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# On Love waves in a stratified hypoelastic solid with material boundary

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**Abstract.** Transverse surface waves in a half-space covered with a stratum of a different material are investigated in the context of hypoelasticity of grade zero, taking into account the surface stress effects on the boundary. It is found that the wave motion is possible and that unlike in the corresponding problem with classical boundary, the variation of amplitude in the stratum may be either periodic or exponential. When the amplitude in the stratum is periodic, the motion is similar to that in a stratum bounded on both sides by very deep layers of different elastic or hypoelastic materials.

**Keywords.** Love waves; hypoelasticity; material boundary; wave motion; amplitude variation.

## 1. Introduction

The theory of hypoelasticity proposed and developed by Truesdell (1955a,b, 1956) has aroused much interest due to its more realistic approach in studying the elasticity theory, compared to the classical (Hookean) approach. This theory is non-linear in nature and is based on the hypothesis that the components of the stress rate are homogeneous linear functions of the components of the rate of deformation. If we restrict our attention to the neighbourhood of the stress-free state and linearize the constitutive equations of the hypoelasticity theory, we obtain the classical elasticity theory. Several problems revealing interesting phenomena which characterize the hypoelasticity theory have been considered by Noll (1955), Green (1956a,b), Thomas (1957), Erickson (1958), Verma (1956a,b 1958), Paria (1958), Grioli (1962), Noll and Truesdell (1964) and Chandrasekharaiah (1976).

In this paper, we consider one more interesting problem of hypo-elasticity. We investigate here transverse surface waves (Love waves) in a half-space covered with a stratum of uniform thickness of a different material, taking into account the surface stress effects on the boundary.* This is precisely a reinvestigation of the problem considered by Paria (1958) and is of considerable geophysical interest. We find that the wave motion of the desired type is possible and that there exist two different ranges for the phase velocity; for one of these ranges the variation of amplitude in the stratum is exponential and for the other it is periodic. This is in contrast to the result of Paria (1958) that the variation of amplitude in the stratum is necessarily periodic for the waves to exist. We also find that when the variation of amplitude

*The corresponding problem in pure elasticity has been considered by the author in a separate paper which is under publication.

in the stratum is periodic, the motion is similar to the one in an infinite body consisting of a stratum bounded on both sides by half-spaces of different material, and conclude that the effect of surface stresses on the boundary is equivalent to replacing the free-space on the other side of the stratum by an appropriate half-space.

## 2. Phase velocity equation

The equations governing the deformation of an isotropic hypo-elastic body of grade zero, in the notation of cartesian tensors, are (Truesdell 1955a)

$$\frac{\partial s_{ij}}{\partial t} + v_k s_{ij,k} - s_{ik} v_{j,k} - s_{kj} v_{i,k} + s_{ij} v_{k,k}$$

$$= \frac{\nu}{1-2\nu} \delta_{ij} v_{k,k} + \tfrac{1}{2} (v_{i,j} + v_{j,i}) \tag{1}$$

$$2\mu s_{ij,j} = \rho \left( \frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) \tag{2}$$

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{3}$$

In these equations $v_i$ is the velocity vector, $2\mu s_{ij}$ is the stress tensor with $\mu$ as the shear modulus of classical elasticity, $\nu$ is the Poisson's ratio, $\rho$ is the mass per unit volume and $t$ is time. We have assumed that the body forces are absent.

