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INTEGRATION OF POLYNOMIALS OVER *N*-DIMENSIONAL LINEAR POLYHEDRA

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Abstract—This paper is concerned with explicit integration formulae for computing integrals of n-variate polynomials over linear polyhedra in n-dimensional space $\mathbb{R}^n$. Two different approaches are discussed; the first set of formulae is obtained by mapping the polyhedron in n-dimensional space $\mathbb{R}^n$ into a standard n-simplex in $\mathbb{R}^n$, while the second set of formulae is obtained by reducing the n-dimensional integral to a sum of $n-1$ dimensional integrals which are $n+1$ in number. These formulae are followed by an application example for which we have explained the detailed computational scheme. The symbolic integration formulae presented in this paper may lead to an easy and systematic incorporation of global properties of solid objects, such as, for example, volume, centre of mass, moments of inertia etc., required in engineering design problems. © 1997 Elsevier Science Ltd

1. INTRODUCTION

The computation of area, volume, centre of mass, moment of inertia and other geometrical properties of rigid homogeneous solids are of central interest in a large number of engineering applications such as CAD/CAE/CAM, geometric modelling and, in addition, a variety of other disciplines including modern developments in robotics. Though most of these applications are three-dimensional in nature, interest in multi-dimensional modelling is growing. Some applications of geometric modelling higher than three-dimensional space are: the efficient representations of moving three-dimensional objects (in the four-dimensional space–time domain), simulation and robotics. Computation of physical quantities for such applications is defined by multiple integrals over domains of three-dimensional Euclidean spaces and higher-dimensional spaces. This has aroused great interest in analytical and numerical methods used in the development of integration formulae for multiple integrals.

A good overview of various methods for evaluating volume (triple) integrals in this context is given by Lee and Requicha [1]. These authors observed that most computational studies in multiple integration often deal with calculations over very simple domains, such as a cube or a sphere, while the integrating function is very complicated; on the contrary, in most engineering applications, the converse problem usually arises. In such problems, the integration domain may have a non-convex shape and the function inside the integral sign is a trivariate polynomial. The same authors [2] outlined a family of approximate algorithms for computing inertial properties of solids. Such algorithms are based on a representation conversion from CSG to octree via recursive subdivision. Using a different approach based on the concept of finite-element coordinate transformations, O'Leary [3] developed integration formulae based on a quasi-disjoint decomposition of the solid in volume elements of simple, predefined shape. Wilson and Farrior [4] presented a large number of formulas for the computation of the main geometrical and inertial properties of planar polygons and of rotational solids. Timmer and Stern [5] discussed a theoretical approach to the evaluation of the volume integral by transforming it to a sum of surface integrals over the boundary of the integration domain. Lien and Kajiya [6] presented an outline of a closed formula of volume integration for a tetrahedron and suggested that volume integration for a linear polyhedron can be obtained by decomposing it into a set of solid tetrahedrons. Cattani and Paoluzzi [7, 8] gave a symbolic solution to both the surface and volume integration of trivariate polynomials in $\mathbb{R}^3$ by using a triangulation of the polyhedral shaped solid based on the concepts proposed by Timmer and Stern [5]. In a recent paper, Rathod and Govinda Rao [9] presented some explicit integration formulae for computing integrals of polynomials over an arbitrary tetrahedron in Euclidean three-dimensional space. In another recent work, Bernardini [10] presented the evaluation of integrals over linear polyhedra in an n-dimensional space. The related work in this area, by Ferrucci and Paouluzzi [11], discusses a method that permits the simplical complex associated with an n-dimensional polyhedron to be obtained by 'extruding' an n-1 dimensional polyhedron with simple combinatorial rules. An application of this method

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to the motion planning of a robot is shown by Paoluzzi [12].

In the present paper, we have developed closed form integration formulae which mainly follow the concepts developed in our earlier work [9], but these concepts are further generalized in this paper to compute integrals in an n-dimensional space. Two different approaches are considered. The first set of formulae is based on the fact that an arbitrary polyhedron in $\mathbb{R}^n$ can always be transformed into a standard n-simplex in $\mathbb{R}^n$ by means of an appropriate mapping; the second set of formulae is based on the proof of a generalized form of a divergence theorem for a standard n-simplex in $\mathbb{R}^n$, according to which an n-dimensional integral for standard n-simplex in $\mathbb{R}^n$ reduces to a sum of $n + 1$ integrals of dimension $n - 1$ for a standard $(n - 1)$-simplex in $\mathbb{R}^{n-1}$. In these derivations, we have made reference to the well-known theorem on differentiation of integrals (Leibnitz's Rule), Leibnitz's theorem on differentiation and Taylor series expansions [13, 14]. It is very clear from the present derivations that the explicit formulae obtained in this paper as well as in our previous work [9] are more compact than other researchers [7, 8]. These explicit integration formulae are followed by an application example for which we have explained the detailed computational scheme with reference to both sets of formulae.

2. INTEGRATION OVER A STANDARD N-SIMPLEX IN $\mathbb{R}^n$

The standard n-simplex $\tilde{\sigma}_n$ in $\mathbb{R}^n$ is defined mathematically by the following inequalities:

$$u_1 \geq 0,\ u_2 \geq 0, \ldots, u_n \geq 0,\ \sum_{i=1}^{n} u_i \leq 1\ . \qquad (1)$$

Hence, the $n + 1$ vertices of the standard n-simplex have the coordinates:

$$\begin{aligned} V_0 &= (0, 0, 0, \ldots, 0) \\ V_1 &= (1, 0, 0, \ldots, 0) \\ V_2 &= (0, 1, 0, \ldots, 0) \\ &\ \ \vdots \\ V_n &= (0, 0, 0, \ldots, 1) \end{aligned} \qquad \ldots\ (2)$$

A closed formula for the integration of monomials over a standard simplex is well known. Here we give the formula with a simple proof, but the integrand is a complex expression slightly different from a monomial. Let us introduce, for the sake of brevity, the notation:

$$\Phi_k = 1 - u_1 - u_2 - ,\ldots, - u_k,\ k = 1,2,\ldots,n-1$$

so that $u_n = \Phi_{n-1}$ is the equation of the hyperplane containing the points $V_1,\ldots,V_n$. Let us consider the following integral over $\tilde{\sigma}_n = \tilde{\sigma}_n(\underline{u})$, the standard n-simplex in $\mathbb{R}^n$:

$$I_0^{n+1}(h_0,h_1,\ldots,h_n) \text{ def } \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-1}} \times u_0^{h_0}u_1^{h_1}u_2^{h_2},\ldots,u_{n-1}^{h_{n-1}}u_n^{h_n}\mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_2\mathrm{d}u_1 \quad (3)$$

where

$$u_0 = 1 - u_1 - u_2 - ,\ldots, - u_n\ . \qquad (4)$$

Now we have

$$\begin{aligned} u_0 &= 1 - u_1 - u_2 - ,\ldots, - u_n \\ &= (1 - u_1 - u_2 - ,\ldots, - u_{n-1}) - u_n \\ &= \Phi_{n-1}\left(1 - \frac{u_n}{\Phi_{n-1}}\right). \end{aligned} \qquad (5)$$

Substituting from eqn (5) and integrating eqn (3) successively, we obtain:

$$I_0^{n+1}(h_0,h_1,h_2,\ldots,h_n) = \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-1}} \Phi_{n-1}^{h_0}\left(1 - \frac{u_n}{\Phi_{n-1}}\right)^{h_0} \times u_n^{h_n}u_1^{h_1},\ldots,u_{n-1}^{h_{n-1}}\mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_2\mathrm{d}u_1\ . \quad (6)$$

Now letting

$$t = \frac{u_n}{\Phi_{n-1}} \qquad (7)$$

and substituting in eqn (6), we obtain:

$$\begin{aligned} I_0^{n+1}&(h_0,h_1,h_2,\ldots,h_n) \\ &= \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-2}}\int_0^1 \Phi_{n-1}^{h_0}\left(1 - t\right)^{h_0} \\ &\quad \times \Phi_{n-1}^{h_n} t^{h_n} u_1^{h_1}u_2^{h_2},\ldots,u_{n-1}^{h_{n-1}}\Phi_{n-1}\mathrm{d}t\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_2\mathrm{d}u_1 \\ &= \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-2}} u_1^{h_1}u_2^{h_2},\ldots,u_{n-1}^{h_{n-1}} \\ &\quad \times \left[\Phi_{n-1}^{h_0+h_n+1}\frac{\lfloor h_0 \lfloor h_n}{\lfloor h_0+h_n+1}\right]\mathrm{d}u_{n-1}\mathrm{d}u_{n-2},\ldots,\mathrm{d}u_2\mathrm{d}u_1 \end{aligned}$$

$$= \frac{\lfloor\underline{h_0}\,\lfloor\underline{h_n}}{\lfloor\underline{h_0+h_n+1}} \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-3}} u_1^{h_1}u_2^{h_2},\ldots,u_{n-2}^{h_{n-2}}$$

$$\times\left(\int_0^{\Phi_{n-2}} u_{n-1}^{h_{n-1}}\Phi_{n-1}^{h_0+h_n+1}\mathrm{d}u_{n-1}\right)\mathrm{d}u_{n-2},\ldots,\mathrm{d}u_2\mathrm{d}u_1\,. \quad (8)$$

Now writing

$$\Phi_{n-1} = \Phi_{n-2}\left(1 - \frac{u_{n-1}}{\Phi_{n-2}}\right) \quad (9)$$

and then with the substitution

$$t = \frac{u_{n-1}}{\Phi_{n-2}} \quad (10)$$

we can evaluate the last integral in eqn (8), getting

$$\int_0^{\Phi_{n-2}} u_{n-1}^{h_{n-1}}\Phi_{n-1}^{h_0+h_n+1}\mathrm{d}u_{n-1}$$

$$= \int_0^1\left(t\Phi_{n-2}\right)^{h_{n-1}}\Phi_{n-2}^{h_0+h_n+1}(1-t)^{h_0+h_n+1}\Phi_{n-2}\mathrm{d}t$$

$$= \Phi_{n-2}^{h_{n-1}+h_n+h_0+2}\int_0^1 t^{h_{n-1}}(1-t)^{h_0+h_n+1}\mathrm{d}t$$

$$= \frac{\lfloor\underline{h_{n-1}}\,\lfloor\underline{h_0+h_n+1}}{\lfloor\underline{h_0+h_n+h_{n-1}+2}}\,\Phi_{n-2}^{h_0+h_n+h_{n-1}+2}. \quad (11)$$

Where we have utilized the well known formula

$$\int_0^1 t^{\alpha}(1-t)^{\beta}\mathrm{d}t = \frac{\lfloor\underline{\alpha}\,\lfloor\underline{\beta}}{\lfloor\underline{\alpha+\beta+1}} \quad (12)$$

substituting from eqn (11) into eqn (8) we obtain:

$$I_0^{n+1}(h_0,h_1,\ldots,h_n)$$

$$= \frac{\lfloor\underline{h_0}\,\lfloor\underline{h_n}}{\lfloor\underline{h_0+h_n+1}}\cdot\frac{\lfloor\underline{h_{n-1}}\,\lfloor\underline{h_0+h_n+1}}{\lfloor\underline{h_0+h_n+h_{n-1}+2}}$$

$$\cdot\int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-3}} u_{n-2}^{h_{n-2}}\Phi_{n-2}^{h_0+h_n+h_{n-1}+2}\mathrm{d}u_{n-2}\cdot$$

$$du_{n-3},\, du_{n-4},\ldots du_2du_1$$

Iterating the method, we finally get

$$I_0^{n+1}(h_0,h_1,\ldots,h_n) = \frac{\prod_{i=0}^{n}\lfloor\underline{h_i}}{\lfloor\underline{(\sum_{i=0}^{n} h_i + n)}} \quad (13a)$$

and substituting $h_0 = 0$ in eqn (13a), we obtain

$$I_0^{n+1}(h_0,h_1,h_2,\ldots,h_n)\ \underline{\underline{\mathrm{def}}}\int_0^1\int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-1}}$$

$$\times\, u_1^{h_1}u_2^{h_2},\ldots,u_n^{h_n}\mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_2\mathrm{d}u_1\,\underline{\underline{\mathrm{def}}}\,I_1^n(h_1,h_2,\ldots,h_n)$$

$$= \frac{\prod_{i=1}^{n}\lfloor\underline{h_i}}{\lfloor\underline{(\sum_{i=1}^{n} h_i + n)}}\,. \quad (13b)$$

3. INTEGRATION OVER AN N-POLYHEDRON IN $\mathbb{R}^n$

Suppose we have an n-polyhedron P in $\mathbb{R}^n$ described by the coordinates of its $n+1$ vertices,

$$V_i = (x_{1i},x_{2i},\ldots,x_{ni})\ (i = 0,1,2,\ldots,n)\,. \quad (14)$$

We want to compute the integral:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)\ \underline{\underline{\mathrm{def}}}\int,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\mathrm{d}\tau \quad (15)$$

where $\mathrm{d}\tau$ is the differential (n-dimensional) element.

A parametric representation for P is [10]:

$$x_i = c_{i0} + c_{i1}u_1 + c_{i2}u_2 + ,\ldots, + c_{in}u_n \quad (16a)$$

where

$$c_{i0} = x_{i0}\ (i = 1,2,\ldots,n),$$

$$c_{ij} = x_{ij} - x_{i0}\ (j = 1,2,\ldots,n,\ i = 1,2,\ldots,n)\,. \quad (16b)$$

We can also express eqn (16a) and eqn (16b) in an alternative form as:

$$x_i = x_{i0}u_0 + x_{i1}u_1 + x_{i2}u_2 + ,\ldots, + x_{in}u_n \quad (17a)$$

where

$$u_0 = 1 - u_1 - u_2 - u_3 - ,\ldots, - u_n,\ i = 1,2,\ldots,n\,. \quad (17b)$$

We can now substitute either equations (16) or (17) into eqn (15) and perform intergration to obtain $I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$. Let us first consider the following theorem which uses eqn (16a) and eqn (16b).

