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Multiportfolio Optimization with CVaR Risk Measure

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by

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ABSTRACT

The vast majority of studies in portfolio optimization problem are conducted under a single portfolio framework. In the financial industry, the trading of multiple portfolios is usually aggregated and optimized simultaneously. When multiple portfolios are managed together, unique issues such as market impact costs must be dealt with properly.

Conditional Value-at-Risk (CVaR) is a coherent risk measure with the computationally friendly feature of convexity. In this thesis, we propose the novel combination of CVaR with multiportfolio optimization (MPO) problem. To the best of our knowledge, this is the first work to use CVaR to measure risks in MPO problem and investigate the impact of CVaR on MPO problem.

This thesis uses mathematical programming approaches to model MPO problem with CVaR. Four MPO models are developed considering fairness. The models are solved by GAMS software. Numerical experiments are conducted and analysed. The comparisons with existing methods and sensitivity analysis are reported.

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Chapter 1 Introduction

1.1 General Overview

Ever since the breakthrough of Harry Markowitz's publication on theory of portfolio selection in 1952, the concept of portfolio optimization has been fundamental in the understanding, development and implementation of decision making in the financial industry. Popularly referred to as the Modern Portfolio Theory, Markowitz's topic of portfolio optimization has received huge attention from both academic and industrial area. Markowitz's idea of incorporating risk in portfolio investment decisions and applying a disciplined quantitative framework to the management of portfolio investment was novel when it was first introduced. Ever since the introduction of the theory, researchers have been exploring and studying different facets and extensions of portfolio optimization theory for decades. Among which, the problem of multiple portfolio optimization needs further study, given the small amount of existing studies and its closeness to real life financial industry practice.

To address the portfolio selection problem with the tool of optimization, Markowitz formulated the classic single-period single-account mean-variance optimization (MVO) problem, suggesting that the investor should choose the portfolio with the smallest amount of risk measured by variance of the return of the portfolio to achieve a particular target return objective. This idea by Markowitz is revolutionary for taking diversification into consideration and regards financial decision-making as quantitative trade-off between portfolio return and risk. Markowitz's famous MVO model addresses the decision-making process of portfolio selection through means of mathematical optimization. However, it is crucial to mention that while diversification in the portfolio position could help with reducing risk, it could not generally and thoroughly eliminate risk. Through ensuring a diversified portfolio position, risk can be reduced without changing the expected portfolio return [Markowitz, 1952].

Before the introduction of Markowitz's modern portfolio optimization theory, financial risk was considered as a correcting factor of expected return, and risk-adjusted returns were defined on an ad-hoc basis. At that time, the investment industry's main focus when making financial decisions was on how to find out and invest in investment assets that have lower price relative to their financial potential, or to put in other way, have high expected returns. Markowitz argued that not only the expected return should be included, it is equally if not more important to take risk from the investment into consideration. In his work, Markowitz used the variance of an asset's future return as risk measure. Markowitz's work shows that the riskiness of a single asset is not what is important to the total expected return, but it is its covariance with all other investable assets in the portfolio that really matters. The decisions concerning whether to hold certain assets or not depend on what other assets the investor choose to hold. To acquire the covariance between each assets, however, requires huge amount of data (historical or simulated), which hinder its widespread in practice. Latter models managed to reduce the size of data requirements by eliminating the estimation of correlation between different assets. Furthermore, Markowitz's traditional model is limited only to the case with elliptical distributions such as normal or t-distributions.

In the past 60 years alongside the development of portfolio theory in academic area, many attempts had been made to try to overcome the shortcomings of

Markowitz's traditional model and move the research topic closer to the real-world financial industry practice, introducing several new different risk measures which are more computationally attractive, and taking several facets of significant real-world impact in portfolio optimization problems into consideration.

Topics concerning portfolio optimization, such as dealing with the optimization problem of multiple accounts simultaneously or addressing the portfolio optimization problem in a multi-period framework, came into sight and draw attention from both the academia and financial industry in the past decade. After reviewing a great amount of literature and reports, we believe that it is conclusive to say that up till today, after more than 60 years of its introduction, the classical framework of portfolio optimization still needs modification when used in practice, and the topic of portfolio optimization problem still deserves more research efforts into [Kolm and Tütüncü. et al, 2013].

1.2 Proposed Research

1.2.1 Research Topic

For a portfolio investment and management process, if independent investors choose to authorize a financial service provider to manage the process on their behalves, they give the financial institutions or the portfolio manager access to their investment account with a certain amount of initial capital (investment fund, cash or existing portfolio position) and personalized investment preferences or targets. These providers range from different sizes and scales, from wealth management firms serving few individual investors to large investment firms performing on behalf of more than hundreds of accounts. The vast majority of the existing studies in the area of portfolio optimization problem was done based on the assumption that portfolios (or accounts) are being managed in isolation by the advisors without considering any interrelationship between each account [Iancu and Trichakis, 2014]. However, in practice, those financial advisers in charge with the investment activities of several clients rarely mange a single portfolio (or account) in isolation for the consideration of efficiency in operation. Regardless of the size and scale of such financial institution, they usually serves multiple investment clients, thus multiple investment accounts would be allocated to a single financial adviser. Since one adviser would end up managing more than one investment account, in reality they provide services to multiple accounts simultaneously and act on behalf of multiple portfolios in optimal selection of assets, rebalance, or liquidation of the portfolio. In this thesis, we regard such problem as the multiportfolio optimization problem. The closeness to industry practice alongside with the lack of sufficient existing research focus on multiportfolio optimization problem certainly draws our research interests hence the proposition of this thesis.

When the advisors offer services to multiple portfolios with either similar or different sizes, compositions, potentially different targets and requirements, and levels of risk preferences, etc., they need to address issues such as the uncertain returns, portfolio position constraints, and level of risks involved. These issues are quite common and arguably similar with the classical single portfolio optimization problem. The realization of the industrial practice brings a natural question to ask. The question is that can the existing models and results for single portfolio problem proposed by previous studies be implemented when the optimization of more than one portfolio is dealt with simultaneously. It is realized by leading practitioners from top financial institutions that the answer to the suggested question is provably negative: because the number and/or size of portfolios being managed in a whole result in unique issues that do not exist in single portfolio optimization problem. One of the unneglectable differences from the classical model is the transaction cost incurred when pooling multiple trades together, which if not dealt with properly can counteract net investment returns. A small amount of researchers from the academia or the industry realized this need and did research into multiportfolio optimization with transaction cost, trying to capture all the relevant aspects concern with the multiportfolio optimization problem.

Although some researches addressed and studied multiportfolio optimization under transaction cost with a considerable degree of thoroughness, we think there remains improvements to be done, which will be fully discussed and studied later in the thesis. Instead of using more advanced and more computationally attractive risk measures, almost all the existing research on this topic are agnostic as to how risk is measured, and the usage of variance as a risk measure is employed in those researches. No existing research has studied the impact of risk measure in the multiportfolio optimization problem. Conditional Value-at-Risk (*CVaR*) has several attractive mathematical properties, such as convexity and it is a coherent risk measure [Sarykalin, Serraino and Uryasev, 2008]. To the best of our knowledge, the combination of *CVaR* and multiportfolio optimization problem was not studied by previous works. Considering the increasingly important role the Conditional Value-at-Risk (*CVaR*) is playing, for reason of regulatory requirements from Basel III (2012) and for reason of its advanced mathematical feature as a risk measure, we propose in this thesis to integrate the more advanced risk measure of *CVaR* into the multiportfolio optimization problem.

1.2.2 Research Methodology and Solution Approach

This thesis uses operations research approaches to formulate and solve the problem. Specifically, linear programming, nonlinear programming, mixed integer linear programming and multi-objective optimization are utilized in the research.

Numerical experiments using real life financial market data are conducted to test the proposed models and the results are analysed in later part of the thesis. Numerical tests with different problem size (number of investment accounts and number of investable assets) is designed and run and its results analysed and verified. The comparisons with existing models are conducted and sensitivity analysis is reported to highlight the impact of different parameters.

The software to be used would include but not limited to the follow: GAMS (General Algebraic Modelling System) and its solvers. MS Excel is used to pre-process the data collected from the real world financial market, and MATLAB is used in numerical analysis of the results from GAMS.

1.2.3 Organization of Thesis

The structure of this thesis is as follow. Following this introductory part is Chapter 2 Literature Review, which presents background information to facilitate the studies in this thesis. We first provide a brief review on Markowitz's classical portfolio optimization model, as well as a brief introduction on some existing models. Secondly, we introduce existing literatures on multiportfolio optimization problem and the main focus and contributions of those works. We then provide definitions and comparisons from previous literature of two popular risk measures: Value-at-Risk and Conditional Value-at-Risk, focusing on the mathematical and computational advantage of *CVaR* over *VaR*. Last two sections of this Chapter focus on reviewing the studies in market impact costs and different approaches to ensure fairness in multiportfolio optimization.

Chapter 3 is the modelling for the multiportfolio optimization problem we proposed. This chapter discusses in details of the five-step optimization scheme we developed and the model notations and assumptions, the formulation of functions for each accounts and how we allocate the mark impact costs incurred during the optimization to each account. This chapter is highlighted with the model we developed which integrates Conditional Value-at-Risk as risk measure in the constraints. The formulation with variance as risk measure is presented as well for the comparison with the proposed *CVaR* method in later numerical experiments. In this chapter, a total of four models are developed and they all use the five-step optimization scheme we propose.

Chapter 4 contains the solution approach and preliminary numerical results using real life financial market data from the New York Stock Exchange (NYSE). We provide a brief introduction on the optimization software GAMS and its integrated solvers, followed by a detailed explanation into how the real world stock data from NYSE is chosen and prepared. Discussion on the choices of values of all crucial parameters in all four models, and scenario generation procedure are provided. We run the numerical tests for all four models, perform sensitivity analysis of the results, and conduct comparisons of performances of different risk measures and approaches of splitting the market impact cost.

Chapter 5 presents a summary of the work in this thesis and the conclusion of the thesis. We once again summarize and highlight the contribution of our research, and provide outlook of the possible extensions of the thesis and recommended future works.

Chapter 2 Literature Review

Modern Portfolio Theory starts with the seminal work by Harry Markowitz published in 1952. In the paper, Markowitz formulated the mathematical model which has then been regarded as the foundation of modern portfolio model. From an investor's point of view, the whole purpose of portfolio managing is to gain the highest return possible under a limited amount of capital. To optimally allocate the limited capital between different investable assets seems an easy and solvable problem, however, several factors have to be taken into account, making the portfolio optimization problem more complicated to solve.

2.1 Markowitz's MVO model

The basic concept and essence of Markowitz's modern portfolio theory lies in the balance between expected returns and risk. Markowitz presented several types of hypothesis or rule when choosing a portfolio: 1) the investor should strives to maximize the discounted value of expected future returns. 2) The investor should seek maximized expected return while insuring diversification. The rule, to be more specific, requires the investors to invest the funds among diversified securities with highest expected return. 3) The investor should attempt to maximize expected returns at a given risk, or equivalently at a given expected return level try to minimize portfolio risk [Markowitz, 1952].

The first and second hypothesis were then proven to be wrong or inadequate later in the paper, for the reasons of either ignoring the superiority of diversification (the first hypothesis) or neglecting the effect of variance of future returns(the second hypothesis). It is easily understandable that, although the most desirable option, the portfolio with maximum anticipated return may not necessarily be the one with minimum variance. In practice, the investor must consider the trade-off between expected return and variance (E-V); to gain expected return by tolerating the variance, or to give up some expected returns to reduce the risk. However, the E-V rule does agree with any undiversified security which have an extremely higher return and lower risk than all other securities. The E-V rule is the fundament mentioned above for further research studies in the area of finance and portfolio, with a formal name Modern Portfolio Theory (MPT). The model formulated following the E-V rule is based on mean of the return and variance of the return, hence the name Mean-Variance Optimization Model (MVO). In MVO model, risk is associated with the variance (standard deviation) of the distribution of portfolio return, the deviation from the expected return of the portfolio. Out of the set of n investable assets S, assuming the uncertain future return of asset *j* (*j*=1,…, n) is r_j, and the standard deviation of the uncertain return is σ_j, so that the vector of the expected return of all the assets is $\mu = [\mu_1, \mu_2, ..., \mu_n]^T$, where $\mu_j = E(r_j)$. Let vector $x = [x_1, x_2, ..., x_n]^T$ represent the proportion of the total funds invested in asset *j*, and $\sum_j x_j = 1$. Then for a certain portfolio combination, the variance of total expected return is

$$V(x) = E[(\sum_j x_j r_j - \sum_j x_j E(r_j))^2] = \sum_{i,j} x_i x_j E[(r_i - E(r_i))(r_j - E(r_j))] = \sum_i \sum_j x_i x_j Cov(r_i, r_j)$$

And the standard deviation of the future return is $\sigma_p(x) = \sqrt{V_p(x)}$. Using the variance of the future return as a measure of risk of the portfolio optimization model, as mentioned above, with the expected return (mean) the objective function of Markowitz's model is to choose among a number of investment combination and choose a portfolio with the least risk (variance) of return that achieve a certain expected return (mean) [Markowitz, 1952]. There are three equivalent formulations of the Mean-Variance Model: (1) the portfolio variance minimization formulation, subject to target return value *R*, $\{\min V(x) \mid s.t: \mu^T x \geq R, x \in X\}$; (2) the expected return maximization formulations, subject to certain risk constraints, $\{\max \mu^T x \mid s.t: V(x) \leq \lambda\sigma^*, x \in X\}$; and (3) the risk aversion formulation. $\{\max \mu^T x - \lambda V_p(x) \mid s.t: x \in X\}$ (**λ** here is a parameter of risk aversion determined by investors representing trade-off between expected portfolio return and risks. *X* is the set of all feasible portfolio positions) [F.J.Fabozzi, 2000].

2.2 Multiportfolio Optimization Problem

In practice, financial service providers rarely manage a single portfolio (or account) because they typically offer their services to multiple clients simultaneously. These providers could be from wealth management firms having few individual clients to large investment firms serving a large number of pension, mutual, and insurance funds. An investment manager may need to take charge of multiple portfolios from different clients, with either similar or different sizes or compositions, reflecting potentially different objectives and requirements, levels of risk aversion, etc. [Iancu and Trichakis, 2014].

From a regulatory viewpoint, when providing financial services to clients, investment advisers are obligated to follow the best execution rules, which states that: "As a fiduciary, an adviser has an obligation to obtain 'best execution' of client's transactions. In meeting this obligation, and adviser must execute securities transactions for clients in such a manner that the clients' total cost or proceeds in each transaction is the most favourable under the circumstances" [Securities and Exchange Commission, 2011]. Research paper from Deutsche Asset Management in New York points out that to provide financial investment management services to large numbers of clients as efficiently as possible relies increasingly on large-scale quantitative portfolio construction methods. Ensuring efficiency in practice usually dictates pooling trades and performing execution of several different investing accounts together.

Goal of multiple portfolio selection problem is fundamentally different from that of the classical single portfolio selection problem. It is a crucial knowing that when being managed simultaneously, investment decisions made for single client affect others' investment outcomes. As a result, instead of simply optimizing each investment accounts independently, advisors must implement a process different from existing ones that serves to mediate between accounts in decision-making [O'Cinneide et al., 2006].

O'Cinneide et al. (2006) propose that multiportfolio optimization combines the objectives of all clients in a simple way and evaluates transaction costs according to aggregate trading needs. The multiportfolio optimization framework optimize all positions and trades for all participating accounts in one optimization model. They argue that multiportfolio optimization is the correct answer to the problem of pooled trading because it addresses the unique competitive equilibrium between participating accounts in the market for liquidity. O'Cinneide et al. believe that multiportfolio optimization makes the same decisions for the clients as they would make for themselves if they were trading competitively in the market for liquidity. This conclusion here might not be necessarily true, since individual clients do not have access to the trading decision made by other competitors in the market as a financial advisor managing several accounts simultaneously could have acquired. They also put emphasis on the issue that when making decisions concerning trading, fairness and the common good of all clients must be considered. They formulated an optimization problem that optimizes the portfolios of all clients in an overall sense, which means the objective is to maximize social welfare, i.e., the sum of the objectives functions of individual accounts. The authors argued that through this process fairness for each client ensured, and call this process multi-account optimization (in this thesis we regards multi-account and multiportfolio as the same).

The firm Axioma argues that multiportfolio optimization is the next stage in the progressive evolution of modern investment technologies and platforms, and this technique benefits all parts by making the aggregated trades optimal and fair under given information. Unlike other naïve strategies that sacrifice optimality to achieve fairness, such as randomization and representative accounts, multiportfolio optimization achieves both optimality and fairness during a pooled-trade execution. [Axioma Advisor, 2006]

Savelsbergh et al. (2010) emphasized that the simultaneous optimization of multiple portfolios needs to be conducted carefully due to possible unintended inequalities in the distribution of investment returns among portfolios, favouring one investment account over another. They examined both collusive solution, in which the total welfare is maximized, and Cournot-Nash equilibrium solution, in which objectives of each account is optimized under the assumption that the trading of all other accounts are known and fixed. The paper concludes that both solution method have corresponding advantages and disadvantages, thus no specific preference over a certain solution is made. However, later work by Iancu et al. argues that A Cournot-Nash equilibrium solution is neither necessarily Pareto optimal nor fair, for the fact that accounts are made to participate in artificial game which probably violates the Securities and Exchange Commission rules.

Stubbs and Vandenbussche (2009) did a thorough review on the topics of multiportfolio optimization techniques and properties. They studied the advantages and disadvantages of two economic approaches: the Cournot-Nash equilibrium, and the collusive solution, and presented a unified framework which is able to solve both problems. The focus of this research paper can be justified as fairness between individual investment accounts, for the authors argued that multiportfolio framework can be bias if the issue of fairness is not addressed properly. They also mentioned that definitions of fairness over multiple investment accounts vary among portfolio managers depending on each specific case of investment offering.

Yang et al.(2013) address the multiportfolio optimization problem from a non-cooperative game theory approach; they model the problem as a Nash Equilibrium problem and hence consider a generalized NEP for the case where global constraints are imposed on all accounts, and total welfare is maximized as objective function.

In the paper published in 2014 by Iancu and Trichakis, a thorough discussion of the existing methods employed in the financial industry as well as introduced in the literature is provided. The authors summarize previous works and bring up three unique issues faced by financial service providers compared to the classical single portfolio model. Firstly, the benefits of rebalancing could be sharply reduced if the problematic interactions between trading activities of multiple accounts is ignored. Secondly, there are potential gains from a joint optimization framework and the coordination of the rebalancing trades of individual portfolios. Lastly, when and what information to share to ensure an unbiased distribution of the resulting gains among all the portfolios. They proposed a novel, tractable approach by introducing a model addresses all three above mentioned challenges taking general market impact cost into consideration.

2.3 Risk Measures: VaR and CVaR

Ever since the introduction of the classical model, multiple alternative methods of risk management have been studied in the vast majority of literature of modern portfolio theory. The MVO model are only the very basic measures in a portfolio selection. The concept of risk management involves various perspectives, from a mathematical perspective in financial industry, risk management is a procedure for shaping a loss distribution (such as an investor's risk profile). Though widely studied, among a great deal of innovations in the risk measurements, only a few have been accepted and adapted in real life financial daily operations by practitioners. Beside the implementation of variance or standard deviation as the measurement of risks, other well-known and widely used measure of risk including Value-at-risk and Conditional Value-at-risk also draw attention as practical methods of risk management in portfolio optimization problem.

2.3.1 Value-at-Risk

Value-at-Risk works on a given investment time horizon and confidence level α. Given a specified confidence level α (commonly set at 0.90, 0.95, and 0.99), the VaR_a value of a portfolio is the lowest amount of loss L such that the loss will not exceed this threshold value L with a probability of α. Let L be the random variable with a cumulative distribution function $F_L(l) = P\{L \leq l\}$, here L stands for loss.

