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A physical approach to spherical stars.

Samuel Pedro. Goldman
University of Windsor

A PHYSICAL APPROACH TO SPHERICAL STARS

by

Samuel Pedro Goldman

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ABSTRACT

The problem of constructing stellar models consisting of static spherical perfect fluids is studied. The method of generating solutions of Einstein's field equations, first developed by Tolman, is improved by a priori physical constraints on the generating functions.

A physical approach to generating solutions is developed which features the isotropic pressure p and mass energy density μ as primary quantities. Working in isotropic coordinates, a single ordinary differential equation relating p and μ is obtained, which is independent of metric components. The metric is obtained by quadrature after a p, μ solution is found.

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1. INTRODUCTION

The line element for a static, spherically symmetric spacetime is given by[1]:

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{1}$$

where θ and φ are the standard coordinates on the sphere and r is the area coordinate of the surfaces (r,t) = constant.

A perfect fluid is taken as the source of the spacetime with energy - momentum tensor given by[2]:

$$T^{\alpha}{}_{\beta} = (p+\mu c^2) \, u^{\alpha} u_{\beta} - p\delta^{\alpha}{}_{\beta} \tag{2}$$

where μ and p are the proper density and isotropic pressure, and u^{α} is the unit vector along the timelike Killing field

$$u^{\alpha} = e^{-\nu/2} \, \delta^{\alpha}{}_{t} \tag{3}$$

In the following, the units c = G = 1 will be used.

The application of Einstein's field equations

$$R^{\alpha}{}_{\beta} - \frac{1}{2} R\delta^{\alpha}{}_{\beta} = -8\pi \, T^{\alpha}{}_{\beta} \tag{4}$$

leads to the following expressions[3]:

$$8\pi \, \mu = e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} \tag{5-a}$$

$$8\pi \, p = e^{-\lambda} \left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r} \right) \tag{5-b}$$

$$8\pi\, p = e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} . \qquad (5\text{-c})$$

This system can be rewritten as one equation for μ, another for p, and one relating only λ, ν and r, by substracting (5-c) from (5-b):

$$- \frac{e^{\lambda}}{r^2} + \frac{1}{4}\nu'\lambda' - \frac{1}{4}\nu'^2 - \frac{1}{2}\nu'' + \frac{\nu'+\lambda'}{2r} + \frac{1}{r^2} = 0 , \qquad (6\text{-a})$$

$$8\pi\, \mu = e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} , \qquad (6\text{-b})$$

$$8\pi\, p = e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} . \qquad (6\text{-c})$$

The system (6) represents three ordinary differential equations for the four unknown functions of r which describe the geometry and physics of the system: $\nu(r)$, $\lambda(r)$, $p(r)$ and $\mu(r)$.

A well posed problem requires one further independent equation. From a physical point of view the additional equation should appear as a specific choice for $p(r)$ or $\mu(r)$, or an equation relating p and μ with r, or as an equation of state $p = p(\mu)$.

But, as remarked by R. C. Tolman[4]: ''From a mathematical point of view, however, the derivatives of λ and ν occur in equations (6) in such a complicated and nonlinear manner that we cannot in general expect to obtain explicit analytic solutions when we complete the set by adding a further equation connecting p with μ or p and μ with r ''.....''In order to obtain explicit analytic solutions, it proves more advantegeous to introduce the additional equation necessary to give a determinate problem in the form of some relation connecting λ or ν or both with r ....''.

These lines were written in 1939 and since then, this was the method of attack on equations (6). Research toward finding explicit analytic solutions was directed mainly to rewriting equation (6-a) such that for a generating function $f = f(\nu,\lambda,\nu',\lambda',r)$, another function $g = g(\nu,\lambda,\nu',\lambda',r)$

could be obtained from (6-a) by quadratures.

Afterwards p and μ are obtained from (6-b-c).

Using this indirect procedure it remains to check whether the solution is physically reasonable, in particular if:

$$p(r) \geq 0, \ \mu(r) \geq 0, \ p'(r) < 0 \tag{7}$$

The excellence of one method over the other being the ease of the quadratures.

A method of obtaining numerical solutions was developed by J. R. Oppenheimer and G. M. Volkoff[5]. It is worth examining this method even if it does not give analytic solutions, because since 1939 it is the most widely used to obtain models for stars.

2. THE OPPENHEIMER - VOLKOFF METHOD

By using computer methods, it is not difficult in general to solve a system of differential equations given suitable initial parameters.

In our case two considerations are important:

a) The method;

b) The choice and use of a fourth equation to make system (6) determinate and the approach physical.

Oppenheimer and Volkoff[5] (O-V) choose as a fourth equation:

$$\mu = \mu\,(p), \tag{8}$$

and made use of the identity[6]

$$\frac{dp}{dr} = -\frac{(p+\mu)}{2}\frac{d\nu}{dr}. \tag{9}$$

The pressure must be $p > 0$ inside the star, with the boundary of the star defined by

$$r_b : \qquad p(r_b) = 0 . \tag{10}$$

By Birkhoff's theorem[7], the solution in the vacuum region surrounding the star, must be Schwarzschild's exterior solution:

$$e^{-\lambda} = 1 - \frac{2m}{r} \; ; \; e^{\nu} = 1 - \frac{2m}{r} \quad , \quad (r > r_b) \tag{11}$$

thus, the boundary conditions are:

$$e^{-\lambda(r_b -)} = e^{-\lambda(r_b +)} = 1 - \frac{2m}{r_b} \quad , \tag{12-a}$$

$$e^{\nu(r_b -)} = e^{\nu(r_b +)} = 1 - \frac{2m}{r_b} \tag{12-b}$$

where $r_b > 2m$. (12) must be satisfied to make the gravitational field continuous across the boundary.

By (8) and (9):

$$\nu(r) - \nu(r_b) = -2\int_{p(r_b)}^{p(r)} \frac{dp}{p + \mu(p)} = -2\int_{0}^{p(r)} \frac{dp}{p + \mu(p)} \quad . \tag{13}$$

The constant $\nu(r_b)$ is obtained from (12-b), and so:

$$e^{\nu(r)} = (1 - \frac{2m}{r}) \exp\left(- \int_{0}^{p(r)} \frac{2dp}{p + \mu(p)}\right) \quad . \tag{14}$$

Thus, the functional form of $\nu(r)$ is known if p is known as a function of r. The choice $p = p(r)$ is not arbitrary since Eq. (8) exhausted the only degree of freedom left by system (6).

O-V define a new variable:

$$u(r) := \frac{1}{2} r(1 - e^{-\lambda}) \quad \text{or} \quad e^{-\lambda} = 1 - 2\frac{u}{r} \quad , \tag{15}$$

and by (12-a) it follows that

$$u(r_b) = m \quad . \tag{16}$$

Using (14) and (15) in (6-b) and (6-c):

$$u' = 4\pi\,\mu(p)\,r^2 ; \tag{17-a}$$

$$p' = -\frac{p+\mu(p)}{r(r-2u)}\,(4\pi\,pr^3 + u) . \tag{17-b}$$

Equations (17) form a system of two first-order equations in u and p.