Theorem 1. A structure product $I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ over an n-polyhedron is a polynomial combination of the coordinates of vertices $V_i = (x_{1i},x_{2i},\ldots,x_{ni})$ $(i = 0,1,2,\ldots,n)$:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)\ \underline{\underline{\mathrm{def}}}\int,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\mathrm{d}x_1\mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

$$= |J|\left[\frac{c_{10}^{\lambda_1}c_{20}^{\lambda_2},\ldots,c_{n0}^{\lambda_n}}{\lfloor\underline{n}} + \left(\prod_{i=1}^{n}\lfloor\underline{\lambda_i}\right)\sum_{k=1}^{\lambda_1+\lambda_2+,\ldots,\lambda_n}\times\sum_{k_1+k_2+,\ldots,k_n=k}\right.$$

$$\left.\times\, \mathrm{I}_1^n\left(\mathrm{k}_1,\mathrm{k}_2,\ldots,\mathrm{k}_n\right)\mathrm{G}_1\left(\mathrm{k}_1,\mathrm{k}_2,\ldots,\mathrm{k}_n\right)\right] \quad (18)$$

where

$$G_1(k_1,k_2,\ldots,k_n) = \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1}$$

$$\sum_{r_1^2+r_2^2+,\ldots,+r_n^2=k_2},\ldots,\sum_{r_1^n+r_2^n+,\ldots,+r_n^n=k_n}\left\{\prod_{i=1}^{n}F_i\left(r_i^0,r_i^1,\ldots,r_i^n\right)\right\}$$

$$r_i^0 = \lambda_i - r_i^1 - r_i^2 - ,\ldots, - r_i^n \geq 0 \ (i = 1,2,\ldots,n) \quad (19)$$

$$F_i(r_i^0,r_i^1,\ldots,r_i^n) = \frac{c_{i0}^{r_i^0}c_{i1}^{r_i^1},\ldots,c_{in}^{r_i^n}}{\lfloor r_i^0 \ \lfloor r_i^1 \ \cdots \ \lfloor r_i^n} \quad (20)$$

$|J| = |\det J| =$ absolute value of det J,

$$\det J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix}^T$$

$$= \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix}^T,$$

$$c_{ij} = x_{ij} - x_{i0}, j = 1,2,\ldots,n, \ i = 0,1,2,\ldots,n \quad \text{and}$$
$$c_{i0} = x_{i0}, \ i = 1,2,\ldots,n \quad (21)$$

and $I_1^n(k_1,k_2,\ldots,k_n)$ is the structure product:

$$I_1^n(k_1,k_2,\ldots,k_n)$$
$$= \int_0^1\int_0^{\Phi_1}\int_0^{\Phi_2},\ldots,\int_0^{\Phi_{n-1}} u_1^{k_1}u_2^{k_2},\ldots,u_n^{k_n}\mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_2\mathrm{d}u_1$$

$$= \frac{\prod_{i=1}^{n}\lfloor k_i}{\lfloor \left(\sum_{i=1}^{n}k_i + n\right)} \text{ [from eqn (13b)]},$$

$$\Phi_i = 1 - u_1 - u_2 - ,\ldots, - u_i \ (i = 1,2,\ldots,n-1). \quad (22)$$

Proof. The natural coordinates of the standard n-simplex $\tilde{\sigma}_n = \tilde{\sigma}_n(\underline{u}) = \tilde{\sigma}_n(u_1,u_2,\ldots,u_n)$ in $\mathbb{R}^n$ are related to the coordinates of n-polyhedron P in $\mathbb{R}^n$ by eqn (16a) and (16b):

$$x_i = x_i(\underline{u}) = x_{i0} + c_{i1}u_1 + c_{i2}u_2 + ,\ldots, + c_{in}u_n$$

with

$$c_{i0} = x_{i0},$$
$$c_{ij} = x_{ij} - x_{i0}$$
$$(i = 1,2,\ldots,n, j = 1,2,\ldots,n). \quad (23)$$

If we now consider the mapping between the n-dimensional space $x_1, x_2,\ldots, x_n$ and the n-dimensional space $u_1, u_2,\ldots,u_n$ by the parametric eqn (23), we have for the differential element:

$$\mathrm{d}\tau = \mathrm{d}x_1\mathrm{d}x_2,\ldots,\mathrm{d}x_n = |\det J|\mathrm{d}u_1\mathrm{d}u_2,\ldots,\mathrm{d}u_n \quad (24)$$

where $|\det J|$ is defined in eqn (21).

Therefore, if we change the coordinates according to eqn (23) and express consistently the differential element by eqn (24), we obtain:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) \ \underline{\text{def}} \int\int,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\mathrm{d}x_1\mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

$$= \int\int,\ldots,\int_{\tilde{\sigma}} x_1^{\lambda_1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u})|\det J|\mathrm{d}u_1\mathrm{d}u_2,\ldots,\mathrm{d}u_n$$

$$= \int_0^1\int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-1}}$$

$$x_1^{\lambda_1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u})|\det J|\mathrm{d}u_1\mathrm{d}u_2,\ldots,\mathrm{d}u_n \quad (25)$$

where

$$\Phi_i = 1 - u_1 - u_2 - ,\ldots, - u_i \ (i = 1,2,\ldots,n-1)$$

Letting

$$X_i(\underline{u}) = x_i^{\lambda_i}(\underline{u}) \ (i = 1,2,\ldots,n),$$
$$f(\underline{u}) = X_1(\underline{u})X_2(\underline{u}),\ldots,X_n(\underline{u})$$
$$= x_1^{\lambda_1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u}) \quad (26)$$

we can now write eqn (25) as:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$$
$$= \int_0^1\int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-1}} f(\underline{u})|\det J|\mathrm{d}u_1\mathrm{d}u_2,\ldots,\mathrm{d}u_n. \quad (27)$$

We can now use the well-known Taylor's theorem to expand the function $f(\underline{u})$ in powers of $u_1,u_2,\ldots,u_{n-1},u_n$; we then obtain:

$$f(\underline{u}) = f(\underline{o}) + \sum_{k=1}^{\lambda_1+\lambda_2+,\ldots,+\lambda_n}\frac{1}{\lfloor k}$$

$$\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + ,\ldots, + u_n\frac{\partial}{\partial u_n}\right)_{(\underline{o})}^k f(\underline{u}). \quad (28a)$$

Now, by application of binomial theorem, we can write:

$$f(\underline{u}) = c_{10}^{\lambda_1}c_{20}^{\lambda_2},\ldots,c_{n0}^{\lambda_n} + \sum_{k=0}^{\lambda_1+\lambda_2+,\ldots,\lambda_n} \frac{1}{\lfloor k}$$

$$\times \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1},\ldots,\sum_{k_{n-1}=0}^{k-k_1-k_2-,\ldots,-k_{n-2}}$$

$$\times \frac{\lfloor k}{\lfloor k_1 \lfloor k_2 \cdots \lfloor k_n}\left[\left(u_1\frac{\partial}{\partial u_1}\right)^{k_1}\left(u_2\frac{\partial}{\partial u_2}\right)^{k_2},\ldots,\right.$$

$$\left.\times \left(u_n\frac{\partial}{\partial u_n}\right)^{k-k_1-,\ldots,-k_{n-1}} f(\underline{u})\right]_{(\varrho)}$$

$$= c_{10}^{\lambda_1}c_{20}^{\lambda_2},\ldots,c_{n0}^{\lambda_n} + \sum_{k=0}^{\lambda_1+\lambda_2+,\ldots,\lambda_n}\sum_{k_1=0}^{k}$$

$$\times \sum_{k_2=0}^{k-k_1},\ldots,\sum_{k_{n-1}=0}^{k-k_1-,\ldots,-k_{n-2}} \cdot \frac{u_1^{k_1}u_2^{k_2},\ldots,u_{n-1}^{k_{n-1}}u_n^{k_n}}{\lfloor k_1 \lfloor k_2 \cdots\cdots \lfloor k_{n-1} \lfloor k_n}$$

$$\times \left(\frac{\partial^k f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}}\right)_{(\varrho)} \quad (28b)$$

where

$$k_1 + k_2 + ,\ldots, + k_n = k,$$
$$k_n = k - k_1 - k_2 - ,\ldots, - k_{n-1}. \quad (29)$$

We can also write eqn 28a in the alternative form

$$f(\underline{u}) = (c_{10}^{\lambda_1}c_{20}^{\lambda_2},\ldots,c_{n0}^{\lambda_n}) + \sum_{k=0}^{\lambda_1+\lambda_2+,\ldots,+\lambda_n}\ \sum_{k_1+k_2+,\ldots,+k_n=k}$$

$$\frac{u_1^{k_1}u_2^{k_2},\ldots,u_n^{k_n}}{\lfloor k_1 \lfloor k_2 \cdots \lfloor k_n}\left(\frac{\partial^k f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}}\right)_{(\varrho)}. \quad (30)$$

We shall now determine the coefficients

$$\left[\frac{\partial^k f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}}\right]_{(0,0,\ldots,0)} /(\lfloor k_1 \lfloor k_2 \cdots \lfloor k_n)$$

of eqn (30).

Using Leibnitz's theorem on differentiation and eqn (26) we can write

$$\frac{\partial^{k_1} f(\underline{u})}{\partial u_1^{k_1}} = \sum_{r_1^1=0}^{k_1}\sum_{r_2^1=0}^{k-r_1^1}\sum_{r_3^1=0}^{k-r_1^1-r_2^1},\ldots,\sum_{r_{n-1}^1}^{k_1-r_1^1-r_2^1-,\ldots,-r_{n-2}^1}$$

$$\times \binom{k_1}{r_1^1}\binom{k_1-r_1^1}{r_2^1}\binom{k_1-r_1^1-r_2^1}{r_3^1},\ldots,$$

$$\times \binom{k_1-r_1^1-r_2^1-,\ldots,-r_{n-2}^1}{r_{n-1}^1}$$

$$\cdot\left(\frac{\partial^{r_1^1}X_1}{\partial u_1^{r_1^1}}\right)\left(\frac{\partial^{r_2^1}X_2}{\partial u_1^{r_2^1}}\right)\left(\frac{\partial^{r_3^1}X_3}{\partial u_1^{r_3^1}}\right),\ldots,$$

$$\times \left(\frac{\partial^{r_{n-1}^1}X_{n-1}}{\partial u_1^{r_{n-1}^1}}\right)\left(\frac{\partial^{k_1-r_1^1-r_2^1-,\ldots,r_{n-1}^1}}{\partial u_1^{k_1-r_1^1-r_2^1-,\ldots,-r_{n-1}^1}}\right). \quad (31)$$

Now letting

$$r_n^1 = k_1 - r_1^1 - r_2^1 - ,\ldots, - r_{n-1}^1 \quad (32)$$

we can also write, in short notations, the partial derivatives:

$$\frac{\partial^{r_1^1}X_1}{\partial u_1^{r_1^1}} = X_1,r_1^1,\ \frac{\partial^{r_2^1}X_2}{\partial u_1^{r_2^1}} = X_2,r_2^1,\ldots,$$

$$\frac{\partial^{r_{n-1}}X_{n-1}}{\partial u_1 r_{n-1}^1} = X_{n-1},r_{n-1}^1 \text{ and } \frac{\partial^{r_n^1}X_n}{\partial u_1^{r_n^1}} = X_n,r_n^1. \quad (33)$$

From eqns (32) and (33), we now have:

$$\frac{\partial^{k_1} f(\underline{u})}{\partial u_1^{k_1}} = \lfloor k_1 \sum_{r_1^1=0}^{k_1}\sum_{r_2^1=0}^{k-r_1^1}\sum_{r_3^1=0}^{k-r_1^1-r_2^1},\ldots,$$

$$\times \sum_{r_{n-1}^1}^{k-r_1^1-r_2^1-,\ldots,-r_{n-2}^1} \frac{(X_1,r_1^1)(X_2,r_2^1)(X_3,r_3^1),\ldots,(X_n,r_n^1)}{\lfloor r_1^i \lfloor r_2^i \lfloor r_3^i \cdots\cdots \lfloor r_n^i} \quad (34)$$

$$= \lfloor k_1 \sum_{r_1^1+r_2^1+r_3^1+,\ldots,+r_n^1=k_1}$$

$$\times \frac{(X_1,r_1^1)(X_2,r_2^1)(X_3,r_3^1),\ldots,(X_n,r_n^1)}{\lfloor r_1^i \lfloor r_2^i \lfloor r_3^i \cdots\cdots \lfloor r_n^i}. \quad (35)$$

Continuing in this manner, we derive:

$$\frac{\partial^{k_1} f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}} \Big/ \left(\lfloor k_1 \lfloor k_2 \lfloor k_3 \cdots\cdots \lfloor k_n\right)$$

$$= S^{k_1}_{r_1^1,r_2^1,\ldots,r_{n-1}^1} S^{k_2}_{r_1^2,r_2^2,\ldots,r_{n-1}^2},\ldots,S^{k_i}_{r_1^i,r_2^i,\ldots,r_{n-1}^i},\ldots,S^{k_n}_{r_1^n,r_2^n,\ldots,r_{n-1}^n}$$

$$\cdot \frac{(X_1,r_1^1,r_1^2,r_1^3,\ldots,r_1^n)}{(\lfloor r_1^1 \lfloor r_1^2 \lfloor r_1^3 \cdots \lfloor r_1^n)} \cdot \frac{(X_2,r_2^1,r_2^2,\ldots,r_2^n)}{(\lfloor r_2^1 \lfloor r_2^2 \cdots \lfloor r_2^n)},\ldots,$$

$$\times \frac{(X_n,r_n^1,r_n^2,\ldots,r_n^n)}{(\lfloor r_n^1 \lfloor r_n^2 \cdots \lfloor r_n^n)} \quad (36)$$

where

$$X_i,r_i^1,r_i^2,r_i^3,\ldots,r_i^n = \left(\frac{\partial^{r_i^1+r_i^2+,\ldots,+r_i^n}X_i}{\partial u_1^{r_i^1}\partial u_2^{r_i^2},\ldots,\partial u_n^{r_i^n}}\right),$$

$$r_n^i = k_n - r_1^i - r_2^i - ,\ldots, - r_{n-1}^i,$$

$$S^{k_i}_{r_1^i,r_2^i,\ldots,r_n^i} = \sum_{r_1^i=0}^{k_i}\sum_{r_2^i=0}^{k_i-r_1^i}\sum_{r_3^i=0}^{k_i-r_1^i-r_2^i},\ldots,\sum_{r_{n-1}^i=0}^{k_i-r_1^i-r_2^i-,\ldots,-r_{n-2}^1}. \quad (37)$$

