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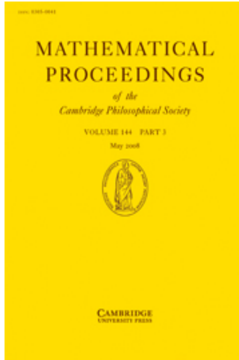

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Character amenability of Banach algebras

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Abstract

We introduce the notion of character amenable Banach algebras. We prove that character amenability for either of the group algebra $L^1(G)$ or the Fourier algebra $A(G)$ is equivalent to the amenability of the underlying group G. Character amenability of the measure algebra $M(G)$ is shown to be equivalent to G being a discrete amenable group. We also study functorial properties of character amenability. For a commutative character amenable Banach algebra A, we prove all cohomological groups with coefficients in finite-dimensional Banach A-bimodules, vanish. As a corollary we conclude that all finite-dimensional extensions of commutative character amenable Banach algebras split strongly.

1. *Introduction*

The concept of amenable Banach algebras was first introduced by B. E. Johnson in [**16**]. This notion also appeared in the work of A. Ya. Helemskii [**10**] which was published in the same year. A Banach algebra is called amenable if its first cohomological groups $\mathcal{H}^1(A, E')$ vanish for all dual Banach A-bimodules E'. We recall that if A is a Banach algebra and E is a Banach A-bimodule, then E', the dual of E, has a natural A-bimodule structure defined by

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle, \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle, \quad (a \in A, x \in E, f \in E').$$

A derivation $d\colon A \to E'$ is a linear map such that $d(ab) = a \cdot d(b) + d(a) \cdot b$ for all $a, b \in A$. Given $f \in E'$, the inner derivation $\delta_f\colon A \to E'$, is defined by $\delta_f(a) = a \cdot f - f \cdot a$. Amenability of A is equivalent to the property that every continuous derivation $d\colon A \to E'$ is an inner derivation, for every Banach A-bimodule E. A substantial amount of research has been done to link amenability to various properties of Banach algebras. For example an early result of Johnson showed that amenability of the group algebra $L^1(G)$ for G a locally compact group, is equivalent to the amenability of the underlying group G. Results of Connes and Haagerup showed that a C*-algebra is amenable if and only if it is nuclear [**19**].

In this paper we introduce the concept of *character amenable Banach algebras* (see Definition 2·1). The definition requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are considered where either of the left or right module action is defined by characters of A (including the zero character). As such, character amenability is *weaker* than amenability in the sense of Johnson. In defining this concept, we have been motivated by the notion of left amenability of A. Lau [**18**] for the class of F-algebras. Left character amenability as defined here is however stronger than left amenability of Lau.

We briefly summarize the results in this paper. Theorem 2·3 characterizes character amenability in terms of bounded approximate identities and certain topological invariant elements of the second dual. As a consequence in Corollary 2·4 we show that the character amenability for each of the Banach algebras $L^1(G)$ and $A(G)$ is equivalent to the amenability of G. The corollary in fact deals with the generalized Fourier algebras $A_p(G)$ (which for the particular case of $p = 2$ coincide with $A(G)$). Our result on character amenability of the group algebra $L^1(G)$ and the result of Dales, Ghahramani and Helemskii [**4**, theorem 3·2, p. 224] on the existence of non-zero point derivations on the measure algebra $M(G)$ (when G is non-discrete) imply that $M(G)$ is character amenable if and only if G is a discrete amenable group (Corollary 2·5). Theorem 2·6 gives the main functorial properties of character amenability. Section 2 ends with a characterization of character amenability of the Lau products of Banach algebras [**21**].

In Section 3, we discuss splitting properties of modules over character amenable Banach algebras. In our main result of this section, that is, Theorem 3·1, we show triviality of cohomological groups with coefficients in finite-dimensional Banach modules over character amenable commutative Banach algebras. As a consequence we conclude that all finite-dimensional extensions of commutative character amenable Banach algebras split strongly. The section ends with another splitting property of short exact sequences over character amenable Banach algebras.

The results in Section 2 were announced in 2006 Canadian Symposium on Abstract Harmonic Analysis, in Winnipeg. In this conference the author was informed by Professor A. Lau of similar unpublished results concerning ϕ-amenability obtained by himself, J. Pym, and independently, by E. Kaniuth. The work of these authors has been accepted for publication in these proceedings [**17**]. Due to many natural overlaps, and following suggestions by the referee, I have presented the results of Section 2 briefly, referring instead to [**17**] for details.

