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Abstract

This paper proposes a distance-based consensus model for fuzzy preference relations where the weights of fuzzy preference relations are automatically determined. Two indices, an individual to group consensus index (ICI) and a group consensus index (GCI), are introduced. An iterative consensus reaching algorithm is presented and the process terminates until both the ICI and GCI are controlled within predefined thresholds. The model and algorithm are then extended to handle multiplicative preference relations. Finally, two examples are illustrated and comparative analyses demonstrate the effectiveness of the proposed methods.

Keywords: Group decision-making; consensus; fuzzy preference relations; multiplicative preference relations; distance.

1. Introduction

Group decision making (GDM) is concerned with deriving a solution from a group of independent decision-makers' (DMs') heterogeneous preferences over a set of alternatives. Before the final choice is identified, two processes are usually carried out: (1) a consensus process and (2) a selection process. The first process addresses how to obtain a maximum degree of consensus or agreement among the DMs over the alternative set, while the second process handles the derivation of the alternative set based on the DMs' individual judgment on alternatives [24].

Numerous approaches have been put forward for consensus measures based on different types of preference relations, including consensus models for ordinal preference [14-16,19], linguistic preference relations [3,4,7-10,17,26-28,58], multi-attribute GDM problems [5,20,21,37,50,59], intuitionistic multiplicative preference relations [29], and other preference relations [1,24,35,38].

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The consensus reaching process has been widely studied for multiplicative preference relations (MPRs). Van den Honert [45] proposed a model to represent a consensus-seeking GDM process based on the analytic hierarchy process (AHP) framework, where group preference intensity judgments are expressed as random variables with associated probability distributions. Dong et al. [18] developed AHP consensus models by using a row geometric mean prioritization method. Wu and Xu [48] presented a consistency and consensus-based model for GDM with MPRs. Gong et al. [22] developed a group consensus deviation degree optimization model for MPRs that minimizes the weighted arithmetic mean of individual consistency deviation degrees. Xu [60] put forward a consensus reaching process for GDM with incomplete MPRs.

For fuzzy preference relations (FPRs), Kacprzyk and Fedrizzi [30] devised a 'soft' measure of consensus. Chiclana et al. [12] furnished a framework for integrating individual consistency into a consensus model. The paradigm consists of two processes: an individual consistency control process and a consensus reaching process. Based on this work, Zhang et al. [67] proposed a set of linear optimization models to address certain consistency issues on FPRs, such as individual consistency construction, consensus modeling and management of incomplete fuzzy preference relations. Herrera-Viedma et al. [23] presented a new consensus model for GDM problems with incomplete fuzzy preference relations. The key feature is to introduce a feedback mechanism for advising DMs to change or complete their preferences so that a solution with high consensus and consistency degrees can be reached. Parreiras et al. [36] proposed a dynamical consensus scheme based on a nonreciprocal fuzzy preference relation modeling. Wu and Xu [46] developed a consistency consensus based decision support model for GDM. Recently, Xu and Cai [62] put forth a number of goal programming and quadratic programming models to maximize group consensus. The main purpose is to determine importance weights for FPRs and MPRs. However, as pointed out in Section 2, a significant drawback exists for their quadratic programming models as the derived weight is always the same for each expert. Furthermore, for existing consensus models for improving consensus indices, it is often the case that the final improved preference relations significantly differ from the DMs' original judgment information, as testified by examples in [1,3-10,12,17,18,20-23,26-28,46-50,59,60,62,67,68]. It is the authors' belief that GDM should utilize the DMs' opinions on the alternatives to find a solution. If DMs' opinions are significantly distorted, the derived solution is likely questionable. In order to obtain a reliable solution, the decision model should retain the DMs' opinions as much as possible. To address these deficiencies, a new consensus measure should be designed to make use of group judgments.

This paper first puts forward a distance-based consensus model for FPRs to derive each DM's individual weight vector, then an aggregation operator is developed to obtain a collective FPR. An individual to group consensus index (*ICI*) and a group consensus index (*GCI*) are subsequently introduced, followed by an iterative algorithm for consensus reaching with a stoppage condition when both *ICI* and *GCI* are lower than predefined thresholds. The model and algorithm are then extended to MPRs.

The remainder of this paper is organized as follows. Section 2 briefly reviews group consensus models introduced by Xu and Cai [62] for FPRs with comments on their drawbacks. Section 3 develops a distance-based model to determine DMs' weights for GDM with FPRs, and puts forward an algorithm for the consensus reaching process. Section 4 extends the model and algorithm to solve consensus problems with MPRs. In Section 5, two illustrative examples are developed and the results are compared with those obtained with existing approaches. Concluding remarks are furnished in Section 6.

2. A review of group consensus based on fuzzy preference relations

For a GDM problem, let $X=\{x_1,x_2,...,x_n\}$ ($n\geq 2$) be a finite set of alternatives and $E=\{e_1,e_2,...,e_m\}$ ($m\geq 2$) be a finite set of DMs. In a multi-criteria decision making problem, a DM e_k often compares each pair of alternatives in X and provides his/her preference degree $p_{ij,k}$ of alternative x_i over x_j on a 0-1 scale, where $0\leq p_{ij,k}\leq 1$, $p_{ij,k}=0.5$ denotes e_k's indifference between x_i and x_j, $p_{ij,k}=1$ denotes that x_i is definitely preferred to x_j by e_k, and $0.5<p_{ijk}<1$ (or $0<p_{jik}<0.5$) denotes that x_i is preferred to x_j by e_k with a varying degree of likelihood. All preference values $p_{ij,k}$ ($i,j=1,2,...,n$) provided by DM e_k are denoted as an FPR $P_k=(p_{ij,k})_{n\times n}$ [11,25,31,33,40-44,46,51-57]

$$0\leq p_{ij,k}\leq 1,\ p_{ii,k}=0.5,\ \ p_{ij,k}+p_{ji,k}=1,\ i,j=1,2,...,n \tag{1}$$

In a GDM problem, let $w=(w_1,w_2,...,w_m)^T$ be the unknown weight vector for FPRs $P_k=(p_{ij,k})_{n\times n}$ ($k=1,2,...,m$), where

$$\sum_{k=1}^{m}w_k=1,\ w_k\geq 0,\ k=1,2,...,m \tag{2}$$

To obtain a collective judgment for the group, Xu and Cai [62] employed the Weighted Arithmetic Averaging (WAA) operator:

$$p_{ij}=\sum_{k=1}^{m}w_k p_{ij,k}\ , \qquad i,j=1,2,...,n \tag{3}$$

to aggregate individual FPRs $P_k=(p_{ij,k})_{n\times n}$ ($k=1,2,...,m$) into a collective preference relation $P=(p_{ij})_{n\times n}$. It can be easily shown that P satisfies condition (1), and is thus also an FPR.

Clearly, a key issue in applying the WAA operator is to determine the weight vector w. If an individual FPR P_k is consistent with the collective FPR P, then $P_k = P$, i.e., $p_{ij,k} = p_{ij}$, for all $i, j = 1,2,...,n$. Using (3), we have

$$p_{ij,k} = \sum_{l=1}^{m} w_l p_{ij,l} \text{, for all } i, j = 1,2,...,n \tag{4}$$

However, generally speaking, Eq.(4) does not always hold. Let

$$\varepsilon_{ij,k} = \left| p_{ij,k} - \sum_{l=1}^{m} w_l p_{ij,l} \right|, \text{ for all } i, j = 1,2,...,n,\ k = 1,2,...,m \tag{5}$$

It follows from (1) that (5) is equivalent to the following:

$$\varepsilon_{ij,k} = \left| p_{ij,k} - \sum_{l=1}^{m} w_l p_{ij,l} \right|, \text{ for all } i = 1,2,...,n-1,\ j = i+1,...,n,\ k = 1,2,...,m \tag{6}$$

where $\varepsilon_{ij,k}$ ($i = 1,2,...,n-1$, $j = i+1,...,n$; $k = 1,2,...,m$) are the absolute deviation between individual and collective FPRs. To reach a consensus among the group, these values should be kept as small as possible. Thus, Xu and Cai [62] constructed the following quadratic programming model:

(M-1) $$\min\ F_1 = \sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon_{ij,k}^2 = \sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(p_{ij,k} - \sum_{l=1}^{m} w_l p_{ij,l} \right)^2$$

$$\text{s.t. } \sum_{k=1}^{m} w_k = 1,\ w_k \geq 0,\ k = 1,2,...,m$$

The solution to this model yields a weight vector for all DMs e_k ($k = 1,2,...,m$) and can be derived as follows [62]:

$$w = \frac{D^{-1}e(1 - e^T D^{-1} p)}{e^T D^{-1} e} + D^{-1} p \tag{7}$$

where

$$p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} p_{ij,k} p_{ij,1}, \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} p_{ij,k} p_{ij,2}, ..., \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m} p_{ij,k} p_{ij,m} \right)^T,\quad e = (1,1,...,1)^T \tag{8}$$

and

$$D = \begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,1}^2 & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,1} p_{ij,2} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,1} p_{ij,m} \\ \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,1} p_{ij,2} & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,2}^2 & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,2} p_{ij,m} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,1} p_{ij,m} & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,2} p_{ij,m} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n} m p_{ij,m}^2 \end{pmatrix}_{m \times m} \tag{9}$$

Xu and Cai [62] employed the aforesaid model (Eqs.(7)-(9)) to derive an optimal weight vector $w=(w_1,w_2,...,w_m)^T$ for the FPRs $P_k=(p_{ij,k})_{n\times n}$ ($k=1,2,...,m$).