We can also write eqn (36) in the alternative form:

$$\left[\frac{\partial^{k_1}f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}}\Bigg/\left(\underline{|k_1}\,\underline{|k_2}\,\underline{|k_3}\cdots\underline{|k_n}\right)\right]$$

$$= \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1}\ \sum_{r_1^2+r_2^2+,\ldots,+r_n^2=k_2},\ldots,\sum_{r_1^n+r_2^n+,\ldots,+r_n^n=k_n}$$

$$\cdot\frac{\left(\prod_{i=1}^{n} X_i, r_i^1, r_i^2,\ldots,r_i^n\right)}{\prod_{i=1}^{n}\left(\underline{|r_i^1}\,\underline{|r_i^2}\,\underline{|r_i^3}\cdots\underline{|r_i^n}\right)}. \quad (38)$$

From eqn (16a), (16b) and eqn (26), we have

$$X_i = x_i^{\lambda_i} = (x_{i0} + c_{i1}u_1 + c_{i2}u_2 + ,\ldots, c_{in}u_n)^{\lambda_i}. \quad (39)$$

Differentiating eqn (39) partially, with respect to $u_1,u_2,\ldots,u_n$, we obtain:

$$\left(\frac{\partial^{r_i^1+r_i^2+,\ldots,+r_i^n}X_i}{\partial u_1^{r_i^1}\partial u_2^{r_i^2},\ldots,\partial u_n^{r_i^n}}\right)$$

$$= \frac{\underline{|\lambda_i}\quad (c_{i1})^{r_i^1}(c_{i2})^{r_i^2},\ldots,(c_{in})^{r_i^n}}{\underline{|\lambda_i - r_i^1 - r_i^2 - \cdots\cdots - r_i^n}}\, x_i^{\lambda_i - r_i^1 - r_i^2 - ,\ldots, r_i^n}. \quad (40)$$

Thus, from eqn (40), we further derive

$$\left[\left(\frac{\partial^{r_i^1+r_i^2+,\ldots,+r_i^n}X_i}{\partial u_1^{r_i^1}\partial u_2^{r_i^2},\ldots,\partial u_n^{r_i^n}}\right)\Bigg/\left(\underline{|r_i^1}\,\underline{|r_i^2}\cdots\underline{|r_i^n}\right)\right]_{(0,0,\ldots,0)}$$

$$=(X_i,r_i^1,r_i^2,\ldots,r_i^n)_{(0,0,\ldots,0)}\left(\underline{|r_i^1}\,\underline{|r_i^2}\cdots\underline{|r_i^n}\right)$$

$$= \frac{\underline{|\lambda_i}\quad (c_{i0})^{\lambda_i - r_i^1 - r_i^2 - ,\ldots, - r_i^n} c_{i1}^{r_i^1} c_{i2}^{r_i^2},\ldots,c_{in}^{r_i^n}}{\underline{|\lambda_i - r_i^1 - r_i^2 - \cdots - r_i^n}\quad \underline{|r_i^1}\,\underline{|r_i^2}\cdots\cdots\underline{|r_i^n}}. \quad (41)$$

Let us define

$$F_i(r_i^0,r_i^1,r_i^2,\ldots r_i^n) = \frac{c_{i0}^{r_i^0}c_{i1}^{r_i^1},\ldots,c_{in}^{r_i^n}}{\underline{|r_i^0}\,\underline{|r_i^1}\cdots\underline{|r_i^n}} \quad (42a)$$

where

$$i = 1,2,\ldots,n,\ r_i^0 = \lambda_i - r_i^1 - r_i^2 - ,\ldots, - r_i^n \geq 0. \quad (42b)$$

Using eqn (42a) and (42b), we can write eqn (41) as:

$$(X_i,r_i^1,r_i^2,\ldots,r_i^n)_{(0,0,\ldots,0)} = \lambda_i \quad F_i(r_i^0,r_i^1,\ldots,r_i^n). \quad (43)$$

From eqns (38),(43), we obtain:

$$\left[\frac{\partial^{k_1}f(\underline{u})}{\partial u_1^{k_1}\partial u_2^{k_2},\ldots,\partial u_n^{k_n}}\Bigg/\left(\underline{|k_1}\,\underline{|k_2}\cdots\underline{|k_n}\right)\right]_{(0,0,\ldots,0)}$$

$$= \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1}\ \sum_{r_1^2+r_2^2+,\ldots,+r_n^2=k_2},\ldots,\sum_{r_1^n+r_2^n+,\ldots,+r_n^n=k_n}$$

$$\cdot\left(\underline{|\lambda_1}\,\underline{|\lambda_2}\cdots\underline{|\lambda_n}\right)\prod_{i=1}^{n}F_i\left(r_i^0,r_i^1,r_i^2,\ldots,r_i^n\right)$$

$$= \underline{|\lambda_1}\,\underline{|\lambda_2}\cdots\underline{|\lambda_n}\,G_1(k_1,k_2,\ldots,k_n)\ \text{(say)} \quad (44)$$

where

$$G_1(k_1,k_2,\ldots,k_n) = \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1}\ \sum_{r_1^2+r_2^2+,\ldots,+r_n^2=k_2},\ldots,$$

$$\sum_{r_1^n+r_2^n+,\ldots,+r_n^n=k_n}\cdot\left\{\prod_{i=1}^{n}F_i\left(r_i^0,r_i^1,\ldots,r_i^n\right)\right\}. \quad (45)$$

Using eqn (44), we can rewrite eqn (30) as:

$$f(\underline{u}) = (c_{10}^{\lambda_1}c_{20}^{\lambda_2},\ldots,c_{n0}^{\lambda_n}) + (\underline{|\lambda_1}\,\underline{|\lambda_2}\cdots\underline{|\lambda_n})\sum_{k=0}^{\lambda_1+\lambda_2,\ldots,+\lambda_n}$$

$$\sum_{k_1+k_2+,\ldots,+k_n=k} u_1^{k_1}u_2^{k_2},\ldots,u_n^{k_n}G_1\left(k_1,k_2,\ldots,k_n\right). \quad (46)$$

Substituting the expansion for $f(\underline{u})$ from eqn (36) into eqn (27) and performing integration, we obtain the result stated in eqn (18). This completes the proof of Theorem 1. □

Theorem 2. A structure product $I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ over an n-polyhedron is a polynomial combination of the coordinates of vertices $V_i = (x_{1i},x_{2i},\ldots,x_{ni})$ $(i = 0,1,2,\ldots,n)$:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)\ \underline{\text{def}} \int,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\mathrm{d}x_1\mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

$$= \left(\underline{|\lambda_1}\,\underline{|\lambda_2}\,\underline{|\lambda_3}\cdots\underline{|\lambda_n}\right)|J|\sum_{k_0+k_1+k_2+,\ldots,+k_n=k=\sum_{i=1}^{n}\lambda_i}$$

$$I_0^{n+1}\left(k_0,k_1,k_2,\ldots,k_n\right)G_0(k_0,k_1,\ldots,k_n) \quad (47)$$

where

$$I_0^{n+1}(k_0,k_1,k_2,\ldots,k_n) = \frac{\underline{|k_0}\,\underline{|k_1}\cdots\underline{|k_n}}{\underline{|(\sum_{i=0}^{n}k_i + n)}},$$

$$G_0(k_0,k_1,k_2,\ldots,k_n) = \sum_{r_1^0+r_2^0+,\ldots,+r_n^0=k_0}\ \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1},\ldots,$$

$$\times \sum_{r_1^n + r_2^n + ,\ldots, + r_n^n = k_n} \cdot \prod_{i=1}^{n} F_i\left(r_i^0, r_i^1, r_i^2, \ldots, r_i^n\right),$$

$$F_i(r_i^0, r_i^1, r_i^2, \ldots, r_i^n) = \frac{x_{i0}^{r_i^0} x_{i1}^{r_i^1} x_{i2}^{r_i^2}, \ldots, x_{in}^{r_i^n}}{\lfloor r_i^0 \; \lfloor r_i^1 \; \lfloor r_i^2 \; \ldots \; \lfloor r_i^n},$$

$$r_i^0 + r_i^1 + r_i^2 + ,\ldots, + r_i^n = \lambda_i \; (i = 1,2,\ldots,n) \qquad (48)$$

and $|J|$ = absolute value of det J = |det J| and det J is same as defined in eqn (21).

Proof. The natural coordinates of the standard n-simplex $\tilde{\sigma}_n = \tilde{\sigma}_n(\underline{u}) = \tilde{\sigma}_n(u_1,u_2,\ldots,u_n)$ in $\mathbb{R}^n$ are also related to the coordiantes of n-polyhedron P in $\mathbb{R}^n$ by the eqn 17a, b):

$$x_i = x_i(\underline{u}) = u_0 x_{i0} + u_1 x_{i1} + u_2 x_{i2} + ,\ldots, + u_n x_{in},$$

$$\tilde{\sigma}_n = \tilde{\sigma}_n(\underline{u}) = \tilde{\sigma}_n(u_1,u_2,\ldots,u_n) \qquad (49)$$

where

$$u_0 = 1 - u_1 - u_2 - ,\ldots, - u_n, \; i = 1,2,\ldots,n$$

$$(\underline{u}) = (u_0,u_1,\ldots,u_n). \qquad (50)$$

Now, proceeding in a way similar to the proof of Theorem 1, we can write

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) \; \underline{\underline{\mathrm{def}}} \; \int\int,\ldots,\int_P x_1^{\lambda_1} x_2^{\lambda_2},\ldots,x_n^{\lambda_n} \mathrm{d}x_1 \mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

$$= \int_0^1 \int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-1}}$$

$$x_1^{\lambda_1}(\underline{u}) x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u}) |\det J| \mathrm{d}u_1 \mathrm{d}u_2,\ldots,\mathrm{d}u_n \qquad (51a)$$

where

$$\Phi_i = 1 - u_1 - u_2 - ,\ldots, - u_i, \; i = 1,2,3,\ldots,n-1 \qquad (51b)$$

and $x_i(\underline{u})$ are as expressed in eqn (49).

Letting

$$X_i(\underline{u}) = x_i^{\lambda_i}(\underline{u}) \; (i = 1,2,\ldots,n),$$

$$f(\underline{u}) = X_1(\underline{u}) X_2(\underline{u}),\ldots,X_n(\underline{u}) \qquad (52)$$

we can write eqn (51a) as:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$$

$$= \int_0^1 \int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-1}} f(\underline{u}) |\det J| \mathrm{d}u_1 \mathrm{d}u_2,\ldots,\mathrm{d}u_n. \qquad (53)$$

We can now use the Taylor's theorem to expand the function $f(\underline{u})$ in powers of $u_0,u_1,u_2,\ldots,u_{n-1},u_n$ and then obtain:

$$f(\underline{u}) = \frac{1}{\lfloor \lambda_0 + \lambda_1 + \ldots + \lambda_n}$$

$$\times \left[\left(u_0 \frac{\partial}{\partial u_0} + u_1 \frac{\partial}{\partial u_1} + ,\ldots, + u_n \frac{\partial}{\partial u_n}\right)^{\lambda_0 + \lambda_1 + ,\ldots, + \lambda_n} f(\underline{u})\right]$$

$$(u_0 = 0, u_1 = 0, \ldots, u_n = 0). \qquad (54)$$

The use of the multinomial theorem in eqn (54) now yields:

$$f(\underline{u}) = \sum_{k_0 + k_1 + k_2 + ,\ldots, + k_n = k = \sum_{i=1}^{n} \lambda_i} \cdot \frac{u_0^{k_0} u_1^{k_1},\ldots,u_n^{k_n}}{\lfloor k_0 \; \lfloor k_1 \ldots \lfloor k_n}$$

$$\left(\frac{\partial^k f(\underline{u})}{\partial u_0^{k_0} \partial u_1^{k_1},\ldots,\partial u_n^{k_n}}\right)_{(u_0 = 0, u_1 = 0,\ldots,u_n = 0)}. \qquad (55)$$

We shall now determine the coefficients

$$\left[\frac{\partial^k f(\underline{u})}{\partial u_0^{k_0} \partial u_1^{k_1},\ldots,\partial u_n^{k_n}} \Big/ \left(\lfloor k_0 \; \lfloor k_1 \ldots \lfloor k_n\right)\right]_{(u_0 = 0, u_1 = 0,\ldots,u_n = 0)}$$

where

$$k = k_0 + k_1 + k_2 + ,\ldots, + k_n = \sum_{i=1}^{n} \lambda_i.$$

By the use of the multinomial theorem and from eqn (52), it can be shown that:

$$\left[\frac{\partial^k f(u_0,u_1,u_2,\ldots,u_n)}{\partial u_0^{k_0} \partial u_1^{k_1},\ldots,\partial u_n^{k_n}}\right]_{(u_0 = 0, u_1 = 0,\ldots,u_n = 0)}$$

$$= \sum_{r_1^0 + r_2^0 + ,\ldots, + r_n^0 = k_0} \; \sum_{r_1^1 + r_2^1 + ,\ldots, + r_n^1 = k^1},\ldots, \sum_{r_1^n + r_2^n + ,\ldots, + r_n^n = k_n}$$

$$\times \frac{\left(\prod_{i=1}^{n} X_{i,r_i^0,r_i^1,\ldots,r_i^n}\right)_{u_0 = 0, u_1 = 0,\ldots,u_n = 0}}{\left(\prod_{i=1}^{n} r_i^0\right)\left(\prod_{i=1}^{n} r_i^1\right),\ldots,\left(\prod_{i=1}^{n} r_i^n\right)} \qquad (56a)$$

where

$$X_i = X_i(\underline{u}) = x^{\lambda_i}(\underline{u}), \; \underline{u} = (u_0,u_1,u_2,\ldots,u_n),$$

$$X_{i,r_i^0,r_i^1,\ldots,r_i^n} = \frac{\partial^{r_i^0 + r_i^1 + ,\ldots, + r_i^n} X_i}{\partial u_0^{r_i^0} \partial u_1^{r_i^1},\ldots,\partial u_n^{r_i^n}} \; (i = 1,2,\ldots,n). \qquad (56b)$$

