, v) = z^{\gamma+1}(u, v)$$

and

$$f(u, v) = X(u, v)Y(u, v)Z(u, v) \quad (57)$$

Using Eq. (57), we can write

$$II_{T_{i,j,k}}^{\alpha, \beta, \gamma+1} = (2\Delta_{ijk}^{xy}) \int_0^1 \int_0^1 uf(u, v) du dv \quad (58)$$

We can write, from Eqs. (57)

$$f(u, v) = \sum_{m=0}^{\alpha+\beta+\gamma+1} \sum_{n=0}^m \frac{u^m v^n}{|m|n} \left[\frac{\partial^{m+n} f(u, v)}{\partial u^m \partial v^n} \right]_{(u=0, v=0)} \quad (59)$$

We shall now determine the term

$$\left[\frac{\partial^{m+n} f(u, v)}{\partial u^m \partial v^n} \right]_{(0,0)}$$

Using Leibnitz's theorem on differentiation,

$$\begin{aligned}
\frac{\partial^m f(u, v)}{\partial u^m} &= \frac{\partial^m}{\partial u^m} \{X(u, v)Y(u, v)Z(u, v)\} \\
&= \sum_{m_1+m_2+m_3=m} \frac{|m|}{|m_1|m_2|m_3|} \frac{\partial^{m_1} X(u, v)}{\partial u^{m_1}} \frac{\partial^{m_2} Y(u, v)}{\partial u^{m_2}} \frac{\partial^{m_3} Z(u, v)}{\partial u^{m_3}} \\
&= \sum_{p=0}^m \sum_{q=0}^{m-p} \frac{|m|}{|p|q|m-p-q|} \left(\frac{\partial^p X(u, v)}{\partial u^p} \right) \left(\frac{\partial^q Y(u, v)}{\partial u^q} \right) \left(\frac{\partial^{m-p-q} Z(u, v)}{\partial u^{m-p-q}} \right). \tag{60} \\
\frac{\partial^{m+n} f(u, v)}{\partial u^m \partial v^n} &= \frac{\partial^{m+n}}{\partial u^m \partial v^n} \{X(u, v)Y(u, v)Z(u, v)\} \\
&= \sum_{p=0}^m \sum_{q=0}^{m-p} \frac{|m|}{|p|q|m-p-q|} \left\{ \sum_{\lambda=0}^n \sum_{\mu=0}^{n-\lambda} \left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} \right) \left(\frac{\partial^{q+\mu} Y(u, v)}{\partial u^q \partial v^\mu} \right) \right. \\
&\quad \times \left. \left(\frac{\partial^{(m-p-q)+(n-\lambda-\mu)} Z(u, v)}{\partial u^{m-p-q} \partial v^{n-\lambda-\mu}} \right) \times \frac{|n|}{|\lambda|\mu|n-\lambda-\mu|} \right\}.
\end{aligned}$$

Thus, from Eq. (60), we obtain

$$\begin{aligned}
\left[\frac{\partial^{m+n} f(u, v)}{\partial u^m \partial v^n} \right]_{(0,0)} &= \sum_{p=0}^m \sum_{q=0}^{m-p} \frac{1}{|p|q|m-p-q|} \left\{ \sum_{\lambda=0}^n \sum_{\mu=0}^{n-\lambda} \frac{1}{|\lambda|\mu|n-\lambda-\mu|} \right. \\
&\quad \times \left. \left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} \right)_{(0,0)} \left(\frac{\partial^{q+\mu} Y(u, v)}{\partial u^q \partial v^\mu} \right)_{(0,0)} \left(\frac{\partial^{(m-p-q)+(n-\lambda-\mu)} Z(u, v)}{\partial u^{m-p-q} \partial v^{n-\lambda-\mu}} \right)_{(0,0)} \right\} \tag{61}
\end{aligned}$$

Let us now illustrate the determination of quantities

$$\left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} \right)_{(0,0)} \left(\frac{\partial^{q+\mu} Y(u, v)}{\partial u^q \partial v^\mu} \right)_{(0,0)} \left(\frac{\partial^{(m-p-q)+(n-\lambda-\mu)} Z(u, v)}{\partial u^{m-p-q} \partial v^{n-\lambda-\mu}} \right)_{(0,0)}$$

we have

$$\begin{aligned}
X(u, v) &= x^\alpha(u, v) = [a_{00} + u(a_{10} + a_{11}v)]^\alpha \\
&= \sum_{r=0}^{\alpha} \binom{\alpha}{r} a_{00}^{\alpha-r} u^r (a_{10} + a_{11}v)^r \\
\therefore \frac{\partial^p X(u, v)}{\partial u^p} &= \sum_{r=p}^{\alpha} \binom{\alpha}{r} a_{00}^{\alpha-r} (a_{10} + a_{11}v)^r r(r-1)\dots(r-p+1) u^{r-p} \\
&= \binom{\alpha}{p} |p| (a_{10} + a_{11}v)^p a_{00}^{\alpha-p} \\
&\quad + \binom{\alpha}{p+1} |p+1| (a_{10} + a_{11}v)^{p+1} a_{00}^{\alpha-(p+1)} u \\
&\quad \vdots \\
&\quad \vdots \\
&+ \binom{\alpha}{\alpha-1} \frac{|\alpha-1|}{|\alpha-1-p|} (a_{10} + a_{11}v)^{\alpha-1} a_{00} u^{\alpha-1-p} \\
&+ \binom{\alpha}{\alpha} \frac{|\alpha|}{|\alpha-p|} (a_{10} + a_{11}v)^\alpha u^{\alpha-p} \\
&= \sum_{r=p}^{\alpha} \binom{\alpha}{r} a_{00}^{\alpha-r} (a_{10} + a_{11}v)^r \frac{|r|}{|r-p|} u^{r-p} \tag{62}
\end{aligned}$$

clearly, we have from Eq. (62)

$$\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} = 0, \quad \text{if } p < \lambda \quad (63)$$

Let $p > \lambda$, i.e. $\lambda < p$, then we have

$$\begin{aligned} \frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} &= \sum_{r=p}^{\alpha} \binom{\alpha}{r} a_{00}^{\alpha-r} \frac{|r|}{|r-p|} u^{r-p} (a_{10} + a_{11}v)^{r-\lambda} + r(r-1)\dots(r-\lambda+1) a_{11}^\lambda \\ &= \sum_{r=p}^{\alpha} \binom{\alpha}{p} a_{00}^{\alpha-r} \frac{|r|}{|r-p|} u^{r-p} (a_{10} + a_{11}v)^{r-\lambda} \frac{|r|}{|r-\lambda|} a_{11}^\lambda \end{aligned} \quad (64)$$

from Eq. (63), we thus have

$$\left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} \right)_{(0,0)} = \binom{\alpha}{p} a_{00}^{\alpha-p} a_{10}^{p-\lambda} \frac{|p|}{|p-\lambda|} a_{11}^\lambda, \quad \text{if } p > \lambda \quad (65)$$

Let $\lambda > p$, i.e. $p < \lambda$, so that

$$\begin{aligned} \frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} &= \frac{\partial^\lambda}{\partial v^\lambda} \left\{ \sum_{r=p}^{\alpha} \binom{\alpha}{p} \frac{|r|}{|r-p|} u^{r-p} (a_{10} + a_{11}v)^r \right\} \\ &= \frac{\partial^\lambda}{\partial v^\lambda} \left\{ \sum_{r=\lambda}^{\alpha} \binom{\alpha}{p} \frac{|r|}{|r-p|} u^{r-p} (a_{10} + a_{11}v)^r \right\} \\ &= \sum_{r=\lambda}^{\alpha} \binom{\alpha}{p} \frac{|r|}{|r-p|} u^{r-p} \frac{|r|}{|r-\lambda|} (a_{10} + a_{11}v)^{r-\lambda} a_{11}^\lambda \end{aligned} \quad (66)$$

From Eq. (66), we thus have

$$\left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} \right)_{(0,0)} = 0, \quad \text{if } p < \lambda \quad (67)$$

let $p = \lambda$, so that Eq. (62), we have

$$\begin{aligned} \frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda} &= \frac{\partial^{2p} X(u, v)}{\partial u^p \partial v^p} \\ &= \frac{\partial^p}{\partial v^p} \left\{ \sum_{r=p}^{\alpha} \binom{\alpha}{r} a_{00}^{\alpha-r} (a_{10} + a_{11}v)^r \frac{|r|}{|r-p|} u^{r-p} \right\} \\ &= \sum_{r=p}^{\alpha} \left\{ \binom{\alpha}{r} a_{00}^{\alpha-r} \frac{|r|}{|r-p|} u^{r-p} (a_{10} + a_{11}v)^{r-p} (a_{11})^p r(r-1)\dots(r-p+1) \right\} \end{aligned}$$

Thus, we have from the above equation

$$\begin{aligned} \left(\frac{\partial^{2p} X(u, v)}{\partial u^p \partial v^p} \right)_{0,0} &= \binom{\alpha}{p} a_{00}^{\alpha-p} \frac{|p|}{|p|} u^0 a_{10}^0 a_{11}^p |p| \\ &= \binom{\alpha}{p} a_{00}^{\alpha-p} a_{11}^p (\underline{p})^2, \quad \text{if} \end{aligned} \quad (68)$$

From Eqs. (63), (65) and (68), we thus have

$$\frac{\left(\frac{\partial^{p+\lambda} X(u, v)}{\partial u^p \partial v^\lambda}\right)_{(0,0)}}{|\underline{p}| \underline{\lambda}} = \begin{cases} 0, & \text{if } p < \lambda \\ \binom{\alpha}{p} a_{00}^{\alpha-p} a_{11}^p, & \text{if } p = \lambda \\ \binom{\alpha}{p} \binom{p}{\lambda} a_{00}^{\alpha-p} a_{10}^{p-\lambda} a_{11}^\lambda, & \text{if } p > \lambda \end{cases} \stackrel{\text{def}}{=} F(p, \lambda) \quad (69)$$

In a similar manner, we can derive

$$\frac{\left(\frac{\partial^{q+\mu} Y(u, v)}{\partial u^q \partial v^\mu}\right)_{00}}{|\underline{q}| \underline{\mu}} = \begin{cases} 0, & \text{if } q < \mu \\ \binom{\beta}{q} b_{00}^{\beta-q} b_{11}^q, & \text{if } q = \mu \\ \binom{\beta}{q} \binom{\beta}{\mu} b_{00}^{\beta-q} b_{10}^{q-\mu} b_{11}^\mu, & \text{if } q > \mu \end{cases} \stackrel{\text{def}}{=} G(q, \mu) \quad (70)$$

$$\frac{\left(\frac{\partial^{m^*+n^*} Z(u, v)}{\partial u^{m^*} \partial v^{n^*}}\right)_{(0,0)}}{|\underline{m^*}| \underline{n^*}} = \begin{cases} 0, & \text{if } m^* < n^* \\ \binom{\gamma+1}{m^*} c_{00}^{\gamma+1-m^*} c_{11}^{m^*}, & \text{if } m^* = n^* \\ \binom{\gamma+1}{m^*} \binom{m^*}{n^*} c_{00}^{\gamma+1-m^*} c_{10}^{m^*-n^*} c_{11}^{n^*}, & \text{if } m^* > n^* \end{cases} \quad (71)$$

where $m^* = m - p - q$, $n^* = n - \lambda - \mu$.

Hence, from Eqs. (61), (69), (70) and (71), we obtain

$$\begin{aligned} \Lambda(m, n) &\stackrel{\text{def}}{=} \left(\frac{\partial^{m+n} f(u, v)}{\partial u^m \partial v^n}\right)_{00} / (|\underline{m}| |\underline{n}|) \\ &= \sum_{p=0}^m \sum_{q=0}^{m-p} \sum_{\lambda=0}^n \sum_{\mu=0}^{n-\lambda} F(p, \lambda) G(q, \mu) H(m-p-q, n-\lambda-\mu) \end{aligned} \quad (72)$$

From Eqs. (59) and (72), we obtain

$$f(u, v) = \sum_{m=0}^{\alpha+\beta+\gamma+1} \sum_{n=0}^m u^m v^n \Lambda(m, n) \quad (73)$$

Substituting from Eq. (73) into Eq. (58), we obtain

$$\begin{aligned} II_{T_{ijk}^{xy}}^{\alpha, \beta, \gamma+1} &= (2\Delta_{ijk}^{xy}) \int_0^1 \int_0^1 u f(u, v) \, du \, dv \\ &= (2\Delta_{ijk}^{xy}) \sum_{m=0}^{\alpha+\beta+\gamma+1} \sum_{n=0}^m \Lambda(m, n) \left(\int_0^1 \int_0^1 u^{m+1} v^n \, du \, dv \right) \\ &= (2\Delta_{ijk}^{xy}) \sum_{m=0}^{\alpha+\beta+\gamma+1} \sum_{n=0}^m \frac{\Lambda(m, n)}{(m+2)(n+1)} \end{aligned} \quad (74)$$

This completes the proof of Lemma 3. \square

3. Volume integration

In most computational studies, we recognize the importance of obtaining practical explicit formulas for the exact evaluation integrals.

$$\int \int \int_p f(x, y, z) \, dx \, dy \, dz \quad (75)$$

where P is a three polyhedra in R^3 , $dx dy dz$ is the differential volume and $f(x, y, z)$ is a simple function:

$$f(x, y, z) = \sum_{\alpha=0}^n \sum_{\beta=0}^m \sum_{\gamma=0}^p a_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma \quad (76)$$

where α, β, γ are non-negative integers (including zero). However, the present paper is focused on the calculation of the following integral of monomials.

$$III_{H_{1,2,3,\dots,8}}^{\alpha,\beta,\gamma} \stackrel{\text{def}}{=} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz \quad (77)$$

where $H_{1,2,3,\dots,8}$ is an arbitrary linear hexahedron with vertices at $((x_\lambda, y_\lambda, z_\lambda), \lambda = 1, 2, 3, 4, \dots, 8)$.

