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PREDICTION AND MEASUREMENT OF TURBULENCE IN THE DEVELOPING REGION OF AXISYMMETRIC ISOTHERMAL FREE JETS.

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PREDICTION AND MEASUREMENT OF TURBULENCE IN THE
DEVELOPING REGION OF AXISYMMETRIC ISOTHERMAL FREE JETS

by

S.M. Nazrul Islam

A Dissertation
submitted to the Faculty of Graduate Studies
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ABSTRACT

Flow in the developing region of axisymmetric tur-
bulent jets in still air has been studied both theoretically
and experimentally.

) Jets produced by three nozzles having three different
turbulent boundary layer velocity profiles at the nozzle
exit were investigated to determine the influence of the
initial conditions on the flow. The displacement thicknesses
at the exit of the nozzles were δ_o^*/r_o = 0.0064, 0.0179 and
0.0628, and for all three nozzles, the Reynolds number based
on the outlet diameter and the average velocity was R_{eD} =
1.53 x 10^5.

For each case, the mean velocities, the turbulent
intensities in x, r and θ-direction and the shear stresses
were measured by the hot-wire anemometer. They were found
to be self-preserving in the initial region except close to
the nozzle, but not in the transition region. The increased
boundary layer thickness at the nozzle exit moves the self-
preserving point downstream, although the potential core
length remains the same.

Numerical solutions of the differential momentum,
continuity and turbulent kinetic energy equations together
with an empirical length scale were performed by using a
finite-difference method to predict the mean and turbulent
quantities in the initial region for any arbitrary initial
condition. Prandtl's mixing length model of shear stress

was used for calculation and other shear stress models were examined. Prandtl's model is not applicable to the transition region and an empirical model of turbulent diffusivity was used to calculate the mean flow quantities from the continuity and momentum equations. The calculated results are in satisfactory agreement with the experimental measurements.

For the thin boundary layer at the nozzle exit, the measured mean velocities are shown to fit a polynomial form velocity profile which is used to determine the jet boundaries in the initial region by solving the integral momentum and mean kinetic energy equations in conjunction with the Prandtl's model of shear stress. Agreement with the measurements and numerical solution is shown.

The turbulence level in the core of the jets was high near 5% intensity. The effect of this turbulence on the shear layer was not investigated, but might be important.

Dedicated to

Prophet Muhammad (Peace Be Upon Him)

ACKNOWLEDGEMENTS

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NOMENCLATURE

A, B, B_L	Constants
a_1	Constant in Energy Equation (3.13)
a_2	Constant in Harsha's Model
a_p	Constant in Rotta's Model
a_t	Tollmien Constant for the Similarity Variable
b	Length Scale for the Similarity Variable η
b_1	Width of the Shear Layer
b_m	Mean Width of the Shear Layer
c	Exponent of the Hot-Wire Equation
c_r, c_L	Constants in Rotta's Transport Model
c_1	Constant in the Energy Model of Shear Stress
c_2, c_3, c_D, c_μ	Constants in Launder's Transport Model
c_4	Proportionality Constant of Length Scale and Width of the Shear Layer
c_a, c_K, c_s, c_t	Constants for Different Mean Velocity Models for the Developed Region
$\dfrac{D}{Dt}$	Material Derivative of a Dependent Variable
d	Hot-Wire Diameter
$\bar{E}$	Mean Voltage of the Hot-Wire Anemometer Without Linearizer.
E'	Fictitious Voltage
$\bar{E}_L$	Mean Voltage with Linearizer
e, e_L	Fluctuating Voltages
f	Characteristic Frequency
H	Shape Factor
k	Turbulent kinetic Energy
k_2	Hot-Wire Constant
$K^* = K/u_o^2$	Non-Dimensional Kinetic Energy

$k_q, k_{qL}, k_{q\tau}$	Constants in Rotta's Transport Model
L	Prandtl's Mixing Length
L^O	Integral Length Scale
L^1	Characteristic Length Scale for the Transport Model
$L^* = L u_o / \nu$	Non-Dimensional Length Scale
ℓ	Hot-Wire Length
n_1, n_2	Slopes of the Normal and Slanting Wires
P	Production of the Turbulent Energy
p	Mean Static Pressure
p'	Fluctuating Pressure
p^O	Instantaneous Pressure
Pr	Prandtl's number in Harsha's Model
$R = r u_o / \nu$	Non-Dimensional Radial Distance
r_O	Radius of Nozzle
$r_{\frac{1}{2}}$	Radial Distance at Half-Velocity ($u/u_c = 0.5$)
r_1	Radius of the Potential Core
r_2	Outer Radius of the Jet
r_m	Mean Jet Radius
R_{eD}	Reynolds Number Based on the Exit Diameter and Average Velocity
$Re\delta^*$	Reynolds Number Based on the Exit Displacement Thickness and the Average Velocity
S	Secondary source Term in Launder's Model
SPL	Sound Pressure Level
s_1, s_2	Hot-Wire Constants
$U = u/u_o$	Non-Dimensional Mean Velocity
u	Axial Mean Velocity
u_c	Centre Line Mean Velocity
u_o	Area Average Velocity at Exit

$u_\tau = \tau_{max}/\rho$ Shear Velocity

U_C Convection Velocity

U_E Cooling Velocity

U_T Tangential Cooling Velocity

U_N Normal Cooling Velocity

$V = \upsilon/u_o$ Non-Dimensional Radial Mean Velocity

υ Radial Mean Velocity (See note at end of nomen-clature)

υ_e Effective Velocity in Bradshaw's Model

u^o, υ^o, w^o Instantaneous Velocities in x, r and θ-Directions

u', υ', w' Fluctuating Velocities in x, r and θ-Directions

$X = x\, u_o/\nu$ Non-Dimensional Axial Distance

X_t Laminar Transition Length

x, y, z A Coordinate System

x, r, θ An Axisymmetric Coordinate System

z A Dummy Variable for Launder's Model

α Angle of Hot-Wire Inclination

α_L Constant in Rotta's Length Scale Model

β A Constant in the Intermittency Equation

γ Intermittency Factor

δ Error in Velocity in Stability Analysis

δ_o Boundary Layer Thickness at the Exit

Δ Difference

δ^* Displacement Thickness

θ_o Momentum Thickness at the Exit

σ Standard Deviation

σ_k, σ_z Constants in Launder's Transport Model

μ_{eff}	Effective Viscosity
υ	Molecular Diffusivity
υ_t	Turbulent or Eddy Diffusivity
η	Self-Preserving Variable
η_t, η_H	Tollmien's and Hatta's Similarity Variable
ρ	Density
ε	Dissipation of Turbulent Energy
ξ	A Constant in Rotta's Length Scale Equation
$\tau = \tau_{rx}$	Shear Stress

Subscripts

0	Exit Condition
RMS	Root Mean Square
i	Axial Direction
j	Radial Direction
Max	Maximum Value
Min	Minimum Value

Note: the symbol, υ, was inadvertently used in place of the usual symbol, v, during typing throughout the Dissertation. Because of extensive changes, the symbol, υ was retained, and the author apologizes for any inconvenience this may cause to the reader. —

CHAPTER I

INTRODUCTION

1.1 General

A jet is formed when a fluid discharges through an opening or nozzle from a container under higher pressure into a region of lower pressure. The neighbouring fluid surrounding the jet may itself be in motion or at rest. The formation of a boundary layer within the nozzle and its separation at the outlet is the initiation of a shear layer in the free jet flow. Turbulence originates as an instability in the laminar flow, and the flow becomes turbulent when the Reynold's number exceeds some critical value.

Jets are of three kinds: 1) bounded, 2) wall, and 3) free jets. In bounded jets, the flow is confined by solid boundaries; wall jets flow over solid surfaces. Free jets flow without contacting solid surfaces and are axisymmetric when formed by discharging through a circular nozzle or orifice.

A detailed knowledge of the mean and turbulent quantities within a jet is essential to many problems of diffusion, aerodynamic noise and airplane design. The diffusion phenomena in free jets is associated with flows in the exhaust of rocket engines, atomized fuel injection systems, sprays, waste disposal plumes and jets in numerous cleaning devices.

Wall and bounded jets also have a number of physical applications. In airplanes, jets may be blown along aerofoil surfaces to energise the boundary layer and prevent separation. This is usually used to reduce the take-off and landing speed of the airplane. It has been applied with success to carrier-based naval aircraft; for example, the Blackburn NA39 has blowing applied to the leading edge of the wing, the flaps and the leading edge of the tail plane which reduces landing and take-off speeds by tens of knots [1] [1]. Directly related, although restricted by boundaries (i.e., free surfaces), is the diffusion of plunging nappes from spillways or of submerged jets from sluice gates. Bounded jets are applied to the relatively new technology of fluidics which include switches and proportional amplifiers.

Free jets in the exhausts of rocket engines create aerodynamic noise which in many cases is objectionable and should be controlled. The aerodynamic noise is generated by Reynolds stresses associated with either subsonic or supersonic flows. It is necessary to know the turbulence structure within such jets for better design related to noise reduction.

Recently, research on free jets has received considerable attention. One kind of jet, the axisymmetric, isothermal jet is a practical flow which embodies many of

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1 Numbers in brackets designate references.

the essential features of shear flow turbulence, and it remains a fertile ground for investigation.

1.2 Axisymmetric Jets in Still Surroundings

1.2.1 Turbulent Jet Formation

The turbulence in jet flow is caused by the generation of vortices and their breakdown due to interaction. Using flow visualisation technique [2], it has been shown that waves starting from the nozzle outlet grow with unstable amplitude within two-to-three wave lengths from the exit. The wave crest in contact with the stationary ambient air folds back into the following trough. This folding engulfs the surrounding fluid and forms a ring vortex core which rolls downstream. After one or two revolutions, the vortices interact strongly with the waves behind and break down into turbulent eddies [2]. The interactions (like vortex pairing) of turbulent eddies cause large scale vortical motions; small scale vortical motions also evolve through breakdown of the large eddies. When the initial boundary layer is laminar, the initial vortical structure results from the instability and rollup of the free shear layer, and the initial size and spacing of vortices can be determined by the unstable eigenmode of the profile. For this case, the laminar-turbulent transition length from the outlet depends on Reynolds number, R_{eD}. The higher the Reynolds number, the smaller the laminar-turbulent

transition length.

A free shear layer resulting from an initially turbulent boundary layer can also roll up into an organized vortical structure from which evolve large and small scale motions not unlike the initially laminar case. This kind of jet flow is shown in Figure 1.1, with three distinguisable layers: 1) shear, 2) ambient, and 3) potential core. For convenient analysis, the turbulent jet flow is divided into three principal regions: 1) initial region, 2) transition region, and 3) developed region, shown in Figure 1.1. The initial region applied to jets with an initial core of uniform velocity or potential core and extends until the core disappears. The transition region starts after the initial region. Further downstream, there exists a developed region where the flow variables, i.e., mean velocity, turbulent intensity, etc. become approximately self-preserving. The combined initial and transition regions is called the developing region of the jet.

1.2.2 Flow in Axisymmetric Turbulent Jets

A continuous transfer of momentum and energy takes place from the jet fluid to the surrounding fluid. A difference in velocity between a jet and the region into which it is discharged forms a pronounced degree of instability due to the intensive shearing of the peripheral jet fluid with the ambient fluid. The shearing action initially occurs over a small lateral region; but, the fluid near

the axis remains unsheared. For turbulent jets, the shearing phenomenon steadily converts kinetic energy of the oncoming mean flow into kinetic energy of turbulence, and the latter decays through viscous shear. Such a conversion of energy occurs throughout the jet flow. On the other hand, the reduction of kinetic energy of the mean flow represents a decrease in the flow velocity. The elementary consideration of continuity indicates that the area of flow section must increase as the flow velocity decreases. Newton's principle of action and reaction would suggest that deceleration of the fluid in the jet can occur only through a simultaneous acceleration of the surrounding fluid, so that the total rate of flow past successive sections of the jet increases with distance from the outlet. This phenomena flattens the velocity profile in the shear layer and reduces centre line velocity after the potential core region.

The initial and boundary conditions play an important role and influence the flow and mixing process in the initial region. The boundary conditions are related to the shape of the nozzle and the space surrounding the shear layer. The initial conditions for steady jets are the flow properties at the exit plane of the nozzle which depend on the boundary layer thickness inside the nozzle. The flow variables in the initial region are dominated by the large scale structure which can be expected to achieve independence

of the initial conditions in a finite flow length and the flow may become self-preserving depending on the boundary conditions.

1.3 Energy Transfer in Turbulent Jets

Turbulent flow represents a continuous interaction, mixing or superposition of turbulent eddies of various sizes which cause fluctuation of flow variables (velocity, pressure, etc.) about their mean values. Energy transfer in a turbulent flow field depends on the interaction of eddies, and correlations between various quantities of the turbulence and the mean motion. For simplicity, turbulence may be considered to consist mainly of eddies of two kinds, depending on scale: 1) large eddies, and 2) small eddies. The large eddies are energy containing eddies, and they are strained by both the mean and turbulent stresses present in the flow field. The small eddies contain less energy and they are invariant to mean and turbulent stresses in the flow field. The small scale eddies exist in the field of large scale eddies and dissipate turbulent energy to heat. The kinetic energy related to small eddies in the field of the big eddies may be called the internal energy of big eddies.

Forms of energy in a turbulent shear flow are:

1)   turbulent kinetic energy (for both large or small eddies);

2)   mean kinetic energy;

3) energy due to pressure.

These forms of energy transform from one form to another

in the following processes:

1) energy transfer from mean motion to turbulence;

2) turbulent energy dissipation to heat;

3) energy transfer due to interaction, between

eddies of different size.

The relative magnitude of energy transfer with res-

pect to wave number is shown in Figure 1.2.

The major portion of turbulent energy production

takes place at relatively low wave numbers which correspond

to the larger scale eddies of the turbulence. The large

eddies are stretched by the mean motion which performs work

on them and this causes turbulence production. Tucker

[3] interpreted turbulence production through vortex stretch-

ing for an irrotational flow field. Experimentally and

analytically, he has shown that the vortex stretching in-

creases turbulent kinetic energy which comes from the mean

motion. Though this analysis is not applicable to jet flow,

it gives an insight of energy transfer from the mean motion

to turbulence.

Dissipation means the decay of turbulence which

occurs through the transfer of energy from the large eddies

to the small eddies. This kind of energy transformation is

large at high wave numbers which represent the small scale

eddies, Figure 1.2.

The energy transfer due to eddy interaction depends on the size of eddies, the correlation of pressure and velocity fluctuations and the triple correlations of the fluctuating velocity components. This transformation is called the diffusion of turbulent energy and its magnitude is high at intermediate wave numbers, Figure 1.2. A detailed discussion of energy transformation related to axisymmetric jets will be presented later.

1.4 Statement of the Problem

The flow in the developing region of axisymmetric incompressible turbulent jets will be studied, both theoretically and experimentally. Jets produced by three different nozzles having different boundary layer velocity profiles at the nozzle exit will be investigated. The exit Reynolds number based on the average velocity and the outlet diameter will be 1.53×10^5 for all three cases.

The mean velocity and turbulent intensity at the nozzle exit will be measured to identify the initial conditions of the jet. The mean velocity, the turbulent intensities in x, r and θ-direction, and the shear stress will be measured in the initial and transition regions.

A differential method will be used to determine mean flow properties by solving the differential momentum and continuity equations for the initial and transition regions with an arbitrary initial condition. This method will be extended with a differential transport equation for

turbulent kinetic energy together with an algebraic equation

for length scale to determine turbulent properties in the

initial region.

An integral method will be used as a special case to

determine jet boundaries in the initial region for a thin

boundary layer at the nozzle exit.

CHAPTER II

LITERATURE SURVEY

2.1 General

Turbulent flows can be expressed mathematically by the conservation of mass equation and the Navier-Stokes equations. Since the Navier-Stokes equations are non-linear, each individual flow pattern has certain unique characteristics that are associated with its initial and boundary conditions. The equations have been analysed by researchers for various flow patterns, but it is still not possible to make quantitative predictions concerning turbulent quantities without relying greatly on empirical data because, in the time averaged turbulent equations, there are more unknown dependent variables than there are equations. In order to obtain a useful set of closed equations, it is necessary to make crucial assumptions concerning the flow, with physical concepts based on experimental data and experience. In this way, many authors have developed empirical and semi-empirical equations to obtain a set of closed equations. Progress in this line of research, as related to axisymmetric jets, is presented in this chapter.

The developed region of axisymmetric jets, sufficiently far downstream, is not affected by the initial conditions. Much work has been done both theoretically and experimentally in this region by Wygnanski [4], Heskestad

[5], Newman [6], Townsend [7], Rotta [8], Launder [9], Roshko [83] and others. On the other hand, much less work has been done in the initial and transition regions except lately in relation to noise by Bradshaw et al [10], Ko and Devies [11], Lau [12] and others. Some relevant experimental work in these regions of turbulent jets is presented in chronological order in Table 2.1, with information on jet size, exit conditions, Reynolds numbers and the variables measured. Relevant theoretical work with information on the equations used is shown in Table 2.2.

2.2 Mean Velocity

2.2.1 Developed Region

In 1925, Prandtl [13] published the concept of mixing length for free turbulent shear flow and it was used by Tollmien [14] in 1926 to calculate mean velocities in an axisymmetric jet without any special consideration of the region of the jet close to the nozzle. The investigation was based on the assumptions that: (a) the effective force was the tangential shear expressed in terms of the lateral momentum transport and mixing length, (b) the mixing length varies as the first power of the axial distance from the efflux section, (c) turbulence velocity was proportional to the mixing length and mean velocity gradient. Tollmien [14] established a series solution for mean velocity with variable, $\eta_t = r/(a_t x)$, and the Prandtl's mixing length, $L = c_t x$ where a_t and c_t are empirical con-

TABLE 2.1:   Relevant Experimental Work

	Authors	Nozzle Size And Jet	Reynolds No.	Exit Conditions	Variables Measured	Model	Initial Region	Transition Region
1	Trupel [27] (1915)	45cm Air jet	5×10^6	Not Identified	u	---	X	X
2	Kuethe [16] (1935)	10cm Air jet	1.75×10^4	"	$u, \sqrt{u'^2}$	u	X	
3	Albertson et al [18] (1948)	2.54, 1.27 & 0.64cm Air jet	9.17×10^4	"	$u\sqrt{u'^2}$	u	X	X
4	Abramovich [27] (1948)	10cm Air jet	1.64×10^4	"	u	u	X	X
5	Laurence [31] (1954)	8.89cm Air jet	1.92, 3.0, 5.0 and 7.5 x104	"	$u, \sqrt{u'^2}, LO, \overline{u'v'}$	---	X	X
6	Davies et al [30] (1963)	2.54cm Air jet	1.92×10^5	"	$u, \sqrt{u'^2}, LO, \overline{u'v'}, U_c$ SPL	LO, SPL	X	X
7	Bradshaw [10] (1964)	5.08cm Air jet	3×10^5	$\delta_o = 0.0003$in $U_o = 2.2$	$u, \sqrt{u'^2}, \sqrt{w'^2}$ $\overline{u'v'}, \sqrt{v'^2}$	---	X	X
8	Kolpin [33] (1964)	2.54, 1.25cm Air jet	1.82×10^5 and more	Not Identified	$u, \sqrt{u'^2}, \sqrt{v'^2}$ $\overline{u'v'}$	---	X	X
9	Bradshaw [36] (1966)	5.08cm Air jet	3×10^5	Various θ_o	$u, \sqrt{u'^2} \sqrt{v'^2}, \overline{u'v'}$, SPL	---	X	X

TABLE 2.1 (Cont'd)

	Authors	Nozzle Size And Jet	Reynolds No.	Exit Conditions	Variables Measured	Model	Initial Region	Transition Region
10	Sami et al [24] (1967)	30.48cm Air jet	2.2×10^5	Not Identified	$u, \sqrt{u'^2}, \dfrac{\sqrt{u'^2}}{u'v'}, \sqrt{p'^2}$	---	X	X
11	Von Frank [25] (1970)	30.48cm Air jet	5×10^5	"	$u, \sqrt{u'^2}, \sqrt{u'^2}$	---	X	X
12	Nilsen [71] (1970)	30.48cm Air jet	6.25×10^5	"	$u, \sqrt{u'^2}, \sqrt{p'^2}$	---	X	X
13	Crow [37] (1971)	2.54cm Air jet	1.05×10^4 to 5.14×10^4	Various δ_o	$u, \sqrt{u'^2},$ spectra	---	X	X
14	Ko [1] (1971)	2.54cm Air jet	2×10^5 10^5	Not Identified	$u, \sqrt{u'^2}, u'u'$ U_c, SPL	---	X	
15	Hatta [23] (1975)	3.6 cm Air jet	10^5	"	u	u	X	X
16	Yule [38] (1977)	5.08cm Air jet	9×10^3 to 2×10^5	"	$u, \sqrt{u'^2}, \sqrt{u'^2},$ $\sqrt{w'^2}, u'u'$	---	X	
17	Yuu [34] (1978)	0.8 cm Air jet	2.2×10^5	"	u	u	X	

TABLE 2.2: Relevant Theoretical Work

	Authors	Equations	Shear Stress Model	Initial Region	Transition Region	Comment
1	Squire [17] (1944)	Integral Form a) Mass b) Momentum	$L^2(\partial u/\partial r)^2$ $L = c_4 b_1$	X	X	Assumed a cosine velocity profile and determined jet boundaries. Shear layer width, b_1, was assumed be-tween $u/u_c=0$ and $u/u_c=1$
2	Kuethe [16] (1935)	Integral Form a) Mass b) Momentum	$L^2(\partial u/\partial r)^2$ $L = c_4 b_1$	X		Assumed a velocity profile and improved by iteration. The width of shear layer was defined between $u/u_c=0$ and $u/u_c=1$.
3	Kirshner [20] (1968)	Integral Form a) Mass b) Momentum	$\nu_t(\partial u/\partial r)$, $\nu_t = $ const.		X	Similar to that of laminar flow case. The solution is not applicable to the ini-tial region.
4	Peters [66] (1972)	Integral Form a) Mass b) Momentum C) Turbulent K.E.	$c_1 k$ $c_1 = $ const.	X	X	Assumed a cosine velocity profile for the initial region, and solved for kine-tic energy, velocity in the transition region. Width of shear layer is the length scale.

TABLE 2.2 (Cont'd)

	Authors	Equations	Shear Stress Model	Initial Region	Transition Region	Comment
5	Harsha [44] (1972)	Diff. Form a) Mass b) Momentum c) Turbulent K.E.	$c_1 k$		X	Calculation was performed for the transition and developed region.
6	Madni [26] (1975)	Diff. Form a) Mass b) Momentum	$L^2 (\partial u/\partial r)^2$ $L = c_4 b_1$	X	X With Different Shear Stress Model	Flat mean velocity profile at the nozzle exit. Sufficient data were not reported.
7	Hatta [23] (1975)	Diff. Form a) Mass b) Momentum c) Energy	$\nu_t (\partial u/\partial r)$ $\nu_t \sim (r_m - r_1) u_c$	X	X	Obtained a polynomial form velocity profile. Linear jet boundaries were calculated. Length scale is b_1.
8	Rotta [8] [41] (1972) (1975)	Diff. Form a) Mass b) Momentum c) Turbulent K.E. d) L'k-Equation	$K_0 \sqrt{k} L^1 (\partial u/\partial r)$			For developed region of jets and other shear flows.
9	Launder [9] [59] (1974) (1977)	Diff. Form a) Mass b) Momentum c) Turbulent K.E. d) Dissipation	$c_\mu \dfrac{k^2}{\epsilon} (\partial u/\partial r)$	X	X	Used two-equation model.

stants. The series solution agreed fairly well with measurements [14] for the developed region of jets. In 1930, Schlichting [15] used Prandtl's hypothesis to express turbulent shear stress, $\tau/\rho = c_s b_1 (U_{max} - U_{min})(\partial u/\partial r)$ where c_s is an empirical constant and b_1 is the width of shear layer, and solved the same problem as Tollmien [14] did in 1926 using similar assumptions. The solution shows satisfactory agreement with measurements except in the region of low velocity near the jet boundary.

2.2.2 Initial and Transition Regions

The solutions given by Tollmien [14] and Schlichting [15] do not apply to the initial region of a jet issuing from a nozzle because of the presence of the potential core and the effects of the initial conditions, although Prandtl's mixing length is approximately linear for this region. Kuethe [16], in 1935, applied Prandtl's mixing length to the initial and transition regions and worked out an approximate method for computing the mean velocity for a round jet. He assumed Prandtl's mixing length to be proportional to the width of shear layer and performed successive approximations introducing a profile for the first approximation of the form:

$$(u)_{1st.app.} = \left[1 - \left(\frac{r - r_1}{b_1} \right)^{\frac{3}{2}} \right]^2 \qquad (2.1)$$

where b_1, the width of shear layer and r_1, the radius of potential core were calculated by successive approximations.

In this analysis, shear stress was expressed as:

$$\overline{u'v'} = L^2 \left|\frac{\partial u}{\partial r}\right|\frac{\partial u}{\partial r} \tag{2.2}$$

where $L = c_4 b_1$; c_4 is an empirical constant determined by experiment.

The theoretical calculations of the mean velocity for both initial and transition regions showed agreement with experimental results from pitot tube measurements.

In 1944, Squire and Trouncer [17] developed a mean velocity model for the initial and transition regions of co-flowing jets with assumptions similar to those that Kuethe [16] used for jets in still air. The Reynolds shear stress was expressed by Prandtl's mixing length hypothesis given by Equation (2.2). The empirical constant, c_4, used to determine the mixing length was found to be different from that predicted by Keuthe [16]. As a special case, the length of the potential core was calculated to be $x/r_o = 7.75$ for jets in still air. This value does not agree with that of Kuethe [16].

Albertson et al [18], in 1948, measured mean axial velocities for both axisymmetric and two-dimensional jets, and their measurements of mean velocity in the initial and transition regions were found to follow the normal probability function:

$$\frac{u}{u_o} = e^{-(r^2/2\sigma^2)} \tag{2.3}$$

where $o = c_a x$; c_a is an empirical constant which has two
values, one for the initial region and the other for the
transition region. This model of mean axial velocity
showed satisfactory agreement for the two-dimensional case,
but it was not wholly satisfactory for the axisymmetric
case. The experimental value of the potential core length
was found to be $x/r_o = 12$, but the authors, using $x/r_o = 9$
achieved agreement with measurements of the velocity in the
downstream sections. The measurements were carried out by
using a hypodermic needle as a stagnation tube connected
directly to a manometer.

In 1957, Miller and Comings [19] used Prandtl's
mixing length hypothesis for a two-dimensional jet flow
and developed a self-preserving empirical model for mean
axial velocity in the transition region after $x/r_o = 10$:

$$\frac{u}{u_c} = \exp \left[- \frac{\pi}{8} (\frac{r}{b_m})^2 \right] \qquad (2.4)$$

where b_m was defined by: $b_m = \int_0^\infty (u/u_c) dr$.

The measured values of b_m showed a linear relation-
ship with axial distance. Experimentally, it was shown
that the axial component of Reynolds shear stress was
balanced by the static pressure. They applied similar
assumptions to those of Kuethe [16] for the theoretical
analysis. Reynolds shear stress was expressed by Equation
(2.2). This model does not apply to the initial region.

Kirshner [20] used a constant viscosity model for

the transition region and solved the momentum integral equation in the same way as for laminar flow. The expression for the mean axial velocity of the turbulent jet was the same as that for the laminar jet with the molecular diffusivity replaced by constant eddy diffusivity. The constant eddy diffusivity was defined by: $\nu_t = c_k b_1 u_c$, where c_k is an empirical constant. The results of this solution agreed with the measurements of Kirshner [20] for the transition region. Though this model is simple to use, the turbulent diffusivity is not constant throughout the jet flow, specially in intermittent regions [21, 22].

Hatta and Nozaki [23], in 1975, developed a model for axial mean velocity by an approximate solution of the axial momentum integral equation for the initial and transition regions. Turbulent diffusity, ν_t was defined by:

$$\nu_t \propto (r_m - r_1) u_c$$

where r_m is the mean radius of the jet expressed by:

$$u_o r_m^2 = r_1^2 u_o + 2 \int_{r_1}^{r_2} urdr$$

Neglecting the pressure gradient in the momentum equation, a model for the axial mean velocity was established:

$$\frac{u}{u_o} = (1 - \eta_H)^4 (1 + 4\eta_H) \qquad (2.5)$$

where $\eta_H = r/b_1$.

This investigation also derived expressions for b_m, b_1, and r_1 as linear functions of axial distance. Though this approximate model showed agreement with Hatta and Nozaki's [23] measurements, it does not agree wholly with the measurements of Bradshaw et al [10], Sami et al [24] and Von-Frank [25]. This discrepancy likely comes from the initial conditions which were not documented by the authors.

Madni and Pletcher [26] calculated the mean velocity by an explicit finite-difference method from the differential forms of mass and momentum conservation equations derived by using the boundary layer approximations. Reynolds shear stress was defined by Equation (2.2) for the initial region, and for the transition region by:

$$\overline{u'v'} = \nu_t \cdot \frac{\partial u}{\partial r} \qquad (2.6)$$

where,

$$\nu_t = \frac{0.03}{r_o} \int_r^\infty u r \, dr \qquad (2.7)$$

The calculation started with uniform exit velocity and showed satisfactory agreement with measurements for the main portion of the shear layer. This model underpredicts the mean velocity of the outer region where velocity is low.

Many experimental investigations, without identifying initial conditions, showed self-preservation of the mean velocity in the initial and transition regions of the turbulent jets. Some empirical equations for the

initial region are:

$$\text{Abramovich [27]}: \quad \frac{u}{u_c} = (1 - \eta^{3/2})^2$$

$$\text{Simson [28]} \quad : \quad \frac{u}{u_c} = (1 - \eta^{7/4})^2 \qquad (2.8a)$$

$$\text{Kimura [23]} \quad : \quad \frac{u}{u_c} = 1 - 3\eta^2 + 2\eta^3$$

where,

$$\eta = r/b_1$$

and for the transition region is:

$$\text{Nayer et al [29]}: \quad \frac{u}{u_c} = e^{-1.415(\eta + 0.7)} \qquad (2.8b)$$

where,

$$\eta = (r - r_{\frac{1}{2}})/b_1$$

In the first three models, the shear layer width, b_1, was chosen between $u/u_c = 1$ and $u/u_c = 0$, and Nayer measured the width, b_1 between $u/u_c = 0.99$ and $u/u_c = 0.01$. Nayer's definition is realistic, but this model does not apply to the initial region. Though these models are developed from similar experimental data, they do not agree with each other, probably because of different initial conditions.

Davies et al [30] measured the axial mean velocity and showed self-preservation against the independent variable $(r - r_o)/x$ in the initial and transition regions. Measurements of Bradshaw et al [10], Von-Frank [25], Sami et al [24], Laurence [31], Lau [32], Kolpin [33] and others

show that the axial mean velocity is self-preserving when
it is plotted against the variable $(r - r_\frac{1}{2})/b_1$. These
measurements do not show wholly self-preservation against
the variable $(r - r_o)/x$. Yuu et al [34] used a polynomial
form as an empirical model for the axial velocity in the
initial and transition regions with independent variable
$(r - r_1)/x$, where r_1 is potential core radius. This model
does not show satisfactory agreement with measurements in
the initial region, but it is fairly applicable to the
developed region. It is interesting to note here that
measurements of Heskestad [5], Wygnanski [4], Gortler [15],
Reed [35], and others show self-preservation of axial mean
velocity when it is plotted against the variable r/x for
the developed region.

Recently, Bradshaw [36], Crow and Champagne [37],
Yule [38] and Hussain and Zedan [39,40] published some
experimental results for jet flow, identifying the initial
conditions. Bradshaw [36] studied the effects of exit
boundary layer on turbulence with boundary layer untripped
or tripped by concentric rings. Experimental results have
shown that the shear layer becomes fully turbulent at an
approximate distance $x_t = 7 \times 10^5(\frac{\nu}{u_o})$ from the exit for
any boundary layer thickness used in the experiment. The
investigation was carried out with maximum momentum thick-
ness 0.15 mm. at the nozzle exit. He also noticed a
significant change of virtual origin and an increase of
noise emission with boundary layer thickness. Yule [38]

studied the jet flow for a wide range of Reynolds numbers
$(9 \times 10^3 - 10^5)$ by using flow visualisation and hot-wire
techniques. He found the shear layer fully turbulent at
an approximate distance $x_t = 1.2 \times 10^5 (\frac{\nu}{u_o})$ from the nozzle
exit which is different from Bradshaw's [36] results. This
work did not mention the exit boundary layer thickness and
turbulence level.