For transverse waves which propagate with wave length $2\pi/\gamma$ and phase velocity $V$ in the positive $x$-direction, with displacement $u_i$ in the positive $y$-direction, we assume

$$u_i = \delta_{i2} F(z) \exp [i\gamma (x - Vt)] \tag{4}$$

The corresponding velocity vector is given by

$$v_i = \frac{\partial u_i}{\partial t} + u_{i,j} u_j = - i\gamma V u_i. \tag{5}$$

Equation (3) is now satisfied for $\rho =$ constant. Also, eq. (1) yields

$$s_{11} = s_{13} = s_{33} = 0 \tag{6}$$

$$s_{12} = \tfrac{1}{2} i\gamma F \exp [i\gamma (x - Vt)] \tag{7}$$

$$s_{22} = -\tfrac{1}{2} [\gamma^2 F^2 - (F')^2] \exp [2i\gamma (x - Vt)] \tag{8}$$

$$s_{23} = \tfrac{1}{2} F' \exp [i\gamma (x - Vt)] \tag{9}$$

Equation (2) yields with the help of eqs. (4) to (9), the general solution for the amplitude function $F(z)$ in the form

$$\left.\begin{aligned} & F(z) = A \exp(pz) + B \exp(-pz) \\ \text{where} \quad & p = \gamma\left(1 - \frac{V^2}{S^2}\right)^{\frac{1}{2}}, \ S^2 = \mu/\rho \end{aligned}\right\} \tag{10}$$

and $A$, $B$ are arbitrary constants; $S$ represents the speed of solenoidal waves in the body.

We assume that the half-space occupies the region $z>0$ and the stratum occupies the space $-T \leqslant z \leqslant 0$. We denote these two bodies by $L_1$ and $L_2$ respectively and suffix all quantities associated with the body $L_\alpha$ with $\alpha$, $\alpha = 1, 2$. Then for waves whose intensity falls off at a large distance from the stratum, we take the general solution for $F(z)$ in $L_\alpha$ in the form (vide, eqs (10))

$$F(z) = \begin{cases} F_1(z) = B_1 \exp(-p_1 z) & \text{in } L_1 \\ F_2(z) = A_2 \exp(p_2 z) + B_2 \exp(-p_2 z) & \text{in } L_2 \end{cases} \tag{11}$$

where

$$p_\alpha = \gamma\left(1 - \frac{V^2}{S_\alpha{}^2}\right)^{\frac{1}{2}} \text{ with } p_1 > 0 \tag{12}$$

and $B_1$, $A_2$ and $B_2$ are arbitrary constants.

With the expressions for $F(z)$ given by (11), eqs (4) to (9) determine the displacements, velocities and stresses developed in the bodies $L_\alpha$, when $B_1$, $A_2$ and $B_2$ are known.

We assume that there is welded contact between the bodies along the interface $z = 0$ and that $z = -T$ is a 'material boundary' — boundary which supports surface stresses (Gurtin and Murdoch 1975; Murdoch 1976). Then the appropriate boundary conditions for the problem are:

(i) $F(z)$ and $2\,\mu s_{i3}$ are continuous on $z = 0$, and

(ii) $2\,\mu s_{i3} + \Sigma_{i\alpha,\alpha} = \rho_0 \left[\dfrac{\partial v_i}{\partial t} + v_{i,j}\, v_j\right]$ on $z = -T$

where $\rho_0$ is the mass per unit area and $\Sigma_{i\alpha}$ are the surface stresses on $z = -T$; $\Sigma_{i\alpha}$ are given by (Murdoch 1976)

$$\begin{aligned} \Sigma_{\alpha\beta} &= \delta_{\alpha\beta}\,[\sigma + (\lambda_0 + \sigma)\, u_{\kappa,\kappa}] + \mu_0\, u_{\alpha,\beta} + (\mu_0 - \sigma)\, u_{\beta,\alpha} \\ \Sigma_{3\beta} &= \sigma\, u_{3,\beta} \end{aligned} \tag{13}$$

where $\sigma$ is the 'residual surface tension' and $\lambda_0$, $\mu_0$ are the Lame' moduli of the material boundary at $z = -T$.

In the absence of $\Sigma_{ia}$ and $\rho_0$ our problem reduces to that considered by Paria (1958).