Using eqn (50) and treating $u_0,u_1,\ldots,u_n$ as independent variables, we obtain:

$$(X_i,r_i^0,r_i^1,\ldots,r_i^n)_{(u_0=0,u_1=0,\ldots,u_n=0)}$$

$$=\begin{cases}\lfloor\underline{\lambda_i}\,(x_{i0})^{r_i^0}(x_{i1})^{r_i^1},\ldots,(x_{in})^{r_i^n},\ r_i^0+r_i^1+,\ldots,+r_i^n=\lambda_i\\ 0,\ \text{otherwise},\end{cases} \tag{56c}$$

From eqns (56a)–(56c), we obtain:

$$\{G_0(k_0,k_1,k_2,\ldots,k_n)\}(\lfloor\underline{\lambda_1}\ \lfloor\underline{\lambda_2}\cdots\lfloor\underline{\lambda_n})$$

$$\underline{\underline{\text{def}}}\left[\left(\frac{\partial^k f(u_0,u_1,\ldots,u_n)}{\partial u_0^{k_0}\partial u_1^{k_1},\ldots,\partial u_n^{k_n}}\right)\Big/(\lfloor\underline{k_1}\ \lfloor\underline{k_2}\cdots\lfloor\underline{k_n})\right]_{(0,0,\ldots,0)}$$

$$=\left(\prod_{i=1}^{n}\lfloor\underline{\lambda_i}\right)\sum_{r_1^0+r_2^0+,\ldots,r_n^0=k_0}\ \sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k^1},\ldots,\sum_{r_1^n+r_2^n+,\ldots,+r_n^n}$$

$$=k_n\left(\prod_{i=1}^{n}F_i\left(r_i^0,r_i^1,r_i^2,\ldots,r_i^n\right)\right) \tag{57}$$

where

$$F_i(r_i^0,r_i^1,\ldots,r_i^n)$$

$$=\frac{x_{i0}^{r_i^0}x_{i1}^{r_i^1}x_{i2}^{r_i^2},\ldots,x_{in}^{r_i^n}}{\lfloor\underline{r_i^0}\ \lfloor\underline{r_i^1}\ \lfloor\underline{r_i^2}\cdots\lfloor\underline{r_i^n}},\ r_i^0+r_i^1+,\ldots,+r_i^n=\lambda_i\,. \tag{58}$$

Now substituting from eqn (57) into eqn (55), we obtain:

$$f(\underline{u})=\left(\prod_{i=1}^{n}\lfloor\underline{\lambda_i}\right)\sum_{k0+k1+,\ldots,+kn=k=\sum_{i=1}^{n}\lambda_i}$$

$$u_0^{k_0}u_1^{k_1},\ldots,u_n^{k_n}G_0\left(k_0,k_1,\ldots,k_n\right). \tag{59}$$

Using eqn 13a|eqns (53),(50) and performing integration, we obtain a result claimed in the statement of this theorem viz. eqn (47). This completes the proof of Theorem 2. □

4. SURFACE INTEGRATION OVER AN N-DIMENSIONAL POLYHEDRON

The integration of the scalar function $f(\underline{P})=x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}$ ($\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n\geq 0$ and positive integers) can be easily derived by using the divergence theorem for a standard n-simplex $\tilde{\sigma}_n=\tilde{\sigma}_n(\underline{u})=\tilde{\sigma}_n(u_1,u_2,\ldots,u_n)$ in $\mathbb{R}^n$.

Theorem 3. Let $\tilde{\sigma}_n=\tilde{\sigma}_n(\underline{u})-\tilde{\sigma}_n(u_1,u_2,\ldots,u_n)$ be the standard n-simplex in $\mathbb{R}^n$ defined by the inequalities $u_1\geq 0,u_2\geq 0,\ldots,u_n\geq 0$, $\Sigma_{i=1}^n u_i\leq 1$ and the $n+1$ vertices $V_0=(0,0,0,\ldots,0)$, $V_1=(1,0,0,\ldots,0)$, $V_2=(0,1,0,\ldots,0),\ldots,$ $V_n=(0,0,0,\ldots,1)$.

Then the integral over $\tilde{\sigma}_n=\tilde{\sigma}_n(\underline{u})=\tilde{\sigma}_n(u_1,u_2,\ldots,u_n)$ of the divergence of a vector function $\hat{A}=\hat{\mathbf{i}}_1A_1(\underline{u})+\hat{\mathbf{i}}_2A_2(\underline{u})+,\ldots,+\hat{\mathbf{i}}_nA_n(\underline{u})$, with $A_i(\underline{u})$, $i=1,2,\ldots,n$ as scalar functions in n-independent variables, $u_1,u_2,\ldots,u_n$ can be expressed as:

$$\iint,\ldots,\int_{\tilde{\sigma}_n\left(u_1,u_2,\ldots,u_n\right)}\nabla_n\cdot\hat{A}\,du_1du_2,\ldots,du_n\ \underline{\underline{\text{def}}}\ I^n(\hat{A})$$

$$=\iint,\ldots,\int_{\tilde{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})}$$

$$\times\left\{\sum_{i=1}^{n}A_i(u_1,u_2,\ldots,u_{n-2},1-u_1-u_2-,\ldots,-u_{n-1})\right\}$$

$$\times\,du_1du_2,\ldots,du_{n-1}-\sum_{i=1}^{n}\iint,\ldots,\int_{\tilde{\sigma}_{n-1}(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_n)}$$

$$[A_i(u_1,u_2,\ldots,u_n)]_{ui=0}\prod_{\substack{K=1\\K\neq i}}^{n}du_K \tag{60}$$

where

$$\nabla_n=\hat{\mathbf{i}}_1\frac{\partial}{\partial u_1}+\hat{\mathbf{i}}_2\frac{\partial}{\partial u_2}+,\ldots,+\hat{\mathbf{i}}_n\frac{\partial}{\partial u_n}\ \text{and}\ \hat{\mathbf{i}}_1,\hat{\mathbf{i}}_2,\ldots,\hat{\mathbf{i}}_n$$

are the unit vectors in the $u_1,u_2,\ldots,u_n$ space. (61)

Proof. We shall give proof of this theorem by using the principle of mathematical induction. Let us verify this theorem for $n=2$. We have from the left hand side of eqn (60):

$$\iint_{\tilde{\sigma}_2}\nabla_2\cdot\hat{A}\,du_1du_2\ \underline{\underline{\text{def}}}\ I^2(\hat{A})$$

$$=\int_0^1\int_0^{1-u_1}\left(\hat{\mathbf{i}}_1\frac{\partial}{\partial u_1}+\hat{\mathbf{i}}_2\frac{\partial}{\partial u_2}\right)$$

$$\cdot(\hat{\mathbf{i}}_1A_1(u_1,u_2)+\hat{\mathbf{i}}_2A_2(u_1,u_2))du_1du_2$$

$$=\int_0^1\int_0^{1-u_1}\left[\frac{\partial A_1(u_1,u_2)}{\partial u_1}+\frac{\partial A_2(u_1,u_2)}{\partial u_2}\right]du_2du_1 \tag{62}$$

$$=\int_0^1\int_0^{1-u_1}\frac{\partial A_1(u_1,u_2)}{\partial u_1}du_1du_2$$

$$+\int_0^1\left[A_2\left(u_1,1-u_1\right)-A_2\left(u_1,0\right)\right]du_1\,. \tag{63}$$

To find a reduction to first integral of eqn (63), let us recall the well-known result on differentiation under integral sign: see Ref. [13]:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx$$

$$= f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx\,. \quad (64)$$

Using eqn (64), we can write:

$$\int_0^1\left[\frac{\partial}{\partial u_1}\left(\int_0^{1-u_1} A_1\left(u_1,u_2\right)du_2\right)\right]du_1$$

$$= \int_0^1\left[-A_1\left(u_1,1-u_1\right) + \int_0^{1-u_1}\frac{\partial A_1(u_1,u_2)}{\partial u_1}du_2\right]du_1\,. \quad (65)$$

From eqn (65), we thus obtain:

$$\int_0^1\int_0^{1-u1}\frac{\partial A_1(u_1,u_2)}{\partial u_1}du_2du_1$$

$$= \int_0^1 A_1\left(u_1,1-u_1\right)du_1 - \int_0^1 A_1\left(0,u_2\right)du_2\,. \quad (66)$$

Substituting from eqn (66) into eqn (63), we obtain:

$$\iint_{\partial_2} \nabla_2\cdot\hat{A}du_1du_2 = \int_0^1\sum_{i=1}^{2}A_i\left(u_1,1-u_1\right)du_1$$

$$-\int_0^1 A_1(0,u_2)du_2 - \int_0^1 A_2(u_1,0)du_1\,. \quad (67)$$

From eqn (67), we see that the theorem is true for $n=2$. Let us now verify the theorem for $n=3$. We again have, from the left hand side of eqn (60):

$$\iiint_{\partial_3}\nabla_3\cdot\hat{A}du_1du_2du_3 \text{ def } I^3(\hat{A}) = \int_0^1\int_0^{1-u_1}\int_0^{1-u_1-u_2}$$

$$\times\left[\frac{\partial A_1(u_1,u_2,u_3)}{\partial u_1} + \frac{\partial A_2(u_1,u_2,u_3)}{\partial u_2} + \frac{\partial A_3(u_1,u_2,u_3)}{\partial u_3}\right]$$

$$\times\, du_3du_2du_1\,. \quad (68)$$

Let us now reduce each of the integrals on the right hand side of eqn (68), so that we have for the last term in eqn (68):

$$\int_0^1\int_0^{1-u_1}\int_0^{1-u_1-u_2}\frac{\partial A_3}{\partial u_3}du_1du_2du_3$$

$$= \int_0^1\int_0^{1-u_1}\left[A_3\left(u_1,u_2,1-u_1-u_2\right)\right.$$

$$\left.-A_3\left(u_1,u_2,0\right)\right]du_1du_2\,. \quad (69)$$

On using the well known result on integration that is stated in eqn (64), we obtain:

$$\int_0^1\int_0^{1-u_1}\int_0^{1-u_1-u_2}\frac{\partial A_1}{\partial u_1}du_3du_2du_1$$

$$= \int_0^1\int_0^{1-u_1}A_1\left(u_1,u_2,1-u_1-u_2\right)du_2du_1$$

$$+\int_0^1\int_0^{1-u_1}\frac{\partial}{\partial u_1}\left(\int_0^{1-u_1-u_2}A_1\left(u_1,u_2,u_3\right)du_3\right)du_2du_1 \quad (70)$$

and

$$\int_0^1\int_0^{1-u_1}\int_0^{1-u_1-u_2}\frac{\partial A_2}{\partial u_2}du_3du_2du_1$$

$$= \int_0^1\int_0^{1-u_1}A_2\left(u_1,u_2,1-u_1-u_2\right)du_2du_1$$

$$+\int_0^1\int_0^{1-u_1}\frac{\partial}{\partial u_2}\left(\int_0^{1-u_1-u_2}A_2\left(u_1,u_2,u_3\right)du_3\right)du_2du_1\,. \quad (71)$$

Substituting from eqns (69) – (71) into eqn (68), we obtain:

$$\iiint_{\partial_3}\nabla_3\cdot\hat{A}du_1du_2du_3 \text{ def } I^3(\hat{A})$$

$$= \int_0^1\int_0^{1-u_1}\left[\sum_{i=1}^{3}A_i\left(u_1,u_2,1-u_1-u_2\right)\right]du_2du_1$$

$$-\int_0^1\int_0^{1-u_1}A_3\left(u_1,u_2,0\right)du_2du_1$$

$$+\int_0^1\int_0^{1-u_1}\frac{\partial}{\partial u_1}\left(\int_0^{1-u_1-u_2}A_1\left(u_1,u_2,u_3\right)\mathrm{d}u_3\right)\mathrm{d}u_2\mathrm{d}u_1$$

$$+\int_0^1\int_0^{1-u_1}\frac{\partial}{\partial u_2}\left(\int_0^{1-u_1-u_2}A_2\left(u_1,u_2,u_3\right)\mathrm{d}u_3\right)\mathrm{d}u_2\mathrm{d}u_1\,. \tag{72}$$

Letting

$$A_1^*(u_1,u_2)=\int_0^{1-u_1-u_2}A_1\left(u_1,u_2,u_3\right)\mathrm{d}u_3,$$

$$A_2^*(u_1,u_2)=\int_0^{1-u_1-u_2}A_2\left(u_1,u_2,u_3\right)\mathrm{d}u_3 \tag{73}$$

the sum of the last two integrals in eqn (72) can be written as:

$$\int_0^1\int_0^{1-u_1}\left[\frac{\partial}{\partial u_1}\left(\int_0^{1-u_1-u_2}A_1\left(u_1,u_2,u_3\right)\mathrm{d}u_3\right)+\frac{\partial}{\partial u_2}\left(\int_0^{1-u_1-u_2}A_2\left(u_1,u_2,u_3\right)\mathrm{d}u_3\right)\right]\mathrm{d}u_2\mathrm{d}u_1$$

$$=\int_0^1\int_0^{1-u_1}\left(\frac{\partial A_1^*}{\partial u_1}+\frac{\partial A_2^*}{\partial u_2}\right)du_2du_1$$

$$=\int_0^1\sum_{i=1}^{2}A_i^*\left(u_1,1-u_1\right)\mathrm{d}u_1-\int_0^1A_1^*\left(0,u_2\right)\mathrm{d}u_2-\int_0^1A_2^*\left(u_1,0\right)\mathrm{d}u_1$$