Hussain and Zedan [39,40] measured mean velocity
and determined the jet boundaries, varying both the laminar
and turbulent boundary layer thicknesses, and controlling
the turbulence at the nozzle exit. Experimental results
showed the mean velocity to be self-preserving at a dis-
tance from the exit which varies with the initial turbu-
lence level and boundary layer thickness. For a turbulent
boundary layer at the nozzle exit, the virtual origin was
found to be very close to the geometric origin at the
nozzle exit. This investigation was carried out over a
range of Reynolds numbers $(6 \times 10^4 - 1.4 \times 10^5)$ with
maximum momentum thickness $\theta_o/r_o = 0.011$ at the exit plane.

All the theoretical work referenced to the above,
used the width of the shear layer, b_1, as the scale for
the similarity variable η. In most cases, b_1 is the width
between iso-velocity lines $u/u_c = 1$ and $u/u_c = 0$. Physi-
cally it is impossible to determine iso-velocity line
$u/u_c = 0$.

Numerical calculations of the differential momentum

equation together with the semi-empirical transport equations presented later, were performed by Rotta [8,41] Biringen [42], Launder and Morse [43], Launder and Spalding [9], Harsha [44] and others for mean velocity and turbulence quantities. This approach will be described in the following sections in some detail.

2.3 Differential Transport Equations

Turbulent shear flows can be expressed mathematically by the continuity equation, the momentum equation, and a set of semi-empirical transport equations for one or more dependent variables, such as turbulent kinetic energy, length scale, dissipation energy, etc. If the set of empirical constants are known, these differential equations may be solved by numerical methods to simulate the behaviour of turbulent quantities in the shear flow. Kolmogorov [45], in 1940, first proposed the semi-empirical transport equations for the turbulent kinetic energy, k, and the characteristic frequency, f. These equations describe how the dependent variables are influenced by basic forms of energy transfer, i.e., production, dissipation, diffusion and convection. The length scale of turbulence was defined by: $L^1 = \sqrt{k}/f$ in this model. These equations were difficult to use because of their complexity.

A few years after Kolmogorov's [45] work, Prandtl [46] independently suggested a similar differential transport equation for the turbulent kinetic energy, k, but he

used a linear algebraic equation for the length scale of
turbulence which was different from Kolmogorov's equation.
Glushko [47] applied this model to turbulent boundary layer
calculations.

Recently, Rotta [8], Launder and Spalding [9], Rodi
and Spalding [48], Bradshaw [49], Daly and Harlow [50],
Kovasznay[51], Launder et al [52] and others have developed
independently a number of useable differential transport
equations. The equations have been applied to a variety of
flows, including the developed region of turbulent jets, by
Rotta [8], Launder and Spalding [9] and others. These mathe-
matical models can be described as one-equation, two-equation,
three-equation models, etc., depending on the number of
semi-empirical transport equations used. Usually, increasing
the number of equations means longer computer time and
greater effort in determining a larger number of empirical
constants. On the other hand, the model having the large
number of equations can usually be applied to a wider variety
of flows, although the accuracy of the calculated values may
not be superior.

Bradshaw, Ferriss and Atwell [53] calculated turbulent
quantities in a turbulent boundary layer using a differential
transport equation for turbulent kinetic energy which was
similar to that of Kolmogorov and Prandtl. The equation is:

$$\underbrace{\rho(u\frac{\partial k}{\partial x} + \upsilon\frac{\partial k}{\partial y})}_{\text{Transportation}} - \underbrace{\tau\frac{\partial u}{\partial y}}_{\substack{\text{Pro-}\\\text{duction}}} + \underbrace{\frac{\partial}{\partial y}(\overline{p'\upsilon'} + \overline{\rho k \upsilon'})}_{\text{Diffusion}} + \underbrace{\rho\epsilon}_{\text{Dissipation}} = 0 \qquad (2.9)$$

where $\tau = \rho\overline{u'\upsilon'}$ and $\varepsilon \simeq \nu\overline{\left(\dfrac{\partial u_i}{\partial x_j}\right)^2}$

The Reynolds shear stress, $\rho\overline{u'\upsilon'}$ was expressed by:

$$-\rho\overline{u'\upsilon'} = c_1\rho k \tag{2.10}$$

and the diffusion term was simplified to:

$$\overline{\rho'\upsilon'} + \overline{\rho k\upsilon'} = \overline{\rho k\upsilon_e} \tag{2.11}$$

where c_1 is supposedly a universal constant and υ_e is the effective velocity at which turbulent energy is transported in the transverse direction by the large eddies. The effective velocity, υ_e, was expressed in terms of shear stress and an unknown function, G:

$$\upsilon_e = \left(\frac{\tau_{max}}{\rho}\right)^{\frac{1}{2}} G\left(\frac{y}{\delta}\right) \tag{2.12}$$

where δ is boundary layer thickness and τ_{max} is wall shear stress. The dissipation term was expressed as:

$$\varepsilon = L^1(\overline{u'\upsilon'})^{3/2} \tag{2.13}$$

The length scale L^1 was assumed to be a universal function of y/δ. This function was similar to that employed by Glushko [47] for developing Prandtl's model. Bradshaw, Ferriss and Atwell [53] calculated turbulence quantities from the continuity equation and a momentum equation and a differential transport equation for turbulent kinetic energy together with an empirical length scale model.

Launder and Spalding [9], Rodi and Spalding [48], Ng

and Spalding [54], Jones and Launder [55] and Spalding [56].
have developed two-equation transport models for turbulence
and applied these to a variety of flows. In all models, one
dependent variable in the transport equations is the turbu-
lent kinetic energy, k. The second dependent variable is
the length scale itself. However, since an equation for k
is prescribed, its simultaneous solution with an equation
for a variable, Z of the form $k^m L^{1h}$ has been used to find
the length scale, L^1. The above authors have developed
three principal models of this type, which resemble each
other. The differential transport equation for Z for two
dimensional flow is:

$$\rho \frac{DZ}{Dt} = \frac{\partial}{\partial y} \left(\frac{\mu_{eff}}{\sigma_z} \frac{\partial Z}{\partial y} \right) + Z \left[\frac{c_2 \mu_{eff}}{k} \left(\frac{\partial u}{\partial y} \right)^2 - c_3 \rho^2 k / \mu_{eff} \right] + S$$

(2.14)

where the term, S represents a secondary source term; its
form is included in Table 2.3.

The equation for turbulent kinetic energy, k, which
is the same for all the models of Launder-Spalding et al,
may be written in a form:

$$\rho \frac{Dk}{Dt} = \frac{\partial}{\partial y} \left(\frac{\mu_{eff}}{\sigma_k} \frac{\partial k}{\partial y} \right) + k \left[\frac{\mu_{eff}}{k} \left(\frac{\partial u}{\partial y} \right)^2 - c_D \rho^2 k / \mu_{eff} \right]$$

(2.15)

Equations (2.14) and (2.15) express the supposition that
convective changes in Z and k may occur through three agen-
cies:

i)   Spatial gradients in turbulent quantities which

give rise to diffusive transfer.

ii)   interactions between the turbulence and the
mean velocity field.

iii) direct action by turbulent motion itself.

The differential equations for Z and k contain the
terms c_2, c_3, c_D, σ_z and σ_k which are assumed to be a set
of empirical constants. The expressions for Z and S, and
the values of constants are presented in Table 2.3.

All these models use the same expression for turbu-
lent shear stress, $\rho \overline{u'v'}$.

$$-\rho \overline{u'v'} = \mu_{eff} \frac{\partial u}{\partial y}$$ (2.16)

where the effective viscosity μ_{eff} is:

$$\mu_{eff} = \rho c_D \sqrt{k}\, L^1$$ (2.17)

These models were applied to turbulent boundary
layers, internal flows and many other flows including the
developed region of turbulent jets and satisfactory results
have been shown in Refs. [9,43,57,58,59].

Rotta [8] also developed two-equation transport
models for turbulence. One equation has been formed with
the dependent variable either the turbulent kinetic energy
or the shear stress, and the other equation with the vari-
able, kL^1. In these equations, it is assumed that the con-
vective changes in dependent variables are balanced by
production, dissipation, and diffusion of turbulence. The
differential equations are:

TABLE 2.?: Two-Equation Models

Reference	z	Physical Meaning	c_2	c_3	c_D	σ_k	σ_z	s
Rodi & Spalding [48] Ng & Spalding [54]	kL^1	Energy And Length Scale Product	0.98	0.059	0.09	1.0	1.0	$(\frac{L^1}{y})^6 k^{3/2}(702)$
Spalding [56]	$k/(L^1)^2$	Square of Ch. Frequency	1.04	0.17	0.09	0.9	0.9	$3.5\mu_{eff}(\frac{\partial^2 u}{\partial y^2})^2$
Jones & Launder [55] Launder & Spalding [9]	$k^{3/2}/L^1$	Dissipation Energy	1.45	0.14	0.09	1.0	1.1	---

the K.E. equation,

$$\frac{Dk}{Dt} = \overline{u'v'} \frac{\partial u}{\partial y} - c_r \frac{k^{3/2}}{L^1} + \frac{1}{y^j} \frac{\partial}{\partial y}\left(y^j k_q \sqrt{k} L^1 \frac{\partial k}{\partial y}\right)$$

(2.18)

the length scale equation,

$$\frac{DkL^1}{Dt} = \overline{u'v'} \, \xi L^1 \frac{\partial u}{\partial y} - c_L c_r k^{3/2}$$

$$+ \frac{1}{y^j} \frac{\partial}{\partial y} \left(y^j k_{qL} \sqrt{kL^1} (L^1 \frac{\partial k}{\partial y} + \alpha_L k \frac{\partial k}{\partial y})\right)$$

(2.19)

where $j = 1$ for axisymmetric flow and $j = 0$ for two dimensional flow. These equations contain the terms c_r, k_q, ξ, c_L, k_{qL} and α_L which are assumed to be empirical constants. Rotta [8] determined these constants for the developed region of turbulent jets to be:

$$c_r = 0.165 \; ; \quad c_L = 0.8 \; ; \; k_q = 0.6$$

$$\xi = 0.99 \; ; \; k_{qL} = 0.6 \; ; \; \alpha_L = 0.5$$

Rotta [8] further used this model replacing the kinetic energy equation by the turbulent shear stress equation in the form:

$$\frac{D}{Dt} (\rho\overline{u'v'}) = \rho k \, a_p \frac{\partial u}{\partial y} - k_p \, c_r \frac{\sqrt{k}}{L^1} (\rho\overline{u'v'})$$

$$+ k_{qT} \frac{\partial}{\partial y}\left(\sqrt{kL^1} \frac{\partial \rho\overline{u'v'}}{\partial y}\right)$$

(2.20)

The terms a_p, k_p and k_{qT} are empirical constants and the values of these constants for developed region of jets are:

$$a_p = 0.09 \; , \; k_p = 1 \;\; \text{and} \;\; k_{q\tau} = k_q$$

For these models, the relationship between turbulent kinetic energy and shear stress is the same as that of Launder and Spalding [9] shown in Equations (2.16) and (2.17). These differential transport equations together with the mass continuity equation and momentum equation were solved numerically for many cases of shear flows [8], including the developed region of turbulent jets.

Harsha [44] formed a one-equation transport model for calculating shear flow turbulence. He used a differential transport equation for the turbulent kinetic energy and an algebraic expression for the length scale. The energy equation is:

$$\rho \frac{Dk}{Dt} = \frac{1}{y^j} \frac{\partial}{\partial y} \left(\frac{\mu_{eff} \, y^j}{p_r} \frac{\partial k}{\partial y} \right) + \rho \overline{u'v'} \frac{\partial u}{\partial y} - a_2 \frac{\rho k^{3/2}}{L^1} \qquad (2.21)$$

for axisymmetric flow j = 1.

The length scale L^1 was chosen to be twice the half-velocity width, $r_{\frac{1}{2}}$, for axisymmetric flow, and the effective viscosity was expressed as:

$$\mu_{eff} = 0.0125 L^1 (u_{max} - u_{min}) \qquad (2.22)$$

The empirical constants p_r and a_2 were determined for axisymmetric flows:

$$p_r = 0.70 \; ; \; a_2 = 1.69$$

Harsha [44] used Equation (2.10) for relating turbu-

lent shear stress and kinetic energy in the same way as that of Bradshaw et al [53]; and Harsha and Lee [60] established the constant, $c_1 = 0.3$ experimentally. Harsha's [44] calculation of the differential transport equation together with the differential form of mass and momentum conservation equations showed satisfactory results everywhere in jet flow except the initial region.

At the Los Alamos laboratory of The University of California, Harlow and co-workers have developed transport models for turbulence which are similar to that of Kolmogorov [45]. They have formed the differential transport equations with two (or more) turbulent quantities as dependent variables. All models developed by this research group used the turbulent kinetic energy as one dependent variable for a transport equation. The other dependent variable was chosen to be the length scale by Harlow and Nakayama [61] and local dissipation rate by Harlow and Nakayama [62] to form another differential transport equation. Recently, Harlow and Hirt [63] and Harlow and Daly [64] have employed differential equations for turbulent shear stresses and normal stresses in the three individual directions of the co-ordinate system. Each model contains a set of empirical constants, and the shear stresses were expressed by similar expressions to those given in Equations (2.16) and (2.17). These models have been applied to a variety of turbulent flows, but their application to axisymmetric jets in still surroundings has not been reported.

The calculation of turbulence by using the differential transport equations has become routine for many turbulent flows. Recently, Heck and Smith [65], Peters and Phares [66], Biringen [42], McGuirk and Rodi [67], Launder et al [52,59,98] and others applied the transport models described above to a variety of flows and obtained satisfactory results. Each worker used a different numerical method of solution and a different set of empirical constants, yet each achieved satisfactory results.

From a survey of the literature, it is apparent that not much theoretical work has been done in the initial and transition regions of axisymmetric jets. This dissertation presents some theoretical work for these regions using a single transport equation together with the continuity and momentum equations.

2.4 Turbulence

Jet shear flows are aniostropic in character and a continuous transfer of energy takes place from mean motion to the turbulence due to the interaction of the Reynolds stresses and the mean velocity gradients. This kind of flow has high Reynolds stresses which are the major cause of jet noise [68]. The turbulence in shear flows is determined either by measurements or by empirical/semi-empirical relations.

Bradshaw et al [10], Davies et al [30,69], Ko and Davies [11], Lau [12,32], Laurence [31], Lau and Fisher [70], Arndt and Nilsen [71] and others measured turbulence in

different jet geometries and at different Reynolds numbers
as shown in Table 2.1. Though most of the measurements were
made under similar conditions using the hot-wire anemometer,
the results do not wholly agree with each other in the ini-
tial region. The agreement of different measurements is
satisfactory in the transition and developed regions. The
disagreement of measurements in the initial region may be
due to different initial conditions which were unspecified.

Rotta [8], Launder et al [9], Harsha [44] and others
applied transport models for calculating turbulence in the
developed and transition regions of jets. Rotta [8] and
Launder et al [9] calculated turbulent kinetic energy and
length scale from a two-equation transport model and deter-
mined the shear stress by Equations (2.16) and (2.17).
Harsha [44] calculated turbulent kinetic energy from a one-
equation transport model and determined the shear stress by
Equation (2.10). Harsha [44] also applied the following
model for shear stress in jet flow:

$$\rho \overline{u'v'} = c_1 \rho \left[\left(\frac{\partial u}{\partial r}\right) / \left(\frac{\partial u}{\partial r}\right)_{max} \right] k \qquad (2.23)$$

This model of shear stress does not show satisfactory agree-
ment with measurements for the initial region.

Rudy and Bushnell [72] calculated Prandtl's mixing
length for many shear flows, and twenty-two out of twenty-
four cases in Langley Workshop conference were tested by
Prandtl's mixing length. The present research examines the
Prandtl's mixing length model of shear stress in addition to

that given in Equations (2.10) and (2.16) for the initial region.

The authors referred to in Section 2.1 above did not report relationship between individual components of Reynolds normal stresses and turbulent kinetic energy. Townsend [7] and Csanady [73] approximated the normal stresses in r and θ directions as $\upsilon'^2 = w'^2$, and in x direction as $\overline{u'^2} = k$. Measurements of Bradshaw et al [10], Sami et al [24], Von-Frank [25] and others do not agree with the assumption, $\upsilon'^2 = w'^2$ in the developing region of jets. The present research develops semi-empirical relations between the individual components of Reynolds normal stresses and the turbulent kinetic energy.

CHAPTER III

THEORY

3.1 General

Turbulent motion is governed by the Navier-Stokes differential equations. The general solution of the non-linear Navier-Stokes equations is not available. In order to apply Navier-Stokes equations to practical cases, hypothesis and empirical assumptions have to be introduced in order to obtain a closed set of equations with time average dependent variables. The early theoretical work uses Boussinesq's eddy viscosity hypothesis and Prandtl's mixing length equation to calculate mean velocities with the eddy viscosity and the mixing length as empirical quantities. The assumptions, necessary to build the empirical relations depend on the boundary conditions of the flow field and have to be readjusted for each particular flow case.

Theoretical work on turbulence can be developed in any of the four levels:

1) Empirical algebraic formulae connecting the turbulent and mean flow quantities. The most common example is eddy viscosity formulae.

2) Empirical differential equations (transport equations) for one-point turbulent quantities. Examples are the Reynolds stress transport equations, length scale transport equation, etc.

3) Empirical differential equations for two-point correlations or spectrum tensor. Examples are Kraichnan

"Direct Interaction" model [74] and Deissler's Linearized models [75].

4) Direct numerical solution of the time-dependent Navier-Stokes equations. Examples are Deardorff [76] and Schumann [77].

Studies at level 3 refer mainly to isotropic turbulence. The mathematical treatment to the shear flow equations simplifies level 3 almost to level 2. Progress at level 4 is possible for low Reynolds no. flows, but consumes much computer time producing results inferior to that of level 3, even for the small scale motion. Therefore, engineering calculations of turbulent flows are confined to levels 1 and 2 for the immediate future.

Theoretical analyses at levels 1 and 2 are most simply explained as attempts to 'close' the exact Reynolds stress transport equations for $-\rho \overline{u_i' u_j'}$ which are derived by manipulating the Navier-Stokes equation and time averaging as shown in Appendix A. Up to the present, the most successful method of closure has been to obtain an empirical turbulent energy transport equation and relate the Reynolds stresses to energy equation by using physical concepts.

3.2 Governing Equation

3.2.1 Mean Flow Equations

Assuming steady, incompressible flow without contacting any solid surface and constant fluid properties and zero pressure gradient, the equations for mean motion are derived in Appendix A.

the mass conservation equation is:

$$\frac{\partial (ur)}{\partial x} + \frac{\partial (\upsilon r)}{\partial r} = 0 \qquad (3.1)$$

the momentum conservation equation is:

$$u \frac{\partial u}{\partial x} + \upsilon \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\tau}{\rho}) \qquad (3.2)$$

where τ is the shear stress and ρ is the density of the fluid. The co-ordinate system is shown in Figure 1.1

The shear stress, τ, includes both viscous and turbulent contributions, and it was written as:

$$\frac{\tau}{\rho} = \nu \frac{\partial u}{\partial r} - \overline{u'\upsilon'} \qquad (3.3)$$

where ν is the molecular diffusivity. The turbulent part of shear stress is $-\rho\overline{u'\upsilon'}$ and it is usually expressed by Boussinesq's hypothesis:

$$-\rho\overline{u'\upsilon'} = \rho\nu_t \frac{\partial u}{\partial r} \qquad (3.4)$$

ν_t is the turbulent diffusivity. For the initial region of turbulent jets, the diffusivity, ν_t is expressed in terms of Prandtl's mixing length, L and mean velocity gradient, $\partial u/\partial r$.

for the initial region:

$$\nu_t = L^2 |\frac{\partial u}{\partial r}| \qquad (3.5)$$

where L is Prandtl's mixing length. This model is not applicable to the transition region. Schetz's [21] and Madni-Pletcher's [26] models are combined for calculating

mean flow properties in the transition region.

Madni-Pletcher's model is:

$$\nu_t = \frac{0.03}{r_o} \int_r^\infty ur\partial r \tag{3.6}$$

Schetz's model is:

$$\nu_t = \frac{0.024\gamma}{r_o} \int_0^\infty ur\partial r \tag{3.7}$$

where γ is an intermittency factor. An expression was developed by the author for γ as follows:

$$\left.\begin{aligned} \gamma &= 1.0 \quad ; \; 0 \le r/r_{\frac{1}{2}} \le 0.8 \\[2mm] \gamma &= (0.5)^\beta \; ; \; r/r_{\frac{1}{2}} > 0.8 \\[2mm] \text{where,} \quad \beta &= (r/r_{\frac{1}{2}} - 0.8)^{2.5} \end{aligned}\right\} \tag{3.7a}$$

This gives a distribution for γ which agrees well with the experimentally determined curve in ref. [87].

A variable $\eta = (r - r_{\frac{1}{2}})/b$ is defined to be a self-preserving variable for the initial region, where the scale, b, is defined as:

$$b/r_o = \frac{-1}{\left[\dfrac{\partial(u/u_c)}{\partial(r/r_o)}\right]_{r=r_{\frac{1}{2}}}} \tag{3.8}$$

where r_o is the jet exit radius, $r_{\frac{1}{2}}$ is the radial distance for $u/u_c = 0.5$ and u_c is the centre line velocity. The co-ordinate at $r_{\frac{1}{2}}$ and the scale, b, were chosen as the most convenient for establishing self-preserving qualities of the jet in the initial region.

3.2.2 General Turbulent Kinetic Energy Equations

The general turbulent kinetic energy equation derived in Appendix A is:

$$\underbrace{u\frac{\partial k}{\partial x} + \upsilon\frac{\partial k}{\partial r}}_{\text{Transport}} = -\Bigg[\underbrace{\overline{u'^2}\frac{\partial u}{\partial x} + \overline{u'\upsilon'}(\frac{\partial u}{\partial r}+\frac{\partial\upsilon}{\partial x}) + \frac{\overline{u'w'}}{r}\frac{\partial u}{\partial\upsilon}}_{\text{Production}}$$

$$+\underbrace{\frac{\overline{w'\upsilon'}}{r}\frac{\partial\upsilon}{\partial\theta} + \overline{w'^2}\frac{\upsilon}{r}}_{}\Bigg] - \upsilon\Bigg[\overline{(\frac{\partial u'}{\partial x})^2} + \overline{(\frac{\partial\upsilon'}{\partial x})^2} + \overline{(\frac{\partial w'}{\partial x})^2}$$

$$+\overline{(\frac{\partial u'}{\partial r})^2} + \overline{(\frac{\partial\upsilon'}{\partial r})^2} + \overline{(\frac{\partial w'}{\partial r})^2} + \overline{(\frac{1}{r}\frac{\partial u'}{\partial\theta})^2} + \underbrace{\overline{(\frac{1}{r}\frac{\partial\upsilon'}{\partial\theta}-\frac{w'}{r})^2}}_{\text{Dissipation}}$$

$$+\overline{(\frac{1}{r}\frac{\partial w'}{\partial\theta}+\frac{\upsilon'}{r})^2}\Bigg] - \Bigg[\underbrace{\overline{\frac{\partial}{\partial x}\{u'(p'/\rho+k)\}}}_{}$$

$$+\underbrace{\frac{1}{r}\overline{\frac{\partial}{\partial r}\{\upsilon'r(p'/\rho+k)\}} + \frac{1}{r}\overline{\frac{\partial}{\partial\theta}\{w'(p'/\rho+k)\}}}_{\text{Diffusion}}$$

$$-\upsilon\{\underbrace{\frac{\partial^2 k}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial k}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 k}{\partial\theta^2}}_{}\Bigg] \qquad (3.9)$$

where $k = (\overline{u'^2} + \overline{\upsilon'^2} + \overline{w'^2})/2$. From physical concepts, this equation contains four basic terms: 1) convective transport, 2) production, 3) dissipation and 4) diffusion transport. A correct interpretation of the terms in the transport equation (3.9) is important. For practical purposes, Equation (3.9) is simplified by introducing empirical assumptions. This means that the terms are approximated by algebraic relations.

3.3 Formation of Empirical Turbulent Energy Equation

3.3.1 Production

The main source of turbulence is the production term. Production means transfer of energy from mean motion to turbulence. This occurs through interaction of turbulent motion with mean flow. In other words, it can be defined as the work done by Reynolds stresses due to the deformation caused by mean flow. It may be seen from Equation (3.9), that the production term is the product of the Reynolds stresses and the rate of deformation caused by mean flow. Mathematically, the production term is:

$$\text{Production} = -\overline{u'_i u'_j}\, S_{i,j}$$

where $u'_i = (u', \upsilon', w')$ and $S_{i,j} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $u_i = (u, \upsilon, w)$ and $x_i = (x, y, z)$. $S_{i,j}$ is the deformation tensor due to mean velocity, and $\overline{u'_i u'_j}$ is Reynolds stress tensor.

Production of turbulence causes the flow to be anisotropic. A vortical motion exists in the flow; the mean motion stretches the vortices and performs work on them which causes the generation of turbulence.

For axisymmetric flow, the azimuthal gradient of any property is zero; hence, $\partial u/\partial\theta$, $\partial\upsilon/\partial\theta$ and $\partial k/\partial\theta$ are equal to zero in Equation (3.9). From the order of magnitude principal, the transverse gradient of any property is very high compared to the longitudinal gradient for a thin shear flow. The production by the transverse mean velocity gradient, $\overline{u'\upsilon'}\,\frac{\partial u}{\partial r}$, in Equation (3.9) dominates the whole turbulence

generation except near the centre of the transition region.
Therefore, the production term in Equation (3.9) is expressed
as,

$$\text{Production} = a_1 \, \overline{u'v'} \, \frac{\partial u}{\partial r} \tag{3.10}$$

where a_1 is an empirical constant which accounts for the
contribution of all the mean velocity gradients to the produc-
tion term.

3.3.2 Dissipation

Turbulent dissipation means the transfer of turbulent
energy to molecular motion. It is the rate at which the
viscous stresses perform deformation work against the fluc-
tuating strain rate. This term given in Equation (3.9) is
the product of the molecular diffusivity and the quadratic
fluctuating deformation. It can be simplified by restrict-
ing the flow to high Reynolds numbers and assuming that
the dissipation is independent of the viscosity, and hence
the Reynolds number. The idea behind this assumption is
that for high Reynolds numbers, the turbulent spectra ex-
tend over a very wide range of wave numbers, and that the
contribution to dissipation comes mainly from the small
eddies or large wave number spectra. The motion of the
large wave number components is statistically independent
of the energy containing part of the spectra. Although
the small eddies are mainly responsible for dissipation,
and independent of the behaviour of the large eddies, the
viscous dissipation is governed by the rate of energy trans-
fer from large to small eddies and not by the viscosity,

i.e., the rate of energy dissipation is independent of viscosity. The magnitude may be expressed empirically [78]:

$$\text{Dissipation} = c_r \frac{k^{3/2}}{L^1} \qquad (3.11)$$

where L^1 is a length scale for energy containing eddies and c_r is an empirical constant. Although this simplification is made for high Reynolds number flows, its extension to low Reynolds number flows is also found to be valid [55] except for the laminar-turbulent transition.

3.3.3 Diffusion

Turbulent diffusion is a method of energy transport caused by turbulent mixing or interaction of eddies. It is high for the lower wave numbers corresponding to the eddies which contain the major portion of the turbulent energy. The diffusion term has two parts: (1) inertia, and (2) viscous diffusion. Viscous diffusion is the second part of diffusion term in Equation (3.9), and it is negligible in high Reynolds number flows. Inertia diffusion is the first part of diffusion term in Equation (3.9) and is triple correlations of the fluctuating velocities and correlations between fluctuating pressures and fluctuating velocities.

In general, the flow field is less affected by the diffusion transport than by the other terms, although diffusion may predominate in certain parts of the flow field. Therefore, some crude assumptions are admissible. Launder [9], Rotta [8] and others have used the simple gradient type diffusion as suggested by Prandtl [46], and expressed

diffusion in the transverse direction by:

$$\overline{(p'/\rho + k)\upsilon'} = -k_q \sqrt{k}\ L^1\ \frac{\partial k}{\partial r} \qquad (3.12)$$

where k_q is an empirical constant. By the order of magnitude principle, the longitudinal rate of diffusion is negligible compared to that in the transverse direction for thin shear layers.

3.3.4 Empirical Turbulent Energy Equation

Using the above interpretations and Equations (3.10), (3.11) and (3.12) for the production, dissipation and diffusion terms respectively, together with axisymmetry and thin shear layer conditions, Equation (3.9) becomes:

$$u\ \frac{\partial k}{\partial x} + \upsilon\ \frac{\partial k}{\partial r} = a_1\ \overline{u'\upsilon'}\ \frac{\partial u}{\partial r} - c_r\ \frac{k^{3/2}}{L^1}$$

$$+ \frac{1}{r}\ \frac{\partial}{\partial r}\{k_q L^1 \sqrt{k}\ r\ \frac{\partial k}{\partial r}\} \qquad (3.13)$$

The closure of the equations are achieved relating the turbulent shear stress and the kinetic energy in any of the following forms:

$$-\overline{u'\upsilon'} = c_1 k \qquad (3.14)$$

$$-\overline{u'\upsilon'} = c_D \sqrt{k}\ L^1\ \frac{\partial u}{\partial r} = c_\mu \frac{k^2}{\epsilon}\ \frac{\partial u}{\partial r} \qquad (3.15)$$

Besides these two models, Equation (2.2) may be used for closure.

3.4 Differential Method

3.4.1 Equations and Boundary Conditions

For mean flow properties, Equation (3.1) and Equation (3.2) were solved numerically using the shear stress Equation (3.4). The turbulent diffusivity, ν_t, was expressed by Equation (3.5), for the initial region and by a combination of Equations (3.6) and (3.7), for the transition region. The appropriate boundary conditions applicable to Equations (3.1) and (3.2) are:

$$\frac{\partial u(x,0)}{\partial r} = 0 \; ; \; \nu(x,0) = 0 \; ; \; \text{Lim } u(x,r) = 0 \qquad (3.16)$$
$$r \to \infty$$

In addition, the measured initial distribution for $u(x,r)$ was used:

$$u(x_o,r) = f(r) \qquad (3.17)$$

where x_o is the exit section of the jet.

This calculation was extended with turbulent kinetic energy Equation (3.13) in addition to Equation (3.1), and Equation (3.2) to calculate turbulent properties in the initial region. Equation (3.13) is subject to boundary conditions.:

$$\frac{k(x,0)}{\partial r} = 0 \; ; \; \text{Lim } k(x,r) = 0 \qquad (3.18)$$
$$r \to \infty$$

and the measured initial distribution:

$$k(x_o,r) = g(r) \qquad (3.19)$$

All calculations use an empirical equation for length scale given by:

$$L^1/b = L/b = 0.10625$$

Here L^1 is made equal to L by adjusting the constants in the shear stress and dissipation equations. Experimental determination of L/b will be presented later. The fluid properties ρ and ν are assumed constant.

3.4.2 Calculation Technique

An explicit finite difference technique of the Dufort-Frankel [94] type was applied here to the momentum Equation (3.2), the continuity Equation (3.1) and the transport Equation (3.13) associated with boundary conditions (3.16), (3.17), (3.18) and (3.19) to calculate the turbulent kinetic energy and the mean velocities. How the derivatives are approximated for this method and their truncation errors are given in Appendix D. The finite difference grid used for the calculation are shown in Figure 3.1. The computer programme developed for this purpose had the capability of handling non-uniform grid spacings in both r- and x-direction.

Von Neuman's method of stability analysis [79] with first order error was applied to the momentum equation and found to generate a mild stability constraint given in Appendix E. For calculating the mean flow properties from the continuity and momentum equations, non-uniform grid spacings were used in x-direction which was restricted by the stability constraint. The grid spacings in the r-direction were chosen to be uniform, dividing the discharge radius r_o into 30 equal divisions to attain convergence of the solution. The calculation with uniform grid spacings ($\Delta x/\Delta r < 0.5$) in both directions also attains convergence but consumes more computer time.

Von-Neuman's method of stability analysis [70] was found to be ineffective with the finite difference equation of the turbulent kinetic energy. For calculating turbulence in addition to mean flow properties from the continuity, momentum and turbulent energy equations, small uniform grid spacings in both directions were used with $\Delta x/\Delta r < 0.5$ which satisfies the stability constraint generated with the momentum equation. All calculations reported here for turbulence use $\Delta x/\Delta r = 0.5$ dividing the discharge radius r_0 into 50 equal divisions to attain convergence of the solution in the initial region. Convergence was not obtained in the transition region, and hence, no turbulence predictions are presented for the region.

The Dufort-Frankel technique requires information from two previous stations for the calculations to proceed in the streamwise direction. The associated boundary conditions give information at one station and information for the second station is calculated by a simple finite difference technique that requires only the previous step values. Switching of the calculations from the initial to the transition region takes place when the centre line velocity in the initial region decreases by 1%.