Subsequently, by using (3), Xu and Cai [62] obtained a collective FPR P. In addition, based on Eq. (6) and the optimal weight vector w, Xu and Cai [62] calculated the deviation (referred to as an individual to group consensus index ICI in this paper) between the individual FPR P_k and the collective FPR P by

$$ICI(P_k)=d(P_k,P)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\varepsilon_{ij,k}=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|p_{ij,k}-\sum_{l=1}^{m}w_l p_{ij,l}\right| \quad (10)$$

Accordingly, the weighted sum of all the deviations $d(P_k,P)$ ($k=1,2,...,m$) (referred to as a group consensus index GCI hereafter) can be defined as

$$GCI=\Delta_1=\sum_{k=1}^{m}w_k d(P_k,P) \quad (11)$$

From Eqs. (10) and (11), one can see that if $d(P_k,P)=0$, then the individual FPR P_k is consistent with the collective fuzzy preference relation P. If $\Delta_1=0$, then the group reaches complete consensus. In addition, Xu and Cai [62] assumed that if $\Delta_1\le\lambda_1$, then the group reaches an acceptable level of consensus, where λ_1 is a pre-specified acceptable threshold of group consensus.

Xu and Cai [62] then developed algorithms for GDM with FPRs based on the quadratic programming model (M-1).

In the following, a further analysis is furnished for the model (M-1).

Theorem 1. For FPRs $P_k=(p_{ij,k})_{n\times n}$ ($k=1,2,...,m$), the optimal solution to (M-1) model is

$$w=(1/m,1/m,....,1/m)^T \quad (12)$$

Proof. From Eqs. (8) and (9), the relationship between p and D can be expressed as follows:

$$p=\frac{De}{m} \quad (13)$$

Plugging (13) into (7), one has

$$w=\frac{D^{-1}e(1-e^TD^{-1}p)}{e^TD^{-1}e}+D^{-1}p$$

$$=\frac{D^{-1}e(1-\frac{e^TD^{-1}De}{m})}{e^TD^{-1}e}+\frac{D^{-1}De}{m}$$

$$=\frac{D^{-1}e(1-\frac{e^Te}{m})}{e^TD^{-1}e}+\frac{e}{m}$$

$$= \frac{D^{-1}e(1-1)}{e^T D^{-1} e} + \frac{e}{m}$$

$$= \begin{pmatrix} \frac{1}{m} \\ \frac{1}{m} \\ \vdots \\ \frac{1}{m} \end{pmatrix} \tag{14}$$

This result indicates that (M-1) always yields an equal weight of $1/m$ for each DM as long as there does not exist complete consensus among the group. This theorem also explains why the numerical examples in [61,62] always give an equal weight of $1/m$ for all DMs.

The aforesaid analysis reveals the following limitations for the algorithms in Xu and Cai [62]:

(1) Xu and Cai [62] applied the quadratic programming model (M-1) to determine an optimal weight vector $w^{(t)} = (w_1^{(t)}, w_2^{(t)}, ..., w_m^{(t)})^T$. Theorem 1 shows that the optimal weight vector is always $w^{(t)} = (1/m, 1/m, ..., 1/m)^T$. The implication is that all the DMs' fuzzy preference relations play an equal role in the aggregated fuzzy preference relations. The unexpected constant weight vector resulting from (M-1) does not serve the original modeling idea of determining the weight vector w in the WAA operator [62] and makes this model redundant.

(2) As per Xu and Cai's Algorithm 1, if the group does not reach an acceptable level of consensus, some DMs need to reassess their preferences over the alternatives. As Xu and Cai [62] pointed out, this trial-and-error process can be time-consuming, or DMs are unable or unwilling to reevaluate the alternatives. Algorithm 2 is then developed to address these cases. New FPRs $P_k^{(t+1)}$ ($k = 1, 2, ..., m$) are obtained by the following equation automatically without the DMs' direct intervention (except for the parameter η) at each iteration.:

$$p_{ij,k}^{(t+1)} = \eta p_{ij,k}^{(t)} + (1-\eta) p_{ij}^{(t)},\ i, j = 1, 2, ..., n,\ k = 1, 2, ..., m,\ 0 < \eta < 1 \tag{15}$$

It is apparent that the revised FPRs $P_k^{(t+1)}$ ($k = 1, 2, ..., m$) are different from the original ones P_k ($k = 1, 2, ..., m$), all elements $p_{ij,k}^{(t+1)}$ (except for diagonal elements $p_{ii,k}^{(t+1)}$, which are always equal to 0.5) are modified. These changes inevitably distort the DMs' original judgment as reflected in their fuzzy preference values (This distortion is illustrated in the example in Xu and Cai [62]). In addition, for the key parameter η in Eq.(15)), no guideline is furnished by Xu and Cai [62] about how to set its value except for its range [0,1].

(3) Xu and Cai [62] employed Eq.(11) to measure the overall deviation, which is then used to measure the group consensus degree. Without explicitly considering

individual deviations, this treatment may lead to undesirable situations. For instance, if some DMs' deviations (determined by Eq.(10)) are negligible, say $d(P_k, P) = 0$ ($k = 1, 2, ..., l$, $l < m$), but remaining DMs' deviations are very high as reflected in large values of $d(P_k, P)$ ($k = l+1, ..., m$). In this case, as long as the weighted sum of all the deviations $d(P_k, P)$ is small enough such that $\Delta_1 < \lambda_1$, Xu and Cai [62] still considered the group reaches an acceptable consensus. However, those large deviation variables $d(P_k, P)$ ($k = l+1, ..., m$) indicate that some DMs $e_{l+1}, ..., e_m$ still hold preferences far away from the group consensus. Therefore, it is reasonable to impose a threshold for individual deviations as well.

To address the aforesaid deficiencies, new models and algorithms will be developed below for reaching acceptable levels of consensus in GDM with FPRs.

3. Distance-based group consensus models for fuzzy preference relations

To reach a group consensus, the approach in Xu and Cai's [62] adjusts FPRs P_k to make them as close to the collective FPR P as possible. Instead of modifying decision input, the proposed method takes a different angle and examines decision output. It is highly likely that individual FPRs are largely dispersed if their weights are not considered. Therefore, the weighs should be incorporated into each FPR. In order to achieve maximum consensus, the weighted FPRs should come closer to each other. This is the basic principle for generating an aggregated decision result. Built upon this idea, a distance-based least-square aggregation optimization model is proposed to integrate different DMs' decision input.

The general modeling idea is to minimize the sum of the squared distance from one decision input to another, thereby achieving maximum agreement. Define the squared distance between each pair of individual FPRs (P_k, P_l) as

$$d^2(w_k P_k, w_l P_l) = \left(\sqrt{(w_k P_k - w_l P_l)^2}\right)^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}(w_k p_{ij,k} - w_l p_{ij,l})^2 \tag{16}$$

Based on this definition, the following optimization model is constructed to minimize the sum of squared distances between all pairs of weighted fuzzy preference judgments:

$$\textbf{(M-2)} \quad \min\ J_1 = \sum_{k=1}^{m}\sum_{l=1, l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_k p_{ij,k} - w_l p_{ij,l})^2 \tag{17}$$

$$\text{s.t.} \quad \sum_{l=1}^{m} w_l = 1 \tag{18}$$

$$w_l \geq 0,\ l = 1, 2, ..., m \tag{19}$$

Theorem 2. Model (M-2) is equivalent to (M-3) below in a matrix form

$$\textbf{(M-3)} \quad \min \quad J_1 = w^T G w \tag{20}$$

$$\text{s.t.} \quad e^T w = 1 \tag{21}$$

$$w \geq 0 \tag{22}$$

where $w = (w_1, w_2, ..., w_m)^T$, $e = (1, 1, ..., 1)^T$,

$$G = (g_{kl})_{m\times m} = 2\begin{bmatrix} (m-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,1}^2\right) & -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,1}p_{ij,2} & \cdots & -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,1}p_{ij,m} \\ -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,2}p_{ij,1} & (m-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,2}^2\right) & \cdots & -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,2}p_{ij,m} \\ \cdots & \cdots & \cdots & \cdots \\ -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,m}p_{ij,1} & -\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,m}p_{ij,2} & \cdots & (m-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,m}^2\right) \end{bmatrix} \tag{23}$$

Proof.

$$J_1 = \sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(w_k p_{ij,k} - w_l p_{ij,l})^2$$

$$= \sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(w_k^2 p_{ij,k}^2 + w_l^2 p_{ij,l}^2\right) - 2\sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} w_k w_l p_{ij,k} p_{ij,l}$$

$$= \sum_{k=1}^{m}\left[2(m-1)\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij,k}^2\right]w_k^2 + \sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(-2p_{ij,k}p_{ij,l})w_k w_l \tag{24}$$

As for J_1 represented by (20), we have

$$J_1 = w^T G w$$

$$= \sum_{k=1}^{m}\sum_{l=1}^{m} g_{kl} w_k w_l$$

$$= \sum_{k=1}^{m} g_{kk} w_k^2 + \sum_{k=1}^{m}\sum_{l=1,l\neq k}^{m} g_{kl} w_k w_l \tag{25}$$

Comparing (24) and (25), we obtain (23).

Theorem 3. For the model (M-3), if for any i, j, k and l, there exists at least one inequality $p_{ij,k} \neq p_{ij,l}$, then matrix G determined by (23) is positive definite and, hence, non-singular and invertible.

Proof. Obviously, $J_1 = w^T G w \geq 0$. Now, we prove that $J_1 \neq 0$ if there exists at least one inequality $p_{ij,k} \neq p_{ij,l}$.