With the help of eqs (4) to (9), (11) and (13), the above boundary conditions yield the following three equations.

$$B_1 = A_2 + B_2 \tag{14}$$

$$\mu_1 p_1 B_1 + \mu_2 p_2 (A_2 - B_2) = 0 \tag{15}$$

$$[\gamma^2(\mu_0 - \rho_0 V^2) - \mu_2 p_2] A_2 e^{-p_2 T} + [\gamma^2(\mu_0 - \rho_0 V^2) + \mu_2 p_2] B_2 e^{p_2 T} = 0. \tag{16}$$

These equations determine the required constants $B_1$, $A_2$ and $B_2$, provided these are consistent. The condition for consistency simplifies to

$$\tanh p_2 T = -\frac{\mu_2 p_2 [\mu_1 p_1 + \gamma^2(\mu_0 - \rho_0 V^2)]}{[\mu_2^2 p_2^2 + \mu_1 p_1 (\mu_0 - \rho_0 V^2)]} \tag{17}$$

This equation determines $V$ when the expressions for $p_1$ and $p_2$ are substituted from eq. (12).

We have assumed that $p_1 > 0$ or equivalently, $V < S_1$ which is necessary for the waves of the desired type to exist. Equation (12) suggests that the following cases are to be considered: (i) $p_2 = 0$ which corresponds to $V = S_2$, (ii) $p_2 > 0$ which corresponds to $V < S_2$, and (iii) $p_2$ is purely imaginary, which corresponds to $V > S_2$. It follows from eq. (11) that the amplitude in the stratum is constant in case (i), exponential in case (ii) and periodic in case (iii), provided the waves exist.

We readily verify that $p_2 = 0$ is a root of eq. (17) and corresponding to this root we get from eqs (14) and (15), $B_1 = 0$. It follows therefore that in case (i) the waves can exist only in the stratum and can propagate with uniform amplitude and phase velocity.

We consider the cases (ii) and (iii) separately in the following sections.

## 3. Waves with exponential amplitude in the stratum

We first consider the case $p_2 > 0$. In this case the waves, if they exist, propagate with exponential amplitude in the stratum.

In the problem with classical boundary we have (Murdoch 1976) $\mu_0 = \rho_0 = 0$ and our problem reduces to that considered by Paria (1958). We readily verify that in this case eq. (17) yields no relevant solution for $V$. Accordingly, the Love waves with exponential amplitude in the stratum cannot exist under the classical boundary condition.

We now assume that $\mu_0 \neq 0$ and $\rho_0 \neq 0$ and rewrite eq. (17) in the form

$$\gamma^2(\mu_0 - \rho_0 V^2) = -\frac{\mu_2 p_2 [\mu_1 p_1 + \mu_2 p_2 \tanh(p_2 T)]}{\mu_1 p_1 \tanh(p_2 T) + \mu_2 p_2} \tag{18}$$

We easily see that this equation *admits* a real root for $V$ that is greater than $S_0$, where $S_0 = (\mu_0/\rho_0)^{\frac{1}{2}}$ which may be called the 'speed of solenoidal waves on a material boundary', in view of its similarity with $S$. Accordingly, the waves with exponential amplitude in the stratum *can exist* whenever $S_0$ is less than both $S_1$ and $S_2$. We may therefore write down the range for phase velocity in the present case in the form

$$S_0 < V < \min(S_1, S_2) \tag{19}$$

It may be noted that $S_2$ can be greater than $S_1$ in the present case which is in contrast to the Paria's (1958) case where $S_2$ is to be less than $S_1$ for the waves to exist.

From eq. (18) we also note that $V$ depends on $\gamma$ in general and therefore there is dispersion.

We now consider some limiting cases.

(i) When the wave length is very small compared to the thickness $T$ of the stratum, we have $\gamma T \to \infty$. Equation (18) then tends to

$$\gamma \mu_0 \left(1 - \frac{V^2}{S_0^2}\right) + \mu_2 \left(1 - \frac{V^2}{S_2^2}\right)^{\frac{1}{2}} = 0 \tag{20}$$

which yields exactly one root in the range $S_0 < V < S_2$. Accordingly in this particular case the wave motion can take place only if $S_2 \leqslant S_1$.