The finite difference equations for the continuity, momentum and turbulent kinetic energy are shown in Appendix D for both the Dufort-Frankel scheme and the simple finite-difference scheme.

Any calculation reported here did not require more than 1.5 minutes of computation time on IBM 370 computer.

3.5 Integral Method

The integral method is an alternative to the differential method for solving a set of differential equations by assuming an empirical mean velocity profile. Mass, momentum and energy are conserved over an integrated finite length to develop the governing equations. This is an approximate method, preferable for ease of calculation.

It was possible to establish an empirical self-preserving velocity profile for the initial region of a jet having a thin boundary layer at the nozzle exit. This velocity profile was used in the integral equations to determine jet geometry.

3.5.1 Integral Equations

The integral momentum and energy equations for the mean flow are derived in Appendix A. The momentum integral equation is:

$$\frac{\partial}{\partial x} \int_0^\infty u^2 r \partial r = 0 \qquad (3.21)$$

The energy integral equation is:

$$\frac{1}{2} \frac{\partial}{\partial x} \int_0^\infty u^3 r \partial r = -\int_0^\infty \nu_t r (\frac{\partial u}{\partial r})^2 \partial r \qquad (3.22)$$

The eddy diffusivity, ν_t is given by Equation (3.5).

3.5.2 Empirical Mean Velocity Profile

Using the present mean velocity measurements for the initial region, which will be presented in Chapter V, in a third order polynomial for the main portion of the mixing layer and superimposing Townsend's [7] hyperbolic

tangent velocity profile at the outer layer side, the follow-
ing velocity profile was established:

$$\frac{u}{u_c} = 1.0 \; ; \qquad\qquad \eta < -0.736$$

$$\frac{u}{u_c} = 0.5 - \eta + 0.05\eta^2 + 0.66\eta^3 ;- 0.736 < \eta < 0.5$$

$$\frac{u}{u_c} = A\left[1 - \tanh B(\eta - 0.5)\right] ; \; \eta < 0.5$$

$$(3.23)$$

where $\eta = (r - r_{\frac{1}{2}})/b$.

The constants $A = 0.095$ and $B = 4.79$ were determined
by using continuity of velocity and matching the slope of the
velocity profile at $\eta = 0.5$. This composite velocity profile
behaves asymptotically at the outer part of the jet with RMS
deviation of $(\Delta u/u_c)_{RMS} = 0.0112$.

3.5.3 Jet Boundaries

Equation (3.23) shows that the inner radius, r_1
corresponds to $\eta = -0.736$ where $u/u_c = 1.0$, and the outer
radius is defined here as corresponding to $\eta = 0.8$ where
$u/u_c = 0.01$. So, the inner and outer radii of the jet in
terms of $r_{\frac{1}{2}}$ and b are:

$$r_1 = r_{\frac{1}{2}} - 0.736b \qquad\qquad (3.24)$$

$$r_2 = r_{\frac{1}{2}} + 0.8b \qquad\qquad (3.25)$$

The width of the shear layer b_1 can be obtained by subtract-
ing r_1 from r_2:

$$b_1 = 1.536b \qquad\qquad (3.26)$$

Introducing Equation (3.5) into Equation (3.22), transforming variable r to η, and using empirical velocity Equation (3.23) in the fundamental integral equations (3.21) and (3.22), and integrating, the following two ordinary differential equations are obtained (details are shown in Appendix B):

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + 0.547578 \ r_{\frac{1}{2}}b - 0.215408b^2\right] = 0 \qquad (3.27)$$

$$\frac{1}{2} \frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + 0.45101941 \ r_{\frac{1}{2}}b - 0.20326464b^2\right]$$

$$= \frac{L^2}{b^2}\left[-0.65829521 \ r_{\frac{1}{2}} + 0.01311954b\right] \qquad (3.28)$$

For thin boundary layer at the nozzle exit, these equations are subject to the approximate boundary conditions:

$$\text{at } x = 0 \ , \ b = 0 \ , \ r_1 = r_o$$

Physically, these boundary conditions mean that the potential core fills the nozzle and there is no boundary layer at the nozzle exit. Integrating Equation (3.27) with the boundary conditions at x = 0, and using Equation (3.24), the following relation is obtained:

$$\frac{r_{\frac{1}{2}}}{r_o} = 0.18842198b/r_o + \sqrt{1.0 - 0.075377(b/r_o)^2}$$

$$(3.29)$$

Equation (3.27) and Equation (3.29) are used in Equation (3.28) to obtain:

$$\frac{d(b/r_O)}{d(x/r_O)} = 2(L/b)^2 \left[0.658296 - 0.049620(b/r_O)^2 \right.$$

$$+ 0.1109177 \; b/r_O \; \sqrt{1 - 0.075377(b/r_O)^2} \left. \right] /$$

$$\left[0.0965586 - 0.0145566(b/r_O)^2 \right.$$

$$+ 0.01210 \; b/r_O \; \sqrt{1 - 0.075377(b/r_O)^2} \left. \right] \qquad (3.30)$$

The ratio L/b was determined experimentally by measuring the shear stresses and velocity gradients, and using the following relation:

$$\overline{u'v'} = L^2 \left| \frac{\partial u}{\partial r} \right| \frac{\partial u}{\partial r} \qquad (3.31)$$

Transforming the variable r into η, Equation (3.31) is written as:

$$(L/b)^2 = \frac{\overline{u'v'}}{\left| \frac{du}{d\eta} \right| \frac{\partial u}{\partial \eta}} \qquad (3.32)$$

The experimental values of L/b presented in Chapter V, lie within the range 0.099 to 0.109 with an average, L/b=0.10625 given by Equation (3.20). Using Equation (3.20), Equation (3.30) can be integrated numerically with the starting conditions at x = 0, to obtain b, and hence, r_1, r_2 and b_1 from Equations (3.24), (3.25) and (3.26) respectively, as functions of x.

CHAPTER IV

EXPERIMENTS AND UNCERTAINTIES

4.1 General

Of many experimental investigations on jet flows that have been published, only a very few have identified the nozzle exit conditions.

The aim of the present investigation was to provide information on turbulence and mean flow quantities in the initial and transition regions of jets with varying flow conditions at the nozzle exit. Of the many flow quantities, only the boundary layer thickness and the turbulent intensity were considered as identifying the exit conditions. In all the experiments, the boundary layers were turbulent at the nozzle exit, being considered turbulent when the following two conditions were satisfied simultaneously:

i)   the exit fluctuation intensity profile showed characteristic peak intensity near the wall with $\sqrt{\overline{u'^2}}/u_c$ near 10%, a value typical of turbulent boundary layers.

ii)   the shape factor was close to 1.40. The shape factor, H_o, is defined as the ratio of the displacement thickness to the momentum thickness of the mean velocity profile.

4.2 Experimental Facilities

Axisymmetric turbulent jets were produced by discharging air through a series of circular nozzles from a large settling chamber as shown in the schematic diagram

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of the experimental set-up and nozzles in Figure 4.1.

Air from the compressed air facility in the labora-
tory was passed through a 3-micron filter, pressure regula-
tor and needle valve before entering the large settling
chamber which contained a set of flow straighteners followed
by 3 screens. Figure 4.1a shows a short circular nozzle
fitted to the large settling chamber. The displacement
thickness near the outlet of this nozzle ($x = -1.0$ mm) was
$\delta_O^*/r_O = 0.0064$, for $R_{eD} = 1.53 \times 10^5$. The jet exit condi-
tions were changed by using nozzles with extensions shown
in Figures 4.1b and 4.1c and tripping the boundary layer by
projections of 0.397 mm and 1.59 mm. The displacement
thicknesses for these cases were $\delta_O^*/r_O = 0.0179$ and $\delta_O^*/r_O =$
0.0628 respectively, for $R_{eD} = 1.53 \times 10^5$. The turbulent
intensity profiles at the exit were different for each case.
Each jet contained a core of uniform velocity at the exit,
which was found to be axisymmetric from measurements of the
axial mean velocity and turbulent intensity at $x/r_O = 1.0$
in both the vertical and the horizontal central planes,
as shown in Figures 5.1,5.2,5.3. The mean velocity profile
within the nozzle was also found to be symmetric about its axis.

Turbulence and mean flow quantities were measured by
using a DISA constant temperature hot-wire anemometer (type
55M10) with a linearizer (type 55D10), signal conditioner
(type 55D31). A DISA miniature probe (55P11) with a 5μ
tungsten normal wire was used for the mean axial velocity
and the axial turbulent intensity. The shear stress and

the turbulent intensities in r- and θ-directions were measured using the same normal wire and a single 5μ tungsten slanting wire with a DISA miniature probe (55P12).

4.3 Hot-Wire Calibration

Calibration of the hot wires was performed by making simultaneous measurements of the voltage with the hot wire connected to the anemometer, and the velocity with a pitot tube connected to a Lambrecht inclined tube manometer. The hot wire and the pitot tube were set up side-by-side in the uniform velocity of the potential core and the velocity was varied in steps over a range of values 2.5 m/s to 100 m/s. The inclined tube manometer was calibrated against a Meriam Micromanometer having a precision of 0.0254 mm; the results are shown in Figure C.3 of Appendix C. The hot-wire equations and the typical calibration curves for both normal and slanting wires without linearizer are shown in Figures C.4 and C.5 of Appendix C. Figure C.6 of Appendix C is a typical calibration curve of the hot wires with linearizer for high velocities (above 6 m/s), and Figure C.7 shows its calibration for low velocities (below 6 m/s). The low velocity calibration was performed as outlined in Ref. [80]. The calibration constants were determined by simple curve fitting principles [86] and their values are given in Appendix C.

The calibration curve was checked at the beginning and end of each series of transverses. If the curve had changed then the whole series of transverses were repeated.

The hot-wire was traversed in the radial and axial directions by a motor operated traversing mechanism with a scale having precision of 0.0254 mm.

4.4 Measurements

All measurements were performed for a Reynolds number, $R_{eD} = 1.53 \times 10^5$ based on the exit diameter and the average velocity. The main air supply was adjusted until the required value of the nozzle exit velocity, as measured by pitot tube, was obtained.

The velocity profile, about 1 mm inside the exit was measured by using a normal hot wire and recording the anemometer output voltage. The hot wire was traversed over a scale with precision 0.0254 mm, and a large number of readings were taken where the velocity began to deviate from the centre line velocity. At each station, at least five readings were taken within a time interval of 2 minutes. For higher fluctuating values, ten observations were taken at a station. The mean velocity and axial turbulence were calculated by using Equations (C.17) and (C.20) of Appendix C.

For the axial mean velocity and the turbulent intensity within the jet, a normal hot wire was positioned on the centre line of the nozzle, and the output from the linearizer was recorded. The hot wire was traversed vertically through the jet, with a large number of readings being taken where the velocity began to deviate from the jet centre line velocity. Within the turbulent shear layer

which corresponded to high fluctuating values, ten observa-
tions were noted for a station within a time interval of 2
minutes. After the upper half of the jet had been completely
traversed, the hot wire was returned to jet centre line. The
linearizer output was compared with the value obtained at
this location before traversing upwards through the jet; this
provided a check on whether the linearizer output had drifted
or the main flow rate had varied. If the two values were
the same, then the hot wire was traversed vertically through
the lower portion of the jet, and its axisymmetry was checked.
This procedure was followed at the downstream locations:

$$x/r_o = 2,4,6,8,10,12,15 \text{ and } 18$$

Traversing and taking readings, similar to those des-
cribed above were followed to measure the shear stress and
radial trubulent intensity using the same normal wire and
a slanting wire. The mean velocity and turbulence were
calculated by using the calibration constants and Equations
(C.17) and (C.20) derived in Appendix C.

4.5 Uncertainty Statement

The variation of the slopes of any calibration curve
was found to lie within ± 1% for at least 4 calibrations.
The angle of inclination of the slanting wire was measured
by positioning the probe in a slide projector and project-
ing the wire image on a paper screen. To minimize image
distortion, the wire was located near the centre of the
lens and the screen was positioned perpendicular to the
projector. Twenty values of the angle of the slanting

wire fell within the range $45° \pm 36'$.

. The uncertainty of the measurements of mean velocity, turbulent intensity and shear stress are influenced by variations of the ambient temperature, characteristics of the linearized circuit, contamination of the hot wires, the accuracy of the angle of inclination of the hot wire to the mean flow direction and the accuracy of the calibration equipment. All uncertainties presented here were calculated by using the standard procedure given in Ref. [81,82]. In the calibration unit, the uncertainty of mean velocity, u/u_c was estimated to be less than $\pm 0.3\%$[*]. The mean velocities which were obtained by using a linearized circuit showed uncertainty of $\pm 1.5\%$ in the main portion of shear layer and $\pm 4\%$ in the outer layer where velocity is near 6 m/s. The uncertainty of $\sqrt{u'^2}/u_c$ was found to be less than $\pm 5.5\%$ for the major portion of the shear layer where the mean velocity is greater than 6 m/s. The uncertainty of the shear stress was estimated to be less than $\pm 4.5\%$ neglecting the wire angle uncertainty which could cause an additional $\pm 3\%$ bringing total uncertainty less than $\pm 7.5\%$ for the major portion of the shear layer where the mean velocity is greater than 6 m/s. The uncertainties of $\sqrt{v'^2}/u_c$ and $\sqrt{w'^2}/u_c$ for the major portion of shear layer are less than $\pm 7\%$. These values are exceeded at the outer layer side where mean velocities are low. All uncertainties reported here exclude the error in the hot-wire equations by assuming

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[*] All uncertainties and repeatabilities were calculated assuming 20:1 odds.

weak turbulence. Estimated from Ref. [22], this could cause additional uncertainty in the shear and intensity values of ±5% in the main portion of the shear layer and ±8% near the outer boundary.

The repeatability of the mean velocity, turbulent intensities and shear stresses were tested by taking 300 readings at each of 18 locations in the mixing layer at planes normal to the axis of the jet and at distances $x/r_o = 4,6,8,$ 12 and 18 from the nozzle exit. Repeatability of the measurements was estimated to be ±5%.

The uncertainty in measuring the linear distance was negligible. Temperature variation during the experiments was within 0.3°C.

CHAPTER V

RESULTS AND DISCUSSION

5.1 General

Properties of the flow in the initial and transition regions of jets were determined both theoretically and experimentally. Three different nozzles were used to produce jets with three different exit conditions. For all three cases, the momentum and continuity equations were solved by a finite difference method for the mean flow quantities. In addition, a simple integral method was applied to the initial region to determine the jet properties for the first case, the very thin exit boundary layer. Turbulent quantities in the initial region were calculated by solving one differential transport equation for the turbulent kinetic energy in conjunction with the momentum and continuity equations. All calculations for the initial region use an empirical equation for turbulent length scale.

This chapter presents comparisons of experimental and theoretical results of the mean and turbulent motions in the developing region of the jets.

5.2 Nozzle Exit Conditions

The present limited measurements of mean velocity within the boundary layer shown in Figure 5.1 give the displacement thickness $\delta^{*}_{o}/r_{o} = 0.0064$ at the nozzle exit, $x/r_{o} = -0.08$, the boundary layer was assumed turbulent.

The measured velocity distribution, as shown in Figure 5.1 fits to the following velocity profile inside the boundary layer:

thin boundary layer: $u/u_c = \left[\dfrac{r_o - r}{\delta_o}\right]^{1/6.66}$ for $\delta_o^*/r_o = 0.0064$

The shape factor and the Reynolds number based on the displacement thickness at the exit were calculated to be $H_o = 1.31$ and $R_{e\delta}^* = 503$ for thin boundary layer. Similar measurements at the nozzle exit were performed for the two other nozzles having intermediate and thick boundary layers of displacement thicknesses $\delta_o^*/r_o = 0.0179$ and 0.0628, respectively. The velocity distributions for these two cases are shown in Figures 5.2 and 5.3, and they fit the following boundary layer velocity profiles:

intermediate boundary layer:

$$u/u_c = \left[\dfrac{r_o - r}{\delta_o}\right]^{1/5.38} \quad \text{for } \delta_o^*/r_o = 0.0179$$

thick boundary layer:

$$u/u_c = \left[\dfrac{r_o - r}{\delta_o}\right]^{1/5.29} \quad \text{for } \delta_o^*/r_o = 0.0628$$

The shape factors and the Reynolds numbers based on the displacement thicknesses at the nozzle exits were calculated to be $H_o = 1.372$, $R_{e\delta}^* = 1387$ for intermediate boundary layer and $H_o = 1.378$, $R_{e\delta}^* = 4824$ for thick boundary layer. In all the three cases, the distributions of turbulent intensity showed peaks close to the wall with values of $\sqrt{\overline{u'^2}}/u_c$ close to 10%, typical of turbulent boundary layers.

·The velocity distributions in Figures 5.1, 5.2 and 5.3 were used to start the finite difference calculations with the momentum and continuity equations for mean flow properties.

The turbulence in the jet core was not controlled, but did not vary widely for the three nozzles; the intensity of turbulence in the core was near 5% which is high when compared to that existing near shear layers investigated in low turbulence wind tunnels.

5.3 Mean Motion in the Initial Region

5.3.1 Thin Exit Boundary Layer

Figure 5.4 shows that the measurements of the axial mean velocity in the initial region has an approximate self-preserving distribution when plotted against the similarity variable, $\eta = (r - r_{\frac{1}{2}})/b$. It is also shown that the empirical velocity profile as expressed by the composite Equation (3.23) is in good agreement with the measurements. The empirical velocity profile given by Equation (3.23) was obtained by curve fitting based on least squares. The overall RMS deviation of the combined experimental data at sections, $x/r_O = 2, 4, 6$ and 8 from the empirical velocity profile was calculated to be $(\Delta u/u_c)_{RMS} = 0.0112$. The axial mean velocity obtained from the finite difference solution is approximately self-preserving in the u/u_c vs. η plane. The results of the finite difference solutions at sections, $x/r_O = 2$ and 8 are plotted in Figure 5.4 to show a comparison with the empirical model. The RMS deviation of the empirical model and the finite difference solution from the experimental data are shown in Table 5.1. The similarity variable, η was chosen to force agreement with experimental

measurements and self-preservation in the central region of the mixing layer. It is not surprising that the slight disagreement of the measurements with the model shows up in the outer and inner parts of the mixing layer, and the major part of the RMS deviation shown in Table 5.1 comes from the outer part of the mixing layer. The comparison shows that Equation (3.23) describes the axial mean velocity in the initial region with deviation from the experimental values of not more than ±1.5%. It is interesting to note that the conventional similarity variable $(r - r_o)/x$ does not produce satisfactory self-preservation of the axial mean velocity in the present experimental measurements or in those of Bradshaw et al [10] and Sami et al [24].

The axial mean velocity calculated by finite difference technique is plotted against the radial distance r/r_o in Figure 5.5 for comparison with the measurements of Bradshaw et al [10], Sami et al [24] and the present measurements. Satisfactory agreement is shown. Simpson's integration rule [84] is used to calculate the scale b/r_o from the first order differential Equation (3.30) with starting condition at $x = 0$. The scale, b/r_o, may be considered as a measure of the width of the shear layer. These results together with measurements and those obtained from the finite difference solution are shown in Figure 5.6. The root mean square deviation of the result of Equation (3.30) from those of the finite difference solution was calculated using values of 100 points of equally spaced intervals in the axial direction. Similarly

TABLE 5.1: A Comparison of Mean Velocity RMS Deviations

Axial Distance x/r_o	Finite Difference Solution $(\Delta \hat{u}/u_c)$ RMS			Empirical Solution $(\Delta u/u_c)$ RMS
	Thin	Intermediate	Thick	Thin
2	0.0067	0.0088	0.0110	0.0154
4	0.0119	0.0103	0.0128	0.0137
6	0.0150	0.0131	0.0166	0.0101
8	0.0180	0.0210	0.0229	0.0098

the RMS deviation of the radial distance, $r_{\frac{1}{2}}$, for the half velocity, $u/u_c = 0.5$ was also calculated. The resulting values are:

$$(\Delta b/r_o)_{RMS} = 0.001285 \text{ and } (\Delta r_{\frac{1}{2}}/r_o)_{RMS} = 0.000953$$

Figure 5.6 shows that the scale, b, is only approximately a linear function of x. The nonlinearity of the scale, b, would generate the width of the shear layer, b_1, and the Prandtl's mixing length, L, as nonlinear functions of x. The linear approximation of Prandtl's mixing length given by Rotta [8] has also been compared with the present results and measurements in Figure 5.6. The non-self-preservation of the mean axial velocity, and hence, the nonlinearity of b arises when using the variable, $\eta = (r - r_o)/x$ as already noted.

The radii $r_{\frac{1}{2}}$, r_1 and r_2 for $u/u_c = 0.5$, $u/u_c = 0.99$ and $u/u_c = 0.01$ respectively were calculated by using Equations (3.29), (3.24) and (3.25), and plotted in Figure 5.7 to show a comparison with the measurement and the results of the finite difference solution. The iso-velocity lines $u/u_c = 0.99$ and $u/u_c = 0.5$ obtained from the results of the finite difference solution are in good agreement with the measurement and the deviation from that of the integral method is insignificant for the major part of the shear layer. The deviation that is observed in the outer and inner boundaries shown in Figure 5.7 may be expressed in terms of the RMS difference which was calculated by choosing 100 points at equally spaced intervals in the axial

direction and found to be $(\Delta r_2/r_o)_{RMS} = 0.0193$ for the outer
boundary and $(\Delta r_1/r_o)_{RMS} = 0.00215$ for the inner boundary.
Here it is to be noted that the jet boundaries are not quite
linear with the axial distance, x. Laurence's [31] approxi-
mation of outer boundary is also drawn in Figure 5.7 to show
a comparison with the present outer boundary for $u/u_c = 0.01$.
The major contribution to uncertainty in the velocity mea-
surement occurred in the outer boundary. If one wishes to
compare theoretical results with experimental values, it is
preferable not to formulate in terms of the outer radius and
the width of the mixing layer as Abramovich [27] and others
have done. Hatta and Nazaki [23] approximated r_1 and r_2 to
be linear functions of x, and developed a self-preserving
model of axial mean velocity in terms of r_1 and r_2 which pre-
dicts constant axial mean velocity at $r/r_o = 1.0$ for any
axial distance in this region. Figure 5.8 shows the short-
coming of Hatta and Nazaki's [23] model and the agreement
of present calculations and measurements with the experi-
mental measurements of Bradshaw et al [10] and Von Frank
[25].

Using the empirical model of the axial mean velocity
in the mass conservation equation and integrating by Simp-
son's rule [84], the radial mean velocity was calculated.
The RMS deviation of these results from the results obtained
by the finite difference technique was calculated by using
100 points at equally spaced intervals in the radial direc-
tion at sections, $x/r_o = 4,6$ and 8; its value was found to

be $(\Delta u/u_c)_{RMS} = 0.0196$. A comparison of the present cal-
culations and the measurements of Sami et al, [24] for the
radial mean velocity is presented in Figure 5.9 with the
results of the finite difference method and shows good agree-
ment.

The length of the initial region calculated by the
finite difference method was close to $x/r_o = 9.0$, which
agrees with the prediction of Madni and Pletcher [26]. It
was calculated here to be $x/r_o = 8.8$ by the integral method.
Albertson et al [18] measured its value to be less than 10.0
and Davies et al [30] found it to be 9.0.

The experimental measurements of the turbulent shear
stresses, presented in Figures 5.22 to 5.25 were used to
calculate the L/b ratio from Equation (3.32) for axial dis-
tances $x/r_o = 2,4,6,8$ and their variations with the radial
direction are shown in Figure 5.10. It may be observed that
the ratio L/b lies in the range of 0.099 to 0.109 for the
initial region of the jet. The calculations used L/b =
0.10625 as an average of all observations obtained for
$x/r_o = 2,4,6$ and 8.

5.3.2 Intermediate and Thick Exit Boundary Layers

The mean velocities obtained by experimental measure-
ments and by finite difference calculations are shown in
Figures 5.11 and 5.12 in the plane u/u_c vs. η for the inter-
mediate ($\delta_o^*/r_o = 0.0179$) and the thick ($\delta_o^*/r_o = 0.0628$)
boundary layer velocity profiles at the nozzle exits. Figure
5.11, for the intermediate boundary layer, shows that the

mean axial velocity profiles are approximately self-preserving in the initial region starting from $x/r_O = 2.5$ with an RMS deviation less than 2% from the measured values at $x/r_O = 2.5$, 4,6 and 8. In comparison, Figure 5.4, for the thin exit boundary layer, shows satisfactory self-preservation starting from $x/r_O = 2$ with an RMS deviation of 1.12% from the measured values at $x/r_O = 2,4,6$ and 8. The mean velocity profiles for the thick exit boundary layer, in Figure 5.12, also show an approach to self-preservation to the initial region at an increased distance downstream, $x/r_O = 3.5$ and the RMS deviation of the results of finite difference solution from the measured values at $x/r_O = 3.5$, 4, 6 and 8 was found to be less than 1.5%.

Mean velocities obtained by the finite difference method are plotted in Figure 5.13, for $\delta_O^*/r_O = 0.0179$ and in Figure 5.14, for $\delta_O^*/r_O = 0.0628$ and show that the radial distribution of u/u_C and the present measurements are in satisfactory agreement. The RMS deviation of the calculated and experimental results at sections, $x/r_O = 2,4,6$ and 8 are given in Table 5.1 for the intermediate and thick boundary layers.

Iso-velocity lines $u/u_C = 0.01$, $u/u_C = 0.5$ and $u/u_C = 0.99$ obtained by the finite difference solution are plotted in Figure 5.15, and satisfactory agreement is shown with the experimental measurements for $u/u_C = 0.5$ and 0.99. The experimental values on line, $u/u_C = 0.99$ where intermittency occurs, were approximated by taking several readings near the inner boundary of the shear layer. Figure

5.15 also shows that the radial distances r_2 and r_{1_2}, for $u/u_c = 0.01$ and $u/u_c = 0.5$ respectively, decrease with the increase of the exit boundary layer thickness. The value of the potential core radius, r_1, decreases with the increase of the thickness of the exit boundary layer at the beginning of the shear layer growth and becomes independent of the exit boundary layer thickness after some axial distance, x/r_0. For the intermediate boundary layer, the radius of the potential core, r_1, becomes equal to that of the thin exit boundary layer at $x/r_0 = 4.5$, and for the thick boundary layer $(\delta_0^*/r_0 = 0.0628)$ at $x/r_0 = 6$. It may be observed from Figure 5.15 that the jet boundaries are not exactly linear for any exit conditions and the results of the integral method for the thin exit boundary layer shown in Figure 5.7 also agree with this finding.

The scale, b, for the jets with the intermediate and thick exit boundary layers are plotted in Figure 5.6 and compared with the experimental results. Figure 5.6 shows that the scale, b, is nonlinear in x and its value is higher than that of the thin exit boundary layer at the beginning of the shear layer and becomes smaller after some axial distance, x/r_0.

The values of L/b for the jet with the intermediate boundary layer $(\delta_0^*/r_0 = 0.0179)$ were calculated by using shear stresses given in Figure 5.33. All the calculated values of L/b at sections $x/r_0 = 2,4,6$ and 8 fall within the range 0.099 to 0.109 which is the same as shown in Figure

5.10 for the thin boundary layer. Though the values for L/b lie within the same range, the average of all calculated values in this case is found to be 0.1076 which is slightly higher than the value, 0.10625 for the case given in Figure 5.10. The finite difference calculation for this case was performed with L/b = 0.10625.

The experimental values of shear stresses are shown in Figure 5.34, for the jet with thick boundary layer ($\delta_o^*/r_o = 0.0628$). These experimental values were used to calculate the ratio, L/b at sections, $x/r_o = 2,4,6$ and 8, and all the calculated values lie between 0.102 and 0.12, having an average value of 0.1135, as shown in Figure 5.16. This value differs significantly from those obtained in the cases of thin and intermediate boundary layers with the displacement thicknesses, $\delta_o^*/r_o = 0.0064$ and 0.0179 respectively. The finite difference calculation for this jet was performed by using L/b = 0.1135.

5.4 Mean Motion in the Transition Region

The jet flow is approximately self-preserving in the initial region, but does not remain self-preserving in the transition region which starts from the end of potential core. This is probably due to the interaction of the shear layers along the centre line of the jet. The experimental values of the mean velocities at sections, $x/r_o = 10,12,15$ and 18 are plotted in Figure 5.17 in the u/u_c vs. η plane. The flow maintains its self-preservation in the central part of the shear layer as required by the scales chosen, but becomes non-self-preserving near the centre line and

the outer boundary of the jet. The results of finite difference calculations at sections, $x/r_o = 10$ and 18 are also presented in Figure 5.17 and show agreement with the experimental values.

The flow develops towards a new mean velocity profile and again becomes self-preserving in the developed region far downstream according to the findings in Figures 5.4, 5.17 and 5.18. The length of the transition region was determined by measuring the mean velocity at different axial distances until the experimental values fitted the velocity profile given by Tollmien [14] for the developed region. The experimental values of the mean axial velocities of various workers at axial distances, $x/r_o = 18, 24, 32, 40, 64, 80$ and 100 are compared with Tollmien's [14] and Schlichting's [15] velocity profiles in Figure 5.18. It is observed from Figure 5.18 that the mean axial velocity achieves approximate self-preservation at $x/r_o = 24$ which may be considered to be the beginning of the developed region. Therefore, the transition region extends from $x/r_o = 9$ to $x/r_o = 24$ as determined from the mean axial velocity. The turbulent quantities may not achieve self-preservation at $x/r_o = 24$.

The experimental results of shear stress are plotted in Figure 5.41 for the transition region. These experimental values were used to calculate the ratio of Prandtl's length scale, L, and the length scale, b, for the transition region. The ratio, L/b, is plotted against the variable, $\eta = (r - r_{\frac{1}{2}})/b$ in Figure 5.19 and shows that the values

of L/b do not lie within as narrow a range as that shown in
Figure 5.10 for the initial region. Therefore, the ratio,
L/b, cannot be treated as a constant for the transition
region, but must be considered as a function of the axial
and the radial distances, and can no longer be expressed
by a simple algebraic relation. Rotta [8] determined the
length scale of turbulence by solving a differential trans-
port equation for the developed region and he found it to
be self-preserving as shown in Figure 5.19. Rotta's [8]
calculation agrees with experimental observations that the
length scale of turbulence is a function of both axial and
radial distances in the developed region, though Townsend
[7] and others used it as a function of x only which gives,
L/b = constant, as in Figure 5.19.

The present finite-difference method replaces the
turbulent diffusivity, ν_t, by combining the turbulent dif-
fusivity models of Madni-Pletcher [26] and Schetz [21],
given by Equations (3.6) and (3.7). This approach is not
based on any valid physical grounds, but only on empiri-
cal agreement. The calculated results of the mean velocity
obtained by using the Madni-Pletcher's [26] model of tur-
bulent diffusivity gives values less than the experimental
values near the outer boundary. The RMS deviation of these
results from the experimental values at sections, $x/r_o = 10$,
12,15 and 18 was calculated to be about 12%. The application
of Schetz's [21] model of turbulent diffusivity to the momen-
tum equation and the finite difference calculation predicted

the mean axial velocity to be higher than that of the experimental values near the outer boundary with RMS deviation at $x/r_o = 10,12,15$ and 18 to be about 9%. The present calculation used Madni-Pletcher's [26] model of turbulent diffusivity from the centre line to the half-radius line and continued by using constant turbulent diffusivity until it became equal to that of Schetz's [21] model and then followed the latter. Both Madni-Pletcher's model and Schetz's model together with the present combination are presented in Figure 5.20. The present results of the mean velocity are plotted in Figure 5.21 and show satisfactory agreement with experimental measurements. The RMS deviation between the experimental and calculated results at sections, $x/r_o = 10,12,15$ and 18 was calculated to be about 2%.

5.5 Turbulent Motion in the Initial Region

5.5.1 Thin Exit Boundary Layer

The momentum equation, the continuity equation and the differential transport equation for the turbulent kinetic energy were solved for turbulent kinetic energy by the finite difference method. The length scale and the Reynolds shear stress were expressed by Equations (3.20) and (3.31). The measured velocity and turbulent kinetic energy distributions for the thin boundary layer at the nozzle exit, in Figure 5.1 was used to start the finite difference solution from $x/r_o = 0$.

The shear stresses obtained from Prandtl's mixing length model, Equation (3.31), are in good agreement with the measurements of Bradshaw et al [10], Sami et al [24] and the

present measurements for $x/r_o = 2,4$ and 6 as shown in Figures 5.22, 5.23 and 5.24, respectively. In Figure 5.25, the model for $x/r_o = 8$ shows somewhat less agreement than the agreement shown for $x/r_o = 2,4$ and 6.