Assume that there exists a weight vector w, for all i, j, k and l, such that $J_1 = 0$. Then,

$$w_k p_{ij,k} = w_l p_{ij,l}, \text{ and } w_k p_{ji,k} = w_l p_{ji,l}$$

thus, by Eq.(1), one can obtain

$$\frac{w_k}{w_l} = \frac{p_{ij,l}}{p_{ij,k}} = \frac{p_{ji,l}}{p_{ji,k}} = \frac{1 - p_{ij,l}}{1 - p_{ij,k}}$$

yielding

$$p_{ij,k} = p_{ij,l}, \text{ for all } i, j, k \text{ and } l$$

This contradicts with the assumption that there exists at least one inequality $p_{ijk} \neq p_{ijl}$. Therefore, $J_1 > 0$ and the symmetry of matrix G and the definition of positive definiteness confirm that G is positive definite, and, hence, nonsingular and invertible, i.e., G^{-1} exists. This completes the proof of Theorem 3.

Remark 1. Theorem 3 shows that G is positive definite as long as not all FPRs are identical. If all DMs' pairwise comparison judgments are the same, a complete consensus is reached and the optimal weight vector to (M-3) is obtained as $(1/m, 1/m, ..., 1/m)^T$. In reality, this complete consensus rarely happens. If it does happen, the consensus building process automatically terminates. In the following, the general case of non-identical FPRs is considered, and it is always assumed that there exits at least one inequality $p_{ij,k} \neq p_{ij,l}$.

Let Ω be the feasible set of (M-3). The following result can be established.

Lemma 1. The convex set Ω of (M-3) is closed, and (M-3) is a convex quadratic program.

Proof. According to the definition of convex set [2], obviously, Ω is a closed convex set. As G is positive definite, J_1 is strictly convex. Since the constraints of (M-3) are linear, (M-3) is a convex quadratic programming. The proof of Lemma 1 is thus completed.

To solve (M-3), the following Lagrangian function is constructed by ignoring the non-negativity constraint (22):

$$L(w, \lambda) = w^T G w + 2\lambda(e^T w - 1) \tag{26}$$

where λ is the Lagrangian multiplier. Let $\partial L / \partial w = 0$ and $\partial L / \partial \lambda = 0$, then

$$Gw + \lambda e = 0 \tag{27}$$

$$e^T w = 1 \tag{28}$$

By Theorem 3, matrix G is invertible. Thus, solutions to (27) and (28) are given as

$$w^* = \frac{G^{-1} e}{e^T G^{-1} e} \tag{29}$$

$$\lambda^* = -\frac{1}{e^T G^{-1} e} \tag{30}$$

Lemma 2 [32]**.** Let $F=(f_{ij})_{m\times m}$ be an $m\times m$ symmetric matrix such that $f_{ij}\le 0$ for $i\ne j$ and $f_{ii}>0$. Then, $F^{-1}\ge[0]_{m\times m}$ (i.e., F^{-1} is a nonnegative matrix) if and only if F is positive definite.

Theorem 4. For model (M-3), if for any i,j,k and l, there exists at least one inequality $p_{ij,k}\ne p_{ij,l}$, then $G^{-1}\ge(0)_{m\times m}$, i.e., G^{-1} is a nonnegative matrix.

Proof. According to Theorem 3, G is a positive definite matrix such that $g_{kl}\le 0$ ($k\ne l$) and $g_{kk}>0$. By Lemma 2, it follows that $G^{-1}\ge(0)_{m\times m}$, i.e., G^{-1} is a nonnegative matrix.

As per Theorems 3 and 4, G is a positive definite and non-singular matrix, and G^{-1} is nonnegative. Therefore, $w^*\ge 0$, implying that the weight vector (29) satisfies the non-negativity constraint (22).

Section 2 comments on the limitations of Xu and Cai's methods. To address these issues, an improved method is put forward and its key features are depicted as follows: (1) The proposed method entertains both group consensus and individual consensus degrees as opposed to Xu and Cai's methods where only the group consensus degree (see Eq.(11)) is considered. The purpose is to handle cases where the group consensus degree is satisfactory, but some individual consensus degrees significantly differ from the group consensus. This is accomplished by setting a separate threshold α_1 for the individual consensus degree $d(P_k,P)\le\alpha_1$ in addition to a group consensus level λ_1. (2) The proposed method modifies only each DM's fuzzy preference values that differs the most from the corresponding group preference at each iteration. The conception aims to retain DMs' original preference information. But in Xu and Cai's methods, when the group does not reach an acceptable level of consensus, the adjustment process (by returning the original FPRs to DMs to reevaluate) often results in significantly different FPRs than the original judgments. (3) In contrast to Xu and Cai's methods that always yield the same weight vector for all DMs, the proposed method is able to obtain an optimal weight vector defined by Eq. (29).

The improved consensus process for GDM problems is detailed in Algorithm 1.

Algorithm 1.

Input: $P_k=(p_{ij,k})_{n\times n}$ ($k=1,2,...,m$), the maximum number of iterations t^*, the thresholds α_1, λ_1 for individual and group consensus indices, respectively.

Output: Improved FPRs $\overline{P}_k$ ($k=1,2,...,m$), the iteration step t, individual consensus index $ICI(\overline{P}_k)$ ($k=1,2,...,m$) and group consensus degree GCI.

Step 1. Let $t=0$, $P_k^{(0)}=P_k$ ($k=1,2,...,m$).

Step 2. Apply the quadratic program (M-3) to determine the optimal weight vector $w^{(t)}=(w_1^{(t)},w_2^{(t)},...,w_m^{(t)})^T$ as per Eq. (29) for $P_k^{(t)}=(p_{ij,k}^{(t)})_{n\times n}$ ($k=1,2,...,m$).

Step 3. Utilize the WAA operator Eq. (3) to aggregate individual FPRs $P_k^{(t)}=(p_{ij,k}^{(t)})_{n\times n}$ ($k=1,2,...,m$) into a collective FPR $P^{(t)}=(p_{ij}^{(t)})_{n\times n}$.

Step 4. Calculate individual consensus indices $ICI(P_k^{(t)})=d(P_k^{(t)},P^{(t)})$ ($k=1,2,...,m$) and the group consensus index $\Delta_1(t)$ using Eqs.(10) and (11), respectively. If $\Delta_1(t)\le\lambda_1$ and $ICI(P_k^{(t)})\le\alpha_1$ (for all $k=1,2,...,m$) or $t=t^*$, go to Step 6. Otherwise, find the FPR $P_k^{(t)}$ such that $ICI(P_k^{(t)})>\lambda_1$. Go to Step 5.

Step 5. Find the position of the elements $d_{i_\tau j_\tau,k}^{(t)}$ for DM e_k such that $ICI(P_k^{(t)})>\lambda_1$, where $d_{i_\tau j_\tau,k}^{(t)}=\max_{i,j}\left|p_{ij,k}^{(t)}-p_{ij}^{(t)}\right|$, modify DM e_k's FPR. Let $P_k^{(t+1)}=(p_{ij,k}^{(t+1)})_{n\times n}$, where

$$p_{ij,k}^{(t+1)}=\begin{cases}p_{ij}^{(t)}, & if\ i=i_\tau,\ j=j_\tau\\ p_{ij,k}^{(t)}, & otherwise\end{cases} \tag{31}$$

and $t=t+1$. Then, go to Step 2.

Step 6. Let $\overline{P}_k=P_k^{(t)}$. Output the modified FPRs $\overline{P}_k$ ($k=1,2,...,m$), the individual consensus index $ICI(P_k^{(t)})$ ($k=1,2,...,m$), the group consensus index GCI, and the number of iterations t.

Remark 2. Generally, for the two thresholds α_1 and λ_1, it is sensible to set $\alpha_1>\lambda_1$. Otherwise, if $\alpha_1\le\lambda_1$, and $ICI(P_k)\le\alpha_1\le\lambda_1$, it follows that $GCI=\Delta_1=\sum_{k=1}^{m}w_k ICI(P_k)\le \sum_{k=1}^{m}w_k\alpha_1=\alpha_1\le\lambda_1$. By setting $\alpha_1>\lambda_1$, the individual to group consensus index ($ICI(P_k)$) is allowed to be somewhat larger than the group consensus index (GCI), giving each expert room for deviating from the group judgment. Furthermore, the two thresholds α_1 and λ_1 in the algorithm have to be carefully chosen to avoid an excessive number of iterations. A survey of the literature showed that these parameters are often subjectively determined by the experts in the group or by a super expert [26]. While there is no specific rule to determine the threshold values, they can generally be specified by a trial-and-error process. If the decision problem is urgent and has to be resolved expeditiously, less restrictive values can be adopted, otherwise, more restrictive values can be introduced. The two thresholds thus provide a flexible choice for the group to control the decision process. Once these thresholds are specified, Step 4 furnishes the condition for the expert to adjust his/her opinion as reflected in his/her fuzzy preference relation (i.e., when his/her *ICI* exceeds the specified threshold) and Step 5 gives a specific scheme to make the adjustment. After the expert opinion $P_k^{(t)}$ is modified, the quadratic program (M-3) is reapplied to determine a new optimal weight vector with this updated information. By iteratively updating the expert opinion and weights, the consensus level is gradually increased.