2. *Definitions and basic properties*

Let A be a Banach algebra and $\sigma(A)$ be the spectrum of A, that is, the set of all non-zero multiplicative linear functionals on A. If $\psi \in \sigma(A) \cup \{0\}$ and E is an *arbitrary* Banach space, then E can be viewed as Banach left or right A-module by the following actions $(a \in A,\ x \in E)$:

$$a \cdot x = \psi(a)x, \tag{2·1}$$

$$x \cdot a = \psi(a)x. \tag{2·2}$$

If the right action of A on E is given by (2·2), then it is easily verified that the left action of A on the dual module E' is given by

$$a \cdot f = \psi(a)f, \quad (f \in E', a \in A).$$

Definition 2·1. Let A be a Banach algebra. We call A *left character amenable* (LCA) if for all $\psi \in \sigma(A) \cup \{0\}$ and all Banach A-bimodules E for which the right module action is given by $x \cdot a = \psi(a)x$ $(a \in A, x \in E)$, every continuous derivation $d\colon A \to E'$ is inner.

Right character amenability (RCA) is defined similarly by considering Banach A-bimodules E for which the left module action is given by $a \cdot x = \psi(a)x$.

We call A **character amenable** (CA) if it is both left and right character amenable.

Any statement about left character amenability turns into an analogous statement about right character amenability (and vice versa) by simply replacing A by its opposite algebra. Right character amenability of A is equivalent to ϕ-amenability in the sense of [**17**] for all $\phi \in \sigma(A)$ together with 0-amenability.

As the definition indicates, character amenability is weaker than amenability; so all amenable Banach algebras are automatically character amenable.

Definition 2·2. Let A be a Banach algebra, $\psi \in \sigma(A) \cup \{0\}$, and $\Psi \in A''$.

(i) Ψ is called ψ-**topologically left invariant** (ψ-TLI) if

$$\langle \Psi, a \cdot \phi \rangle = \psi(a)\langle \Psi, \phi \rangle \qquad (a \in A, \phi \in A'). \tag{2·3}$$

(ii) Ψ is called ψ-**topologically right invariant** (ψ-TRI) if

$$\langle \Psi, \phi \cdot a \rangle = \psi(a)\langle \Psi, \phi \rangle \qquad (a \in A, \phi \in A'). \tag{2·4}$$

The products $a \cdot \phi$ and $\phi \cdot a$ are of course the natural left and right module actions of A on A'. If Ψ is both ψ-TLI and ψ-TRI, it is called ψ-topologically invariant, or for short ψ-TI.

For $\phi \in \sigma(A)$, the notion of a ϕ-*mean* as defined in [**17**] is equivalent to a ϕ-TRI element Φ such that $\Phi(\phi) = 1$.

THEOREM 2·3. *A Banach algebra A is left character amenable if and only if the following two conditions hold*:

(i) *A has a bounded left approximate identity*;

(ii) *for every $\psi \in \sigma(A)$ there exists a ψ-TLI element $\Psi \in A''$ such that $\Psi(\psi) \neq 0$.*

Similar statements hold for right character amenability.

Proof. It follows from [**16**, propositions 1·5 and 1·6, pp. 10–11] that the existence of a bounded left approximate identity for A is equivalent to $\mathcal{H}^1(A, E') = \{0\}$ for every Banach A-bimodule E for which the right module action of A is the trivial multiplication $x \cdot a = 0$. Now the theorem follows from [**17**, theorem 1·1] and our remark following Definition 2·1.

For a locally compact group G, the Fourier algebra $A(G)$ and the generalized Fourier algebra $A_p(G)$ $(1 < p < \infty)$ were introduced in [**5**] and [**7**], respectively (see also [**6**]). The algebras $A(G)$ and $L^1(G)$ are considered dual objects in harmonic analysis, since $A(G) \cong L^1(\widehat{G})$ via the Fourier transform, when G is abelian. The generalized Fourier algebra $A_p(G)$ coincide with $A(G)$ if $p = 2$. The next corollary shows that the character amenability of these algebras are completely characterized by the amenability of their underlying group G.