Although the Prandtl's mixing length model adequately represents the turbulent shear stress in the initial region of the jet, it is interesting to examine an energy model for the shear stress given by Equation (3.14), based on the assumption that the ratio of the turbulent shear stress and the kinetic energy in the flow remains constant, i.e.,

$$-\frac{\overline{u'\upsilon'}}{k} = c_1 \qquad\qquad (3.14)$$

This energy model has been used for boundary layers by Bradshaw et al [53] and for the developed region of jets by Harsha and Lee [60]. In the latter flow, the value of c_1 was reported to be 0.3.

The present measurements in Figure 5.26 show that $-\overline{u'\upsilon'}/k$ remains constant at a value near 0.375 in the central portion of the mixing layer, but decreases near both boundaries. The shear stresses, based on the energy model, $-\overline{u'\upsilon'}/k = 0.375$, are shown in Figures 5.22, 5.23, 5.24 and 5.25. As expected, agreement with measurements are good in the central region of the mixing layer with less agreement near the boundaries. It may be noted in passing that the energy model may be improved by using the form $\overline{u'\upsilon'}/k = 0.375\gamma$, where γ is an intermittency factor. However, this introduces another empirical quantity into the model which is a disadvantage when compared with Prandtl's mixing length

model.

The empirical model of shear stress used by Launder and Spalding [9] and Rotta [8] has already been mentioned in Chapter III and it is rewritten here in the form:

$$-\overline{u'u'} = c_\mu \frac{k^2}{\varepsilon} \frac{\partial u}{\partial r} \qquad (3.15)$$

where k is the turbulent kinetic energy, and L^1 is a length scale which is the dependent variable in an empirical differential transport equation. ε is the rate of dissipation of turbulent energy. The two equations, k-ε, transport model of Launder and Spalding [8] was solved by using the above empirical model of shear stress and satisfactory results were found by them for the developed region of jets and also for other turbulent flows. Figures 5.22, 5.23, 5.24 and 5.25 show this model is in reasonable agreement with measurements when k and ε are obtained from a single transport Equation (3.13) for the turbulent kinetic energy and an algebraic Equation (3.11) for the dissipation of energy together with the length scale Equation (3.20). The value of the constant c_μ is 0.09 as proposed by Launder and Spalding [9]. The empirical constants a_1=1.09, c_r=0.211, and k_q=0.86 which were used in Equation (3.13) were obtained from the experimental values of Sami [85]. This will be discussed later.

The calculated values of turbulent kinetic energy are plotted in Figure 5.27 and show satisfactory agreement with measurements. The experimental values of $\overline{u'^2}/2k$ and $\overline{v'^2}/2k$ are plotted in Figure 5.26 and show that these ratios remain approximately constant throughout the shear layer. The values of $\overline{u'^2}/2k$ lie between 0.42 and 0.45 and those of $\overline{v'^2}/2k$ between 0.27 and 0.30. Values of $\overline{w'^2}/2k$ were very

close to those of $\overline{u'^2}/2k$ and are not shown. The ratios of
the normal stresses to the turbulent kinetic energy were
estimated experimentally by taking 300 readings at each
of the nine stations at distances $x/r_o = 4$, 6 and 8. By
drawing frequency distributions and using 20:1 odds, the
ratios are selected to be:

$$\overline{u'^2}/2k = 0.44$$

$$\overline{v'^2}/2k = 0.29 \qquad\qquad (5.1)$$

These ratios together with the calculated values of the
kinetic energy were used to determine turbulent intensities.
Figures 5.28 and 5.29 show comparisons of the calculated
intensities with the measurements of Sami et al [24], Brad-
shaw et al [10], Laurence [31] and the present measurements.
These show reasonable agreement between the calculated in-
tensities and the measurements.

Rotta [8] estimated the constants a_1, c_r and k_q in
Table 5.2, associated with turbulent energy Equation (3.13)
and obtained good agreement with the measurements of Wygnan-
ski and Fiedler [4] for the developed region of jets.

TABLE 5.2   Constant of Energy Transport Equation

	a_1	c_r	k_q	c_1
Rotta [8]	1.0	0.165	0.60	---
Present	1.09	0.211	0.86	0.375

The calculated values using these constants do not show satisfactory agreement with measured diffusion energy, transport energy, etc. of Sami [85] for the initial region. To achieve satisfactory agreement with Sami's measurements, the constants a_1, c_r and k_q are adjusted to the values shown in Table 5.2. This agreement is shown in Figures 5.30, 5.31 and 5.32. The constants are chosen on the basis of minimum RMS deviation of the calculated values from the measured values of Sami [85].

The choice of constant $a_1 = 1$ used by Rotta [8] ignores the contribution of the other velocity gradients to the production term. Figure 5.31 shows the energy production term obtained by using $a_1 = 1.09$ is in good agreement with the measurement of Sami [85]. This indicates that the contribution of other velocity gradients to the production term is about 9% of $\overline{u'v'}\, \partial u/\partial r$ for this region.

The comparisons show that one transport equation for turbulent kinetic energy associated with an algebraic length scale used here is adequate to simulate turbulent flow in the initial region of jets. The term by term energy balance indicates useful performance of this model. This calculation saves computer time necessary for simultaneous solution of two transport equations and shows satisfactory agreement with experimental measurements.

The results of the turbulent quantities presented here for the initial region show that these are approximately self-preserving, starting from $x/r_o = 2$ for the jet

with the exit displacement thickness $\delta_o^*/r_o = 0.0064$.

5.5.2 Intermediate and Thick Exit Boundary Layers

The turbulent kinetic energy equation (3.13), the momentum equation (3.2) and the continuity equation (3.1) together with an empirical length scale model were solved for the turbulent kinetic energy. The measured velocity and kinetic energy distributions at the nozzle exit, in Figure 5.2, for $\delta_o^*/r_o = 0.0179$ and, in Figure 5.3, for $\delta_o^*/r_o = 0.0628$ were used to start the solutions from the exit section of the nozzles. The length scale models used in this calculation were: $L/b = 0.10625$, for $\delta_o^*/r_o = 0.0179$ and $L/b = 0.113$, for $\delta_o^*/r_o = 0.0628$. The shear stress was related in terms of the length scale, L, and the mean velocity gradient given by Equation (3.31). The empirical constants associated with the energy equation already determined for the thin boundary layer were used for the intermediate and thick boundary layers.

The turbulent shear stresses calculated by using Prandtl's mixing length and the present experimental measurements are shown in Figures 5.33 and 5.34 for the intermediate and thick boundary layers with displacement thicknesses at exit of $\delta_o^*/r_o = 0.0179$ and 0.0628 respectively. An approximate self-preservation is also achieved in the initial region for the intermediate and thick exit boundary layer velocity profiles after $x/r_o = 4$. The energy model of shear stress is also presented in Figures 5.33 and 5.34 and shows a reasonable agreement with the

experimental measurement except near the boundaries.

The calculated values of turbulent kinetic energy are plotted in Figures 5.35 and 5.36 and show good agreement with measurements. The turbulent intensities, $\sqrt{\overline{u'^2}}/u_c$ and $\sqrt{\overline{v'^2}}/u_c$ were calculated as for the case with the thin boundary layer. The calculated results of $\sqrt{\overline{u'^2}}/u_c$ and $\sqrt{\overline{v'^2}}/u_c$ plotted in Figures 5.37 and 5.38, for $\delta_o^*/r_o = 0.0179$ and in Figures 5.39 and 5.40 for $\delta_o/r_o = 0.0628$, and the experimental results show satisfactory agreement with the calculated results. Figures 5.37 to 5.40 show the turbulent intensities are approximately self-preserving in the initial region at $x/r_o = 4$, for the intermediate and thick boundary layers.

5.6   Turbulent Motion in the Transition Region

As already mentioned, the mean velocity is not self-preserving in the transition region of axisymmetric jets. The turbulent quantities were measured in this region, for the thin boundary layer at the nozzle exit, to examine whether they possess self-preservation. The experimental values of shear stress, $-\overline{u'v'}/u_o^2$, and the intensities $\sqrt{\overline{u'^2}}/u_o$, $\sqrt{\overline{v'^2}}/u_o$ and $\sqrt{\overline{w'^2}}/u_o$ are plotted in Figures 5.41, 5.42, 5.43 and 5.44 respectively. Figures 5.41 to 5.44 show turbulent quantities are not self-preserving in this region. Similar plots for the intermediate and thick boundary layers, not included here, show that the turbulent quantities for these jets are also not self-preserving in the transition region.

CHAPTER VI

CONCLUSION

Numerical calculations were performed for the mean and turbulent properties in the initial and transition regions of axisymmetric turbulent jets, and the results were compared with experimental measurements. Jets produced by three different nozzles having displacement thicknesses at the nozzle exit of $\delta_o^*/r_o = 0.0064, 0.0179$ and 0.0628, and Reynolds number, $R_{eD} = 1.53 \times 10^5$ were investigated.

Prandtl's mixing length model with a simple algebraic equation for length scale, L, adequately represents the shear stress in the initial region. The length scale, L, is found to be a nonlinear function of x and its dependency on radial distance is negligible.

Both the mean and turbulent properties are approximately self-preserving in the initial region except close to the outlet for all three nozzles. The increase of the displacement thickness and the turbulent intensity at the exit of the nozzle cause the self-preserving point to move downstream. Self-preservation of mean axial velocity starts from the axial distances, $x/r_o = 2, 2.5$ and 3.5 for the displacement thicknesses, $\delta^*/r_o = 0.0064, 0.0179$ and 0.0628, respectively, with the length of potential core the same for all three nozzles.

The width of the shear layer obtained by defining outer and inner boundaries is not the most suitable as a length scale for developing the self-preserving model

because the highest experimental uncertainty occurs near the outer boundary where the mean velocity is low. Therefore, in developing the self-preserving model it is found convenient to define the length scale on the basis of the measurements along the half-radius (u/u_c = 0.5) where experimental uncertainty is the lowest. The length scale, b, was chosen as inversely proportional to the mean velocity gradient at half-radial line.

Calculations using the one-equation transport model together with an empirical length scale gives satisfactory results for the mean and turbulent properties in the initial region and consumes less computer time than that required when using the two-equation model.

The ratios of the turbulent intensities to turbulent kinetic energy remain constant at values close to $\overline{u'^2}/2k$ = 0.44 and $\overline{u'^2}/2k$ = 0.29 throughout the whole initial region shear layer. The ratio of the shear stress to turbulent kinetic energy $\overline{u'u'}/k$ remains constant at 0.375 in the central portion of the shear layer, but decreases near the boundaries where intermittency also occurs. It is concluded that the structure of turbulence is uniform in the central region of the shear layer.

For the thin boundary layer at the nozzle exit, a polynomial form mean velocity profile fits the experimental measurements in the initial region. A simple integral method is applied to determine jet boundaries which agree with the results of the finite difference solution and measurements.

The length scale model used for the initial region is not applicable to the transition region because the shear layers interact along the centre line causing a change in the boundary conditions and the length scale. A simple algebraic equation cannot be adopted for the length scale. So, Boussinesq's model, with an empirical equation for turbulent diffusivity, is preferred for the shear stress. The mean and turbulent properties are not self-preserving in the transition region which starts near $x/r_o = 9$.

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FIGURES
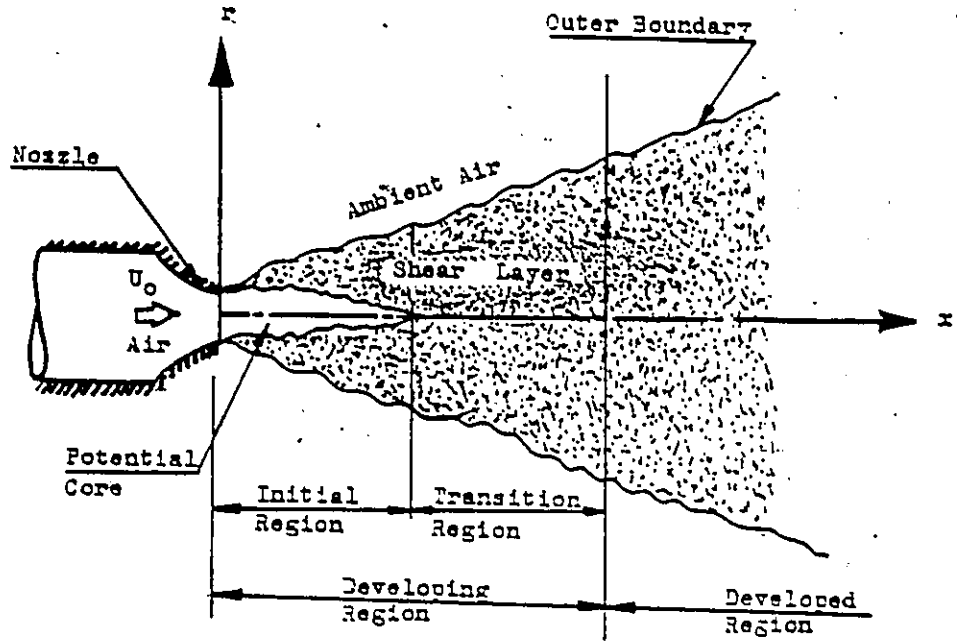
Fig.1.1

Jet Geometry and Nomenclature.
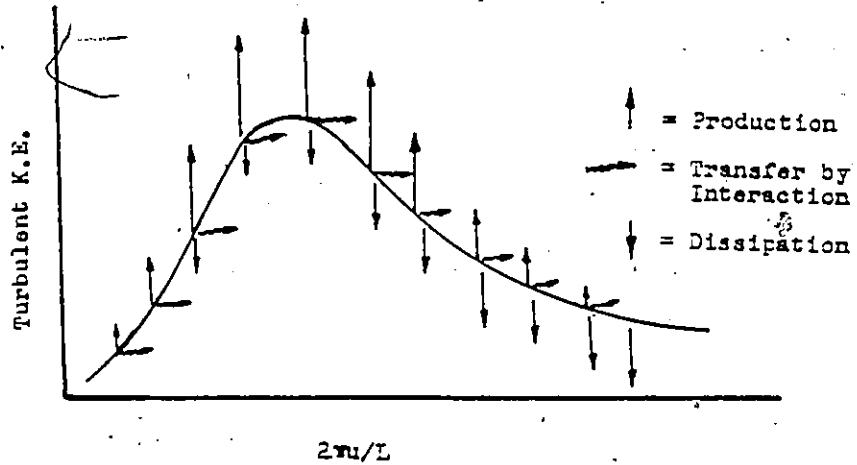
Fig.1.2

Representation of Energy Transfer with

Wave Numbers.
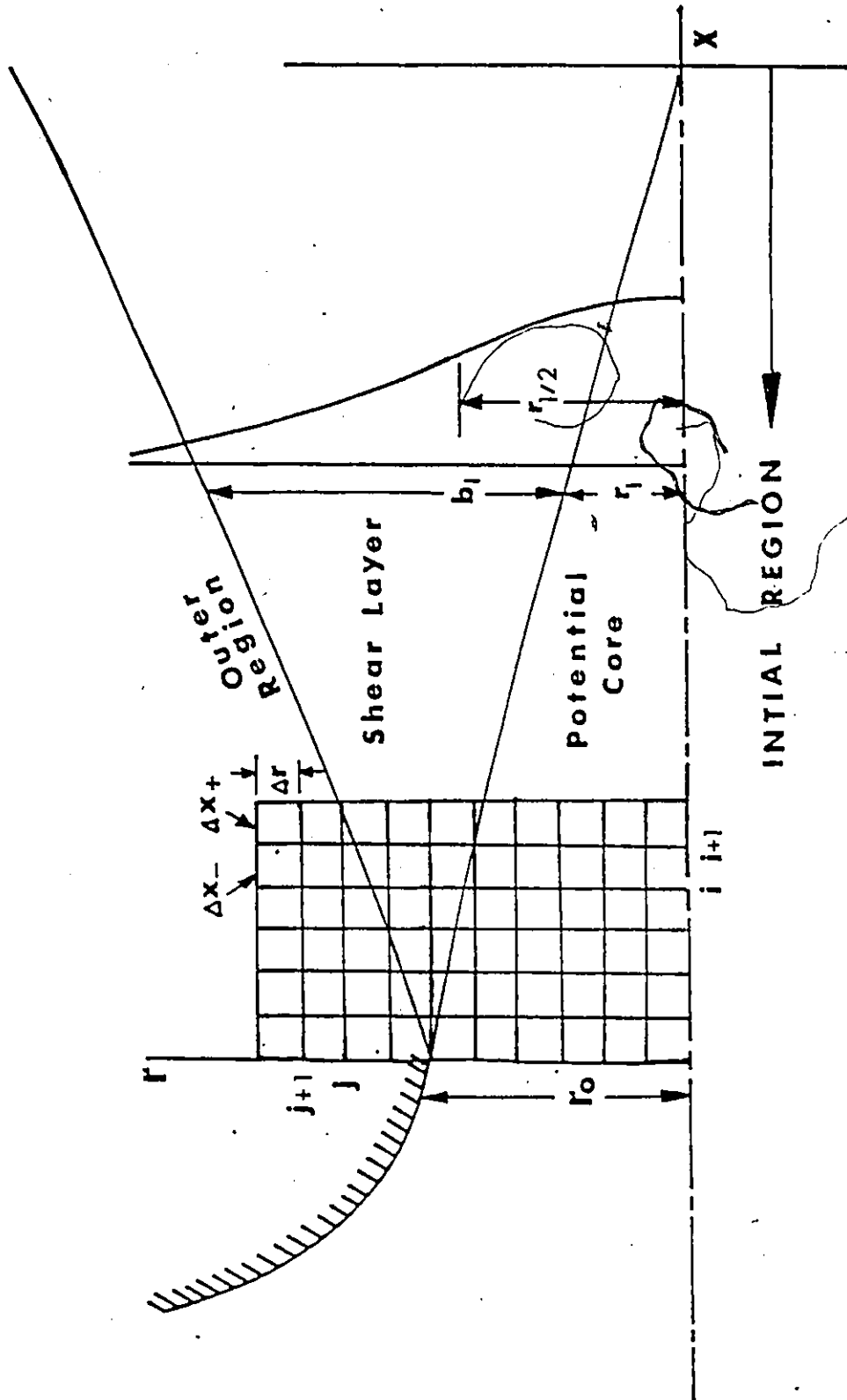
Fig.3.1.

Co-ordinate System and Finite-Difference Grid.
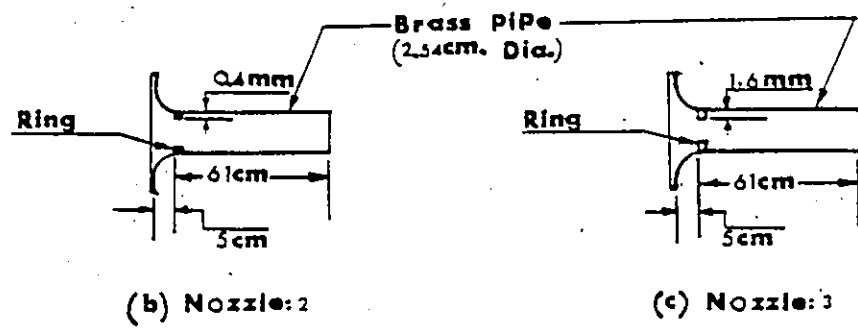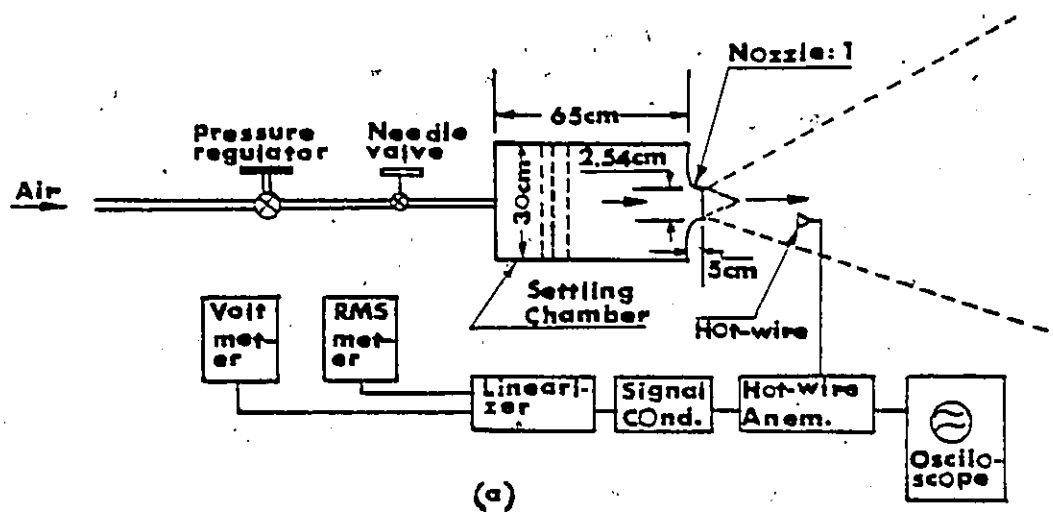
Fig.4.1

Schematic Diagrams of Experimental Set-up and

Round Axisymmetric Nozzles.
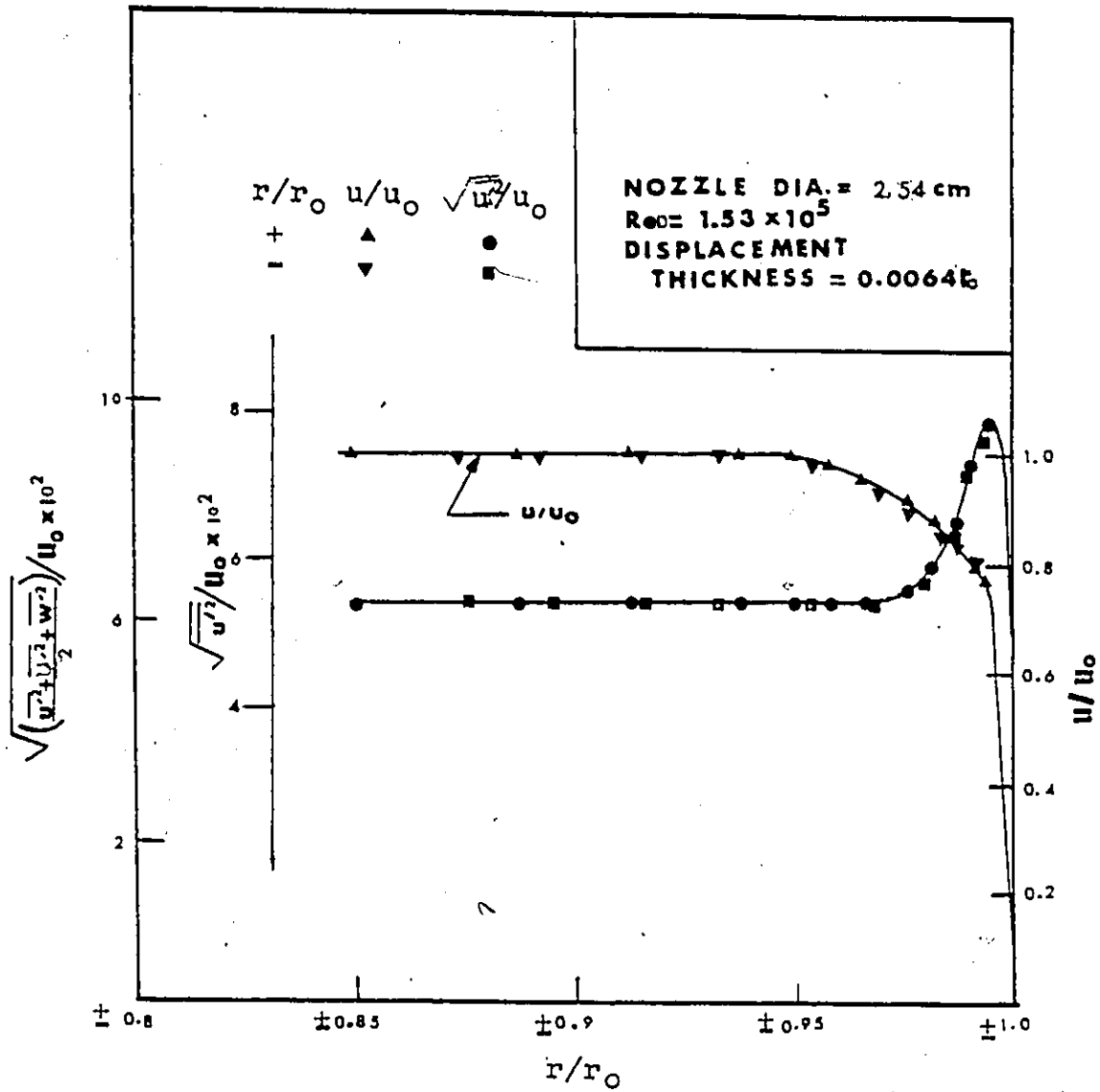
Fig.5.1

Measured Axial Velocity and Turbulent Kinetic
Energy Distributions at exit Plane, Thin Boundary Layer.
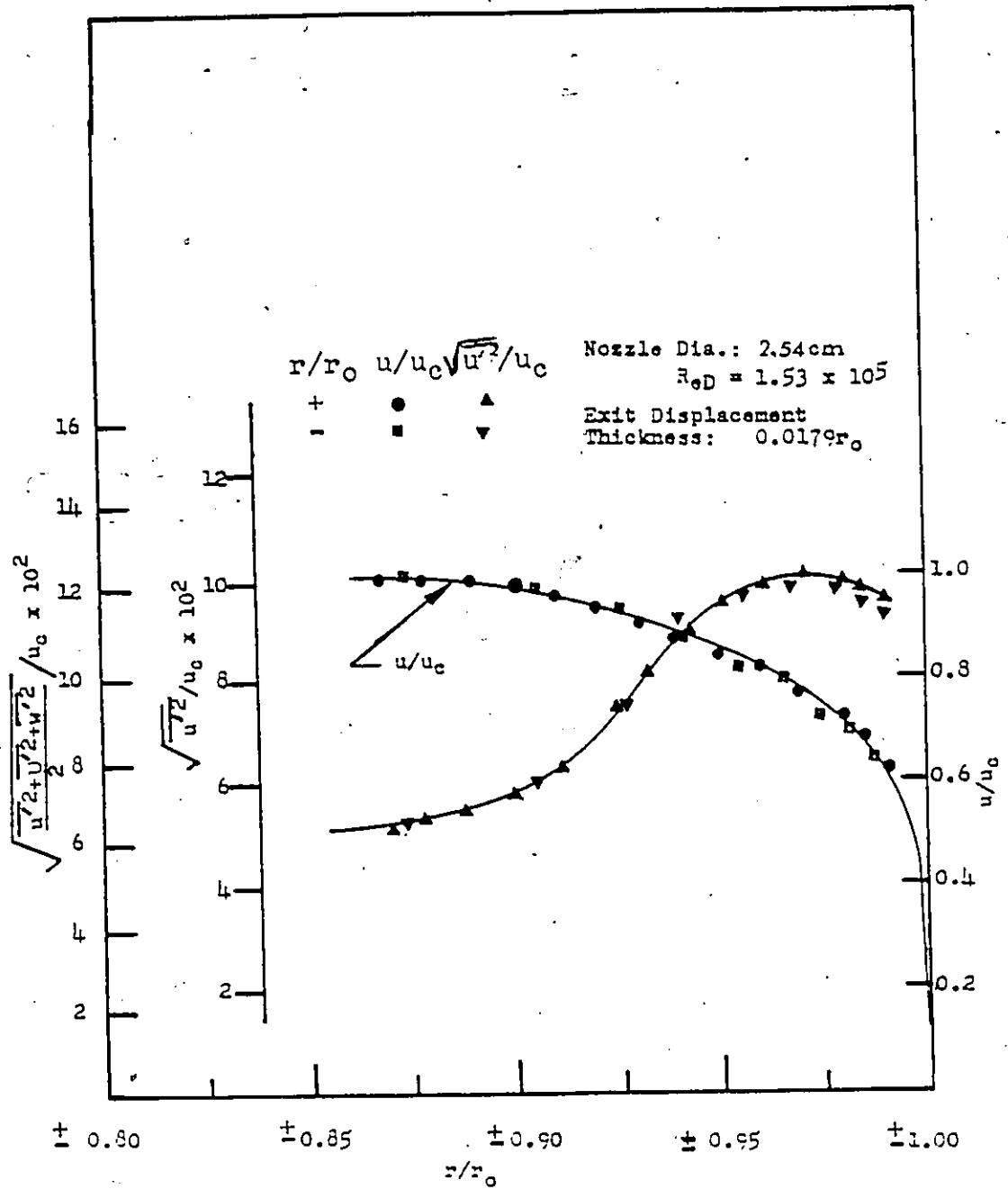
Fig.5.2

Measured Velocity and Turbulent Kinetic Energy

Distributions at Exit Plane, Intermediate Boundary
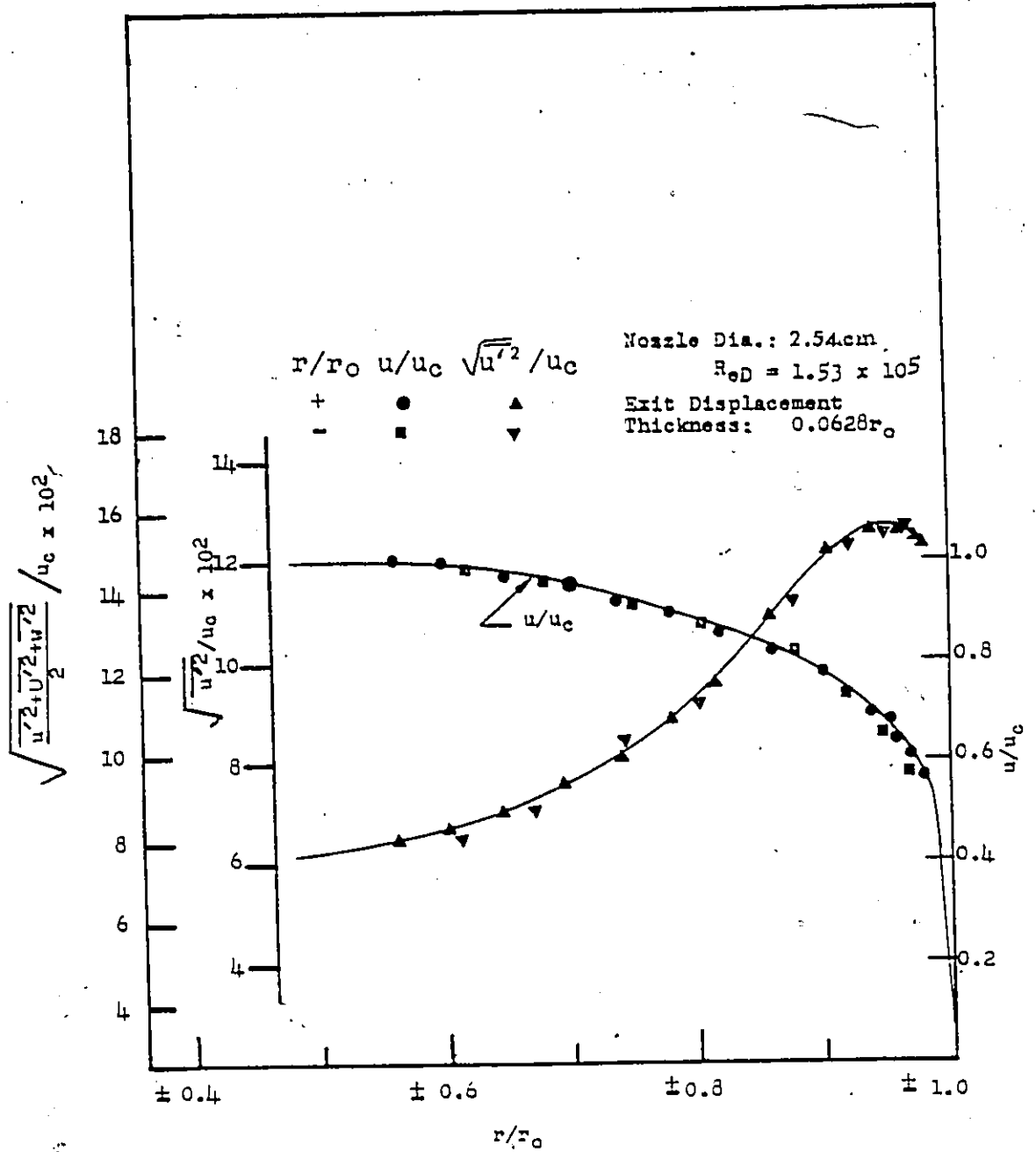
Layer.