Starting with initial values u(0) and p(0), the two equations are integrated simultaneously (numerically this is a straight-forward step by step procedure) to the value r_b at which p = 0. From Eq. (16), $r = r_b$ will define the mass of the spherical distribution as measured by a distant observer.

In principle it should be possible to integrate (17) in the reverse way: by taking $p(r_b) = 0$, $u(r_b) = m$, for some values of m and r_b, but with the equation of state chosen and used in (14) and in (17) this may give unphysical values at the centre (r=0), for example, negative pressure.

Both ways are possible and even fast using computers, and in both it is possible to change the initial parameters or $\mu(p)$ until a reasonable result is obtained.

The following restrictions are still to be satisfied:

(a) $p(r=0) \geq 0$;

(b) $u(r=0) = 0$ for finite values of $e^{-\lambda(r=0)}$. This restriction makes the first method convenient.

(c) O-V showed[5] that for $u(0) < 0$ (The case in which $e^{-\lambda(0)} \to \infty$) p(0) must be zero, but remarked that for any particular equation of state, a special investigation must be made to see whether solutions exist in which $u(0) < 0$ and $p \to \infty$ when $r \to 0$.

3. METHODS OF OBTAINING EXPLICIT ANALYTIC SOLUTIONS

I. - R. C. Tolman's approach.

As stated above, the time honored method consists of obtaining solutions to (6-a), and then substituting the results into (6-b) and (6-c) to obtain p and μ.

R. C. Tolman[4] rewrote (6-a) such that it is easier to find relations connecting λ, ν or both with r, so that (6-a) will be easily integrable.

Tolman wrote (6-a) as:

$$\frac{d}{dr}\left(\frac{e^{-\lambda}-1}{r^2}\right) + \frac{d}{dr}\left(\frac{e^{-\lambda}\nu'}{2r}\right) + e^{-\lambda-\nu}\frac{d}{dr}\left(\frac{e^{\nu}\nu'}{2r}\right) = 0 \qquad (6\text{-}a')$$

Tolman assumed an equation:

$$f(\nu,\nu',\lambda,\lambda',r) = 0 \;, \qquad (18)$$

such that (6-a) would be directly integrable.

He obtained eight types of solutions (8 assumed equations (18)). Three of them will be presented here.

i) (Case I in Tolman's article. 'Einstein Universe')

The assumed equation is:

$$f := e^{\nu} - \text{const.} = e^{\nu} - c^2 = 0 \;. \qquad (19\text{-}a)$$

From (6-a'):

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} \;, \qquad (19\text{-}b)$$

where R is a constant of integration, and:

$$8\pi\,\mu = \frac{3}{R^2} \;, \qquad (19\text{-}c)$$

$$8\pi\, p = -\frac{1}{R^2} . \tag{19-d}$$

This solution is unphysical (negative pressure).

In fact Tolman assumed the cosmological constant $\Lambda \neq 0$, in which case:

$$8\pi\, \mu = \frac{3}{R^2} - \Lambda \quad ,$$

$$8\pi\, p = -\frac{1}{R^2} + \Lambda \, ,$$

and so, for $\frac{1}{R^2} < \Lambda < \frac{3}{R^2}$, (19) can describe an actual distribution of matter.

ii) (Case III in Tolman's article. 'Schwarzschild interior solution'.)

$$f := e^{-\lambda} - \left(1 - \frac{r^2}{R^2} \right) = 0 . \tag{20-a}$$

Defining:

$$x := 1 - \frac{r^2}{R^2} \quad , \tag{20-b}$$

the resulting solution is:

$$e^{-\lambda} = x \; , \tag{20-c}$$

$$e^{\nu} = (\alpha - \beta x^{1/2})^2 \; , \tag{20-d}$$

$$8\pi\, \mu = \frac{3}{R^2} \quad , \tag{20-e}$$

$$8\pi\, p = \frac{1}{R^2} \left(\frac{3\beta x^{1/2} - \alpha}{\alpha - \beta x^{1/2}} \right) . \tag{20-f}$$

iii) (Case VI in Tolman's article)

$$f := e^{-\lambda} - (2 - n^2)^{-1} = 0 . \tag{21-a}$$

The resulting solution is:

$$e^{\lambda} = 2 - n^2 , \tag{21-b}$$

$$e^{\nu} = (\alpha r^{1-n} - \beta r^{1+n})^2 , \tag{21-c}$$

$$8\pi\, \mu = \frac{1 - n^2}{2 - n^2} \frac{1}{r^2} , \tag{21-d}$$

$$8\pi\, p = \frac{1}{2 - n^2} \frac{1}{r^2} \frac{(1-n^2)\alpha - (1+n)^2 \beta r^{2n}}{\alpha - \beta r^{2n}} , \tag{21-e}$$

The solutions must now satisfy the boundary conditions (10) and (12).

II. - Other methods of obtaining analytic solutions.

Since 1939, the quest for analytic solutions has followed Tolman's general approach, rewriting (6-a) in term of new functions $f_1(\nu,\nu',\lambda,\lambda',r)$ and $f_2(\nu,\nu',\lambda,\lambda',r)$, using one as a generator and obtaining the second by two explicit quadratures. Equations (10) and (12) are then used to determine constants.

Only two recent methods will be presented here.

i) One was proposed by R. J. Adler[8]. He made the following substitutions:

$$\gamma(r) = e^{\nu/2} , \quad (23\text{-a})$$

$$\tau(r) = e^{-\lambda} . \quad (23\text{-b})$$

Replacing in (6-a) he obtained:

$$\tau' + f(r)\tau = g(r) , \quad (24\text{-a})$$

where:

$$f(r) = - \frac{2\,(\gamma + r\gamma' - r^2\gamma'')}{r\,(\gamma + r\gamma')} , \quad (24\text{-b})$$

$$g(r) = - \frac{2\,\gamma}{r(\gamma + r\gamma')} . \quad (24\text{-c})$$

Using $\gamma(r)$ as a generating function, f and g are obtained above, and then τ is found by the general formula:

$$\tau(r) = e^{-F(r)} \left(\int^{r} e^{F(u)}\, g(u)\, du + C\right) , \quad (25\text{-a})$$

with:

$$F(r) = \int^{r} f(w)\, dw . \quad (25\text{-b})$$

It is clear that conditions (7) remain to be satisfied for a solution to be physically reasonable.