(by the use of the statement of this theorem for $n = 2$)

$$=0-\int_0^1\int_0^{1-u_2}A_1\left(0,u_2,u_3\right)\mathrm{d}u_3\mathrm{d}u_2-\int_0^1\int_0^{1-u_1}A_2\left(u_1,0,u_3\right)\mathrm{d}u_3\mathrm{d}u_1\,. \tag{74}$$

Now, on substituting from eqn (74) into eqn (72), we obtain:

$$\iiint_{\partial_3}\nabla_3\cdot\hat{A}\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 \;\underline{\underline{\mathrm{def}}}\; I^3(\hat{A})$$

$$=\int_0^1\int_0^{1-u_1}\left[\sum_{i=1}^{3}A_i\left(u_1,u_2,1-u_1-u_2\right)\right]\mathrm{d}u_2\mathrm{d}u_1$$

$$-\int_0^1\int_0^{1-u_2}A_1\left(0,u_2,u_3\right)\mathrm{d}u_3\mathrm{d}u_2$$

$$-\int_0^1\int_0^{1-u_1}A_2\left(u_1,0,u_3\right)\mathrm{d}u_3\mathrm{d}u_1$$

$$-\int_0^1\int_0^{1-u_1}A_3\left(u_1,u_2,0\right)\mathrm{d}u_2\mathrm{d}u_1\,. \tag{75}$$

From eqn (75), we find that the theorem is true for $n = 3$. Now let us assume that the theorem is true for $n = m$; we shall then prove that the theorem is also true for $n = m + 1$. To prove this, let us consider:

$$\iiint,\ldots,\int_{\partial_{m+1}}\nabla_{m+1}\cdot\hat{A}\mathrm{d}u_1\mathrm{d}u_2,\ldots,\mathrm{d}u_{m+1}\;\underline{\underline{\mathrm{def}}}\; I^{m+1}(\hat{A})$$

$$=\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_m}\left[\frac{\partial A_1}{\partial u_1}+\frac{\partial A_2}{\partial u_2}+,\ldots,+\frac{\partial A_{m+1}}{\partial u_{m+1}}\right]\mathrm{d}u_{m+1}\mathrm{d}u_m,\ldots,\mathrm{d}u_1\,. \tag{76}$$

Clearly, the last term in the above integral [i.e. eqn (76)] can be reduced to:

$$\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_m}\frac{\partial A_{m+1}}{\partial u_{m+1}}\mathrm{d}u_{m+1}\mathrm{d}u_m,\ldots,\mathrm{d}u_2\mathrm{d}u_1$$

$$=\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m+1}}$$

$$\times\left[A_{m+1}\left(u_1,u_2,\ldots,u_m,1-u_1-u_2-,\ldots,-u_m\right)-A_{m+1}\left(u_1,u_2,\ldots,u_m,0\right)\right]\mathrm{d}u_m\mathrm{d}u_{m-1},\ldots,\mathrm{d}u_2,\mathrm{d}u_1\,. \tag{77}$$

Now, on using the well known result on integration which we have stated in eqn (64), we obtain:

$$\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_m}\frac{\partial A_1}{\partial u_1}\mathrm{d}u_{m+1}\mathrm{d}u_m,\ldots,\mathrm{d}u_2\mathrm{d}u_1$$

$$=\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times A_1\left(u_1,u_2,\ldots,1-u_1-u_2-,\ldots,-u_m\right)$$

$$\times\, du_m du_{m-1},\ldots,du_1 + \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times \frac{\partial}{\partial u_1}\left(\int_0^{1-u_1-u_2-,\ldots,-u_m} A_1(u_1,u_2,\ldots,u_{m+1})\right)$$

$$\times\, du_{m+1}\cdot du_m du_{m-1},\ldots,du_1\,. \tag{78}$$

Proceeding in a similar manner, we can show that:

$$\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-u_3,\ldots,-u_m}\frac{\partial A_i}{\partial u_i}\,du_{m+1}du_m,\ldots,du_2du_1$$

$$= \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times A_i\left(u_1,u_2,\ldots,1-u_1-u_2-,\ldots,-u_m\right)$$

$$\times\, du_m du_{m-1},\ldots,du_1 + \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times \frac{\partial}{\partial u_i}\left(\int_0^{1-u_1-u_2-,\ldots,-u_m} A_i(u_1,u_2,\ldots,u_{m+1})\right)$$

$$\times\, du_{m+1}\cdot du_m du_{m-1},\ldots,du_1\ (i=1,2,\ldots,m)\,. \tag{79}$$

Letting

$$A_i^*(u_1,u_2,\ldots,u_m) = \int_0^{1-u_1-u_2-,\ldots,-u_m}$$

$$\times A_i\left(u_1,u_2,\ldots,u_{m+1}\right)du_{m+1} \tag{80}$$

we can now write the sum of first m-integrals in eqn (76) as:

$$\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_m}\left(\sum_{i=1}^m \frac{\partial A_i}{\partial u_i}\right)$$

$$\times\, du_{m+1}du_m,\ldots,du_2du_1$$

$$= \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times\left[\sum_{i=1}^m A_i\left(u_1,u_2,\ldots,u_m,1-u_1-u_2-,\ldots,-u_m\right)\right]$$

$$\cdot du_m du_{m-1},\ldots,du_2du_1 + \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times\left(\sum_{i=1}^m \frac{\partial A_i^*}{\partial u_i}\right)du_m du_{m-1},\ldots,du_2du_1$$

$$= \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}\left[\sum_{i=1}^m A_i\right.$$

$$\left.\times\left(u_1,u_2,\ldots,u_m,1-u_1-u_2-,\ldots,-u_m\right)\right]$$

$$\cdot du_m du_{m-1},\ldots,du_2du_1 + \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-2}}$$

$$\times \sum_{i=1}^m A_i^*\left(u_1,u_2,\ldots,u_{m-1},1-u_1-u_2-,\ldots,-u_{m-1}\right)$$

$$\cdot du_{m-1}du_{m-2},\ldots,du_2du_1$$

$$-\sum_{i=1}^m \int\int,\ldots,\int_{\bar\sigma_{m-1}(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_m)}$$

$$\times\left[A_i^*(u_1,u_2,\ldots,u_m)\right]_{u_i=0}\prod_{\substack{K=1\\K\neq i}}^m du_k\,. \tag{81}$$

From eqn (80), we find that

$$A_i^*(u_1,u_2,\ldots,u_{m-1},1-u_1-u_2-,\ldots,-u_{m-1}) = 0\,. \tag{82}$$

Substituting from eqn (82) into eqn (81), we obtain:

$$\int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_m}\left(\sum_{i=1}^m\frac{\partial A_i}{\partial u_i}\right)$$

$$\times\, du_{m+1}du_m,\ldots,du_2du_1$$

$$= \int_0^1\int_0^{1-u_1},\ldots,\int_0^{1-u_1-u_2-,\ldots,-u_{m-1}}$$

$$\times\left[\sum_{i=1}^m A_i\left(u_1,u_2,\ldots,u_m,1-u_1-u_2-,\ldots,-u_m\right)\right]$$

$$\cdot du_m du_{m-1},\ldots,du_2du_1 - \sum_{i=1}^m$$

$$\times \int\int\int,\ldots,\int_{\bar\sigma_m\left(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_m u_{m+1}\right)}$$

$$\times A_i(u_1,u_2,\ldots,u_{i-1},0,u_{i+1},\ldots,u_{m+1})$$

$$\cdot\left(\prod_{\substack{K=1\\K\neq i}}^m du_k\right)du_{m+1}\,. \tag{83}$$

Adding eqns (77),(83), we see that this sum is equal

$I^{m+1}(\hat{A})$. This sum proves that the theorem is true for $n = m + 1$. Thus, by using the principle of mathematical induction, we find that the theorem is true for all n. This completes the proof of Theorem 3. □

Theorem 4. Let P be an n-dimensional polyhedron with $n + 1$ vertices V_i with each V_i defined in terms of coordinates as: $V_i = (x_{1i}, x_{2i}, \ldots, x_{ni})$ $(i = 1,2,3,\ldots,n)$, then the structure product:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) \underline{\underline{\text{def}}} \iint,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\mathrm{d}x_1\mathrm{d}x_2,\ldots,\mathrm{d}x_n \tag{84}$$

is reducible to a sum of $n + 1$ integrals over polyhedral surfaces of dimension $n - 1$,

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{|\det J_0^n|}{(\lambda_1 + 1)\det J_0^n}$$

$$\times \left[\iint,\ldots,\int_{\partial_{n-1}(u_1,u_2,\ldots,u_{n-1})}\right.$$

$$\times \left\{J_1^{n-1} - J_2^{n-1} + ,\ldots, + (-1)^{n-1}J_n^{n-1}\right\}$$

$$\times f\left(u_1,u_2,\ldots,u_{n-1},1 - u_1 - u_2 - ,\ldots,u_{n-1}\right)$$

$$\cdot \mathrm{d}u_{n-1}\mathrm{d}u_{n-2}\mathrm{d}u_{n-3},\ldots,\mathrm{d}u_2\mathrm{d}u_1 - \iint,\ldots,\int_{\partial_{n-1}(u_2,u_3,\ldots,u_n)} J_1^{n-1}$$

$$\times f\left(0,u_2,u_3,\ldots,u_n\right)\mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_3\mathrm{d}u_2$$

$$+ \left(\iint,\ldots,\int_{\partial_{n-1}(u_1,u_3,u_4,\ldots,u_n)} J_2^{n-1} f\left(u_1,0,u_3,u_4,\ldots,u_n\right)\right.$$

$$\left.\times \mathrm{d}u_n\mathrm{d}u_{n-1},\ldots,\mathrm{d}u_3\mathrm{d}u_1\right),\ldots + \left((-1)^{n-1}\iint,\ldots,\int_{\partial_{n-1}}\right.$$

$$\times \left(u_1,u_2,\ldots,u_{n-1}\right)J_n^{n-1} f\left(u_1,u_2,\ldots,u_{n-1},0\right)$$

$$\left.\times \mathrm{d}u_{n-1}\mathrm{d}u_{n-2},\ldots,\mathrm{d}u_2\mathrm{d}u_1\right) \tag{85}$$

where $\det J_0^n = \det J$ as defined in eqn (21),

$$J_k^{n-1} = \begin{vmatrix} \frac{\partial x_2}{\partial u_1}, & \frac{\partial x_2}{\partial u_2} & ,\ldots, & \frac{\partial x_2}{\partial u_{k-1}}, & \frac{\partial x_2}{\partial u_{k+1}} & ,\ldots, & \frac{\partial x_2}{\partial u_n} \\ \vdots & & & & & & \\ \frac{\partial x_{k-1}}{\partial u_1}, & \frac{\partial x_{k-1}}{\partial u_2} & ,\ldots, & \frac{\partial x_{k-1}}{\partial u_{k-1}}, & \frac{\partial x_{k-1}}{\partial u_{k+1}} & ,\ldots, & \frac{\partial x_{k-1}}{\partial u_n} \\ \frac{\partial x_{k+1}}{\partial u_1}, & \frac{\partial x_{k+1}}{\partial u_2} & ,\ldots, & \frac{\partial x_{k+1}}{\partial u_{k-1}}, & \frac{\partial x_{k+1}}{\partial u_{k+1}} & ,\ldots, & \frac{\partial x_{k+1}}{\partial u_n} \\ \vdots & & & & & & \\ \frac{\partial x_n}{\partial u_1}, & \frac{\partial x_n}{\partial u_2} & ,\ldots, & \frac{\partial x_n}{\partial u_{k-1}}, & \frac{\partial x_n}{\partial u_{k+1}} & ,\ldots, & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$

$$= \begin{vmatrix} c_{21}, & c_{22} & ,\ldots, & c_{2k-1}, & c_{2k+1} & ,\ldots, & c_{2n} \\ \vdots & & & & & & \\ c_{k-1,1}, & c_{k-1,2} & ,\ldots, & c_{k-1,k-1}, & c_{k-1,k+1} & ,\ldots, & c_{k-1,n} \\ c_{k+1,1}, & c_{k+1,2} & ,\ldots, & c_{k+1,k-1}, & c_{k+1,k+1} & ,\ldots, & c_{k+1,n} \\ \vdots & \vdots & & & & & \\ c_{n1}, & c_{n2} & ,\ldots, & c_{n,k-1}, & c_{n,k+1} & ,\ldots, & c_{nn} \end{vmatrix}$$

$$(k = 1,2,\ldots,n) \tag{86}$$

$$f(u_1,u_2,\ldots,u_n) = x_1^{\lambda_1+1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u})$$

where we have from eqn (16a) and eqn (16b):

$$x_i = (c_{i0} + c_{i1}u_1 + c_{i2}u_2 + ,\ldots, + c_{in}u_n)\ (i = 1,2,\ldots,n),$$

$$c_{i0} = x_{i0}\ (i = 1,2,3,\ldots,n),\ c_{ij} = x_{ij} - x_{i0}\ (i,j = 1,2,\ldots,n)$$

and also, from eqn (17a) and eqn (17b),

$$x_i = u_0x_{i0} + u_1x_{i1} + ,\ldots, + u_nx_{in}\ (i = 1,2,3,\ldots,n)$$

and

$$u_0 = 1 - u_1 - u_2 - ,\ldots, - u_n\,. \tag{87}$$

Proof. We have, from Eqns (24-26):

