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Effect of Buoyancy on the Free Surface Flow Past a Permeable Bed

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Abstract. Laminar steady free surface flow having one permeable bounding wall is investigated in the presence of buoyancy force. The experimental results of Rajasekhara [1] were found to be in good agreement with our theoretical results based on a model which admits slip-velocity at the porous material. The effect of buoyancy force is to increase the velocity distribution in the case of greater heat addition ($N_0 > 0$) and to decrease it by a greater cooling ($N_0 < 0$). As a result, the mass flow rate increases and the friction factor decreases for $N_0 > 0$ and the opposite is true for $N_0 < 0$. We further find that the effect of buoyancy force on the temperature distribution is to increase its magnitude. In particular, we find that the rate of heat transfer at its nominal surface is increased in the case of heating ($N_0 > 0$) of flow.

Auftrieb und Wärmeübertragung an laminar parallel angeströmten Oberflächen poröser Körper

Zusammenfassung. Die laminare Strömung entlang poröser Grenzflächen wird in Anwesenheit von Auftriebskräften theoretisch untersucht. Die Übereinstimmung zwischen Theorie und Experimenten von Rajasekhara [1] ist dann gut, wenn Strömungsgleitung an der porösen Oberfläche vorausgesetzt wird. Die Auftriebskräfte erhöhen die Geschwindigkeitsverteilung bei Wärmezufuhr ($N_0 > 0$) und verringern sie bei Kühlung ($N_0 < 0$). Im ersten Fall erhöht sich der Massenfluß bei abnehmenden Widerstandsbeiwert ($N_0 > 0$). Umgekehrte Verhältnisse liegen für $N_0 < 0$ vor. Insbesondere stellt sich heraus, daß der Wärmeübergang mit steigender Erwärmung der Strömung zunimmt.

Symbols

u	= the velocity
T	= the entrance temperature
T'	= the temperature of fluid
Q	= the Darcy velocity
ρ	= the density of fluid
μ	= the viscosity of fluid
N_0	= Buoyancy parameter
k	= the permeability of porous media
σ	= $h/\sqrt{k}$
α	= the slip parameter
η	= y/h
x, y	= cartesian coordinates
h	= the depth of flow above the bed
Re	= $2\bar{u}h/\nu$ the Reynolds number

1 Introduction

A common technique in chemical industry for obtaining extended solid-fluid interfacial areas or good fluid mixing is to pass the fluid through and past a bed of solid particles. Such systems as pebble type of heat exchangers, packed filters, packed absorption and distillation towers depend on this technique. The design of these units is based upon the mechanisms of heat and mass transfer, fluid flow, and pressure drop of a fluid perculating through a porous bed. In these cases the heat transfer is very sensitive to the boundary layer that develops just at the nominal surface [2]. The available literature [3] shows that much attention has not been given to the theoretical development of the boundary layer that exists either at the nominal surface or at the rigid surfaces bounding a porous bed. The purpose for this study is to develop, using the no-slip boundary condition of Beavers and Joseph [2] (here after called BJ) a theoretical model for obtaining the rate of heat transfer between the fluid and the nominal surface. The required boundary layer equation for flow is discussed in section 2 below.

Recently Rajasekhara [1] has investigated, both theoretically and experimentally, the flow past a porous bed with free upper surface neglecting the buoyancy effect. The work of Rajasekhara [1] is connected only with the measurements of flow (see section 3 below) and not on the determination of rate of heat transfer between the fluid and the nominal surface. He found a deviation between his experimental and theoretical results in the mass flow rate. This deviation may be due to assuming the free surface, in the analytical evaluation of mass flow rate, as horizontal and neglecting the effect of buoyancy. Sparrow et. al [4] and Gill and Casol [5] have shown, in the absence of porous bed, that even in the case of parallel flows the buoyancy force significantly affects the flow field. To demonstrate whether this is true also in the case of porous beds we consider in this paper the effect of buoyancy force. Further, since we are dealing with free surface flow, the fluid has to be treated as having an unknown upper boundary as free surface. This free surface makes the problem much more intractable and progress is often be made under certain approximations. In this paper we try to find exact solutions under the approximation that the slope of the free surface is everywhere small compared with unity. We find that our theoretical results are in good agreement with the experimental results of Rajasekhara [1] and thus valididates the assumptions made above.

2 Mathematical Formulation of the Problem

A physical model is shown in Fig. 1 which consists of an infinitely long channel one of whose bounding walls is a naturally permeable bed while the other boundary is a free surface $y = h(x)$ which varies smoothly. For concreteness, the flow regime is divided into two zones. The region above the porous bed is called "zone 1" and is governed by the Navier-Stokes equation. Below the nominal surface, in the permeable bed, the region is called "zone 2" and the flow is governed by the Darcy law which is the statistical average of the modified Navier-stokes equation. The axial and transverse coordinates are, respectively x and y, the latter being measured vertically upwards from the porous bed. The free surface is represented by the curve $y = h(x)$ where $h(x)$ is continuous and positive for all x.