As stated by Adler: ''A judicious choice of $\gamma(r)$ is thus necessary ...''. There is no way to conect $\gamma(r)$ with the physics of the problem before integrations (25) are performed and equations (6-b-c) solved.

ii) Another method was proposed by R. C. Adams and J. M. Cohen[9]. They defined

$$x = r^2,\ b = e^{\lambda/2},\ a = e^{\nu/2}\ , \tag{26-a}$$

and

$$f = \frac{1}{x}\ (\ 1 - b^{-2}\ )\ . \tag{26-b}$$

Using (26), (6-a) can be written as:

$$f' + 2f\ (a' + 2xa'')(a + 2xa')^{-1} = 4a''(a + 2xa')^{-1}\ . \tag{27}$$

Now, a(r) is assumed as a generating function, and defining

$$g = (a' + 2xa'')(a + 2xa')^{-1}\ , \tag{28-a}$$

$$h = 4a''\ (a + 2xa')^{-1}\ , \tag{28-b}$$

$$G = \int^{x} g(u)\ du\ , \tag{28-c}$$

f is obtained by

$$f(x) = e^{-G(x)}\ (\ f_o + \int^{x} e^{G(t)}\ h(t)\ dt\ )\ . \tag{28-d}$$

Once f(x) is obtained, μ and p are found by

$$8\pi\ \mu = 3f + 2xf'\ ,$$

$$8\pi\ p = 4\ \frac{a'}{a} - f - 4xf\ \frac{a'}{a}\ .$$

Constraints are then placed on the arbitrary constants in f and a and it is then possible to see if conditions (7) are satisfied.

To quote from their paper, the main point is "... to assume a reasonable but definite form for a = a(x) ...".

We see the strong similarity between the last two methods presented.

4. ISOTROPIC COORDINATES

We rewrite the line element (1) in curvature coordinates as:

$$ds^2 = a^2\, dt^2 - b^2\, dr^2 - r^2\, (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{29}$$

In isotropic coordinates, the static spherically symmetric line element, is written as[10]

$$ds^2 = A^2(\rho)dt^2 - B^2(\rho)\, (\, d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2))\ , \tag{30}$$

where the two metrics are related by

$$b(r)\, dr = B(\rho)\, d\rho\ , \tag{31-a}$$

$$r = B(\rho)\ , \tag{31-b}$$

$$a(r) = A(\rho)\ . \tag{31-c}$$

Using (31), we can rewrite the system (6) with the isotropic coordinate . A prime is denoted in all the following as a derivative with respect to ρ:

$$f' = \frac{df}{d\rho} .$$

We obtain for the metric (30):

$$\frac{A''}{A} + \frac{B''}{B} - \frac{1}{\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right) - 2\,\frac{A'B'}{AB} - 2\left(\frac{B'}{B}\right)^2 = 0 \quad , \qquad (32\text{-a})$$

$$8\pi\, p = \frac{1}{B^2}\left(2\,\frac{A'B'}{AB} + \left(\frac{B'}{B}\right)^2 + \frac{2}{\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right)\right) \quad , \qquad (32\text{-b})$$

$$8\pi\, \mu = \frac{1}{B^2}\left(\left(\frac{B'}{B}\right)^2 - \frac{4}{\rho}\,\frac{B'}{B} - 2\,\frac{B''}{B}\right) \quad . \qquad (32\text{-c})$$

5. SYMMETRIC FIELD VARIABLES

Making the substitutions

$$D = (B)^{1/2} \quad , \qquad (33\text{-a})$$

$$E = A(B)^{1/2} \quad , \qquad (33\text{-b})$$

we can rewrite equations (32) as

$$DE'' + ED'' - 6E'D' - \frac{1}{\rho}\,(DE' + ED') = 0 \quad , \qquad (34\text{-a})$$

$$8\pi\, p = \frac{2}{D^4}\left(2\,\frac{D'E'}{DE} + \frac{1}{\rho}\left(\frac{D'}{D} + \frac{E'}{E}\right)\right) \quad , \qquad (34\text{-b})$$

$$8\pi\,\mu = -\frac{2}{D^4}\left(2\,\frac{D''}{D} + \frac{4}{\rho}\,\frac{D'}{D}\right) \quad . \tag{34-c}$$

Let us define an operator Ψ_x such that

$$\Psi_x(f,g) := f\,\frac{d^2g}{dx^2} + g\,\frac{d^2f}{dx^2} - 6\,\frac{df}{dx}\,\frac{dg}{dx} - \frac{1}{x}\left(f\,\frac{dg}{dx} + g\,\frac{df}{dx}\right) \quad . \tag{35}$$

If $\Psi_\rho(E,D) = 0$, then E and D satisfy the field equation (34-a).

The operator Ψ has the following properties:

$$\text{i)}\quad \Psi_x(f,g) = \Psi_x(g,f) \quad . \tag{36-a}$$

$$\text{ii)}\quad \Psi_x(c_1(f_1+f_2),(g_1+c_2g_2)) =$$

$$= c_1(\Psi_x(f_1,g_1) + \Psi_x(f_2,g_1)) + c_1c_2(\Psi_x(f_1,g_2) + \Psi_x(f_2,g_2)). \tag{36-b}$$

where c_1 and c_2 are constants.

So that Ψ_x is a symmetric and bilinear second order differential operator.

For $y = y(x)$, choose

$$y = (c_1 + c_2x^2)^{1/2} \quad , \tag{36-c}$$

with c_1, c_2 constants, then

iii) $\Psi_x(f,g) = c_2 \frac{x}{y} \Psi_y(f,g)$. (36-d)

It follows that

Proposition I

If

$$\Psi_x(f,g) = 0 ,$$

and

$$y = (c_1 + c_2 x^2)^{1/2} , \text{ with } c_1, c_2 \text{ constants,}$$

then

$$\Psi_y(f,g) = 0 . \quad (37)$$

This implies that if $E(\rho)$, $D(\rho)$ are a solution to (34-a), then $E((c_1+c_2\rho^2)^{1/2})$, $D((c_1+c_2\rho^2)^{1/2})$ also form a solution of (34-a), which in turn implies that any solution of (34-a) is in fact a two-parameter family of solutions just by replacing ρ by $(c_1+c_2\rho^2)^{1/2}$ with c_1 and c_2 arbitrary.

Proposition II

If

$$\Psi_\rho(D,E_1) = \Psi_\rho(D,E_2) = 0 ,$$

then

$$\Psi_\rho(D,E_1 + E_2) = 0 . \quad (38)$$

This follows trivialy from the bilinearity of (35).

Proposition III

If

$$\psi_\rho(f,f) = \psi_\rho(g,g) = 0 \quad ,$$

then

$$\psi_\rho(f+g,f-g) = 0 \quad . \tag{39}$$

This follows directly from $\psi(f,g) = \psi(g,f)$.

Equation (39) yields a new solution to (34-a) from a solution E = D.

In fact, if $f((c_1+c_2\rho^2)^{1/2})$ is a solution of $\psi_\rho(f,f)= 0$, then g can be taken as $g = f((c_3+c_4\rho^2)^{1/2})$.