Definition 1 (Value-at-Risk). With a given confidence level α, $VaR_\alpha(L)$ is a lower α-percentile of the random variable L :

$VaR_\alpha(L) = \min\{\, l \mid F_x(l) \geq \alpha\}$

If loss is a normally distributed random variable $L \sim N(\mu, \sigma^2)$, then *VaR* is proportional to the standard deviation [Sarykalin, Serraino and Uryasev, 2008]:

$$VaR_\alpha(L) = F_L^{-1}(\alpha) = \mu + f_L^{-1}(\alpha)\sigma$$

However, the easiness and intuitiveness in the formulation of *VaR* is counterbalanced by unfavourable mathematical properties; a lack of both convexity and continuity as a result of being a function of the confidence level α bring numerical difficulties into the problem. It will be discussed and analysed in later part of this paper.

2.3.2 Conditional Value-at-Risk

Conditional-Value-at-Risk (*CVaR*) was introduced as a new approach to reduce the risk of high losses during portfolio optimization, its other names includes mean excess loss, mean shortfall, or tail *VaR*. As defined above, given a probability α as confidence level, VaR_a is the threshold value of loss of a portfolio, so that the loss will not exceed this value with a probability α. The $CVaR_\alpha$ is the conditional expectation of losses above the threshold value of loss. $CVaR_\alpha$ is, by definition, always no less than the VaR_a . Consider a random loss function $f(x, y)$ associated with the decision vector x and a random vector of y of risk factors. x can represent a portfolio selection decision, just as defined above, however, other interpretations are also possible. The vector y represents uncertainties such as uncertain returns, or market variables that can affect the loss, with a probability density function $p(y)$.

Definition 2 (Conditional-Value-at-Risk): With $p(y)$ given, the $CVaR_{\alpha}$ can be denoted by

$$CVaR_{\alpha} = (1-\alpha)^{-1} \int_{f(x,y) \geq VaR_{\alpha}} f(x,y)\, p(y)\, dy$$

However, an analytical expression $p(y)$ for the implementation of the approach is not needed. It is sufficient to have an algorithm (code) which generates random samples from $p(y)$. A two-step procedure can be used to derive analytical expression for $p(y)$ or construct a Monte Carlo simulation code for drawing samples from $p(y)$ [Rockafellar and Uryasev, 2000].

Rockafellar and Uryasev proposed an alternative approach for $CVaR$ calculation, it is a minimization formula that works as a replacement for $CVaR_{\alpha}$. Define a function

$$F_{\alpha}(x,l) = l + (1-\alpha)^{-1} \int_{y} [\, f(x,y) - l\,]^{+}\, p(y)\, dy$$

Such that

$$CVaR_{\alpha} = \min_{l \in \Re} F_{\alpha}(x,l)$$

Function $F_{\alpha}(x,l)$ has a favourable mathematical feature, as a function of loss, $F_{\alpha}(x,l)$ is convex and continuously differentiable, so that a local minimum equals to a global minimum, which is crucial in optimization problems.

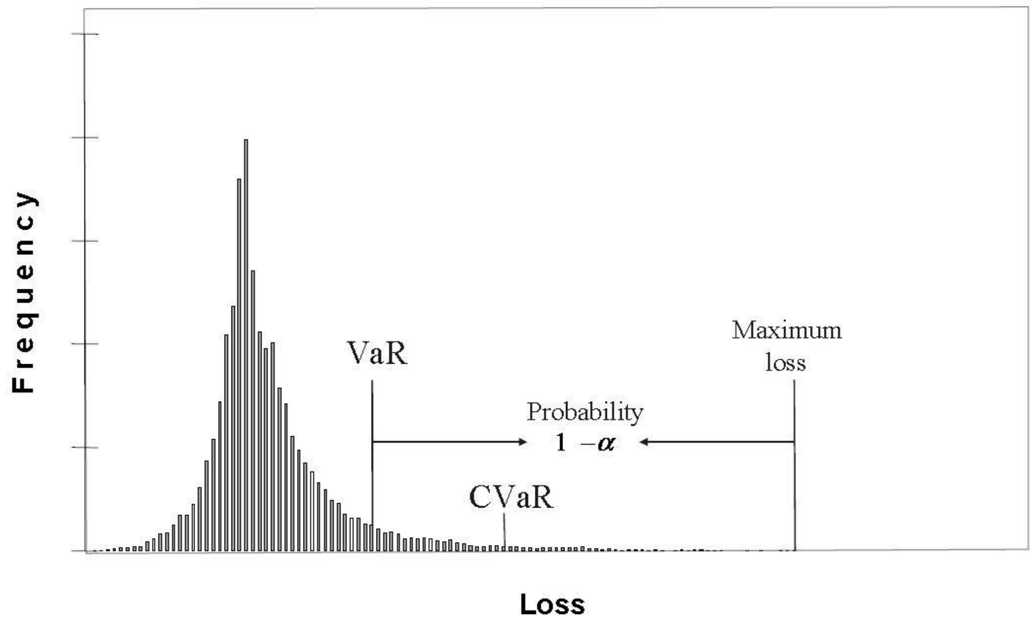

Figure 2.1 Graphical Representation of Maximum Loss, CVaR, and VaR (Uryasev and Rockafellar, 2000)

According to the definition of *CVaR* and *VaR*, Figure 2.1 shows a graphical representation of the relationship between the value of *CVaR*, *VaR* and maximum loss.

Rockafellar and Uryasev (2000) argues that *CVaR* is a superior risk measure to *VaR* in optimization applications in many ways. When returns of the portfolio R is discretely distributed, *VaR* is nonconvex and discontinuous with respect to portfolio positions x^T , these properties makes the *VaR* hard to optimize computationally. *VaR* does not consider scenarios where loss exceeds *VaR*. This property of failing to consider extreme tails, however, is considered to be an advantage with a poorly constructed models with inaccurate estimation, where the use of *VaR* can disregards the tail part of the distribution. Also, another complaints on the shortcoming of *VaR* is the violation of subadditivity, hence not being a coherent risk measure. One of the attractive mathematical properties of *CVaR* is that it is a coherent measure of risk (according to the classification scheme proposed by Artzner (1999), four axioms of subadditivity, homogeneity, monotonicity and risk free condition holds). It is a continuous function with respect to confidence level α, and a convex function of portfolio positions vector x^T .

CVaR measures outcomes that hurt the most, which gives itself a clear engineering interpretation. It can be reduced to convex programming, in some cases, to linear programming (i.e., for discrete distributions). This attractive feature can greatly reduce the computational complexity in optimization problem [Sarykalin, Serraino and Uryasev, 2008].

From the point of regulatory requirements, advantages of *CVaR* are recognized by financial supervision committees. Basel Committee (2012) propose in the Basel III regulations to move the quantitative risk metrics system from *VaR* to Expected Shortfall (also known as *CVaR* or *tail-VaR*).

The conclusion, on the different usage of *VaR* and *CVaR* in different situation, is that *CVaR* is preferable with an accurately constructed model for tail loss, while *VaR* is a better choice when an acceptable good model for tail loss is not available. But it is still important not to ignore the properties of *VaR* that bring difficulties into optimization.

2.4 Transaction Cost in Multiportfolio Optimization Model

In financial and other related area, transaction cost is "costs of using the price mechanism" [Coase, 1937] or "costs of market transactions" [Coase, 1960]. It is a cost incurred when monetary exchange occurs. Transaction costs can be generally divided into two categories: explicit (such as bid-ask spreads, commissions and fees), and implicit (such as price movement risk costs and market impact costs). Among which, as the implicit part of the transaction cost, market impact cost is mostly widely accounted for in existing literatures of multiportfolio optimization problem. Market impact is the effect a trader has on the market price of an asset when it sells or buys the asset. It is the extent to which the price moves up or down in response to the trader's activities. For example, the selling of a large number of shares of a particular stock may drive down the stock's market price [Fabozzi et al. 2010].

An important component of the objective function of modern portfolio rebalancing techniques that rely on optimization is the trading costs. As a result of the buying or selling activities which may drive the asset's market price down or up, the actual price of a certain asset usually differs from the expectation (usually worse than expected price) [Savelsbergh et al. 2010].

Under the multiportfolio optimization settings, the transactions costs incurred by each portfolio heavily depend on the trading activity of other portfolios. That is to say that the transaction costs for a given account may depend on not only the account's own trading requirements but also the overall level of trading. In a multiple portfolio setting, transaction costs typically increase for each account when trading of the accounts are pooled [O'Cinneide et al., 2006].

One of the primary type of transaction costs is the market impact costs, it is where the core of the difference between single and multiportfolio selection problem lies. These market impact cost originate from price impact and limited "at-the-money" liquidity [Iancu and Trichakis, 2014].

The critical problem of how to keep track of transaction costs and mediate between accounts to ensure fairness arise. In practice, the market impact costs is commonly split over all accounts proportional to its holding of the total trade for a particular asset, which is called the *pro rata* scheme. Even though this scheme is easily understandable and applicable and sometimes regarded as fair [Fabozzi et al. 2007], it works only under the assumption that the market impact costs are separable across assets, and it also fails to properly reflect all interactions between the accounts which leads to unfairness. In literature, the issue of splitting market impact costs is seldom discussed, the market impact cost is either not considered or split in the *pro rata* fashion [Iancu and Trichakis, 2014].

Assume w_j^0 is the initial portfolio holding of an account on behalf of asset *j*, then w_j is the optimal portfolio holdings of this account. There are many different models for the transaction cost t.

1) The simplest one is the linear transaction costs, which is under the assumption that the costs are proportional to the trading size. Given a certain percentage c_j, the transaction cost function could be formulated as: $\sum_{j=1}^{n} c_j \left| w_i - w_j^0 \right|$.

2) To take a step further from the linear model, a piecewise-linear transaction model is more realistic, especially for large trades. The costs increase alongside with the increase in the trading size. Here we do not include the formulation because piecewise-linear transaction cost is not the main focus of this thesis.

3) A more general formulation of the transaction cost is to assume that the transaction cost takes form: $t = \sum_j \theta_j \left| w_i - w_j^0 \right|^{\gamma}$, where θ_j is a coefficient calibrated from the data, and γ generally takes the value more than 1. If $\gamma = 2$ then the transaction cost takes a quadratic form.

2.5 Fairness in Multiportfolio Optimization Model

In multiportfolio optimization, a central problem associated with the optimal solution is the fairness issue. Because the trading decision for one account affect the outcomes for other accounts, the advisor must take into consideration fairness and the common good of all clients [O'Cinneide et al., 2006]. Iancu and Trichakis (2014) points out that when one of the accounts is much larger in size than the others, smaller accounts can suffer from a shortage of liquidity. For those small accounts, the socially optimal solution is not fair in the sense that they can achieve a better return profile by acting alone. If the separate accounts belong to individual clients who care about their own utilities only, those "smaller" clients may not be satisfied with the socially optimal solution.

It is understandable that the primary goal of optimization process is to strive for optimality, but under the multiportfolio setting, it is more than just necessary to obtain fairness in the allocation of trades across portfolios [Iancu and Trichakis, 2014]. Consider a simple example in which all accounts are optimized in isolation which means no sharing of information across the investors, if fairness is not ensured, then investment returns of the accounts can probably be very disproportionate [Savelsbergh et al. 2010]. Accounts that obtain less gains than that under the independent optimization setting would rightfully refuse to share information and participate in multiportfolio optimization.

2.6 General Literatures on Portfolio Optimization Models

The search for literature conducted using different combinations of the above keywords provided many papers related with these topics. In this section of the chapter we introduce the general review of literature conducted while searching for desirable research topic.

Fang and Lai (2006) considered liquidity to treat the uncertain expected return and risk as fuzzy numbers and proposed a linear programming model for portfolio rebalancing with transaction costs. Furthermore, based on fuzzy decision theory, a portfolio rebalancing model with transaction costs is proposed.

Tanaka and Gotoh (2010) studied and implemented the constant rebalancing strategy for multi-period portfolio optimization via *CVaR* under nonlinear transaction costs. They quoted that to solve a multi-period portfolio optimization with a constant rebalancing strategy problem is considerably easy for log-optimal portfolio. But when a risk measure is taken into consideration in the model, the problem becomes nonconvex, plus if the size of the question is large, then even the state-of-the art NLP solvers would have difficulties finding local optimal solution. Furthermore, if transaction costs are introduced, these costs cannot be easily dealt with because transaction costs would prevent the problem from having a compact representation. The authors developed a local search algorithm for solving the constant rebalanced portfolio optimization problem under nonlinear transaction costs. In this algorithm, linear approximation problems and nonlinear equations are iteratively solved via a linear programming (LP) solver and Newton's method, respectively.

Skaf and Boyd (2009) formulated the multi-asset multi-period portfolio optimization problem as a stochastic control problem with linear dynamics and a convex quadratic objective, the mean-square error in achieving a desired final wealth. Without the consideration of transaction costs the optimal solution could be solved by dynamic programming. With transaction costs, however, the optimal solution is hard to reach. To deal with the difficulty, the author then proposed two suboptimal policies based on the optimal policy for unconstrained cases.

Wang and Li (2014) considered V-shaped transaction cost in rebalancing model with self-finance strategy, meaning that the investor will not supply any additional investment amount.. They pointed out the main contribution of the paper to be the introduction of a new constraint that confirms the rebalancing necessity of the existing portfolio needs to be adjusted. *CVaR* as risk measure is used in the objective function to be minimized.

Yu and Lee (2009) considered several criteria including risk, return, short selling, skewness, and kurtosis. They studied a total of five portfolio rebalancing models to determine the important design criteria for portfolio model. They implemented a fuzzy multi-objective programming approach to found out that the rebalancing models that consider transaction cost, including short selling cost, are more flexible and their results can reflect real transactions. For future study, they suggested that rather than a portfolio selection based on historical return, a portfolio selection that is able to predict future return can be developed in order to meet this fast-changing environment.

Fabian (2008) proposed decomposition frameworks to solve two-stage stochastic portfolio optimization models with *CVaR* in the objectives function or as constraints. The two-stage decomposition framework has the decision/observation/decision/observation pattern.

Zhang and Zhang (2009) improved the stochastic programming model with simulated paths proposed by Hibiki (2001) by applying genetic algorithm to solve a multi-period portfolio optimization model with *CVaR* as risk measure to be minimized in the objective function. Moreover, proportional transaction costs and market imperfections are also considered in the model. The authors also mentioned that their genetic algorithm can solve the stochastic optimization model with transaction cost and large simulated paths very efficiently, while existing papers reported that large dimension of the stochastic model results in difficulty in computation and only a small number of simulated paths being considered for the brevity of computation.

Meng and Jiang (2010) presented a time-consistent dynamic risk measure: the sum of *CVaR* of each period in the multi-period model. A Markov decision process model is used in getting the optimality equation. The model and the result was then applied in a multi-period portfolio optimization problem with the *CVaR* in the objective functions to be minimized.

Najafi and Mushakhian (2015) characterized their multi-period portfolio selection model with three parameters: the expected value, semivariance and *CVaR* at a given confidence level α. The authors' hybrid Genetic Algorithm (GA) and particle swarm optimization (PSO) algorithm to solve the multi-period model. Taguchi experimental design method is applied to ensure the parameters of the model are wisely chosen for the sake of the performance of the hybrid GA-PSO meta-heuristic algorithm.

Kocak (2014) designed a portfolio selection method using a canonic coalition game, in which the players are the stock certificates traded in FTSE-100 (Financial Times and Stock Exchanges). Risk return values of the stock certificates were treated with clustering analysis technique based on the data for 330 days with the help of SPSS software. The proposed method is able to get the optimal solution out of 15 players (stock certificates) with different risk abilities, the obtained return was distributed in accordance with the weight of each player in the portfolio using Shapley Vector.

Yang and Rubio (2013) considered the case of multiportfolio optimization, in which in practice individual investment accounts are usually pooled together for execution, so the aggregated effects such as market impact must be treated carefully. Multiportfolio optimization aims at finding optimal rebalancing between different investing accounts. The paper implemented non-cooperative game theory and presented a Nash Equilibrium problem.

Wu and Chen (2015) consider a multi-period MV portfolio optimization under a dynamic risk aversion assumption (regime switching). According to the authors, in the real world, it is quite usual that the decision-making process in different portfolio selection period is conducted by different decision-makers (players), hence they treat the problem as a non-cooperative game and proposed that the decision-maker n can only choose the control the portfolio position strategy π_n to maximize objective function given that the successors choose the equilibrium strategy. The authors derived the subgame perfect Nash equilibrium strategy and equilibrium value function in closed-form.

In brief conclusion, both stochastic programming of multi-period model and *CVaR*/*VaR* are used in the area of portfolio optimization for a relatively long period of time with many solution method including decomposition of the model, linear approximation, heuristic algorithms, etc. After searching and reviewing the literatures in these topics, we draw the conclusion that both multi-period portfolio optimization and stochastic programming method are well-studied topics in the area of portfolio optimization, and the risk measures of *CVaR*, *VaR*, semivariance etc., are frequently seen in the objective functions or in the constraints.

Chapter 3 Modelling for MultiPortfolio Optimization Problem

As discussed in Chapter 2, the uniqueness of multiportfolio optimization problem compared with the classical single portfolio optimization problem inevitably render both the academy and industry in search for mathematical models that can accurately and efficiently address the differences. To address the problem of multiportfolio optimization, based on existing literature we introduce our MPO models with Conditional Value-at-Risk (*CVaR*) as risk measures. And we also focus on the allocation of trading incurred market-impact costs. Compared with researches done in the past on the MPO problem, we mainly focus on two topics, namely, the measurement of risks and the allocation of costs between portfolios. Among the existing literatures on the MPO problem, the question of how risk is measured has never been given enough emphasis on. The introduction of the risk measure of *CVaR* in our model distinguishes our method from the existing researches. In terms of splitting the market impact costs, we implement both the industrial standard approach of splitting the market impact cost in a *pro rata* fashion, and the solution method by Iancu and Trichaskis (2014) to treat market impact cost as decision variables.

In this chapter, we present the formulation of our multiportfolio optimization problem with *CVaR* as risk measure. The formulation with variance as risk measure will also be constructed.

3.1 Introduction of Multiportfolio Optimization Modelling

3.1.1 Problem Description

We propose a multiportfolio optimization framework, where one financial advisor provides advisory services regarding portfolio selection and positions to *n* accounts simultaneously. Thus, the problem of optimizing the portfolio selections of *n* accounts simultaneously from an investment pool consists of *m* assets is regarded as the Multiportfolio Optimization Problem. Note that one account represents one client served by the financial advisor. The trading activities of an account act on behalf of the client's portfolio investment preferences and target, while properties such as total available investment funds represents the client's monetary input. The three terms account, client, and portfolio are used interchangeably in our problem. The investment pool consists of a total number of *m* risky assets. As introduced above, when one financial advisor manages multiple accounts, all trading activities of the *n* accounts are pooled together in a whole by the advisor during the optimization process.

To be more specific on the executions of trading of the assets under the multiportfolio framework proposed above, the term "pooling trades" indicates that the portfolio advisor combines all buying orders of a certain asset by all participating portfolios into one order, and submits the aggregated trades to the market at once, the same with all selling orders of a certain asset by all portfolios as well.

The aggregation of trades under the multiportfolio framework inevitably leads to market impact costs that take as arguments the aggregated buying and aggregated selling orders submitted by the financial advisor. The costs is calculated on the aggregated trading activities of all accounts, and thus not split and charged to each accounts intuitively. This raises the question of how to appropriately split the cost between all accounts. The unique issue of transaction costs induced by the aggregation of trades across accounts distinguishes the MPO problem from the classical single portfolio optimization problem, and needs to be reckoned with. The MPO problem requires the issue of splitting the cost across accounts be addressed properly. Our thesis consider both the implicit and explicit part of the transaction costs. For the implicit market impact cost, we use two different approaches to split the costs across the accounts, namely the *pro rata approach* and the *decision variable approach.* The explicit part of the transaction costs is modelled as linear transaction cost proportional to the trading size.