Fig.5.3

Measured Velocity and Turbulent Kinetic Energy

Distributions at exit Plane, Thick Boundary Layer.
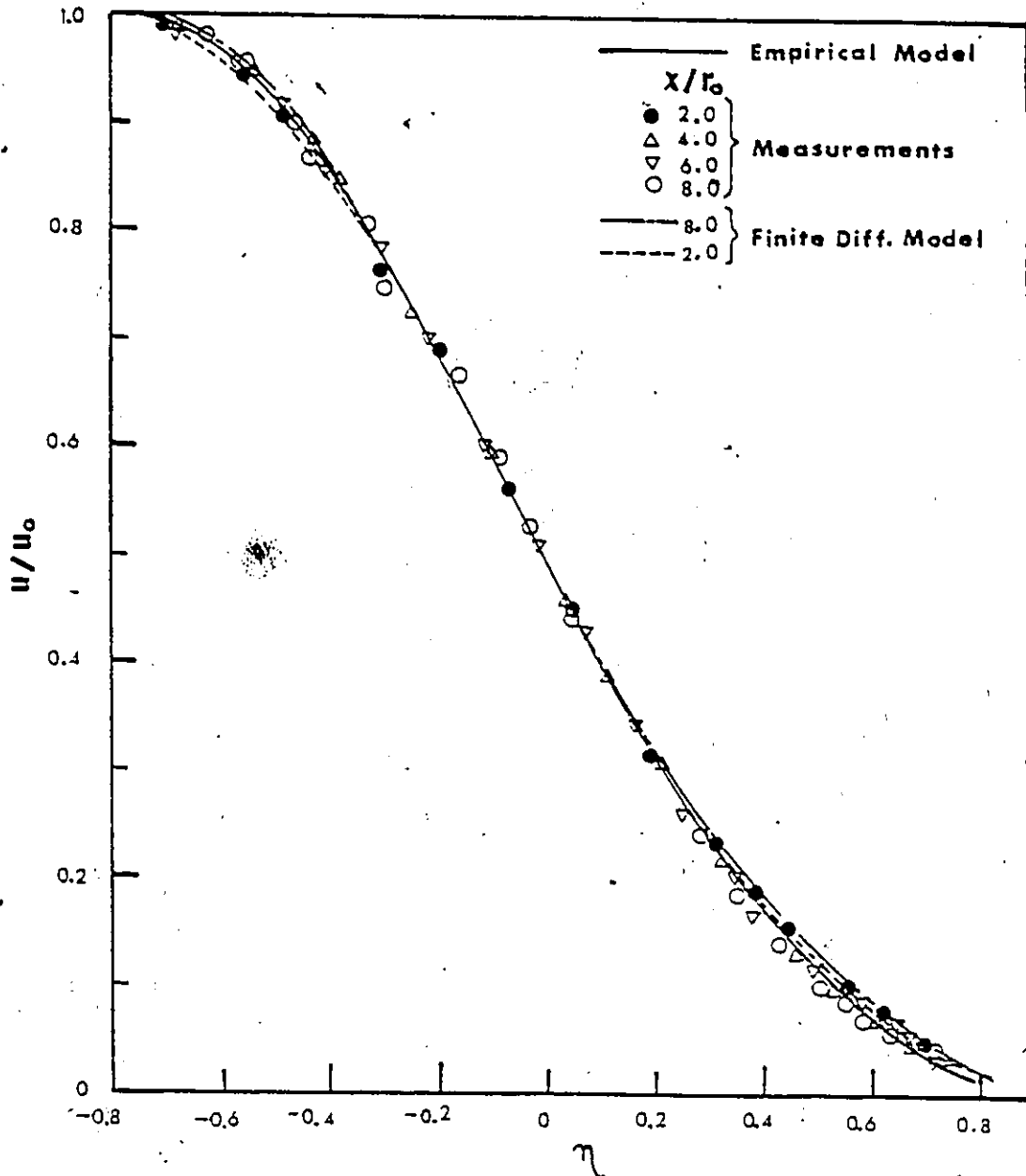
Fig.5.4.

Calculated and Experimental Self-Preserving Velocity
Profiles in the Initial Region for Thin Boundary Layer.
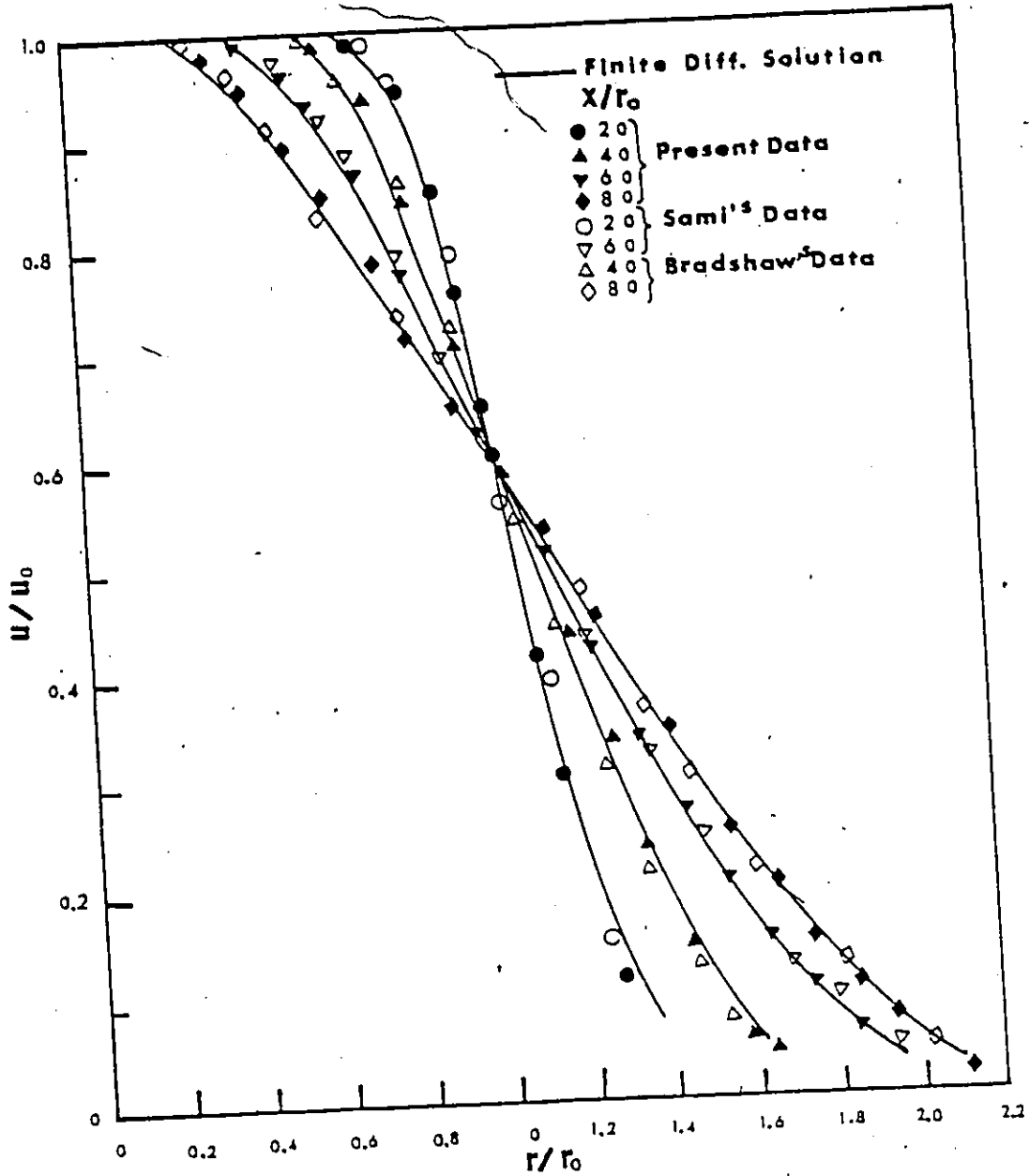
Fig.5.5.

Calculated and Experimental Mean Axial Velocity

Distribution,Thin Boundary Layer.
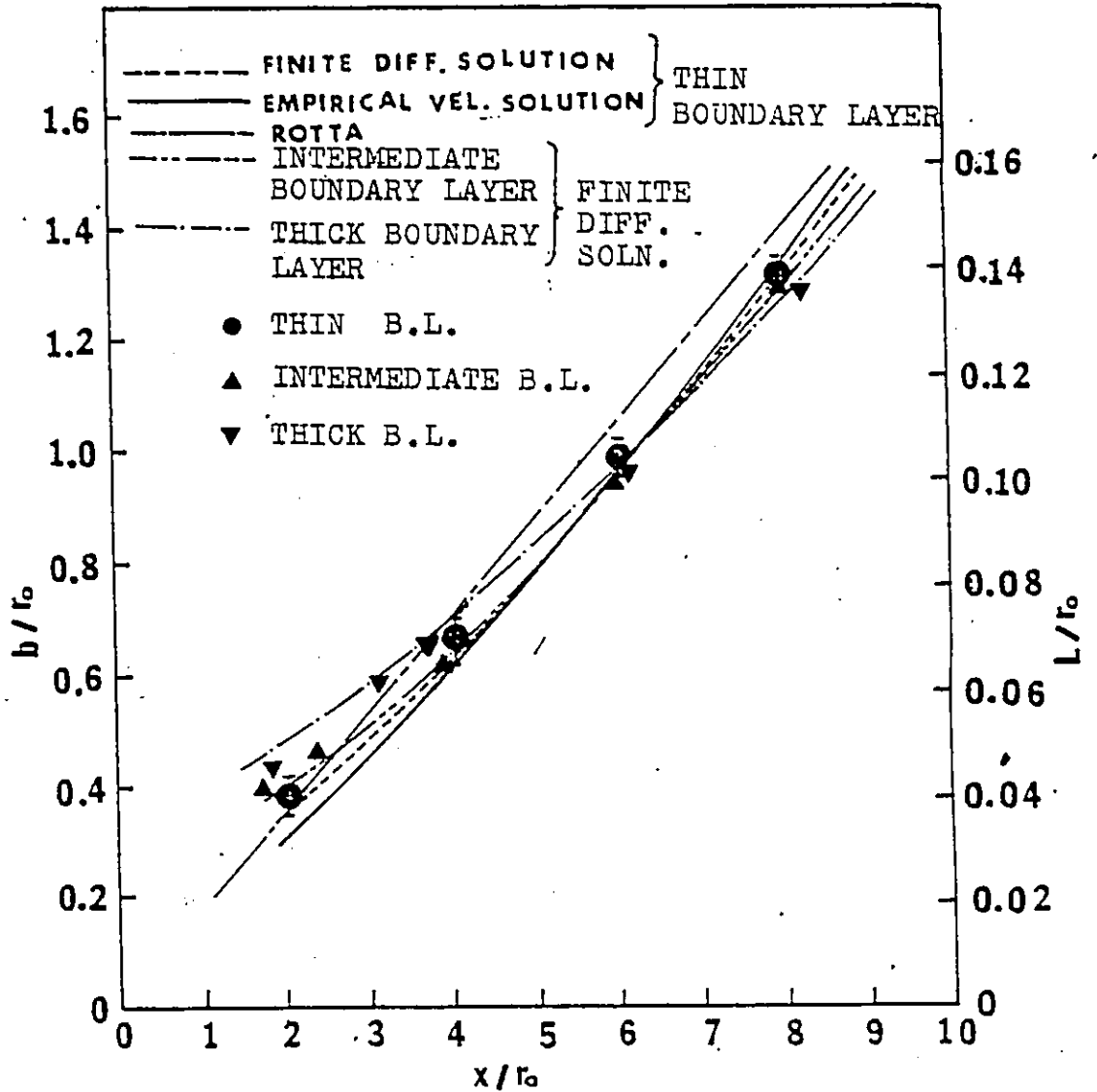
Fig.5.6

Length Scale, b/r_o and Prandtl's Mixing Length,
L/r_o for Thin, Intermediate and Thick Boundary
Layers.
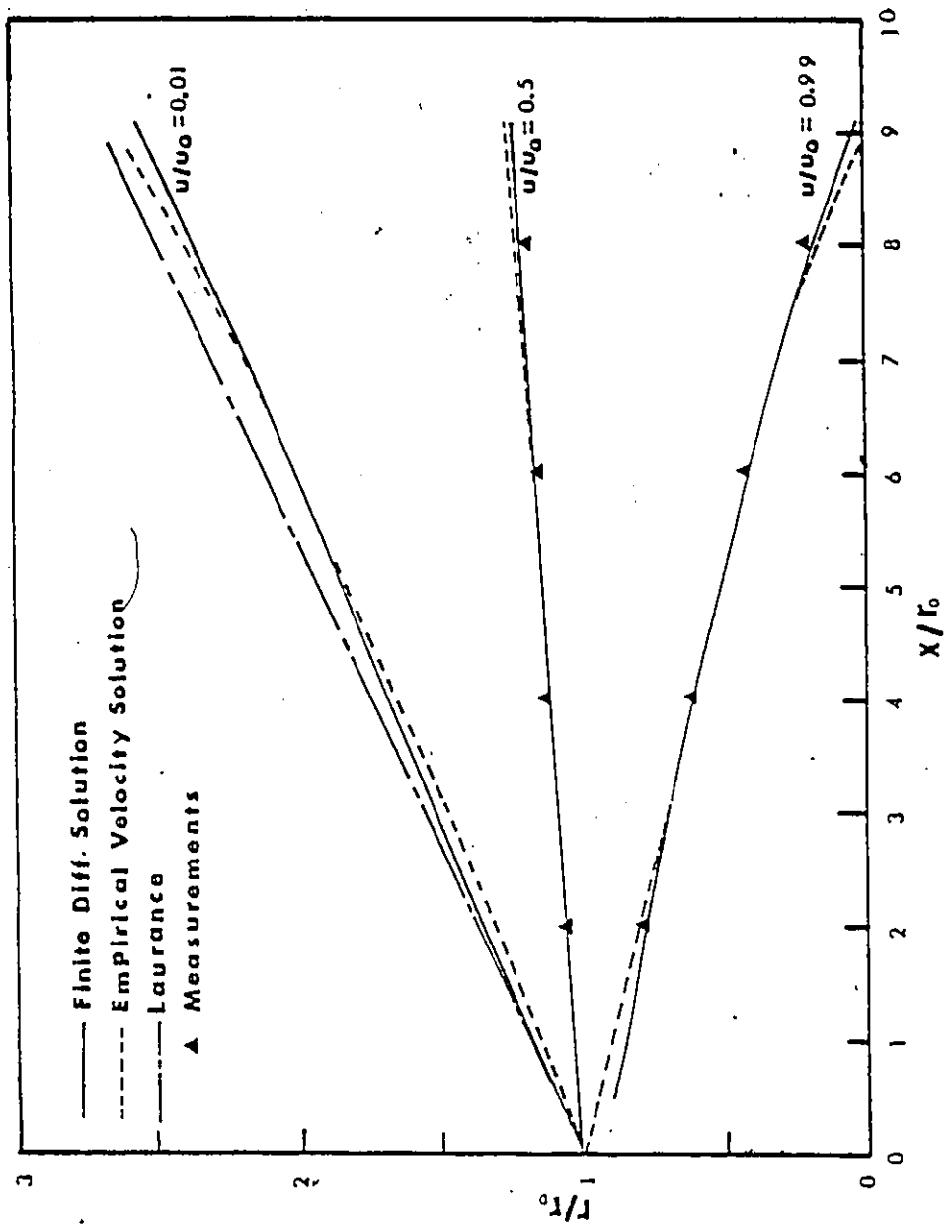
Fig.5.7

Calculated and Experimental Iso-Velocity Lines, Thin Boundary Layer.
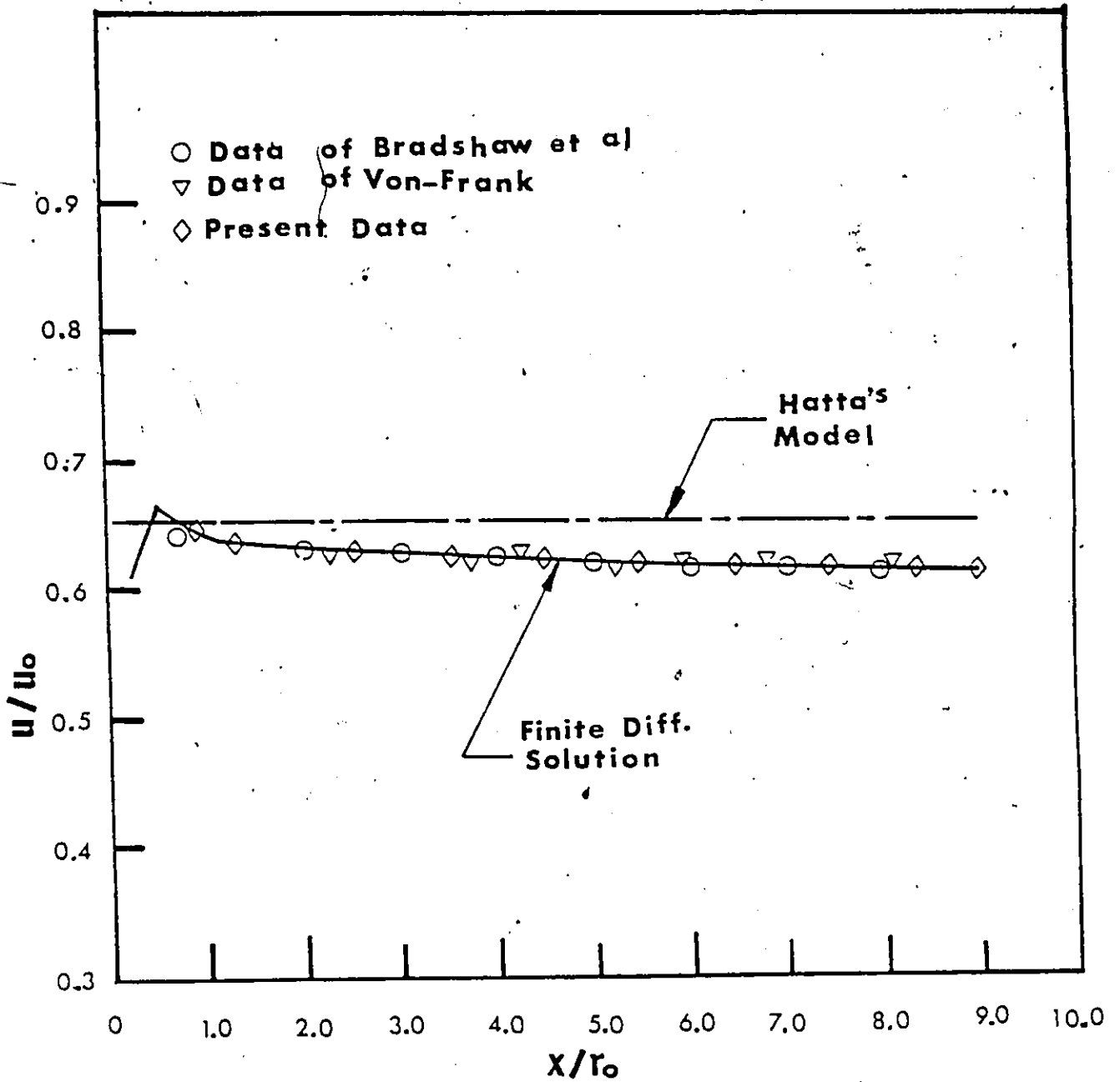
Fig.5.8.

Calculated Mean Velocity at $r/r_o = 1$ and Its Comparisons
with Hatta's Model and Experiments, Thin Boundary Layer.
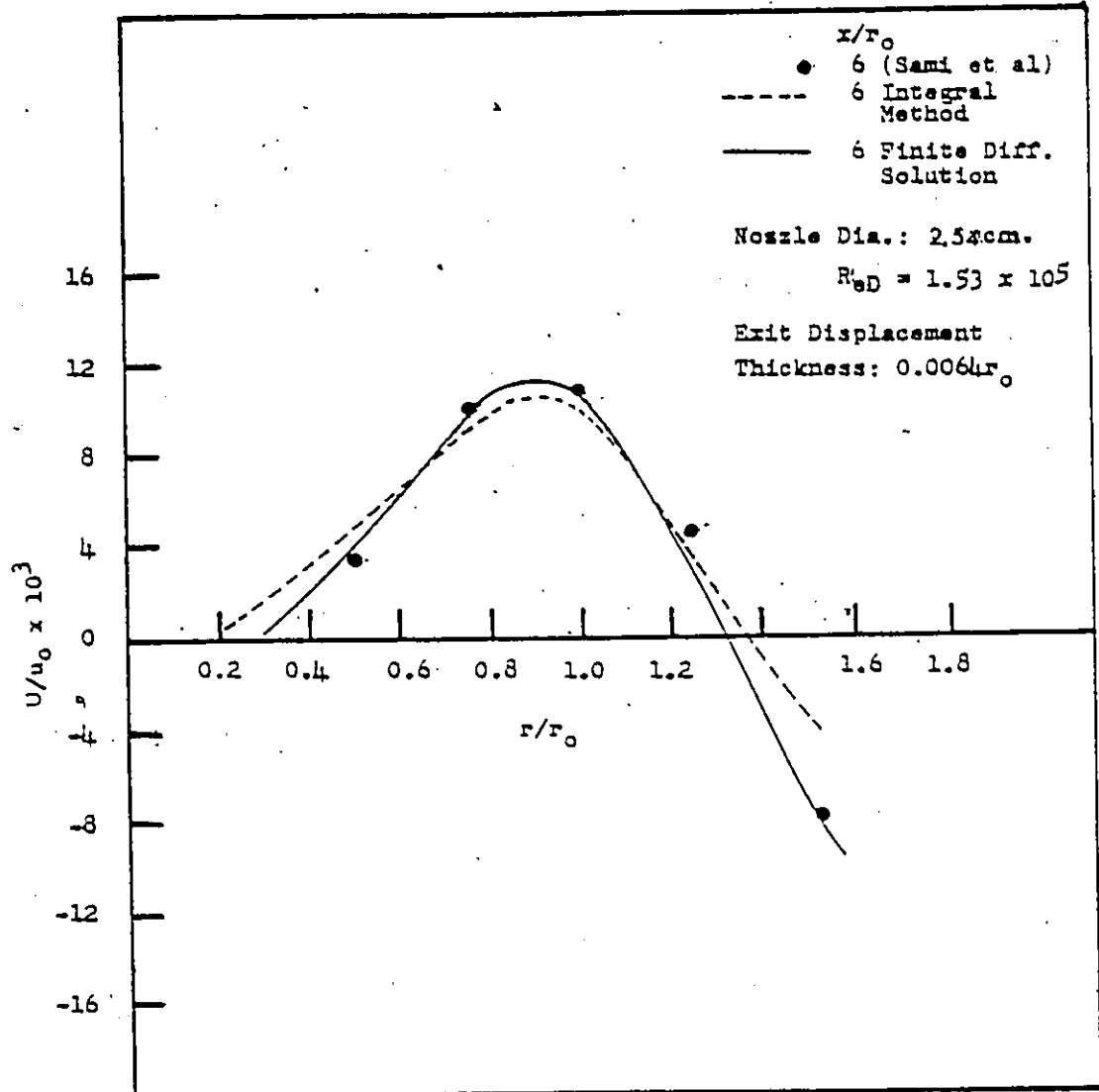
Fig.5.9

Mean Radial Velocity Distribution in the Initial
Region, Thin Boundary Layer.
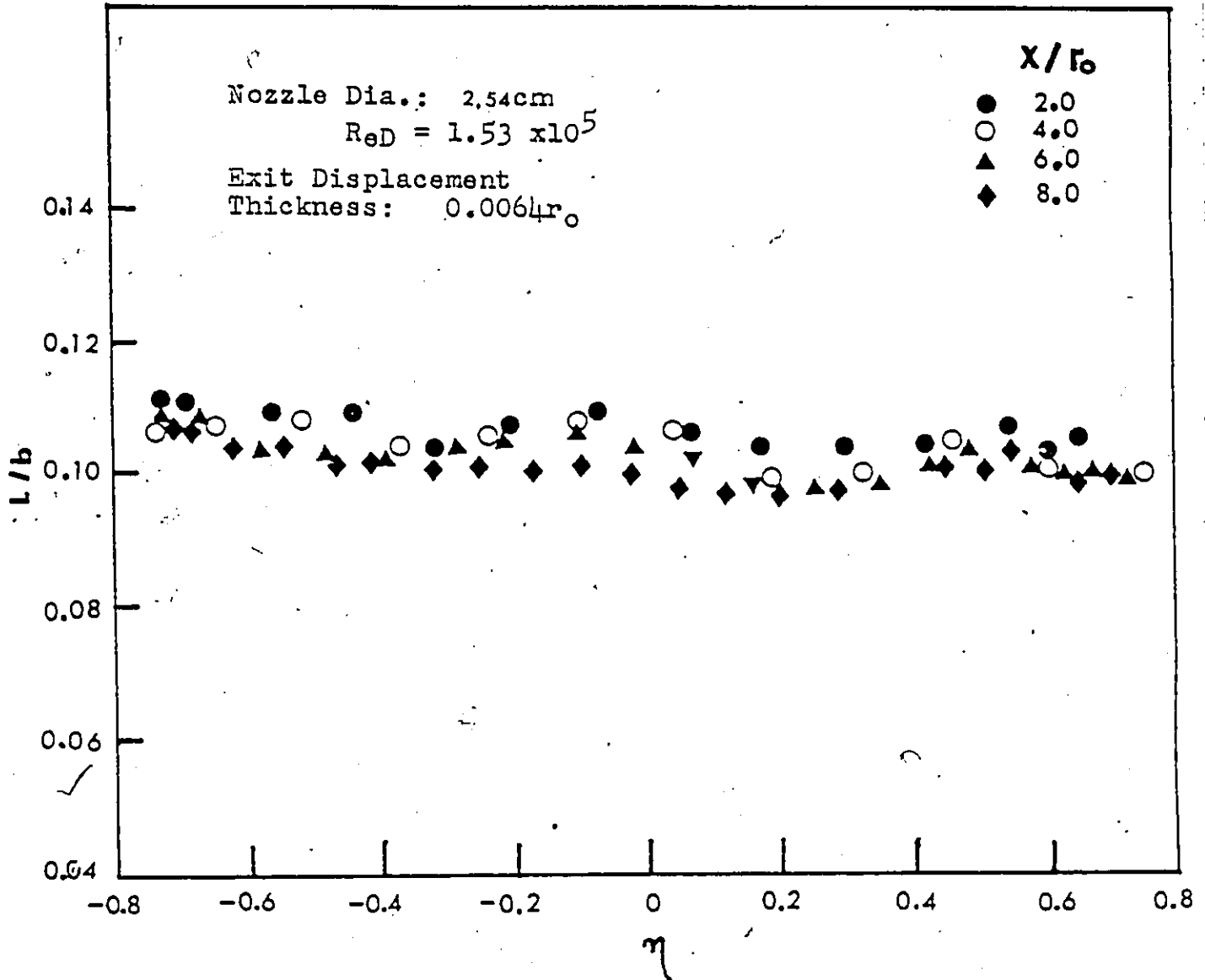
Fig.5.10

Experimental Values of the Ratio, L/b, Thin Boundary Layer.
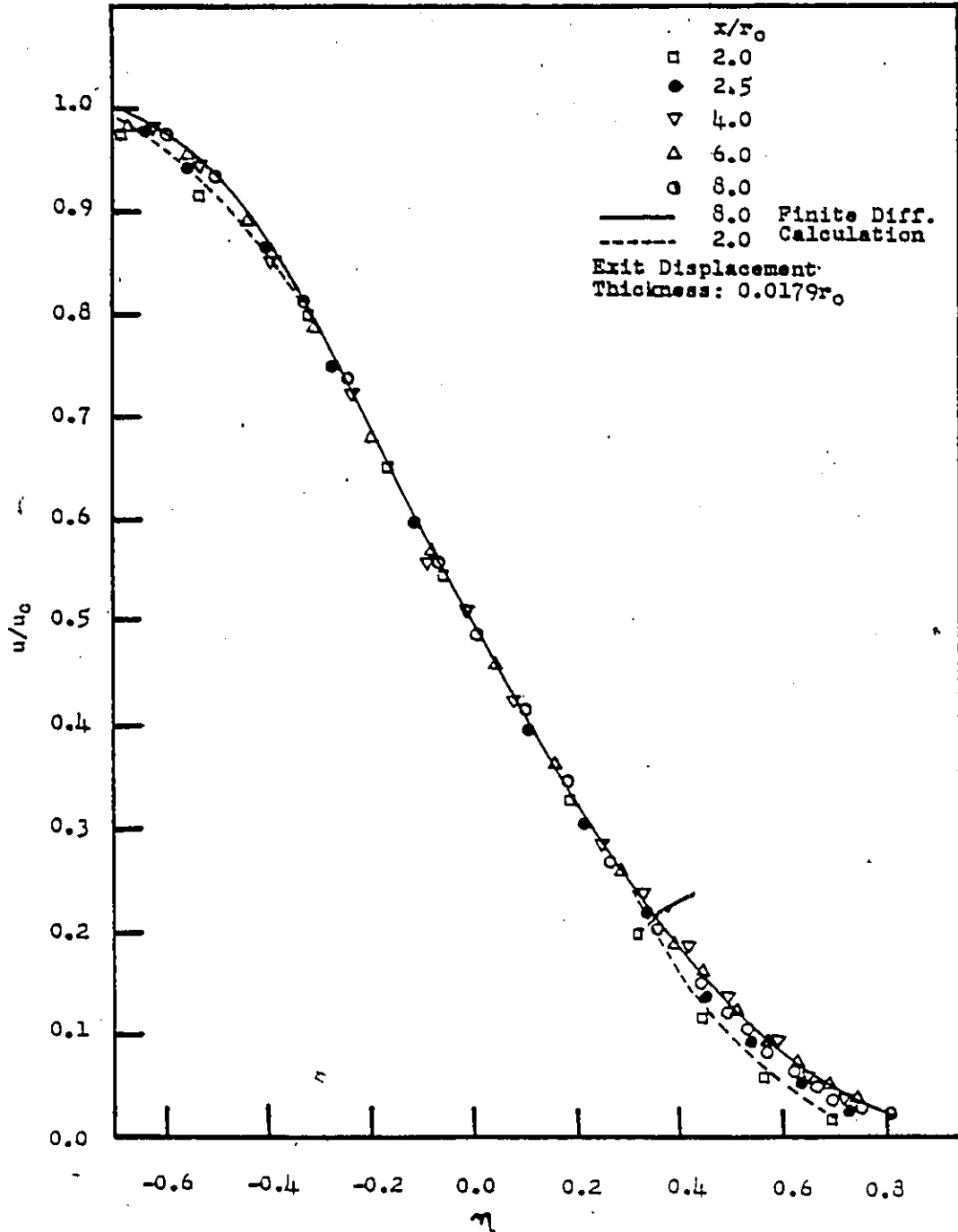
Fig.5.11

Velocity Profiles in the Initial Region, Intermediate
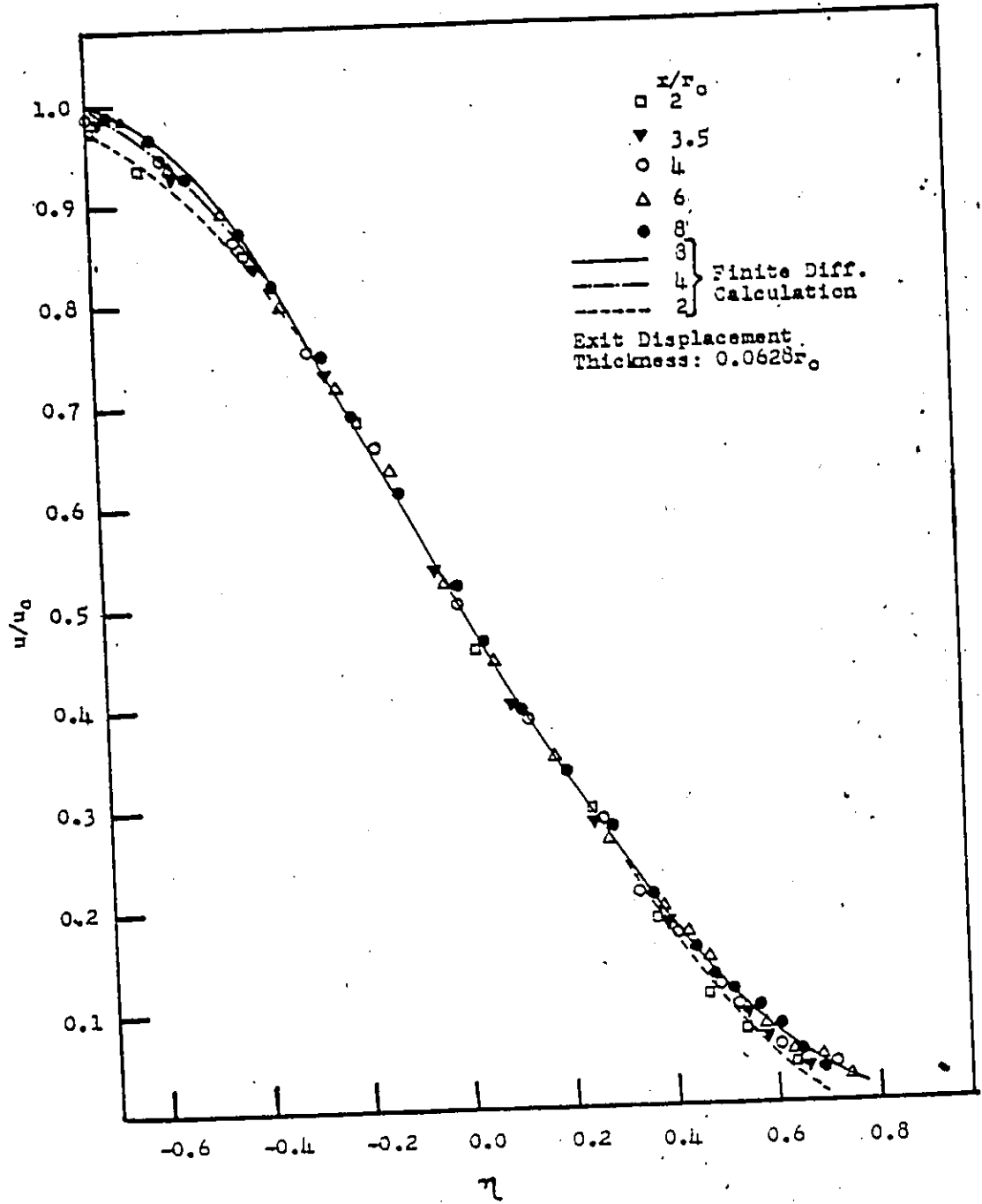
Boundary Layer.