In our earlier works [10,11], it is stated that an extension to the integral $\int \int \int_p f(x, y, z) dx dy dz$ can be obtained by using the integral:

$$III_{T_{i,j,k,l}}^{\alpha,\beta,\gamma} \stackrel{\text{def}}{=} \int \int \int_{T_{i,j,k,l}} x^\alpha y^\beta z^\gamma dx dy dz \quad (78)$$

where $T_{i,j,k,l}$ is an arbitrary linear tetrahedron with vertices at $((x_\lambda, y_\lambda, z_\lambda), \lambda = i, j, k, l)$ and the linearity property of integral. We shall show in the present section of this paper, how this can be achieved most efficiently for a linear arbitrary hexahedron. This process is gradually developed in Theorems 1–5. This has led us to the important reset that a finite integration formula can be developed for a linear arbitrary hexahedron in terms of integral over the linear quadrilateral faces. This has, in turn, opened the possibility of explicit integration for a linear arbitrary hexahedron by using Lemmas 1 and 2 developed in the previous section.

THEOREM 1. A structure product:

$$III_{H_{1,2,\dots,8}}^{\alpha,\beta,\gamma} = \int \int \int_{H_{1,2,\dots,8}} x^\alpha y^\beta z^\gamma dv$$

over a three-hexahedron $H_{1,2,3,\dots,8}$ ($H_{1,2,\dots,8}$ = a hexahedron in 3-space with vertices at $((x_p, y_p, z_p), p = 1, 2, 3, \dots, 8)$) is a polynomial combination of structure products of suitable order over triangulation of the hexahedral boundary $\partial H_{1,2,3,\dots,8}$

$$III_{H_{1,2,\dots,8}}^{\alpha,\beta,\gamma} = \frac{1}{(\gamma + 1)} \sum_{H_{1,2,\dots,8}} \int \int_T x^\alpha y^\beta z^{\gamma+1} \hat{k} \hat{n} dT$$

where T is a linear triangle in the three-dimensional space, \hat{n} is outward unit normal vector to T and \hat{k} is the unit normal vector along z -axis.

PROOF. The proof follows from Rathod and Govinda Rao [10]. \square

THEOREM 2. A structure product

$$III_{T_e}^{\alpha,\beta,\gamma} = \int \int \int_{T_e} x^\alpha y^\beta z^\gamma dx dy dz$$

where T_e is a tetrahedron with vertices at $((x_p, y_p, z_p), p = i, j, k, l)$ is expressible as polynomial combination of suitable order over triangulation of the tetrahedron boundary:

$$III_{T_e}^{\alpha,\beta,\gamma} = \frac{\Omega(i, j, k, l)}{(\gamma + 1)} [II_{T_{ijk}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{ilj}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{ilk}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{lji}^{xy}}^{\alpha,\beta,\gamma+1}] \quad (79)$$

where

$$\Omega(i, j, k, l) = \frac{|\det J|}{(\det J)},$$

$$\det J = \begin{bmatrix} x_{il} & x_{jl} & x_{kl} \\ y_{il} & y_{jl} & y_{kl} \\ z_{il} & z_{jl} & z_{kl} \end{bmatrix},$$

$$x_{pq} = x_p - x_q, \quad y_{pq} = y_p - y_q, \quad z_{pq} = z_p - z_q,$$

$$(p, q) \in \{(i, l), (j, l), (k, l)\}$$

and

$$II_{T_{ijk}^{xy}}^{\alpha, \beta, \gamma+1} = \int \int_{T_{ijk}^{xy}} x^\alpha y^\beta (l_x + m_y + h)^{\gamma+1} dx dy \quad (80)$$

where $z(x, y) = lx + my + h$ refers to the equation of the plane spanning points $((x_p, y_p, z_p), p = i, j, k)$ and T_{ijk}^{xy} is the oriented triangle in the xy -plane obtained on projecting the linear tetrahedral boundary T_{ijk} ($T_{ijk} = a$ triangle in three spaces with vertices at $((x_p, y_p, z_p), p = i, j, k)$ on the xy -plane.

PROOF. The proof follows from Rathod and Govinda Rao [11]. \square

THEOREM 3. A structure product

$$III_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} \stackrel{\text{def}}{=} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz \quad (81)$$

over a three-hexahedron $H_{1,2,3,\dots,8}$ with vertices at $((x_p, y_p, z_p), p = 1, 2, 3, \dots, 8)$ is a polynomial combination of structure products of suitable order over quadragulation of the hexahedral boundary $\partial H_{1,2,3,\dots,8}$

$$III_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} = \frac{1}{(\gamma+1)} \sum_{Q \in \partial H_{1,2,3,\dots,8}} \int \int_Q x^\alpha y^\beta z^{\gamma+1} \hat{k} \hat{n} dQ \quad (82)$$

where Q is a linear quadrilateral in three-dimensional space, \hat{n} is outward unit normal vector to Q and \hat{k} is the unit normal vector along Z-axis.

PROOF. The proof follows from Rathod and Govinda Rao [10]. \square

THEOREM 4. A structure product over the volume of a linear arbitrary hexahedron $H_{1,2,3,\dots,8}$ with $((x_p, y_p, z_p), p = 1, 2, 3, \dots, 8)$

$$III_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} \stackrel{\text{def}}{=} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz$$

is expressible as the sum of structure products of volume integrals over all tetrahedra formed by disjoint decomposition of the linear hexahedron $H_{1,2,3,\dots,8}$. That is

$$\int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz = \sum_{p,q,r,s \in S} \int \int \int_{T_{p,q,r,s}} x^\alpha y^\beta z^\gamma dx dy dz \quad (83)$$

where

$$S = \{(p_i, q_i, r_i, s_i), i = 1, 2, \dots, n\}$$

and n refers to the number of tetrahedrons $Tpqrs$ that can be obtained by a disjoint decomposition of the linear arbitrary hexahedron $H_{1,2,3,\dots,8}$.

PROOF. Follows from the conditions on regularity of integration domain and continuity of integrating function. \square

3.1. Hexahedron as a assemblage of tetrahedra

The division of space volume into individual tetrahedra sometimes presents difficulties of visualisation and could easily lead to errors in node numbering, etc. A more convenient subdivision of space is into eight cornered arbitrary hexahedra. Such elements could easily be assembled automatically from several tetrahedra and process of creating these tetrahedra left to a simple logical program. It will be readily appreciated from the exploded view, that an hexahedron element could be built in two and only two distinct ways from the five tetrahedral shaped elements. This has been proposed in [12]. Both possible divisions of a hexahedron into five tetrahedral shaped elements are illustrated in Figs. 5 and 6 and this concept could easily be used in Computer Aided Design and stress analysis, etc.

From Figs. 5 and 6 and Theorem 4, we can write

$$\int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz = \sum_{p,q,r,s \in S_0} \int \int \int_{T_{p,q,r,s}} x^\alpha y^\beta z^\gamma dx dy dz \quad (84)$$

where $S_0 = \{(1, 4, 2, 6), (1, 4, 3, 7), (6, 7, 5, 1), (6, 7, 8, 4), (1, 4, 6, 7)\}$.

THEOREM 5. The structure product over the volume of a linear arbitrary hexahedron $H_{1,2,3,\dots,8}$ as defined in Eq. (77) is expressible as

$$\begin{aligned} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz &\stackrel{\text{def}}{=} III_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} \\ &= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [-II_{Q_{1,2,4,3}^{xy}}^{\alpha, \beta, \gamma+1} + II_{Q_{5,6,8,7}^{xy}}^{\alpha, \beta, \gamma+1} \\ &\quad - II_{Q_{1,3,7,5}^{xy}}^{\alpha, \beta, \gamma+1} + II_{Q_{2,4,8,6}^{xy}}^{\alpha, \beta, \gamma+1} - II_{Q_{1,5,6,2}^{xy}}^{\alpha, \beta, \gamma+1} + II_{Q_{3,7,8,4}^{xy}}^{\alpha, \beta, \gamma+1}] \end{aligned} \quad (85)$$

where

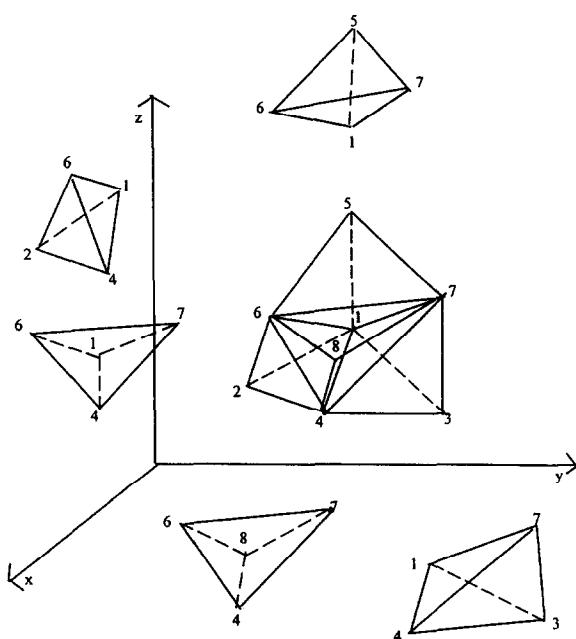


Fig. 5. A systematic way of splitting an eight-cornered hexahedron-shaped brick into five tetrahedra.

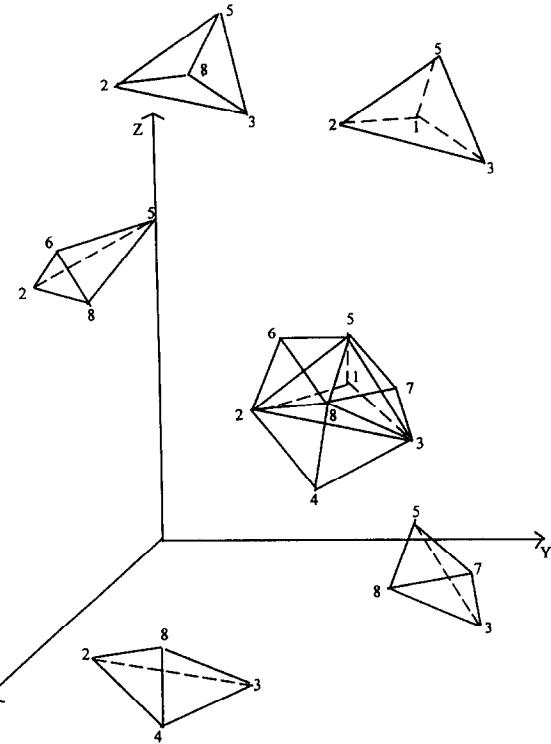


Fig. 6. An alternative systematic way of splitting an eight-cornered hexahedron-shaped brick into five tetrahedra.

$$II_{Q_{i,j,k,l}^{xy}}^{\alpha, \beta, \gamma+1} \stackrel{\text{def}}{=} \int \int_{Q_{i,j,k,l}^{xy}} x^\alpha y^\beta z^{\gamma+1} (x, y) dx dy$$

$$\begin{aligned} Q_{i,j,k,l}^{xy} &= \text{linear quadrilateral spanned by point } ((x_p, y_p, z_p), p = i, j, k) \text{ in the } xy\text{-plane} \\ z(x, y) &= \text{equation of plane spanning point } ((x_p, y_p, z_p), p = i, j, k) \text{ in three space and } \Omega(1, 4, 6, 7) \quad (86) \\ &\text{is as defined in Eq. (80).} \end{aligned}$$

PROOF. One of the district ways to build an arbitrary linear hexahedron from five tetrahedral shaped elements is fully illustrated in the book by Chandrupatla and Belegundu [14]. Now, using the notation (i, j, k, l) to refer to nodal number of corner points for a tetrahedron T_{ijkl} with vertices $((x_p, y_p, z_p), p = i, j, k)$ and $(k, i, j), (l, k, j), (l, i, k)$ and (l, j, i) to refer the four triangular faces of the tetrahedron, we can form the following table for the above subdivision of a linear hexahedron into five tetrahedra.