To address the MPO problem, we designed four different optimization models, each with different decision variables or risk measures, for the above mentioned problem. The five steps optimization schemes are designed to perform from the advisor's point of view and to help the advisor in the portfolio selection decision making process by providing the optimal portfolio position for each account participating in the multiportfolio optimization. Notations and assumptions used in the schemes are introduced and discussed in details in the following section of this chapter.

3.1.2 Notations

In this section we introduce indices, parameters, variables, and expressions that are used in the later part of the thesis.

Model Indices, Parameters and Variables:

Indices

i - Index for portfolio (or accounts), $i \in I = \{1,...,n\}$;

j - Index for assets, $j \in J = \{1,...,m\}$;

s - Index for scenarios, $s \in S$ where S is a finite set of scenarios;

Model Parameters:

C_i - Total available capital for the i^{th} account;

$\mathbf{w}_i$ - The vector to denote the initial holding of the i^{th} account, $\mathbf{w}_i \in \Re^m$, i.e., w_{ij} denotes the initial holding in the j^{th} asset on behalf of the i^{th} account;

$y_j^{(s)}$ - The rate of return of the j^{th} asset on the s^{th} scenario;

μ - The vector of expected return, $\mu \in \Re^m$, i.e., μ_j denotes the expected return of the j^{th} asset. μ_j is the mean of y_j^s across all scenarios;

Σ - The covariance matrix of the return of the assets, $\Sigma \in \Re^{mm}$;

κ_i - The risk preference coefficients for each account (client's risk tolerance), $\kappa_i \geq 1, \kappa_i \in \Re^n$;

σ_i - The minimum risk level for the i^{th} account, this value is a result from the first optimization step in our optimization scheme.

θ_j - Market impact cost coefficients for the j^{th} asset. Calibrated from data, satisfying $\theta_j > 0$

γ - Constant for transaction cost model, $\gamma \geq 1$.

Decision Variables:

$\mathbf{x}_i$ - The vector to denote the portfolio position (in units of currency) of the i^{th} account. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n) \in \Re^{mn}$ be the matrix containing portfolio position for all accounts. $\mathbf{x}_i \in \Re^m$, i.e., x_{ij} denotes the portfolio position in the j^{th} asset on behalf of the i^{th} account;

Auxiliary Variables:

x_{ij}^+ - The buy order of the i^{th} account on the j^{th} asset, where

$$x_{ij}^+ = \max(x_{ij} - w_{ij}, 0);$$

x_{ij}^- - The sell order of the i^{th} account on the j^{th} asset, where

$$x_{ij}^- = \max(-(x_{ij} - w_{ij}), 0);$$

$\mathbf{x}_i^+$ - The buy order vector of the i^{th} account;

$\mathbf{x}_i^-$ - The sell order vector of the i^{th} account;

x_{ij}^+ and x_{ij}^- are positive variables.

Functions:

$t(\mathbf{x}_i^+, \mathbf{x}_i^-)$ - The market impact costs resulting from the execution of trades $\mathbf{x}_i$;

$u_i(\mathbf{x}_i)$ - The utility derived by the i^{th} account; the functions $\{u_i(\mathbf{x}_i)\}_{i \in I}$ are required to be concave and expressed in units of currency for all accounts;

U_i - The net utility derived by the i^{th} account;

$f(U_1,...,U_n)$ - The welfare function $f:\Re^n \to \Re$. This function is assumed to be component wise increasing;

The expressions for the functions above are given in later part of this section.

Assumptions:

a. The problem is considered under a stylized, single-period rebalancing framework;
b. In this problem, the financial adviser provide portfolio selection, rebalancing or liquidation services to *n* distinct portfolio accounts;
c. There exist a same pool of *m* risky assets that are investable for all the accounts. The available pool of assets could be the entire universe of stocks in the Standard & Poor 500, or New York Stock Exchange;
d. All trading in this single-period framework is assumed to be not frictionless for all accounts, i.e., the transaction costs incurred during monetary transactions of all *n* portfolios are nonzero. This assumption is relaxed in Model IV;
e. There exist both explicit (linear transaction costs) and implicit (market impact cost) part of transaction costs in the models. Only the market impact costs is considered in the first three models, and both market impact costs and linear transaction costs are considered in the last model;
f. To follow the common practice in the financial industry, during one rebalancing period, the financial adviser pool all the buy and sell orders from all *n* accounts together into a single buy and single sell order, respectively;
g. Possibility of cross-trading, where the financial advisor net buy and sell orders for the same asset offset without recording the trade, is forbidden. That is to

say, any trades on behalf of all the accounts must be operated through the market, no in-house trading is allowed in our model;

h. The market impact costs is separable across assets, i.e., the buying and selling of a particular asset does not affect the market impact costs incurred during the buying and selling of the other assets. The expression of this assumption will be provided below;

i. In our models, the market impact cost is split across the accounts after the optimization problem is solved. We employ two means of splitting the cost, to split it in *pro rata* scheme, or as decision variables set by the solver.

j. The portfolio selection problem under a single-portfolio setting is formulated as maximizing the net utility U_i, which is represented by the portfolio return less market impact cost. Under a multiportfolio setting, the net utility U_i is then jointly optimized by solving a multi-objective optimization problem;

k. Even if the financial adviser makes rebalancing decisions and places buying and selling orders for each portfolio separately, the transaction costs incurred by each portfolio would still depend on the activity of other portfolio. To put it in the form of the market impact cost, $\gamma > 1$;

l. Shorting selling of any asset by any account is prohibited in the thesis.

3.1.3 Market Impact Costs and the Pro Rata Scheme

As is proposed in the assumption, we take into consideration both the implicit and explicit part of the transaction costs. To model the implicit part of the transaction costs, we use a nonlinear formulation which takes as arguments the buy and sell orders for the j^{th} asset by the i^{th} account.

Let the market impact costs due to the execution of trades x_{ij}^{+} and x_{ij}^{-} be

$$t_j(\sum_{i\in I} x_{ij}^{+}, \sum_{i\in I} x_{ij}^{-}) = \theta_j \left\{ (\sum_{i\in I} x_{ij}^{+})^{\gamma} + (\sum_{i\in I} x_{ij}^{-})^{\gamma} \right\}$$

As described in *assumption (h)*, the total market impact cost is separable across assets, the expression for this assumption is as follow

$$t(\sum_{i\in I} x_i^{+}, \sum_{i\in I} x_i^{-}) = \sum_{j\in J} t_j(\sum_{i\in I} x_{ij}^{+}, \sum_{i\in I} x_{ij}^{-})$$

The pro rata scheme

The most common approach of splitting market impact costs incurred during pool trading of multiportfolio optimization is the pro rata approach, which indicates that each account is charged a cost proportional to its share of the total trade for a particular asset. In a pro rata fashion, for market impact costs that are separable across the assets, the trades for the j^{th} asset are $\{x_{ij}\}_{i\in I}$, the i^{th} account is charged a cost of

$$\frac{x_{ij}}{\sum_{a\in I} x_{aj}} t_j(\sum_{a\in J} x_{aj}^{+}, \sum_{a\in J} x_{aj}^{-}), \ \forall i \in I, j \in J$$

Which brings the total market impact cost charged to a particular portfolio i is expressed as follow

$$t_i = \sum_{j\in J} \frac{x_{ij}}{\sum_{a\in I} x_{aj}} t_j(\sum_{a\in J} x_{aj}^{+}, \sum_{a\in J} x_{aj}^{-}), \quad \forall i \in I$$

3.1.4 Utility Functions

To express the total utility generated from the rebalancing trades for the i^{th} account, the expected utility is $u_i(x_i)$. The most widely-used expression to quantify the utility is in units of currency, as follow

$$u_i(\mathbf{x}_i) = \mu^T \mathbf{x}_i \,, \ \forall i \in I$$

And if the risk is considered, the risk-adjusted expected utility function is as follow,

$$u_i(\mathbf{x}_i) = \mu^T \mathbf{x}_i - \lambda_i \cdot Risk \,, \qquad \forall i \in I$$

Note that the risk measure in the above expression can be replaced by *CVaR*, variance, which will be introduced as a major part of the model.

The net expected utility U_i for the i^{th} account, is the total expected return $u_i(x_i)$ for the i^{th} account deducted the amount charged from that account as market impact costs.

$$U_i = u_i(\mathbf{x}_i) \text{-} t_i \,, \qquad \forall i \in I$$

3.1.5 Risk Measure

Conditional Value-at-Risk (CVaR) as risk measure

Mathematically, we follow the definitions and theorems proofed by Rockafellar and Uryasev (2000) to define our *CVaR* model in this thesis.

Known that the return on a portfolio is the sum of return made through individual assets invested in the portfolio being $\mu^T x$, the loss of the portfolio is then the negative of the return, taking the form

$$f(x, \mu_s) = -\mu^T x$$

Introducing a function $F_\alpha(x, \zeta) = \zeta + \frac{1}{S(1-\alpha)} \sum_{s \in S} [f(x, \mu_s) - \zeta]^+$, function $F_\alpha(x, \zeta)$ is piecewise linear with respect to ζ .

Rockafellar and Uryasev (2000) proved the following theorems;

THEOREM 1 *As a function of* ζ , $F_\alpha(x, \zeta)$ *is convex and continuously differentiable. The* $CVaR_a$ *of the loss associated with* $x \in X$ *can be determined from the formula*

$$CVaR_\alpha = \min_\zeta F_\alpha(x, \zeta)$$

THEOREM 2 *Minimizing the* $CVaR_a$ *of the loss associated with x over all* $x \in X$ *is equivalent to minimizing* $F_\alpha(x, \zeta)$ *over all* $(x, \zeta) \in X \times R$, *in the sense that*

$$\min_x CVaR_\alpha = \min_{x,\zeta} F_\alpha(x, \zeta)$$

The minimization of $F_\alpha(x, \zeta)$ *over all* $(x, \zeta) \in X \times R$ *produce a pair* (x^*, ζ^*) *,not necessarily unique, such that* x^* *minimize the* $CVaR_a$ *and* ζ^* *gives the corresponding* VaR_a *. Furthermore,* $F_\alpha(x, \zeta)$ *is convex w.r.t* (x, ζ) *, and* $CVaR_a$ *is convex w.r.t* **x**, *when $f(x, \mu_s)$ is convex w.r.t* ***x****, in which case, if the constraints are such that X is a convex set, the joint minimization is an instance of convex programming.*

To make the function of $CVaR_a$ more optimization-solver friendly, we introduce auxiliary variables $y_1,..., y_s$ for all S scenarios. And $y_s \geq -\mu_{(s)}^T x - \zeta$, $y_s \geq 0$, for all s.

The introduction of function $F_\alpha(x, \zeta)$ makes the calculation of $CVaR_a$ easier for optimization software. For the formulation of our model, we apply this approach to calculate $CVaR_a$.

The Variance as risk measure

Variance σ of the portfolio is formulated as follow;

$$\sigma = x^T \Sigma x$$

where Σ is the covariance matrix calculated from data.

3.2 Modelling

In this section, we introduce four different multiportfolio optimization models with 5-step optimization schemes, focusing on different approaches to measure risk and different approaches to model the market impact cost. In terms of risk measures, the above introduced variance and *CVaR* are utilized in the models as measurement of risks, respectively. The usage of *CVaR* as risk measure in the multiportfolio framework is novel to the existing research focusing on the problem of portfolio optimization and is one of the main contributions of our thesis.

The execution of dividing the market impact costs incurred during multiportfolio optimization practice and charge the costs to each individual portfolio according to certain rules is also a major focus of this section. Market impact costs in our models, due to its categorization as the implicit type of transaction costs, is estimated using a nonlinear, quadratic function which takes the trading of the assets as arguments. To split the costs, we implement two different approaches, namely the *pro rata* approach and the *decision variable* approach.

3.2.1 Model Ⅰ: Multiportfolio optimization scheme with variance risk measure

The following part of this section discusses the modelling of the above mentioned 5-step scheme with the notations and assumptions introduced in Section 3.1. We start simple and explain our 5-step optimization scheme with the classical risk measure variance. Model I takes variance as risk measure, and the market impact cost is split in a *pro rata* fashion across accounts. Detailed explanations of the objective functions and constraints for all five steps are provided below.

Step1. Solve the following portfolio optimization problem for each account *i* independently with variance as objective function to be minimized.

$$\min \ \mathbf{x}_i^T \Sigma \mathbf{x}_i \tag{1}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{2}$$

In this step, we solve the variance minimization problem subject to a set of feasible trade constraint $\boldsymbol{X}_i$ in order to obtain the minimum value of the dispersion of the expected return of the portfolio. We regard the optimal objective value as the lower bound of the portfolio variance for the i^{th} account. Here, Σ is the covariance matrix of all the assets calculated from historical data.

Let $\{\mathbf{x}_i^{opt}\}_{i\in I}$ denote the optimal solution obtained. Then the optimal value of the objective function is $\sigma_i = \mathbf{x}_i^{opt^T}\Sigma\mathbf{x}_i^{opt}$.

Step2. Solve the following independent optimization problem for each account *i*, with net utility as objective function to be maximized subject to constraint for upper bound for variance.

$$\max\{u_i(\mathbf{x}_i) - t(\mathbf{x}_i^+, \mathbf{x}_i^-)\} \tag{3}$$

$$s.t \quad \mathbf{x}_i^T\Sigma\mathbf{x}_i \le \kappa_i \cdot \sigma_i \tag{4}$$

$$\mathbf{x}_i \in \mathbf{X}_i \tag{5}$$

where $u_i(\mathbf{x}_i)$ is the expected portfolio return, and $t(\mathbf{x}_i^+, \mathbf{x}_i^-)$ is the market impact cost.

In this step, we still consider the standard single account setting and maximize the expected portfolio net utility, subject to a constraint of the variance of the expected portfolio return.

A more detailed formulation of this step is as below,

$$\max\{\mu^T\mathbf{x}_i - \sum_{j=1}^{m}\theta_j[(x_{ij}^+)^\gamma + (x_{ij}^-)^\gamma]\} \tag{6}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{7}$$

$$\mathbf{x}_i^T\Sigma\mathbf{x}_i \le \kappa_i \cdot \sigma_i \tag{8}$$

$$\mathbf{x}_i^+, \quad \mathbf{x}_i^- \ge 0 \tag{9}$$

As previously mentioned, market impact costs charged to the i^{th} account in trading of the assets are described as a nonlinear function, and impacts from the trading of assets by any other accounts ($a \in I \setminus \{i\}$) are neglected in this step. This step is to solve a maximization problem of the expected return of the i^{th} account with a constraint to limit variance of the portfolio return relative to a benchmark $\kappa_i \sigma_i$. The value of κ_i, where $\kappa_i \geq 1$, is set by either the client or by the financial advisor.

Note: The optimal solution $\mathbf{x}_i^{IND}$ differs from the optimal solution $\mathbf{x}_i^{opt}$

Step3. Aggregate optimal buy and sell orders for each asset from Step 2 across all *n* accounts

$$\sum_i (\mathbf{x}_i^{IND})^+, \sum_i (\mathbf{x}_i^{IND})^- \tag{10}$$

Step 2 solves the individual net utility maximization problem for all n accounts, and as a result acquires the solution of n optimal solution of vector $\mathbf{x}_i$. However, the single portfolio optimization model of Step 2 overlooks the presence of other accounts participating in the investment markets, buying and selling the assets. The ignorance of the existence of other accounts can cause significant underestimation of the true market impact costs incurred by the trading activity of every account. To take into account the effects of aggregated trading of all accounts managed by the advisor, the buy and sell orders of each asset *j* are aggregated to calculate the total market impact cost. For the j^{th} asset, the aggregated buy and sell orders from all accounts are $\sum_{i=1}^{n} (x_{ij}^{IND})^+$ and $\sum_{i=1}^{n} (x_{ij}^{IND})^-$, respectively. The resulting market impact cost for the j^{th} asset is then formulated as $t_j(\sum_{i=1}^{n} (x_{ij}^{IND})^+, \sum_{i=1}^{n} (x_{ij}^{IND})^-) = \theta_j\{(\sum_{i=1}^{n} (x_{ij}^{IND})^+)^\gamma + (\sum_{i=1}^{n} (x_{ij}^{IND})^+)^\gamma\}$, and the total aggregated market impact cost across all m assets is $t=\sum_{j=1}^{m} t_j(\sum_{i=1}^{n}(x_{ij}^{IND})^+, \sum_{i=1}^{n}(x_{ij}^{IND})^-)$.

Step4. Split the aggregated market impact cost in a pro rata fashion

The realized net utility of the i^{th} account is,

$$U_i^{IND}=u_i(\mathbf{x}_i^{IND})-\sum_{j\in J}\frac{x_{ij}^{IND}}{\sum_{a\in I} x_{aj}^{IND}}\cdot t_j(\sum_{a\in I}(x_{aj}^{IND})^+, \sum_{a\in I}(x_{aj}^{IND})^-),\quad \forall i\in I \tag{11}$$

After the buy and sell orders for the j^{th} asset are aggregated as $\{(x_{ij}^{IND})^+\}_{i\in I}$ and $\{(x_{ij}^{IND})^-\}_{i\in I}$, respectively, the i^{th} account is charged a market impact cost proportional to its share of the total trade for that particular asset [O'Cinneide et al., 2006], which is $\frac{x_{ij}^{IND}}{\sum_{a\in I} x_{aj}^{IND}}\cdot t_j(\sum_{a\in I}(x_{aj}^{IND})^+, \sum_{a\in I}(x_{aj}^{IND})^-),\quad \forall i\in I, j\in J$.

The realized net utility U_i^{IND} here is the expected return of the i^{th} portfolio derived from Step 2, subtracts the proportionally split market impact costs charged to the i^{th} portfolio. Note that the U_i^{IND} derived in this step is the net utility under the independent framework, where no information is shared across the accounts and each account is optimized in isolation.

Step5. Optimize multiportfolio simultaneously using *maxmin* objective function

$$\max\{f(u_1(\mathbf{x}_1)-t_1, u_2(\mathbf{x}_2)-t_2, ..., u_n(\mathbf{x}_n)-t_n)\} \tag{12}$$

$$s.t \quad \mathbf{x}_i\in\mathbf{X}_i,\quad \forall i\in I \tag{13}$$

$$\mathbf{x}_i^T\Sigma\mathbf{x}_i\le\kappa_i\cdot\sigma_i,\quad \forall i\in I \tag{14}$$

$$t_i = \sum_{j \in J} \frac{x_{ij}}{\sum_{a \in I} x_{aj}} \cdot t_j \left(\sum_{a \in I} (x_{aj})^+ , \sum_{a \in I} (x_{aj})^- \right), \quad \forall i \in I \tag{15}$$

$$u_i(\mathbf{x}_i) - t_i \geq U_i^{IND}, \quad \forall i \in I \tag{16}$$

$$\mathbf{x}_i^+, \quad \mathbf{x}_i^- \geq 0, \quad \forall i \in I \tag{17}$$

In this step the advisor optimize the portfolio selection problem of all the accounts jointly and at the same time split the market impact cost across all the accounts. Solution provided in Step 5 differs from the independent solution from the previous four steps, where trading information of individual accounts are not accessible by other participants. The objective function $f(U_1, U_2, ..., U_n)$ is a welfare function which takes the form of $f(U_1, U_2, ..., U_n) = \min\{ \frac{U_i - U_i^{IND}}{U_i^{IND}} \}$. U_i^{IND} denotes the realized net utility for the i^{th} account derived from the independent framework, while the realized net utility derived from the joint optimization framework is denoted by $U_i = u_i(\mathbf{x}_i) - t_i$.

The *maxmin* objective function is to maximize the minimum increase in realized net utility relative to the realized net utility U_i^{IND} under the independent solution across all accounts. The *maxmin* function has well established fairness properties that provides trade-off between social welfare (sum of utilities) and fairness (equitable allocation of utilities) [Iancu et al., 2014].