(ii) When the wave length is very large compared to the thickness of the stratum, we have $\gamma T \to 0$. Equation (18) then tends to

$$\left(1 - \frac{V^2}{S_2^2}\right)^{\frac{1}{2}} \left[\mu_1 \left(1 - \frac{V^2}{S_1^2}\right)^{\frac{1}{2}} + \gamma \mu_0 \left(1 - \frac{V^2}{S_0^2}\right)\right] = 0 \tag{21}$$

This equation yields two roots ; one of these is equal to $S_2$ and the other lies in the range $S_0 < V < S_1$. Accordingly in this particular case the wave motion can take place in two modes; one of these propagates with phase velocity tending to $S_2$, provided $S_2 \leqslant S_1$ and the other propagates with phase velocity greater than $S_0$ provided $S_1 \leqslant S_2$. Both these modes can exist only when $S_1 = S_2$.

(iii) When the stratum and the half-space are of the same material, we have $\mu_1 = \mu_2$ and $\rho_1 = \rho_2$ and the entire body reduces to a single half-space in the region $z \geqslant -T$. Equation (18) then reduces to

$$\mu_1 \left(1 - \frac{V^2}{S_1^2}\right)^{\frac{1}{2}} + \gamma \mu_0 \left(1 - \frac{V^2}{S_0^2}\right) = 0 \tag{22}$$

which admits a root for $V$ in the range $S_0 < S < S_1$ and therefore the wave motion is possible in this case also. This is analogous to the result obtained by Murdoch (1976) on the existence of Love waves in a pure elastic half-space.

## 4. Waves with periodic amplitude in the stratum

We now consider the case of imaginary $p_2$ and put $p_2 = iq_2$ where

$$q_2 = \gamma \left(\frac{V^2}{S_2^2} - 1\right)^{\frac{1}{2}} > 0 \tag{23}$$

In this case the waves, if they exist, propagate with periodic amplitude in the stratum.

Equation (17) now reduces to

$$\tan q_2 T = \frac{\mu_2 q_2 [\mu_1 p_1 + \gamma^2 (\mu_0 - \rho_0 V^2)]}{[\mu_2^2 q_2^2 - \mu_1 p_1 \gamma^2 (\mu_0 - \rho V^2)]} \tag{24}$$

In the case of classical boundary condition ($\mu_0 = \rho_0 = 0$) this reduces to the phase velocity equation obtained by Paria (1958), viz.,

$$\mu_2 q_2 \tan q_2 T = \mu_1 p_1 \tag{25}$$

which yields a real root for $V$ in the range

$$S_2 < V < S_1 \tag{26}$$

showing that the wave motion is possible only when $S_2 < S_1$.

We now assume that $\mu_0 \neq 0$, $\rho_0 \neq 0$ and rewrite eq. (24) in the form

$$\tan (q_2 T) = \frac{\rho_2 q_2 S_2^2 [\rho_1 p_1 S_1^2 + \bar{\rho}_0 p_0 S_0^2]}{[\rho_2^2 q_2^2 S_2^4 - \bar{\rho}_0 \rho_1 p_0 p_1 S_0^2 S_1^2]} \tag{27}$$

where

$$p_0 = \gamma \left(1 - \frac{V^2}{S_0^2}\right)^{\frac{1}{2}} \cdot \bar{\rho}_0 = \rho_0 \gamma \left(1 - \frac{V^2}{S_0^2}\right)^{\frac{1}{2}} \tag{28}$$