The above result follows from the fact that

$$II_{T_{pqr}}^{\alpha, \beta, \gamma+1} = II_{T_{grp}}^{\alpha, \beta, \gamma+1} = II_{T_{rpq}}^{\alpha, \beta, \gamma+1} = II_{T_{qpr}}^{\alpha, \beta, \gamma+1} = II_{T_{rqp}}^{\alpha, \beta, \gamma+1} = II_{T_{prq}}^{\alpha, \beta, \gamma+1} \quad (87)$$

$$2\Delta_{pqr}^{xy} = 2 \times \text{area of triangle } T_{pqr}^{xy}, \quad (88)$$

where the corner node p, q, r in anticlockwise orientation, and

$$\begin{aligned} 2\Delta_{pqr}^{xy} &= 2\Delta_{grp}^{xy} = 2\Delta_{rpq}^{xy} \\ &= -2\Delta_{qpr}^{xy} = -2\Delta_{rqp}^{xy} = -2\Delta_{prq}^{xy} \quad (89) \end{aligned}$$

using Eqs. (87) and (89) we find

$$\begin{aligned} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz &= \sum_{p,q,r,s \in S_0} \int \int \int_{T_{p,q,r,s}} x^\alpha y^\beta z^\gamma dx dy dz \\ S_0 &= \{(1, 4, 2, 6), (1, 4, 3, 7), (6, 7, 5, 1), (6, 7, 8, 4), (1, 4, 6, 7)\} \quad (90) \end{aligned}$$

Table 1

Subdivision of an arbitrary linear hexahedron into five tetrahedra with the corresponding triangular faces

Element e	Corner node $ijkl$	Triangular faces			
		(k, i, j)	(l, k, j)	(l, i, k)	(l, j, i)
1	1426	(2, 1, 4)	(6, 2, 4)	(6, 1, 2)	(6, 4, 1)
2	1437	(3, 1, 4)	(7, 3, 4)	(7, 1, 3)	(7, 4, 1)
3	6751	(5, 6, 7)	(1, 5, 7)	(1, 6, 5)	(1, 7, 6)
4	6784	(8, 6, 7)	(4, 8, 7)	(4, 6, 8)	(4, 7, 6)
5	1467	(6, 1, 4)	(7, 6, 4)	(7, 1, 6)	(7, 4, 1)

We have from Eq. (84)

$$\int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz = \frac{1}{(\gamma + 1)} \sum_{p,q,r,s \in S_1} \int \int_{Q_{p,q,r,s}} x^\alpha y^\beta z_{\gamma+1} \hat{k} \hat{n} dx dy \quad (91)$$

where $S_1 = \{(1, 2, 4, 3), (5, 6, 8, 7), (1, 3, 7, 5), (2, 4, 8, 6), (1, 5, 6, 2), (3, 7, 8, 4)\}$. We can form a similar table for alternative distinct subdivision of an arbitrary linear hexahedron into five tetrahedra (see Table 1).

In Theorem 2, we have referred to the integrals of type

$$II_{T_{p,q,r}^{xy}}^{\alpha, \beta, \gamma+1} \stackrel{\text{def}}{=} \int \int_{T_{p,q,r}^{xy}} x^\alpha y^\beta (lx + my + h)^{\gamma+1} dx dy$$

where $T_{p,q,r}^{xy}$ is an oriented triangle in the xy plane. By using usual area coordinates, we can map the arbitrary triangle $T_{p,q,r}^{xy}$ to a unit triangle in the new plane, say uv as (see [10]):

$$\begin{aligned} II_{T_{p,q,r}^{xy}}^{\alpha, \beta, \gamma+1} &= \int \int_{T_{p,q,r}^{xy}} x^\alpha y^\beta (lx + my + h)^{\gamma+1} dx dy \\ &= 2\Delta_{pqr}^{xy} \int \int_{\bar{T}_{p,q,r}^{xy}} x^\alpha(u, v) y^\beta(u, v) z^{\gamma+1}(u, v) du dv \\ &= 2\Delta_{pqr}^{xy} II_{T_{p,q,r}^{xy}}^{\alpha, \beta, \gamma+1} \end{aligned} \quad (92)$$

It can be further shown that

$$II_{T_{p,q,r}^{xy}}^{\alpha, \beta, \gamma+1} = II_{T_{q,r,p}^{xy}}^{\alpha, \beta, \gamma+1} = II_{T_{r,p,q}^{xy}}^{\alpha, \beta, \gamma+1} = -II_{T_{q,p,r}^{xy}}^{\alpha, \beta, \gamma+1} = -II_{T_{r,p,q}^{xy}}^{\alpha, \beta, \gamma+1} = -II_{T_{p,r,q}^{xy}}^{\alpha, \beta, \gamma+1} \quad (93)$$

We know from Gauss's divergence theorem that volume integral is equal to the sum of surface integral taken over the entire boundary surface of the volume V .

When we use this concept in Eq. (90) and further make use of Eqs. (91)–(93), (79) and Table 1:

$$\begin{aligned} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz &= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [(2\Delta_{214}^{xy} II_{T_{2,1,4}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{624}^{xy} II_{T_{6,2,4}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{612}^{xy} II_{T_{6,1,2}}^{\alpha, \beta, \gamma+1})] \\ &\quad - \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [(2\Delta_{314}^{xy} II_{T_{3,1,4}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{734}^{xy} II_{T_{7,3,4}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{713}^{xy} II_{T_{7,1,3}}^{\alpha, \beta, \gamma+1})] \\ &\quad + \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [(2\Delta_{567}^{xy} II_{T_{5,6,7}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{157}^{xy} II_{T_{1,5,7}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{165}^{xy} II_{T_{1,6,5}}^{\alpha, \beta, \gamma+1})] \\ &\quad - \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [(2\Delta_{867}^{xy} II_{T_{8,6,7}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{487}^{xy} II_{T_{4,8,7}}^{\alpha, \beta, \gamma+1}) + (2\Delta_{468}^{xy} II_{T_{4,6,8}}^{\alpha, \beta, \gamma+1})] \end{aligned} \quad (94)$$

We know that

$$\begin{aligned}
II_{Q_{i,j,k,l}^{xy}}^{\alpha,\beta,\gamma+1} &= \int \int_{Q_{i,j,k,l}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y) dx dy \\
&= \left(\int \int_{T_{i,j,k}^{xy}} + \int \int_{T_{j,k,l}^{xy}} \right) x^\alpha y^\beta z^{\gamma+1}(x, y) dx dy \\
&= II_{T_{i,j,k}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{j,k,l}^{xy}}^{\alpha,\beta,\gamma+1} \\
&= \left(\int \int_{T_{i,j,l}^{xy}} + \int \int_{T_{j,k,l}^{xy}} \right) x^\alpha y^\beta z^{\gamma+1}(x, y) dx dy \\
&= II_{T_{i,j,l}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{j,k,l}^{xy}}^{\alpha,\beta,\gamma+1}
\end{aligned} \tag{95}$$

We should note that

$$II_{T_{p,q,r}^{xy}}^{\alpha,\beta,\gamma+1} = II_{T_{q,r,p}^{xy}}^{\alpha,\beta,\gamma+1} = II_{T_{r,p,q}^{xy}}^{\alpha,\beta,\gamma+1} = -II_{T_{r,p,r}^{xy}}^{\alpha,\beta,\gamma+1} = -II_{T_{r,q,p}^{xy}}^{\alpha,\beta,\gamma+1} = -II_{T_{p,r,q}^{xy}}^{\alpha,\beta,\gamma+1} \tag{96}$$

Using the above results of Eqs. (95) and (96) and rearranging terms in Eq. (94), we now obtain

$$\begin{aligned}
III_{H_{1,2,3,\dots,8}}^{\alpha,\beta,\gamma} &\stackrel{\text{def}}{=} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz \\
&= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [-II_{Q_{1,2,4,3}^{xy}}^{\alpha,\beta,\gamma+1} + II_{Q_{3,6,8,7}^{xy}}^{\alpha,\beta,\gamma+1} - II_{Q_{1,3,7,5}^{xy}}^{\alpha,\beta,\gamma+1} + II_{Q_{2,4,8,6}^{xy}}^{\alpha,\beta,\gamma+1} - II_{Q_{1,5,6,2}^{xy}}^{\alpha,\beta,\gamma+1} + II_{Q_{3,7,8,4}^{xy}}^{\alpha,\beta,\gamma+1}]
\end{aligned} \tag{97}$$

This completes the proof of the theorem. \square

4. Application example

Consider the evaluation of volume integral:

$$III_{H_{1,2,3,\dots,8}}^{2,1,0} = \int \int \int_{H_{1,2,3,\dots,8}} x^2 y dx dy dz \tag{98}$$

where $H_{1,2,3,\dots,8}$ is an arbitrary linear hexahedron bounded by the hexahedral surface $\partial H_{1,2,3,\dots,8}$ with vertices ($V_s = (x_s, y_s, z_s)$, $s = 1, 2, 3, \dots, 8$) (see Fig. 7)

$$\begin{aligned}
V_1 &= (0, 0, 0), \quad V_2 = \left(1, \frac{1}{4}, 0\right), \quad V_3 = (0, 1, 0) \\
V_4 &= \left(1, \frac{3}{4}, 0\right), \quad V_5 = \left(0, \frac{3}{16}, \frac{3}{4}\right), \quad V_6 = \left(\frac{3}{4}, \frac{7}{16}, 1\right) \\
V_7 &= \left(0, \frac{19}{16}, \frac{3}{4}\right), \quad V_8 = \left(\frac{3}{4}, \frac{17}{16}, 1\right)
\end{aligned} \tag{99}$$

We know that the integral

$$II_{Q_{i,j,k,l}^{xy}}^{\alpha,\beta,\gamma+1} = \int \int_{Q_{i,j,k,l}^{xy}} x^\alpha y^\beta z^{\gamma+1} dx dy = 0$$

Whenever the equation of the plane spanning points $((x_a, y_a, z_a), a = i, j, k, l)$ is $x = 0$ or $y = 0$ or $z = 0$.

Using this fact and the statement of Theorem 5, we have for the application examples stated in Eqs. (98) and (99) (see Fig. 7).

$$III_{H_{1,2,3,\dots,8}}^{2,1,0} = II_{Q_{2,4,8,6}^{xy}}^{2,1,1} + II_{Q_{1,2,6,5}^{xy}}^{2,1,1} - II_{Q_{3,4,8,7}^{xy}}^{2,1,1} + II_{Q_{5,6,8,7}^{xy}}^{2,1,1} \tag{100}$$

ALGORITHM 1. We shall now illustrate the use of Lemma 1 to evaluate the integrals appearing on the right-hand side of Eq. (100).

We have on using Lemma 1 for linear arbitrary quadrilateral, it can be shown that

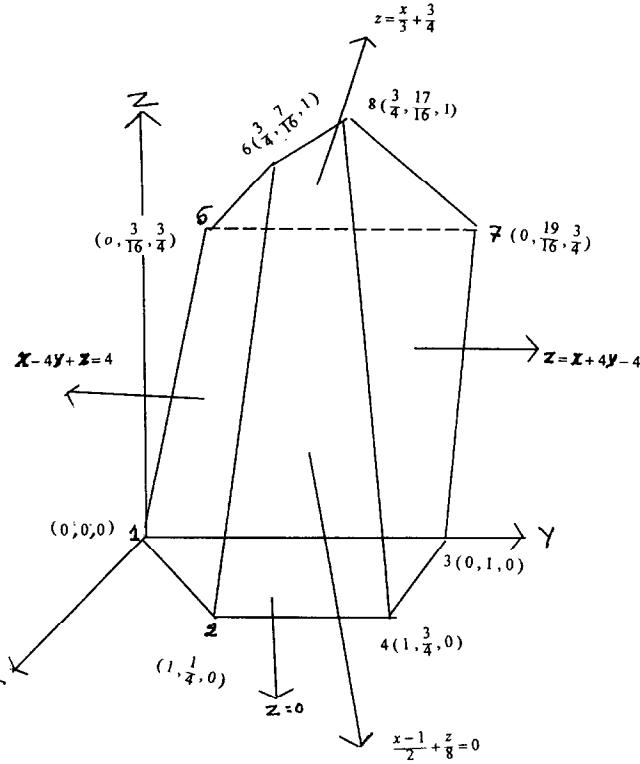


Fig. 7. Arbitrary linear hexahedron for application example. Note: Eq. of plane $\langle 1, 2, 4, 3 \rangle$ is $z = 0$; Eq. of plane $\langle 5, 6, 8, 7 \rangle$ is $z = x/3 + 3/4$; Eq. of plane $\langle 1, 3, 7, 5 \rangle$ is $x = 0$; Eq. of plane $\langle 2, 4, 8, 6 \rangle$ is $(x - 1)/2 + z/8 = 0$; Eq. of plane $\langle 1, 2, 6, 5 \rangle$ is $x - 4y + z = 0$; Eq. of plane $\langle 3, 4, 8, 7 \rangle$ is $z = x + 4y - 4$.