2.1 Basic Equations

To obtain the basic equations, we make the following approximations:

(i) The flow in the zones is steady and is driven by a common pressure gradient $\frac{\partial p}{\partial x}$ and the buoyancy force $\frac{\partial T'}{\partial x}$.

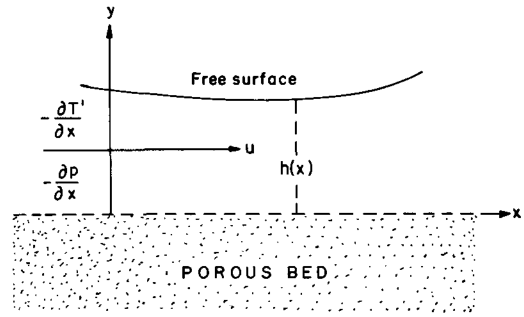

Fig. 1. Physical model

(ii) The fluid is viscous and satisfies the Boussinesq approximation which means that fluctuations in density occur principally as a result of thermal rather than pressure variations which is valid in the case of fluid considered in this paper.

(iii) Since we consider creeping flow, the inertia terms are negligible.

(iv) The porous medium is homogeneous and isotropic so that its permeability k is assumed to be constant. We also assume that the viscosity of the fluid is constant although density varies with temperature. This is because we consider Boussinesq fluids.

Under these approximations, the basic equations of flow are:

For Zone 1:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$\nabla^2 u - \frac{1}{\mu}\frac{\partial p}{\partial x} = 0 \tag{2.2}$$

$$\nabla^2 v - \frac{1}{\mu}\frac{\partial p}{\partial y} - \frac{\rho}{\mu} g = 0 \tag{2.3}$$

$$\rho = \rho_0[1 - \beta(T' - T)] \tag{2.4}$$

$$\rho_0 C_p\left[u\frac{\partial T'}{\partial x} + v\frac{\partial T'}{\partial y}\right] = K\nabla^2 T' + \Phi \tag{2.5}$$

where u and v are the x and y components of velocity, T is the entrance temperature i.e. at $x = 0$, T' the temperature of fluid, ρ the density, p the pressure, μ the viscosity, K the thermal conductivity, C_p the specific heat at constant temperature, β the thermal expansion coefficient, ρ_0 the density at

$T' = T$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and

$$\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right].$$

For Zone 2:

$$Q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{2.6}$$

$$Q_y = -\frac{k}{\mu}\left(\frac{\partial p}{\partial y} + \rho g\right) \tag{2.7}$$

where Q_x and Q_y are the x and y components of Darcy velocity. They are the components of mean filter velocity rather than the components of true velocity.

It is of interest to know that the velocity components given by (2.6) and (2.7) are irrational under homogeneous conditions and hence valid only in the potential flow region away from the nominal surface. However, very near to it there should be a thin boundary layer for the existence of the slip velocity at the nominal surface as postulated by BJ. In this region, the velocity should be of the form

$$\vec{q} = -\nabla\Phi + \nabla x \vec{\Psi}$$

where the first term on the right hand side represents the usual Darcy term and the second term represents the rotational velocity. The expression for this rotational part can be obtained by considering the effect of drag due to solid material of the porous media on the flow (see Tam [6], Lundgren [7]). In that case the momentum equations for the flow through porous media are

$$\nabla^2 u - \frac{k}{\mu}u + \frac{1}{\mu}\frac{\partial p}{\partial x} = 0 \tag{2.8}$$

$$\nabla^2 v - \frac{k}{\mu}v + \frac{1}{\mu}\frac{\partial p}{\partial y} - \frac{\rho}{\mu}g = 0 \tag{2.9}$$

and the energy equation is

$$\rho_0 C_p\left(u\frac{\partial T'}{\partial x} + v\frac{\partial T'}{\partial y}\right) = K\nabla^2 T'. \tag{2.10}$$

Where the viscous dissipation terms are neglected.

As stated in the introduction we consider here the solutions of the above equations under the approximation that the slope of the free surface is everywhere negligible. This means that if $h(x)$ represents the free surface then $h'(x)$ is everywhere small compared with unity.

In this approximation, Eqs.(2.1) to (2.10) reduce, at each value of x, approximately to the Eqs.:

Zone 1:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{2.11}$$

$$\frac{\partial p}{\partial y} = -\rho g \tag{2.12}$$

$$\rho = \rho_0[1 - \beta(T' - T)] \tag{2.13}$$

$$\rho_0 C_p u\frac{\partial T'}{\partial x} = K\frac{\partial^2 T'}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{2.14}$$