COROLLARY 2·4. *Let $1 < p < \infty$, G be a locally compact group, and A be either of the Banach algebras $L^1(G)$ or $A_p(G)$. Then the following are equivalent*:

(i) *A is left character amenable*;

(ii) *A is right character amenable*;

(iii) *G is amenable.*

Proof. First let $A = A_p(G)$. For $x \in G$, define $\phi_x : A_p(G) \to \mathbf{C}$, $\phi_x(u) = u(x)$, to be the evaluation functional at x. In that case $\sigma(A_p(G)) = \{\phi_x : x \in G\}$ [**13**, theorem 3, p. 102]. It is also known that, given an arbitrary locally compact group G, for every $x \in G$, there exits a ϕ_x-TI element Φ_x such that $\Phi_x(\phi_x) = 1$ [**20**, lemma 3·1, p. 417] (for the case of $x = e$ this result is due to Granirer [**9**, theorem 5, p. 123]). But since amenability of G is equivalent to the existence of a bounded approximate identity for $A_p(G)$ [**13**, theorem 6, p. 120], it follows that character amenability of $A_p(G)$ is equivalent to the amenability of G. For the case of $A = L^1(G)$, our corollary follows from the Theorem 2·3 and the famous result of Johnson [**16**, theorem 2·5, p. 32] to the effect that G is amenable if and only if $L^1(G)$ is amenable.

Let $M(G) = M_d(G) \oplus M_c(G)$ be the direct sum decomposition of the measure algebra $M(G)$ into its closed subalgebra of discrete measures $M_d(G)$ and its closed ideal of continuous measures $M_c(D)$ [**15**, theorem 19·20, p. 273]. Every measure $\mu \in M(G)$ has a unique decomposition $\mu = \mu_d + \mu_c$, where $\mu_d \in M_d(G)$ and $\mu_c \in M_c(G)$. The map defined by

$$\phi\colon M_d(G) \longrightarrow \mathbf{C}, \quad \sum_{s\in G} \alpha_s \delta_s \longmapsto \sum_{s \in G} \alpha_s,$$

is a character on $M_d(G)$. This character has an extension to the entire $M(G)$ defined by

$$\tilde{\phi}\colon M(G) \longrightarrow \mathbf{C}, \quad \mu \longmapsto \phi(\mu_d),$$

which is called the *discrete augmentation character* on $M(G)$ [**4**, p. 215].

COROLLARY 2·5. *The measure algebra $M(G)$ is character amenable if and only if G is a discrete amenable group.*

Proof. Sufficiency is clear in view of Corollary 2·4 and the fact that if G is discrete then $M(G) = l^1(G)$. To prove the converse statement, we note that in [**4**, theorem 3·2, p. 224] Dales, Ghahramani and Helemskii have shown that if G is a non-discrete locally compact group, then there exists a non-zero point derivation on $M(G)$ at the discrete augmentation character $\tilde{\phi}$. Thus character amenability of $M(G)$ implies that G is discrete, and $M(G) = l^1(G)$, in which case Corollary 2·4 once again applies to show that G must be amenable.

It follows from Theorem 2·3 that radical Banach algebras with bounded left approximate identities are automatically left character amenable, since the spectrum of such Banach algebras are empty. However it is easy to show that

$$A = \left\{ \begin{bmatrix} \lambda I_n & C \\ 0 & \lambda I_n \end{bmatrix} : \lambda \in \mathbf{C}, C \in \mathbb{M}_n(\mathbf{C}) \right\}$$

is an example of a commutative, non-radical, non-semisimple Banach algebra that is *not* character amenable. Such examples justify our approach of not restricting the notion of character amenability to merely semisimple Banach algebras.

The next theorem summarizes the functorial properties of left character amenability.

THEOREM 2·6. *Let A and B be Banach algebras and I a closed two sided ideal of A.*

(i) *If A is left character amenable and $\mu\colon A \to B$ is a continuous homomorphism with $\overline{\mu(A)} = B$, then B is left character amenable.*

(ii) *If A is left character amenable then I is left character amenable if and only if I has a bounded left approximate identity.*

(iii) *If both I and A/I are left character amenable, then A is left character amenable.*

(iv) *The unitization algebra $A^\#$ is left character amenable if and only if A is left character amenable.*

(v) *$A \times B$ is left character amenable if and only if both A and B are left character amenable.*

Proof. (i) and (iv) follow from [**17**, proposition 3·5 and lemma 3·2], respectively.