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) \underline{\underline{\text{def}}} \iint,\ldots,\int_p x_1^{\lambda_1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}dx_1dx_2,\ldots,dx_{n-1}dx_n$$

$$= |\det J_0^n| \iiint_{\bar{\sigma}_n(u_1,u_2,\ldots,u_n)} x_1^{\lambda_1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u})du_1du_2,\ldots,du_n \quad (88)$$

where $\bar{\sigma}_n(u_1,u_2,\ldots,u_n)$ is the standard n-simplex $\bar{\sigma}_n$ defined in eqns (1) and (2) and det J_0 is the same as det J; we can also write eqn (88) in an alternative form as:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{|\det J_0^n|}{(\lambda_1+1)} \iint,\ldots,\int_{\bar{\sigma}_n(u)} \frac{\partial}{\partial x_1} \times \left\{x_1^{\lambda_1+1}x_2^{\lambda_2},\ldots,x_n^{\lambda_n}\right\}du_1du_2,\ldots,du_{n-1}du_n$$

$$= \frac{|\det J_0^n|}{(\lambda_1+1)(\det J_0^n)} \iint,\ldots,\int_{\bar{\sigma}_n(u)} \left\{\frac{\partial}{\partial u_1} \times \left[f(\underline{u})\frac{\partial^T(x_2,x_3,\ldots,x_n)}{\partial(u_2,u_3,\ldots,u_n)}\right] + \frac{\partial}{\partial u_2}\left[-f(\underline{u})\frac{\partial^T(x_2,x_3,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}\right] +,\ldots,+(-1)^{n-1}\frac{\partial}{\partial u_n} \times \left[f(\underline{u})\frac{\partial^T(x_2,x_3,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_{n-1})}\right]\right\}du_1du_2,\ldots,du_{n-1}du_n \quad (89)$$

where

$f(\underline{u}) = x_1^{\lambda_1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u})$ and $\partial^T(x_2,x_3,\ldots,x_n)/\partial(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_n)$ $(i = 1,2,\ldots,n)$

are cofactors in

$$J^T = \begin{vmatrix} \frac{\partial x_1}{\partial u_1}, & \frac{\partial x_1}{\partial u_2} & ,\cdots, & \frac{\partial x_1}{\partial u_{n-1}}, & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1}, & \frac{\partial x_2}{\partial u_2} & ,\cdots, & \frac{\partial x_2}{\partial u_{n-1}}, & \frac{\partial x_2}{\partial u_n} \\ \vdots & & ,\cdots, & & \\ \frac{\partial x_n}{\partial u_1}, & \frac{\partial x_n}{\partial u_2} & & \frac{\partial x_n}{\partial u_{n-1}}, & \frac{\partial x_n}{\partial u_n} \end{vmatrix}. \quad (90)$$

Clearly,

$$J_0^n = \det J^T = \det J,$$

$$J_i^{n-1} = \frac{\partial^T(x_2,x_3,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_n)} \quad (i = 1,2,\ldots,n). \quad (91)$$

We can also rewrite eqn (89) as:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{1}{(\lambda_1+1)}\frac{|\det J|}{(\det J)} \times \iint,\ldots,\int_{\bar{\sigma}_n(u)} \nabla\cdot\hat{F}du_1du_2,\ldots,du_{n-1}du_n \quad (92)$$

where

$$F_i(\underline{u}) = (-1)^{i-1}J_i^n x_1^{\lambda 1}(\underline{u})x_2^{\lambda_2}(\underline{u}),\ldots,x_n^{\lambda_n}(\underline{u}) = (-1)^{i-1}J_i^n f(\underline{u}) \; (i = 1,2,3,\ldots,n). \quad (93)$$

$\nabla = \Sigma_{k=1}^n \mathbf{i}_k\partial/\partial u_k$ with $\mathbf{i}_k$ as unit-normal vectors along the u_k $(k = 1,2,\ldots,n)$ directions. Now, using the statement on divergence theorem for a standard n-simplex proved in Theorem 4 via eqns (60) and (61), we can write, on using eqns (92) and (93) as:

$$I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{1}{(\lambda_1+1)}\frac{|\det J|}{(\det J)}\Bigg[\iint,\ldots,\int_{\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})} \sum_{i=1}^n F_i\left(u_1,u_2,\ldots,u_{n-1},1-u_1-u_2-,\ldots,-u_{n-1}\right) \cdot du_1du_2,\ldots,du_{n-2}du_n - \sum_{i=1}^n \iint,\ldots,\int_{\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{i-1},u_{i+1},u_n)} \left\{F_i\left(u_1,u_2,\ldots,u_n\right)\right\}_{ui\neq 0} \prod_{\substack{k=1\\k\neq i}}^n du_k\Bigg]$$

$$= \frac{1}{(\lambda_1+1)}\frac{|\det J|}{(\det J)}\Bigg[\iint,\ldots,\int_{\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})} \times \left\{J_1^{n-1} - J_2^{n-1} +,\ldots,+(-1)^{n-1}J_n^{n-1}\right\} \cdot f\left(u_1,u_2,\ldots,u_{n-1},1-u_1-u_2-,\ldots,-u_{n-1}\right) \times du_{n-1}du_{n-2},\ldots,du_2du_1 - \iint,\ldots,\int_{\bar{\sigma}_{n-1}(u_2,u_3,\ldots,u_n)} J_1^{n-1}$$

$$\times f\left(0,u_2,u_3,\ldots,u_{n-1},u_n\right)du_n du_{n-1},\ldots,du_3 du_2$$

$$+\left(\iint,\ldots,\int_{\bar{\sigma}_{n-1}\left(u_1,u_3,u_4,\ldots,u_n\right)} J_2^{n-1}\right.$$

$$\left.\times f\left(u_1,0,u_3,\ldots,u_{n-1},u_n\right)du_n du_{n-1},\ldots,du_2 du_1\right),\ldots,$$

$$+\left((-1)^{n-1}\iint,\ldots,\int_{\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})} J_n^{n-1}\right.$$

$$\left.\left.\times f\left(u_1,u_2,\ldots,u_{n-1},0\right)du_{n-1}du_{n-2},\ldots,du_2 du_1\right)\right]. \quad (94)$$

This completes the proof of Theorem 4. □

We could now use Theorems 1 and 2 with some partial modifications to compute all the $n+1$ integrals to find $I_p^n(\lambda_1,\lambda_2,\ldots,\lambda_n)$.

5. APPLICATION EXAMPLE

We shall illustrate an application example which was previously considered in Refs [9, 10] by using the algorithm proposed in Theorem 2. The illustration of the same example by the use of Theorems 1 and 4 can be easily worked out following Ref. [9] in which the concepts were developed by use of finite-element coordinate transformations and the Gauss's divergence theorem for a three-dimensional Euclidean space.

Let us consider:

$$I_p^3(2,1,0) = \iiint_p x_1^2 x_2 dx_1 dx_2 dx_3 \quad (95a)$$

where P is the tetrahedron in $\mathbb{R}^3$ with vertices

$$< V_1 = (5,5,0),\ V_2 = (10,10,0),\ V_3(8,7,8),\ V_0 = (10,5,0) > . \quad (95b)$$

5.1. Volume Integration

Using the statement of Theorem 2 for $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$, $V_1 = (5,5,0)$, $V_2 = (10,10,0)$, $V_3 = (8,7,8)$ and $V_0 = (10,5,0)$, we can compute the integral of eqn (95a) and eqn (95b) by the following equation:

$$I_p^3(2,1,0) = |J|\,\underline{|2}\ \underline{|1}\ \underline{|0} \sum_{k_0+k_1+k_2+k_3=3} I_0^4\left(k_0,k_1,k_2,k_3\right)G_0(k_0,k_1,k_2,k_3)$$

$$= 2|J|[I_0^4(0,0,0,3)G_0(0,0,0,3) + I_0^4(0,0,1,2)G_0(0,0,1,2)$$
$$+ I_0^4(0,0,2,1)G_0(0,0,2,1) + I_0^4(0,0,3,0)G_0(0,0,3,0)$$
$$+ I_0^4(0,1,0,2)G_0(0,1,0,2) + I_0^4(0,1,1,1)G_0(0,1,1,1)$$
$$+ I_0^4(0,1,2,0)G_0(0,1,2,0) + I_0^4(0,2,0,1)G_0(0,2,0,1)$$
$$+ I_0^4(0,2,1,0)G_0(0,2,1,0) + I_0^4(0,3,0,0)G_0(0,3,0,0)$$
$$+ I_0^4(1,0,0,2)G_0(1,0,0,2) + I_0^4(1,0,1,1)G_0(1,0,1,1)$$
$$+ I_0^4(1,0,2,0)G_0(1,0,2,0) + I_0^4(1,1,0,1)G_0(1,1,0,1)$$
$$+ I_0^4(1,1,1,0)G_0(1,1,1,0) + I_0^4(1,2,0,0)G_0(1,2,0,0)$$
$$+ I_0^4(2,0,0,1)G_0(2,0,0,1) + I_0^4(2,0,1,0)G_0(2,0,1,0)$$
$$+ I_0^4(2,1,0,0)G_0(2,1,0,0) + I_0^4(3,0,0,0)G_0(3,0,0,0)]. \quad (96)$$

From eqn (13a) and eqn (48), we obtain:

$$I_0^4(k_0,k_1,k_2,k_3) = \frac{\underline{|k_0}\ \underline{|k_1}\ \underline{|k_2}\ \underline{|k_3}}{\underline{|6}}, \quad (97a)$$

$$G_0(k_0,k_1,k_2,k_3) = \sum_{r_1^0+r_2^0=k_0}\ \sum_{r_1^1+r_2^1=k_1}\ \sum_{r_1^2+r_2^2=k_2}\ \sum_{r_1^3+r_2^3=k_3} \cdot \prod_{i=1}^{3} F_i\left(r_i^0,r_i^1,r_i^2,r_i^3\right)$$

(since $r_i^0 + r_i^1 + r_i^2 + r_i^3 = \lambda_i$, $i = 1,2,3$ and $\lambda_3 = 0$, we have $r_3^0 = r_3^1 = r_3^2 = r_3^3 = 0$)

$$F_i(r_i^0,r_i^1,r_i^2,r_i^3) = \frac{x_{i0}^{r_i^0} x_{i1}^{r_i^1} x_{i2}^{r_i^2} x_{i3}^{r_i^3}}{\underline{|r_i^0}\ \underline{|r_i^1}\ \underline{|r_i^2}\ \underline{|r_i^3}}. \quad (97b)$$

From eqn (97b), we obtain:

$$G_0(0,0,0,3) = F_1(0,0,0,2)F_2(0,0,0,1)$$
$$G_0(0,0,1,2) = \{F_1(0,0,1,1)F_2(0,0,0,1) + F_1(0,0,0,2)F_2(0,0,1,0)\}$$
$$G_0(0,0,2,1) = \{F_1(0,0,2,0)F_2(0,0,0,1) + F_1(0,0,1,1)F_2(0,0,1,0)\}$$
$$G_0(0,0,3,0) = F_1(0,0,2,0)F(0,0,1,0)$$
$$G_0(0,1,0,2) = \{F_1(0,1,0,1)F_2(0,0,0,1) + F_1(0,0,0,2)F_2(0,1,0,0)\}$$
$$G_0(0,1,1,1) = \{F_1(0,1,1,0)F_2(0,0,0,1) + F_1(0,1,0,1)F_2(0,0,1,0) + F_1(0,0,1,1)F_2(0,1,0,0)\}$$
$$G_0(0,1,2,0) = \{F_1(0,1,1,0)F_2(0,0,1,0) + F_1(0,0,2,0)F_2(0,1,0,0)\}$$
$$G_0(0,2,0,1) = \{F_1(0,2,0,0)F_2(0,0,0,1) + F_1(0,1,0,1)F_2(0,1,0,0)\}$$
$$G_0(0,2,1,0) = \{F_1(0,2,0,0)F_2(0,0,1,0) + F_1(0,1,1,0)F_2(0,1,0,0)\}$$
$$G_0(0,3,0,0) = F_1(0,2,0,0)F_2(0,1,0,0)$$
$$G_0(1,0,0,2) = \{F_1(1,0,0,1)F_2(0,0,0,1) + F_1(0,0,0,2)F_2(1,0,0,0)\}$$

$$
\begin{aligned}
G_0(1,0,1,1) &= \{F_1(1,0,1,0)F_2(0,0,0,1) \\
&\quad + F_1(1,0,0,1)F_2(0,0,1,0) \\
&\quad + F_1(0,0,1,1)F_2(1,0,0,0)\} \\
G_0(1,0,2,0) &= \{F_1(1,0,1,0)F_2(0,0,1,0) \\
&\quad + F_1(0,0,2,0)F_2(1,0,0,0)\} \\
G_0(1,1,0,1) &= \{F_1(1,1,0,0)F_2(0,0,0,1) \\
&\quad + F_1(1,0,0,1)F_2(0,1,0,0) \\
&\quad + F_1(0,1,0,1)F_2(1,0,0,0)\} \\
G_0(1,1,1,0) &= \{F_1(1,1,0,0)F_2(0,0,1,0) \\
&\quad + F_1(1,0,1,0)F_2(0,1,0,0) \\
&\quad + F_1(0,1,1,0)F_2(1,0,0,0)\} \\
G_0(1,2,0,0) &= \{F_1(1,1,0,0)F_2(0,1,0,0) \\
&\quad + F_1(0,2,0,0)F_2(1,0,0,0)\} \\
G_0(2,0,0,1) &= \{F_1(2,0,0,0)F_2(0,0,0,1) \\
&\quad + F_1(1,0,0,1)F_2(1,0,0,0)\} \\
G_0(2,0,1,0) &= \{F_1(2,0,0,0)F_2(0,0,1,0) \\
&\quad + F_1(1,0,1,0)F_2(1,0,0,0)\} \\
G_0(2,1,0,0) &= \{F_1(2,0,0,0)F_2(0,1,0,0) \\
&\quad + F_1(1,1,0,0)F_2(1,0,0,0)\} \\
G_0(3,0,0,0) &= F_1(2,0,0,0)F_2(1,0,0,0)\,. \qquad (97c)
\end{aligned}
$$