Fig.5.12

Velocity Profiles in the Initial Region, Thick

Boundary Layer.
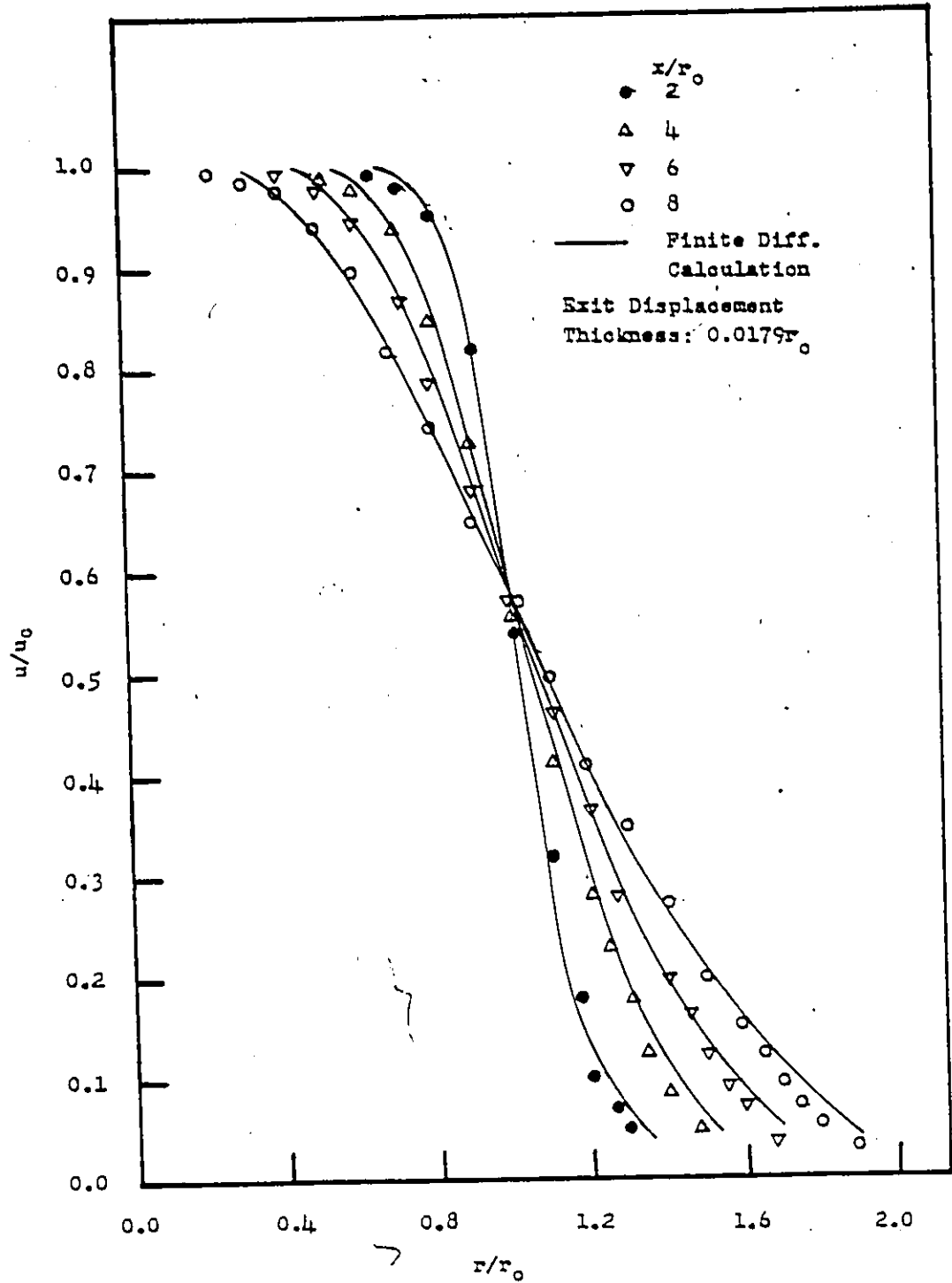
Fig.5.13

Calculated and Experimental Mean Axial Velocity

Distribution, Intermediate Boundary Layer.
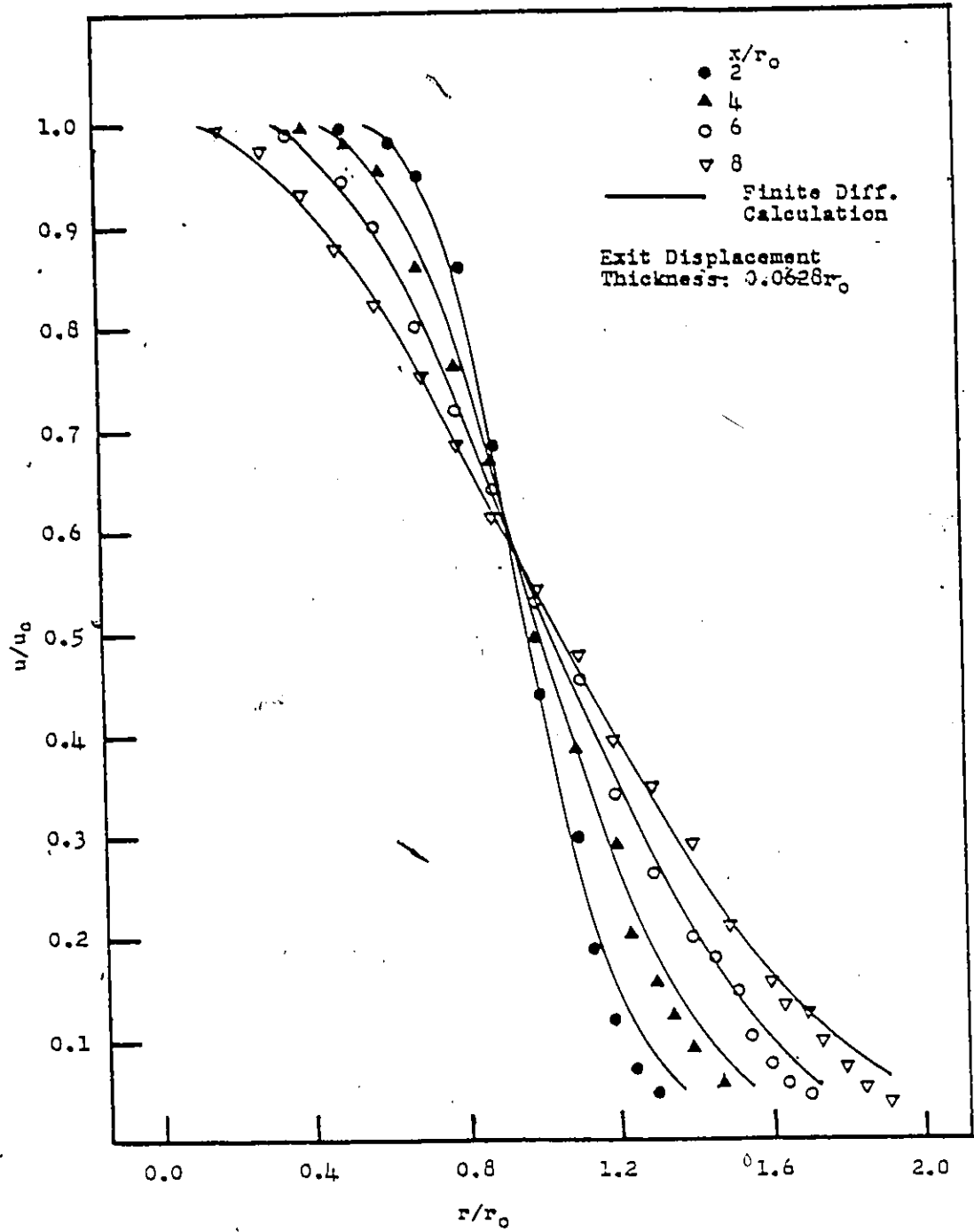
Fig.5.14

Calculated and Experimental Mean Axial Velocity

Distribution, Thick Boundary Layer.
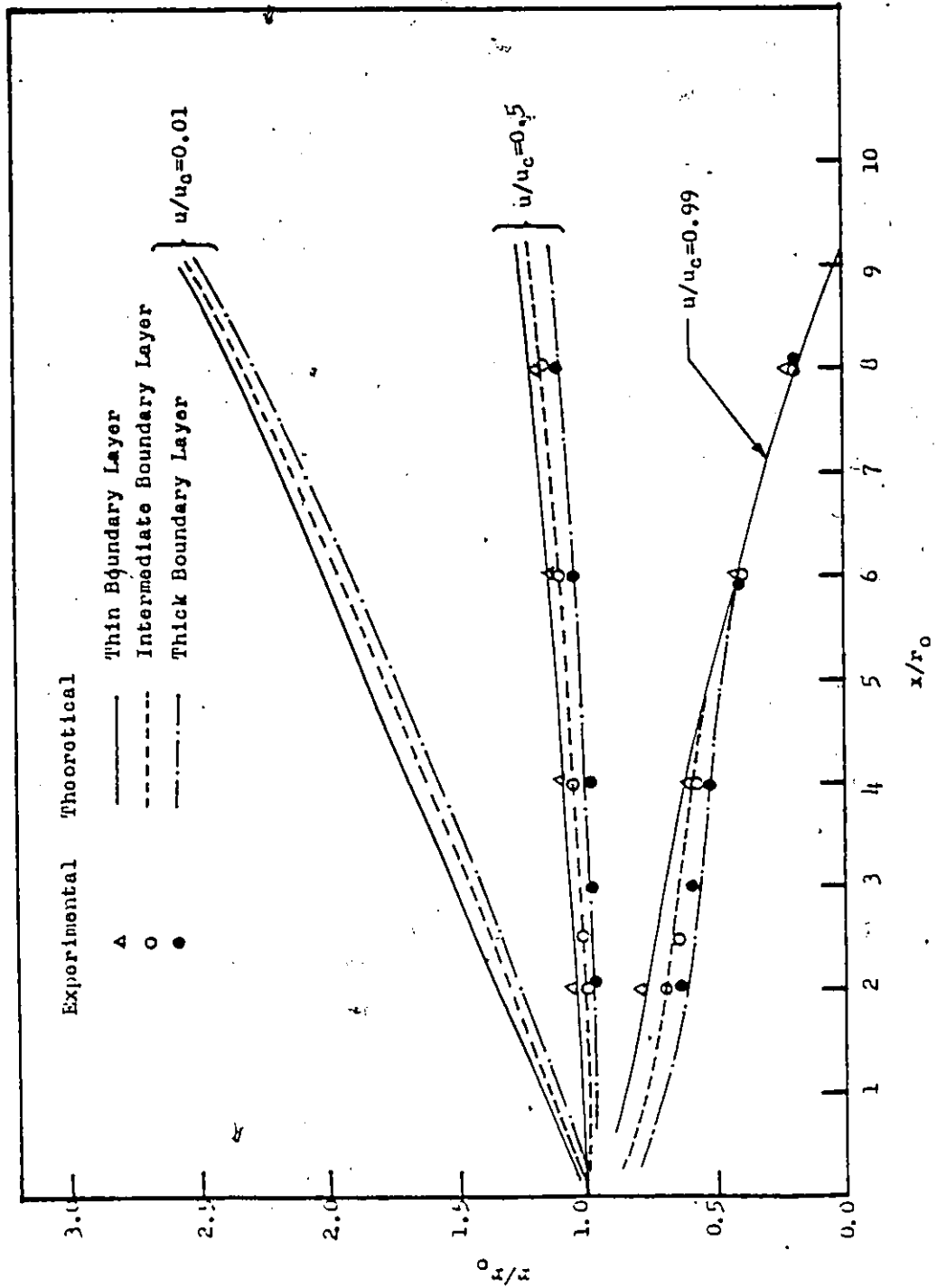
Fig.5.15

Calculated and Experimental Iso-Velocity Lines, Thin, Intermediate and

Thick Boundary Layers.
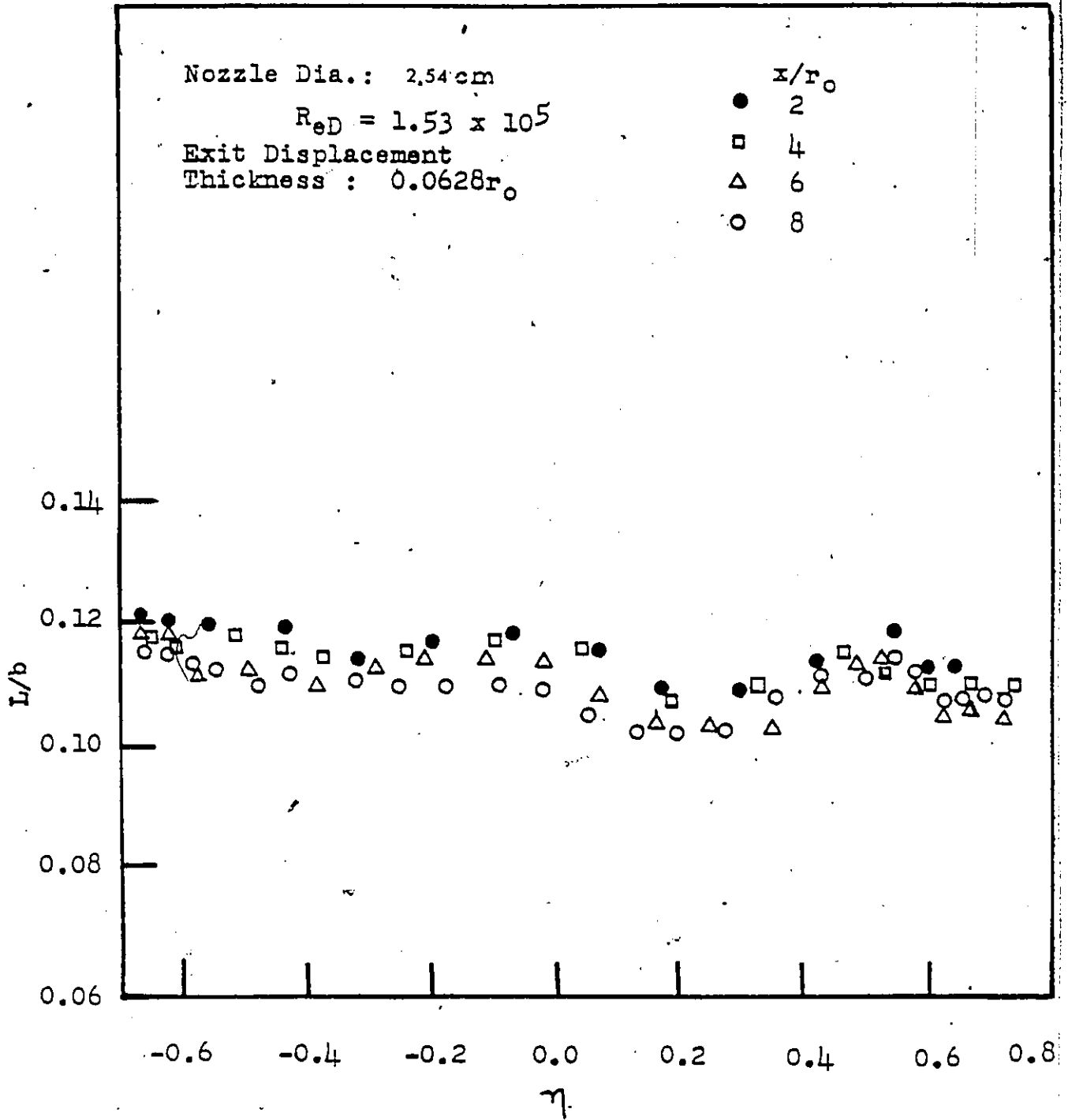
Fig.5.16

Experimental Values of the Ratio, L/b, Thick Boundary

Layer.
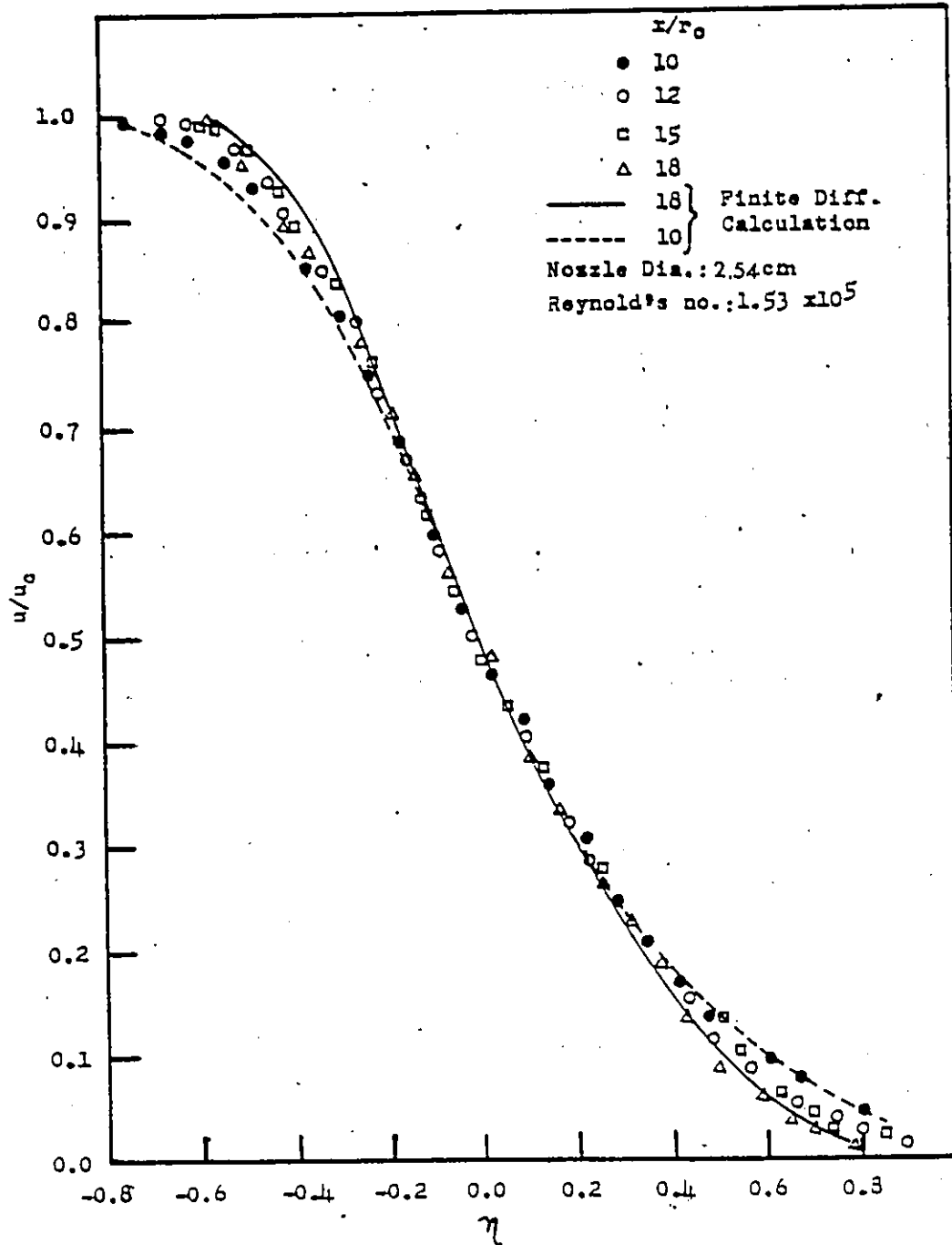
Fig.5.17

Calculated and Experimental Mean Axial Velocity

Distributions in u/u_c vs η Plane for the

Transition Region.
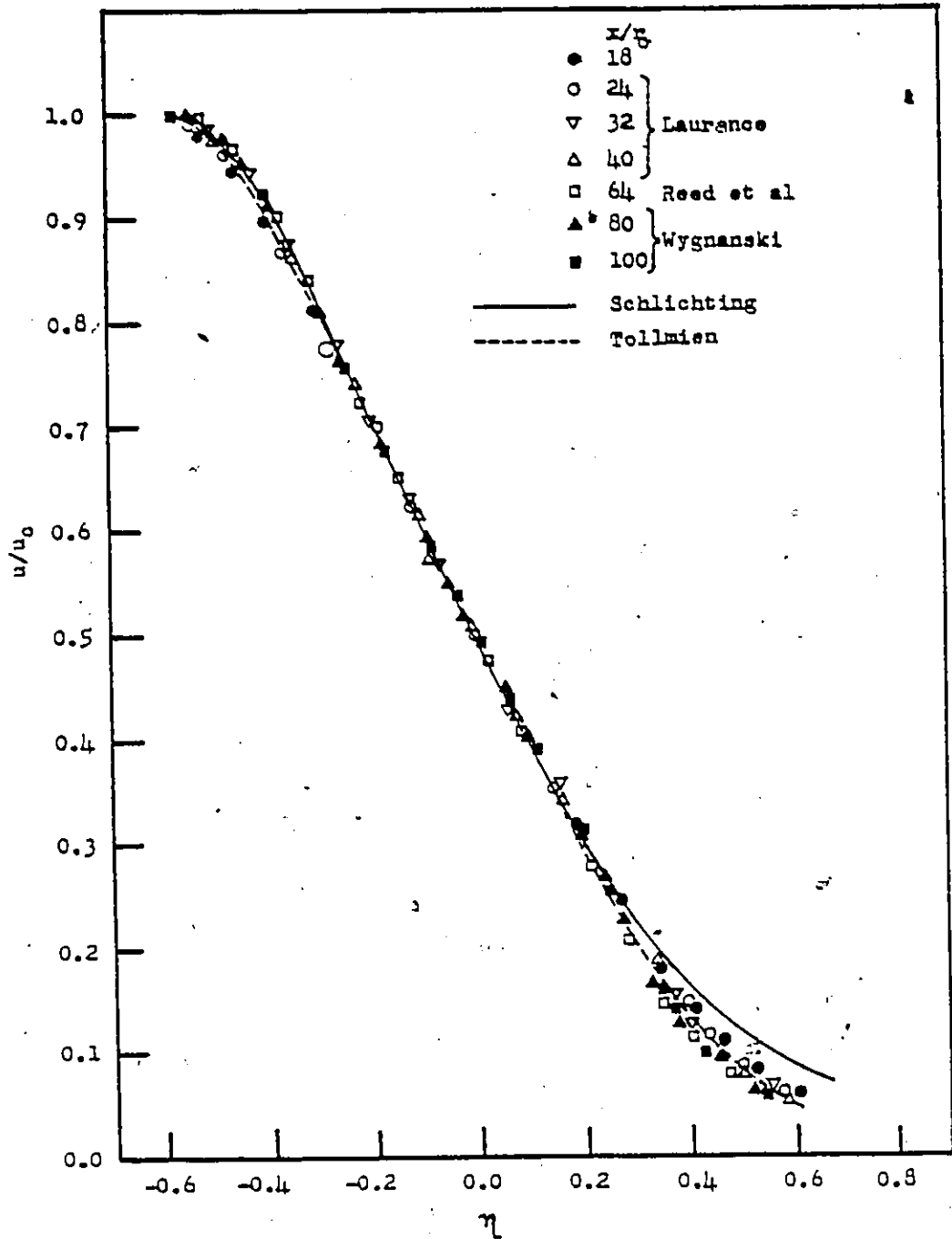
Fig.5.18

Comparison of Experimental Mean Axial Velocity

with Tollmien's and Schlichting's Models for

the Developed Region.
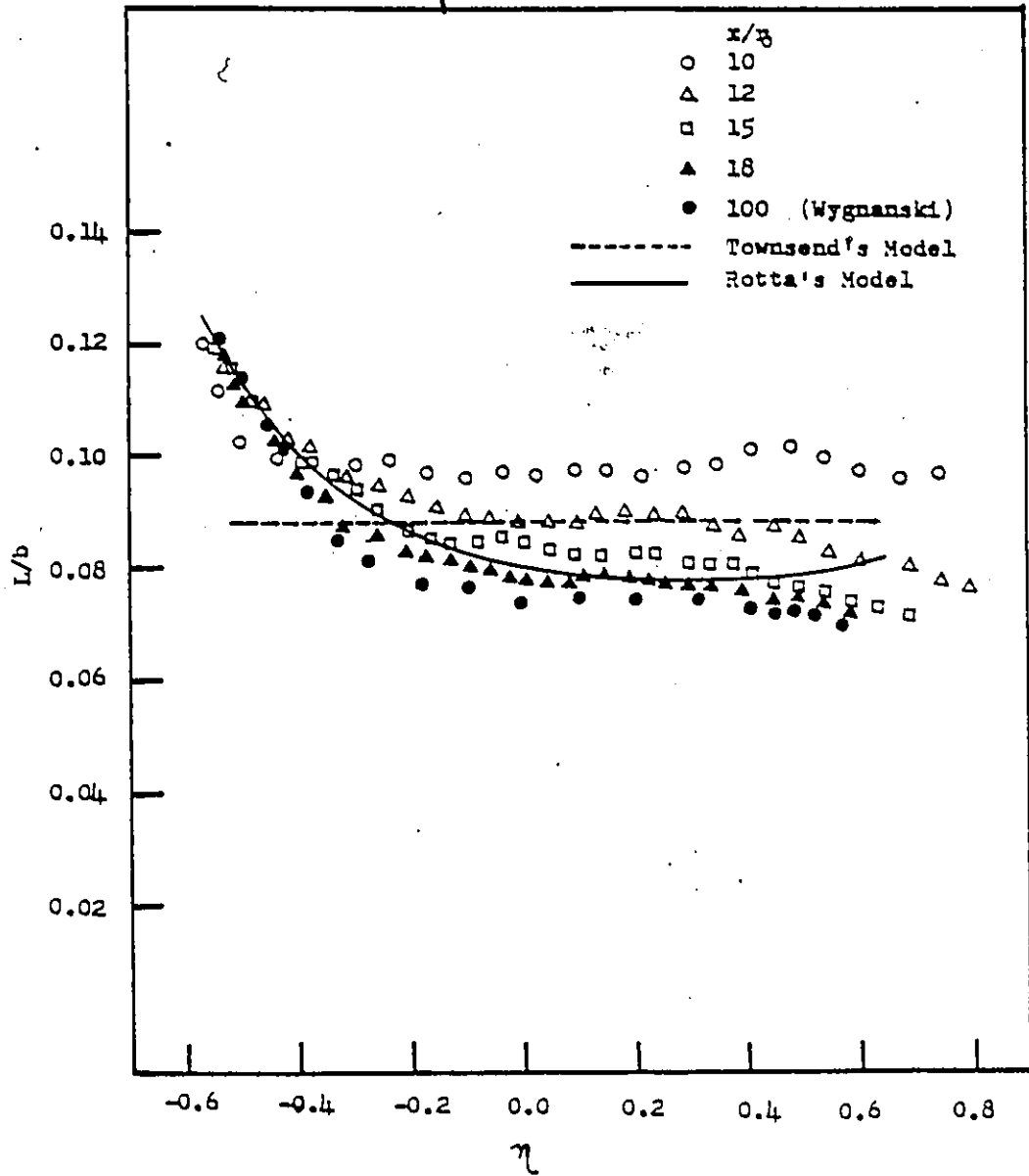
Fig.5.19

Prandtl's Mixing Length in the Transition
Region.
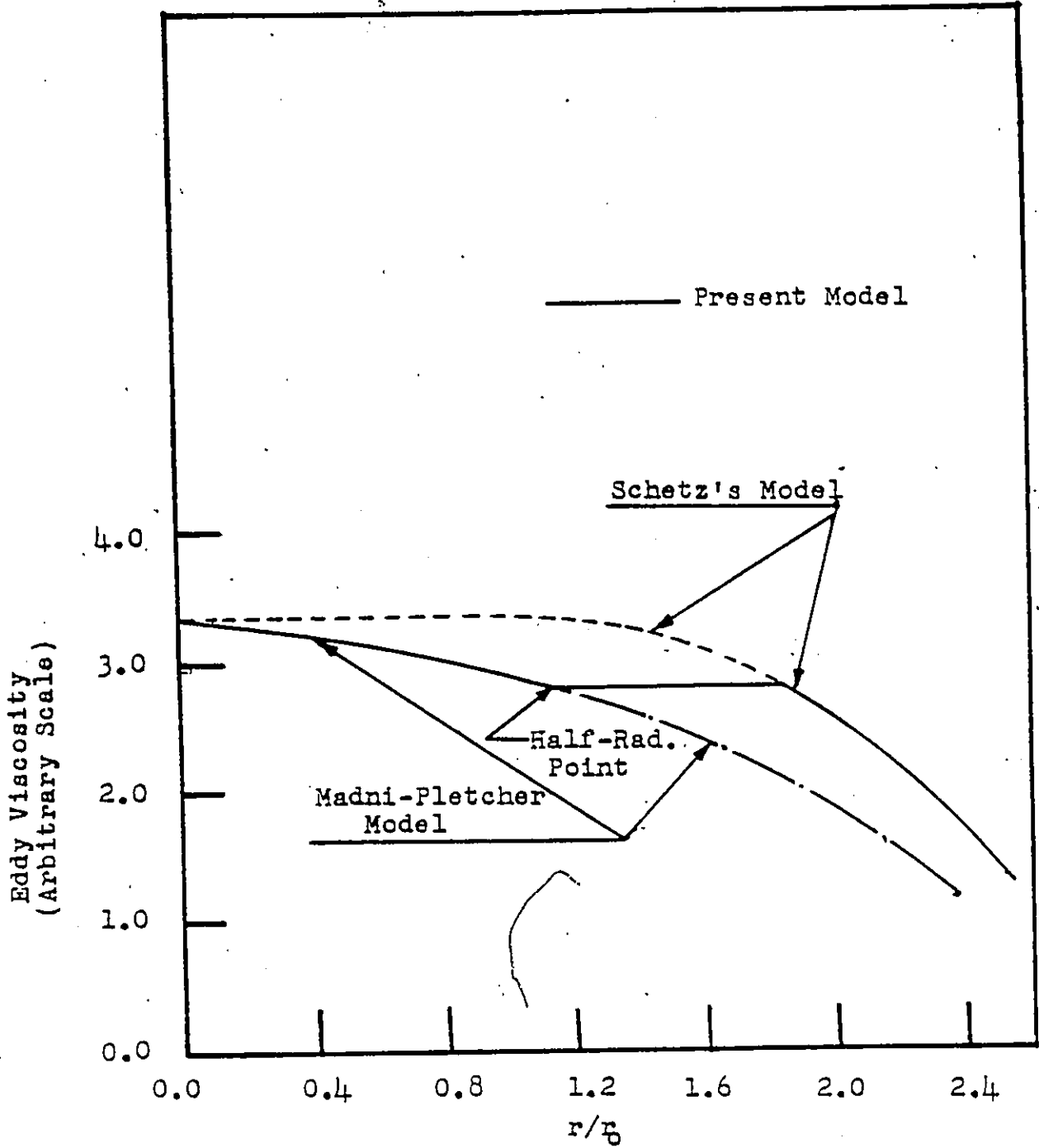
Fig.5.20

Eddy Viscosity vs Radial Distance in the Transition
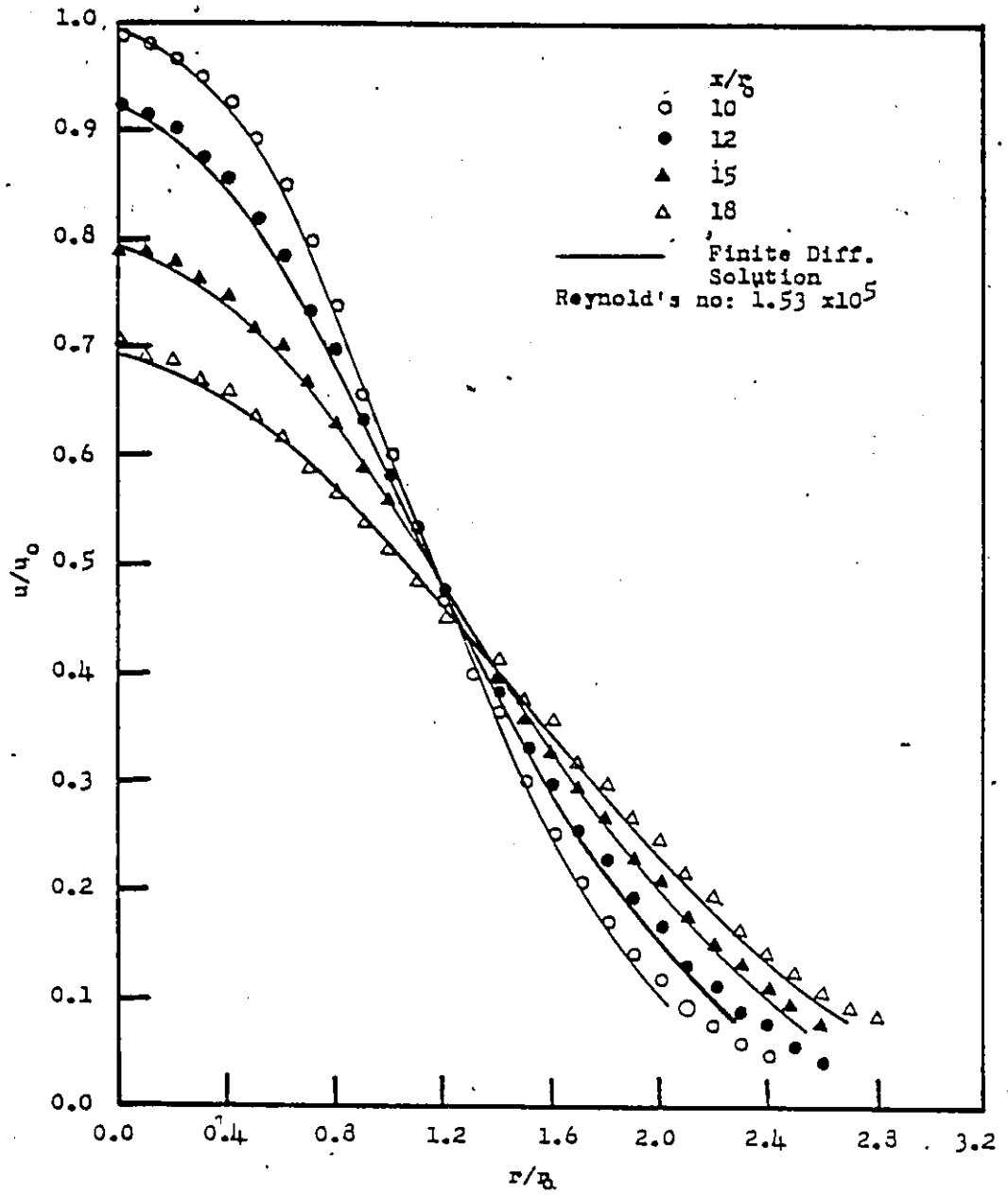
Region.