$$II_{Q_{i,j,k,l}^{xy}}^{\alpha,\beta,\gamma+1} = II_{T_{j,o,i}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{k,o,j}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{l,o,k}^{xy}}^{\alpha,\beta,\gamma+1} + II_{T_{i,o,l}^{xy}}^{\alpha,\beta,\gamma+1} \quad (101)$$

We can apply Eq. (101) to each of the integrals appearing in the right-hand side of Eq. (100) we have for the application example $\alpha = 2$, $\beta = 1$, $\gamma = 0$, hence we can write

$$II_{Q_{2,4,8,6}^{xy}}^{2,1,1} = II_{T_{4,0,2}^{xy}}^{2,1,1} + II_{T_{8,0,4}^{xy}}^{2,1,1} + II_{T_{6,0,8}^{xy}}^{2,1,1} + II_{T_{2,0,6}^{xy}}^{2,1,1} \quad (102)$$

the other integrals on the right-hand side of Eq. (100) can be similarly written on using Eq. (101). From Lemma 1, we can show that

$$\begin{aligned} II_{T_{k,o,i}^{xy}}^{2,1,1} &= (A_{koj}^{xy})[F(2, 0)G(1, 0)H(1, 0)] \\ &+ \frac{1}{2}[F(1, 1)G(1, 0)H(0, 1) + F(2, 0)G(0, 1)H(1, 0) + F(2, 0)G(1, 0)H(0, 1)] \\ &+ \frac{1}{3}[F(1, 1)G(1, 0)H(0, 1) + F(2, 0)G(0, 1)H(0, 1) + F(0, 2)G(1, 0)H(1, 0)] \\ &+ F(1, 1)G(0, 1)H(1, 0)] \\ &+ \frac{1}{4}[F(0, 2)G(1, 0)H(0, 1) + F(1, 1)G(0, 1)H(0, 1) + F(2, 0)G(0, 1)H(1, 0)] \\ &+ \frac{1}{5}[F(0, 2)G(0, 1)H(0, 1)] \end{aligned} \quad (103)$$

where

$$\begin{aligned}
F(2, 0) &= x_k^2, \quad F(1, 1) = 2x_k x_{ik}, \quad F(0, 2) = x_{ik}^2 \\
G(1, 0) &= y_k, \quad G(0, 1) = y_{ik}, \\
H(1, 0) &= \begin{cases} z_k, & \text{if } z(0, 0) = h = Z_0 = 0 \\ \frac{Z_{ko}}{6} + \frac{Z_0}{5}, & \text{if } z(0, 0) = h = z_0 \neq 0 \end{cases} \\
H(0, 1) &= \begin{cases} z_{ik}, & \text{if } z(0, 0) = h = z_0 = 0 \\ \frac{z_{ik}}{6}, & \text{if } z(0, 0) = h = z_0 \neq 0 \end{cases} \\
A_{koi}^{xy} &= \begin{cases} 2\Delta_{koi}^{xy}/6, & \text{if } z_0 = h = 0 \\ 2\Delta_{koi}^{xy}, & \text{if } z_0 = h \neq 0 \end{cases} \\
2\Delta_{koi}^{xy} &= x_i y_k - x_k y_i
\end{aligned} \tag{104}$$

We shall also note from Eq. (99) and Fig. 7 that:

$$\begin{aligned}
II_{Q_{2,4,8,6}^{xy}}^{2,1,1} &= \int \int_{Q_{2,4,8,6}^{xy}} x^2 y (4 - 4x) dx dy = \int \int_{Q_{2,4,8,6}^{xy}} x^2 y z dx dy \\
&= \left(\int \int_{T_{4,0,2}^{xy}} + \int \int_{T_{8,0,4}^{xy}} + \int \int_{T_{6,0,8}^{xy}} + \int \int_{T_{2,0,6}^{xy}} \right) x^2 y (4 - 4x) dx dy
\end{aligned} \tag{105}$$

Now, on using Eqs. (99), (103), (104) and (105) we find that

$$\begin{aligned}
II_{Q_{2,4,8,6}^{xy}}^{2,1,1} &= \left(\int \int_{T_{4,0,2}^{xy}} + \int \int_{T_{8,0,4}^{xy}} + \int \int_{T_{6,0,8}^{xy}} + \int \int_{T_{2,0,6}^{xy}} \right) x^2 y (4 - 4x) dx dy \\
&= \frac{1}{30} + \frac{1127}{16^2 \times 60} - \frac{243}{16^3} - \frac{143}{16^2 \times 40} = \frac{2053}{16^3 \times 15}
\end{aligned} \tag{106}$$

In a similar manner, we obtain

$$\begin{aligned}
II_{Q_{5,6,8,7}^{xy}}^{2,1,1} &= \int \int_{T_{5,6,8,7}^{xy}} x^2 y \left(\frac{x}{3} + \frac{3}{4} \right) dx dy \\
&= \left(\int \int_{T_{6,0,5}^{xy}} + \int \int_{T_{8,0,6}^{xy}} + \int \int_{T_{7,0,8}^{xy}} + \int \int_{T_{5,0,7}^{xy}} \right) x^2 y \left(\frac{x}{3} + \frac{3}{4} \right) dx dy \\
&= \left(-\frac{945}{16^4 \times 8} + \frac{621}{16^3 \times 4} + \frac{86697}{16^4 \times 40} + 0 \right) \\
&= \frac{45333}{16^4 \times 10}
\end{aligned} \tag{107}$$

$$\begin{aligned}
II_{Q_{1,2,6,5}^{xy}}^{2,1,1} &= \int \int_{Q_{1,2,6,5}^{xy}} x^2 y (4y - x) dx dy \\
&= \left(\int \int_{T_{2,0,1}^{xy}} + \int \int_{T_{6,0,2}^{xy}} + \int \int_{T_{5,0,6}^{xy}} + \int \int_{T_{1,0,5}^{xy}} \right) x^2 y (4y - x) dx dy \\
&= \left(0 + \frac{247}{16^2 \times 180} + \frac{4077}{16^4 \times 40} + 0 \right) \\
&= \frac{163157}{16^4 \times 360}
\end{aligned} \tag{108}$$

$$\begin{aligned}
II_{Q_{3,4,8,7}^{xy}}^{2,1,1} &= \int \int_{Q_{3,4,8,7}^{xy}} x^2 y(x + 4y - 4) dx dy \\
&= \int \int_{T_{4,0,3}^{xy}} + \int \int_{T_{8,0,4}^{xy}} + \int \int_{T_{7,0,8}^{xy}} + \int \int_{T_{3,0,7}^{xy}} x^2 y(x + 4y - 4) dx dy \\
&= \frac{13}{360} - \frac{841}{16^2 \times 180} - \frac{3629}{16^4 \times 120} + 0 \\
&= \frac{519359}{16^4 \times 360}
\end{aligned} \tag{109}$$

Substituting from Eqs. (106)–(109) into Eq. (100), we obtain

$$\begin{aligned}
III_{H_{1,2,3,\dots,8}}^{2,1,0} &= \int \int \int_{H_{1,2,3,\dots,8}} x^2 y dx dy dz \\
&= II_{Q_{2,4,8,6}^{xy}}^{2,1,1} + II_{Q_{1,2,6,5}^{xy}}^{2,1,1} - II_{Q_{3,4,8,7}^{xy}}^{2,1,1} + II_{Q_{3,6,8,7}^{xy}}^{2,1,1} \\
&= \frac{2053}{16^3 \times 15} + \frac{163157}{16^4 \times 360} - \frac{519359}{16^4 \times 360} + \frac{45333}{16^4 \times 10} \\
&= \frac{344023}{16^4 \times 60}
\end{aligned} \tag{110}$$

ALGORITHM 2. From Lemma 2, we can show that ($\alpha = 2$, $\beta = 1$, $\gamma = 0$)

$$\begin{aligned}
II_{Q_{ijkl}^{xy}}^{2,1,1} &= \int \int_{Q_{ijkl}^{xy}} x^2 yz dx dy \\
&= 2 \sum_{r=0}^4 \sum_{s=0}^4 \Phi(r, s) I(r, s) \\
&= (a_{00}^2 b_{00} c_{00}) \left[J_{00} + \frac{1}{2} (J_{10} + J_{01}) \right] + 2[\{\Phi(1, 1)I(1, 1) + \Phi(2, 2)I(2, 2) \right. \\
&\quad \left. + \Phi(3, 3)I(3, 3) + \Phi(4, 4)I(4, 4)\} + 2[(\{\Phi(0, 1) + \Phi(1, 0)\} + \{\Phi(0, 2) + \Phi(2, 0)\} \right. \\
&\quad \left. + \{\Phi(0, 3) + \Phi(3, 0)\} + \{\Phi(0, 4) + \Phi(4, 0)\} + \{\Phi(1, 2) + \Phi(2, 1)\} + \{\Phi(1, 3) + \Phi(3, 1)\} \right. \\
&\quad \left. + \{\Phi(1, 4) + \Phi(4, 1)\} + \{\Phi(2, 3) + \Phi(3, 2)\} + \{\Phi(2, 4) + \Phi(4, 2)\} + \{\Phi(3, 4) + \Phi(4, 3)\}\}] \quad (111)
\end{aligned}$$

where

$$\begin{aligned}
\Phi(1, 1) &= G(0, 0)\{E(1, 1)F(0, 0) + E(1, 0)F(0, 1) + E(0, 1)F(1, 0) + E(0, 0)F(1, 1)\} \\
&\quad + G(0, 1)\{E(1, 0)F(0, 0) + E(0, 0)F(1, 0)\} + G(1, 0)\{E(0, 1)F(0, 0) + E(0, 0)F(0, 1)\} \\
&\quad + G(1, 1)\{E(0, 0)F(0, 0)\},
\end{aligned}$$

$$\begin{aligned}
\Phi(2, 2) &= G(0, 0)\{E(1, 1)F(1, 1) + E(1, 2)F(1, 0) + E(2, 1)F(0, 1) + E(2, 2)F(0, 0)\} \\
&\quad + G(1, 0)\{E(0, 1)F(1, 1) + E(0, 2)F(1, 0) + E(1, 1)F(0, 1) + E(1, 2)F(0, 0)\} \\
&\quad + G(0, 1)\{E(1, 0)F(1, 1) + E(1, 1)F(1, 0) + E(2, 0)F(0, 1) + E(2, 1)F(0, 0)\} \\
&\quad + G(1, 1)\{E(0, 0)F(1, 1) + E(0, 1)F(1, 0) + E(1, 0)F(0, 1) + E(1, 1)F(0, 0)\},
\end{aligned}$$

$$\begin{aligned}
\Phi(3, 3) &= G(0, 0)\{E(2, 2)F(1, 1)\} + G(0, 1)\{E(2, 2)F(1, 0) + E(2, 1)F(1, 1)\} \\
&\quad + G(1, 0)\{E(2, 2)F(0, 1) + E(1, 2)F(1, 1)\} \\
&\quad + G(1, 1)\{E(2, 2)F(0, 0) + E(2, 1)F(0, 1) + E(1, 2)F(1, 0) + E(1, 1)F(1, 1)\},
\end{aligned}$$