(ii) This statement follows from [**17**, Lemma 3·1] provided we show that every $\phi \in \sigma(I)$ extends to some $\widetilde{\phi} \in \sigma(A)$. The following proof was kindly pointed out to me by the referee. First we note that since I has a bounded left approximate identity the kernel of ϕ, say J, is a closed ideal in A. Let $u \in I$ be such that $\phi(u) = 1$. Then u is an identity of I modulo J, and $\widetilde{\phi}(x) = \phi(xu)$, $x \in A$, defines an element of $\sigma(A)$ extending ϕ. Indeed,

$$\widetilde{\phi}(x_1)\widetilde{\phi}(x_2) = \phi(x_1 u)\phi(x_2 u) = \phi(x_1 u x_2 u) = \phi(x_1 x_2 u) = \widetilde{\phi}(x_1 x_2),$$

since $x_1 u x_2 u - x_1 x_2 u = x_1(u x_2 u - x_2 u) \in J$.

The proof of (iii) is similar to the analogous result for amenability of short exact sequences [**16**, proposition 5·1, p. 51]. Verification of (v) is routine.

The following corollary now follows from [**17**, propositions 2·1 and 2·2].

COROLLARY 2·7. *A Banach algebra A is left [right] character amenable if and only if* $\ker\phi$ *has a bounded left [right] approximate identity for every $\phi \in \sigma(A) \cup \{0\}$.*

It follows from the above corollary that every C^*-algebra is character amenable.

Note that left character amenability is not automatically inherited by closed ideals. In fact if G is an amenable group, an ideal I in the Fourier algebra $A(G)$ has bounded approximate identity if and only if $I = I(E) = \{u \in A(G) \colon u(E) = \{0\}\}$ where E belongs to the coset ring of G [**8**, theorem 2·3]. Thus if F a closed set which does not belong to the coset ring of G, then $I(F)$ is not left character amenable.

Next we would like to look at certain *twisted* products between Banach algebras, and characterize when they are left character amenable. Let A and B be Banach algebras, $\sigma(B) \neq \emptyset$, and $\theta \in \sigma(B)$. Then the θ-Lau product, denoted by $A \times_\theta B$, is defined as the set $A \times B$ equipped with the multiplication

$$(a, b) \cdot (a', b') = (aa' + \theta(b)a' + \theta(b')a, bb'),$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$. Then A is a closed two sided ideal of $A \times_\theta B$ and $A \times_\theta B/A \cong B$, in other words one can view $A \times_\theta B$ as a strongly splitting Banach algebra extension of B by A. We note that in the special case that $B = \mathbf{C}$ is the complex numbers and θ is the identity map on $\mathbf{C}$, then $A \times_\theta \mathbf{C} = A^\#$ is the unitization of the algebra A. Lau products have been studied in [**18**] and [**21**]. In particular it has been shown that [**21**, proposition 2·4]

$$\sigma(A \times_\theta B) = \sigma(A) \times \{\theta\} \cup \{0\} \times \sigma(B).$$

We denote the first and second Arens product of the second dual of a Banach algebra by $\Box$ and $\Diamond$, respectively.

PROPOSITION 2·8. *Let $(\Phi, \Psi) \in (A \times_\theta B)'' = A'' \times_\theta B''$.*

(i) *(Φ, Ψ) is (ϕ, θ)-TLI with $\langle(\Phi, \Psi), (\phi, \theta)\rangle \neq 0$ if and only if $\Psi = 0$ and Φ is ϕ-TLI with $\Phi(\phi) \neq 0$.*

(ii) *(Φ, Ψ) is $(0, \theta)$-TLI with $\langle(\Phi, \Psi), (0, \theta)\rangle \neq 0$ if and only if Ψ is θ-TLI with $\Psi(\theta) \neq 0$ and $-1/\Psi(\theta)\Phi$ is a left identity for $(A'', \Diamond)$.*

(iii) *For $\psi \neq \theta$, (Φ, Ψ) is $(0, \psi)$-TLI with $\langle(\Phi, \Psi), (0, \psi)\rangle \neq 0$ if and only if $\Phi = 0$, $\Psi(\theta) = 0$, and Ψ is ψ-TLI with $\Psi(\psi) \neq 0$.*

Similar results hold for topologically right invariant elements.