The result of the multiportfolio optimization scheme with variance risk measure, provided by Step 5, is the optimal portfolio position (how should the total available capital be allocated among assets) for each account *i* as well as the amount of split market impact cost charged to each account *i*, and the derived realized net utility for each account *i*.

3.2.2 Model II: Multiportfolio optimization scheme with CVaR risk measure

As previously emphasized, the integration of $CVaR$ risk measure with the MPO framework is one of the major contributions that distinguish our thesis from the existing researches. In this section, we introduce Model II with the 5-step multiportfolio optimization scheme and $CVaR$ risk measure, and the final market impact cost for each account is split in a pro rata fashion across all participating accounts.

In the notation, we declared the parameter w_{ij} to represent the initial portfolio holding of the j^{th} asset by the i^{th} account, the corresponding decision variables x_{ij} of the model is designed to provide represents the portfolio position at the end of the optimization period. The difference between the initial w_{ij} and final portfolio position x_{ij} is represented by the expression $x_{ij}^{+} - x_{ij}^{-}$. Under close examination of the unique structure of the formulation of $CVaR$, the necessity of separate discussion in Step 1 for the situation with $w_{ij} = 0$ and the situation with $w_{ij} \neq 0$ raises. There are several major differences in the formulation of constrains for the first step of the 5-step scheme from the one in Model I. We provide two different cases of formulation, one for non-zero initial holding ($w_{ij} \neq 0$) and one for zero initial holding ($w_{ij} = 0$). The differences in the type of models between the two cases lie in the introduction of a new set of binary decision variables. Detailed explanations of the objective functions and constraints for each step are provided below.

For the formulation for Step 1, we introduce new notation:

Binary variables:

v_{ij} - Takes value 1 if the initial holding of asset *j* is sold by account *i*, and 0 if not sold.

Step1. Solve the single portfolio optimization model for each account *i*

Case1. Non-zero initial holdings $w_{ij} \neq 0$

The $CVaR$ model is formulated as follow:

$$\min\ CVaR_{\alpha}(\mathbf{x}_i) \tag{18}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{19}$$

$$x_{ij} - w_{ij} = x_{ij}^{+} - x_{ij}^{-}, \quad \forall j \tag{20}$$

$$x_{ij}^{-} \leq w_{ij} \cdot v_{ij}, \quad \forall j \tag{21}$$

$$x_{ij}^{+} \leq C_i \cdot (1 - v_{ij}), \quad \forall j \tag{22}$$

$$\mathbf{x}_i^{+}, \mathbf{x}_i^{-} \geq 0 \tag{23}$$

$$v_{ij} \quad binary, \quad \forall j \tag{24}$$

The purpose of this step is to get the value of the objective function at the optimal point, and we regard this minimum value of $CVaR$ as the lower bound of the average portfolio tail loss. The model is a Mix Integer Programming problem, because v_{ij} is a binary variable.

Objective Function: to be specific on the formulation of $CVaR$, the objective function is

$$\min\{\zeta + \frac{1}{(1-\alpha)\cdot S}\sum_{s=1}^{S}\left[-y^{(s)T}\mathbf{x}_i - \zeta\right]^{+}\}, \tag{25}$$

where the m-dimensional vector $y^{(s)}$ is the vector containing all assets' rate of return in the s^{th} scenario. $-y^{(s)^T}\mathbf{x}_i$ represents the loss of the portfolio in the s^{th} scenario. This formulation of $CVaR$ is the one introduced in Section 3.1.5.

The formulation of $CVaR$ is a nonlinear function, which renders its minimization of risk for the above single portfolio optimization model in this step difficult to solve. To make the minimization problem more computationally friendly, we follow the method of Rockafellar and Uryasev (2000) and introduce a vector of auxiliary variables $k=\{k_1,k_2,...,k_s\}$ to substitute the nonlinear expression $\left[-y^{(s)^T}\mathbf{x}_i-\zeta\right]^+$, one for each scenario.

The optimization problem then can be written as,

$$\min \zeta+\frac{1}{(1-\alpha)\cdot S}\sum_{s=1}^{S}k_s \tag{26}$$

$$s.t \quad \mathbf{x}_i\in\mathbf{X}_i \tag{27}$$

$$k_s\geq 0, \qquad \forall s\in S \tag{28}$$

$$k_s\geq -y^{(s)^T}\mathbf{x}_i-\zeta, \qquad \forall s\in S \tag{29}$$

$$x_{ij}-w_{ij}=x_{ij}^+-x_{ij}^-, \qquad \forall j \tag{30}$$

$$x_{ij}^-\leq w_{ij}\cdot v_{ij}, \qquad \forall j \tag{31}$$

$$x_{ij}^+\leq C_i\cdot(1-v_{ij}), \qquad \forall j \tag{32}$$

$$\mathbf{x}_i^+,\mathbf{x}_i^-\geq 0 \tag{33}$$

$$v_{ij} \qquad binary, \qquad \forall j \tag{34}$$

Constraint (27) represents a set of feasible trade constraints, which impose certain requirements on the portfolio position decision variable vector $\mathbf{x}_i$. This set of constraints could include the total available capital constraint $\sum_{j=1}^{m}x_{ij}=C_i$.

By substituting the $k = \left[-y^{(s)^T}\mathbf{x}_i - \zeta\right]^+$, auxiliary variable k takes the value of $\max(0, -y^{(s)^T}\mathbf{x}_i - \zeta)$, hence constraints (28) and (29). The formulation of $CVaR_a$ then becomes a linear expression after the introduction vector of auxiliary variable k. The mathematical feature of linearity makes the minimization problem of $CVaR$ computationally friendly and easier for analysis.

Constraints (30) define the relationship between x_{ij} and w_{ij}. The difference between the initial w_{ij} and final portfolio position x_{ij} is the trading of asset j by account i in currency units, the expression is $x_{ij}^+ - x_{ij}^-$.

Constraints (31) and (32): By introducing the binary variable v_{ij}, we can guarantee that between the buy order x_{ij}^+ and sell order x_{ij}^-, there can be one and only one nonzero variable. It ban be interpreted as that during a single optimization period, we forbid any accounts to operate the buy order and sell order of the same asset at the same time. The model given by (26)-(34) is a Mixed Integer Linear Program problem.

Step 1 requires a number of n executions, each time with different feasible trade set $\mathbf{X}_i$ corresponding to each account. To be more specific on the differences between each account's $\mathbf{X}_i$, total available capital C_i varies across accounts. Outputs of this model are the optimal value of the objective function $CVaR_\alpha(\mathbf{x}_i^{opt})_{\min} = \sigma_i$, σ_i is then treated as input in the next step. Let $\{\mathbf{x}_i^{opt}\}_{i\in I}$ denote the optimal solution obtained.

Case2. No initial holding $w_{ij} = 0$

Under the hypothesis that there are no initial holding in any assets at the beginning of the optimization period, the minimization of risk is as follow:

$$\min \; CVaR_{\alpha}(\mathbf{x}_i) \tag{35}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{36}$$

Though Case 2 can be regarded as a special case for Case 1, we treat the two case separately because of the elimination of constraints (30)-(34) from the formulation. By eliminating the binary variable the model is reduced from a MILP problem in Case1 to a LP problem. A more detailed formulation can be written as below:

$$\min \; \zeta + \frac{1}{(1-\alpha)\cdot S}\sum_{s=1}^{S} k_s \tag{37}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{38}$$

$$k_s \geq 0, \qquad \forall s \in S \tag{39}$$

$$k_s \geq -y^{(s)T}\mathbf{x}_i - \zeta, \qquad \forall s \in S \tag{40}$$

Step2. Solve the single portfolio optimization problem for each account *i*, with net utility as objective function to be maximized subject to constraint for upper bound for $CVaR_a$.

$$\max\{u_i(\mathbf{x}_i) - t(\mathbf{x}_i^{+}, \mathbf{x}_i^{-})\} \tag{41}$$

$$s.t \quad CVaR_{\alpha}(\mathbf{x}_i) \leq \kappa_i \cdot \sigma_i \tag{42}$$

$$\mathbf{x}_i \in \mathbf{X}_i \tag{43}$$

$$\mathbf{x}_i^{+}, \mathbf{x}_i^{-} \geq 0 \tag{44}$$

The objective function consists of two parts: the expression $u_i(\mathbf{x}_i)$ represents the expected portfolio return for the i^{th} portfolio, and $t(\mathbf{x}_i^+, \mathbf{x}_i^-)$ is the market impact cost charged to aforesaid portfolio due to trading of assets in the available pool.

In this step, we still consider the standard single portfolio setting and formulate an optimization problem of maximizing the expected portfolio net utility (expected portfolio return less market impact cost), subject to a constraint of the portfolio $CVaR_a$ and the feasible trading. This is the expression of the advisor's duty to achieve "best execution" for a single client, i.e. the maximum net utility. A more detailed formulation of this step is as below,

$$\max\{\mu^T \mathbf{x}_i - \sum_{j=1}^{m} \theta_j [(x_{ij}^+)^\gamma + (x_{ij}^-)^\gamma]\} \tag{45}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{46}$$

$$x_{ij} - w_{ij} = x_{ij}^+ - x_{ij}^-, \qquad \forall j \in J \tag{47}$$

$$\zeta + \frac{1}{(1-\alpha)\cdot S} \sum_{s=1}^{S} k_s \le \kappa_i \sigma_i, \tag{48}$$

$$k_s \ge 0, \qquad \forall s \in S \tag{49}$$

$$k_s \ge -y^{(s)^T} \mathbf{x}_i - \zeta, \qquad \forall s \in S \tag{50}$$

$$\mathbf{x}_i^+, \mathbf{x}_i^- \ge 0 \tag{51}$$

From this point, we start to include market impact costs charged to the i^{th} account in trading of the assets. The cost is calculated using the expression $\sum_{j=1}^{m} \theta_j [(x_{ij}^+)^\gamma + (x_{ij}^-)^\gamma]$, which is a summation over the cost of trading each asset *j*.

To be specific, the term "trading" includes the action of both buying (x_{ij}^+) and selling (x_{ij}^{-}) of certain asset. Impacts from the trading of assets by any other accounts ($a \in I \setminus \{i\}$) are neglected in this step. The market impact cost takes nonlinear form because the total cost of a trade is a nonlinear function of the size of the trade. Nonlinear market impact costs are the rule rather than the exception [O'Cinneide et al., 2006].

Constraint (46) is identical to constraints (36) and (38). In the following content of this chapter, if no additional explanation notice given, the feasible trade constraints are the same as (36).

Constraint (48) is the expanded form of the $CVaR$ risk constraint. The left hand side of the inequality is the expression for $CVaR$, and the $\kappa_i \sigma_i$ on the right hand side is the upper bound of i^{th} account's (client's) tolerance of the average loss in the tail. The value of κ_i takes value greater than one and is customized by either the client herself if she has a certainty risk preference, or by the advisor.

Constraints (49) and (50) have the same function as (28), (29), and (39),(40).

Same as Step1, Step2 also requires a number of n executions, each time with different total available capital C_i in constraint (46) and σ_i (known as output from Step1) in inequality (48).

Note: The optimal solution $\mathbf{x}_i^{IND}$ differs from the optimal solution $\mathbf{x}_i^{opt}$ from Step 1.

Step3. Aggregate optimal buy and sell orders for each asset from Step 2 across all *n* accounts

$$\sum_i (\mathbf{x}_i^{IND})^+, \sum_i (\mathbf{x}_i^{IND})^- \qquad (52)$$

Step 3 is the same with the Step 3 in Model I. The single portfolio net utility maximization problem for all n accounts are solved in Step 2, generating the solution of *n* optimal values of vector $\mathbf{x}_i$. In this step, optimal solution $\mathbf{x}_i^{IND}$ for all the n accounts from Step 2 are categorized into two types and then aggregated as the buying $\sum_i (\mathbf{x}_i^{IND})^+$ and selling $\sum_i (\mathbf{x}_i^{IND})^-$, respectively.

Step4. Split the aggregated market impact cost in a pro rata fashion for each account

We introduce the following realized net utility of the *i^th^* account:

$$U_i^{IND} = u_i(\mathbf{x}_i^{IND}) - \sum_{j \in J} \frac{x_{ij}^{IND}}{\sum_{a \in I} x_{aj}^{IND}} \cdot t_j(\sum_{a \in I} (x_{aj}^{IND})^+, \sum_{a \in I} (x_{aj}^{IND})^-), \quad \forall i \in I \tag{53}$$

Results of the aggregated buy $\{(x_{ij}^{IND})^+\}_{i \in I}$ and sell $\{(x_{ij}^{IND})^-\}_{i \in I}$ orders from Step 3 are then taken as input in the above function of U_i^{IND}. The i^{th} account is charged a market impact cost proportional to its share of the total trade for that particular asset which is $\frac{x_{ij}^{IND}}{\sum_{a \in I} x_{aj}^{IND}} \cdot t_j(\sum_{a \in I} (x_{aj}^{IND})^+, \sum_{a \in I} (x_{aj}^{IND})^-), \quad \forall i \in I, j \in J$.

Same as the Step 4 in Model Ⅰ, we calculate the realized net utility U_i^{IND} for the i^{th} account using the optimal solution $\mathbf{x}_i^{IND}$ from Step 2 and aggregated buy and sell order from Step 3. The realized net utility U_i^{IND} here is the expected return of the i^{th} portfolio derived from Step 2, minus the market impact costs charged to the i^{th} portfolio in a *pro rata* fashion. Note that same as Model Ⅰ, the U_i^{IND} derived in this step is under the independent solution, where no information is shared across the accounts.

Step5. Optimize multiportfolio simultaneously using *maxmin* objective function

$$\max\{f(u_1(\mathbf{x}_1)-t_1,u_2(\mathbf{x}_2)-t_2,...,u_n(\mathbf{x}_n)-t_n)\} \tag{54}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i\,, \quad \forall i \in I \tag{55}$$

$$x_{ij}-w_{ij}=x_{ij}^{+}-x_{ij}^{-}, \quad \forall i \in I, j \in J \tag{56}$$

$$CVaR_{\alpha}(\mathbf{x}_i) \le \kappa_i \cdot \sigma_i\,, \quad \forall i \in I \tag{57}$$

$$t_i=\sum_{j\in J}\frac{x_{ij}}{\sum_{a\in I}x_{aj}}\cdot t_j(\sum_{a\in I}(x_{aj})^{+},\sum_{a\in I}(x_{aj})^{-}), \quad \forall i \in I \tag{58}$$

$$u_i(\mathbf{x}_i)-t_i \ge U_i^{IND}, \quad \forall i \in I \tag{59}$$

$$\mathbf{x}_i^{+},\mathbf{x}_i^{-} \ge 0, \quad \forall i \in I \tag{60}$$

Step 5 employs a joint optimization framework for all n accounts. All participating accounts are optimized simultaneously within a single run of the model, maximizing the welfare function $f(U_1,U_2,...,U_n)=\min\{\frac{U_i-U_i^{IND}}{U_i^{IND}}\}$ same as that in Model Ⅰ and splitting market impact cost in a *pro rata* fashion.

The result of the multiportfolio optimization scheme with *CVaR* risk measure, provided by Step 5, is the optimal portfolio position (how should the total available capital be allocated among assets) for each account *i* as well as the amount of split market impact cost charged to each account *i*, and the derived realized net utility for each account *i*.

3.2.3 Model III: Split of market impact cost as decision variables

Instead of following the method implemented in Model I and Model II to split the market impact cost across each account in a *pro rata* fashion, a set of decision variables for each account is introduced to assist the advisor in allocating the amount of market impact cost charged to each account. The model not only provides the optimal portfolio position $\mathbf{x}_i$, the corresponding split of market impact cost among the n accounts is provided as a result as well.

We introduce the following new decision variable notation:

τ_{ij} - The amount of market impact cost charged to the i^{th} account due to trading the j^{th} asset;

The decision of how to split the market impact cost among the n accounts under the proposed multiportfolio framework is made in the last step in the optimization scheme, i.e. Step 5 in both Model I and Model II. Since it is the unique method of treating market impact cost as decision variables that distinguishes Model III from the above two models, we choose to discuss Step 5 in Model III in detail in the following paragraph. Step 1 to Step 4 take the same form as they are in the above two models with the only difference lies in how risk is measured. Let the following model take *CVaR* as risk measure for example.

Step 5. Optimize multiportfolio simultaneously using *maxmin* objective function with split of market impact cost as decision variables.

$$\max\{f(u_1(\mathbf{x}_1)-\tau_1, u_2(\mathbf{x}_2)-\tau_2, ..., u_n(\mathbf{x}_n)-\tau_n)\} \tag{61}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i\,, \quad \forall i \in I \tag{62}$$

$$x_{ij} - w_{ij} = x_{ij}^{+} - x_{ij}^{-}, \quad \forall i \in I, j \in J \tag{63}$$

$$CVaR_{\alpha}(\mathbf{x}_i) \le \kappa_i \cdot \sigma_i\,, \quad \forall i \in I \tag{64}$$

$$\tau_i = \sum_{j=1}^{m} \tau_{ij}\,, \quad \forall i \in I \tag{65}$$

$$\sum_{i=1}^{m} \tau_i = \sum_{j=1}^{m} t_j \Big(\sum_{i=1}^{n} x_{ij}^{+}, \sum_{i=1}^{n} x_{ij}^{-}\Big) \tag{66}$$

$$\tau_{ij} \ge t_j(x_{ij}^{+}, x_{ij}^{-}), \quad \forall i \in I, j \in J \tag{67}$$

$$u_i(\mathbf{x}_i) - \tau_i \ge U_i^{IND}, \quad \forall i \in I \tag{68}$$

$$\mathbf{x}_i^{+}, \mathbf{x}_i^{-}, \tau_{ij} \ge 0, \quad \forall i \in I, j \in J \tag{69}$$

Values of σ_i as well as U_i^{IND} were calculated in Step 1 and Step 4, respectively. Under the multiportfolio optimization framework, the advisor pools the trading of a certain asset in to a single buy and/or sell order in practice. Therefore, given a certain market impact cost model the total market impact cost incurred by the aggregated trading of the j^{th} asset can be calculated accordingly. The market impact cost for the j^{th} asset is denoted by the expression $t_j(\sum_{i=1}^{n} x_{ij}^{+}, \sum_{i=1}^{n} x_{ij}^{-})$, and then the total market impact cost for all m assets is the summation over set *J*, i.e. $\sum_{j=1}^{m} t_j(\sum_{i=1}^{n} x_{ij}^{+}, \sum_{i=1}^{n} x_{ij}^{-})$.

The fourth constraint ensures that the decisions of the amount of market impact cost charged to the n accounts add up to the total market impact cost for all m assets.

3.2.4 Model Ⅳ: Adding real life constraints to the multiportfolio optimization Model

The daily practice of MPO problems often require more specifications on investment policies and preferences that result in more complicated constraints in the optimization model than in the above introduced three models. In the formulation of Model I, Model II, and Model III, the feasible portfolio set $\mathbf{X}_i$ constraints implemented is the total available capital constraints $\sum_j x_{ij} = C_i, \quad \forall i \in I$. Model IV is formulated to capture other real world constraints, such as the total turnover constraints and the rebalancing constraints, etc., to make the model closer to the daily portfolio optimization practice. The constraints, either imposed by the clients according to their investment preferences or by financial regulations, when added actively to the optimization model we designed, can render different portfolio position decisions from previous ones.

Model IV follows the 5-step optimization scheme structure used in the previous three models. Since both *CVaR* and variance have been integrated in the previous three models and the main focus of Model IV is the modelling of real life portfolio optimization common practice constraints, no specific preference is made towards how risk is measured in this model. The implementation of both variance and *CVaR* as risk measures in Model IV will be studied through numerical experiments in the following chapter. Here, for the consideration of keeping this section within an appropriate length, we only provide the formulation using *CVaR*.