From Eqs.(2.11), using Eqs.(2.12) to (2.13), we get

$$\frac{d^3 u}{dy^3} = \frac{\beta g}{\gamma}\frac{\partial T'}{\partial x} \tag{2.15}$$

where $\gamma = \frac{\mu}{\rho_0}$ is the kinematic viscosity. This Equation describes the fully developed flow only when the left hand-side is independent of x. To satisfy this condition, we assume that the temperature varies linearly in the direction of flow. Physically this means that the heat flux is constant in the direction of flow. Mathematically, this can be expressed as

$$T'(x,y) = Ax + T(y) \tag{2.16}$$

where A is the axial constant temperature gradient with $A < 0$ for favourable thermal gradient and $A > 0$ for adverse temperature gradient. Eqs.(2.14) and (2.15), using (2.16), give the momentum and energy Equations respectively in the form

$$\frac{d^3 u}{dy^3} = \frac{A\beta g}{\gamma} = G \tag{2.17}$$

$$\rho_0 C_p A u = K\frac{d^2 T'}{dy^2} + \left(\frac{du}{dy}\right)^2 \tag{2.18}$$

Eq.(2.17) may be written in the form

$$\frac{d^2\zeta}{dy^2} = -G$$

where $\zeta = -\frac{du}{dy}$ = vorticity vector in the direction of z-axis. This shows that the potential energy balances with the dissipation of vorticity. G may be positive, zero or negative depending on the entrance temperature of the fluid. If $T' - T < 0$, then the temperature of the fluid (for $x > 0$) in the channel is less than the entrance temperature and the free convection current flows into the channel. Hence $G < 0$, corresponds to the external heating of flow. Similarly, $G > 0$, corresponds to the absence of free convection currents.

For Zone 2:

The basic Equations away from the nominal surface are

$$Q = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{2.19}$$

$$\frac{\partial p}{\partial y} + \rho g = 0 \ . \tag{2.20}$$

The momentum Equation very near to the nominal surface now becomes

$$\frac{\partial^2 u}{\partial y^2} - \frac{1}{k}u = \frac{1}{\mu}\frac{\partial p}{\partial x} \ . \tag{2.21}$$

This Equation is useful to determine the boundary layer thickness postulated by BJ in their slip boundary condition at the nominal surface. Eq.(2.11) using Eqs.(2.20), (2.13) and (2.16), takes the form

$$\frac{d^3u}{dy^3} - \frac{1}{k}\frac{du}{dy} = G \tag{2.22}$$

or

$$\frac{d^2\zeta}{dy^2} - \frac{\zeta}{k} = -G \ .$$

Although this Equation is independent of temperature, to know the variation of density, we should determine the temperature distribution by solving the energy equation. For unidirectional steady flow, Eq.(2.18) takes the form

$$\frac{d^2T'}{dy^2} = \frac{\rho_0 C_p A u}{K} \tag{2.23}$$

where $\rho_0 C_p$ is the heat capacity per unit volume of the fluid.

2.2 The Boundary Conditions

For Zone 1:

Eqs.(2.17), (2.18), (2.22) and (2.23) have to be solved using proper boundary conditions. Until recently, it was generally assumed that the conventional no-slip velocity boundary condition is valid at the porous walls which leads to the parabolic velocity profile in the channel. However, recent experiments of BJ and Beavers et al. [8] involving laminar flow of water or oil in flat rectangular ducts having one porous wall demonstrated the existence of a streamwise slip-velocity at the permeable bounding surface and proposed that the boundary conditions at the nominal surface of the porous medium can be expressed as

$$\frac{du}{dy} = \frac{\alpha(u_B - Q)}{\sqrt{k}} \tag{2.24}$$

where α is a dimensionless constant which depends on the structure of the porous material and not on the geometry of flow (see Taylor [9]) and u_B is the slip-velocity. The existence of this slip-velocity is connected with the presence of a very thin layer of streamwise moving fluid just beneath the surface of the porous material. The fluid in this layer is pulled along by the flow in the channel. The effect of this velocity slip is to cause a skewing of the main flow velocity profile in the channel. When $k \to 0$ (solid wall), Eq.(2.24) tends to the no-slip boundary condition $u = 0$.

The second boundary condition to be valid at the free surface, in the absence of deformation, is that no tangential stress acts at the free surface. This leads to

$$u = u_s \quad \text{at} \quad y = h \ . \tag{2.25}$$

For the third boundary condition, we impose that the pressure gradients in the channel and in the porous bed must have the same value at the interface i.e.

$$\frac{d^2u}{dy^2} = -\frac{Q}{k} \quad \text{at} \quad y = 0 \ . \tag{2.26}$$

The boundary condition on temperature is

$$T' = T_1 \quad \text{at} \quad y = h \tag{2.27}$$

which means that the free surface is maintained at a constant temperature T_1. The other boundary condition on temperature is obtained from the basic physical consideration of heat balance for an element at the nominal surface. The heat conducted away from channel through the nominal surface must be equal to the heat absorbed from the porous medium and hence

$$K \frac{\partial T'}{\partial y} = h_e(T_B - T_0)$$

or

$$\frac{\partial T'}{\partial y} = \frac{H(T_B - T_0)}{\sqrt{k}} \tag{2.28}$$

where $H = \frac{h_e \sqrt{k}}{K}$ is the Biot number, h_e is the heat transfer coefficient from the porous medium into the channel, T_B is the temperature at the interface and T_0 is the ambient temperature of the porous medium. Physically, H represents the rate of heat loss through the channel relative to the conductors in the porous media. If H is large, then the interface must be a nominal surface in order to supply the heat lost from the porous media. If H is small, then the heat losses are small and the interface is really the free boundary. In other words, H is the controlling parameter because of its relation to the over all thermal balance. Since the boundary condition (2.24) is based on the assumption of the nominal boundary (see BJ), H has to be large in our analysis.