Proof. (i) Suppose that (Φ, Ψ) is (ϕ, θ)-TLI with $\langle(\Phi, \Psi), (\phi, \theta)\rangle \neq 0$. Then

$$\langle(\Phi, \Psi), (\phi, \theta)\rangle = \Phi(\phi) + \Psi(\theta) \neq 0, \tag{2·5}$$

$$(\Phi \cdot a + \theta(b)\Phi + \Psi(\theta)a, \Psi \cdot b) = (\Phi, \Psi) \cdot (a, b) = (\phi(a) + \theta(b))(\Phi, \Psi). \tag{2·6}$$

From (2·6) we get

$$\Phi \cdot a + \Psi(\theta)a = \phi(a)\Phi, \tag{2·7}$$

$$\Psi \cdot b = \phi(a)\Psi + \theta(b)\Psi. \tag{2·8}$$

Choosing $b = 0$ and a such that $\phi(a) \neq 0$, we conclude from the last equation that $\Psi = 0$. It follows from (2·5) and (2·7), that Φ is ϕ-TLI and $\Phi(\phi) \neq 0$. The converse is clear from (2·5) and (2·6).

(ii) If (Φ, Ψ) is $(0, \theta)$-TLI with $\langle(\Phi, \Psi), (0, \theta)\rangle \neq 0$, then $\Psi(\theta) \neq 0$ and

$$(\Phi, \Psi) \cdot (a, b) = (\Phi \cdot a + \theta(b)\Phi + \Psi(\theta)a, \Psi \cdot b) = \theta(b)(\Phi, \Psi).$$

From this equation we conclude that $\Psi \cdot b = \theta(b)\Psi$ for all $b \in B$, and thus Ψ is θ-TLI. Furthermore we get

$$\left(\frac{-\Phi}{\Psi(\theta)}\right) \cdot a = a \quad \text{(for all } a \in A).$$

From the continuity of the second Arens multiplication from the left-hand side, it follows that $-\Phi/\Psi(\theta)$ is a left identity for $(A'', \Diamond)$.

We leave the verification of the rest of the statements to the reader.

COROLLARY 2·9. *The Banach algebra $A \times_\theta B$ is left [right] character amenable if and only if both A and B are left [right] character amenable.*

Proof. In view of Theorem 2·6, (i) and (iii), it suffices to show that A is left character amenable, whenever $A \times_\theta B$ is left character amenable. Considering $(0, \theta) \in \sigma(A \times_\theta B)$, we must have a $(0, \theta)$-TLI element $(\Phi, \Psi) \in A'' \times_\theta B''$ such that $\langle(\Phi, \Psi), (0, \theta)\rangle = \Psi(\theta) \neq 0$. Therefore by part (ii) of the previous proposition, $-(1/\Psi(\theta)\Phi)$ is a left identity for $(A'', \Diamond)$ which implies that A has a bounded left approximate identity.

Next, for every $\phi \in \sigma(A)$, $(\phi, \theta) \in \sigma(A \times_\theta B)$, and by our assumption there exists a (ϕ, θ)-TLI element $(\Phi, \Psi) \in A'' \times_\theta B''$ such that $\Phi(\phi) + \Psi(\theta) \neq 0$. By part (i) of the previous result, we must have $\Psi = 0$ and Φ a ϕ-TLI element with $\Phi(\phi) \neq 0$. This proves that A is left character amenable.

3. *Splitting properties of modules over character amenable Banach algebras*

THEOREM 3·1. *Let A be a commutative character amenable Banach algebra, and E a finite-dimensional Banach A-bimodule. Then $\mathcal{H}^n(A, E) = \{0\}$, for every $n \geqslant 1$.*

Proof. Given $\phi \in \sigma(A)$, from the fact that $\ker\phi$ has a bounded approximate identity [corollary 2·7] and by Cohen's factorization theorem [**14**, theorem 32·26, p. 270] it follows that $(\ker\phi)^2 = \ker\phi$. Therefore by [**1**, pp. 56–57] we can write

$$E = \bigoplus_{i=1}^{n} \mathbf{C}_{\phi_i,\psi_i}, \tag{3·1}$$

where $n = \dim E$, and $\phi_i, \psi_i \in \sigma(A) \cup \{0\}$, and $\mathbf{C}_{\phi_i,\psi_i}$ is the set of complex numbers with the A-bimodule multiplications

$$a \cdot z = \phi_i(a)z, \quad z \cdot a = \psi_i(a)z \quad (a \in A, z \in \mathbf{C}).$$