Equation (27) is strikingly similar to the equation

$$\tan (q_2 T) = \frac{\rho_2 q_2 S_2^2 [\rho_1 p_1 S_1^2 + \rho_3 p_3 S_3^2]}{[\rho_2^2 q_2^2 S_2^4 - \rho_1 \rho_3 p_1 p_3 S_1^2 S_3^2]} \tag{29}$$

obtained by Stoneley (1924) in the pure elastic case and by the present author (1976) in the hypoelastic case, apart from a change in notations, for the phase velocity of transverse surface waves in an infinite body with three layers in $z > 0$, $-T \leqslant z \leqslant 0$ and $z < -T$; in this equation $\rho_3$ denotes the mass density of the half-space $L_3$ which occupies the space $z < -T$, $S_3$ denotes the speed of solenoidal waves in $L_3$, $p_3$ is given by

$$p_3 = \gamma \left(1 - \frac{V^2}{S_3^2}\right)^{\frac{1}{2}} \tag{30}$$

and other symbols are as defined in the earlier paragraphs. The associated displacement in $L_3$ is given by

$$\left.\begin{aligned} u_i &= \delta_{i2} F_3(z) \exp [i\gamma (x - Vt)] \\ \text{where} \quad F_3(z) &= A_3 \exp (p_3 z) \end{aligned}\right\} \tag{31}$$

$A_3$ being a constant.

It has been shown (Stoneley 1924) that the waves governed by eq. (29) can exist when $V$ satisfies the inequality

$$S_2 < V < \min(S_1, S_3).$$

The similarity between eqs (27) and (29) enables us to conclude that the waves considered by us can exist when $V$ satisfies the inequality

$$S_2 < V < \min(S_0, S_1). \tag{32}$$

This, however, may also be proved directly.

Thus the wave motion in our problem with periodic amplitude in the stratum is possible whenever $S_2$ is less than both $S_0$ and $S_1$. Further, the waves are dispersive as in the previous case. Comparing the inequality (32) with (26) which holds in the case of classical boundary condition, we may conclude that the surface stresses on the boundary have their effect in reducing the upper bound of $V$ when $S_0 < S_1$.

When the wave motion is possible, we have from (28) and (32), $p_0$ and $\bar{\rho}_0 > 0$. If we define $F_0(z)$ by the relation

$$F_0(z) = A_0 \exp(p_0 z) \tag{33}$$

where $A_0$ is a constant, we readily verify that

$$u_i = \delta_{i2} F_0(z) \exp[i\gamma(x - Vt)] \tag{34}$$

satisfies eq. (2), provided $\mu = \bar{\mu}_0$, where $\bar{\mu}_0 = \mu_0 [1 - (V^2/S_0^2)]^{\frac{1}{2}}$ and $S = S_0$. Accordingly eqs (33) and (34) determine a solution of the form (31) in the fictitious halfspace (elastic or hypo-elastic) of shear modulus $\bar{\mu}_0$ and mass density $\bar{\rho}_0$ and occupying the region $z < -T$.

It follows therefore that when the amplitude in the stratum is periodic the wave propagation considered by us through the half-space $L_1$ and the stratum $L_2$ with material boundary is identical with that considered by Stoneley (1924) in an infinite body consisting of the layers $L_1$, $L_2$ and $L_3$, provided the half-space $L_3$ is replaced by the fictitious half-space $L_0$. In other words, the effect of having a material boundary $z = -T$ is equivalent to having the classical boundary at $z = -T$ together with a half-space occupying the other side of the stratum, viz., $z < -T$, when the amplitude in the stratum is periodic.

We now consider some limiting cases.

(i) For waves of wavelength which is very small compared to the thickness of the stratum ($\gamma T \to \infty$), eq. (24) tends to

$$\mu_1 \mu_0 \gamma \left(1 - \frac{V^2}{S_1^2}\right)^{\frac{1}{2}} \left(1 - \frac{V^2}{S_0^2}\right) + \mu_2^2 \left(1 - \frac{V^2}{S_2^2}\right) = 0$$

In the problem with classical boundary $V = S_2$ is a root of this equation, while in the present problem the root will be greater than $S_2$. Accordingly in this particular case the waves travel with greater phase velocity than their counterparts in the classical problem.