For Zone 2:

The corresponding boundary conditions are:

$$u = u_B \quad \text{at} \quad y = 0 \tag{2.29}$$

$$u = Q \quad \text{at} \quad y = -\delta \tag{2.30}$$

$$\frac{d^2u}{dy^2} = \frac{-Q}{k} \quad \text{at} \quad y = 0 \tag{2.31}$$

$$T' = T_B \quad \text{at} \quad y = 0 \tag{2.32}$$

$$T' = T_0 \quad \text{at} \quad y = -\delta \tag{2.33}$$

where δ is the boundary layer thickness just below the bed and we assume that this δ is the same for both velocity and temperature distributions. Since we are dealing with the velocity of flow much less than the sonic velocity, the error introduced by this approximation is not much. The boundary condition (2.29) will enable us to match the velocity distribution at the interface.

3 Velocity Distribution and Mass Flow Rate

In this section, we determine the velocity and the mass flow rate for zones 1 and 2, and the value of the boundary layer thickness δ.

3.1 Velocity Distribution

The velocity distribution in zone 1, solving the momentum Eq. (2.17) subject to the boundary conditions (2.24) to (2.26) is

$$\frac{u}{\bar{u}} =$$

$$= \frac{4[3\{(1+f_0)(1+\alpha\sigma)-\alpha\sigma(1-\eta)^2\}+(3+3\alpha\sigma\eta-\alpha\sigma\eta^3)N_0]}{[4\{1+(2+3f_0)(1+\alpha\sigma)\}+(12+5\alpha\sigma)N_0]}$$

$$= \frac{6(\eta+f_0)(1-\eta)}{1+3f_0} + N_0F_1(\eta) + U_0F_2(\eta) \tag{3.1}$$

where

$$F_1(\eta) = \frac{2(1-\eta)}{g_0(1+3f_0)}\left\{\left(\frac{1}{1+\alpha\sigma}+\eta+\eta^2\right)(1+3f_0) - - 3f_1(\eta+f_0)\right\},$$

$$F_2(\eta) = 6\left[\frac{1+\alpha\sigma\eta}{(1+\alpha\sigma)(g_0+U_0f_2)} - (1-\eta)g_1(\eta)\right],$$

$$g_0 = 1 + 3f_0 + N_0f_1,$$

$$g_1(\eta) = \frac{\left[\frac{No}{3}\left(\frac{1}{1+\alpha\sigma}+\eta+\eta^2\right)(1+3f_0)+g_0(\eta+f_0)\right]f_2}{g_0(1+3f_0)(g_0+U_0f_2)},$$

$$U_0 = \frac{3+2\alpha\sigma}{3\alpha\sigma}N_0 + \frac{2(\alpha+\sigma)+\alpha\sigma^2}{\alpha\sigma^2} = \frac{2u_s}{Q\sigma^2},$$

$$f_0 = \frac{\sigma + 2\alpha}{\sigma(1+\alpha\sigma)}, \quad f_1 = \frac{3+\alpha\sigma}{2(1+\alpha\sigma)}, \quad f_2 = \frac{3(2+\alpha\sigma)}{1+\alpha\sigma},$$

$$\sigma = \frac{h}{\sqrt{k}}, \quad \eta = \frac{y}{h},$$

$$N_0 = -\frac{Gh^3}{Q\sigma^2} = \frac{\mu G h}{\frac{\delta p}{\delta x}}$$

$\bar{u}$ = Average velocity in the channel.

$$= \frac{1}{h}\int_0^h u\,dy = -\frac{h^2}{12\mu}\frac{\delta p}{\delta x}(1 + 3f_0 + N_0 f_1 + U_0 f_2). \tag{3.2}$$

The right-hand side of (3.2) is exactly the same as Rajasekhara [1] when $N_0 = 0$. The slip-velocity u_B is given by

$$\frac{u_B}{\bar{u}} = \frac{12[\{(1+f_0)(1+\alpha\sigma) - \alpha\sigma\} + N_0]}{[4\{1 + (2+3f_0)(1+\alpha\sigma)\} + (12+5\alpha\sigma)N_0]} \tag{3.3}$$

The slip-velocity ratio (3.3) varies with the channel height h, increasing as h decreases, i.e. increasing with decreasing values of σ. It is of interest to note that when N_0 is negative, the stagnation points occur in the channel for various values of N_0 as shown in Table 1.