Fabozzi et al (2010) summarizes the constraints commonly used in daily practice of portfolio optimization including, but not limited to, no-short-selling constraints, assets holding constraints, portfolio turnover constraints, cardinality constraints, minimum holding and transaction size constraints, and round lot constraints, etc. Those constraints are commonly used by individual investors or advisors representing financial firms, and reflect the above mentioned parties' investment policy. The existence of those constraints in the model results in a more complicated model than previous designed three models.

We use the same set of notations and variables introduced in Section 3.1 for the formulation of constraints in this part, whether one choose to model the risk using *CVaR* or variance. In addition to the already defined decision variables, sets and parameters, we introduce new decision variable and parameters notations as follow;

Binary variables:

z_{ij} - Takes value 1 if asset j is held by account i, and 0 otherwise.

Sets:

D_h - A subset of the total investment universe. Represents the set of assets of similar types or in the same industrial sector. $h \in H$

Parameters:

$P_{i,h}^{u}$ - A vector of positive integers denoting the maximum number of assets in a certain subset D_h that account i can hold. Value of $P_{i,h}^{u}$ is less than or equal to the cardinality of D_h. $P_{i,h}^{u} \leq |D_h|$

$P_{i,h}^{l}$ - Positive integers denoting the minimum number of assets in a certain subset D_h that account *i* needs to hold. $P_{i,h}^{l} \le |D_h|$

b_i - Minimal amount (percentage) to hold an asset for account *i*,

$L_{i,h}^{u}$ - The upper bound of the percentage of holding for asset *j* (or subset D_h of assets) by account *i* among total monetary capital C_i. $0 \le L_{i,h}^{u} \le 1$

$L_{i,h}^{l}$ - The lower bound of the percentage of holding for asset *j* (or subset D_h of assets) by account *i* among total monetary capital C_i. $0 \le L_{i,h}^{l} \le 1$

q_j - Coefficients of the linear transaction costs functions for trading a certain amount of asset *j*

Cardinality Constraints:

Models developed in the previous sections are likely to generate assets selection decisions disregarding the feasibility of the decisions in everyday trading of the assets. To be more specific, the optimal decision variables provided by the models might recommends the account to hold very small amounts of a large number of assets, which can be unnecessarily costly when fixed costs of trading costs are taken into considerations in the daily practice.

On the other hand, we now pay more attentions to the customization of the portfolio position decision, i.e. clients' preferences on how their portfolios are constructed vary from each other.

$$P_{i,j\in D_h}^{l} \le \sum_{j\in D_h} z_{ij} \le P_{i,j\in D_h}^{u}, \qquad \forall i \in I, h \in H \tag{70}$$

$$z_{ij} b_i \leq \frac{x_{ij}}{C_i} \leq z_{ij}, \qquad \forall i \in I, j \in J \tag{71}$$

Constraints (70) provide the upper bound and/or lower bound of the number of assets from subset D_h the portfolio can/must hold. The m assets from the total investment universe can be further classified into multiple subsets of assets according to attributes such as assets type or industrial sector they belonged to. As defined, D_h is a subset of the investment universe, consequently the subscript $j \in D_h$ represents that asset *j* is contained within this particular subset D_h.

Constraints (71) work alongside with Constraints (70), to ensure that if asset *j* is selected into portfolio *i*, then portfolio weight $0 < \frac{x_{ij}}{C_i} \leq 1$, and $z_{ij} = 1$. If asset *j* is not selected, the value of the binary variable z_{ij} is then automatically set to be 0 by the solver in the optimal solution. At the same time, we add the minimal holding constraint of an asset by account *i*: $z_{ij} b_i \leq \frac{x_{ij}}{C_i}$, to avoid the situation where $x_{ij} = 0$ but $z_{ij} = 1$.

Holding Constraints:

$$0 \leq \sum_{j \in D_h} x_{ij} \leq L^u_{i,h} C_i, \qquad \forall i \in I, h \in H \tag{72}$$

or

$$L^l_{i,h} C_i \leq \sum_{j \in D_h} x_{ij}, \qquad \forall i \in I, h \in H \tag{73}$$

While cardinality constraints set requirements on the number of assets, the issue of holding small amount of a large number of assets still exists. The holding constraints perform in conjunction with the cardinality constraints to set limitations on the upper and/or lower bound on the amount of an asset *j* that can/must be held by account *i*, therefore effectively eliminate trading of very small monetary amount.

x_{ij} is the dollar holding of asset *j*, and C_i is the total investment capital of account i , and $0 \le L_{i,h}^{l} < L_{i,h}^{u} \le 1$, thus on both left and right hand side of the inequality are the dollar values.

Linear Transaction Cost Constraints:

We propose that there exist a linear transaction cost for each asset related to the trading of said asset. The linear transaction cost we consider can be commissions, fees and regulatory charges one has to pay for trading in the market. It’s been stressed repeatedly in previous content of our thesis that, our formulation of the MPO model pays great attention to the unique issue of market impact costs incurred under MPO framework. The nonlinear market impact costs are incurred implicitly due to price impact caused by trading of the assets, while the linear transaction costs for trading the can be calculated. For the buying and selling of each asset *j*, the linear transaction cost is calculated as q_j times the transaction $(x_{ij}^{+} + x_{ij}^{-})$.

We propose the balancing constraints that maintain the total value of an account as follow:

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} q_j (x_{ij}^{+} + x_{ij}^{-}) = \sum_{j=1}^{m} w_{ij}, \qquad \forall i \in I \tag{74}$$

w_{ij} on the right hand side of the constraint is the initial holding in asset *j* by account *i*. The value of w_{ij} sum up to equal to the total capital of account *i*. $\sum_{j=1}^{m} w_{ij} = C_i, \quad \forall i \in I$. Constraints (74) also emphasis that linear transaction costs incurred during the optimization period are financed by the total capital C_i of account *i*. Note that this is different from market impact costs, which are split after the optimization period.

Similar to Model III, here we take the example of *CVaR* risk measures and use decision variables for market impact cost allocation to formulate the five-step optimization scheme with the above introduced three sets of constraints.

Step1. Solve the single portfolio optimization model for each account *i*

$$\min\ CVaR_{\alpha}(\mathbf{x}_i) \tag{75}$$

$$s.t \quad \mathbf{x}_i \in \mathbf{X}_i \tag{76}$$

$$P_{i,h}^{l} \le \sum_{j \in D_h} z_{ij} \le P_{i,h}^{u}, \qquad \forall h \in H \tag{77}$$

$$z_{ij} b_i \le \frac{x_{ij}}{C_i} \le z_{ij}, \qquad \forall j \in J \tag{78}$$

$$L_{i,h}^{l} C_i \le \sum_{j \in D_h} x_{ij} \le L_{i,h}^{u} C_i, \qquad \forall h \in H \tag{79}$$

$$\sum_{j=1}^{m} x_{ij} + \sum_{j=1}^{m} q_j (x_{ij}^{+} + x_{ij}^{-}) = \sum_{j=1}^{m} w_{ij} \tag{80}$$

$$x_{ij} - w_{ij} = (x_{ij}^{+} - x_{ij}^{-}), \qquad \forall j \in J \tag{81}$$

$$\mathbf{x}_i^{+}, \mathbf{x}_i^{-} \ge 0, \qquad \forall j \in J \tag{82}$$

$$z_{ij} \quad binary, \quad \forall j \in J \tag{83}$$

Step2. Solve the single portfolio optimization problem for each account *i*, with net utility as objective function to be maximized subject to constraint for upper bound for $CVaR_a$

$$\max\{u_i(\mathbf{x}_i) - t(\mathbf{x}_i^+, \mathbf{x}_i^-)\} \tag{84}$$

$$s.t \quad CVaR_\alpha(\mathbf{x}_i) \le \kappa_i \cdot \sigma_i \tag{85}$$

$$\mathbf{x}_i \in \mathbf{X}_i \tag{86}$$

$$P_{i,h}^l \le \sum_{j \in D_h} z_{ij} \le P_{i,h}^u, \quad \forall h \in H \tag{87}$$

$$z_{ij} b_i \le \frac{x_{ij}}{C_i} \le z_{ij}, \quad \forall j \in J \tag{88}$$

$$L_{i,h}^l C_i \le \sum_{j \in D_h} x_{ij} \le L_{i,h}^u C_i, \quad \forall h \in H \tag{89}$$

$$\sum_{j=1}^m x_{ij} + \sum_{j=1}^m q_j (x_{ij}^+ + x_{ij}^-) = \sum_{j=1}^m w_{ij} \tag{90}$$

$$x_{ij} - w_{ij} = (x_{ij}^+ - x_{ij}^-), \quad \forall j \in J \tag{91}$$

$$\mathbf{x}_i^+, \mathbf{x}_i^- \ge 0, \quad \forall j \in J \tag{92}$$

$$z_{ij} \quad binary, \quad \forall j \in J \tag{93}$$

Step3. Aggregate optimal buy and sell orders for each asset from Step 2 across all *n* accounts

$$\sum_i (\mathbf{x}_i^{IND})^+, \sum_i (\mathbf{x}_i^{IND})^- \tag{94}$$

Step4. Split the aggregated market impact cost in a pro rata fashion

$$U_i^{IND} = u_i(\mathbf{x}_i^{IND}) - \sum_{j\in J} \frac{x_{ij}^{IND}}{\sum_{a\in I} x_{aj}^{IND}} \cdot t_j(\sum_{a\in I}(x_{aj}^{IND})^+, \sum_{a\in I}(x_{aj}^{IND})^-), \quad \forall i \in I \tag{95}$$

Step 5. Optimize multiportfolio simultaneously using *maxmin* objective function with split of market impact cost as decision variables.

$$\max\{f(u_1(\mathbf{x}_1)-\tau_1, u_2(\mathbf{x}_2)-\tau_2, ..., u_n(\mathbf{x}_n)-\tau_n)\} \tag{96}$$

$$s.t \qquad \mathbf{x}_i \in \mathbf{X}_i, \quad \forall i \in I \tag{97}$$

$$CVaR_\alpha(\mathbf{x}_i) \le \kappa_i \cdot \sigma_i, \quad \forall i \in I \tag{98}$$

$$\tau_i = \sum_{j=1}^m \tau_{ij}, \quad \forall i \in I \tag{99}$$

$$\sum_{i=1}^m \tau_i = \sum_{j=1}^m t_j(\sum_{i=1}^n x_{ij}^+, \sum_{i=1}^n x_{ij}^-) \tag{100}$$

$$\tau_{ij} \ge t_j(x_{ij}^+, x_{ij}^-), \quad \forall i \in I, j \in J \tag{101}$$

$$u_i(x_i) - \tau_i \ge U_i^{IND}, \quad \forall i \in I \tag{102}$$

$$P_{i,h}^l \le \sum_{j\in D_h} z_{ij} \le P_{i,h}^u, \quad \forall i \in I, h \in H \tag{103}$$

$$z_{ij} b_i \le \frac{x_{ij}}{C_i} \le z_{ij}, \quad \forall i \in I, j \in J \tag{104}$$

$$L_{i,h}^l C_i \le \sum_{j\in D_h} x_{ij} \le L_{i,h}^u C_i, \quad \forall i \in I, h \in H \tag{105}$$

$$\sum_{j=1}^m x_{ij} + \sum_{j=1}^m q_j(x_{ij}^+ + x_{ij}^-) = \sum_{j=1}^m w_{ij}, \quad \forall i \in I \tag{106}$$

$$x_{ij} - w_{ij} = x_{ij}^{+} - x_{ij}^{-}, \qquad \forall i \in I, j \in J \tag{107}$$

$$\mathbf{x}_i^{+}, \mathbf{x}_i^{-}, \tau_{ij} \geq 0, \qquad \forall i \in I, j \in J \tag{108}$$

$$z_{ij} \qquad binary, \qquad \forall i \in I, j \in J \tag{109}$$

The constraints in the five-step optimization scheme are explained either in the previous models or in the beginning of this section. The last step finds the final optimal portfolio solution with consideration of the real life constraints.

Chapter 4 Solutions and Numerical Results

We present four different optimization models for the MPO problem in Chapter 3. In this chapter, we provide detailed introduction and analysis for the solution method, optimization software programming, numerical tests and numerical analysis.

The solution approach is applied with historical data from the stock market, we conducted numerical tests based on the historical data acquired for the four different models, and performed sensitivity analysis using the preliminary results from the tests to justify the performance and capability the four models we propose.

4.1 Optimization Software: GAMS

4.1.1 GAMS Introduction

The General Algebraic Modelling System (GAMS) is a high-level modelling system for mathematical programming and optimization. It consists of a language compiler and a stable of integrated high-performance solvers. GAMS is tailored for complex, large scale modelling applications, and allows the users to build large maintainable models that can be adapted quickly to new situations. (GAMS Home Page)

GAMS Language is formally similar to commonly used programming languages, which guarantee programming accessibility to users with programming experience. GAMS contains an integrated development environment (IDE) and supports plenty of mathematical programming model types including Linear Programming, Mixed Integer Programming, Mixed Integer Nonlinear Programming, and different forms of Nonlinear Programming.

4.1.2 GAMS Solvers

A large number of solvers for mathematical programming models have been integrated in GAMS. Each solver uses specific algorithms to solve one or more than one types of models. According to the specific optimization model types of the four models developed in this thesis, we choose the CONOPT, CPLEX, and SBB solvers to solve the NLP, MILP, and MINLP models, respectively. Brief introductions of these solvers are provided below from GAMS Solver Manual;

CONOPT: this is a large scale NLP solver. CONOPT is a feasible path solver based on the generalized reduced gradient (GRG) method. CONOPT contains extensions to the GRG method such as a special phase 0, linear mode iterations, and a sequential linear programming component. CONOPT can solve the LP, RMIP, NLP, CNS, DNLP, and RMINLP model types.

CPLEX: GAMS/CPLEX is a GAMS solver that allows users to combine the high level modelling capabilities of GAMS with the power of CPLEX optimizers. CPLEX optimizers are designed to solve large, difficult problems quickly and with minimal user intervention. With proper GAMS licensing, access is provided to CPLEX solution algorithms for linear, quadratically constrained and mixed integer programming problems. While numerous solving options are available, GAMS/CPLEX automatically calculates and sets most options at the best values for specific problems.

SBB: this is a GAMS solver for Mixed Integer Nonlinear Programming (MINLP) models. SBB is based on a combination of the standard Branch and Bound method known from MILP and some of the standard NLP solvers already supported by

GAMS. SBB supports all types of discrete variables supported by GAMS, including binary and integer variables.

The choices of solvers in GAMS also need to take licensing issue and the size of the problems into consideration. Due to the large number of decision variables and constraints in our models, we need to choose the solvers (CONOPT, CPLEX, SBB) with Full License in GAMS in order to get the best performance out of the solvers.

4.1.3 Data Exchange with Excel

We take the advantage of GAMS's ability to exchange data with Excel. GAMS can communicate with Excel via GAMS Data Exchange (GDX) files. A GDX file stores the values of GAMS symbols such as sets, parameters, variables and equations. GDX files act as an intermediate between GAMS and Excel by preparing data for a GAMS model, presenting and storing results of a GAMS model. We use the GDXXRW (in short of GAMS Data Exchange Excel Read and Write) utility to read and write Excel spreadsheet data. GDXXRW is competent to the task of reading and writing multiple ranges in an Excel spreadsheet. The processing speed of GDXXRW utility is satisfying, considering the scale of the current problem.

The exchange of information between GAMS and Excel through GDX files is seamless and only requires few commands in the programming of the model. To import data from an Excel file to our GAMS code, the data in Excel is first written into GDX file then read into GAMS: Excel $\rightarrow$ GDX $\rightarrow$ GAMS. And to export the solution of our model to Excel is the reverse process: GAMS $\rightarrow$ GDX $\rightarrow$ Excel.

4.2 Data Selection and Preparation

To conduct the numerical tests for the four models, we need to choose the values for the parameters in the models. We choose to use historical data of 20 stocks from New York Stock Exchange (NYSE). The adjusted closing price for each stock for 1500 scenarios is processed to get the return rates we need, as well as the covariance matrix between then 20 stocks.

In Table 4.1 below, we present the stock symbols of the 20 stocks we choose as well as the sectors of industries in which they belong to.

Table 4.1 Symbols and industrial sector of the 20 stocks from NYSE

No.	Stock Symbol	Sector
1	BAC	Financial
2	F	Consumer Goods
3	GE	Industrial Goods
4	MSFT	Technology
5	T	Technology
6	GLW	Technology
7	CXW	Financial
8	ORCL	Technology
9	YHOO	Technology
10	KO	Consumer Goods
11	C	Financial
12	PFE	Healthcare
13	EDE	Utilities
14	CNP	Utilities
15	SGMA	Technology
16	RAD	Services
17	SYY	Services
18	AIG	Financial
19	AGM	Financial
20	S	Technology

4.2.1 Scenario Generation

For the models using *CVaR* as risk measure, according to its formulation, we need to consider scenario generation techniques to provide data as input to the models.

We choose to use historical data from twenty stocks from NYSE. We set the time length Δt of the multiportfolio optimization period to be one month (22 trading days). There are many ways the return rate (usually called just return) can be calculated in, the two most common forms are the arithmetic return and the geometric return (also called the log return). We choose the log return over the arithmetic return due to its many merits (more explanation and discussion here).

Denote the closing price for stock *j* of one certain day by $p_j^{t_s}$, here t_s represents the business day. To calculate the monthly log returns for each stock for 1500 scenarios from the historical data, we take logarithms of the ratio of $p_j^{t_s+\Delta t} / p_j^{t_s}$, here $\Delta t = 22$ business days. Once the log returns for 1500 scenarios are calculated, the expected return for stock *j* is calculated as the mean of the 1500 scenarios.

4.3 Numerical Studies

In this section, we present numerical studies that illustrate the performance of the four models we developed. The optimization software GAMS and related solveres with the above introduced options and settings are used to implement the models and solve the problems. The studies involve with five clients trading simultaneously: number of accounts $n = 5$, number of stocks $m = 20$, and length of the optimization period $\Delta t = 22$. The five accounts are with total available funds C_1=100, C_2=150, C_3=200, C_4=250, C_5=300.

4.3.1 Parameters Choices

This section provides brief introduction on how the values of crucial parameters vectors are chosen in the numerical examples. These vectors are risk preference coefficients κ_i , market impact costs coefficients θ_j , linear transaction costs coefficients q_j, and γ in the market impact cost function. Values of these vectors are required to be decided in advance by either the clients or the investment manager, and be treated as input data into the model.

In terms of how risk preference coefficients $\kappa_i, \forall i$ in the models are chosen, it has to be pointed out that the coefficients work as a way to shape the risks in Step 2 and Step 5 from each models. The expression $\kappa_i \sigma_i$ on the right hand side of the risk constraint is the upper bound for the risk that client *i* can tolerate. Because σ_i is the minimum value of risk that portfolio *i* can expect to get, we require that the value of κ_i satisfies the inequity $\kappa_i \geq 1$ to ensure feasibility of the solution in Step 2 and Step 5. We argue that there're upper bounds for coefficients $\kappa_i, \forall i$, and value of κ_i used in the numerical tests has to lie in between the lower and upper bounds for the risk constraints to be effective. To decide the value, we perform tests on Step2 based on the input data for the 20 stocks for both *CVaR* risk measure and variance risk measure. The outcome of the tests suggests that $1 \leq \kappa_i < 3.5, \forall i$ for models using the risk measure *CVaR*, and $1 \leq \kappa_i < 12, \forall i$ for models using variance. The upper bounds are calculated at the solution that maximizes the return for the model in Step 2 but relaxing the risk constraints.

In terms of the choices of linear transaction costs coefficients q_j in Model IV, for simplicity, we set the value of all $q_i, \forall i$ to be 0.01%. This means that a percentage of 0.01% of the transaction size is charged to account *i* for trading any stocks under the

MPO framework. Linear transaction cost isn't the main focus of this thesis, hence the simplicity in deciding the values.