Remark 3. Wu and Xu [46] adopted Eq. (10) to measure the group consensus assuming that a consensus is reached if all DMs' preference relations are sufficiently close to the group preference (deviations are smaller than a given threshold). As commented in Remark 2, this treatment is equivalent to setting $\alpha_1 \le \lambda_1$, and, hence, can be viewed as a special case of the proposed method. On the other hand, Xu and Cai [62] employed Eq. (11) to gauge the consensus level. As long as the weighted sum of group consensus indices for all DMs is less than a given consensus threshold λ, the consensus level is deemed acceptable without considering the individual to group consensus index defined by Eq. (10). This method may treat the consensus level of a group decision situation as acceptable where the majority of the DMs possess fairly close judgments to the group's, but a small number of DMs significantly differ from the group preference judgment. By considering both Eqs. (10) and (11), the proposed method extends the relevant research reported by Wu and Xu [46] and Xu and Cai [62]. In this research, the WAA operator is adopted to aggregate ICIs to GCI as the weights of individual FPRs are determined by the model M-2. On the other hand, an ordered weighted averaging (OWA) [63] operator proves to be an effective way to aggregate ICIs to a GCI. If an OWA operator is used here, the aggregated values have to be ordered and Eq. (3) has to be updated by using an OWA operator to aggregate individual preference relations into a group one. To this end, the parameterized attitude-OWA operator proposed by Palomares et al.[35] can be potentially applied to the proposed consensus models in this article. In addition, *t*-norms such as minimum *t*-norm, product *t*-norm, Łukasiewicz *t*-norm are also possible ways to aggregate the arguments. If minimum and maximum *t*-norm operations are employed to carry out the aggregation process, a key challenge is how to handle the consequent loss of information.

Remark 4. This algorithm automatically updates the experts' preference values in order to reach a group consensus. This treatment helps to relieve the experts from the burden of constantly adjusting their judgments. On the other hand, if the experts are willing to reevaluate their preferences, the algorithm can serve as an invaluable aid to the expert in identifying which preferences values to change so that the highest degree of consensus can be reached expeditiously.

4. Group consensus models for multiplicative preference relations

If DM e_k compares each pair of alternatives in X and provides his/her preference degree $a_{ij,k}$ of x_i over x_j on a 1-9 scale, where $1/9 \le a_{ij,k} \le 9$, $a_{ij,k} = 1$ denotes e_k's indifference between x_i and x_j, $a_{ij,k}=9$ denotes that x_i is definitely preferred to x_j, and $1<a_{ij,k}<9$ (or $1/9<a_{ji,k}<1$) denotes that x_i is preferred to x_j to a varying degree. All preference values $a_{ij,k}$ ($i,j=1,2,...,n$) provided by DM e_k constitute a multiplicative preference relation (MPR) $A_k=(a_{ij,k})_{n\times n}$, if [39]

$$a_{ijk}>0,\ a_{iik}=1,\ a_{ijk}\cdot a_{jik}=1,\quad i,j=1,2,...,n \tag{32}$$

Let $v=(v_1,v_2,...,v_n)^T$ be the implied weight vector of MPRs $A_k=(a_{ij,k})_{n\times n}$ ($k=1,2,...,m$), where $v_k\ge 0$, $k=1,2,...,m$, and $\sum_{k=1}^{m}v_k=1$. To obtain a collective opinion, Xu and Cai [62] adopted the Weighted Geometric Average (WGA) operator:

$$a_{ij}=\prod_{k=1}^{m}(a_{ij,k})^{v_k},\quad i,j=1,2,...,n \tag{33}$$

to aggregate individual MPRs $A_k=(a_{ij,k})_{n\times n}$ ($k=1,2,...,m$) into a collective preference relation $A=(a_{ij})_{n\times n}$. It is easy to verify that A satisfies (32), and is thus an MPR as well.

If an individual MPR A_k is perfectly consistent with the collective MPR A, then $A_k=A$, i.e., $a_{ij,k}=a_{ij}$, for all $i,j=1,2,...,n$. Using (33), we have

$$a_{ij,k}=\prod_{l=1}^{m}(a_{ij,l})^{v_l}\text{, for all } i,j=1,2,...,n \tag{34}$$

If Eq. (34) holds for all $k=1,2,...,m$, then the group reaches a complete consensus. In this case, by taking natural logarithms on both sides of Eq. (34), Xu and Cai [62] transformed it into the following form:

$$\lg a_{ij,k}=\lg\prod_{l=1}^{m}(a_{ij,l})^{v_l}=\sum_{l=1}^{m}v_l\lg a_{ij,l},\qquad \text{for all } i,j=1,2,...,n \tag{35}$$

However, generally speaking, Eq. (35) does not always hold. Define the absolute deviation variables as

$$f_{ij,k}=\left|\lg a_{ij,k}-\sum_{l=1}^{m}v_l\lg a_{ij,l}\right|,\ \text{for all } i,j=1,2,...,n,\ k=1,2,...,m \tag{36}$$

According to Eq. (32) , it is only necessary to check the upper diagonal deviations:

$$f_{ij,k}=\left|\lg a_{ij,k}-\sum_{l=1}^{m}v_l\lg a_{ij,l}\right|,\ \text{for all } i=1,2,...,n-1,\ j=i+1,...,n,\ k=1,2,...,m \tag{37}$$

It is understandable that these absolute deviations should be kept as small as possible. Similar to model (M-1), Xu and Cai [62] constructed the following quadratic program:

(M-4) $$\min\ J_2=\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}f_{ij,k}^2=\sum_{k=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lg a_{ij,k}-\sum_{l=1}^{m}v_l\lg a_{ij,l}\right)^2$$

$$\text{s.t.}\ \sum_{l=1}^{m}v_l=1,\ v_l\ge 0,\ l=1,2,...,m$$

Solving the model yields the DMs' optimal weight vector $v=(v_1,v_2,...,v_m)^T$ [62]:

$$v=\frac{Q^{-1}e(1-e^TQ^{-1}\theta)}{e^TQ^{-1}e}+Q^{-1}\theta \tag{38}$$

where

$$\theta=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\lg a_{ij,k}\,\lg a_{ij,1},\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\lg a_{ij,k}\,\lg a_{ij,2},...,\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\lg a_{ij,k}\,\lg a_{ij,m}\right)^T,$$

$$e=(1,1,...,1)^T \tag{39}$$

and

$$Q=\begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{n}m(\lg a_{ij,1})^2 & \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,1}\lg a_{ij,2} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,1}\lg a_{ij,m}\\ \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,1}\ln a_{ij,2} & \sum_{i=1}^{n}\sum_{j=1}^{n}m(\lg a_{ij,2})^2 & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,2}\lg a_{ij,m}\\ \cdots & \cdots & \cdots & \cdots\\ \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,1}\lg a_{ij,m} & \sum_{i=1}^{n}\sum_{j=1}^{n}m\lg a_{ij,2}\lg a_{ij,m} & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{n}m(\lg a_{ij,m})^2\end{pmatrix}_{m\times m} \tag{40}$$

By plugging the optimal weight vector into Eq. (33), Xu and Cai [62] obtained a collective MPR A. Subsequently, Xu and Cai [62] calculated the sum of absolute deviations (here referred to as the individual to group consensus index ICI) between the individual MPR A_k and the collective MPR A by

$$ICI(A_k)=d(A_k,A)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}f_{ij,k}=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|\lg a_{ij,k}-\sum_{l=1}^{n}v_l\lg a_{ij,l}\right| \tag{41}$$

Accordingly, the weighted sum of deviations $d(A_k,A)$ ($k=1,2,...,m$) (hereafter, referred to as the group consensus index GCI) is defined as

$$GCI=\Delta_2=\sum_{k=1}^{m}v_k d(A_k,A) \tag{42}$$

From Eqs. (41) and (42), it is apparent that if $d(A_k,A)=0$, the individual MPR A_k is perfectly consistent with the collective MPR A. If $\Delta_2=0$, the group reaches a complete consensus. Once again, Xu and Cai [62] assumed that, for a pre-defined threshold λ_2, if $\Delta_2\le\lambda_2$, the group is deemed to reach an acceptable level of consensus. If $\Delta_2>\lambda_2$, the same idea to that of Algorithms 1 and 2 in Xu and Cai [62] is utilized to improve the group consensus.

Similar to the case of FPRs in Theorem 1, the following result is established for MPRs.

Theorem 5. For MPRs $A_k = (a_{ijk})_{n\times n}$ ($k = 1,2,...,m$), if for any i, j and k, there exists at least one inequality $\lg a_{ij,k} \neq \sum_{l=1}^{m} v_l \lg a_{ij,l}$, then the optimal solution to (M-4) is

$$v = (1/m, 1/m, ..., 1/m)^T \tag{43}$$

Proof. The proof is similar to that of Theorem 1 and, hence, is omitted.

As per Proposition 2.1 in [25], an MPR can be transformed into an FPR by the following formula :

$$p_{ij} = \tfrac{1}{2}(1 + \log_9 a_{ij}) \tag{44}$$

Analogous to model (M-2), a squared weighted distance between a pair of individual MPRs (A_k, A_l) can be defined as

$$d^2(v_k A_k, v_l A_l) = \left(\sqrt{(v_k \cdot \tfrac{1}{2}(1+\log_9 A_k) - v_l \cdot \tfrac{1}{2}(1+\log_9 A_l))^2}\right)^2$$

$$= \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(v_k(1+\log_9 a_{ij,k}) - v_l(1+\log_9 a_{ij,l})\right)^2 \tag{45}$$

Following this definition, an optimization model is constructed to minimize the sum of squared weighted distances between all pairs of MPRs:

$$\textbf{(M-5)} \quad \min \ J_2 = \frac{1}{4}\sum_{k=1}^{m}\sum_{l=1, l\neq k}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(v_k(1+\log_9 a_{ij,k}) - v_l(1+\log_9 a_{ij,l})\right)^2 \tag{46}$$

$$\text{s.t.} \quad \sum_{l=1}^{m} v_l = 1 \tag{47}$$

$$v_l \geq 0, \ l = 1,2,...,m \tag{48}$$

Similar to the case of FPRs, (M-5) can be rewritten in a matrix form.