(ii) For waves of wave length which is very large compared to the thickness of the stratum ($\gamma\, T\to 0$), eq. (24) tends to

$$\mu_1\left(1-\frac{V^2}{S_1^2}\right)^{\frac{1}{2}}+\gamma\,\mu_0\left(1-\frac{V^2}{S_0^2}\right)=0$$

This equation does not yield a relevant root for $V$ except in the special case $S_0 = S_1$. Accordingly in this particular case the wave motion is not possible in general. This is in contrast to the corresponding classical case where waves can propagate with speed tending to $S_1$.

(iii) If the two bodies $L_1$ and $L_2$ are of the same material, we easily verify that eq. (24) has no relevant solution and therefore the wave motion of the desired type is not possible.

## 5. Computation of normal stresses

From eqs (6) we have the two normal stresses, viz., $2\mu s_{11}$ and $2\mu s_{33}$ identically zero. The other normal stress, viz., $2\mu s_{22}$ is given, with the help of eqs (8), (11) and (14) to (16), by

$$2\mu\, s_{22}=[(F_a')^2-\gamma^2 F_a^2]\exp\{2i\gamma\,(x-Vt)\}\qquad \text{in } L_a \tag{35}$$

where

$$F_1=\frac{2A_2\,\mu_2\,p_2}{\mu_2\,p_2+\gamma^2\,(\mu_0-\rho_0 V^2)}\exp\,[-(2p_2T+p_1z)] \tag{36}$$

and

$$F_2=A_2\left[\exp\,(p_2z)+\frac{\mu_2\,p_2-\gamma^2(\mu_0-\rho_0V^2)}{\mu_2\,p_2+\gamma^2(\mu_0-\rho_0V^2)}\exp\,\{-p_2(z+2T)\}\right] \tag{37}$$

We easily verify that the normal stress $2\mu s_{22}$ is *not* zero identically in any of the two bodies. This is in contrast to the corresponding pure elastic problem where all the normal stresses vanish identically.

In the case of classical boundary, eqs. (35) to (37) reduce to

$$2\mu s_{22}=[(F_a')^2-\gamma^2 F_a^2]\,\exp\{2i\gamma\,(x-Vt)\}\qquad \text{in } L_a$$

where $\quad F_1 = 2A_2\,\exp\,[-(2p_2T+p_1z)]$

and $\quad F_2 = A_2\,[\exp\,(p_2z)+\exp\,\{-p_2\,(z+2T)\}]$

These are meaningful only when $p_2$ is imaginary and are identical with the expressions obtained by Paria (1958), apart from a change in the notation.

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## References


Chandrasekharaiah D S 1976 *Appl. Sci. Res.* **32** 347
Erickson J L 1958 *Q. J. Mech. Appl. Math.* **11** 67
Green A E 1956a *Proc. R. Soc.* **A234** 46
Green A E 1956b *J. Rat. Mech. Anal.* **5** 637
Grioli G 1962 *Math. theory of elastic equilibrium (Recent results)* (N. Y.: Academic Press)
Gurtin M E and Murdoch A I 1975 *Arch. Rat. Mech. Anal.* **57** 291
Murdoch A I 1976 *J. Mech. Phys. Solids* **24** 137
Noll W 1955 *J. Rat. Mech. Anal.* **4** 335
Noll W and Truesdell C 1964 in *Encyclopedia of Physics*, ed. Flugge (Berlin: Springer) Vol. 3, Part 3
Paria G 1958 *Q. J. Mech. Appl. Math.* **11** 509
Stoneley R 1924 *Proc. R. Soc.* **A106** 416
Truesdell C 1955a *Comm. Pure Appl. Math.* **8**
Truesdell C 1955b *J. Rat. Mech. Anal.* **4** 83
Truesdell C 1956 *J. Appl. Phys.* **27** 441
Thomas T Y 1957 *J. Math. Phys.* **35** 335
Verma P D S 1956a *Proc. Indian Acad. Sci.* **A44** 185
Verma P D S 1956b Proc. II Congr. Theor. Appl. Mech., New Delhi
Verma P D S 1958 *J. Sci. Eng. Res.* **2** 251