Proposition IV

Consider the monomials $F(\rho)$, $G(\rho)$, $H(\rho)$, $I(\rho)$.

For $F(\rho) = \text{const.}\ \rho^k$, let $\deg(F) = k$, and let

$$\psi_\rho(F,G) = \psi_\rho(H,I) = 0 \ ,$$

if

$$\deg(F) + \deg(H) = \deg(G) + \deg(I),$$

then there exists a constant $a \neq 0$, such that

$$\psi_\rho(\ F+aI\ ,\ G+H\ ) = 0 \quad . \tag{40}$$

This is because

$$\psi_\rho(F+aI,G+H) = a\,\psi_\rho(I,J) + \psi_\rho(F,H) \quad ,$$

and by the assumptions, $\psi_\rho(I,G)$ and $\psi_\rho(F,H)$ are monomials of the same degree:

$$\deg(\psi_\rho(I,G)) = \deg(\psi_\rho(F,G)) = \deg(F) + \deg(H) - 2 \quad .$$

We can break the symmetry in equation (34-a) by defining:

$$E = \lambda - \mu \quad , \tag{41-a}$$

$$D = \lambda + \mu \quad , \tag{41-b}$$

and so (34-a) becomes:

$$\lambda\lambda'' - 3\,(\lambda')^2 - \frac{1}{\rho}\,\lambda\lambda' = \mu\mu'' - 3\,(\mu')^2 - \frac{1}{\rho}\,\mu\mu' \quad . \tag{41-c}$$

If

$$x = \alpha + \beta\rho^2 \quad , \quad \text{with } \alpha,\ \beta \text{ constants,} \tag{42-a}$$

then

$$\lambda\lambda_{,xx} - 3\,(\lambda_{,x})^2 = \mu\mu_{,xx} - 3\,(\mu_{,x})^2 \quad . \tag{42-b}$$

If

$$F = \lambda^{-2} \quad , \tag{43-a}$$

$$H = \mu^{-2} \quad , \tag{43-b}$$

then

$$\frac{F_{,xx}}{F^2} = \frac{H_{,xx}}{H^2} \quad . \tag{43-c}$$

So, we have the following set of equations that can be used instead of (32-a):

$$D = (B)^{1/2} \ , \quad E = A\ (B)^{1/2} \ ,$$

$$DE'' + ED'' - 6\ E'D' - \frac{1}{\rho}\ (DE' + ED') = 0 \qquad (34\text{-a})$$

If $x = \alpha + \beta\rho^2$, with α and β constants,

$$\lambda = \frac{1}{2}\ (A + 1)(B)^{1/2} \ , \quad \mu = \frac{1}{2}\ (1 - A)(B)^{1/2} \ ,$$

$$\lambda\lambda_{,xx} - 3\ (\lambda_{,x})^2 = \mu\mu_{,xx} - 3\ (\mu_{,x})^2 = h(x) \ . \qquad (42\text{-b})$$

$$F = \frac{1}{(A + 1)^2\ B} \ , \quad H = \frac{1}{(A - 1)^2\ B} \ ,$$

$$\frac{F_{,xx}}{F^2} = \frac{H_{,xx}}{H^2} = f(x) \ . \qquad (43\text{-c})$$

Each of this equations, although not immediately integrable, will be useful for specific situations.

For a given solution $D(\rho)$, $E(\rho)$, we can always make use of (37), thus giving the solution two more arbitrary parameters.

6. SOLUTION GENERATING WITH SYMMETRIC FIELD VARIABLES

Taking $E(\rho)$ (or $D(\rho)$) as a generating function in (34-a), we remain with a homogeneous, second order, linear equation for D (or E).

This case is very easy to deal with, when

$$E = \rho^k \quad , \tag{44-a}$$

is a generating function.

In this case, use of (38) yields

$$D = (c_1 \rho^{(8k^2+8k+1)^{1/2}} + c_2 \rho^{-(8k^2+8k+1)^{1/2}}) \rho^{(3n+1)} \quad , \tag{44-b}$$

with c_1, c_2 constants. Or by (36-a):

$$D = \rho^n \quad , \tag{45-a}$$

$$E = (b_1 \rho^{(8n^2+8n+1)^{1/2}} + b_2 \rho^{-(8n^2+8n+1)^{1/2}}) \rho^{(3n+1)} \quad , \tag{45-b}$$

with b_1, b_2 arbitrary constants. In (44) and (45), n and k are arbitrary real numbers.

Now, we can make use of property (37) and, for example, equations (45-a-b), become for α and β arbitrary constants:

$$D = (\alpha + \beta\rho^2)^{n/2} \tag{45-c}$$

$$E = (c_1(\alpha+\beta\rho^2)^s + c_2(\alpha+\beta\rho^2)^{-s})(\alpha+\beta\rho^2)^t \quad , \qquad (45\text{-d})$$

with:

$$s = \frac{1}{2}(8n^2 + 8n + 1)^{1/2} \quad ,$$

$$t = \frac{1}{2}(3n + 1) \quad .$$

Some examples of (45-c-d) for n, k integers, are:

$$D = c_1 \quad ,$$

$$E = c_2 + c_3\rho^2 \quad . \qquad (46)$$

$$D = c_1 (\alpha+\beta\rho^2)^{-1/2} \quad ,$$

$$E = c_2 (\alpha+\beta\rho^2)^{-1/2} + c_3 (\alpha+\beta\rho^2)^{-3/2} \quad . \qquad (47)$$

$$D = c_1 (\alpha+\beta\rho^2) \quad ,$$

$$E = c^3 + c_4 (\alpha+\beta\rho^2)^7 \quad .$$

$$D = c_1 (\alpha+\beta\rho^2)^{-3/2} \quad ,$$

$$E = c_2 (\alpha+\beta\rho^2)^{-1/2} + c_3 (d+\beta\rho^2)^{-15/2} \quad .$$

We can construct another family of two binomial solutions, by making use of property (40).

Examples will be:

$$D = c_1 + c_2\ (\alpha+\beta\rho^2)^{-1/2}\ ,$$
$$E = c_1 - c_2\ (\alpha+\beta\rho^2)^{-1/2}\ . \qquad (48)$$

$$D = -17\ \frac{c_1 c_2}{c_3} + c_1\ (\alpha+\beta\rho^2)^{-3/2}\ ,$$
$$E = c_2\ (\alpha+\beta\rho^2\ ) + c_3\ (\alpha+\beta\rho^2\ )^{-1/2}\ ; \qquad (49)$$

or, by using the symmetry (36-a):

$$D = c_1\ (\alpha+\beta\rho^2\ ) + c_2\ (\alpha+\beta\rho^2\ )^{-1/2}\ ,$$
$$E = -17\ \frac{c_1 c_3}{c_2} + c_3\ (\alpha+\beta\rho^2\ )^{-3/2}\ . \qquad (50)$$

Solutions (49) and (50) are new solutions for which pressure and density are given by equations (34-b-c).