Hence

$$\mathcal{H}^n(A, E) = \bigoplus_{i=1}^{n} \mathcal{H}^n(A, \mathbf{C}_{\phi_i,\psi_i}). \tag{3·2}$$

So the problem is reduced to showing

$$\mathcal{H}^n(A, \mathbf{C}_{\phi,\psi}) = \{0\}, \tag{3·3}$$

whenever $\phi, \psi \in \sigma(A) \cup \{0\}$. This can be done in two ways. The short way is to use the character amenability of A and the 'reduction of dimension formula' [**3**, proposition 2·8·22, p. 283] to write

$$\mathcal{H}^{n+1}(A, \mathbf{C}_{\phi,\psi}) = \mathcal{H}^1(A, \mathcal{B}^n(A, \mathbf{C}_{\phi,\psi})) = \mathcal{H}^1(A, \mathcal{B}_n(A, \mathbf{C}_{\phi,\psi})') = \{0\}. \tag{3·4}$$

A more direct, though lengthier, proof of (3·3) consists of exploiting the existence of topological invariant elements of A''. This method does not use the existence of bounded approximate identities in A and as such relies on weaker assumptions compared to our shorter proof (the last identity in (3·4) uses the fact that A is character amenable and hence has bounded approximate identities).

If either $\phi = 0$ or $\psi = 0$, then (3·3) holds by a result of Johnson [**16**, proposition 1·5, p. 10] and from the fact that A has a bounded approximate identity. So we are left to show (3·3) if both ϕ and ψ are non-zero. Let $T \in \mathcal{Z}^n(A, \mathbf{C}_{\phi,\psi})$ and let $\widetilde{T} \in (A\widehat{\otimes}\cdots\widehat{\otimes}A)'$ canonically correspond to T (that is $\widetilde{T}(a_1 \otimes \cdots \otimes a_n) = T(a_1, \ldots, a_n)$). Consider the canonical isometric isomorphism

$$(\underbrace{A\widehat{\otimes}\cdots\widehat{\otimes}A}_{n-\text{times}})' \cong \mathcal{B}(\underbrace{A\widehat{\otimes}\cdots\widehat{\otimes}A}_{(n-1)-\text{times}}, A'), \quad R \longmapsto \Phi_R,$$

given by $\langle \Phi_R(a_1 \otimes \cdots \otimes a_{n-1}), a_0\rangle = R(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})$. Then the n-cocycle identity for $\widetilde{T}$, that is,

$$\begin{aligned}\phi(a_0)\widetilde{T}(a_1 \otimes \cdots \otimes a_n) &+ \sum_{j=0}^{n-1}(-1)^{j+1}\widetilde{T}(a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n)\\ &+ (-1)^{n+1}\psi(a_n)\widetilde{T}(a_0 \otimes \cdots \otimes a_{n-1}) = 0,\end{aligned}$$

can be written as

$$\begin{aligned}&\langle \widetilde{T}(a_1 \otimes \cdots \otimes a_n)\phi, a_0\rangle - \langle \Phi_{\widetilde{T}}(a_2 \otimes \cdots \otimes a_n), a_0 a_1\rangle\\ &+ \sum_{j=1}^{n-1}(-1)^{j+1}\langle \Phi_{\widetilde{T}}(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n), a_0\rangle\\ &+ (-1)^{n+1}\psi(a_n)\langle \Phi_{\widetilde{T}}(a_1 \otimes \cdots \otimes a_{n-1}), a_0\rangle = 0.\end{aligned}$$

Since $a_0 \in A$ is arbitrary, we obtain

$$\begin{aligned}&\widetilde{T}(a_1 \otimes \cdots \otimes a_n)\phi - a_1 \cdot \Phi_{\widetilde{T}}(a_2 \otimes \cdots \otimes a_n)\\ &\quad+ \sum_{j=1}^{n-1}(-1)^{j+1}\Phi_{\widetilde{T}}(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n)\\ &\quad+ (-1)^{n+1}\psi(a_n)\Phi_{\widetilde{T}}(a_1 \otimes \cdots \otimes a_{n-1}) = 0.\end{aligned} \tag{3·5}$$