In terms of the choices of market impact costs coefficients θ_j, they are calibrated from data to fit observed trading costs in the market [O'Cinneide, 2006]. Almgren, Thum and Hauptmann (2005) analysed a large set of data from the Citigroup US equity trading desks and used a simple but realistic theoretical framework to determine value of market impact costs coefficients, and they stressed that their results fit the stocks in NYSE. According to the research by Almgren, et al. (2005), we determine value of θ_j to be 0.0000314. We assume market impact costs coefficients for trading all 20 stocks takes the same value for further simplicity.

As for the formulation of the market impact costs, we follow the numerical studies in the works of O'Cinneide et al. (2006) and Iancu et al. (2014), and set the value of $\gamma = 2$.

4.3.2 Random Number Generation for Initial Holdings

We propose two different cases for the initial holdings and model formulation. For the first one we consider a general case where w_{ij} takes non-zero values, indicating that the accounts have already entered the investment market and traded according to previously made decisions. The second situation is where there are no initial holdings, indicating that the accounts hold no assets at the beginning of the optimization period of the MPO model. Case 1 represents a more general situation, while case 2 is a special case for case 1. One can easily modify the input data of w_{ij} to fit the according assets holding position, and then start with the 5-step MPO scheme we proposed.

In the following numerical examples, the input parameters $w_{ij}, \forall i, j$ we use in the models are random numbers generated by the Random Number Generator (RNG) in GAMS. The series of numbers generated from the RNG are pseudo-random numbers, we make sure in the GAMS code that all initial holdings are non-negative: $w_{ij} \geq 0$ for case 1 and $w_{ij} = 0$ for case 2. And for Case 1, the initial holdings w_{ij} sum up to each account's total available money in the investment: $\sum_j w_{ij} = C_j, \forall i$. Input file is then imported into GAMS using the data exchange utility GDXXRW By ensuring the initial holding parameters w_{ij} and portfolio position decision variables x_{ij} take non-negative value, we ensure that short selling of any stock is prohibited in the model as proposed in the assumption.

4.3.3 Numerical Results

To design the numerical tests for the 4 models proposed in this thesis, we follow the above discussed details in the first half of this chapter including choices of optimization software and solvers, data pre-processing and scenario generation, choices of parameters, etc.

We present the preliminary results of the numerical tests for each model in the following section. The numerical examples are conducted in two different cases according to the above-discussed two situations with initial holdings. In Numerical Case 1, we consider the special case where there are zero initial holdings in all assets *j* for all accounts *i*. In Numerical Case 2, we overthrow the zero initial holding setting, and consider the more general setting of random initial holdings. For Numerical Case 1, we present results of Model I with variance risk measures, Model II with *CVaR* risk measures, Model III with both variance and *CVaR* risk measures. For Numerical Case 2, we present the results of Model I with variance risk measures, Model II with *CVaR* risk measures, Model III with both variance and *CVaR* risk measures, and Model IV with both variance and CVaR risk measures.

Case 1: without initial holding

An investment advisor is in charge of $n = 5$ portfolios, investing in a market of $m = 20$ stocks from NYSE. Assume the manager start from zero holdings in all stocks for all portfolios, i.e. $w_{ij} = 0, \forall i, j$. To put it in a more specific way of explanation, we consider the situation under which all the five accounts enter the market for the first time, so that their holdings on any of the 20 stocks are all zero. The value of κ_i used for risk control is set to 3 for all models. We run the GAMS codes designed for Model I, Model II and Model III, and the results generated by the program is shown in the following separate tables.

- Model I: MPO scheme with variance risk measure and market impact costs split in *pro rata* fashion.

Table 4.2 Value of variance for Model I (zero initial holding)

Account	**Risk (Variance)**		
	Step1	Step2	Step5
1	8.357	25.070	25.070
2	18.803	56.408	56.408
3	33.427	100.282	100.282
4	52.230	156.690	156.690
5	75.211	225.634	225.634

Table 4.3 Value of return, market impact cost, utility, and improvement rate for Model I (zero initial holding)

Account	**Return**			**Market Impact Cost**			**Utility**			**Improve (%)**
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.832	1.606	1.414	0.067	0.582	0.279	1.539	1.025	1.135	10.76%
2	1.248	2.397	2.166	0.136	0.838	0.438	2.261	1.559	1.727	10.76%
3	1.664	3.176	2.951	0.218	1.065	0.613	2.958	2.110	2.337	10.76%
4	2.080	3.943	3.759	0.311	1.272	0.800	3.632	2.671	2.959	10.76%
5	2.496	4.699	4.588	0.412	1.459	0.999	4.287	3.240	3.589	10.76%

Table 4.3 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5. As shown in the table, the final utilities of all accounts increase by 10.76% from the utilities of Step 4, which reflects that optimizing multiportfolio simultaneously can significantly improve the performance. The same improvement value for all five accounts represents the fairness.

- Model II: MPO scheme with *CVaR* risk measure and market impact costs split in pro rata fashion.

Table 4.4 Value of CVaR and VaR for Model II (zero initial holding)

Account	**Risk (*CVaR* & *VaR*)**					
	Step1		Step2		Step5	
	CVaR	VaR	CVaR	VaR	CVaR	VaR
1	5.621	4.457	16.864	11.919	16.574	5.234
2	8.432	6.686	25.296	17.718	23.606	8.376
3	11.243	8.914	33.728	23.440	33.136	15.042
4	14.053	11.143	42.159	29.337	41.926	21.622
5	16.864	13.372	50.591	35.010	50.591	41.384

Table 4.5 Value of return, market impact cost, utility, and improvement rate for Model II (zero initial holding)

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.813	2.189	1.875	0.108	1.065	0.571	2.081	1.124	1.305	16.13%
2	1.219	3.282	2.815	0.242	1.596	0.857	3.041	1.686	1.958	16.13%
3	1.626	4.376	3.755	0.429	2.127	1.143	3.947	2.249	2.612	16.13%
4	2.032	5.462	4.713	0.662	2.642	1.438	4.800	2.819	3.274	16.13%
5	2.438	6.524	5.718	0.921	3.113	1.757	5.603	3.410	3.960	16.13%

Table 4.5 presents the numerical results for returns, impact costs and utility from for Step 1 to Step 5 for Model II. As can be seen in Table 4.5, the improvement rate of all accounts is 16.13%. Returns for all five accounts increase in Step 2 and Step 5, compared with the results in Table 4.3 for Model I. The same increases are seen in utilities and improvement rate for all five accounts.

- Model III: MPO scheme with variance risk measure and market impact costs split as decision variables.

Table 4.6 Value of variance for Model III (zero initial holding)

Account	Risk (Variance)		
	Step1	Step2	Step5
1	8.357	25.070	25.070
2	18.803	56.408	56.408
3	33.427	100.282	100.282
4	52.230	156.690	156.690
5	75.211	225.634	225.634

Table 4.7 Value of return, market impact cost, utility, and improvement rate for Model III with Variance risk measure (zero initial holding)

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.832	1.606	1.492	0.067	0.582	0.356	1.539	1.025	1.136	10.87%
2	1.248	2.397	2.238	0.136	0.838	0.509	2.261	1.559	1.729	10.87%
3	1.664	3.176	2.984	0.218	1.065	0.645	2.958	2.110	2.340	10.87%
4	2.080	3.943	3.730	0.311	1.272	0.769	3.632	2.671	2.961	10.87%
5	2.496	4.699	4.477	0.412	1.459	0.884	4.287	3.240	3.593	10.87%

- Model III: MPO scheme with *CVaR* risk measure and market impact costs split as decision variables.

Table 4.8 Value of CVaR and VaR for Model III with CVaR risk measure (zero initial holding)

Account	Risk(CVaR & VaR)					
	Step1		Step2		Step5	
	CVaR	VaR	CVaR	VaR	CVaR	VaR
1	5.621	4.457	16.864	11.919	16.864	1.257
2	8.432	6.686	25.296	17.718	25.296	5.235
3	11.243	8.914	33.728	23.440	33.833	15.647
4	14.053	11.143	42.159	29.337	42.159	34.231
5	16.864	13.372	50.591	35.010	55.748	36.364

Table 4.9 Value of return, market impact cost, utility, and improvement rate for Model III with CVaR risk measure (zero initial holding)

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.813	2.189	1.474	0.108	1.065	0.169	2.081	1.124	1.305	16.13%
2	1.219	3.282	2.525	0.242	1.596	0.567	3.041	1.686	1.958	16.13%
3	1.626	4.376	3.665	0.429	2.127	1.053	3.947	2.249	2.612	16.13%
4	2.032	5.462	4.450	0.662	2.642	1.176	4.800	2.819	3.274	16.13%
5	2.438	6.524	6.762	0.921	3.113	2.801	5.603	3.410	3.960	16.13%

Table 4.7 shows the numerical results for returns, impact costs and utility from Step 1 to Step 5 for Model III with variance risk measure. A significant increase of utilities for all five accounts from Step 4 to Step 5 is seen, with value of 10.87%. Table 4.9 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5 for Model III with *CVaR* risk measure. The improvement rate for the five accounts is 16.13%, reflecting significant improvement from utilities in Step 4. Comparing the results from Table 4.7 with Table 4.9, the model with *CVaR* risk measure generates relatively higher returns, utilities, and improvement rates for all five accounts.

The improvement rate and utilities from Step 5 in Model III with *CVaR* risk measure (shown in Table 4.9) are the same with their corresponding part in Model II with *CVaR* risk measure (shown in Table 4.5). But, as shown in Tables 4.5 and 4.9, values of return and market impact costs from Step 1 to Step 5 are different for the two models. Model III causes a lower value of costs for account 1,2,3,4 and a higher costs for account 5 compared with Model II. These are caused by the differences in the ways of splitting market impact costs in the two models. The same with Model III with variance risk measure and Model I, utilities and improvement rates are approximately the same while returns and market impact costs are different.

Case 2: with random initial holding

We consider a similar setting to the one in Example 1, where the investment manager/advisor is in charge of $n = 5$ portfolios, investing in a market of $m = 20$ stocks from NYSE. In this example 2, we relax the previous assumption of no initial holding to a general case where there're initial holdings on each asset by each account. The initial holdings $w_{ij}, \forall i, j$ are generated by random number generator in GAMS as previously introduced. We run the GAMS codes designed for Model I, Model II, Model III, and Model IV, and the results generated by the program is shown in the following separate tables.

- Model I: MPO scheme with variance risk measure and market impact costs split in *pro rata* fashion.

Table 4.10 Value of variance for Model I

Account	Risk (Variance)		
	Step1	Step2	Step5
1	8.357	25.070	25.070
2	18.803	56.408	56.408
3	33.427	100.282	100.282
4	52.230	156.690	156.690
5	75.211	225.634	225.634

Table 4.11 Value of return, market impact cost, utility, and improvement rate for Model I

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.832	1.607	1.467	0.046	0.397	0.148	1.560	1.209	1.319	9.10%
2	1.248	2.395	2.177	0.102	0.591	0.209	2.294	1.804	1.968	9.10%
3	1.664	3.171	2.915	0.184	0.810	0.340	2.987	2.360	2.575	9.10%
4	2.080	3.917	3.702	0.254	0.962	0.477	3.663	2.956	3.224	9.10%
5	2.496	4.667	4.518	0.296	1.029	0.550	4.371	3.637	3.968	9.10%

Table 4.11 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5 in Model I with random initial holdings. As shown in the table, the final utilities of all accounts increase 9.1% from the results of Step 4, which reflects that optimizing multiportfolio simultaneously can significantly improve the performance. The same improvement value for all accounts represents the fairness.

- Model II: MPO scheme with *CVaR* risk measure and market impact costs split in pro rata fashion.

Table 4.12 Value of CVaR and VaR for Model II

Account	Risk (CVaR & VaR)					
	Step1		Step2		Step5	
	CVaR	VaR	CVaR	VaR	CVaR	VaR
1	5.621	4.457	16.864	11.553	16.727	5.040
2	8.432	6.686	25.296	17.218	24.181	11.365
3	11.278	9.006	33.833	23.038	33.614	27.238
4	14.053	11.143	42.159	28.687	41.896	25.652
5	18.583	13.875	55.748	38.183	50.260	38.397

Table 4.13 Value of return, market impact cost, utility, and improvement rate for Model II

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.813	2.187	1.899	0.096	0.965	0.444	2.091	1.222	1.454	18.99%
2	1.219	3.280	2.791	0.212	1.429	0.589	3.069	1.851	2.202	18.99%
3	1.683	4.383	3.605	0.406	2.000	0.770	3.977	2.382	2.835	18.99%
4	2.032	5.447	4.511	0.636	2.515	1.022	4.811	2.932	3.489	18.99%
5	2.432	6.934	5.728	1.084	3.272	1.371	5.850	3.662	4.357	18.99%

Table 4.13 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5 in Model II with random initial holdings. As shown in the table, the final utilities of all accounts increase 18.99% from the results of Step 4, which reflects that optimizing multiportfolio simultaneously can significantly improve the performance. The same improvement value for all accounts represents the fairness.

- Model III: MPO scheme with *variance* risk measure and market impact costs split as decision variables.

Table 4.14 Value of variance for Model III with Variance risk measure

Account	Risk(Variance)		
	Step1	Step2	Step5
1	8.357	25.070	25.070
2	18.803	56.408	56.408
3	33.427	100.282	100.282
4	52.230	156.690	156.690
5	75.211	225.634	225.634

Table 4.15 Value of return, market impact cost, utility, and improvement rate for Model III with Variance risk measure

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	

1	0.832	1.607	1.482	0.046	0.397	0.162	1.560	1.209	1.320	9.13%
2	1.248	2.395	2.223	0.102	0.591	0.254	2.294	1.804	1.969	9.13%
3	1.664	3.171	2.965	0.184	0.810	0.390	2.987	2.360	2.576	9.13%
4	2.080	3.917	3.694	0.254	0.962	0.468	3.663	2.956	3.225	9.13%
5	2.496	4.667	4.444	0.296	1.029	0.475	4.371	3.637	3.970	9.13%

Table 4.15 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5 in Model III using variance risk measure with random initial holdings. As shown in the table, the final utilities of all accounts increase by 9.13% from the results of Step 4. Same with Case 1, utilities and improvement rates in Model III with variance risk measure and Model I (shown in Table 4.11) are approximately the same, but returns and market impact costs are different.

- Model III: MPO scheme with *CVaR* risk measure and market impact costs split as decision variables.

Table 4.16 Value of CVaR and VaR for Model III with CVaR risk measure

Account	**Risk (CVaR & VaR)**					
	Step1		Step2		Step5	
	CVaR	VaR	CVaR	VaR	CVaR	VaR
1	5.621	4.457	16.864	11.553	15.876	5.596
2	8.432	6.686	25.296	17.218	22.470	11.449
3	11.278	9.006	33.833	23.038	33.833	28.873
4	14.053	11.143	42.159	28.687	42.159	32.171
5	18.583	13.875	55.748	38.183	49.667	28.936

Table 4.17 Value of return, market impact cost, utility, and improvement rate for Model III with CVaR risk measure

Account	**Return**			**Market Impact Cost**			**Utility**			**Improve (%)**
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.813	2.187	1.855	0.096	0.965	0.401	2.091	1.222	1.454	18.99%
2	1.219	3.280	2.539	0.212	1.429	0.337	3.069	1.851	2.202	18.99%
3	1.683	4.383	3.839	0.406	2.000	1.004	3.977	2.382	2.835	18.99%
4	2.032	5.447	4.578	0.636	2.515	1.089	4.811	2.932	3.489	18.99%
5	2.432	6.934	5.723	1.084	3.272	1.366	5.850	3.662	4.357	18.99%

Table 4.17 presents the numerical results for returns, impact costs and utility from Step 1 to Step 5 in Model III using *CVaR* risk measure with random initial holdings. As shown in the table, the final utilities of all accounts increase by 18.99% from the results of Step 4. Same with Case 1, Model II and Model III saw a same value of utilities and improvement rate for all five accounts, but different returns and costs values, which are caused by the differences in costs allocation methods used in the two models.

- Model IV: MPO scheme with *variance* risk measure and market impact costs split as decision variables.

Table 4.18 Value of Variance for Model IV with Variance risk measure

Account	Risk (Variance)		
	Step1	Step2	Step5
1	11.314	33.943	33.943
2	25.287	75.860	75.860
3	44.916	134.747	134.747
4	70.776	212.329	212.329
5	101.080	303.240	303.240

Comparing with the results in Table 4.10 for Model I (without the real life constraints), the risks of all accounts in Step1, Step 2, and Step 5 in Model IV increase about 33%, which are caused by the real life constraints.

Table 4.19 Value of return, market impact cost, utility, and improvement rate for Model IV with Variance risk measure

Account	Return			Market Impact Cost			Utility			Improve (%)
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.786	1.736	1.635	0.051	0.457	0.239	1.685	1.279	1.396	9.14%
2	1.187	2.611	2.460	0.104	0.653	0.323	2.507	1.958	2.137	9.14%
3	1.590	3.470	3.228	0.208	0.946	0.473	3.262	2.524	2.755	9.14%
4	1.950	4.284	4.078	0.321	1.184	0.695	3.963	3.100	3.383	9.14%
5	2.381	5.120	4.902	0.399	1.314	0.748	4.721	3.806	4.154	9.14%

Comparing with the results in Table 4.11 for Model I (without the real life constraints), the utilities of all accounts increase about 5% at Step 5. With introduced extra constraints, we expect the utilities would decrease. However, increasing the tolerance of risk can increase our returns and the utilities.

- Model IV: MPO scheme with *CVaR* risk measure and market impact costs split as decision variables.

Table 4.20Value of CVaR and VaR for Model IV with CVaR risk measure

Account	**Risk (CVaR & VaR)**					
	Step1		Step2		Step5	
	CVaR	VaR	CVaR	VaR	CVaR	VaR
1	7.356	4.931	21.930	7.612	21.253	2.892
2	10.849	7.426	28.844	16.269	29.677	6.153
3	14.904	10.377	37.858	27.020	32.371	10.627
4	19.366	13.540	45.417	36.133	36.854	22.310
5	21.899	14.786	60.825	53.291	50.176	32.819

Comparing with the results in Table 4.12 for Model II (without the real life constraints), the risks of all accounts in Step1 and Step 2, and 3 accounts in Step 5 in Model IV increase, which are caused by the real life constraints.

Table 4.21Value of return, market impact cost, utility, and improvement rate for Model IV with CVaR risk measure

Account	**Return**			**Market Impact Cost**			**Utility**			**Improve (%)**
	Step1	Step2	Step5	Step2	Step4	Step5	Step2	Step4	Step5	
1	0.894	2.199	1.771	0.107	1.026	0.371	2.092	1.173	1.401	19.45%
2	1.369	3.423	2.812	0.246	1.582	0.614	3.177	1.841	2.199	19.45%
3	1.779	4.635	3.387	0.480	2.233	0.517	4.156	2.403	2.870	19.45%
4	2.131	5.440	4.233	0.606	2.507	0.730	4.834	2.933	3.504	19.45%
5	2.616	6.887	5.969	1.053	3.299	1.684	5.834	3.588	4.286	19.45%

Comparing with the results in Table 4.13 for Model II (without the real life constraints), the utilities decrease for three accounts and two increase, but all changes are small, which are caused by both risks and real life constraints.

4.3.4 Numerical Analysis

The outcomes of the numerical tests are presented in previous tables. Based on the preliminary results from Section 4.3.3, we design a series of tests for the numerical analysis.

- *Efficient Frontiers*

To start with the basic and classical demonstration of trade-off between risk and return in the portfolio, we draw the graphs of efficient frontiers for both Model I (risk measure: variance) and Model II (risk measure: *CVaR*) under the setting with initial holdings. Figures 4.1 and 4.2 present the graphs of efficient frontiers for Model I and Model II.