Solution (47), with:

$c_1 = 1;\ c_2 = 0;\ \alpha = 1;\ \beta = \dfrac{1}{4R^2}$,

corresponds to Tolman's solution I, i.e.

$$D = (1 + \frac{\rho^2}{4R^2}\ )^{-1/2}\ ,$$
$$E = c\ (1 + \frac{\rho^2}{4R^2}\ )^{-1/2}\ , \qquad (51)$$

corresponds to equation (19). (Eintein Universe).

Solution (47), with:

$c_1 = 1;\ c_2 = \alpha + \beta;\ c_3 = -2\beta;\ \alpha = 1;\ \beta = (4R^2\ )^{-1}$,

corresponds to equation (20). That is,

$$D = \left(1 + \frac{\rho^2}{4R^2}\right)^{-1/2} ,$$

$$E = (\alpha+\beta)\left(1 + \frac{\rho^2}{4R^2}\right)^{-1/2} - 2\beta\left(1 + \frac{\rho^2}{4R^2}\right)^{-3/2} , \qquad (52)$$

corresponds to Tolman's Type III solution. (Schwarzschild constant density interior solution).

Schwarzschild's Exterior solution is obtained from (48) by choosing:

$c_1 = 1$; $c_2 = m/2$; $\alpha = 0$; $\beta = 1$,

thus yielding:

$$D = 1 + \frac{m}{2\rho} ,$$

$$E = 1 - \frac{m}{2\rho} . \qquad (53)$$

Tolman's solution type VI, equations (21), corresponds to solution (45), constrained to n -1/2.

7. ANOTHER SYMMETRIC WAY TO GENERATE SOLUTIONS

The use of equations (42-b), (42-c) seems to be a natural way to generate solutions of the type E=f+g; D=f-g. (It also seems a natural approach the time-dependent spherically symmetric collapse case.-Appendix I-)

Rewriting equations (42), (43):

$$x = \alpha + \beta\rho^2 , \qquad (54)$$

$$\lambda = \frac{1}{2}(D + E) \quad ; \quad \mu = \frac{1}{2}(D - E) \quad ,$$

$$\lambda\lambda_{,xx} - 3\,(\lambda_{,x})^2 = \mu\mu_{,xx} - 3\,(\mu_{,x})^2 = h(x) \qquad (42\text{-b})$$

$$F = \lambda^{-2} \quad ; \quad H = \mu^{-2} \quad ,$$

$$\frac{F_{,xx}}{F^2} = \frac{H_{,xx}}{H^2} = f(x) \quad . \qquad (43\text{-c})$$

In (42-b) or (43-c), we see that λ and μ or F and H, have the same functional form but differ in at least one of the two constants of integration when h(x) or f(x) are taken as generators. (If F=H then the metric becomes degenerate).

It follows that in (43-c), $H(x, c_1, c_2)$, c_1 and c_2 parameters, can be a solution (physical) if and only if at least one of the parameters does not appear in .

$$\frac{H_{,xx}}{H^2} \quad .$$

In this case, F is obtained by giving a different value to this parameter.

If f(x) is taken in (43-c) as a generator, then a non-degenerate solution is obtained by taking different values for the constants of integration for F and H in:

$$\frac{F_{,xx}}{F^2} = f(x) \quad . \qquad (55)$$

The simplest choice of generating function is:

$$f(x) = 0 \quad . \qquad (56\text{-a})$$

In this case:

$$F_{,xx} = 0 \quad , \tag{56-b}$$

so:

$$F = c_1 x + c_2 \quad ,$$

$$H = c_3 x + c_4 \quad , \tag{56-c}$$

where the constants c_i satisfy:

$$c_1 c_4 - c_2 c_3 \neq 0 \quad .$$

But:

$$E = F^{-1/2} - H^{-1/2} \quad ,$$

and

$$D = F^{-1/2} + H^{-1/2} \quad .$$

Choosing:

$$c_1 = \gamma; \; c_2 = \delta; \; x = \rho^2; \; c_3 = \frac{4\alpha}{\lambda_o^2} \; ; \; c_4 = \frac{4\beta}{\lambda_o^2} \quad ,$$

we obtain the solution (static) found in Glass and Mashhoon's (G-M) work[11], that is:

$$D = \frac{1}{(\gamma\rho^2 + \delta)^{1/2}} + \frac{\lambda_o}{2} \frac{1}{(\alpha\rho^2 + \beta)^{1/2}}$$

$$E = \frac{1}{(\gamma\rho^2 + \delta)^{1/2}} - \frac{\lambda_o}{2} \frac{1}{(\alpha\rho^2 + \beta)^{1/2}} \tag{57}$$

with:

$$B = D^2 \; ; \quad A = E/D$$

We see that with the field equation written as equation

(43-c), then (57) is the simplest of all possible solutions. (In fact the G-M time-dependent extension of (57) is also the simplest of all possible solutions in the sense that in (55), $f(x,t) = 0$, and that their solution is the only one in which E and D are a product of one x-dependent and one t-dependent functions. This is analized in the Appendix.

We can now use (43-c) to obtain one more general solution containing (57). A solution can be obtained in the form:

$$D = D_o \left(c_1^2\, G(\rho,\alpha,\beta,\gamma,\delta,\lambda_o) + c_2\right)^{-1/2} ,$$

$$E = E_o \left(c_1^2\, G(\rho,\alpha,\beta,\gamma,\delta,\lambda_o) + c_2\right)^{-1/2} . \qquad (58)$$

where c_1, c_2 are constants and D_o and E_o are given by (57).

8. A MORE PHYSICAL APPROACH TO A GENERATING FUNCTION

Until now, all the methods proposed made use of mathematical advantages, but none of them had a physical motivation for the selection or rejection of a generating function. In fact only after integration are the metric components substituted into the pressure and density equations, and then it becomes possible to check if the solutions are physically reasonable.

Another approach is the following:

We write the field equation (34-a)

$$\frac{E''}{E} + \frac{D''}{D} - 6\,\frac{E'D'}{ED} - \frac{1}{\rho}\left(\frac{E'}{E} + \frac{D'}{D}\right) = 0 \qquad (34\text{-a})$$

by using:

$$\frac{D''}{D} = \left(\frac{D'}{D}\right)' + \left(\frac{D'}{D}\right)^2 .$$

Introducing the dimensionless variables:

$$u = \rho\left(\frac{D'}{D} + \frac{E'}{E}\right) ,$$

and

$$e = \rho\left(\frac{D'}{D} - \frac{E'}{E}\right) ,$$

the field equation becomes:

$$\rho u' - 2u - u^2 + 2e^2 = 0 . \tag{59}$$

Using u as a generator, e is obtained without any integration. Given e and u, D and E are obtained by:

$$\ln D = \frac{1}{2}\int^{\rho} \frac{(u(t) + e(t))}{t}\, dt + \text{const.} ,$$
$$\ln E = \frac{1}{2}\int^{\rho} \frac{(u(t) - e(t))}{t}\, dt + \text{const.} , \tag{60}$$

or the metric components themselves are:

$$\ln B = \int^{\rho} (u + e)\, \frac{dt}{t} + \text{const.} ,$$
$$\ln A = -\int^{\rho} e\, \frac{dt}{t} + \text{const.} \tag{61}$$

The physics enters in the fact that physical constraints are placed on u and e <u>before</u> any integration is made.