Theorem 6. Model (M-5) is equivalent to (M-6) below in a matrix form

$$\textbf{(M-6)} \quad \min \quad J_2 = v^T B v \tag{49}$$

$$\text{s.t.} \quad e^T v = 1 \tag{50}$$

$$v \geq 0 \tag{51}$$

where $v = (v_1, v_2, ..., v_m)^T$, $e = (1,1,...,1)^T$, and $B = (b_{kl})_{m\times m}$. The elements in matrix B are

$$b_{kk} = \frac{(m-1)}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\log_9 a_{ij,k})^2, \quad k = 1,2,...,m \tag{52}$$

$$b_{kl} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(1+\log_9 a_{ij,k})(1+\log_9 a_{ij,l}), \quad k,l = 1,2,...,m, \ k \neq l. \tag{53}$$

Similar to Theorem 3, the following result is obtained for MPRs.

Theorem 7. For model (M-6), if for any i, j, k and l, there exists at least one inequality $a_{ijk} \neq a_{ijl}$, then matrix B determined by (52) and (53) is positive definite and, hence, non-singular and invertible.

Proof. Obviously, $J_2 = v^T B v \geq 0$. Now, we prove that $J_2 \neq 0$ if there exists at least one inequality $a_{ij,k} \neq a_{ij,l}$.

Assume that there exists a weight vector v, for all i, j, k and l, such that $J_2 = 0$. Then,

$v_k(1+\log_9 a_{ij,k}) = v_l(1+\log_9 a_{ij,l})$, and $v_k(1+\log_9 a_{ji,k}) = v_l(1+\log_9 a_{ji,l})$

thus, by Eq. (32), one can obtain

$$\frac{v_k}{v_l} = \frac{1+\log_9 a_{ij,l}}{1+\log_9 a_{ij,k}} = \frac{1+\log_9 a_{ji,l}}{1+\log_9 a_{ji,k}} = \frac{1-\log_9 a_{ij,l}}{1-\log_9 a_{ij,k}}$$

which yields

$a_{ij,k} = a_{ij,l}$, for all i, j, k and l

This contradicts with the assumption that there exists at least one inequality $a_{ij,k} \neq a_{ij,l}$. Therefore, $J_2 > 0$, implying that B is positive definite and, hence, nonsingular and invertible, i.e., B^{-1} exists. This completes the proof of Theorem 7.

Remark 5. Theorem 7 indicates that B is positive definite as long as A_k is not identical for all DMs. If all the judgment matrices are the same, then $|B| = 0$, and the weight vector for (M-6) is $(1/m, 1/m, ..., 1/m)^T$. In this case, a complete consensus is reached and no further process is needed. As such, only the general case is considered where there exits at least one inequality $p_{ij,k} \neq p_{ij,l}$.

Similarly, the Lagrangian multiplier method is employed to solve (M-6) as follows

$$v^* = \frac{B^{-1} e}{e^T B^{-1} e} \tag{54}$$

$$\lambda^* = -\frac{1}{e^T B^{-1} e} \tag{55}$$

It is trivial to verify that Theorems 3 and 4 hold for model (M-6) where G is replaced with B. As such, B is positive definite, B^{-1} is non-negative. Therefore, $v^* \geq 0$.

Based on the aforesaid models, similar to Algorithm 1, a consensus algorithm is devised for GDM with MPRs.

Algorithm 2.

Input: Each DM e_k 's MPR $A_k = (a_{ij,k})_{n \times n}$ ($k = 1, 2, ..., m$), the maximum number of iterations t^*, the thresholds α_2, λ_2 for individual and group consensus indices, respectively. Generally, $\alpha_2 > \lambda_2$.

Output: Improved MPRs $\overline{A}_k$ ($k = 1, 2, ..., m$), terminal iterative step t, individual consensus index $ICI(\overline{A}_k)$ ($k = 1, 2, ..., m$) and group consensus degree GCI.

Step 1. Let $t=0$, $A_k^{(0)}=A_k$ ($k=1,2,...,m$).

Step 2. Apply the quadratic program (M-6) to determine the optimal weight vector $v^{(t)}=(v_1^{(t)},v_2^{(t)},...,v_m^{(t)})^T$ as per Eq. (54) for $A_k^{(t)}=(a_{ij,k}^{(t)})_{n\times n}$ ($k=1,2,...,m$).

Step 3. Utilize the WGA operator Eq. (33) to aggregate individual MPRs $A_k^{(t)}=(a_{ij,k}^{(t)})_{n\times n}$ ($k=1,2,...,m$) into a collective MPR $A^{(t)}=(a_{ij}^{(t)})_{n\times n}$.

Step 4. Calculate individual consensus index $ICI(A_k^{(t)})$ by the following formula:

$$ICI(A_k)=d(A_k,A)=\frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|\tfrac{1}{2}(1+\log_9 a_{ij,k})-\sum_{l=1}^{n}v_l\cdot\tfrac{1}{2}(1+\log_9 a_{ij,l})\right|$$

$$=\frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|\log_9 a_{ij,k}-\sum_{l=1}^{n}v_l\log_9 a_{ij,l}\right| \tag{56}$$

and

$$GCI=\Delta_2=\sum_{k=1}^{m}v_k d(A_k,A) \tag{57}$$

respectively. If $\Delta_2\le\lambda_2$ and $ICI(A_k^{(t)})\le\alpha_2$ (for all $k=1,2,...,m$) or $t=t^*$, then go to Step 6. Otherwise, find MPRs $A_k^{(t)}$ such that $ICI(A_k^{(t)})>\lambda_2$. Go to Step 5.

Step 5. Find the position i_τ and j_τ of the maximum elements $d_{i_\tau j_\tau,k}^{(t)}$ ($k=1,2,...,m$), such that $ICI(A_k^{(t)})>\lambda_2$, where $d_{i_\tau j_\tau,k}^{(t)}=\max_{i,j}\left|\log_9 a_{ij,k}^{(t)}-\log_9 a_{ij}^{(t)}\right|$ for each DM e_k, and adjust the corresponding preference value as per

$$a_{ij,k}^{(t+1)}=\begin{cases}a_{ij}^{(t)}, & if\ i=i_\tau,\ j=j_\tau\\ a_{ij,k}^{(t)}, & otherwise\end{cases} \tag{58}$$

and $t=t+1$. Then, go to Step 2.

Step 6. Let $\overline{A}_k=A_k^{(t)}$. Output the modified MPRs $\overline{A}_k$ ($k=1,2,...,m$), the terminal iteration step t, individual consensus index $ICI(A_k^{(t)})$ ($k=1,2,...,m$), and group consensus index GCI.

5. Illustrative examples

Example 1. Consider a GDM problem that is concerned with evaluating and selecting suitable locations for a shopping center as shown in [62] and [46]. Five experts e_k ($k=1,2,...,5$) are commissioned to assess six potential locations (adapted from [34]), denoted by x_i ($i=1,2,...,6$). After carrying out pairwise comparisons, the experts e_k ($k=1,2,...,5$) furnish their assessments as the following FPRs $P_k=P_k^{(0)}=(p_{ij,k})_{6\times 6}$ ($k=1,2,...,5$):

$$P_1 = P_1^{(0)} = \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.6 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.4 & 0.6 & 0.9 & 0.7 \\ 0.8 & 0.6 & 0.5 & 0.6 & 0.8 & 1.0 \\ 0.4 & 0.4 & 0.4 & 0.5 & 0.7 & 0.6 \\ 0.3 & 0.1 & 0.2 & 0.3 & 0.5 & 0.3 \\ 0.4 & 0.3 & 0.0 & 0.4 & 0.7 & 0.5 \end{bmatrix}, \; P_2 = P_2^{(0)} = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.5 & 0.8 & 0.7 \\ 0.7 & 0.5 & 0.4 & 0.7 & 1.0 & 0.8 \\ 0.7 & 0.6 & 0.5 & 0.5 & 0.9 & 0.9 \\ 0.5 & 0.3 & 0.5 & 0.5 & 0.6 & 0.7 \\ 0.2 & 0.0 & 0.1 & 0.4 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.1 & 0.3 & 0.6 & 0.5 \end{bmatrix},$$

$$P_3 = P_3^{(0)} = \begin{bmatrix} 0.5 & 0.5 & 0.6 & 0.6 & 0.7 & 0.9 \\ 0.5 & 0.5 & 0.3 & 0.8 & 0.7 & 0.8 \\ 0.4 & 0.7 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.4 & 0.2 & 0.3 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.3 & 0.3 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.4 & 0.8 & 0.5 \end{bmatrix}, P_4 = P_4^{(0)} = \begin{bmatrix} 0.5 & 0.2 & 0.1 & 0.5 & 0.8 & 0.9 \\ 0.8 & 0.5 & 0.2 & 0.9 & 0.6 & 1.0 \\ 0.9 & 0.8 & 0.5 & 0.8 & 0.6 & 0.6 \\ 0.5 & 0.1 & 0.2 & 0.5 & 1.0 & 0.8 \\ 0.2 & 0.4 & 0.4 & 0.0 & 0.5 & 0.4 \\ 0.1 & 0.0 & 0.4 & 0.2 & 0.6 & 0.5 \end{bmatrix},$$

$$P_5 = P_5^{(0)} = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & 0.5 \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & 0.9 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ 0.5 & 0.4 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix}.$$

Algorithm 1 is employed to obtain a solution to the GDM problem. Assume that the maximum number of iterations $t^* = 10$, the individual consensus degree threshold $\alpha_1 = 0.065$. To facilitate a comparison with the results in [46] and [62], the group consensus degree threshold is set at $\lambda_1 = 0.05$.