Now using eqns (97a)–(97c), we obtain:

$$
\begin{aligned}
I_p^3(2,1,0) = |J|\Bigg[& \frac{x_{13}^2 x_{23}}{120} + \frac{1}{360}\left(2x_{12}x_{13}x_{23} + x_{13}^2 x_{12}\right) \\
& + \frac{1}{360}\left(x_{12}^2 x_{23} + 2x_{12}x_{13}x_{22}\right) + \frac{x_{12}^2 x_{22}}{120} \\
& + \frac{1}{360}\left(2x_{11}x_{13}x_{23} + x_{13}^2 x_{21}\right) \\
& + \frac{1}{360}\left(x_{11}x_{12}x_{23} + x_{11}x_{13}x_{22} + x_{12}x_{13}x_{21}\right) \\
& + \frac{1}{360}\left(2x_{11}x_{12}x_{22} + x_{12}^2 x_{21}\right) \\
& + \frac{1}{360}\left(x_{11}^2 x_{23} + 2x_{11}x_{13}x_{21}\right) \\
& + \frac{1}{360}\left(x_{11}^2 x_{22} + 2x_{11}x_{12}x_{21}\right) + \frac{1}{120}\left(x_{11}^2 x_{21}\right) \\
& + \frac{1}{360}\left(2x_{10}x_{13}x_{23} + x_{13}^2 x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}x_{12}x_{23} + x_{10}x_{13}x_{22} + x_{12}x_{13}x_{20}\right) \\
& + \frac{1}{360}\left(2x_{10}x_{12}x_{22} + x_{12}^2 x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}x_{11}x_{23} + x_{10}x_{13}x_{21} + x_{11}x_{13}x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}x_{11}x_{22} + x_{10}x_{12}x_{21} + x_{11}x_{12}x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}x_{11}x_{21} + x_{11}^2 x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}^2 x_{23} + 2x_{10}x_{13}x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}^2 x_{22} + 2x_{10}x_{12}x_{20}\right) \\
& + \frac{1}{360}\left(x_{10}^2 x_{21} + 2x_{10}x_{11}x_{20}\right) + \frac{1}{120}\left(x_{10}^2 x_{20}\right)\Bigg]. \qquad (98)
\end{aligned}
$$

For the application example of eqn (95a) and (95b),

$$
\begin{aligned}
V_0 &= (x_{10},x_{20},x_{30}) = (10,5,0), \\
V_1 &= (x_{11},x_{21},x_{31}) = (5,5,0), \\
V_2 &= (x_{12},x_{22},x_{32}) = (10,10,0), \\
V_3 &= (x_{13},x_{23},x_{33}) = (8,7,8), \quad |J| = |\det J| = 200\,. \qquad (99)
\end{aligned}
$$

Using eqn (99) and rewriting eqn (98) as:

$$
\begin{aligned}
I_p^3(2,1,0) = |J|\Bigg[& \frac{1}{120}\left(x_{13}^2 x_{23} + x_{11}^2 x_{21} + x_{10}^2 x_{20} + x_{12}^2 x_{22}\right) \\
& + \frac{1}{360}\Bigg\{\left(x_{11}^2 x_{20} + 2x_{11}x_{10}x_{21}\right) + \left(x_{12}^2 x_{20} + 2x_{12}x_{10}x_{22}\right) \\
& + \left(x_{13}^2 x_{20} + 2x_{13}x_{10}x_{23}\right) + \left(x_{10}^2 x_{21} + 2x_{11}x_{10}x_{20}\right) \\
& + \left(x_{12}^2 x_{21} + 2x_{11}x_{12}x_{22}\right) + \left(x_{13}^2 x_{21} + 2x_{11}x_{13}x_{23}\right) \\
& + \left(x_{10}^2 x_{22} + 2x_{12}x_{10}x_{20}\right) + \left(x_{11}x_{22} + 2x_{11}x_{12}x_{21}\right) \\
& + \left(x_{13}^2 x_{22} + 2x_{12}x_{13}x_{23}\right) + \left(x_{10}^2 x_{23} + 2x_{13}x_{10}x_{20}\right) \\
& + \left(x_{11}^2 x_{23} + 2x_{11}x_{13}x_{21}\right) + \left(x_{12}^2 x_{23} + 2x_{12}x_{13}x_{22}\right) \\
& + \left(x_{11}x_{12}x_{20} + x_{12}x_{10}x_{21} + x_{11}x_{10}x_{22}\right)
\end{aligned}
$$

$$+\left(x_{11}x_{13}x_{20}+x_{13}x_{10}x_{21}+x_{11}x_{10}x_{23}\right)$$

$$+\left(x_{12}x_{13}x_{20}+x_{13}x_{10}x_{22}+x_{12}x_{10}x_{23}\right)$$

$$+\left(x_{11}x_{13}x_{22}+x_{12}x_{13}x_{21}+x_{11}x_{12}x_{23}\right)\Big\}\Big]$$

$$=(200)\left[\frac{1}{120}(500+125+1000+448)\right.$$

$$+\left(\frac{1}{360}\right)\Big\{625+2500+1440+1000+1500$$

$$+880+2000+750+1760+1500+575$$

$$\left.+2300+1250+950+1900+1150\Big\}\right]$$

$$=\left(\frac{25}{15}\right)\left[(2073)+\frac{1}{3}(22\,080)\right]=\frac{25}{15}\left(\frac{28\,299}{3}\right)$$

$$=\frac{47\,165}{3}\,. \qquad (100)$$

The result obtained in eqn (100) is in agreement with Ref. [10]. We see that the present algorithm is on par with the one illustrated in Ref. [9]. Hence, it is also economical in terms of arithmetic operations, compared to Ref. [10], by about 60%.

5.2. Surface Integration

We shall again illustate the application example of eqn (95a) and (95b) by the second algorithm based on the concept of surface integration, stated in Theorem 4. Following the method outlined in Theorems 1 and 2, we can also state the following two corollories without proof:

Corollary 1. A structure product over a standard $(n-1)$-simplex $\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})=\bar{\sigma}_{n-1}(\underline{u})$ in $\mathbb{R}^{n-1}$ defined by:

$$I^n_{\bar{\sigma}_{n-1}(\underline{u})}(\mu_1,\mu_2,\ldots,\mu_n)\ \underline{\underline{\mathrm{def}}}\ \int_0^1\int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-2}} U_1^{\mu_1}(\underline{u})U_2^{\mu_2}(\underline{u}),\ldots,U_n^{\mu_n}(\underline{u})\mathrm{d}u_n,\ldots,\mathrm{d}u_1$$

and is expressible as

$$I^n_{\bar{\sigma}_{n-1}(\underline{u})}(\mu_1,\mu_2,\ldots,\mu_n)=\frac{D_{10}^{\mu_1}D_{20}^{\mu_2},\ldots,D_{n0}^{\mu_n}}{\underline{|n}}$$

$$+\ \underline{|\mu_1}\,\underline{|\mu_2}\cdots\underline{|\mu_n}\sum_{k=1}^{\mu_1+\mu_2+,\ldots,+\mu_n}\ \sum_{k_1+k_2+,\ldots,k_{n-1}=k}$$

$$I_1^{n-1}\left(k_1,k_2,\ldots,k_{n-1}\right)G(k_1,k_2,\ldots,k_{n-1}) \qquad (101)$$

where

$$I_1^{n-1}(k_1,k_2,\ldots,k_{n-1})=\frac{\underline{|k_1}\,\underline{|k_2}\cdots\underline{|k_{n-1}}}{\underline{|\{\sum_{i=1}^{n}k_i+(n-1)\}}}\ ,$$

$$\Phi_i=1-u_1-u_2-,\ldots,-u_i,\ i=1,2,\ldots,n-2,$$

$$U_i(\underline{u})=D_{i0}+D_{i1}u_1+D_{i2}u_2+,\ldots,+D_{in-1}u_{n-1}$$

D_{ij} depends upon C_{ij} $(i=1,2,\ldots,n,\ j=0,1,2,\ldots,n)$

$$G_1(k_1,k_2,\ldots,k_{n-1})=\sum_{r_1^1+r_2^1+,\ldots,+r_n^1=k_1}\ \sum_{r_1^2+r_2^2,\ldots,+r_n^2}$$

$$=k_2,\ldots,\sum_{r_1^{n-1}+r_2^{n-1}+,\ldots,+r_n^{n-1}=k_{n-1}}$$

$$\cdot\left(\prod_{i=1}^{n}S_i\left(r_i^0,r_i^1,\ldots,r_i^{n-1}\right)\right),$$

$$r_i^0=\lambda_i-r_i^1-r_i^2-,\ldots,-r_i^{n-1}\geq 0\ (i=1,2,\ldots,n),$$

$$S_i(r_i^0,r_i^1,r_i^2,\ldots r_i^{n-1})=\frac{D_{i0}^{r_i^0}D_{i1}^{r_i^1}D_{i2}^{r_i^2},\ldots,D_{in-1}^{r_i^{n-1}}}{\underline{|r_i^0}\,\underline{|r_i^1}\,\underline{|r_i^2}\cdots\underline{|r_i^{n-1}}}\,. \qquad (102)$$

Corollary 2. A structure product over a standard $(n-1)$-simplex $\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})=\bar{\sigma}_{n-1}(\underline{u})$ in $\mathbb{R}^{n-1}$ is defined by:

$$I^n_{\bar{\sigma}_{n-1}(\underline{u})}(\mu_1,\mu_2,\ldots,\mu_n)\ \underline{\underline{\mathrm{def}}}\ I^n_{\bar{\sigma}_{n-1}(u_1,u_2,\ldots,u_{n-1})}(\mu_1,\mu_2,\ldots,\mu_n)$$

$$=\int_0^1\int_0^{\Phi_1},\ldots,\int_0^{\Phi_{n-2}}U_1^{\mu_1}(\underline{u})U_2^{\mu_2}(\underline{u}),\ldots,$$

$$\times\ U_n^{\mu_n}(\underline{u})\mathrm{d}u_{n-1}\mathrm{d}u_{n-2},\ldots,\mathrm{d}u_2\mathrm{d}u_1$$

$$=\left(\underline{|\mu_1}\,\underline{|\mu_2}\cdots\underline{|\mu_n}\right)\sum_{k_0+k_1+k_2+,\ldots,k_{n-1}=k=\sum_{i=1}\mu_i}$$

$$\times\ I_0^n\left(k_0,k_1,k_2,\ldots,k_{n-1}\right)G_0(k_0,k_1,k_2,\ldots,k_{n-1}) \qquad (103)$$

where

$$U_i(\underline{u})=U_i(u_1,u_2,\ldots,u_{n-1})$$

$$=u_{i0}u_0+u_{i1}u_1+,\ldots,+u_{i,n-1}u_{n-1},$$

U_{ij} $(i=1,2,\ldots,n;\ j=0,1,2,\ldots,n)$ depend on x_{ij},

$$I_0^{n-1}(k_0,k_1,k_2,\ldots,k_{n-1})=\frac{\underline{|k_0}\,\underline{|k_1}\,\underline{|k_2}\cdots\underline{|k_{n-1}}}{\underline{|\{\sum_{i=0}^{n-1}k_i+(n-1)\}}}$$

$$=\frac{\underline{|k_0}\,\underline{|k_1}\,\underline{|k_2}\cdots\underline{|k_{n-1}}}{\underline{|\{\sum_{i=1}^{n}\mu_i+(n-1)\}}}\ ,$$

$$\Phi_i = 1 - u_1 - u_2 - ,\ldots, - u_i,\ (i = 1,2,\ldots,n-2),$$

$$G_0(k_0,k_1,k_2,\ldots,k_{n-1}) = \sum_{r_1^0 + r_2^0 + ,\ldots, + r_n^0 = k_0} \sum_{r_1^1 + r_2^1,\ldots, + r_n^1 = k_1} ,\ldots,$$

$$\times \sum_{r_1^{n-1} + r_2^{n-1} + ,\ldots, + r_n^{n-1} = k_{n-1}} \cdot\left(\prod_{i=1}^{n} S_i\left(r_i^0,r_i^1,r_i^2,\ldots,r_i^{n-1}\right)\right),$$

$$S_i(r_i^0,r_i^1,r_i^2,\ldots r_i^{n-1}) = \frac{U_{i0}^{r_i^0}U_{i1}^{r_i^1}U_{i2}^{r_i^2},\ldots,U_{in-1}^{r_i^{n-1}}}{\lfloor r_i^0 \ \lfloor r_i^1 \ \lfloor r_i^2 \cdots \lfloor r_i^{n-1}},$$

$$r_i^0 + r_i^1 + r_i^2 + ,\ldots,r_i^{n-1} = \mu_i\ (i = 1,2,\ldots,n)\,. \quad (104)$$

We shall now illustrate the computation of eqn (95a) and (95b) by the use of Theorem 4 and Corollory 2, since the use of Corollory 1 has appeared in a different context in the earlier paper by the authors [9]. By the use of Theorem 4, we can now write, from eqn (85) and eqns (95a), (95b):

$$I_p^3(2,1,0) = \iiint_p x_1^2 x_2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

$$= \frac{1}{3}\frac{|\det J|}{(\det J)}\left[\iint_{\bar\sigma_2\left(u_1,u_2\right)} \times \left(\sum_{i=1}^{3} F_i\left(u_1,u_2,1-u_1-u_2\right)\right)\mathrm{d}u_2\mathrm{d}u_1 - \iint_{\bar\sigma_2\left(u_2,u_3\right)} F_1\left(0,u_2,u_3\right)\mathrm{d}u_3\mathrm{d}u_2 - \iint_{\bar\sigma_2\left(u_1,u_3\right)} F_2\left(u_1,0,u_3\right)\mathrm{d}u_3\mathrm{d}u_1 - \iint_{\bar\sigma_2\left(u_1,u_2\right)} F_3\left(u_1,u_2,0\right)\mathrm{d}u_2\mathrm{d}u_1\right]. \quad (105)$$