By equation (34-b):

$$8\pi\ p = \frac{1}{\rho^2 D^4}(u^2 - e^2 + 2u) \quad , \tag{62-a}$$

and by using (59):

$$8\pi\ p = \frac{1}{\rho^2 D^4}(\rho u' + e^2) \quad . \tag{62-b}$$

The boundary of the star is given by $p(\rho_b) = 0$, so:

$$\rho_b = -\frac{e^2}{u'}\Big|_b \quad . \tag{63-a}$$

(63-a) is an equation from which ρ_b is obtained in terms of the other parameters (and viceversa) and it constrains the derivative of u to be negative on the boundary:

$$u'_b < 0 \quad . \tag{63-b}$$

Now, p must be

$$p \geq 0 \quad \text{for} \quad 0 \leq \rho \leq \rho_b \quad ,$$

so that from (62-a)

$$(u + 1)^2 - e^2 \geq 1 \quad \text{for} \quad 0 \leq \rho \leq \rho_b \quad , \tag{64}$$

or in (62-b)

$$\rho u' + e^2 \geq 0 \quad .$$

So, a sufficient condition to mantain $p > 0$ for $0 \leq \rho \leq \rho_b$ and $p = 0$ in $\rho = \rho_b$ is

$$- e^2 < \rho u' < 0 \quad \text{for } 0 \leq \rho < \rho_b \quad ,$$

$$\rho_b = -\frac{e^2}{u'}\Big|_b \quad . \tag{65}$$

We also ask p to monotonically decrease with ρ. Since by (9):

$$e = \frac{\rho p'}{\mu + p} \; ,$$

it follows that:

$$e < 0 \tag{66}$$

To require $\mu > 0$ for $0 < \rho < \rho_b$ we use the relation μ/p, also assuring that $p/\mu < 1/3$.

From the relation:

$$\frac{\mu}{p} + 3 = \frac{\frac{E''}{E} - \frac{D''}{D} + \frac{2}{\rho}\left(\frac{E'}{E} - \frac{D'}{D}\right)}{2\,\frac{D'E'}{DE} + \frac{1}{\rho}\left(\frac{E'}{E} + \frac{D'}{D}\right)} \; ,$$

we obtain:

$$\frac{\mu}{p} + 3 = -2\,\frac{\rho e' + e(u + 1)}{e^2 + \rho u'} \; . \tag{67}$$

Requiring $\mu/p > 3$, it follows that

$$\frac{\rho e' + e(u + 1)}{e^2 + \rho u'} < -3 \; , \tag{68-a}$$

but $e^2 + \rho u' > 0$, so it is sufficient that

$$\rho e' + e(u + 1) < -3 \tag{68-b}$$

Resuming, we can now write all the physical constraints as constraints on the generating function u by using (59).

From (64):

$$\rho u' + u^2 + 2u > 0 \quad , \quad (0 \leq \rho < \rho_b) \tag{69-a}$$

$$\rho u' + u^2 + 2u = 0 \quad \text{defines } \rho_b , \tag{69-b}$$

from (66)

$$e < 0 \quad , \tag{69-c}$$

which is always possible to take because e^2 is obtained from (59).

Also:

$$u^2 + 2u - \rho u' > 0 \quad , \quad \text{for } e^2 > 0 , \tag{69-d}$$

which is trivialy satisfied because by (65-a) $\rho u' < 0$.

And finally to ensure $0 < p/\mu < 3$ it is sufficient that:

$$\rho e' + e\,(u + 1) < -3 \tag{69-e}$$

Equations (69) constitute a set of constraints to be applied before any integration is performed.

This is a physically motivated approach and is more reasonable than to merely observe ''the generating function should be judiciously chosen''[8].

As an example of (69), (from Tolman's case I) choose:

$$u = a \frac{\rho^2}{1 + \rho^2} \qquad \text{with } a = \text{const.} \ ,$$

from (59):

$$e^2 = \frac{(a^2 + 2a)}{2} \frac{\rho^4}{(1 + \rho^2)^2} \ .$$

a must be $a < -2$ to satisfy (65-a), but $\rho u' < 0$ so $a < 0$.

However, $\rho u' > -e^2$ so $4a + (a^2 + 2a)\rho^2 > 0$ for $0 \le \rho < \rho_b$. but $a < 0$, so the solution is not physically possible.

This situation corresponds to case I in Tolman's paper ('Einstein Universe') which is unphysical as a stellar model.

9. PRESSURE AND DENSITY AS PRIMARY FIELD VARIABLES

In equation (34-a)

$$\frac{D''}{D} + \frac{E''}{E} - 6\frac{E'D'}{ED} - \frac{1}{\rho}\left(\frac{D'}{D} + \frac{E'}{E}\right) = 0$$

we see that by the symmetry in E and D in (36-a) if $\psi(f,g)=0$ then $\psi(g,f)=0$. That is if $D=f$, $E=g$ is a solution, then $D=g$, $E=f$ is also a solution.

But the metric is given by $A=E/D$ and $B=D^2$, so the change $f \rightleftarrows g$ changes the physical situation.

It is interesting to ask what new physical situation is described by $f \rightleftarrows g$, and what are the p and μ corresponding to the new situation.

In fact we look for functions $h_i(p,\mu)$, $j_i(p,\mu)$ such that $f \rightleftarrows g$ implies $h_i(p,\mu) \rightleftarrows j_i(p,\mu)$.