Step 1. Applying the quadratic program (M-3) to determine the optimal weight vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, ..., w_5^{(0)})^T$ for $P_k^{(0)} = (p_{ij,k}^{(0)})_{6\times 6}$ ($k = 1, 2, ..., 5$) as per Eq. (29):

$$w^{(0)} = (0.2041, 0.2005, 0.2025, 0.1886, 0.2042)^T$$

Step 2. Using Eq. (3) to obtain the collective FPR:

$$P^{(0)} = \begin{bmatrix} 0.5 & 0.3421 & 0.3026 & 0.5815 & 0.7593 & 0.7170 \\ 0.6579 & 0.5 & 0.3012 & 0.7376 & 0.8025 & 0.7765 \\ 0.6974 & 0.6988 & 0.5 & 0.6583 & 0.7417 & 0.8232 \\ 0.4185 & 0.2624 & 0.3417 & 0.5 & 0.7976 & 0.6782 \\ 0.2407 & 0.1975 & 0.2583 & 0.2024 & 0.5 & 0.3391 \\ 0.2830 & 0.2235 & 0.1768 & 0.3218 & 0.6609 & 0.5 \end{bmatrix}$$

Step 3. Calculating $ICI(P_k^{(0)})$ ($k = 1, 2, ..., 5$) and $GCI(0)$ based on Eqs. (10) and (11):

$ICI(P_1^{(0)}) = 0.0849$, $ICI(P_2^{(0)}) = 0.0810$, $ICI(P_3^{(0)}) = 0.0821$, $ICI(P_4^{(0)}) = 0.1487$,

$ICI(P_5^{(0)}) = 0.0687$, $GCI(0) = 0.0923$.

Step 4. Since $GCI(0) = 0.0923 > 0.05$, and $ICI(P_k^{(0)}) > 0.065$ ($k = 1,2,...,5$), we need to find the position of elements $d_{i_\tau j_\tau,k}^{(0)}$ ($k = 1,2,...,5$), where $d_{i_\tau j_\tau,k}^{(0)} = \max_{i,j} \left| p_{ij,k}^{(0)} - p_{ij}^{(0)} \right|$. For $P_1^{(0)}$, since $d_{36,1}^{(0)} = d_{63,1}^{(0)} = \max_{i,j} \left| p_{ij,1}^{(0)} - p_{ij}^{(0)} \right| = 0.1768$, replacing these two preference values with the corresponding elements in the collective FPR $P^{(0)}$, $p_{361}^{(0)} = p_{36}^{(0)} = 0.8232$, $p_{631}^{(0)} = p_{63}^{(0)} = 0.1768$. Similarly, the same procedure is used to update the other four DMs' FPRs.

$p_{25,2}^{(0)} = p_{25}^{(0)} = 0.8025$, $p_{52,2}^{(0)} = p_{52}^{(0)} = 0.1975$,

$p_{13,3}^{(0)} = p_{13}^{(0)} = 0.3026$, $p_{31,3}^{(0)} = p_{31}^{(0)} = 0.6974$,

$p_{26,4}^{(0)} = p_{26}^{(0)} = 0.7765$, $p_{62,4}^{(0)} = p_{62}^{(0)} = 0.2235$,

$p_{16,5}^{(0)} = p_{16}^{(0)} = 0.7170$, $p_{61,5}^{(0)} = p_{61}^{(0)} = 0.2830$.

Let $t = 1$, then go to Step 1.

This procedure terminates after 6 iterations, and the detailed iterative processes are depicted in Table 1.

The final improved individual fuzzy preference relations $\overline{P}_k$ ($k = 1,2, \ ...,5$) and group fuzzy preference relation $\overline{P}$ are

$$\overline{P}_1 = \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.6 & 0.7 & 0.7616 \\ 0.6 & 0.5 & 0.3010 & 0.7379 & 0.8005 & 0.7 \\ 0.8 & 0.6990 & 0.5 & 0.6 & 0.8 & 0.8232 \\ 0.4 & 0.2621 & 0.4 & 0.5 & 0.8374 & 0.6 \\ 0.3 & 0.1995 & 0.2 & 0.1626 & 0.5 & 0.3 \\ 0.2384 & 0.3 & 0.1768 & 0.4 & 0.7 & 0.5 \end{bmatrix},$$

$$\overline{P}_2 = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.5 & 0.8 & 0.7 \\ 0.7 & 0.5 & 0.4 & 0.7 & 0.8016 & 0.8 \\ 0.7 & 0.6 & 0.5 & 0.6581 & 0.7415 & 0.9 \\ 0.5 & 0.3 & 0.3419 & 0.5 & 0.7976 & 0.7 \\ 0.2 & 0.1984 & 0.2585 & 0.2024 & 0.5 & 0.4 \\ 0.3 & 0.2 & 0.1 & 0.3 & 0.6 & 0.5 \end{bmatrix},$$

$$\overline{P}_3 = \begin{bmatrix} 0.5 & 0.3418 & 0.3026 & 0.6 & 0.7 & 0.9 \\ 0.6582 & 0.5 & 0.3 & 0.8 & 0.7 & 0.8 \\ 0.6974 & 0.7 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.4 & 0.2 & 0.3 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.3 & 0.3 & 0.2 & 0.5 & 0.3392 \\ 0.1 & 0.2 & 0.2 & 0.4 & 0.6608 & 0.5 \end{bmatrix},$$

$$\overline{P}_4 = \begin{bmatrix} 0.5 & 0.2 & 0.2418 & 0.5 & 0.8 & 0.9 \\ 0.8 & 0.5 & 0.2 & 0.7666 & 0.8016 & 0.7765 \\ 0.7582 & 0.8 & 0.5 & 0.8 & 0.6 & 0.7866 \\ 0.5 & 0.2334 & 0.2 & 0.5 & 0.8374 & 0.8 \\ 0.2 & 0.1984 & 0.4 & 0.1626 & 0.5 & 0.4 \\ 0.1 & 0.2235 & 0.2134 & 0.2 & 0.6 & 0.5 \end{bmatrix},$$

$$\overline{P}_5 = \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.7 & 0.8 & 0.7170 \\ 0.7 & 0.5 & 0.2 & 0.7 & 0.8 & 0.7344 \\ 0.7 & 0.8 & 0.5 & 0.7 & 0.7 & 0.8 \\ 0.3 & 0.3 & 0.3 & 0.5 & 0.9 & 0.7 \\ 0.2 & 0.2 & 0.3 & 0.1 & 0.5 & 0.4 \\ 0.2830 & 0.2656 & 0.2 & 0.3 & 0.6 & 0.5 \end{bmatrix},$$

$$\overline{P} = \begin{bmatrix} 0.5 & 0.3091 & 0.2690 & 0.5803 & 0.7597 & 0.7952 \\ 0.6909 & 0.5 & 0.2808 & 0.7408 & 0.7806 & 0.7621 \\ 0.7310 & 0.7192 & 0.5 & 0.6909 & 0.7090 & 0.8222 \\ 0.4197 & 0.2592 & 0.3091 & 0.5 & 0.8344 & 0.6792 \\ 0.2403 & 0.2194 & 0.2910 & 0.1656 & 0.5 & 0.3676 \\ 0.2048 & 0.2379 & 0.1778 & 0.3208 & 0.6324 & 0.5 \end{bmatrix}.$$

The corresponding $ICI(P_k)$ ($k = 1, 2, ..., 5$) for the final modified FPRs and $GCI(t)$ are:

$ICI(P_1^{(6)}) = 0.0474$, $ICI(P_2^{(6)}) = 0.0472$, $ICI(P_3^{(6)}) = 0.0420$, $ICI(P_4^{(6)}) = 0.0609$,

$ICI(P_5^{(6)}) = 0.0404$, $GCI(6) = 0.0475$, $t = 6$.

Table 1 shows that after two iterations (i.e., $t = 2$), $ICI(P_5^{(2)}) = 0.0476 < 0.05$, indicating that DM e_5's modified FPR has reached an acceptable level of consensus with the collective FPR at this step. Therefore, $P_5^{(2)}$ will not be further updated so that the DM's original judgments can be by and large retained. Similarly, at $t = 3$, $ICI(P_3^{(3)}) = 0.0446 < 0.05$, the updating of P_3 will be stopped at this step. When $t = 6$, the group consensus index $GCI(6) = 0.0475 < 0.05$, and all individual to group consensus indices are less than the threshold 0.065, so the iteration process terminates. The updated FPRs $\overline{P}_1$, $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$ and $\overline{P}_5$ are deemed to reach an acceptable consensus level, and an appropriate selection method can be applied to come up with a recommendation for the group decision problem. As an illustration, the normalizing rank aggregation method [53]

$$\omega_i = \frac{2}{n^2}\sum_{j=1}^{n}\overline{p}_{ij}$$

is adopted to derive a priority vector for the collective FPR $\overline{P}$ as follows

$$\omega = (0.1785, 0.2086, 0.2318, 0.1668, 0.0991, 0.1152)^T$$

As commented earlier, the method in Wu and Xu [46] is equivalent to setting $\alpha_1 = \lambda_1 = 0.05$. Based on their approach, a slightly different priority weight vector is obtained as $\omega = (0.1772, 0.2111, 0.2289, 0.1672, 0.0956, 0.1200)^T$. In both cases, x_3 arises as the best option for the group DMs.