Suppose that Ψ_ϕ is a ϕ-TI element of A'' such that $\Psi_\phi(\phi) \neq 0$. By dividing Ψ_ϕ with $\Psi_\phi(\phi)$ we may assume $\Psi_\phi(\phi) = 1$. Let $S \in \mathcal{B}^{n-1}(A, \mathbf{C})$ be a continuous $(n-1)$-linear map defined by

$$S(a_2, \ldots, a_n) = \Psi_\phi \circ \Phi_{\widetilde{T}}(a_2 \otimes \cdots \otimes a_n),$$

and let

$$\widetilde{S} = \Psi_\phi \circ \Phi_{\widetilde{T}} \in (\underbrace{A \otimes \cdots \otimes A}_{(n-1)-\text{times}})'$$

be the canonical continuous linear functional associated to S. If we apply Ψ_ϕ on the left to the identity (3·5), we get

$$\begin{aligned}&\widetilde{T}(a_1 \otimes \cdots \otimes a_n) - \phi(a_1)\widetilde{S}(a_2 \otimes \cdots \otimes a_n)\\ &+ \sum_{j=1}^{n-1}(-1)^{j+1}\widetilde{S}(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n)\\ &+ (-1)^{n+1}\psi(a_n)\widetilde{S}(a_1 \otimes \cdots \otimes a_{n-1}) = 0.\end{aligned}$$

Rewriting this last equation in terms of T and S we have

$$\begin{aligned}T(a_1, \ldots, a_n) - \phi(a_1)S(a_2, \ldots, a_n) + \sum_{j=1}^{n-1}(-1)^{j+1}S(a_1, \ldots, a_j a_{j+1}, \ldots, a_n)\\ + (-1)^{n+1}\psi(a_n)S(a_1, \ldots, a_{n-1}) = 0,\end{aligned}$$

or equivalently,

$$\begin{aligned}T(a_1, \ldots, a_n) = a_1 \cdot S(a_2, \ldots, a_n) + \sum_{j=1}^{n-1}(-1)^{j}S(a_1, \ldots, a_j a_{j+1}, \ldots, a_n)\\ + (-1)^{n+2}S(a_1, \ldots, a_{n-1}) \cdot a_n = 0.\end{aligned}$$

Therefore $T = \delta^{n-1}S \in \mathcal{N}^n(A, \mathbf{C}_{\phi,\psi})$, as we wanted to show.

It is well known that the obstruction to the splitting of singular admissible extensions lie in the two dimensional cohomology group [**1**, **12**]. For the case of finite-dimensional extensions however, splitting of singular extensions implies the splitting of all extensions [**1**, theorem 1·8(ii), p. 13]. Thus

COROLLARY 3·2. *If A is a commutative character amenable Banach algebra, then any finite-dimensional extension of A splits strongly.*

Remark 3·1. (a) The global dimension theorem of Helemskii [**12**, assertion V·2·21, p. 213] shows that if A is a commutative Banach algebra for which $\sigma(A)$ is infinite, then $dbA \geqslant 2$. Therefore $\mathcal{H}^2(A, E) \neq \{0\}$ for some infinite-dimensional Banach A-bimodule E. Thus Theorem 3·1 cannot be extended to infinite-dimensional Banach A-bimodules E, without restrictive assumptions on A.

(b) The above theorem and its corollary generalize earlier results of H. Steiniger [**22**] and myself [**20**] from $A(G)$ and $A_p(G)$ for G amenable, to the much larger class of commutative character amenable Banach algebras.

In [**3**, proposition 2·8·24, p. 283] Dales proves that if A is a *commutative*, unital Banach algebra and $\phi, \psi \in \sigma(A)$ with $\phi \neq \psi$, then $\mathcal{H}^1(A, \mathbf{C}_{\phi,\psi}) = \mathcal{H}^2(A, \mathbf{C}_{\phi,\psi}) = \{0\}$. An inspection of the proof of Theorem 3·1 shows that the commutativity of A was only used to prove (3·1). In particular in the proof of (3·3), commutativity of A plays no role. So as a corollary we obtain the following variation of Dales' result, in which A is no longer commutative, but is assumed left character amenable.