$$\Phi(4, 4) = G(1, 1)E(2, 2)F(1, 1)$$

$$\Phi(0, 1) = \{E(0, 1)F(0, 0) + E(0, 0)F(0, 1)\}G(0, 0) + \{E(0, 0)F(0, 0)\}G(0, 1),$$

$$\Phi(1, 0) = \{E(1, 0)F(0, 0) + E(0, 0)F(1, 0)\}G(0, 0) + \{E(0, 0)F(0, 0)\}G(1, 0),$$

$$\begin{aligned}
\Phi(0, 2) &= \{E(0, 2)F(0, 0) + E(0, 1)F(0, 1)\}G(0, 0) + \{E(0, 1)F(0, 0) + E(0, 0)F(0, 1)\}G(0, 1), \\
\Phi(2, 0) &= \{E(2, 0)F(0, 0) + E(1, 0)F(1, 0)\}G(0, 0) + \{E(1, 0)F(0, 0) + E(0, 0)F(1, 0)\}G(1, 0) \\
\Phi(0, 3) &= \{E(0, 2)F(0, 1)\}G(0, 0) + \{E(0, 2)F(0, 0) + E(0, 1)F(0, 1)\}G(0, 1), \\
\Phi(3, 0) &= \{3, 0\} = \{E(2, 0)F(1, 0)\}G(0, 0) + \{E(2, 0)F(0, 0) + E(1, 0)F(1, 0)\}G(1, 0), \\
\Phi(0, 4) &= E(0, 2)F(0, 1)G(0, 1), \\
\Phi(4, 0) &= E(2, 0)F(1, 0)G(1, 0), \\
\Phi(2, 1) &= \{E(2, 1)F(0, 0) + E(2, 0)F(0, 1) + E(1, 1)F(1, 0) + E(1, 0)F(1, 1)\}G(0, 0) \\
&\quad + \{E(2, 0)F(0, 0) + E(1, 0)F(1, 0)\}G(0, 1) \\
&\quad + \{E(1, 1)F(0, 0) + E(1, 0)F(0, 1) + E(0, 1)F(1, 0) + E(0, 0)F(1, 1)\}G(1, 0) \\
&\quad + \{E(1, 0)F(0, 0) + E(0, 0)F(1, 0)\}G(1, 1), \\
\Phi(1, 2) &= \{E(1, 2)F(0, 0) + E(1, 1)F(0, 1) + E(0, 2)F(1, 0) + E(0, 1)F(1, 1)\}G(0, 0) \\
&\quad + \{E(1, 1)F(0, 0) + E(1, 0)F(0, 1) + E(0, 1)F(1, 0) + E(0, 0)F(1, 1)\}G(0, 1) \\
&\quad + \{E(0, 2)F(0, 0) + E(0, 1)F(0, 1)\}G(1, 0) + \{E(0, 1)F(0, 0) + E(0, 0)F(0, 1)\}G(1, 1), \\
\Phi(1, 3) &= \{E(1, 2)F(0, 1) + E(0, 2)F(1, 1)\}G(0, 0) \\
&\quad + \{E(1, 2)F(0, 0) + E(1, 1)F(0, 1) + E(0, 2)F(1, 0) \\
&\quad + E(0, 1)F(1, 1)\}G(0, 1) + \{E(0, 2)F(0, 1)\}G(1, 0) \\
&\quad + \{E(0, 2)F(0, 0) + E(0, 1)F(0, 1)\}G(1, 1), \\
\Phi(3, 1) &= \{E(2, 1)F(1, 0) + E(2, 0)F(1, 1)\}G(0, 0) + \{E(2, 0)F(1, 0)\}G(0, 1) \\
&\quad + \{E(2, 1)F(0, 0) + E(2, 0)F(0, 1) + E(1, 1)F(1, 0) + E(1, 0)F(1, 1)\}G(1, 0) \\
&\quad + \{E(2, 0)F(0, 0) + E(1, 0)F(1, 0)\}G(1, 1), \\
\Phi(1, 4) &= \{E(1, 2)F(0, 1) + E(0, 2)F(1, 1)\}G(0, 1) + \{E(0, 2)F(0, 1)\}G(1, 1), \\
\Phi(4, 1) &= \{E(2, 1)F(1, 0) + E(2, 0)F(1, 1)\}G(1, 0) + \{E(2, 0)F(1, 0)\}G(1, 1), \\
\Phi(2, 3) &= \{(E(2, 2)F(0, 1) + E(1, 2)F(1, 1)\}G(0, 0) + \{E(2, 2)F(0, 0) + E(2, 1)F(0, 1) + E(1, 2)F(1, 0) \\
&\quad + E(1, 1)F(1, 1)\}G(0, 1) + \{E(1, 2)F(0, 1) + E(0, 2)F(1, 1)\}G(1, 0) \\
&\quad + \{E(1, 2)F(0, 0) + E(1, 1)F(0, 1) + E(0, 2)F(1, 0) + E(0, 1)F(1, 1)\}G(1, 1), \\
\Phi(3, 2) &= \{(E(2, 2)F(1, 0) + E(2, 1)F(1, 1)\}G(0, 0) + \{E(2, 1)F(1, 0) + E(2, 0)F(1, 1)\}G(0, 1) \\
&\quad + \{E(2, 2)F(0, 0) + E(2, 1)F(0, 0) + E(1, 2)F(1, 0) + E(1, 1)F(1, 1)\}G(1, 0) + \{E(2, 1)F(0, 0) \\
&\quad + E(2, 0)F(0, 1) + E(1, 1)F(1, 0) + E(1, 0)F(1, 1)\}G(1, 1), \\
\Phi(2, 4) &= \{E(2, 2)F(0, 1) + E(1, 2)F(1, 1)\}G(0, 1) + \{E(1, 2)F(0, 1) + E(0, 2)F(1, 1)\}G(1, 1), \\
\Phi(4, 2) &= \{E(2, 2)F(1, 0) + E(2, 1)F(1, 1)\}G(1, 0) + \{E(2, 1)F(1, 0) + E(2, 0)F(1, 1)\}G(1, 1), \\
\Phi(3, 4) &= \{E(2, 2)F(1, 1)\}G(0, 1) + \{E(2, 2)F(0, 1) + E(1, 2)F(1, 1)\}G(1, 1), \\
\Phi(4, 3) &= \{E(2, 2)F(1, 1)\}G(1, 0) + \{E(2, 2)F(1, 0) + E(2, 1)F(1, 1)\}G(1, 1),
\end{aligned}$$

and $(I(r, s), r = 0, 4), s = 0, 4$ are defined in Eq. (30).

From Eqs. (30), (112)–(114), we obtain

$$J_{00} = \frac{3}{4}$$

$$J_{10} = -\frac{9}{32}$$

$$J_{01} = 0$$

$$a_{00} = 0, \quad a_{10} = \frac{3}{4}, \quad a_{01} = 0, \quad a_{11} = 0,$$

$$\begin{aligned}
b_{00} &= \frac{3}{16}, & b_{10} &= \frac{4}{16}, & b_{01} &= 1, & b_{11} &= -\frac{6}{16}, \\
c_{00} &= \frac{3}{4}, & c_{10} &= \frac{1}{4}, & c_{01} &= 0, & c_{11} &= 0 \\
E_{(0,0)} &= 0, & E_{(1,0)} &= 0, & E_{(0,2)} &= 0 \\
E_{(1,0)} &= 0, & E_{(1,1)} &= 0, & E_{(1,2)} &= 0 \\
E_{(2,0)} &= \frac{9}{32}, & E_{(2,1)} &= 0, & E_{(2,2)} &= 0 \\
F_{(0,0)} &= \frac{3}{16}, & F_{(0,1)} &= 1, & F_{(1,0)} &= \frac{4}{16}, & F_{(1,1)} &= -\frac{6}{16}, \\
G_{(0,0)} &= \frac{3}{4}, & G(0,1) &= 0, & G(1,0) &= \frac{1}{4}, & G(1,1) &= 0, \\
\Phi(1,1) &= 0, & \Phi(2,2) &= 0, & \Phi(3,3) &= 0, & \Phi(4,4) &= 0, \\
\Phi(0,1) &= 0, & \Phi(1,0) &= 0, & \Phi(0,2) &= 0, \\
2\Phi(2,0) &= \frac{81}{16 \times 4 \times 32}, & \Phi(0,3) &= 0, \\
2\Phi(3,0) &= \frac{135}{16^2 \times 4}, & \Phi(0,4) &= 0, & 2\Phi(4,0) &= \frac{36}{16^2 \times 4}, \\
\Phi(1,2) &= 0, & 2\Phi(2,1) &= \frac{9 \times 48}{16^2 \times 4}, \\
2\Phi(3,1) &= -\frac{18}{16^2 \times 4}, & \Phi(1,3) &= 0 \\
\Phi(1,4) &= 0, & 2\Phi(4,1) &= \frac{-54}{16^2 \times 4} \\
E(0,0) &= \frac{a_{00}^2}{2} \\
E(0,1) &= a_{01}a_{0,0} \\
E(0,2) &= \frac{a_{01}^2}{2} \\
E(1,0) &= a_{00}a_{10} \\
E(1,1) &= (a_{11}a_{00} + a_{10}a_{01}) \\
E(1,2) &= a_{11}a_{01} \\
E(2,0) &= \frac{a_{10}^2}{2} \\
E(2,1) &= a_{10}a_{11} \\
E(2,2) &= \frac{a_{11}^2}{2} \\
F(0,0) &= b_{00} \\
F(0,1) &= b_{01} \\
F(1,0) &= b_{10} \\
F(1,1) &= b_{11} \\
G(0,0) &= c_{00} \\
G(0,1) &= c_{01}
\end{aligned}$$

$$\begin{aligned} G(1, 0) &= c_{10} \\ G(1, 1) &= c_{11} \end{aligned} \quad (113)$$

and J_{00} , J_{10} , J_{01} , $((a_{ij}, b_{ij}, c_{ij}), i = 0, 1, j = 0, 1)$ are already defined in Eq. (30).

For the application example stated in Eq. (99), let us illustrate the above numerical scheme with reference to Q_{ijkl}^{xy} ($i = 5, j = 6, k = 7, l = 8$).

$$\begin{aligned} (x_i, y_i, z_i) &= \left(0, \frac{3}{16}, \frac{3}{4}\right) \\ (x_j, y_j, z_j) &= \left(\frac{3}{4}, \frac{7}{16}, 1\right) \\ (x_k, y_k, z_k) &= \left(\frac{3}{4}, \frac{17}{16}, 1\right) \\ (x_l, y_l, z_l) &= \left(0, \frac{19}{16}, \frac{3}{4}\right) \end{aligned} \quad (114)$$

$$\begin{aligned} \Phi(2, 3) &= 0, & \Phi(3, 2) &= 0, & \Phi(2, 4) &= 0, \\ \Phi(4, 2) &= 0, & \Phi(3, 4) &= 0, & \Phi(4, 3) &= 0 \end{aligned} \quad (115)$$

Now substituting from Eq. (115) into Eq. (111), we obtain

$$\begin{aligned} II_{Q_{5687}^{xy}}^{2,1,1} &= 2\{\Phi(2, 0)I(2, 0) + \Phi(3, 0)I(3, 0) + \Phi(4, 0)I(4, 0) \\ &\quad + \Phi(2, 1)I(2, 1) + \Phi(3, 1)I(3, 1) + \Phi(4, 1)I(4, 1)\} \\ &= \left(\frac{81}{16^2 \times 4}\right)\left(\frac{23}{128}\right) + \left(\frac{135}{16^2 \times 4}\right)\left(\frac{21}{160}\right) + \left(\frac{36}{16^2 \times 4}\right)\left(\frac{33}{320}\right) + \left(\frac{9 \times 48}{16^2 \times 4}\right)\left(\frac{23}{256}\right) \\ &\quad + \left(\frac{-18}{16^2 \times 4}\right)\left(\frac{21}{320}\right) + \left(\frac{-54}{16^2 \times 4}\right)\left(\frac{33}{640}\right) \\ &= \frac{45333}{16^4 \times 10} \end{aligned} \quad (116)$$

In a similar fashion, we obtain other integrals as

$$\begin{aligned} II_{Q_{1265}^{xy}}^{2,1,1} &= \frac{163157}{16^4 \times 360}, \\ II_{Q_{2486}^{xy}}^{2,1,1} &= \frac{2053}{16^3 \times 15}, \\ II_{Q_{3487}^{xy}}^{2,1,1} &= \frac{519359}{16^4 \times 360} \end{aligned} \quad (117)$$

Substituting from Eqs. (116) and (117) into Eq. (100), we obtain

$$III_{H_{1,2,3,\dots,8}}^{2,1,0} = \frac{344023}{16^4 \times 60} \quad (118)$$

ALGORITHM 3. We shall now illustrate the use of Lemma 3 to compute the integrals

$$II_{T_{ijk}^{xy}}^{\alpha, \beta, \gamma+1} = \int \int_{T_{ijk}^{xy}} x^\alpha y^\beta (Lx + My + h)^{\gamma+1} dx dy$$

On comparing with the statement of Lemma 3, for the application example considered in [10], we have $\alpha = 2$, $\beta = 1$, $\gamma = 0$. Let us first demonstrate the computational scheme with reference to the integral

$$\int \int_{T_{ijk}^{xy}} x^2 y \{-4(x - 10)\} dx dy$$

where

$$\begin{aligned} x_i &= 10, & y_i &= 10 \\ x_j &= 8, & y_j &= 7 \\ x_k &= 10, & y_k &= 5 \end{aligned} \quad (119)$$