Compared with the approaches proposed in [46] and [62], the study here differs in several aspects. Firstly, separate thresholds α_1, λ_1 are set for individual and group consensus indices. In doing so, each expert is allowed to express his/her judgments slightly different from the group opinion, making it sensible to model consensus reaching processes in reality. Secondly, at each iteration, only one pair of judgments, if any, in each DM's individual FPR that deviate the most from the corresponding elements in the collective FPR are adjusted in the proposed consensus reaching process. The rationale is to retain each DM's original preference information. On the other hand, Wu and Xu [46] and Xu and Cai [62] employ Eq. (15) to modify all preference values for all DMs by setting a parameter η. The implication is that the final modified FPRs often significantly differ from the original judgments furnished by the DMs. Thirdly, the proposed quadratic programming models can be used to determine expert weights automatically. Although Xu and Cai [62] aimed to incorporate this idea in their quadratic programs, our theoretic analysis and their illustrative examples demonstrate that the resulting expert weights are always $1/m$ for every DM (m is the number of DMs in the GDM problem). As for Wu and Xu [46], expert weights are arbitrarily set without sufficiently considering each DM's judgment information.

Table 1. The iterative process for Example 1.

t	$w^{(t)}$	$P^{(t)}$	$ICI(P_k^{(t)})\,,GCI(t)$	$p_{ij,k}^{(t)}$
0	0.2041 0.2005 0.2025 0.1886 0.2042	$\begin{bmatrix} 0.5 & 0.3421 & 0.3026 & 0.5815 & 0.7593 & 0.7170 \\ 0.6579 & 0.5 & 0.3012 & 0.7376 & 0.8025 & 0.7765 \\ 0.6974 & 0.6988 & 0.5 & 0.6583 & 0.7417 & 0.8232 \\ 0.4185 & 0.2624 & 0.3417 & 0.5 & 0.7976 & 0.6782 \\ 0.2407 & 0.1975 & 0.2583 & 0.2024 & 0.5 & 0.3391 \\ 0.2830 & 0.2235 & 0.1768 & 0.3218 & 0.6609 & 0.5 \end{bmatrix}$	$ICI(P_1^{(0)}) = 0.0849$, $ICI(P_2^{(0)}) = 0.0810$, $ICI(P_3^{(0)}) = 0.0821$, $ICI(P_4^{(0)}) = 0.1487$, $ICI(P_5^{(0)}) = 0.0687$, $GCI(0) = 0.0923$	$p_{36,1}^{(0)} \to 0.8232$, $p_{63,1}^{(0)} \to 0.1768$, $p_{45,2}^{(0)} \to 0.7976$, $p_{54,2}^{(0)} \to 0.2024$, $p_{13,3}^{(0)} \to 0.3026$, $p_{31,3}^{(0)} \to 0.6974$, $p_{26,4}^{(0)} \to 0.7765$, $p_{62,4}^{(0)} \to 0.2235$, $p_{16,5}^{(0)} \to 0.7170$, $p_{61,5}^{(0)} \to 0.2830$.
1	0.2057 0.1978 0.2020 0.1917 0.2028	$\begin{bmatrix} 0.5 & 0.3418 & 0.2416 & 0.5813 & 0.7592 & 0.7616 \\ 0.6582 & 0.5 & 0.3009 & 0.7380 & 0.8016 & 0.7344 \\ 0.7584 & 0.6991 & 0.5 & 0.6590 & 0.7410 & 0.7862 \\ 0.4187 & 0.2620 & 0.3410 & 0.5 & 0.8376 & 0.6784 \\ 0.2408 & 0.1984 & 0.2590 & 0.1624 & 0.5 & 0.3390 \\ 0.2384 & 0.2656 & 0.2138 & 0.3216 & 0.6610 & 0.5 \end{bmatrix}$	$ICI(P_1^{(1)}) = 0.0746$, $ICI(P_2^{(1)}) = 0.0826$, $ICI(P_3^{(1)}) = 0.0678$, $ICI(P_4^{(1)}) = 0.1243$, $ICI(P_5^{(1)}) = 0.0547$, $GCI(1) = 0.0803$	$p_{16,1}^{(1)} \to 0.7616$, $p_{61,1}^{(1)} \to 0.2384$, $p_{25,2}^{(1)} \to 0.8016$, $p_{52,2}^{(1)} \to 0.1984$, $p_{12,3}^{(1)} \to 0.3418$, $p_{21,3}^{(1)} \to 0.6582$, $p_{25,4}^{(1)} \to 0.8016$, $p_{52,4}^{(1)} \to 0.1984$, $p_{26,5}^{(1)} \to 0.7344$, $p_{62,5}^{(1)} \to 0.2656$.
2	0.2044 0.2010 0.2018 0.1910 0.2018	$\begin{bmatrix} 0.5 & 0.3098 & 0.2419 & 0.5810 & 0.7594 & 0.7946 \\ 0.6902 & 0.5 & 0.3013 & 0.7379 & 0.8009 & 0.7618 \\ 0.7581 & 0.6987 & 0.5 & 0.6585 & 0.7415 & 0.7866 \\ 0.4190 & 0.2621 & 0.3415 & 0.5 & 0.8375 & 0.6785 \\ 0.2406 & 0.1991 & 0.2585 & 0.1625 & 0.5 & 0.3392 \\ 0.2054 & 0.2382 & 0.2134 & 0.3215 & 0.6608 & 0.5 \end{bmatrix}$	$ICI(P_1^{(2)}) = 0.0700$, $ICI(P_2^{(2)}) = 0.0675$, $ICI(P_3^{(2)}) = 0.0554$, $ICI(P_4^{(2)}) = 0.1048$, $ICI(P_5^{(2)}) = 0.0476$, $GCI(2) = 0.0687$	$p_{24,1}^{(2)} \to 0.7379$, $p_{42,1}^{(2)} \to 0.2621$, $p_{35,2}^{(2)} \to 0.7415$, $p_{53,2}^{(2)} \to 0.2585$, $p_{56,3}^{(2)} \to 0.3392$, $p_{65,3}^{(2)} \to 0.6608$, $p_{36,4}^{(2)} \to 0.7866$, $p_{63,4}^{(2)} \to 0.2134$.

Table 1. (Continued)

t	$w^{(t)}$	$P^{(t)}$	$ICI(P_k^{(t)})$, $GCI(t)$	$p_{ij,k}^{(t)}$
3	0.2031 0.2029 0.2028 0.1902 0.2011	$\begin{bmatrix} 0.5 & 0.3098 & 0.2422 & 0.5808 & 0.7594 & 0.7945 \\ 0.6902 & 0.5 & 0.3015 & 0.7660 & 0.8007 & 0.7620 \\ 0.7578 & 0.6985 & 0.5 & 0.6581 & 0.7097 & 0.8224 \\ 0.4192 & 0.2340 & 0.3419 & 0.5 & 0.8374 & 0.6784 \\ 0.2406 & 0.1993 & 0.2903 & 0.1626 & 0.5 & 0.3674 \\ 0.2055 & 0.2380 & 0.1776 & 0.3216 & 0.6326 & 0.5 \end{bmatrix}$	$ICI(P_1^{(3)}) = 0.0643$, $ICI(P_2^{(3)}) = 0.0566$, $ICI(P_3^{(3)}) = 0.0446$, $ICI(P_4^{(3)}) = 0.0889$, $ICI(P_5^{(3)}) = 0.0461$, $GCI(3) = 0.0598$.	$p_{45,1}^{(3)} \to 0.8374$, $p_{54,1}^{(3)} \to 0.1626$, $p_{34,2}^{(3)} \to 0.6581$, $p_{43,2}^{(3)} \to 0.3419$, $p_{45,4}^{(3)} \to 0.8374$, $p_{54,4}^{(3)} \to 0.1984$.
4	0.2016 0.2025 0.2025 0.1926 0.2008	$\begin{bmatrix} 0.5 & 0.3094 & 0.2418 & 0.5806 & 0.7596 & 0.7948 \\ 0.6906 & 0.5 & 0.3011 & 0.7664 & 0.8005 & 0.7621 \\ 0.7582 & 0.6989 & 0.5 & 0.6906 & 0.7093 & 0.8223 \\ 0.4194 & 0.2336 & 0.3094 & 0.5 & 0.8343 & 0.6789 \\ 0.2404 & 0.1995 & 0.2907 & 0.1657 & 0.5 & 0.3675 \\ 0.2052 & 0.2379 & 0.1777 & 0.3211 & 0.6325 & 0.5 \end{bmatrix}$	$ICI(P_1^{(4)}) = 0.0577$, $ICI(P_2^{(4)}) = 0.0480$, $ICI(P_3^{(4)}) = 0.0422$, $ICI(P_4^{(4)}) = 0.0759$, $ICI(P_5^{(4)}) = 0.0441$, $GCI(4) = 0.0534$.	$p_{25,1}^{(4)} \to 0.8005$, $p_{52,1}^{(4)} \to 0.1995$, $p_{13,4}^{(5)} \to 0.2418$, $p_{31,4}^{(5)} \to 0.7582$.
5	0.2023 0.2018 0.2019 0.1938 0.2002	$\begin{bmatrix} 0.5 & 0.3093 & 0.2690 & 0.5805 & 0.7596 & 0.7950 \\ 0.6907 & 0.5 & 0.3010 & 0.7666 & 0.7805 & 0.7621 \\ 0.7310 & 0.6990 & 0.5 & 0.6907 & 0.7092 & 0.8223 \\ 0.4195 & 0.2334 & 0.3093 & 0.5 & 0.8343 & 0.6790 \\ 0.2404 & 0.2195 & 0.2908 & 0.1657 & 0.5 & 0.3675 \\ 0.2050 & 0.2379 & 0.1777 & 0.3210 & 0.6325 & 0.5 \end{bmatrix}$	$ICI(P_1^{(5)}) = 0.0543$, $ICI(P_2^{(5)}) = 0.0476$, $ICI(P_3^{(5)}) = 0.0390$, $ICI(P_4^{(5)}) = 0.0695$, $ICI(P_5^{(5)}) = 0.0435$, $GCI(5) = 0.0507$.	$p_{23,1}^{(5)} \to 0.3010$, $p_{32,1}^{(5)} \to 0.6990$, $p_{24,4}^{(5)} \to 0.7666$, $p_{42,4}^{(5)} \to 0.2334$.
6	0.2016 0.2015 0.2015 0.1954 0.2	$\begin{bmatrix} 0.5 & 0.3091 & 0.2690 & 0.5803 & 0.7597 & 0.7952 \\ 0.6909 & 0.5 & 0.2808 & 0.7408 & 0.7806 & 0.7621 \\ 0.7310 & 0.7192 & 0.5 & 0.6909 & 0.7090 & 0.8222 \\ 0.4197 & 0.2592 & 0.3091 & 0.5 & 0.8344 & 0.6792 \\ 0.2403 & 0.2194 & 0.2910 & 0.1656 & 0.5 & 0.3676 \\ 0.2048 & 0.2379 & 0.1778 & 0.3208 & 0.6324 & 0.5 \end{bmatrix}$	$ICI(P_1^{(6)}) = 0.0474$, $ICI(P_2^{(6)}) = 0.0472$, $ICI(P_3^{(6)}) = 0.0420$, $ICI(P_4^{(6)}) = 0.0609$, $ICI(P_5^{(6)}) = 0.0404$, $GCI(6) = 0.0475$.	