One of these functions is:

$$\omega(\rho) = \frac{p'}{\mu + p} = \frac{D'}{D} - \frac{E'}{E} \quad . \tag{70}$$

for which

$$D \rightleftarrows E \text{ implies } \omega \rightleftarrows -\omega \text{ or } \omega^2 \text{ invariant.} \tag{71}$$

Now,

$$f(\rho) = \frac{p}{\mu + 3p} = \frac{2\frac{D'E'}{DE} + \frac{1}{\rho}\left(\frac{E'}{E} + \frac{D'}{D}\right)}{\frac{E''}{E} - \frac{D''}{D} + \frac{2}{\rho}\left(\frac{E'}{E} - \frac{D'}{D}\right)}. \tag{72}$$

So, we see that

$$D \rightleftarrows E \text{ implies } f \rightleftarrows -f. \tag{73}$$

Using the field equations

$$3f - 1 = -\frac{\mu}{\mu + 3p} = \frac{2\frac{D''}{D} + \frac{4}{\rho}\frac{D'}{D}}{\frac{E''}{E} - \frac{D''}{D} + \frac{2}{\rho}\left(\frac{E'}{E} - \frac{D'}{D}\right)},$$

and by equation (70)

$$\omega' = \frac{D''}{D} - \frac{E''}{E} + \omega^2 - 2\omega\frac{D'}{D},$$

it follows that

$$3f - 1 = \frac{2\frac{D''}{D} + \frac{4}{\rho}\frac{D'}{D}}{\omega^2 - \omega' - 2\omega\frac{D'}{D} - \frac{2}{\rho}\omega},$$

$$2\frac{D''}{D} + 2\left(\frac{2}{\rho} + \omega(3f - 1)\right)\frac{D'}{D} + (3f - 1)\left(\omega' + \frac{2}{\rho}\omega - \omega^2\right) = 0.$$

Introducing a dimensionless function

$$e = \rho\omega = \rho\left(\frac{D'}{D} - \frac{E'}{E}\right) = \frac{\rho p'}{p+\mu} \tag{74}$$

we find

$$2\frac{D''}{D} + \frac{2}{\rho}\left(2 + e(3f - 1)\right)\frac{D'}{D} + \frac{3f - 1}{\rho^2}\left(e' + e - e^2\right) = 0. \tag{75}$$

But $$\frac{D''}{D} = -\frac{E''}{E} + 6\frac{D'E'}{DE} + \frac{1}{\rho}\left(\frac{D'}{D} + \frac{E'}{E}\right)$$

or

$$\frac{D''}{D} = -\frac{D''}{D} + 6\left(\frac{D'}{D}\right)^2 + \frac{2}{\rho}(1 - 2e)\frac{D'}{D} + \frac{1}{\rho^2}(e' - e^2 - 2e) .$$

And so (75) becomes

$$6\left(\frac{D'}{D}\right)^2 + \frac{6}{\rho}(1 + e(f - 1))\frac{D'}{D} + \frac{3f}{\rho^2}(e' + e - e^2) - \frac{3e}{\rho^2} = 0 ;$$

or

$$2\rho\frac{D'}{D} = e(1 - f) - 1 \pm (1 + e^2(f^2 + 1) - 2\rho fe')^{1/2} \tag{76}$$

We obtain a similar formula for E'/E by using (71) and (73),

so

$$2\frac{E'}{E} = -e(1 + f) - 1 \pm (1 + e^2(f^2 + 1) - 2\rho fe')^{1/2} \tag{77}$$

But

$$2\rho\left(\frac{D'}{D} - \frac{E'}{E}\right) = 2e ,$$

so the sign in the square root must be the same in (76) and (77).

As we see, the metric components are given by quadrature, when the pressure and density are given by

$$f = \frac{p}{\mu + 3p} , \qquad e = \frac{\rho p'}{\mu + p} .$$

Now e and f in (76), (77) are not arbitrary. A sufficient condition that must be obeyed by e and f (that is p and μ) can be obtained by asking (76), (77) to satisfy the field equation (34-a), or the equivalent equation (75).

Denote

$$Q = \pm (1 + e^2(f^2 + 1) - 2 fe')^{1/2} \tag{78-a}$$

So

$$2\frac{D'}{D} = \frac{1}{\rho}(e(1-f) - 1 + Q) \tag{78-b}$$

and

$$2\,\frac{D''}{D} = -\frac{1}{\rho^2}\,(e(1-f) - 1 + Q) + \frac{1}{\rho}\,(e'(1-f) - ef' + Q')$$

$$+\frac{1}{2\rho^2}\,(e(1-f) - 1 + Q)^2 \,. \qquad (78\text{-c})$$

We obtain from (75):

$$Q\,(2e^2 - (\tfrac{e}{f})^2 - \rho(\tfrac{e}{f})') = 2\,\frac{e}{f}\,(1 + e^2(1 + f^2)) - \rho^2 f^{-2}(fe')'$$

$$-\rho\, ef(\tfrac{e}{f})' + \rho(\tfrac{e'}{f})(\tfrac{e}{f} - 1) \,. \qquad (79)$$

The positive sign must be taken for Q, as determined in the Newtonian limit below.

10. THE NEWTONIAN LIMIT

Since

$$e = \frac{\rho\, p'}{\mu c^2 + p} \,, \qquad (74)$$

$$f = \frac{p}{\mu c^2 + 3p} \,, \qquad (72)$$

in the limit $c \to \infty$, we obtain

$e \to 0$, $f \to 0$,

$\frac{e}{f} \to \rho\frac{p'}{p}$,

$$(\frac{e}{f})' \rightarrow \frac{p'}{p} + \rho\frac{p''}{p} - \rho\frac{(p')^2}{p^2} \quad ,$$

$$\frac{e'}{f} \rightarrow \frac{p'}{p} + \rho\frac{p''}{p} - \rho\frac{p'\mu'}{p\mu} \quad ,$$

and so equation (79) becomes

$$\pm(-\frac{p''}{\rho p} - \frac{p'}{\rho^2 p}) = -\frac{p'''}{p} + \frac{p'\mu''}{p\mu} + 3\,\frac{p''\mu'}{p\mu} - 3\,\frac{p'(\mu')^2}{p\mu^2}$$

$$\mp \frac{1}{\rho}(4\,\frac{p'\mu'}{p\mu} + \frac{p'}{\rho p} - 3\,\frac{p''}{p}). \qquad (80)$$

But in the Newtonian case, for a star in equilibrium:

$$\frac{dm}{dr} = 4\pi\mu r^2 \quad ,$$

$$\frac{dp}{dr} = -G\,\frac{m}{r^2}\,\mu \quad ;$$

and so

$$-\frac{p''}{\mu^2} + \frac{p'}{\mu^2}(\frac{\mu'}{\mu}) - \frac{2p'}{r\mu^2} = 4\pi G \qquad (81)$$

Differentiating (81) and multiplying by μ^2/p, yields

$$-\frac{p'''}{p} + 3\,\frac{p''\mu'}{p\mu} + \frac{p'\mu''}{p\mu} - 3\,\frac{p'(\mu')^2}{p\mu^2} + \frac{1}{r}\left(4\,\frac{p'\mu'}{p\mu} + \frac{2p'}{rp} - \frac{2p''}{p}\right) = 0\,. \tag{82}$$

We see that (82) will be identical with (80) only when the <u>plus</u> sign is used in (80) and thus in (79). (Note that $\rho \to r$ in the Newtonian limit.)