Now, we have from Eqs. (52) and (53) for $\alpha = 2$, $\beta = 1$, $\gamma = 0$:

$$\begin{aligned} H_{ijk}^{2,1,1} &= (2\Delta_{ijk}^{xy}) \sum_{m=0}^4 \frac{A(m, n)}{(m+2)(n+1)} \\ &= (2\Delta_{ijk}^{xy}) \left[\frac{1}{2} A(0, 0) + \frac{1}{3} A(1, 0) + \frac{1}{6} A(1, 1) + \frac{1}{4} A(2, 0) + \frac{1}{8} A(2, 1) + \frac{1}{12} A(2, 2) + \frac{1}{5} A(3, 0) \right. \\ &\quad + \frac{1}{10} A(3, 1) + \frac{1}{15} A(3, 2) + \frac{1}{20} A(3, 3) + \frac{1}{6} A(4, 0) + \frac{1}{12} A(4, 1) + \frac{1}{18} A(4, 2) \\ &\quad \left. + \frac{1}{24} A(4, 3) + \frac{1}{30} A(4, 4) \right] \end{aligned} \quad (120)$$

where

$$\begin{aligned} A(0, 0) &= F(0, 0)G(0, 0)H(0, 0) \\ A(1, 0) &= F(0, 0)G(0, 0)H(1, 0) + F(0, 0)G(1, 0)H(0, 0) + F(1, 0)G(0, 0)H(0, 0) \\ A(1, 1) &= F(0, 0)G(0, 0)H(1, 1) + F(0, 0)G(1, 1)H(0, 0) + F(1, 1)G(0, 0)H(0, 0) \\ A(2, 0) &= F(0, 0)G(1, 0)H(1, 0) + F(1, 0)G(0, 0)H(1, 0) \\ &\quad + F(1, 0)G(1, 0)H(0, 0) + F(2, 0)G(0, 0)H(0, 0) \\ A(2, 1) &= F(0, 0)G(1, 0)H(1, 1) + F(0, 0)G(1, 1)H(1, 0) \\ &\quad + F(1, 0)G(0, 0)H(1, 1) + F(1, 1)G(0, 0)H(1, 0) \\ &\quad + F(1, 0)G(1, 1)H(0, 0) + F(1, 1)G(1, 0)H(0, 0) + F(2, 1)G(0, 0)H(0, 0) \\ A(2, 2) &= F(0, 0)G(1, 1)H(1, 1) + F(1, 1)G(0, 0)H(1, 1) \\ &\quad + F(1, 1)G(1, 1)H(0, 0) + F(2, 2)G(0, 0)H(0, 0) \\ A(3, 0) &= F(1, 0)G(1, 0)H(1, 0) + F(2, 0)G(0, 0)H(1, 0) + F(2, 0)G(1, 0)H(0, 0) \\ A(3, 1) &= F(1, 0)G(1, 0)H(1, 1) + F(1, 0)G(1, 1)H(1, 0) \\ &\quad + F(1, 1)G(1, 0)H(1, 0) + F(2, 0)G(1, 1)H(0, 0) \\ &\quad + F(2, 1)G(1, 0)H(0, 0) + F(2, 1)G(0, 0)H(1, 0) + F(2, 0)G(0, 0)H(1, 1) \\ A(3, 2) &= F(1, 0)G(1, 1)H(1, 1) + F(1, 1)G(1, 0)H(1, 1) \\ &\quad + F(1, 1)G(1, 1)H(1, 0) + F(2, 1)G(0, 0)H(1, 1) \\ &\quad + F(2, 2)G(0, 0)H(1, 0) + F(2, 1)G(1, 1)H(0, 0) + F(2, 2)G(1, 0)H(0, 0) \\ A(3, 3) &= F(1, 1)G(1, 1)H(1, 1) + F(2, 2)G(1, 1)H(0, 0) \\ A(4, 0) &= F(2, 0)G(1, 0)H(1, 0) \\ A(4, 1) &= F(2, 0)G(1, 0)H(1, 1) + F(2, 0)G(1, 1)H(1, 0) \\ A(4, 2) &= F(2, 0)G(1, 1)H(1, 1) + F(2, 1)G(1, 0)H(1, 1) + F(2, 1)G(1, 1)H(1, 0) \\ A(4, 3) &= F(2, 1)G(1, 1)H(1, 1) + F(2, 2)G(1, 0)H(1, 1) + F(2, 2)G(1, 1)H(1, 0) \\ A(4, 4) &= F(2, 2)G(1, 1)H(1, 1) \\ F(0, 0) &= a_{00}^2 \\ G(0, 0) &= b_{00} \\ H(0, 0) &= c_{00} \\ F(1, 0) &= 2a_{00}a_{10} \end{aligned}$$

$$\begin{aligned}
G(1, 0) &= b_{10} \\
H(1, 0) &= c_{10} \\
F(1, 1) &= 2a_{00}a_{11} \\
G(1, 1) &= b_{11} \\
H(1, 1) &= c_{11} \\
F(2, 0) &= a_{10}^2 \\
F(2, 1) &= 2a_{10}a_{11} \\
F(2, 2) &= a_{11}^2 \\
a_{00} &= x_i, \quad b_{00} = y_i, \quad c_{00} = z_i, \\
a_{10} &= x_j - x_i, \quad b_{10} = y_j - y_i, \quad c_{10} = z_j - z_i, \\
a_{11} &= x_k - x_j, \quad b_{11} = y_k - y_j, \quad c_{11} = z_k - z_j. \tag{121}
\end{aligned}$$

We can now evaluate the integral of Eq. (119) from Eqs. (119)–(121), we obtain

$$\begin{aligned}
a_{00} &= 10, \quad b_{00} = 10, \quad c_{00} = 0 \\
a_{10} &= -2, \quad b_{10} = -3, \quad c_{10} = 8 \\
a_{11} &= 2, \quad b_{11} = -2, \quad c_{11} = -8 \\
2\Delta_{ijk}^{xy} &= (x_j - x_i)(y_k - y_j) - (x_k - x_j)(y_j - y_i) \\
&= a_{10}b_{11} - a_{11}b_{10} = 10 \\
A(0, 0) &= 0 \\
A(1, 0) &= 8000 \\
A(1, 1) &= -8000 \\
A(2, 0) &= -5600 \\
A(2, 1) &= 7200 \\
A(2, 2) &= -1600 \\
A(3, 0) &= 1280 \\
A(3, 1) &= -2240 \\
A(3, 2) &= 640 \\
A(3, 3) &= 640 \\
A(4, 0) &= -96 \\
A(4, 1) &= 32 \\
A(4, 2) &= -96 \\
A(4, 3) &= -96 \\
A(4, 4) &= 64 \tag{122}
\end{aligned}$$

Now, substituting from Eq. (122) into Eq. (120), we obtain for the integral of Eq. (119):

$$\begin{aligned}
II_{T_{ijk}^{xy}}^{2,1,1} &= 10 \left[0 + \frac{(8000)}{3} + \frac{(-8000)}{6} + \frac{(-5600)}{4} + \frac{(7200)}{8} \right. \\
&\quad + \frac{(-1600)}{12} + \frac{(1280)}{5} + \frac{(-2240)}{10} + \frac{(640)}{15} \\
&\quad \left. + \frac{(640)}{20} + \frac{(-96)}{6} + \frac{(32)}{12} + \frac{(-96)}{18} + \frac{(-96)}{24} + \frac{(64)}{30} \right] \\
&= \frac{23584}{3} \tag{123}
\end{aligned}$$

The result obtained in Eq. (123) is again in agreement with that of Rathod and Govinda Rao [10].

Following the numerical scheme outlined in Eqs. (120) and (121), we shall now compute the integrals

$$II_{Q_{ijkl}^{xy}}^{2,1,1} = \int \int_{Q_{ijkl}^{xy}} x^2 yz \, dx \, dy \quad (124)$$

where

$$(i, j, k, l) \in \{(2, 4, 8, 6), (1, 2, 6, 5), (3, 4, 8, 7), (5, 6, 8, 7)\} \quad (125)$$

One of the ways to integrate over the area of quadrilateral is by treating the same as a sum of integrals over two triangles: There are two ways to compute these integrals as sum of the integrals over two triangles. We shall demonstrate the same in one of the ways. Following the computational scheme outlined in Eqs. (120), (121) and the numerical data for the application example as given in Eq. (99), we obtain

$$\begin{aligned} II_{Q_{24,8,6}^{xy}}^{2,1,1} &= II_{T_{248}^{xy}}^{2,1,1} + II_{T_{286}^{xy}}^{2,1,1} \\ &= \frac{753}{16^3 \times 15} + \frac{2080}{16^3 \times 24} = \frac{2053}{16^3 \times 16} \end{aligned} \quad (126)$$

$$\begin{aligned} II_{Q_{1265}^{xy}}^{2,1,1} &= II_{T_{256}^{xy}}^{2,1,1} + II_{T_{215}^{xy}}^{2,1,1} \\ &= \frac{152789}{16^4 \times 360} + \frac{10368}{16^4 \times 360} = \frac{163157}{16^4 \times 360} \end{aligned} \quad (127)$$

$$\begin{aligned} II_{Q_{3487}^{xy}}^{2,1,1} &= II_{T_{348}^{xy}}^{2,1,1} + II_{T_{387}^{xy}}^{2,1,1} \\ &= \frac{393728}{16^4 \times 360} + \frac{125631}{16^4 \times 360} = \frac{519359}{16^4 \times 360} \end{aligned} \quad (128)$$

$$\begin{aligned} II_{Q_{5687}^{xy}}^{2,1,1} &= II_{T_{687}^{xy}}^{2,1,1} + II_{T_{675}^{xy}}^{2,1,1} \\ &= \frac{6849}{16^4 \times 2} + \frac{693}{16^3 \times 10} = \frac{45333}{16^4 \times 10} \end{aligned} \quad (129)$$

Substituting from Eqs. (126)–(129) into Eq. (100), we obtain

$$H_{H_{1,2,\dots,8}}^{2,1,1} = II_{Q_{2486}^{xy}}^{2,1,1} + II_{Q_{1265}^{xy}}^{2,1,1} - II_{Q_{3487}^{xy}}^{2,1,1} + II_{Q_{5687}^{xy}}^{2,1,1} = \frac{344023}{16^4 \times 60} \quad (130)$$

4.1. Analytical integration of the application example

The analytical evaluation of the integral (98) with reference to Fig. 7 and numerical data of Eq. (99), we can write

$$\begin{aligned} II_{H_{1,2,\dots,8}}^{2,1,0} &= \int \int \int_{H_{1,2,\dots,8}} x^2 y \, dx \, dy \, dz \\ &= \int_{z=0}^1 \int_{x=0}^{1-z/4} \int_{y=x+z/4}^{y=z-x+4/4} x^2 y \, dy \, dx \, dz - \int_{x=0}^{3/4} \int_{z=x/3+3/4}^1 \int_{y=x+z/4}^{y=z-x+4/4} x^2 y \, dy \, dz \, dx \end{aligned} \quad (131)$$

$$\begin{aligned} &= \frac{4547}{12 \times 16^3} - \frac{6579}{64^3 \times 5} \\ &= \frac{344023}{16^4 \times 60} \end{aligned} \quad (132)$$

5. Line integration

5.1. In the previous section we have shown that the surface integral of the type $\Pi_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}$ are in fact reducible to line integral over the edge joining points (x_i, y_i) and (x_k, y_k) and the $z(x, y)$ is in fact the equation of the plane spanning points $((x_a, y_a, z_a), a = p, q, r, s)$ and $(i, k) \in (p, q, r, s)$.

In order to identify the equation of the plane with its spanning points, we introduce a new notation. $z(x, y, p, q, r, s) = h + Lx, My$ instead of simple notation $z(x, y) = h + Lx + My$ and the existing notation of $\Pi_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}$ will be redefined as

$$\Pi_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}(p, q, r, s) \stackrel{\text{def}}{=} \int \int_{T_{koi}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, p, q, r, s) dx dy \quad (133)$$

in which $i, k \in (p, q, r, s)$.

We shall now propose a theorem which will express the volume integral over an arbitrary linear hexahedron as a sum of twelve line integrals of type given in Eq. (133) along the straight line edges.