Example 2. The following numerical example was first developed by Yeh et al. [64], and further discussed by Wu and Xu [48]. Suppose that three managers from the design, manufacturing and marketing departments in a firm participate in a group decision to formulate their new product development strategy. Five decision criteria for the new product are identified as cost (c_1), manufacturability (c_2), quality (c_3), technological improvement (c_4) and market share (c_5). The three managers provide their preferences as MPRs A_k ($k=1,2,3$) given below.

$$A_1=\begin{bmatrix} 1 & 5 & 7 & 3 & 1/3 \\ 1/5 & 1 & 3 & 1/3 & 1/5 \\ 1/7 & 1/3 & 1 & 1/7 & 1/9 \\ 1/3 & 3 & 7 & 1 & 1/3 \\ 3 & 5 & 9 & 3 & 1 \end{bmatrix},\quad A_2=\begin{bmatrix} 1 & 1/3 & 7 & 1/2 & 3 \\ 3 & 1 & 3 & 1 & 5 \\ 1 & 1/3 & 1 & 1/3 & 3 \\ 2 & 1 & 3 & 1 & 5 \\ 1/3 & 1/5 & 1/3 & 1/5 & 1 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 1 & 7 & 5 & 4 & 3 \\ 1/7 & 1 & 1/3 & 1/4 & 1/5 \\ 1/5 & 3 & 1 & 1/3 & 1/4 \\ 1/4 & 4 & 3 & 1 & 1 \\ 1/3 & 5 & 4 & 1 & 1 \end{bmatrix}.$$

Now, Algorithm 2 is applied to solve the problem. Assume that the maximum number of iterations $t^*=10$, the individual consensus degree threshold $\alpha_2=0.055$, and the group consensus degree threshold $\lambda_2=0.05$. The iterations terminate after 6 steps. Table 2 lists the iteration time t along with the weight vector $w^{(t)}$, the individual to group consensus degree $ICI(A_k^{(t)})$ and the group consensus index $GCI(t)$ at each iteration.

The terminal improved individual MPRs $\overline{A}_k$ ($k=1,2,3$) and group MPR $\overline{A}$ are

Table 2. t, $w^{(t)}$, $ICI(A_k^{(t)})$, $GCI(t)$ for Example 2.

t	$w^{(t)}$			$ICI(A_k^{(t)})$			$GCI(t)$
0	0.3292	0.3259	0.3449	0.1841	0.2622	0.1556	0.1997
1	0.3270	0.3305	0.3425	0.1355	0.2052	0.1083	0.1492
2	0.3270	0.3317	0.3413	0.1176	0.1567	0.0875	0.1203
3	0.3267	0.3351	0.3382	0.0929	0.1170	0.0799	0.0966
4	0.3274	0.3368	0.3358	0.0821	0.0910	0.0691	0.0807
5	0.3292	0.3342	0.3365	0.0698	0.0697	0.0573	0.0656
6	0.3272	0.3388	0.3340	0.0535	0.0485	0.0484	0.0500

$$\bar{A}_1 = \begin{bmatrix} 1 & 2.5995 & 7 & 1.3916 & 1.4554 \\ 0.3847 & 1 & 3 & 1/3 & 1/5 \\ 1/7 & 1/3 & 1 & 1/7 & 0.4360 \\ 0.7186 & 3 & 7 & 1 & 2.2169 \\ 0.6871 & 5 & 2.2935 & 0.4511 & 1 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 1 & 1.5606 & 7 & 1.3916 & 3 \\ 0.6408 & 1 & 3 & 0.4381 & 0.5710 \\ 1/7 & 1/3 & 1 & 1/3 & 0.4360 \\ 0.7186 & 2.2825 & 3 & 1 & 2.2169 \\ 1/3 & 1.7513 & 2.2935 & 0.4511 & 1 \end{bmatrix},$$

$$\bar{A}_3 = \begin{bmatrix} 1 & 2.2921 & 5 & 1.8268 & 3 \\ 0.4363 & 1 & 2.3250 & 0.4381 & 1/5 \\ 1/5 & 0.4301 & 1 & 1/3 & 1/4 \\ 0.5474 & 2.2825 & 3 & 1 & 1.8155 \\ 1/3 & 5 & 4 & 0.5508 & 1 \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} 1 & 2.0968 & 6.2560 & 1.5240 & 2.3677 \\ 0.4769 & 1 & 2.7552 & 0.4006 & 0.2854 \\ 0.1598 & 0.3629 & 1 & 0.2526 & 0.3621 \\ 0.6562 & 2.4961 & 3.9585 & 1 & 2.0738 \\ 0.4223 & 3.5043 & 2.7617 & 0.4822 & 1 \end{bmatrix}.$$

In order to compare with the results obtained in [48] and [64], we continue the selection process with the eigenvector method to derive a weight vector of $\bar{A}$ as follows:

$$\xi = (0.3525, 0.1162, 0.0568, 0.2745, 0.1999)^T$$

Thus, the ranking of the five criteria is $c_1 \succ c_4 \succ c_5 \succ c_2 \succ c_3$. In [48] and [64], the final weight vector of five criteria are $\xi = (0.3722, 0.0822, 0.0691, 0.2177, 0.2587)^T$ and $\xi = (0.3743, 0.1288, 0.0833, 0.1867, 0.2270)^T$, respectively, resulting in a slightly different ranking with the only difference between c_4 and c_5. However, a closer examination of the original MPRs A_k ($k = 1,2,3$) reveal that, by setting $v = (1/3, 1/3, 1/3)^T$ and applying Eq. (34), Wu and Xu [48] would have obtained $a_{45}^{(0)} = 1.1856$, indicating that c_4 is preferred to c_5 (i.e., $c_4 \succ c_5$). This can also be verified by examining the original weight vector of the collective MPR in Wu and Xu [48], $\xi^{(c)} = (0.3264, 0.1232, 0.0841, 0.2574, 0.2088)^T$, yielding a ranking of $c_1 \succ c_4 \succ c_5 \succ c_2 \succ c_3$ based on the DMs' original judgments. This result would have been identical to the ranking derived from the proposed method in this article. This minor discrepancy in the ranking result based on the modified collective MPR, in our opinion, is due to the different adjustment mechanisms in the consensus reaching process. The approaches in [48] and [64] take a more aggressive manner to rectify preference values in the updating process, resulting in a larger distortion of the DMs' original judgment. On the other hand, this study takes a more progressive approach to adjust at most one pair of preference values in each DM's individual MPR, aiming to preserve DM's original judgment. Therefore, the proposed method here tends to yield a ranking result closer to what is implied in the original judgments than those obtained in [48] and [64].

6. Conclusions

In this paper, distance-based group consensus models are proposed for FPRs and MPRs, respectively. Based on the proposed model, the expert weights can be automatically determined. We define an individual to group consensus index (*ICI*) between the individual FPR P_k (or MPR A_k) and a collective FPR P (or a collective MPR A), and a group consensus index (*GCI*) which is a weighted average of *ICIs*. An *ICI* evaluates how far an individual's judgments differ from the collective judgments and is used to determine whether an individual should adjust his/her judgments in the consensus building stage. A *GCI* measures the group's overall consensus level and is employed to judge whether the group should continue to the next consensus improving stage. Two algorithms are provided for reaching group consensus based on FPRs and MPRs, respectively. Comparing with existing consensus models, the proposed consensus models have the following features:

(1) The distance-based group consensus models can determine expert weights automatically. The weights of DMs would change when DMs adjust their preference values in the consensus reaching stage. This can use the DMs' information sufficiently.
(2) In the consensus reaching process, if an individual's consensus index is larger than a predefined threshold, we only modify one pair of his/her judgments with the largest deviation from the corresponding group judgments at each iteration.
(3) By introducing the *ICI* and *GCI*, the proposed models can monitor both the overall group consensus level and how far each DM deviates from the group in terms of the judgment. Furthermore, in the consensus reaching process, we set *ICI* a little larger than *GCI*, thereby allowing each individual judgment to differ slightly from the group opinion.

The proposed models have potentials to be extended to other types of preference relations and adopting different aggregation schemes. It is also a worthy topic to explore real-world applications in intelligent GDM, such as the selection of advanced technology [13], credit scoring in financial risk management [66], emergency decision support [65], to name a few.

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