We have, clearly:

$$\det J = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

$$\sum_{i=1}^{3} F_i\left(u_1,u_2,1-u_1-u_2\right) = (J_1^2 - J_2^2 + J_3^2)x_1^2(u_1,u_2,1-u_1-u_2) \times x_2(u_1,u_2,1-u_1-u_2),$$

$$F_1(0,u_2,u_3) = x_1^3(0,u_2,u_3)x_2(0,u_2,u_3)J_1^2,$$

$$F_2(u_1,0,u_2) = -x_1^3(u_1,0,u_3)x_2(u_1,0,u_3)J_2^2,$$

$$F_3(u_1,u_2,0) = x_1^3(u_1,u_2,0)x_2(u_1,u_2,0)J_3^2,$$

$$J_1^2 = \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix},\quad J_2^2 = \begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix},\quad J_3^2 = \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix},$$

$$c_{11} = x_{11} - x_{10},\ c_{12} = x_{12} - x_{10},\ c_{13} = x_{13} - x_{10},$$

$$c_{21} = x_{21} - x_{20},\ c_{22} = x_{22} - x_{20},\ c_{23} = x_{23} - x_{20},$$

$$c_{31} = x_{31} - x_{30},\ c_{32} = x_{32} - x_{30},\ c_{33} = x_{33} - x_{30}. \quad (106)$$

Using eqn (95b), we obtain:

$$I_p^3(2,1,0) = -\frac{40}{3}\left[\iint_{\bar\sigma_2\left(u_1,u_2\right)} x_1^3\left(u_1,u_2,1-u_1-u_2\right) \times x_2\left(u_1,u_2,1-u_1-u_2\right)\mathrm{d}u_2\mathrm{d}u_1 - \iint_{\bar\sigma_2\left(u_2,u_3\right)} x_1^3\left(0,u_2,u_3\right)x_2\left(0,u_2,u_3\right)\mathrm{d}u_3\mathrm{d}u_2\right] \quad (107)$$

since

$$\det J = -200,\ J_1^2 = 40,\ J_2^2 = 0,\ J_3^2 = 0\,. \quad (108)$$

We shall now illustate the application of integrals in eqn (10) by use of Corollory 2. We see that from eqn (108):

$$x_i(u_1,u_2,1-u_1-u_2) = u_1x_{i1} + u_2x_{i2} + (1-u_1-u_2)x_{i3} = u_0x_{i3} + u_1x_{i1} + u_2x_{i2},$$

$$x_i(0,u_2,u_3) = (1-u_2-u_3)x_{i0} + u_2x_{i2} + u_3x_{i3} (i = 1,2,3). \quad (109)$$

Using eqn (103), we find that:

$$I^3_{\bar\sigma_2(u_1,u_2)}(3,1,0) = \iint_{\bar\sigma_2\left(u_1,u_2\right)} U_1^3(u_1,u_2)U_2(u_1,u_2)\mathrm{d}u_2\mathrm{d}u_1$$

$$= \lfloor 3 \ \lfloor 1 \ \lfloor 0 \sum_{k_0+k_1+k_2=4} I_0^3\left(k_0,k_1,k_2\right)G_0(k_0,k_1,k_2)$$

$$= \lfloor 3 \ \lfloor 1 \ \lfloor 0\ [I_0^3(0,0,4)G_0(0,0,4) + I_0^3(0,1,3)G_0(0,1,3) + I_0^3(1,0,3)G_0(1,0,3) + I_0^3(2,0,2)G_0(2,0,2) + I_0^3(1,1,2)G_0(1,1,2) + I_0^3(0,2,2)G_0(0,2,2) + I_0^3(3,0,1)G_0(3,0,1) + I_0^3(2,1,1)G_0(2,1,1) + I_0^3(1,2,1)G_0(1,2,1) + I_0^3(0,3,1)G_0(0,3,1) + I_0^3(4,0,0)G_0(4,0,0) + I_0^3(3,1,0)G_0(3,1,0) + I_0^3(2,2,0)G_0(2,2,0) + I_0^3(1,3,0)G_0(1,3,0) + I_0^3(0,4,0)G_0(0,4,0)]\,. \quad (110)$$

We also have, from eqn (104):

$$G_0(k_0,k_1,k_2) = \sum_{r_1^0+r_2^0+r_3^0=k_0} \; \sum_{r_1^1+r_2^1+r_3^1=k_1} \; \sum_{r_1^2+r_2^2+r_3^2=k_2}$$

$$\left[\prod_{i=1}^{3} S_i\left(r_i^0,r_i^1,r_i^2\right)\right],\; r_i^0+r_i^1+r_i^2=\mu_i,\; i=1,2,3.$$

Clearly, since $\mu_1=3$, $\mu_2=1$, $\mu_3=0$, we have $r_3^0=r_3^1=r_3^2=0$ and $S_3(r_3^0,r_3^1,r_3^2)=1$.

Hence, we have

$$G_0(k_0,k_1,k_2) = \sum_{r_1^0+r_2^0=k_0} \; \sum_{r_1^1+r_2^1=k_1} \; \sum_{r_1^2+r_2^2=k_2} \left(\prod_{i=1}^{2} S_i\left(r_i^0,r_i^1,r_i^2\right)\right)$$

$$(r_1^0+r_1^1+r_1^2=3, r_2^0+r_2^1+r_2^2=1)\,. \quad (111)$$

From eqn (111) we obtain:

$$G_0(0,0,4) = S_1(0,0,3)S_2(0,0,1),$$
$$G_0(0,1,3) = S_1(0,1,2)S_2(0,0,1) + S_1(0,0,3)S_2(0,1,0),$$
$$G_0(1,0,3) = S_1(1,0,2)S_2(0,0,1) + S_1(0,0,3)S_2(1,0,0),$$
$$G_0(2,0,2) = S_1(2,0,1)S_2(0,0,1) + S_1(1,0,2)S_2(1,0,0)$$
$$G_0(1,1,2) = S_1(1,1,1)S_2(0,0,1) + S_1(1,0,2)S_2(0,1,0) + S_1(0,1,2)S_2(1,0,0),$$
$$G_0(0,2,2) = S_1(0,2,1)S_2(0,0,1) + S_1(0,1,2)S_2(0,1,0),$$
$$G_0(3,0,1) = S_1(3,0,0)S_2(0,0,1) + S_1(2,0,1)S_2(1,0,0),$$
$$G_0(2,1,1) = S_1(2,0,1)S_2(0,1,0) + S_1(2,1,0)S_2(0,0,1) + S_1(1,1,1)S_2(1,0,0)$$
$$G_0(1,2,1) = S_1(1,2,0)S_2(0,0,1) + S_1(0,2,1)S_2(1,0,0) + S_1(1,1,1)S_2(0,1,0),$$
$$G_0(0,3,1) = S_1(0,3,0)S_2(0,0,1) + S_1(0,2,1)S_2(0,1,0),$$
$$G_0(4,0,0) = S_1(3,0,0)S_2(1,0,0),$$
$$G_0(3,1,0) = S_1(3,0,0)S_2(0,1,0) + S_1(2,1,0)S_2(1,0,0),$$
$$G_0(2,2,0) = S_1(2,1,0)S_2(0,1,0) + S_1(1,2,0)S_2(1,0,0),$$
$$G_0(1,3,0) = S_1(1,2,0)S_2(0,1,0) + S_1(0,3,0)S_2(1,0,0),$$
$$G_0(0,4,0) = S_1(0,3,0)S_2(0,1,0)\,. \quad (112)$$

From eqn (104), we also have:

$$I_0^3(k_0,k_1,k_2) = \frac{\lfloor\underline{k_0}\,\lfloor\underline{k_1}\,\lfloor\underline{k_2}}{\lfloor\underline{6}}, \text{ since } k_0+k_1+k_2=4\,. \quad (113)$$

From eqns (110) – (112) we obtain:

$$I^3_{\partial_2(u_1,u_2)}(3,1,0) = \iint_{\partial_2(u_1,u_2)} U_1^3(u_1,u_2)U_2(u_1,u_2)\mathrm{d}u_2\mathrm{d}u_1$$

$$= \frac{\lfloor\underline{3}}{\lfloor\underline{6}}\,[(4U_{12}^3U_{22}) + \{3U_{11}^2U_{12}^2U_{22} + U_{12}^3U_{21}\}$$
$$+ \{3U_{10}U_{12}^2U_{22} + U_{12}^3U_{20}\}$$
$$+ \{2U_{10}^2U_{12}U_{22} + 2U_{10}U_{12}^2U_{20}\}$$
$$+ \{2(U_{10}U_{11}U_{12})U_{22} + (U_{10}U_{12}^2)U_{21} + (U_{11}U_{12}^2)U_{20}\}$$
$$+ \{2(U_{11}^2U_{12})U_{22} + 2(U_{11}U_{12}^2)U_{21}\}$$
$$+ \{U_{10}^3U_{22} + 3U_{10}^2U_{12}U_{20}\}$$
$$+ \{(U_{10}^2U_{12})U_{21} + (U_{10}^2U_{11})U_{22} + 2(U_{10}U_{11}U_{12})U_{20}\}$$
$$+ \{(U_{10}U_{11}^2)U_{22} + (U_{11}^2U_{12})U_{20} + 2(U_{10}U_{11}U_{12})U_{21}\}$$
$$+ \{U_{11}^3U_{22} + 3(U_{11}^2U_{12})U_{21}\} + \{4U_{10}^3U_{20}\}$$
$$+ \{(U_{10}^3)U_{21} + 3(U_{10}^2U_{11})U_{20}\}$$
$$+ \{2(U_{10}^2U_{11})U_{21} + 2(U_{10}U_{11}^2)U_{20}\}$$
$$+ \{3(U_{10}U_{11}^2)U_{21} + (U_{11}^3)U_{20}\} + \{4U_{11}^3U_{21}\}]\,. \quad (114)$$

Using the explicit expression of eqn (114), we can obtain integrals of eqn (107) by allowing the following two sets of substitutions: first set [to evaluate first integral of eqn (107)],

$$U_{10}=8,\; U_{11}=5,\; U_{12}=10,\; U_{20}=7,$$
$$U_{21}=5,\; U_{22}=10\,. \quad (115)$$

Second set [to evaluate second integral of eqn (107)]:

$$U_{10}=10,\; U_{11}=10,\; U_{12}=8,\; U_{20}=5,$$
$$U_{21}=10,\; U_{22}=7\,. \quad (116)$$

Using eqns (107),(114) and (115), we obtain:

$$I_p^3(2,1,0) = -\frac{40}{3}\left[\iint_{\partial_2(u_1,u_2)} x_1^3\left(u_1,u_2,1-u_1-u_2\right)\right.$$
$$\times\, x_2\left(u_1,u_2,1-u_1-u_2\right)\mathrm{d}u_2\mathrm{d}u_1$$
$$\left. - \iint_{\partial_2(u_1,u_2)} x_1^3\left(0,u_2,u_3\right)x_2\left(0,u_2,u_3\right)\mathrm{d}u_3\mathrm{d}u_2\right]$$

$$= -\frac{40}{3}\,\frac{\lfloor\underline{3}}{\lfloor\underline{6}}\left[\left\{(40\,000) + (20\,000)\right.\right.$$
$$+ (31\,000) + (24\,000) + (15\,500)$$
$$+ (10\,000) + (18\,560) + (12\,000) + (7750)$$
$$+ (5000) + (14\,336) + (9280) + (6000)$$
$$\left. + (3875) + (2500)\right\} - \left\{(14\,336) + (18\,560)\right.$$
$$+ (16\,000) + (17\,600) + (20\,800)$$
$$+ (24\,000) + (19\,000) + (23\,000)$$
$$+ (27\,000) + (31\,000) + (20\,000)$$

$$+ (25\,000) + (30\,000) + (35\,000 + (40\,000)\Big\}\Big]$$

$$= -\frac{1}{9}[219\,801 - 361\,296] = \frac{47\,165}{3}. \quad (117)$$

The result of eqn (117) is again in agreement with that in eqn (100). Both these results are in total confirmity with the previous work of Bernardini [10] and the work of the authors [9]. Clearly the present computational scheme is more efficient than the previous work of Bernardini [10].

7. CONCLUSION

The theorems we have presented in this paper are interesting for various reasons; they provide us with a powerful method to compute the integrals of n-variate polynomials over linear polyhedra in n-dimensional space $\mathbb{R}^n$. We have presented two algorithms that permit us to achieve the exact computation of the integral

$$\int_P x_1^{\lambda_1} x_2^{\lambda_2},\ldots,x_n^{\lambda_n} \mathrm{d}x_1 \mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

where P is a regular n-polyhedron (an n-dimensional polyhedron), eventually non-convex, unconnected and non-manifold, embedded in the n-dimensional space. The first algorithm is well suited to a decompositive representation, the second works well with a boundary representation, where the boundary faces are known or the effort of extracting them is easy. We have developed a new technique to expand the spatial expression

$$x_1^{\lambda_1} x_2^{\lambda_2},\ldots,x_n^{\lambda_n}$$

in terms of the natural coordinates of the transformation. This has clearly demonstrated the use of Taylor series expansion, the generalized form of Leibnitz's theorem on differentiation, multinomial theorem and Leibnitz's rule on differentation of integrals. The first algorithm uses direct mapping to transform an n-polyhedron in $\mathbb{R}^n$ into a standard n-simplex in $\mathbb{R}^n$.

The second algorithm computes the n-dimensional integral

$$\int_P x_1^{\lambda_1} x_2^{\lambda_2},\ldots,x_n^{\lambda_n} \mathrm{d}x_1 \mathrm{d}x_2,\ldots,\mathrm{d}x_n$$

as a sum of $n+1$ integrals of dimension $n-1$ in $\mathbb{R}^{n-1}$. These derivations are followed by a numerical example which, although worked out earlier by the authors, has now been illustrated again with a slightly modified algorithm which we believe is as efficient and accurate as our previous algorithm [9].

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