Fig.5.21

Calculated and Experimental Mean Axial Velocity

Distributions in the Transition Region.
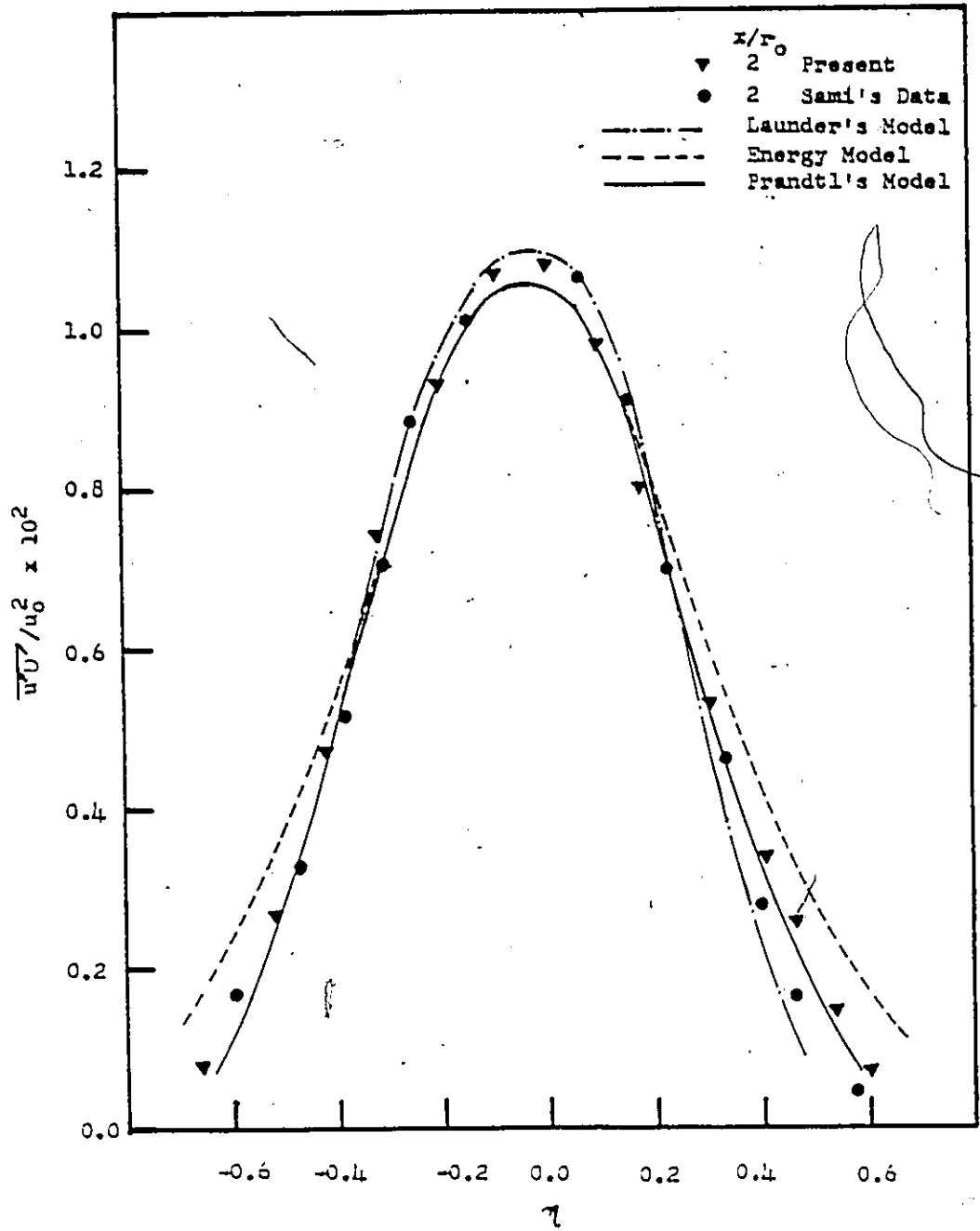
Fig.5.22

Comparison of Shear Stress Models and Measurements,

Thin Boundary Layer, $x/r_0 = 2.$
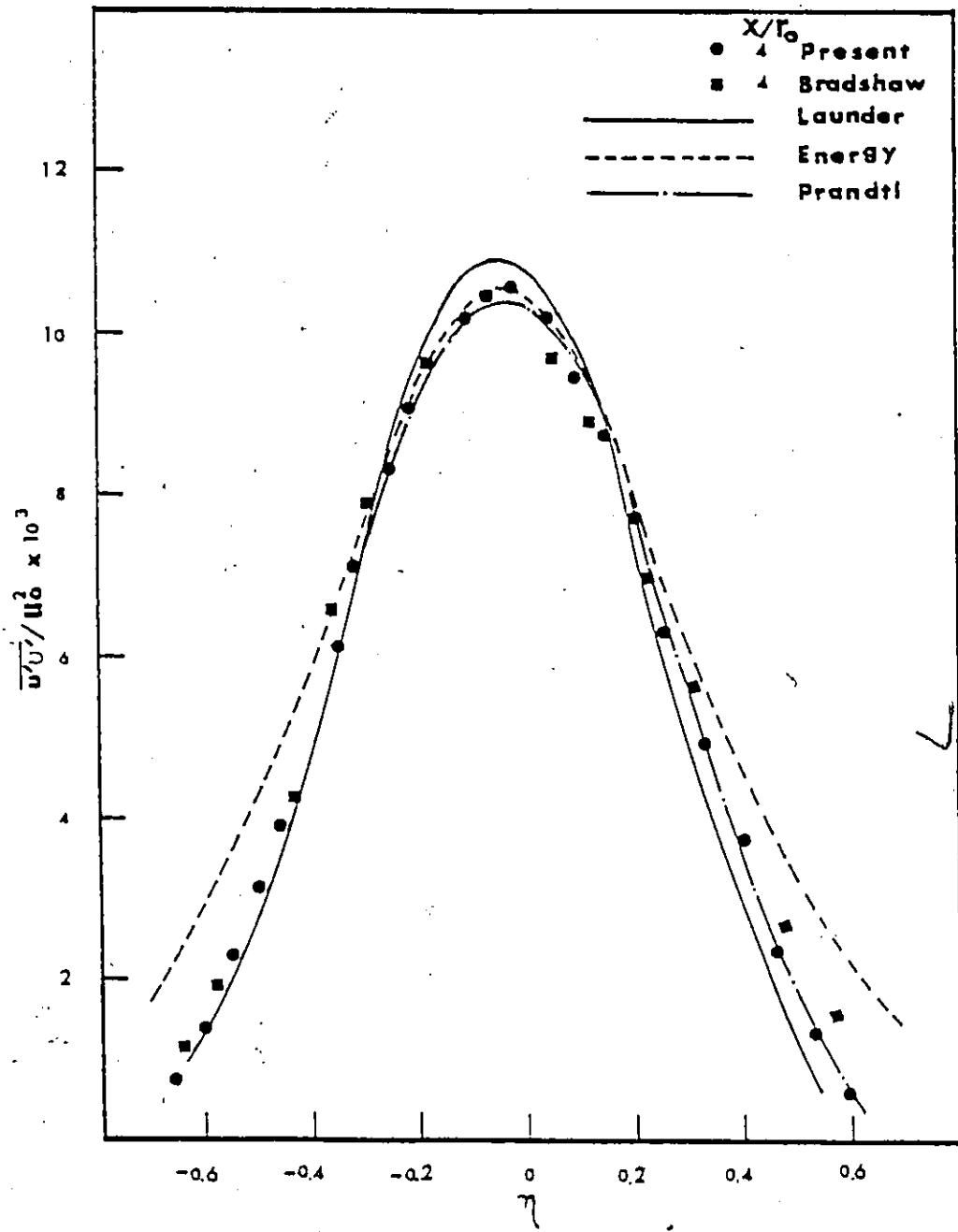
Fig.5.23.

Comparison of Shear Stress   Models and

Measurements, Thin Boundary Layer, $x/r_o = 4$.
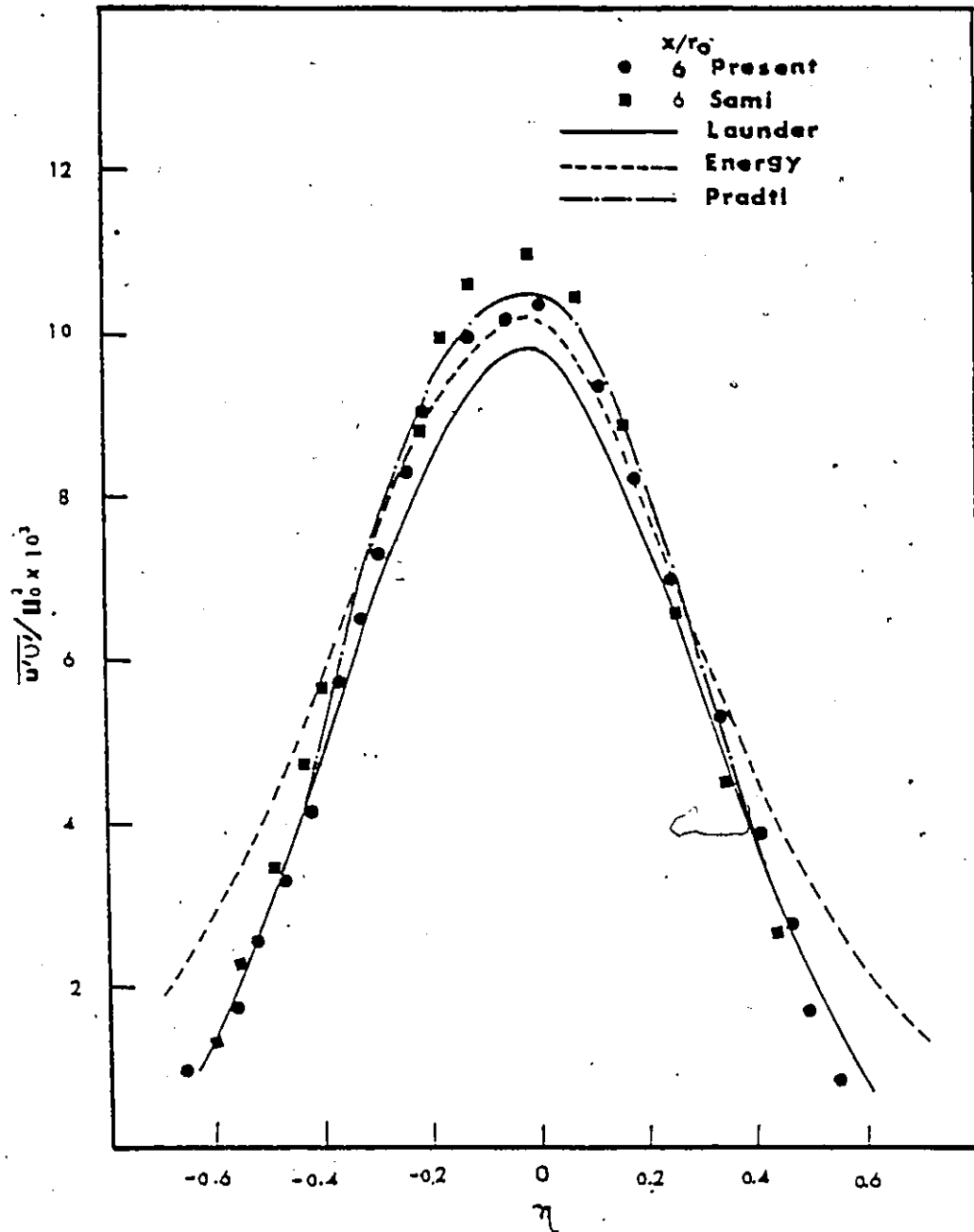
Fig.5.24.

Comparison of Shear Stress Models and

Measurements,Thin Boundary Layer, $x/r_0 = 6$.
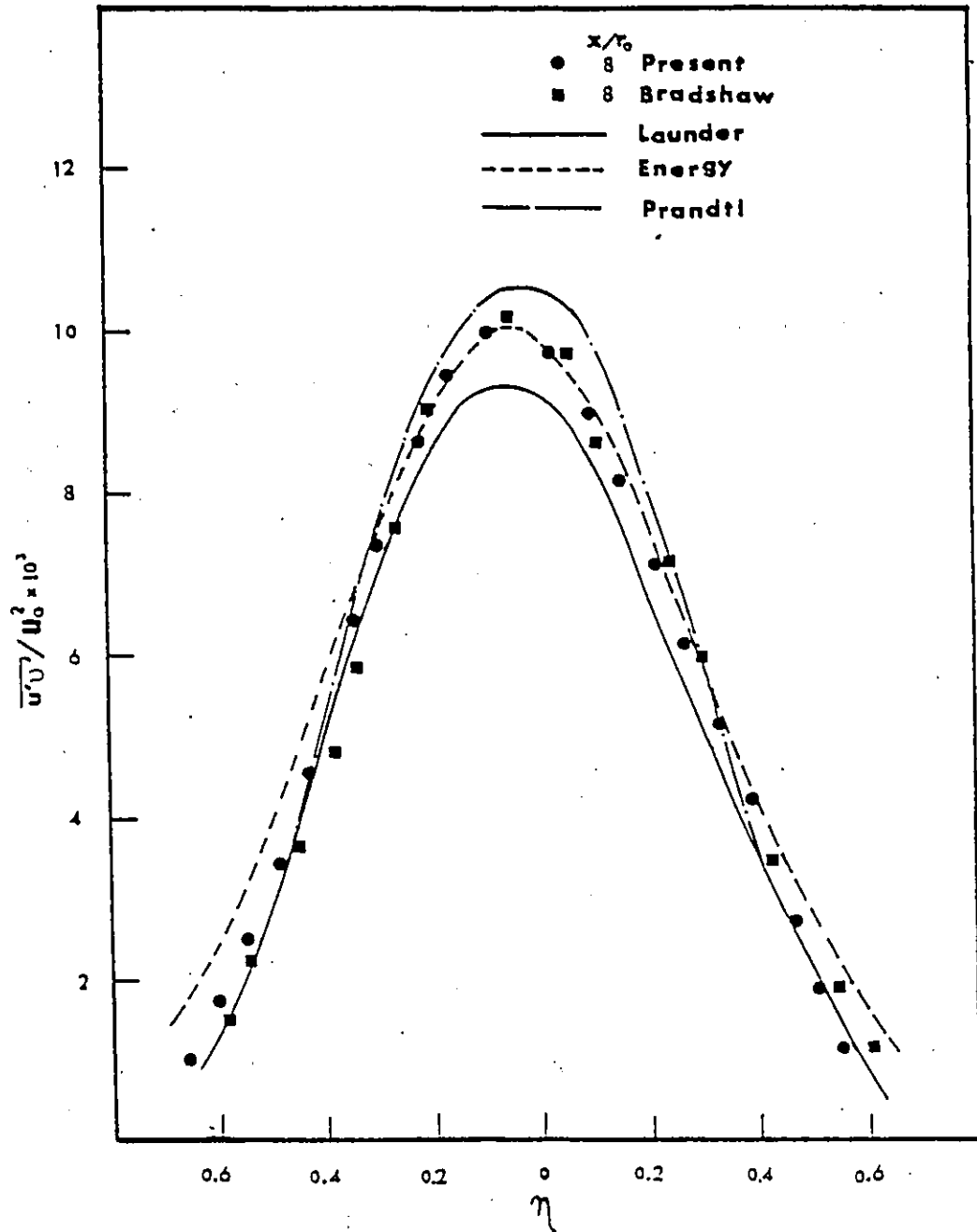
Fig.5.25.

Comparison of Shear Stress Models and

Measurements,Thin Boundary Layer, $x/r_o = 8$.
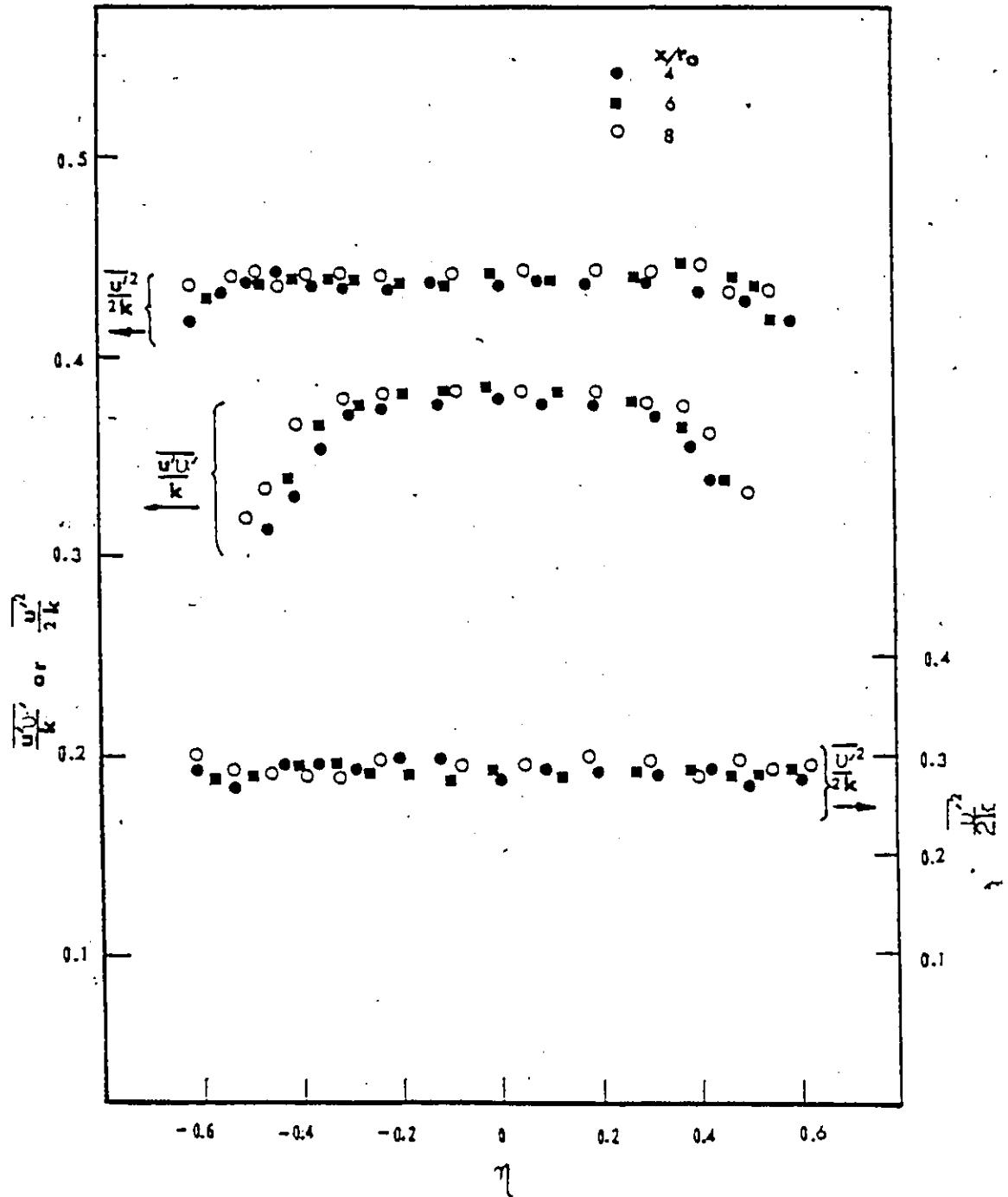
Fig.5.26.

Experimental Values of $\overline{u'^2}/2k$, $\overline{\upsilon'^2}/2k$ and

$\overline{u'\upsilon'}/k$ in the Initial Region.
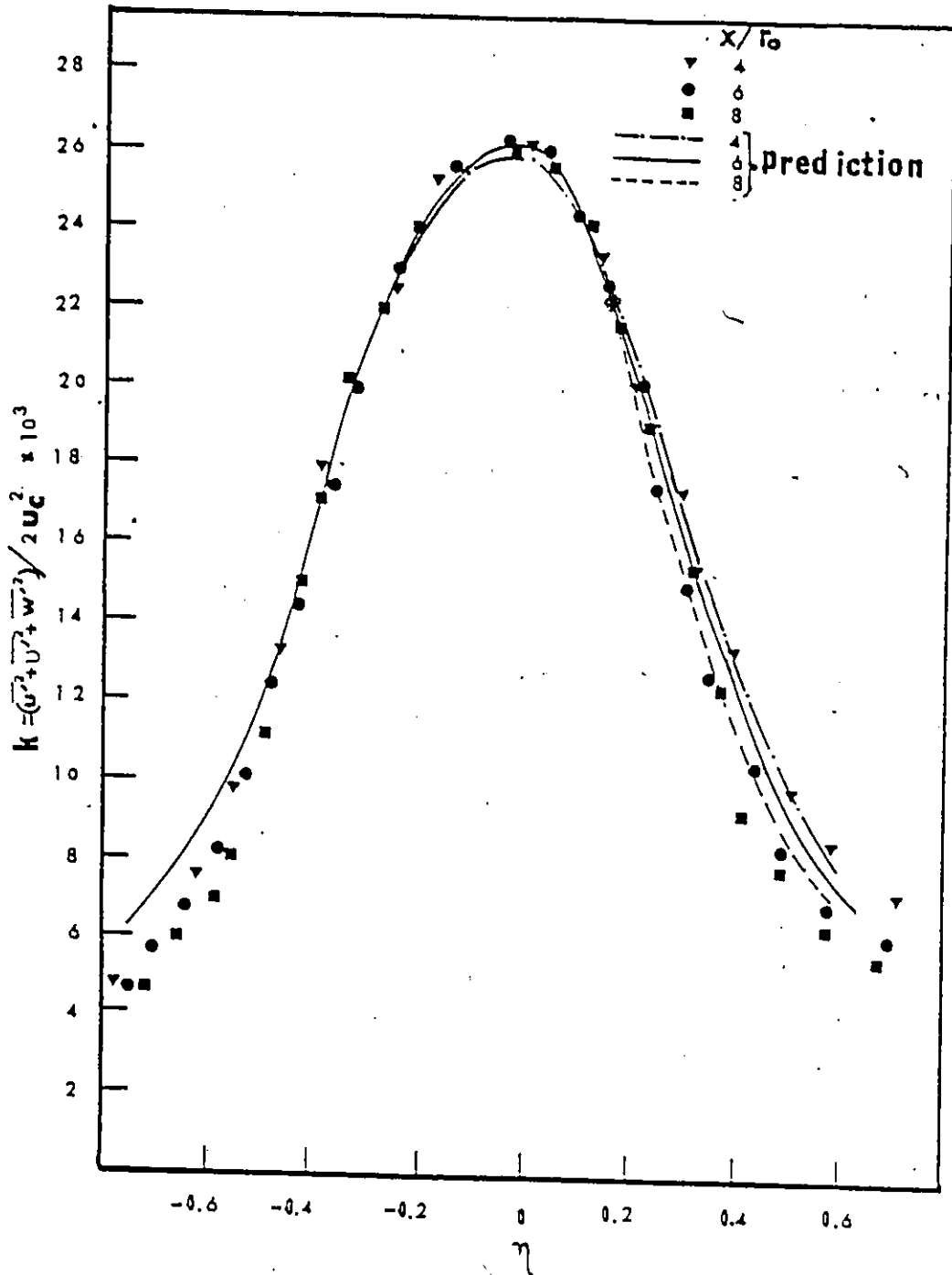
Fig.5.27.

Calculated and Experimental Turbulent Kinetic
Energy Distributions, Thin Boundary Layer.
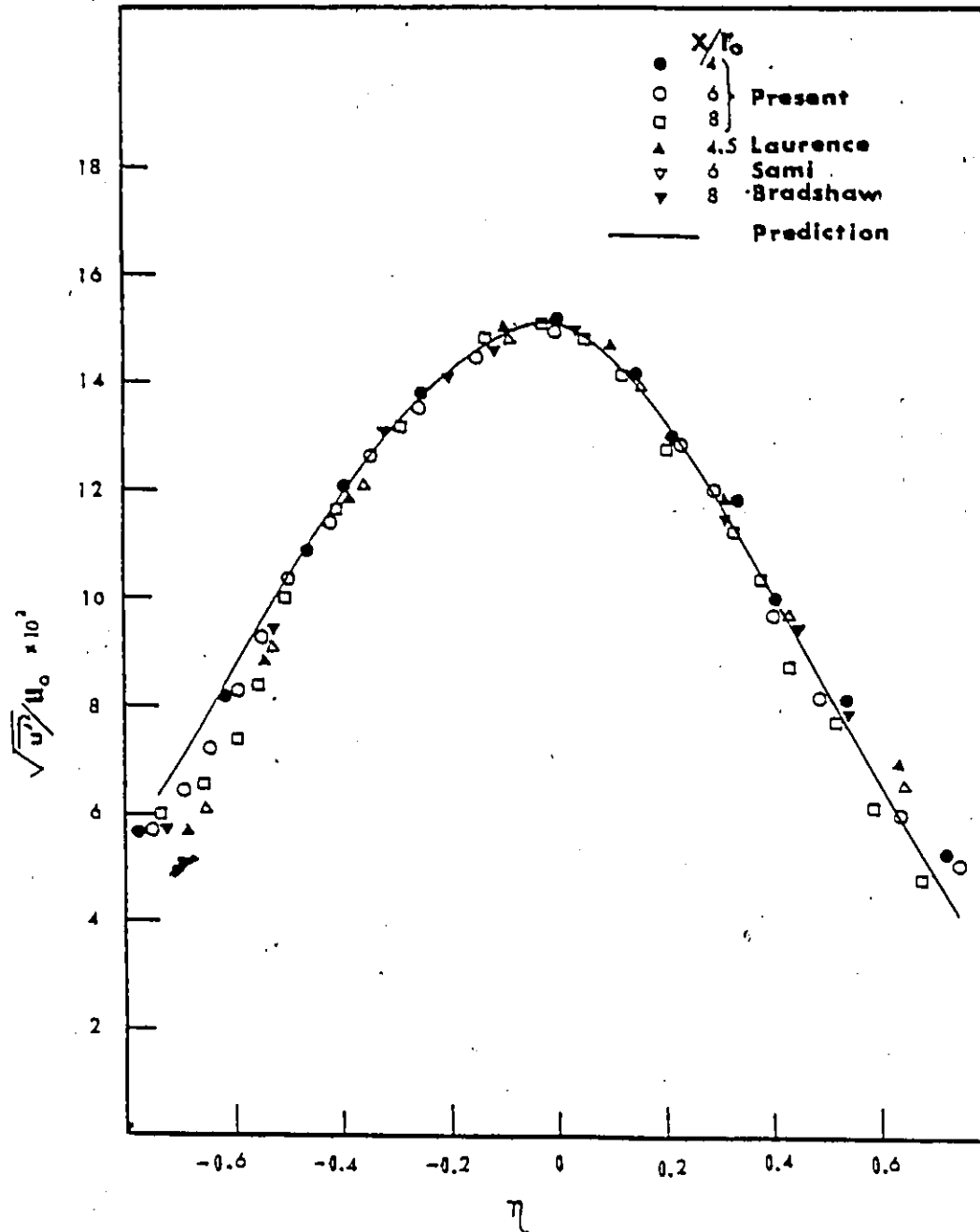
Fig.5.28.