COROLLARY 3·3. *Let A be a (not necessarily commutative) left character amenable Banach algebra. Then for all ϕ, $\psi \in \sigma(A) \bigcup \{0\}$ and all $n \geqslant 1$, we have $\mathcal{H}^n(A, \mathbf{C}_{\phi,\psi}) = \{0\}$.*

The following is an analogue of a well-known splitting property [**11**, theorem II·30], and [**2**, Theorem 2.3, p. 94]. The proof is similar.

THEOREM 3·4. *Let A be a left character amenable Banach algebra and let E be a right Banach A-bimodule such that $x \cdot a = \phi(a)x$ for some $\phi \in \sigma(A) \bigcup \{0\}$ $(x \in E, a \in A)$. Let F and G be Banach left A-bimodules. Then each admissible short exact sequence of Banach left A-modules*

$$0 \longrightarrow E' \xrightarrow{\ S\ } F \xrightarrow{\ T\ } G \longrightarrow 0,$$

splits strongly.

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REFERENCES

[**1**] W. G. BADE, H. G. DALES and Z. A. LYKOVA. Algebraic and strong splittings of extensions of Banach algebras. *Mem. Amer. Math. Soc.* Vol. 656 (Amer. Math. Soc., 1999).

[**2**] P. C. CURTIS and R. J. LOY. The structure of amenable Banach algebras. *J. London Math. Soc.* **40**, (1989), 89–104.

[**3**] H. G. DALES. *Banach Algebras and Automatic Continuity.* (Oxford Uinversity Press, 2000).

[**4**] H. G. DALES, F. GHAHRAMANI and A. YA. HELEMSKII. *The amenability of measure algebras. J. London Math. Soc.* **66** (2002), 213–226.

[**5**] P. EYMARD. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France* **92** (1964), 181–236.

[**6**] P. EYMARD. *Algèbres A_p et convoluteurs de L^p*. Lecture Notes in Math. Vol. **180**, pp. 364–381 (Springer-Verlag, 1971).

[**7**] A. FIGÀ–TALAMANCA. Translation invariant operators in L^p. *Duke Math. J.* **32** (1965), 459–501.

[**8**] B. FORREST, E. KANIUTH, A. LAU and N. SPRONK. Ideals with bounded approximate identities in Fourier algebras. *J. Funct. Anal.* **203** (2003), 268–304.

[**9**] E. E. GRANIRER. On some spaces of linear functionals on the algebra $A_p(G)$ for locally compact groups. *Colloq. Math.* LII (1987), 119–132.

[**10**] A. YA. HELEMSKII. A certain class of flat Banach modules and its applications. *Vest. Mosk. Univ. Ser. Mat. Mekh.* **27** (1972), 29–36.

[**11**] A. YA. HELEMSKII. Flat Banach modules and amenable Banach algebras. *Trans. Moscow Math. Soc.* **47** (1985), 229–250.

[**12**] A. YA. HELEMSKII. The Homology of Banach and Topological Algebras (Kluwer Academic Publishers, 1989).

[**13**] C. HERZ. Harmonic synthesis for subgroups. *Ann. Inst. Fourier (Grenoble)* **23** (1973), 91–123.

[**14**] E. HEWITT and K. A. ROSS. *Abstract Harmonic Analysis*, Vol. 2 (Springer-Verlag, 1970).

[**15**] E. HEWITT and K. A. ROSS. *Abstract Harmonic Analysis*, Vol. 1, Second edition (Springer-Verlag, 1979).

[**16**] B. E. JOHNSON. Cohomology in Banach Algebras. *Mem. Amer. Math. Soc.* Vol. **127** (Amer. Math. Soc., 1972).

[**17**] E. KANIUTH, A. T. LAU and J. PYM. On ϕ-Amenability of Banach algebras. *Math. Proc. Camb. Phil. Soc.* (to appear).

[**18**] A. T.-M. LAU. Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups. *Fund. Math.* **118** (1983), 161–175.
[**19**] V. RUNDE. *Lectures on Amenability* (Springer, 2002).
[**20**] M. SANGANI–MONFARED. Extensions and isomorphims for the generalized Fourier algebras of a locally compact group. *J. Funct. Anal.* **198** (2003), 413–444.
[**21**] M. SANGANI-MONFARED. On certain products of Banach algebras with applications to harmonic analysis. *Studia Math.* **178** (2007), 277–294.
[**22**] H. STEINIGER. *Finite-dimensional extensions of the Fourier algebras*, (preprint).