No stagnation point occurs for positive values of N_0 (i.e. heating of nominal surface with favourable temperature gradient).

The velocity distribution in zone 2, solving Eq. (2.22) using the boundary conditions (2.29) to (2.31) is

$$u = (u_B + Q) - Q\,\mathrm{Cosh}\,\eta\sigma + \frac{\mathrm{Sinh}\,\eta\sigma}{\mathrm{Sinh}\,\delta^*\sigma}[u_B - Q\,\mathrm{Cosh}\,\delta^*\sigma - Q N_0 \delta^*] + QN_0\eta \tag{3.4}$$

where $\delta^* = \frac{\delta}{h}$ is the relative boundary-layer thickness. We know that at the edge of the boundary layer, the shear has to be zero i.e. at $y = -\delta$, the velocity tends to the free stream velocity Q. Using this and condition (2.24), we obtain the equation for δ in the form

$$u_B - Q\,\mathrm{Cosh}\,\delta^*\sigma - Q N_0\delta^* - \left[\alpha(u_B - Q) - \frac{QN_0}{\sigma}\right]\mathrm{Sinh}\,\delta^*\sigma = 0. \tag{3.5}$$

Table 1. Stagnation points, $\frac{u}{\bar{u}} = 0$ for values of N_0

σ	5.0		10.0		η
α	0.1	0.01	0.1	0.01	
N_0	-2.04	-0.58	-2.02	-2.002	0.0
	-2.02	-0.57	-1.99	-1.998	0.2
	-2.03	-0.57	-1.93	-1.991	0.4
	-1.95	-0.55	-1.87	-1.981	0.6
	-1.92	-0.55	-1.83	-1.974	0.8
	-1.91	-0.53	-1.81	-1.971	1.0

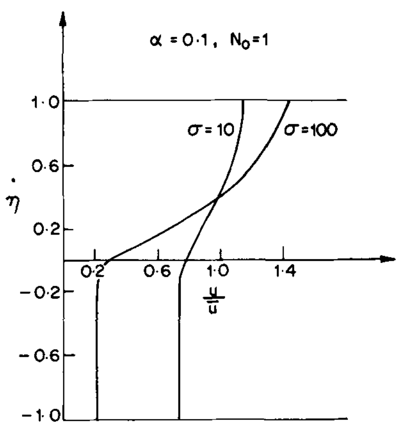

Fig. 2. Velocity distribution

This Equation for δ is a transcendental equation and is difficult to obtain an analytic solution. Since the boundary layer thickness δ is very small compared to the width h of flow, neglecting squares and higher powers of δ^*, we obtain

$$\delta^* = \frac{\delta}{h} = \frac{1}{\alpha\sigma}. \tag{3.6}$$

From this, it is clear that as $\sigma \to \infty$ (i.e. solid wall), $\delta \to 0$ as required. The expression for velocity distribution for flow through porous bed, from Eq. (3.4) using (3.6) and (3.2) is

$$\frac{u}{\bar{u}} = \frac{u_B}{\bar{u}} + \frac{12(1 - \mathrm{Cosh}\,\eta\sigma - \mathrm{Sinh}\,\eta\sigma + N_0\eta)}{\sigma_0^2 g_0} - \frac{12u_0 f_2(1 - \mathrm{Cosh}\,\eta\sigma - \mathrm{Sinh}\,\eta\sigma + N_0\eta)}{\sigma^2 g_0(g_0 + u_0 f_2)}. \tag{3.7}$$

To find qualitatively, the effect of buoyancy force on the flow, Eqs. (3.1) and (3.7) are numerically evaluated for different values of α and σ, and is shown in Fig. 2.

3.2 Mass Flow Rate

To find the quantitative effect of slip-velocity and buoyancy force, it is useful to calculate the mass flow rate in the channel with and without the porous bed. If M_p denotes the mass flow rate per unit channel width with porous bed, then

$$M_p = \int_0^h \rho u \, dy = \rho \bar{u} h$$

$$= -\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\left(1 + 3f_0 + N_0 f_1 + U_0 f_2\right) . \qquad (3.8)$$

On the other hand, for a channel bounded by impermeable walls, the mass flow rate M_i, is given by

$$M_i = -\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\left(1 + \frac{N_0}{2} + 3\,U_0\right) . \qquad (3.9)$$

Therefore, we have

$$\frac{M_p}{M_i} = \frac{2(1 + 3f_0 + N_0 f_1 + U_0 f_2)}{2 + N_0 + 6U_0} . \qquad (3.10)$$

When $N_0 = 0$, then

$$\frac{M_p}{M_i} = 1 + \frac{3(\alpha + \sigma)}{\alpha\sigma^2}$$

which is the result of Rajasekhara. From Eq.(3.10), we observe that the mass flow rate can be stopped for values of N_0 given by

$$N_0 = -\frac{1 + 3f_0 + U_0 f_2}{f_1} . \qquad (3.11)$$

For example, when $\sigma = 10$, $\alpha = 0.1$, Eq.(3.11) gives $N_0 = -1.90$.