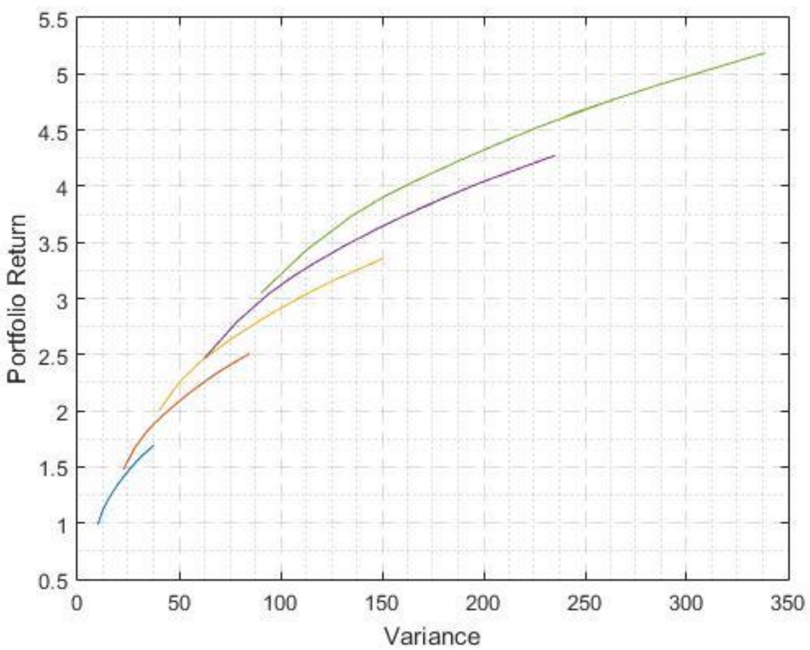

Figure 4.1 Efficient Frontier for five portfolios computed from Model I with variance risk measure

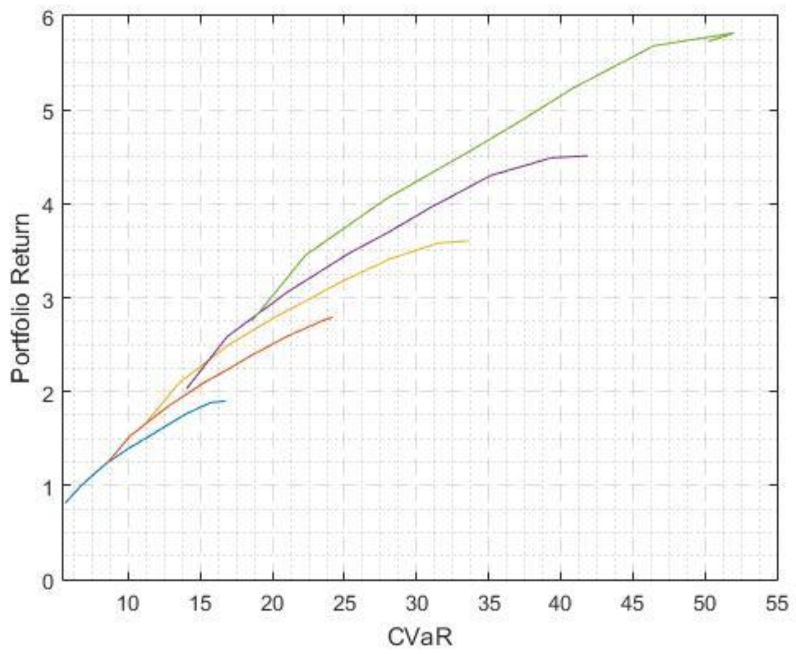

Figure 4.2 Efficient Frontier for five portfolios computed from Model II with CVaR risk measure

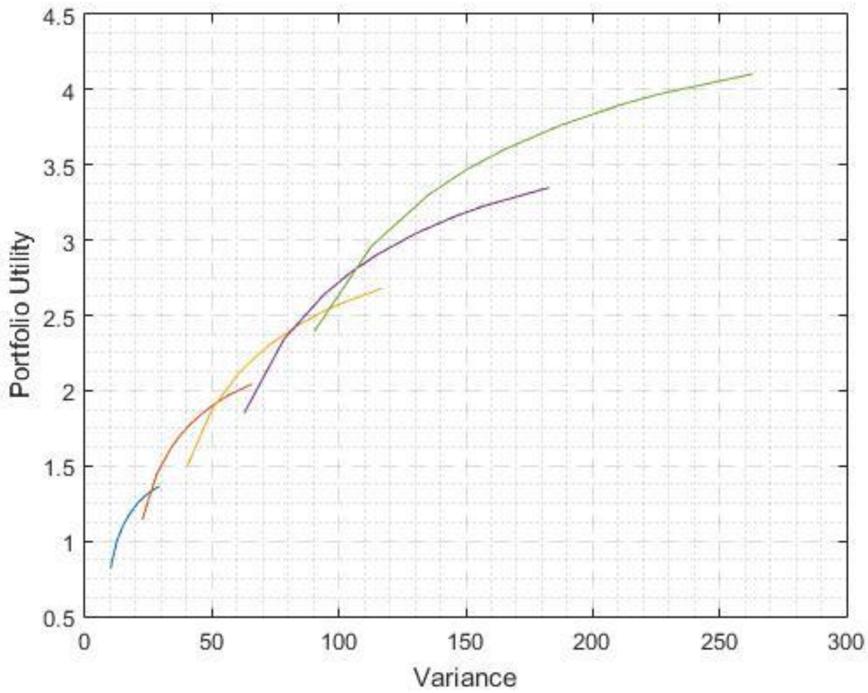

Figure 4.3 Efficient frontier: utility vs variance for five portfolios computed from Model I with variance risk measure

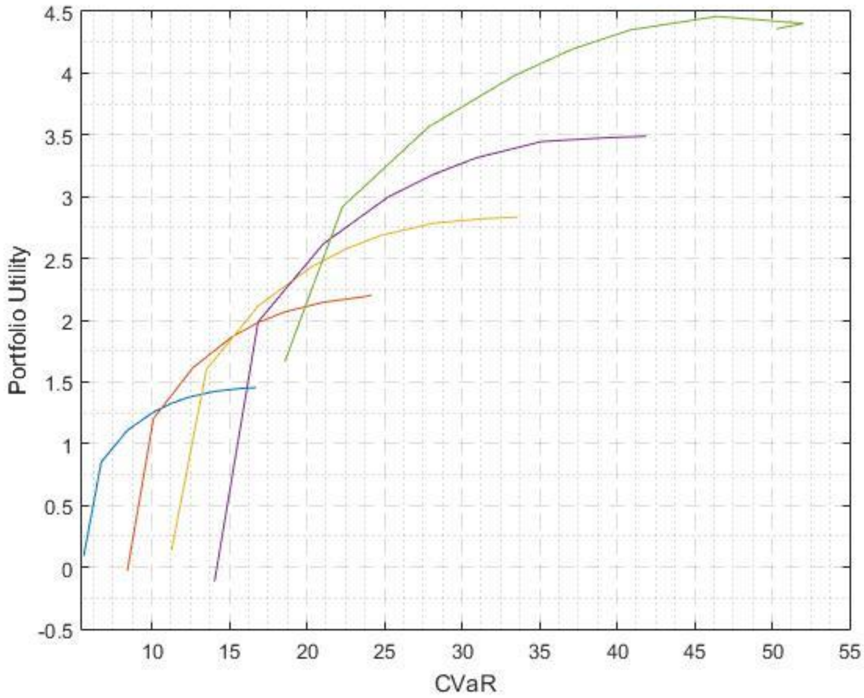

Figure 4.4 Efficient frontier: utility vs CVaR for five portfolios computed from Model II with CVaR risk measure

There are five efficient frontiers in each of the two figures, each line represents one account. The line is the optimal combination of risk and return provided by the optimal solution from GAMS. Each point on the efficient frontier represents an optimal portfolio position that maximizes the return for the given level of risk. The efficient frontier is curved because of a diminishing marginal return to risk. Each minor increase of risk in the portfolio gains a smaller and smaller amount of return.

The difference between Figures 4.3, 4.4 and Figures 4.1, 4.2 is that 4.3 and 4.4 present the frontiers of impact costs adjusted utility vs risks (variance for Figure 4.3 and *CVaR* for Figure 4.4).

We choose Model I and Model II out of the four models for efficient frontier plotting because of the relative simplicity in model structure, such as number of decision variables and constraints, etc.

- *Improvement Rate*

One significant measure of performance of the models is the improvement rate computed in the last step, which is also the value of the objective function $\max_{x,\tau,z}\{f(u_1(x_1)-\tau_1,u_2(x_2)-\tau_2,...,u_n(x_n)-\tau_n)\}$. This objective function is the maxmin function, with function $f(U_1,U_2,...,U_n)=\min\{\frac{U_i-U_i^{IND}}{U_i^{IND}}\}$. We can also call the improvement rate as the relative increase. The expression $\frac{U_i-U_i^{IND}}{U_i^{IND}}$ is what we call the relative increase in utility U_i of Step5 compared with utility U_i^{IND} under independent framework. The independent framework as introduced in Chapter 3 is the case where the accounts do not "cooperate" and are optimized independently, which is Step 1 to Step 4.

Moreover, another important function of the improvement rate is that it is a measure of fairness. We study the improvement rate from the results of numerical tests provided in Section 4.3.3. Using the maxmin objective function, we find out that the values of improvement rate for all accounts are the same. This means that all accounts improve by exactly the same amount in percentage, which is the same as we have expected the maxmin scheme to be. By maximizing the minimum relative increase in utility, the maxmin function demonstrates an attractive feature that optimizes jointly over the trades and split of market impact costs. We consider the maxmin function we use as considerably fair among all accounts with various capital. As discussed in the introduction, the issue of fairness is one of the major considerations in MPO problem, and by utilizing the maxmin function we ensure fairness in our models.

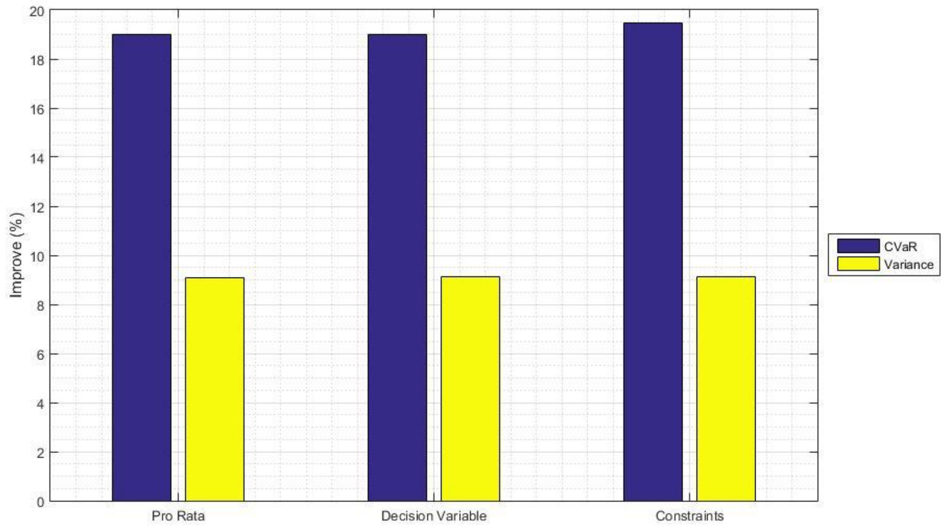

Figure 4.5 Improvement rate (%) in different models using different risk measures

We use two different measures of risk in the model formulations, namely, *CVaR* and variance. To compare the effects they have on the improvement rate, we analysis results from different models. As can be seen from Figure 4.5, we demonstrate the improvement rates of models using *CVaR* or variance as risk measures. For the x-axis tick, pro rata represents Model I and Model II where market impact cost is split proportionally. The label Decision Variable represents Model III, and the label Constraints represents Model IV. Note that for one certain model the values of all other parameters remain the same. It is shown from the bar chart, that in terms of the improvement rate, the models using *CVaR* perform relatively better than the ones using variance for the given same risk preference coefficients κ_i.

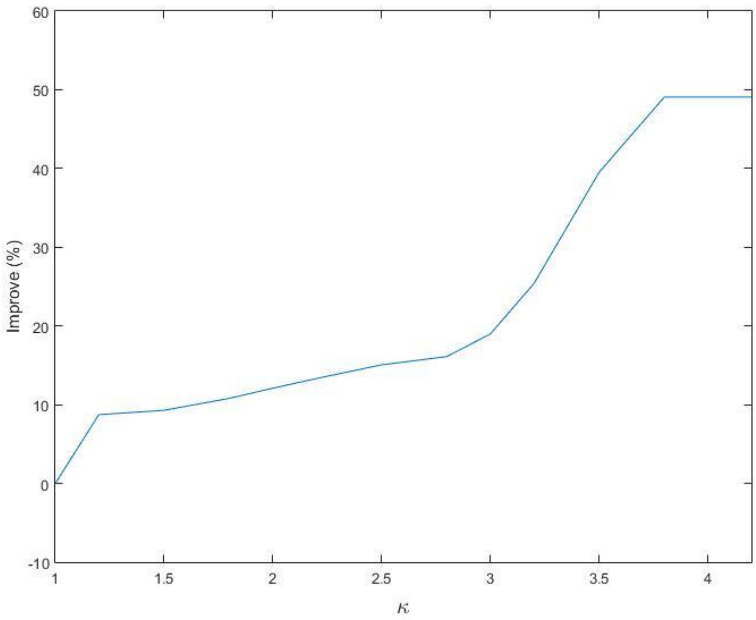

Figure 4.6 Improvement rate of Model II with initial holdings when coefficient κ increase from 1 to 4

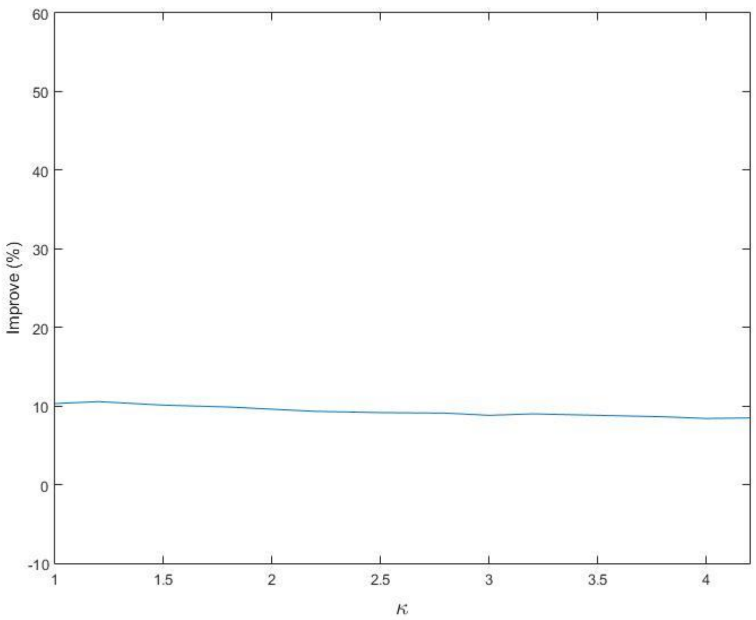

Figure 4.7 Improvement rate of Model I with initial holdings when coefficient κ increase from 1 to 4

Alongside the analysis in Figure 4.5, to further study the improvement rate of the models with different risk measures, we conduct sensitivity analysis on how the changes of the risk coefficients κ_i affect the improvement rate of the accounts. Figure 4.6 and 4.7 show the changes of improvement rate with respect to the increase of κ_i with *CVaR* and variance risk measures, respectively. Figure 4.6 reveals that improvement rate strictly increases when κ_i increase. The growth trend of improvement rate is in an approximate S-curve. Improvement rate increases gradually in the interval of $1 \le \kappa_i \le 3$, and drastically in $3 \le \kappa_i$. Figure 4.7 carries information that the increase of κ_i in the interval of [1, 4] does not affect the improvement rate in a positively attracting way.

From the analysis we conducted on this problem, we draw the conclusion that Model II performs better than Model I in terms of improvement rate, when the coefficients κ_i change in the interval of [1, 4]. Note that such conclusion does not deny the ability of variance as risk measure when it comes to the problem of assisting the model to achieve better return. We can only say that under the specific situation where the investment advisors or the clients lay much stress on the increase of the improvement rate, *CVaR* provides better outcome than variance does. And we also argue that, because improvement rate is an important performance measure under the multiportfolio framework, *CVaR* has an advantage over variance evidenced by this test.

- *Returns and market impact costs under independent optimization and jointly optimization framework*

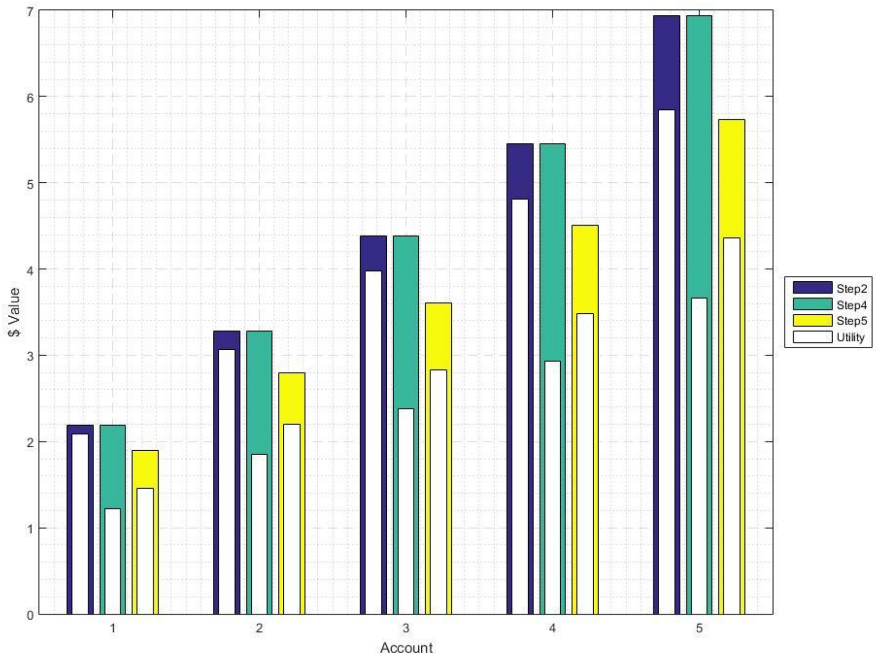

Figure 4.8 Changes of Return and Utility from Step 2 to Step5, taking Model II with initial holding as example

Figure 4.8 reveals how return and utility change from Step 2 to Step 5. Recall that Step 2 acts in an independent optimization framework, under which utilities of the five accounts are been optimized isolated subject to risk constraints. The utilities are the returns of each account less its corresponding market impact costs. As previously mentioned, the formulation of market impact costs charged to each account in Step 2 do not consider impacts from the trading by any other accounts. Due to such unrealistic overlook on interaction of all the accounts, market impact costs charged to each account are relatively low. This phenomenon is reflected by the bar chart. Step 4 computes the utilities by same return of each account from Step 2 subtracts the proportionally split market impact costs charged to the account. Step 5 is under the joint optimization framework, the return generated for each account is relatively less than that from previous two steps. But utilities achieved in Step 5 is relatively more than that in Step 4, which means that the joint optimization framework manages to incur considerably less market impact cost than the independent Step 4. This outcome is consistent with numerical tests results from all the four models, and is exactly the way we expected how the returns, utilities and market impact costs incurred would change. The pattern of the changes in returns, utilities and costs across this three steps conforms to the one we mentioned in previous part of the thesis.