THEOREM 6. A Structure Product:

$$\text{III}_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} = \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz$$

where $H_{1,2,3,\dots,8}$ is a linear arbitrary hexahedron with vertices at $((x_a, y_a, z_a), a = 1, 2, \dots, 8)$ is expressible as a sum of line integrals along the straight line edges joining points $((x_a, y_a), (x_b, y_b))$, $(a, b) \in \{(1, 2), (1, 3), (1, 5), (2, 4), (2, 6), (3, 4), (3, 7), (4, 8), (5, 6), (5, 7), (6, 8), (7, 8)\}$

$$\begin{aligned} \text{III}_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} &= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [\{\Pi_{T_{201}^{xy}}^{\alpha, \beta, \gamma+1}(1, 5, 6, 2) - \Pi_{T_{201}^{xy}}^{\alpha, \beta, \gamma+1}(1, 2, 4, 3)\} \\ &\quad + \{\Pi_{T_{402}^{xy}}^{\alpha, \beta, \gamma+1}(2, 4, 8, 6) - \Pi_{T_{402}^{xy}}^{\alpha, \beta, \gamma+1}(1, 2, 4, 3)\} + \{\Pi_{T_{403}^{xy}}^{\alpha, \beta, \gamma+1}(1, 2, 4, 3) - \Pi_{T_{403}^{xy}}^{\alpha, \beta, \gamma+1}(3, 7, 8, 4)\}] \\ &\quad + \{\Pi_{T_{301}^{xy}}^{\alpha, \beta, \gamma+1}(1, 2, 4, 3) - \Pi_{T_{301}^{xy}}^{\alpha, \beta, \gamma+1}(1, 3, 7, 5)\} + \{\Pi_{T_{605}^{xy}}^{\alpha, \beta, \gamma+1}(5, 6, 8, 7) - \Pi_{T_{605}^{xy}}^{\alpha, \beta, \gamma+1}(1, 5, 6, 2)\} \\ &\quad + \{\Pi_{T_{806}^{xy}}^{\alpha, \beta, \gamma+1}(5, 6, 8, 7) - \Pi_{T_{806}^{xy}}^{\alpha, \beta, \gamma+1}(2, 4, 8, 6)\} + \{\Pi_{T_{807}^{xy}}^{\alpha, \beta, \gamma+1}(3, 7, 8, 4) - \Pi_{T_{807}^{xy}}^{\alpha, \beta, \gamma+1}(5, 6, 8, 7)\} \\ &\quad + \{\Pi_{T_{705}^{xy}}^{\alpha, \beta, \gamma+1}(1, 3, 7, 5) - \Pi_{T_{705}^{xy}}^{\alpha, \beta, \gamma+1}(5, 6, 8, 7)\} + \{\Pi_{T_{703}^{xy}}^{\alpha, \beta, \gamma+1}(3, 7, 8, 4) - \Pi_{T_{703}^{xy}}^{\alpha, \beta, \gamma+1}(1, 3, 7, 5)\} \\ &\quad + \{\Pi_{T_{501}^{xy}}^{\alpha, \beta, \gamma+1}(1, 3, 7, 5) - \Pi_{T_{501}^{xy}}^{\alpha, \beta, \gamma+1}(1, 5, 6, 2)\} + \{\Pi_{T_{804}^{xy}}^{\alpha, \beta, \gamma+1}(2, 4, 8, 6) - \Pi_{T_{804}^{xy}}^{\alpha, \beta, \gamma+1}(3, 7, 8, 4)\} \\ &\quad + \{\Pi_{T_{602}^{xy}}^{\alpha, \beta, \gamma+1}(1, 5, 6, 2) - \Pi_{T_{602}^{xy}}^{\alpha, \beta, \gamma+1}(2, 4, 8, 6)\}] \end{aligned} \quad (134)$$

PROOF. Using Theorem 5, we can write

$$\begin{aligned} \text{III}_{H_{1,2,3,\dots,8}}^{\alpha, \beta, \gamma} &\stackrel{\text{def}}{=} \int \int \int_{H_{1,2,3,\dots,8}} x^\alpha y^\beta z^\gamma dx dy dz \\ &= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} [-\Pi_{Q_{1,2,4,3}^{xy}}^{\alpha, \beta, \gamma+1} + \Pi_{Q_{5,6,8,7}^{xy}}^{\alpha, \beta, \gamma+1} - \Pi_{Q_{1,3,7,5}^{xy}}^{\alpha, \beta, \gamma+1} + \Pi_{Q_{2,4,8,6}^{xy}}^{\alpha, \beta, \gamma+1} - \Pi_{Q_{1,5,6,2}^{xy}}^{\alpha, \beta, \gamma+1} + \Pi_{Q_{3,7,8,4}^{xy}}^{\alpha, \beta, \gamma+1}] \end{aligned} \quad (135)$$

Using Lemma 1, we can write

$$\Pi_{Q_{i,j,k,l}^{xy}}^{\alpha, \beta, \gamma+1} = \Pi_{T_{j,o,i}^{xy}}^{\alpha, \beta, \gamma+1}(i, j, k, l) + \Pi_{T_{k,o,j}^{xy}}^{\alpha, \beta, \gamma+1}(i, j, k, l) + \Pi_{T_{l,o,k}^{xy}}^{\alpha, \beta, \gamma+1}(i, j, k, l) + \Pi_{T_{i,o,l}^{xy}}^{\alpha, \beta, \gamma+1}(i, j, k, l) \quad (136)$$

Also, we note the equivalence of integrals over the oriented triangles $T_{p,o,q}^{xy}$ and $T_{q,o,p}^{xy}$ as

$$\begin{aligned} \Pi_{T_{p,o,q}^{xy}}^{\alpha, \beta, \gamma+1}(i, j, k, l) &= \int \int_{T_{p,o,q}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, i, j, k, l) dx dy \\ &= - \int \int_{T_{q,o,p}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, i, j, k, l) dx dy \end{aligned} \quad (137)$$

in which $p, q \in (i, j, k, l)$.

Using the result of Eqs. (136) and (137) in Eq. (135) and rearranging the terms, we obtain the result of Eq. (134). This completes the proof of Theorem 6. \square

In Theorem 6, we have used the notation

$$II_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(p, q, r, s) = \int \int_{T_{koi}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, p, q, r, s)$$

in which $i, k \in (p, q, r, s)$ and $z(x, y, p, q, r, s)$ is the equation of the plane spanning points $((x_a, y_a, z_a), a = p, q, r, s)$ which includes (x_i, y_i, z_i) and (x_k, y_k, z_k) . Hence, without loss of meanings and concepts involved in Theorem 6, for the sake of uniformity we define

$$JJ_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(r, s, r, s) \stackrel{\text{def}}{=} II_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(p, q, r, s) - II_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(p, q, r, s) \quad (138)$$

in which $p = p' = i$, $q = q' = k$ and certainly noted in previous sections the plane spanned by $((x_a, y_a, z_a), a = p, q, r, s)$ and $((x_b, y_b, z_b), b = p', q', r', s')$ are different. Alternatively, Eq. (138) can be written as

$$\begin{aligned} JJ_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(r, s, r', s') &= II_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r, s) - II_{T_{k,o,i}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r', s') \\ &= \int \int_{T_{koi}^{xy}} x^\alpha y^\beta \{z^{\gamma+1}(x, y, i, k, r, s) - z^{\gamma+1}(x, y, i, k, r', s')\} dx dy \end{aligned} \quad (139)$$

Using Eqs. (138) and (139), we can rewrite Eq. (134) as

$$\begin{aligned} III_{H_{1,2,3,\dots,8}}^{\alpha,\beta,\gamma} &= \frac{\Omega(1, 4, 6, 7)}{(\gamma + 1)} JJ_{T_{2,0,1}^{xy}}^{\alpha,\beta,\gamma+1}(5, 6, 3, 4) + JJ_{T_{4,0,2}^{xy}}^{\alpha,\beta,\gamma+1}(6, 8, 1, 3) + JJ_{T_{4,0,3}^{xy}}^{\alpha,\beta,\gamma+1}(1, 2, 7, 8) \\ &\quad + JJ_{T_{3,0,1}^{xy}}^{\alpha,\beta,\gamma+1}(2, 4, 5, 7) + JJ_{T_{6,0,5}^{xy}}^{\alpha,\beta,\gamma+1}(7, 8, 1, 2) + JJ_{T_{8,0,6}^{xy}}^{\alpha,\beta,\gamma+1}(5, 7, 2, 4) + JJ_{T_{8,0,7}^{xy}}^{\alpha,\beta,\gamma+1}(3, 4, 5, 6) \\ &\quad + JJ_{T_{7,0,5}^{xy}}^{\alpha,\beta,\gamma+1}(1, 3, 6, 8) + JJ_{T_{7,0,3}^{xy}}^{\alpha,\beta,\gamma+1}(4, 8, 1, 5) + JJ_{T_{5,0,1}^{xy}}^{\alpha,\beta,\gamma+1}(3, 7, 2, 6) + JJ_{T_{8,0,4}^{xy}}^{\alpha,\beta,\gamma+1}(2, 6, 3, 7) \\ &\quad + JJ_{T_{6,0,2}^{xy}}^{\alpha,\beta,\gamma+1}(1, 5, 4, 8) \end{aligned} \quad (140)$$

5.2. In the previous section 5.1, viz. Eq. (139), we have defined

$$\begin{aligned} JJ_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(r, s, r', s') &\stackrel{\text{def}}{=} II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r, s) - II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r', s') \\ &= \int \int_{T_{koi}^{xy}} x^\alpha y^\beta \{z^{\gamma+1}(x, y, i, k, r, s) - z^{\gamma+1}(x, y, i, k, r', s')\} dx dy \end{aligned} \quad (141)$$

where $z(x, y, i, \delta, \sigma)$, $(\delta, \sigma) \in \{(r, s), (r', s')\}$ are equations of planes spanning points $((x_a, y_a, z_a), a = i, k, \delta, \sigma)$. Hence, we can write

$$\begin{aligned} z(x, y, i, k, r, s) &= h + Lx + My \\ z(x, y, i, k, r', s') &= h' + L'x + M'y \end{aligned} \quad (142)$$

which further satisfy the relations

$$\begin{aligned} z(x_i, y_i, i, k, r, s) &= z(x_i, y_i, i, k, r', s') = z_i \\ z(x_k, y_k, i, k, r, s) &= z(x_k, y_k, i, k, r', s') = z_k \\ z(x_r, y_r, i, k, r, s) &= z_r \\ z(x_s, y_s, i, k, r, s) &= z_s \\ z(x_{r'}, y_{r'}, i, k, r', s') &= z_{r'} \\ z(x_{s'}, y_{s'}, i, k, r', s') &= z_{s'} \\ z(0, 0, i, k, r, s) &= h = z_0^{irs} \quad (\text{say}) \\ z(0, 0, i, k, r', s') &= h = z_0^{irs'k} \quad (\text{say}). \end{aligned} \quad (143)$$

LEMMA 4. Let T_{koi}^{xy} be an oriented triangle in the xy -plane with vertices at (x_i, y_i) , $(0, 0)$ and (x_k, y_k) then the structure product:

$$\begin{aligned} JJ_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}(r, s, r', s') &\stackrel{\text{def}}{=} \{II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}(i, k, r, s) - II_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}(r, s, r', s')\} \\ &= (A_{koi}^{xy}) \sum_{n=0}^{\alpha+\beta+\gamma+1} \sum_{n_1+n_2+n_3=n} F_{koi}(\alpha-n_1, n_1) G_{koi}(\beta-n_2, n_2) H_{koi}^{rs, r', s'}(\gamma+1-n_3, n_3) \end{aligned} \quad (144)$$

where

$$\begin{aligned} F_{koi}(\alpha-n_1, n_1) &= \binom{\alpha}{n_1} x_k^{\alpha-n_1} x_{ik}^{n_1} \\ G_{koi}(\beta-n_2, n_2) &= \binom{\beta}{n_2} x_k^{\beta-n_2} x_{ik}^{n_2} \\ H_{koi}^{rs, r', s'}(\gamma+1-n_3, n_3) &= H_{koi}^{rs}(\gamma+1-n_3, n_3) - H_{koi}^{r's'}(\gamma+1-n_3, n_3) \\ H_{koi}^{\lambda, \mu}(\gamma+1-n_3, n_3) &= \begin{cases} \binom{\gamma+1}{n_3} z_k^{\gamma+1-n_3} z_{ik}^{n_3}, & \text{if } z_o^{i\lambda\mu k} = 0 \\ \sum_{p=n_3}^{\gamma+1} \frac{\binom{\gamma+1}{p} (z_o^{i\lambda\mu k})^{\gamma+1-p} \binom{p}{n_3} (z_k - z_o^{i\lambda\mu k})^{p-n_3}}{(\alpha+\beta+p+2)} z_{ik}^{n_3}, & \text{if } z_o^{i\lambda\mu k} \neq 0 \end{cases} \\ (\lambda, \mu) &\in \{(rs), (r's')\} \end{aligned} \quad (145)$$

and

$$\begin{aligned} A_{koi}^{xy} &= \begin{cases} \frac{2\Delta_{koi}^{xy}}{(\alpha+\beta+\gamma+3)}, & \text{if } z_o^{i\lambda\mu k} = 0 \\ 2\Delta_{koi}^{xy}, & \text{if } z_o^{i\lambda\mu k} \neq 0 \end{cases} \\ (\lambda, \mu) &\in \{(rs), (r's')\} \end{aligned} \quad (146)$$

PROOF. Let us consider the integral

$$\begin{aligned} II_{T_{kii}^{xy}}^{\alpha, \beta, \gamma+1}(i, k, \lambda, \mu) &\stackrel{\text{def}}{=} \int \int_{T_{kii}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, i, k, \lambda, \mu) dx dy \\ (\lambda, \mu) &\in \{(r, s), (r', s')\}, \quad t = r, r' \end{aligned} \quad (147)$$

where T_{kii}^{xy} is a triangle in xy -plane which is obtained by collapsing one of the side of the quadrilaterals Q_{krsi}^{xy} or $Q_{kr's'i}^{xy}$, i.e. a triangle obtained by letting $(x_r = x_s, y_r = y_s)$ or $(x_{r'} = x_{s'}, y_{r'} = y_{s'})$ clearly the triangle so obtained has vertices at $((x_a, y_a), a = k, t, i)$, $t = r, r'$. The parametric equation of the oriented triangle T_{kii}^{xy} ($t = r, r'$) are given by

$$\begin{aligned} x &= x_k + x_{tk}u + x_{ik}v \\ y &= y_k + y_{tk}u + y_{ik}v \\ z(x, y, i, k, \lambda, \mu) &= w_k + w_{tk}u + w_{ik}v \\ 0 \leq u, v \leq 1, \quad u+v \leq 1 \\ x_{tk} &= x_t - x_k, \quad x_{ik} = x_i - x_k, \text{ etc.} \end{aligned} \quad (148)$$

Using Eqs. (148), we can map the oriented triangles T_{kii}^{xy} in the xy -plane to unit triangle in the uv -plane.