The fractional increase in mass flow rate through the channel with a permeable lower wall over what it would be if the wall were impermeable is

$$\Phi = \frac{3(\sigma + 2\alpha)}{\sigma(1+\alpha\sigma)} - \frac{(\sigma + 6\alpha)N_0 + 12(\sigma + 3\alpha)U_0}{\sigma(1+\alpha\sigma)(2+N_0+6U_0)} .$$

From this it follows that Φ takes the value

$$\frac{2(3 + N_0 + 3U_0)}{2 + N_0 + 6U_0} \quad \text{when} \quad \sigma = \sqrt{\frac{2}{1 + \frac{N_0}{3} + U_0}}$$

independent of α and this occurs when $u_B = Q$ i.e. when the slip-velocity balances with Darcy velocity within the permeable material, and hence the velocity profile in the channel has a zero gradient at the permeable wall. In this particular case, there is no boundary-layer thickness just beneath the permeable bed. In most applications, h will be considerably greater than $\sqrt{\frac{2k}{1 + \frac{N_0}{3} + U_0}}$. It is possible therefore, that for values of σ greater than $\sqrt{\frac{2}{1 + \frac{N_0}{3} + U_0}}$, the average size of the individual pores within the material is at least equal to the height of the channel, and the assumption of rectilinear flow in the channel breaks down.

The ratio $\frac{M_p}{M_i}$ gives the effect of porous bed. However, to find the effect of buoyancy on the flow, we define the ratio

$$\frac{M_p}{M_{p_0}} = 1 + \frac{f_1 N_0}{1 + 3f_0 + U_0 f_2} .$$

Where

$$M_{p_0} = -\frac{\rho h^3}{12\mu}\frac{\partial p}{\partial x}\left(1 + 3f_0 + U_0 f_2\right)$$

is the mass flow rate, in the absence of buoyancy force (i.e. $N_0 = 0$). This shows that the presence of buoyancy, increases the mass flow rate for positive values of N_0 and tends to a constant value for large values of σ and decreases for negative values of N_0. In particular, this will be zero (i.e. mass flow rate can be stopped) for the values of N_0 given by Eq. (3.11). This value of N_0 is different for different values of α and σ, as shown in Table 2.

Table 2. $\frac{M_p}{M_{p_0}} = 0$, for different values of N_0

σ	5.0		10.0	
α	0.1	0.01	0.1	0.01
N_0	-1.891	-1.99	-1.90	-1.98

3.3 Friction Factor

The above theory is applicable only for laminar flow. Therefore, it is of interest to find the Critical Reynolds number at which transition from laminar to turbulent flow occurs. To identify the breakdown of the laminar region for a fixed slip-velocity ratio characterised by a fixed value of σ, we use the friction factor C_f defined by

$$C_f = \frac{\left(-\frac{\partial p}{\partial x}\right) D}{\frac{1}{2}\rho \bar{u}^2} \tag{3.12}$$

where D = 2h. This, using (3.2), becomes

$$(C_f Re)_p = \frac{96}{1 + 3f_0 + N_0 f_1 + U_0 f_2} \tag{3.13}$$

where $Re = \frac{\bar{u}D}{\gamma}$ is the Reynolds number for the flow. This shows that $(C_f Re)_p$ decreases for positive values of N_0 and increases for negative values of N_0 and becomes infinite for the value of N_0 given by Eq.(3.11). Also $(C_f Re)_p$ is constant (independent of Reynolds number) for a channel of fixed height, fixed N_0 for a given porous bounding wall. However, the friction factor for a solid walled channel is given by

$$(C_f Re)_i = \frac{96}{1 + \frac{N_0}{2} + 3U_0} . \tag{3.14}$$

Thus

$$\frac{(C_f Re)_p}{(C_f Re)_i} = \frac{1 + \frac{N_0}{2} + 3U_0}{1 + 3f_0 + N_0 f_1 + U_0 f_2} . \tag{3.15}$$

If $N_0 = 0$, then

$$\frac{(C_f Re)_p}{(C_f Re)_i} = \frac{1}{1 + \frac{3(\alpha + \sigma)}{\alpha\sigma^2}}$$

which is the result of Rajasekhara [1].

Form Eq.(3.15), we observe that this ratio increases with an increase in α and σ. It is of interest to note that although this ratio is independent of the Reynolds number, it depends on the nature of the buoyancy parameter N_0. The ratio given by Eq.(3.13) becomes zero for the values of

$$N_0 = -(1 + 3U_0)2$$

and infinite for the values of N_0 given by Eq.(3.11). In other words, the minimum or maximum value of Eq.(3.13) occurs only for negative values of N_0 but it becomes uniform for positive values of N_0.