11. THE METRIC RECOVERED

Now we can rewrite the sufficient condition for the field equation (34-a) to be obeyed, as

$$\left(2e^2 - \left(\frac{e}{f}\right)^2 - \rho\left(\frac{e}{f}\right)'\right)\left(1 + e^2(1 + f^2) - 2\rho f e'\right)^{1/2} =$$

$$2\,\frac{e}{f}\left(1 + e^2(1 + f^2)\right) - \rho^2 f^{-2}(fe')' - \rho e f\left(\frac{e}{f}\right)' + \rho\left(\frac{e'}{f}\right)\left(\frac{e}{f} - 1\right). \tag{83}$$

Thus the symmetric metric components are given by

$$2\rho\,\frac{D'}{D} = e(1 - f) - 1 + \left(1 + e^2(f^2 + 1) - 2\rho f e'\right)^{1/2},$$

$$2\rho\,\frac{E'}{E} = -e(1 + f) - 1 + \left(1 + e^2(f^2 + 1) - 2\rho f e'\right)^{1/2}. \tag{84}$$

The metric components A and B of metric (30) are obtained directly from e and f by a single integration:

$$\frac{A'}{A} = -\frac{e}{\rho} , \tag{85-a}$$

$$\frac{B'}{B} = \frac{1}{\rho} \left[e(1 - f) - 1 + (1 + e^2(f^2 + 1) - 2\rho f e')^{1/2} \right], \tag{85-b}$$

where (85-a) follows from the equations of motion (Euler's equation) and (85-b) follows from (84) and $B = D^2$. One of the constants of integration of the system (83-85) will be determined by equation (34-b).

APPENDIX

The time-dependent case

In the time-dependent situation, the field equations for shear-free flow are[11]:

$$\frac{A''}{A} + \frac{B''}{B} - \frac{1}{\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right) - 2\,\frac{A'}{A}\,\frac{B'}{B} - 2\left(\frac{B'}{B}\right)^2 = 0$$

and

$$\dot{B} = ABh(t)$$

where h(t) is an arbitrary function of time.

By the transformations:

$$D = (B)^{1/2}$$

$$E = A(B)^{1/2}$$

the system becomes:

$$DE'' + ED'' - 6E'\,D' - \frac{1}{\rho}(DE' + ED') = 0\ ,$$

$$\dot{D} = Eh(t)$$

where $\dot{f} = \frac{\partial f}{\partial t}$.

Introducing the new variables λ and μ

$$E = \lambda - \mu$$

$$D = \lambda + \mu$$

$$x = \alpha\rho^2 + \beta$$

and

$$f' = \frac{\partial f}{\partial x} \quad ,$$

we obtain

$$\lambda\lambda'' - 3(\lambda')^2 = \mu\mu'' - 3(\mu')^2$$

$$\dot{\lambda} - \dot{\mu} = (\lambda + \mu)h(t) \quad .$$

Now, if

$$g(t) = e^{\int h dt}$$

and

$$L = \frac{\lambda(x,t)}{g(t)} \quad ,$$

$$M = \mu(x,t)\ g(t) \ ,$$

then

$$g^2(LL'' - 3(L')^2) = \frac{1}{g^2}(MM'' - 3(M')^2) = f(x,t) \qquad (86\text{-a})$$

$$g(t)L,_t = \frac{1}{g(t)} M,_t \quad , \quad g(t) \ \text{arbitrary} \ . \qquad (86\text{-b})$$

The symmetry between (86-a) and (86-b) is striking and it is seen more explicitly by using (86-b) to obtain:

$$g^2(\dot{L}\dot{L}'' - 3(\dot{L}')^2) = \frac{1}{g^2}(\dot{M}\dot{M}'' - 3(\dot{M}')^2) \quad .$$

If

$$F = L^{-2} \quad , \qquad H = M^{-2}$$

then

$$g^2 \frac{F''}{F^2} = \frac{1}{g^2} \frac{H''}{H^2} = f_1(x,t) \quad . \tag{87}$$

The solution (time-dependent) obtained in G-M[11] is once again the simplest, corresponding to:

$$\dot{L} = \dot{M} = 0 \quad . \tag{88}$$

Now, by (88)

$$\lambda(x,t) = l(x)g(t)$$

$$\mu(x,t) = \frac{m(x)}{g(t)} \tag{89}$$

and using (57) we obtain the solution:

$$D = \frac{g(t)}{(\gamma\rho^2+\delta)^{1/2}} + \frac{\lambda_o}{2g(t)} \frac{1}{(\alpha\rho^2+\beta)^{1/2}}$$

$$E = \frac{g(t)}{(\gamma\rho^2+\delta)^{1/2}} - \frac{\lambda_o}{2g(t)} \frac{1}{(\alpha\rho^2+\beta)^{1/2}} \tag{90}$$

(90) coincides with the G-M solution by taking $g = f^{1/2}(t)$.

Solution (88) is the simplest also in the sense that is the unique solution in which , t are separate. This can be easily seen by looking at:

$$g^2(t) \frac{F''}{F^2} = \frac{1}{g^2} \frac{H''}{H^2} \tag{91}$$

With a separation other than (89), we can always leave in (91) one side of the equality depending only on x and the other as a function of x multiplied by a function of t.

By defining

$$K(x,t) = 2\int g\dot{g}L\,dt$$

we can relate the generating function f(x,t) in (86-a) with the choice made in (86-b) by:

$$4f\,\frac{\ddot{g}}{\dot{g}} = \dot{L}K'' + K\dot{L}'' + 6K'\dot{L}'$$

Now we see explicitly that for the case $\dot{L}=0$, f=0, justifying the solution (90). We recover the static case by $\dot{g}=0$, $\dot{L}=0$, in which case f is not defined.

Another interesting feature of K(x,t) is obtained by:

$$\frac{(KK'' - 3K'^2)_{,t}}{(KK'' - 3K'^2)} = 2g\dot{g}$$

and so:

$$\ln(KK'' - 3K'^2) = -4\ln K + \ln(\frac{K'}{K^3})' = g(t)q(x)\ .$$

where q(x) is some function of x only.

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VITA AUCTORIS

I was born in 1948 in Buenos Aires, Argentina. I graduated from J. J. Urquiza Secondary School in 1965. At the same time, I received my teaching diploma from S. Aleichem High School also in Buenos Aires.

I then entered the Electronic Engineering Faculty at the University of Buenos Aires and also entered at the College of Hebrew Studies, Buenos Aires. After my first year of post-secondary education I went on an Israeli government scholarship to the Greenberg Institute for teachers in Jerusalem. At this institution I received my diploma in Jewish Studies.

I returned to Argentina in 1968 and after three years in the Faculty of Electronic Engineering I transferred to the Faculty of Physics.

Unfortunately, the Faculty of Physics had to close for political reasons. As a result, in 1973 I went to Israel to complete my B. Sc. in Physics at the Technion - Israel Institute of Technology in Haifa. I received my B. Sc. Honours in 1975 and entered the Graduate School at the Technion in the same year. In 1976 I came to Windsor to complete my Master's programme.