We have for the area element

$$\begin{aligned}
dx dy &= \frac{\partial(x, y)}{\partial(u, v)} du dv = x_{ik}y_{ik} - x_{ik}y_{ik} \\
&= (2\Delta_{kri}^{xy}) du dv \\
&= (2 \times \text{area of triangle } T_{kri}^{xy}) du dv \quad (t = r, r')
\end{aligned} \tag{149}$$

Using Eqs. (147)–(149) we obtain

$$\begin{aligned}
JJ_{T_{kr,r'}^y}^{\alpha,\beta,\gamma+1}(r, s, r', s') &\stackrel{\text{def}}{=} II_{T_{kr}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r, s) - II_{T_{kri}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r', s') \\
&= (2\Delta_{kri}^{xy}) \int_0^1 \int_0^{1-u} [x_k + x_{rk}u + x_{ik}v]^\alpha [y_k + y_{rk}u + y_{ik}v]^\beta [z_k + z_{rk}u + z_{ik}v]^{\gamma+1} du dv \\
&\quad - (2\Delta_{kr'i}^{xy}) \int_0^1 \int_0^{1-u} [x_k + x_{r'k}u + x_{ik}v]^\alpha [y_k + y_{r'k}u + y_{ik}v]^\beta \\
&\quad \times [z_k + z_{r'k}u + z_{ik}v]^{\gamma+1} du dv
\end{aligned} \tag{150}$$

We also note from Eq. (142) that

$$\begin{aligned}
z(x, y, i, k, r, s) &= h + Lx + My = z_k + z_{rk}u + z_{ik}v \\
z(x, y, i, k, r', s') &= h' + L'x + M'y = z_k + z_{r'k}u + z_{ik}v
\end{aligned} \tag{151}$$

Let us further use the transformation

$$u = 1 - \xi, \quad v = \xi\eta \tag{152}$$

Using Eq. (152) into Eq. (150) gives

$$\begin{aligned}
JJ_{T_{kr,r'}^y}^{\alpha,\beta,\gamma+1} &= (2\Delta_{kri}^{xy}) \int_0^1 \int_0^1 [x_r + x_{kr}\xi + x_{ik}\xi\eta]^\alpha [y_r + y_{kr}\xi + y_{ik}\xi\eta]^\beta [z_r + z_{kr}\xi + z_{ik}\xi\eta]^{\gamma+1} \xi d\xi d\eta \\
&\quad - (2\Delta_{kr'i}^{xy}) \int_0^1 \int_0^1 [x_{r'} + x_{kr'}\xi + x_{ik}\xi\eta]^\alpha [y_{r'} + y_{kr'}\xi + y_{ik}\xi\eta]^\beta \\
&\quad \times [z_{r'} + z_{kr'}\xi + z_{ik}\xi\eta]^{\gamma+1} \xi d\xi d\eta
\end{aligned} \tag{153}$$

We have from Eqs. (143) and (151)

$$\begin{aligned}
z(0, 0, i, k, r, s) &= h = z_0^{irs k} \\
z(0, 0, i, k, r', s') &= h' = z_0^{ir's'k}
\end{aligned} \tag{154}$$

Choosing $x_r = 0$, $y_r = 0$, we obtain $z_r = z_0^{irs k}$

$$x_{r'} = 0, \quad y_{r'} = 0 \quad \text{We obtain } z_{r'} = z_0^{ir's'k} \tag{155}$$

Let us recall from Eqs. (138) and (139) that

$$II_{T_{koi}^y}^{\alpha,\beta,\gamma+1}(i, k, \lambda, \mu) = \int \int_{T_{koi}^{xy}} x^\alpha y^\beta z^{\gamma+1}(x, y, i, k, \lambda, \mu) dx dy \quad (\lambda, \mu) \in (r, s), (r', s')$$

and

$$JJ_{T_{koi}^y}^{\alpha,\beta,\gamma+1}(r, s, r', s') = II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r, s) - II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r', s') \tag{156}$$

Using Eqs. (154)–(156) in Eq. (153) and (156), we obtain

$$\begin{aligned}
JJ_{T_{koi}^y}^{\alpha,\beta,\gamma+1}(r, s, r', s') &= II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r, s) - II_{T_{koi}^{xy}}^{\alpha,\beta,\gamma+1}(i, k, r', s') \\
&= (2\Delta_{koi}^{xy}) \int_0^1 \int_0^1 \xi^{\alpha+\beta+\gamma+2} (x_k + x_{ik}\eta)^\alpha (y_{ik}\eta)^\beta \{[z_0^{irs k} + \xi(z_k - z_0^{irs k} + z_{ik}\eta)]^{\gamma+1} \\
&\quad - [z_0^{ir's'k} + \xi(z_k - z_0^{ir's'k} + z_{ik}\eta)]^{\gamma+1}\} d\xi d\eta
\end{aligned} \tag{157}$$

Let us now define

$$\begin{aligned} X(\eta) &= (x_k + x_{ik}\eta)^{\alpha} \\ Y(\eta) &= y_k + y_{ik}\eta^{\beta} \\ Z(\eta)^{\lambda\mu k} &= \begin{cases} (z_k + z_{ik}\eta)^{\gamma+1} & \text{if } z_o^{i\lambda\mu k} = 0 \\ \sum_{p=0}^{\gamma+1} \frac{\binom{\gamma+1}{p} (z_o^{i\lambda\mu k})^{\gamma+1-p} \binom{p}{n_3}}{(\alpha+\beta+p+2)} [(z_k - z_o^{i\lambda\mu k}) + z_{ik}\eta]^p, & \text{if } z_o^{i\lambda\mu k} \neq 0 \end{cases} \\ A_{koi}^{xy} &= \begin{cases} \frac{2\Delta_{koi}^{xy}}{(\alpha+\beta+\gamma+3)}, & \text{if } z_o^{i\lambda\mu k} = 0 \quad (\lambda\mu) \in \{(rs), (r's')\} \\ 2\Delta_{koi}^{xy}, & \text{if } z_o^{i\lambda\mu k} \neq 0 \end{cases} \end{aligned} \quad (158)$$

Use of Eq. (158) into Eq. (157) gives us

$$JJ_{T_{koi}^{xy}}^{\alpha, \beta, \gamma+1}(r, s, r', s') = (A_{koi}^{xy}) \int_0^1 X(\eta)Y(\eta)\{Z(\eta)^{i\lambda\mu k} - Z(\eta)^{i\lambda\mu k}\} d\eta \quad (159)$$

Letting $f(\eta) = X(\eta)Y(\eta)\{Z(\eta)^{i\lambda\mu k} - Z(\eta)^{i\lambda\mu k}\}$ and using Taylor's series expansion, we can write

$$f(\eta) = \sum_{q=0}^{\alpha+\beta+\gamma+1} \left\{ \frac{f_{(o)}^{(q)}}{|q|} \right\} s^q \quad (160)$$

where

$$f_{(o)}^{(q)} = \left(\frac{d^q}{d\eta^q} [X(\eta)Y(\eta)\{Z(\eta)^{irs k} - Z(\eta)^{ir's' k}\}] \right)_{\eta=0} \quad (161)$$

We now have, using generalized form Leibnitz's theorem for Eq. (160), obtained the expression:

$$f_{(o)}^{(q)} = \sum_{q_1+q_2+q_3=q} \frac{|q|}{|q_1||q_2||q_3|} \left\{ \frac{d^{q_1}X(\eta)}{d\eta^{q_1}} \right\}_{\eta=0} \left\{ \frac{d^{q_2}Y(\eta)}{d\eta^{q_2}} \right\}_{\eta=0} \left[\left\{ \frac{d^{q_3}Z(\eta)^{irs k}}{d\eta^{q_3}} \right\}_{\eta=0} - \left\{ \frac{d^{q_3}Z(\eta)^{ir's' k}}{d\eta^{q_3}} \right\}_{\eta=0} \right] \quad (162)$$

Now, from Eq. (158), we obtain

$$\begin{aligned} \frac{\left\{ \frac{d^{q_1}X(\eta)}{d\eta^{q_1}} \right\}_{\eta=0}}{|q_1|} &= \binom{\alpha}{q_1} x_i^{\alpha-q_1} x_{ik}^{q_1} \stackrel{\text{def}}{=} F_{koi}^{(\alpha-q_1, q_1)} \\ \frac{\left\{ \frac{d^{q_2}Y(\eta)}{d\eta^{q_2}} \right\}_{\eta=0}}{|q_2|} &= \binom{\alpha}{q_2} y_i^{\beta-q_2} y_{ik}^{q_2} \stackrel{\text{def}}{=} G_{koi}^{(\beta-q_2, q_2)} \\ \frac{\left\{ \frac{d^{q_3}Z(\eta)^{irs k}}{d\eta^{q_3}} \right\}_{\eta=0}}{|q_3|} &\stackrel{\text{def}}{=} H_{koi}^{\lambda\mu}(\gamma+1-q_3, q_3) \end{aligned} \quad (163)$$

$$\begin{aligned} &= \begin{cases} \binom{\gamma+1}{q_3} z_i^{\gamma+1-q_3} z_{ik}^{q_3}, & \text{if } z_o^{irs k} = 0 \\ \sum_{p=q_3}^{\gamma+1} \frac{\binom{\gamma+1}{q_3} (z_o^{irs k})^{\gamma+1-p} \binom{p}{n_3}}{(\alpha+\beta+p+2)} \binom{p}{q_3} \{z_i - z_o^{i\lambda\mu k}\}^{p-q_3} z_{ik}^{q_3}, & \text{if } z_o^{irs k} \neq 0 \end{cases} \\ H_{koi}^{rs, r's'}(\gamma+1-q_3, q_3) &\stackrel{\text{def}}{=} H_{koi}^{rs}(\gamma+1-q_3, q_3) - H_{koi}^{r's'}(\gamma+1-q_3, q_3) \end{aligned} \quad (164)$$

Thus, from Eqs. (157)–(159), (163) and (164), we obtain the desired result stated in the lemma.

5. Conclusions

The theorems and lemmas which we have presented in this paper are interesting for various reasons. We have expressed the integral of spatial expression $x^\alpha y^\beta (h + Lx + My)^{\gamma+1}$ (α, β, γ positive integer including zero) into line integrals, not via the use of Green's theorem for plane which was normally done in all previous works [7,8,10], but by means of a simple transformation which joins the end points of boundary line segments to the origin of the xy -plane. This transforms the area integral over linear plane polygon to a sum of area integrals over triangles joining the origin and the end points of line segments. It is shown that the area integrals over the triangle joining the origin and the end points of line segments are in fact reducible to simple line integrals. These line integrals so obtained have a product of three linear functions as their integral, viz.

$$(x_k + x_{ik}s)^\alpha (y_k + y_{ik}s)^\beta (z_k + z_{ik}s)^{\gamma+1}$$

or

$$(x_k + x_{ik}s)^\alpha (y_k + y_{ik}s)^\beta \left\{ \sum_{p=0}^{\gamma+1} \frac{\binom{\gamma+1}{p} z_0^{\gamma+1-p}}{(\alpha+\beta+p+2)} (z_k - z_o + z_{ik}s) \right\}$$

We have further used the technique developed in our earlier works [10,11] to obtain finite integration formulas for the line integral having the above integrand. We have also developed finite integration formulas for area integrals over linear arbitrary quadrilaterals and triangles by use of isoparametric coordinate transformation of finite element origin [12,13]. We have demonstrated these derivations and the numerical scheme proposed in Theorems 1–5 to evaluate the volume integral of monomial $x^\alpha y^\beta z^\gamma$ over a linear arbitrary hexahedron. We have further developed a finite integration formula for the volume integral of monomial $x^\alpha y^\beta z^\gamma$ over an arbitrary linear hexahedron which is further expressed in terms of twelve area integrals (with two end points of line segment and the origin of xy -plane as corner points of these triangles) over the triangles. It can be easily verified that these proposed algorithms are much simpler and economical in terms of arithmetic operations.

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