3.4 Mass flow rate curve: Comparison of theory and experiment

Rajasekhara [1] has evaluated experimentelly the effect of the slip at the nominal surface on the free flow past a porous bed. His experiments consists of a rectangular duct bounded on the two sides and below by the rigid plates and is free above. This model is exactly the same as the one used by Rajasekhara [10] in the investigation of plane Couette flow past a porous bed except that the upper moving plate is replaced by a free surface. The porous medium employed in his experiments consists of a natural sand which passes through 14 but retained on 18 B.S. sieves for which the calculation of permeability is $k = 1.48 \times 10^{-7}\,cm^2$ with scatter of ± 5 percent. He has measured only the mass flow rate and not the heat flow rate and found that the effect of the slip at the porous bed is to increase the mass flow rate and to decrease the friction factor.

In particular he has shown, in the absence of buoyancy force, that the ratio M_p/M_i is independent of the Reynolds number and found, by introducing experimentally determined values of M_p/M_i, the value of α to be 0,01. When the experimental results are drawn on the theoretical values of M_p/M_i Rajasekhara [1] has found some deviations between them. This deviation may be due to the neglection of buoyancy force on the flow. To bridge the gap between theoretical experimental results, we have calculated, in section 3, the mass flow rate with the buoyancy force. The M_p/M_i ratio given by Eq.(3.10) in the presence of buoyancy force is numerically evaluated for different values of α, σ and N_0, and are compared with the experimental values of Rajasekhara [1] in Fig.3, for $\alpha = 0.01$. From this it is clear that the theoretical results are in excellent agreement with the experimental results for values of $N_0 = \pm 0.5$ and we find that $\frac{M_p}{M_i}$ increases with increasing N_0.

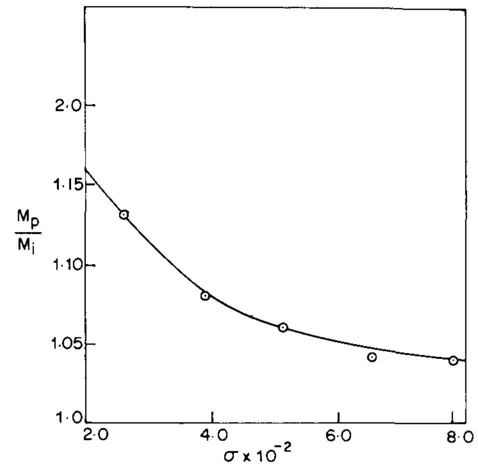

Fig.3. Mass flow rate comparison with experimental results

The crux of the present analysis is the assumption of the boundary condition (2.26). It is satisfying to note that this assumption gets valididated in view of the close agreement between the theoretical and experimental results.

4 Temperature Distribution

Although the momentum Eq.(2.17) is independent of temperature, for completeness, we present the temperature distribution for zones 1 and 2.

The temperature distribution for zone 1, neglecting viscous dissipation terms, solving (2.18), using the boundary conditions (2.27) and (2.28) is

$$\Theta'(\xi, \eta) = a\xi + \Theta(\eta) \tag{4.1}$$

where

$\Theta = \dfrac{T' - T_0}{T_1 - T_0}$, $a = \dfrac{AL}{T_1 - T_0}$, L is the length of the channel.

$\xi = \dfrac{x}{L}$

$$\Theta(\eta) = \frac{1 + \sigma H \eta}{1 + \sigma H} + \frac{Pe}{24}(a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 - \eta^4) + \frac{PeU_0}{6}(b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3) + \frac{N_0 Pe}{360}(c_0 + c_1\eta + c_2\eta^2 + c_3\eta^3 - 3\eta^5) \tag{4.2}$$

$Pe = \dfrac{hU}{k}$ is the Peclet number,

$a_0 = -\dfrac{1 + 4f_0}{1 + \sigma H}$, $b_0 = -\dfrac{f_1}{1 + H\sigma}$, $c_0 = -\dfrac{27 + 7\alpha\sigma}{(1 + \alpha\sigma)(1 + \sigma H)}$

$a_1 = \sigma H a_0$, $b_1 = \sigma H b_0$, $c_1 = \sigma H c_0$

$a_2 = 6 f_0$, $b_2 = \dfrac{3}{1 + \alpha\sigma}$, $c_2 = \dfrac{30}{1 + \alpha\sigma}$

$a_3 = 2(1 - f_0)$, $b_3 = \frac{1}{3}\alpha\sigma b_2$, $c_3 = \frac{1}{3}\alpha\sigma c_2$.

The first term in Eq.(4.2) represents the heat transport due to diffusion and the remaining terms represent the transport of heat due to convection. When $Pe = 0$ i.e. in the absence of convection, Eq. (4.2) becomes

$$\Theta(\eta) = \frac{1 + \sigma H \eta}{1 + \sigma H} . \tag{4.3}$$

This shows that as $H \to \infty$ i.e. for a perfeçtly conducting permeable interface,

$\Theta(\eta) = \eta$.

However, when $H = 0$, i.e. for insulating permeable interface

$\Theta(\eta) = 1$.