- *Utility and market impact costs allocation approach*

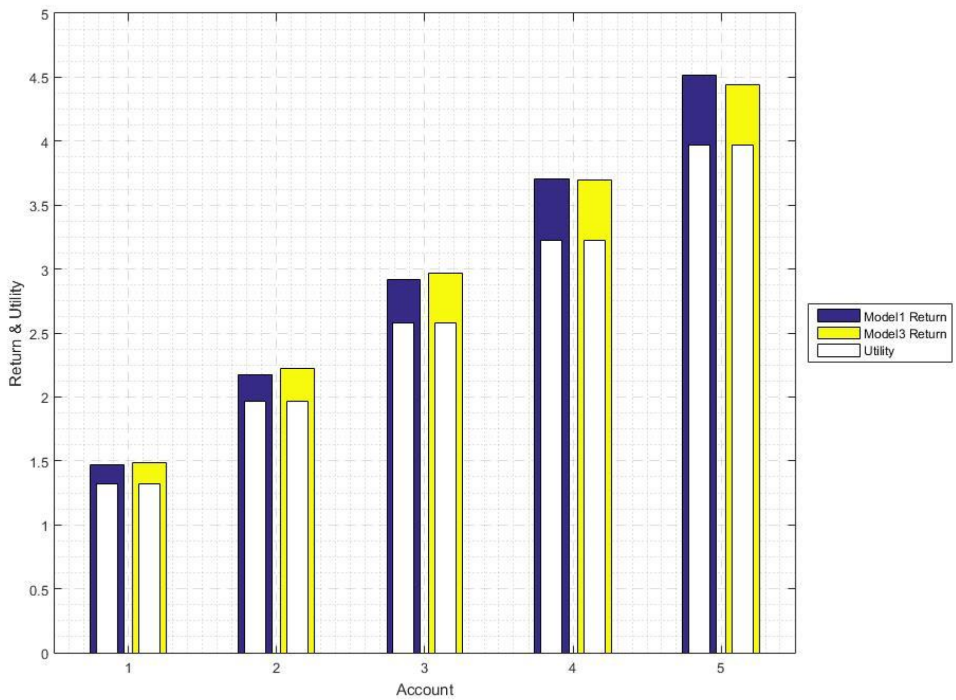

Figure 4.9 Comparisons of return and utility in Model I and Model III using Variance risk measures with initial holdings

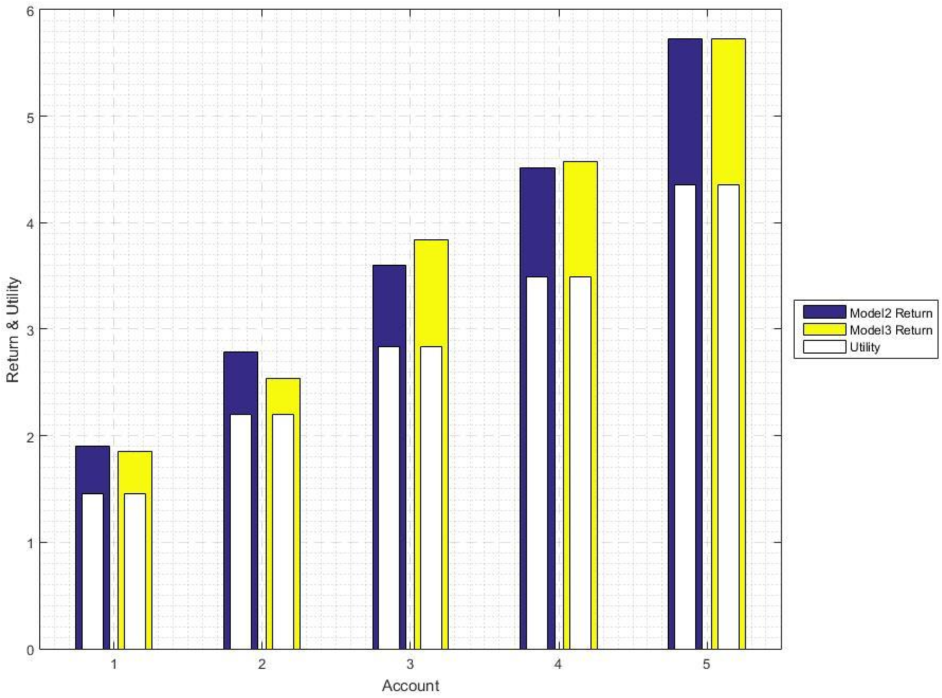

Figure 4.10 Comparisons of return and utility in Model II and Model III using Variance risk measures with initial holdings

To address the allocation of market impact costs across all five accounts, in our thesis we present two approaches, namely, the *pro rata* scheme and decision variable scheme. Figures 4.9 and 4.10 shows the outcomes of return and utility by two different schemes using different risk measures. Figure 4.9 is a comparison of Model I and Model III, both with random initial holdings and variance risk measure. Figure 4.10 is a comparison of Model II and Model III, both with random initial holdings and *CVaR* risk measure. From the numerical tests we conducted, come a result of same utilities in two models using different market impact costs allocation approaches.

- *Results of return and risks under different numbers of scenarios*

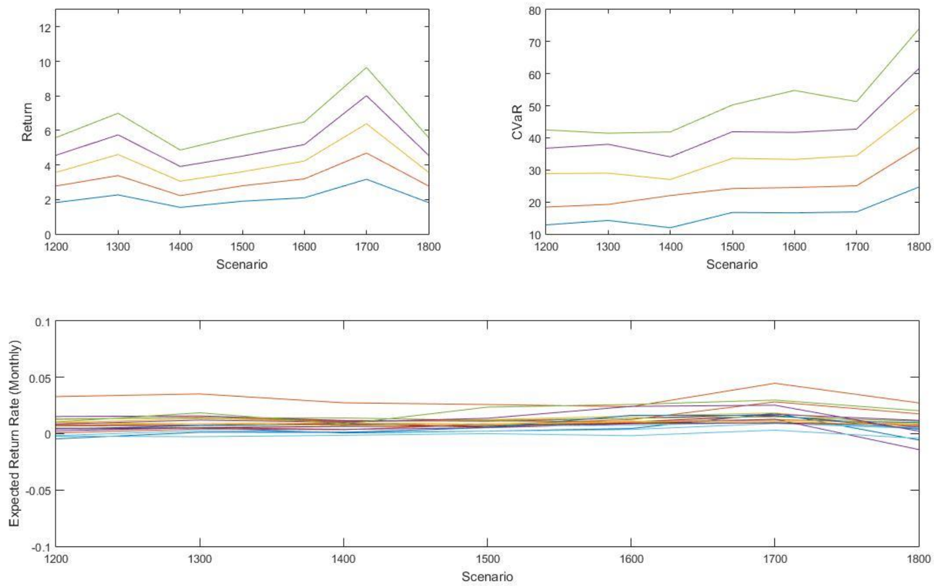

Figure 4.11 Returns (4.11a), risks (4.11b) of Model II with random initial holdings for all 5 accounts across scenarios. Historical expected monthly return (4.11c) across scenarios

To do the analysis with different scenario numbers, we take Model II with random initial holdings input as an example. Number of scenarios changes from 1200 to 1800, and the line graphs in Figure 4.11 show how return and risk of all the accounts vary according to number of scenarios. The subplot below shows how expected monthly return rates from historical data for the twenty stocks change with number of scenarios from 1200 to 1800. Values of return are typically high in number of scenarios 1300 and 1700, and low in number of scenarios 1800. This has the same trend to the changes in monthly expected returns, where there're considerable rise in the historical data in scenario number 1300 and 1700, and a notable fall in 1800. The changes in values of *CVaR* for the five accounts run counter to the changes in expected monthly return rates.

- *Results of return, utility and improvement rate under different value of market impact cost coefficients* θ_j

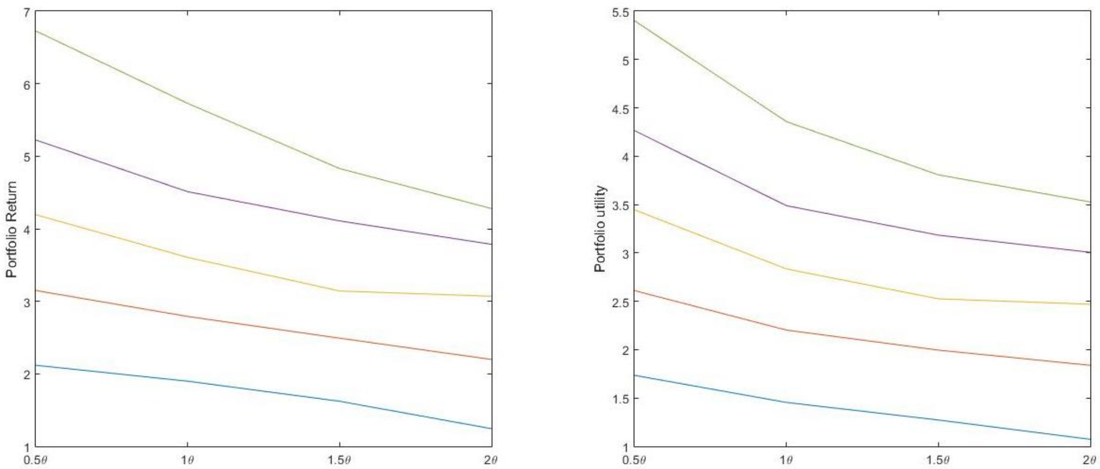

Figure 4.12 Portfolio return and utility of Model II with CVaR risk measure when market impact cost coefficient increase from 0.5θ to 2θ

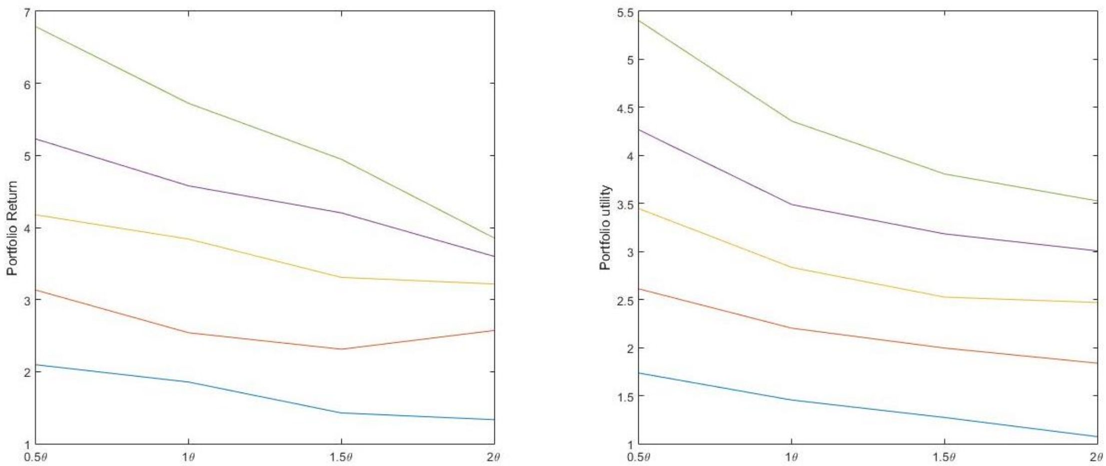

Figure 4.13 Portfolio return and utility of Model III with CVaR risk measure when market impact cost coefficient increase from 0.5θ to 2θ

As declared in Section 4.3.1, in numerical tests the value of market impact cost coefficient $\theta_j = 0.0000314$. To analyse the influence of the coefficient θ_j on portfolio return and utility, numerical analysis is provided. Figures 4.12 and 4.13 show that, in both Model II and Model III, portfolio return and utility for all five accounts under joint optimization framework decrease with the increase of cost coefficient θ_j. Same numerical analysis is performed on Model I and Model III, and the result shows the same decreasing trend of return and utility when cost coefficient θ_j increases. As space is limited, here we only provide the line graph of models under *CVaR* as examples.

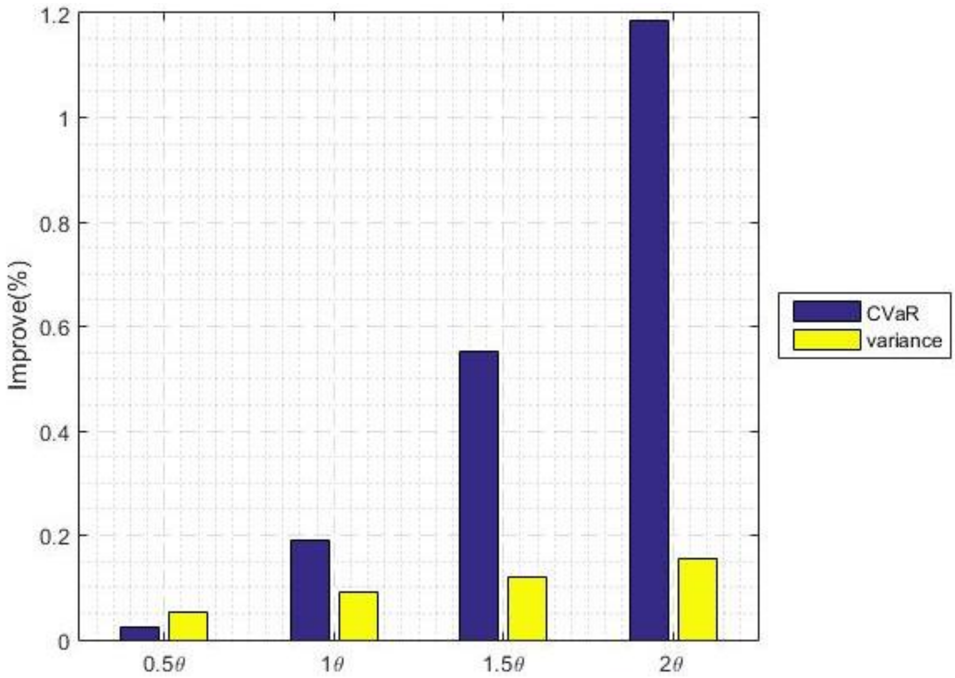

Figure 4.14 Improvement rate (%) for Model I and Model II when market impact cost coefficient increase from 0.5θ to 2θ

Figure 4.14 illustrates that the improvement rate for both Model I and Model II increase when market impact cost coefficient θ_j increases. In terms of the improvement rate, Model II with *CVaR* risk measure performs better than Model I with variance risk measure, except for data point 0.5θ. Improvement rate under *CVaR* risk measure shows a drastically increasing trend. Another set of numerical analysis of improvement rate under risk measure *CVaR* and variance is conducted using Model III, and the increasing trend is the same with Figure 4.14.

Chapter 5 Conclusions and Future Work

5.1 Conclusions

The thesis is set out to study the modelling of risk and allocation of trading incurred costs during the portfolio optimization under the multiportfolio framework. Instead of researching into the classical Markowitz Mean Variance Optimization problem under single portfolio framework, we see the optimization problem from an angle of multiportfolio that is more suitable to the practice of financial firms and investment advisors managing multiple investment accounts simultaneously.

From the vast amount of literature searching and reviewing we conducted in the portfolio optimization related area, we can see that portfolio optimization problem has been a hot topic for a long period of time. But the area of multiportfolio optimization problem has not been regarded with enough academic attention that the topic deserves. To address the portfolio optimization problem under a multiportfolio framework, one must answer the question of how to allocate the market impact costs incurred during the trading. Another old and permanent question of the portfolio optimization problem is how risk is measured. In our thesis, considering the increasingly important role of *CVaR* in regulatory requirements from Basel III (2012) and for reason of its advanced mathematical feature as a risk measure, we propose a novel combination of the risk measure of *CVaR* with multiportfolio optimization problem.

Our thesis focuses on the allocation of the market impact costs and portfolio risk measurements under the multiportfolio optimization framework. To address the problems, a five-step optimization scheme is proposed in the thesis. Following the five-step optimization scheme, we propose four different models. Model I uses variance as risk measure and the *pro rata* fashion to split the incurred market impact costs. Model II uses *CVaR* risk measure and also the *pro rata* fashion to split the market impact costs. Model III focuses on the allocation of market impact costs and regards the split of costs as decision variables, *CVaR* and variance are used separately as two different formulations of this model. In Model IV, we introduce real life portfolio trading constraints such as cardinality constraints and holding constraints. Linear transaction costs are introduced in Model IV as well.

The numerical studies are designed and conducted using the commercial optimization software GAMS. Different from many of the existing literatures using simulated data, we use historical stock data from the NYSE. The four models range from LP and NLP to MILP and MINLP, and we utilize the GAMS solvers CONOPT, CPLEX, and SBB. Two cases of numerical tests are conducted, one with zero initial holdings and the other case with random generated initial holdings. Based on the results from the tests, we design a series of numerical analysis to demonstrate the performance of our models. Although the types of the models make the problem hard to solve, our program is sufficient enough to provide optimal solutions for the problem.

Through the numerical study and analysis with the real stock market data, the following observations and conclusions can be made:

- The proposed five-step frame and models for multiportfolio optimization problem are effective, where two unique features of the problem, market impact costs and fairness, are addressed. Our numerical results show that the joint optimization framework can manage to incur considerably less market impact cost than the independent decision, and therefore increase the utilities of all accounts significantly.

- Comparing with the risk measure of variance, MPO with *CVaR* has the better performance in terms of improvement rate from independent optimization for each account when the given risk preference coefficients are the same for both variance and *CVaR*.
- Our numerical results show that both *pro rata* and decision variable approaches to split market impact costs work well and the resulted utilities are the same for the both models though the returns and impact costs from the two models are different.
- The results from the model with considering extra real life constraints show that the utilities can keep the high level as without those constraints if the customers can take higher risk.

5.2 Contributions

While all existing literatures in the area of multiportfolio optimization are not concerned with the choice of risk measure and use the traditional risk measure variance, our thesis proposes the novel and unique combination of the risk measure of *CVaR* with the MPO problem. In details, the contributions of this thesis include the follows.

- To the best of our knowledge, this is the first work to use *CVaR* as the risk measure in MPO, and the optimization models that combine MPO with *CVaR* are proposed, while *CVaR* is suggested by Basel committee in 2012 to use for market risk management. This also is the first academic research to focus on how risk is measured under the multiportfolio framework. We also build the MPO model using the classical risk measure variance, and provide comparisons for the two risk measures.

- We propose a five-step multiportfolio optimization scheme, and build four models following the scheme. MILP and MINLP models are proposed to address market impact costs, fairness and other factors, which are not seen in the existing researches in the multiportfolio optimization area. Besides, we consider two cases: without initial holding and with initial holding, propose related models, and verify the proposed schemes under the two cases.
- Our thesis uses both *pro rara* and decision variables to allocate the market impact costs incurred during the portfolio optimization process. And numerical results are provided to demonstrate the performance of the two approaches.
- We introduce some real life constraints into MPO, such as transaction costs, cardinality, and holding constraints, which are the first time to be considered in MPO though those have been considered in portfolio optimization.
- We conduct numerical study and analysis by using the real historical data from NYSE to test the proposed models and approaches. Comparisons of two risk measures and allocation methods of impact costs are reported for MPO environment.

5.3 Future Works

Although our thesis provides a complete framework of our research ideas, to further develop the research in the future, we present the following recommendation for future studies:

- Extending the current single-period multiportfolio framework to a multi-period multiportfolio framework is a possible extension of this study. Since multiple period portfolio optimization problem is a topic that draws vast research attention, the combination of multi-period and multiportfolio framework deserves further studies.

- Another further work may develop different models or more accurate ways to measure the transaction costs incurred during the optimization process, especially under MPO framework. Further studies may focus on the modelling of both the implicit and explicit part of the transaction costs. For the explicit part, we recommend studying different formulations of the transaction costs, and taking both fixed, linear and nonlinear transaction costs into consideration. For the implicit part, such as market impact costs, we recommend further studying into the market pricing impact model and experiment with various models of the market impact costs.
- From an application point of view, our five-step optimization scheme and models can be developed into an integrated Decision Support System, to help financial firms and advisors in decision makings of multiportfolio optimization problem.

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