The temperature distribution for zone 2, solving Eq.(2.23), using the boundary conditions (2.32) and (2.33) is

$$\Theta(\eta) = (1 + \alpha\sigma\eta)\Theta_B + \frac{Pe}{\sigma^2} \times \left[\frac{1 - \cosh\sigma\eta - \sinh\sigma\eta}{\sigma^2} + a_1'\eta + a_2'\eta^2\right] + \frac{N_0 Pe}{\sigma^2}[b_1'\eta + b_2'\eta^2 + \frac{1}{6}\eta^3] + \frac{U_0 Pe}{\sigma^2}(c_1'\eta + c_2'\eta^2) \tag{4.5}$$

where

$\Theta_B = \dfrac{1}{1 + \sigma H} + \dfrac{Pe}{360}(15 a_0 + C_0 N_0 + 60 b_0 U_0)$

$a_1' = \dfrac{\alpha}{\sigma} + \dfrac{2\alpha + \sigma}{4\alpha(1 + \alpha\sigma)} + \dfrac{1}{2\alpha\sigma}$, $b_1' = \alpha\sigma^3 - 2(1 + \alpha\sigma)$,

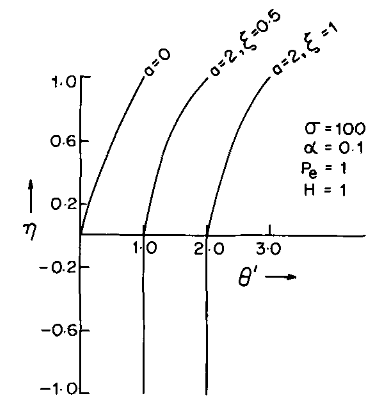

Fig.4. Temperature distribution

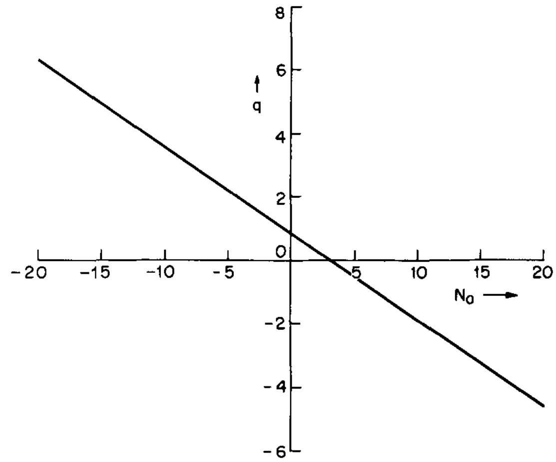

Fig.5. Rate of heat transfer

$$c_1' = \frac{\sigma}{4\alpha(1+\alpha\sigma)}$$

$$a_2' = \frac{2+4\alpha\sigma+\sigma^2}{4(1+\alpha\sigma)}, \quad b_2' = \frac{\sigma^2}{12(1+\alpha\sigma)}, \quad c_2' = \frac{\sigma^2}{4(1+\alpha\sigma)}.$$

In the absence of convection (i.e. Pe = 0), the temperature distribution is linear in η as in the case of flow in the channel and is given by (4.3). Comparing the derivative of this, with the boundary condition (2.28), we find that $\alpha = H$. This means that when heat is transported only by diffusion, the values of α will depend on the values of H. Since H depends only on the structure of the porous media, we conclude that α depends only on the structure of the porous material and not on the geometrical configuration.

Eq.(4.1) together with Eq.(4.5), represent the temperature distribution and is numerically evaluated for different values of a, ξ and N_0, for fixed α and σ, and are shown in Fig.4. For favourable temperature gradient (i.e. $\frac{\partial T}{\partial x} = A < 0$), $a > 0$, corresponds to the heating of the free surface (i.e. $T_1 - T_0 < 0$) because heat flows from the bed towards the free surface. Similarly $a < 0$, corresponds to the cooling of the surface because heat flows from the free surface towards the bed. But $a = 0$ (i.e. $\frac{\partial T}{\partial x} = 0$) corresponds to the absence of buoyancy force in which no heat is transported by convection and the heat is transported only by diffusion. From Fig.4, it is clear that when $a = 0$, there exists a thin thermal boundary layer just beneath the nominal surface with higher temperature in the free flow compared to that in the Darcy flow.

However, when $a \neq 0$, with $N_0 > 0$, heat is transported both by convection and diffusion and Fig.4 shows that the effect of convection is to increase the magnitude of the temperature in the free flow. Similar conclusions are true for $N_0 < 0$.

From the technological point of view, it is of interest to know that the rate of heat transfer q between the fluid and the nominal surface and we get

$$q = \left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0}$$

$$= \frac{\sigma H}{1+\sigma H}\left[1 - \frac{Pe}{24}(1+4f_0) - \frac{NoPe}{360}\,\frac{27+7\alpha\sigma}{1+\alpha\sigma} - \frac{UoPe}{6}f_1\right]. \tag{4.6}$$

This q is shown in Fig.5, for positive and negative values of N_0. We observe that q decreases linearly for positive values of N_0 and increases for negative values of N_0.

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