Calculated and Experimental Turbulent Axial

Intensity Distributions, Thin Boundary Layer.
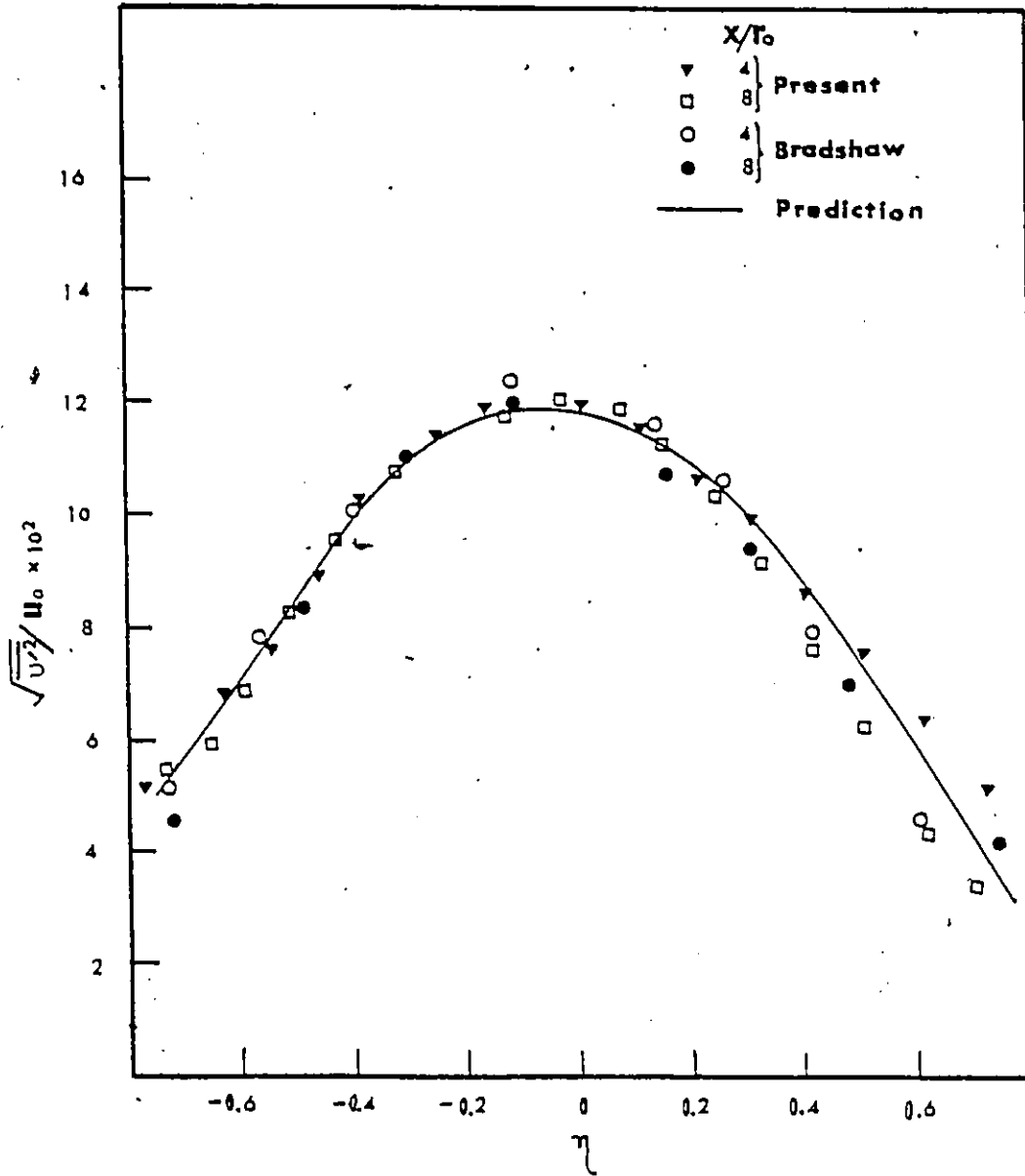
Fig.5.29.

Calculated and Experimental Turbulent Radial
Intensity Distributions,Thin Boundary Layer.
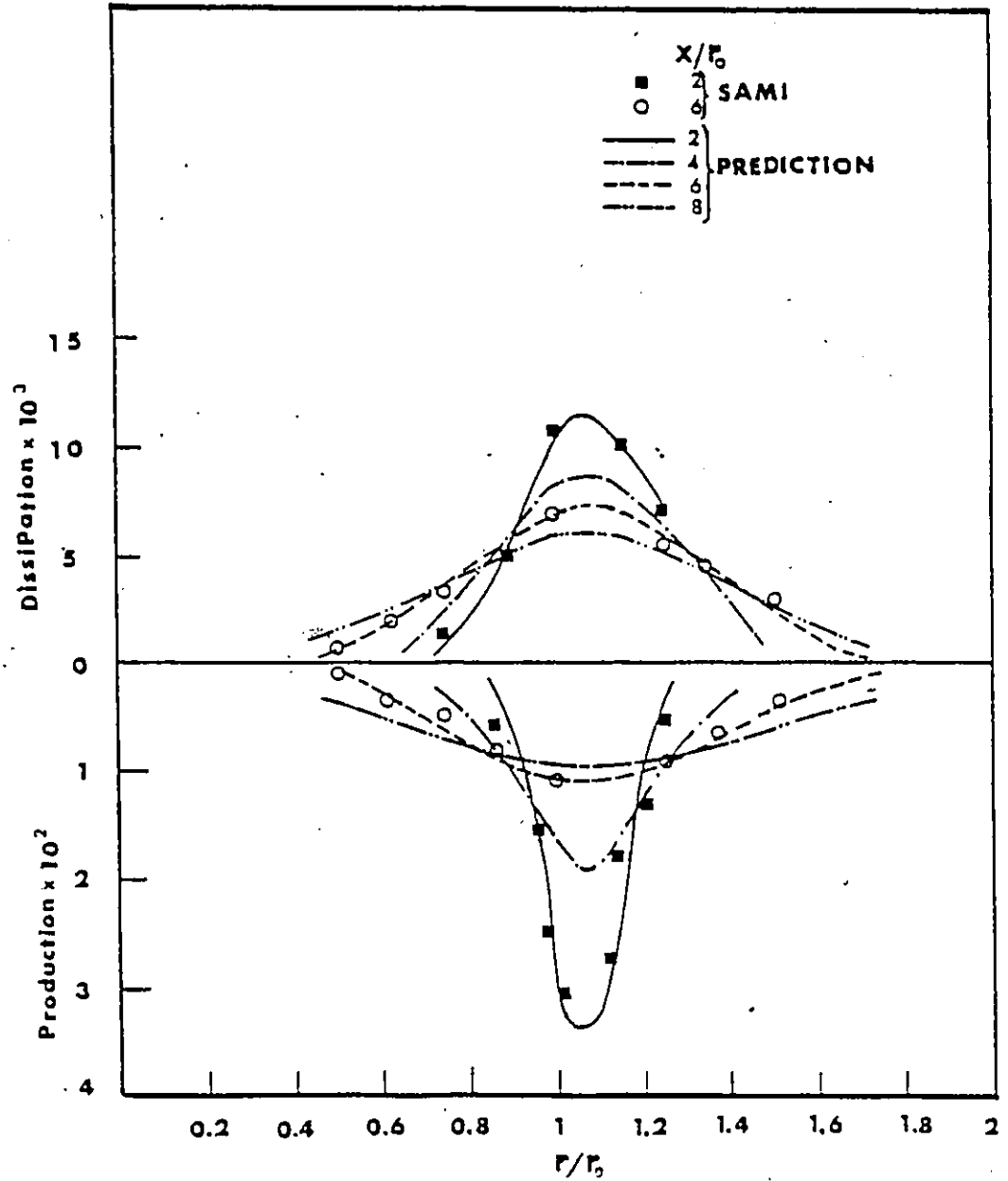
Fig.5.30.

Calculated and Experimental Production and Dissipation
of Turbulent Energy Distributions, Thin Boundary Layer.
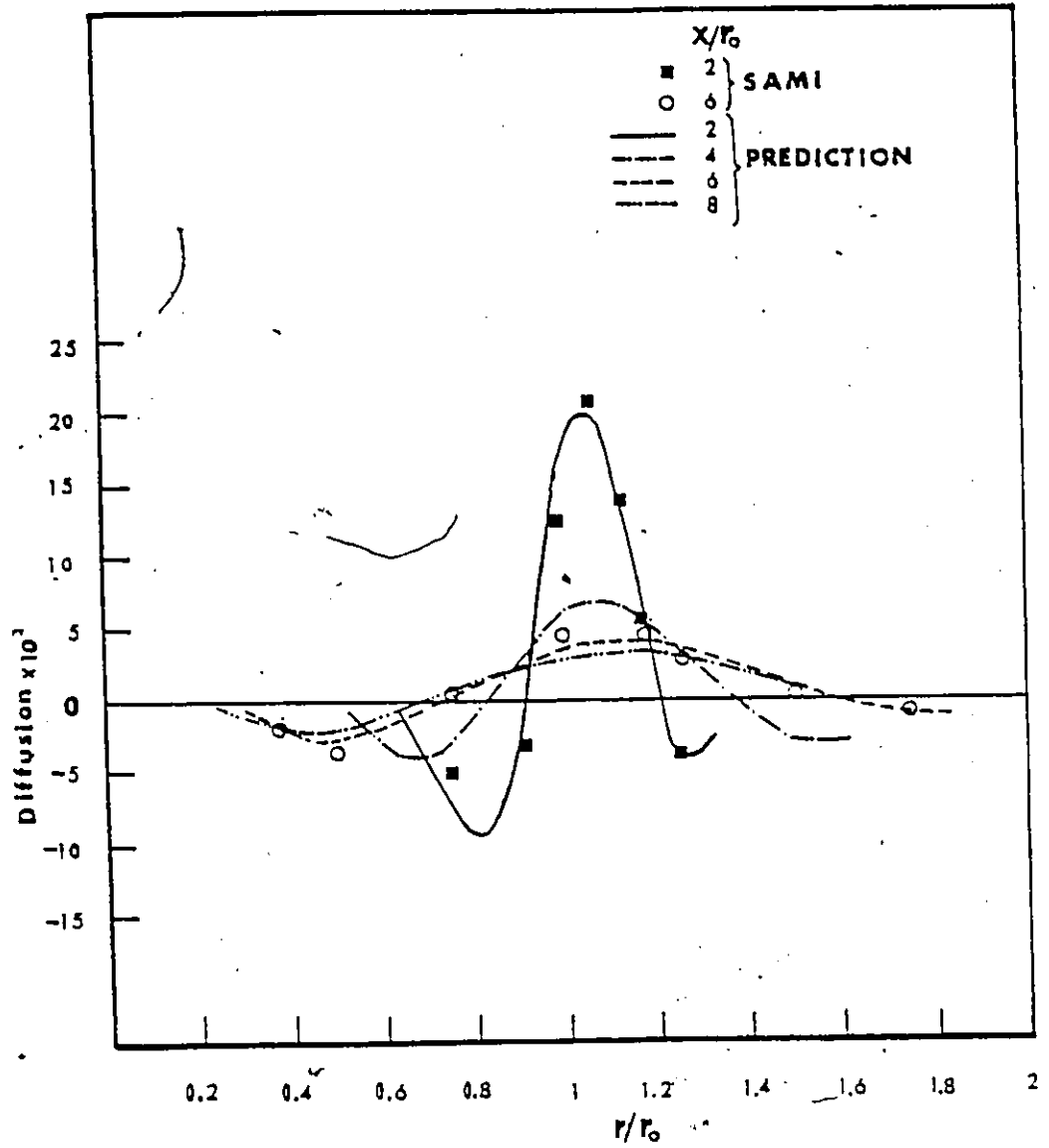
Fig.5.31.

Calculated and Experimental Duffusion of
Turbulent Energy Distributions,Thin Boundary Layer.
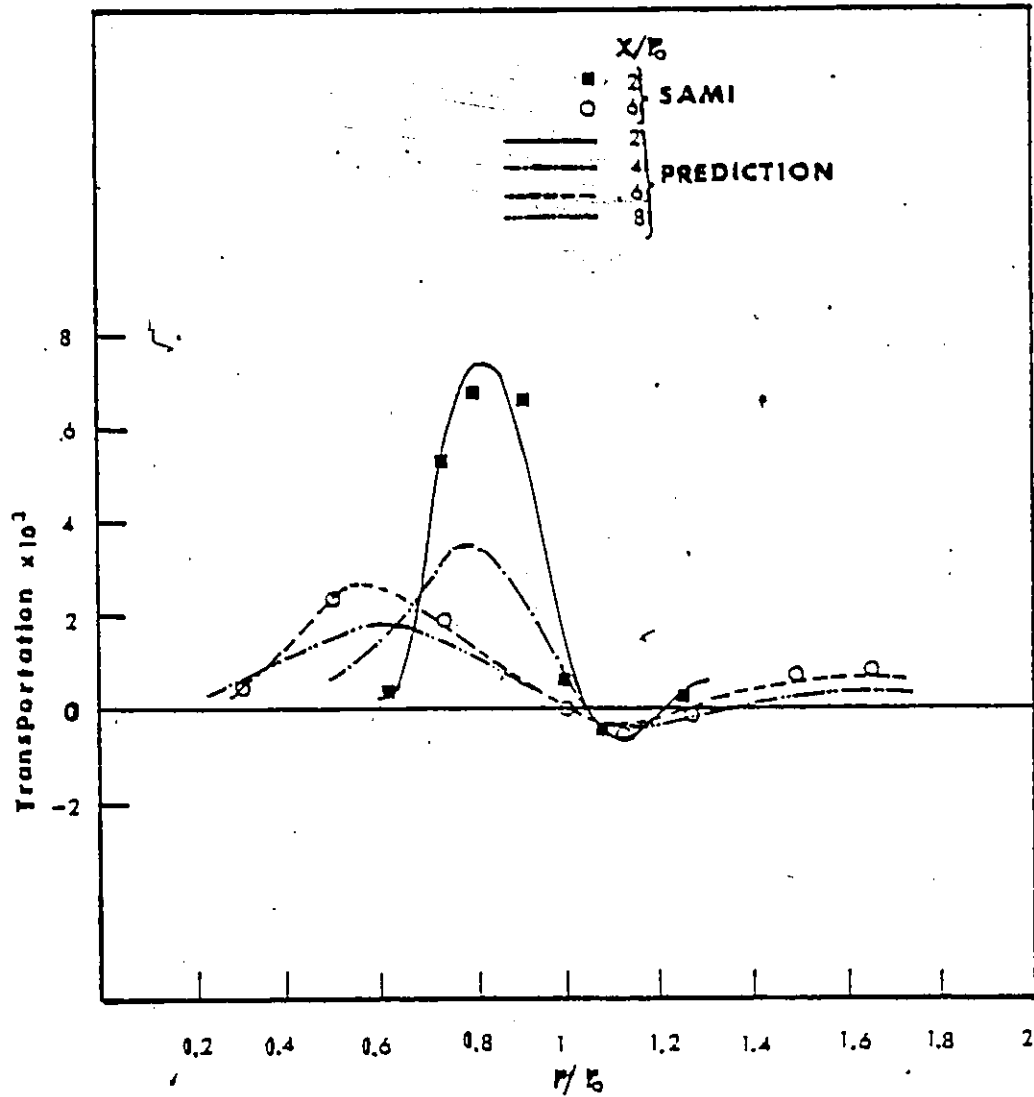
Fig.5.32.

Calculated and Experimental Convective Transport
of Turbulent Energy Distributions,Thin Boundary Layer.
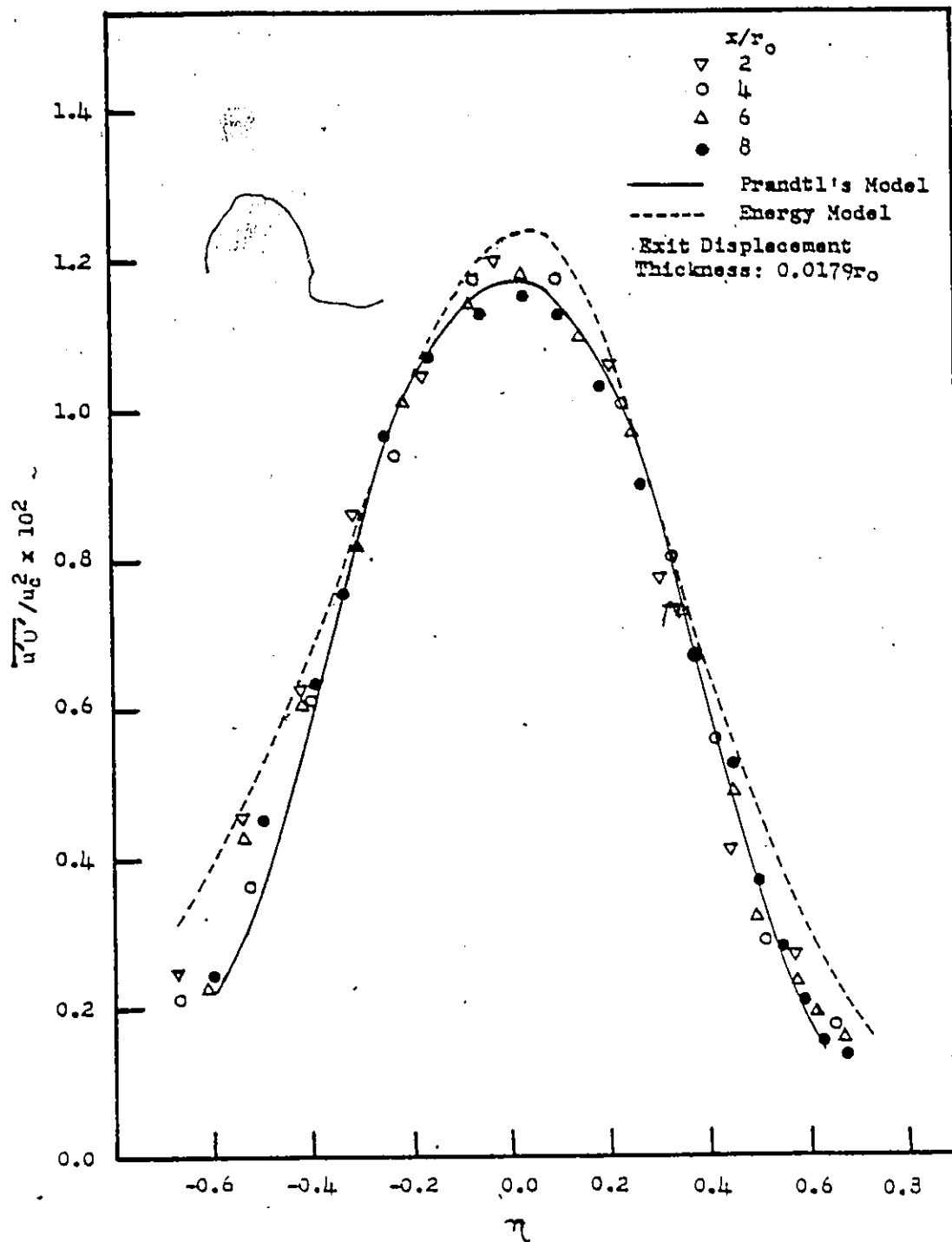
Fig.5.33

Calculated and Experimental Shear Stress Distributions,

Intermediate Boundary Layer.
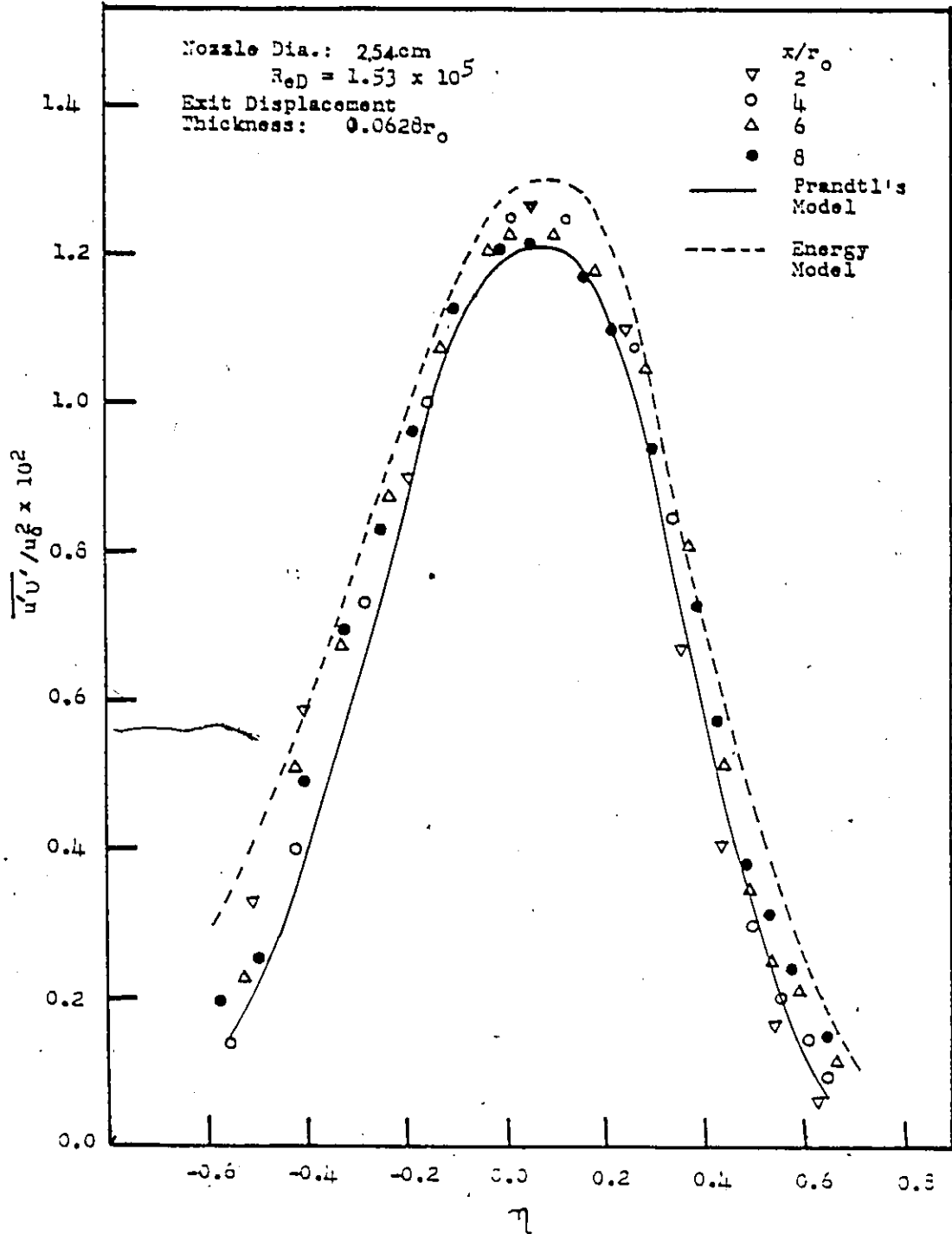
Fig. 5.34

Calculated and Experimental Shear Stress Distributions,

Thick Boundary Layer.
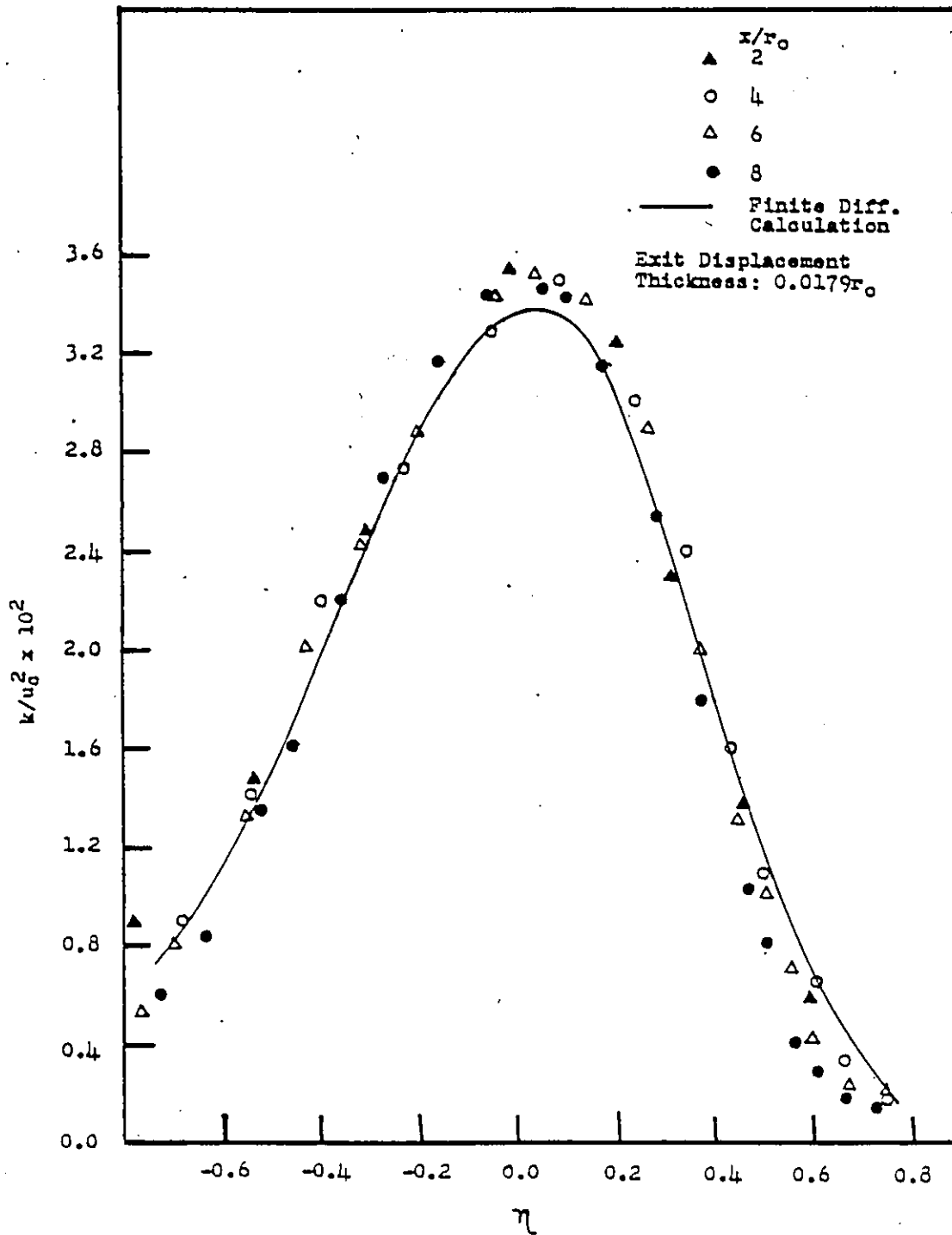
Fig.5.35

Calculated and Experimental Turbulent Kinetic Energy
Distributions, Intermediate Boundary Layer.
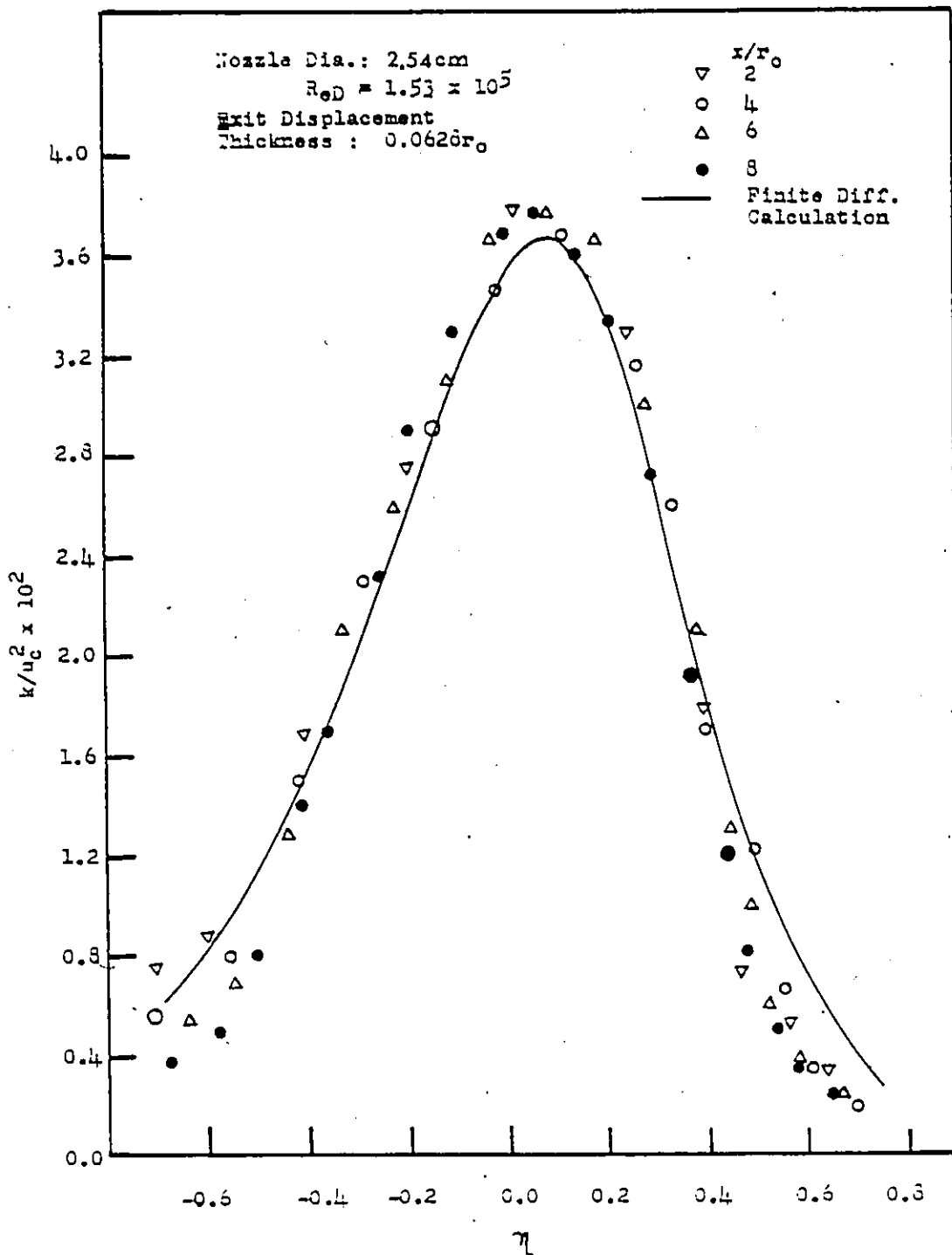
Fig.5.36

Calculated and Experimental Turbulent Kinetic Energy
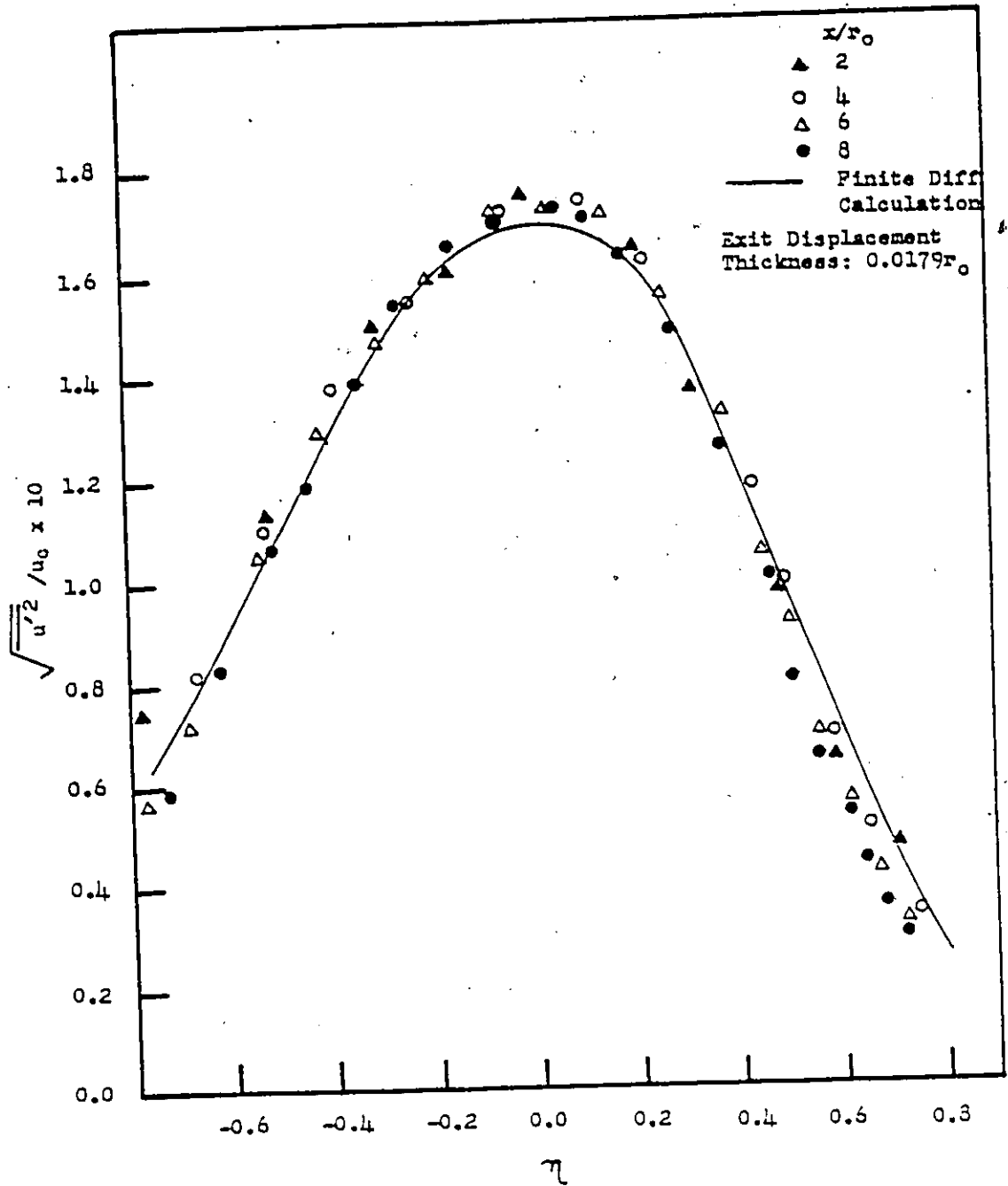
Fig.5.37

Calculated and Experimental Turbulent Axial Intensity

Distributions, Intermediate Boundary Layer.
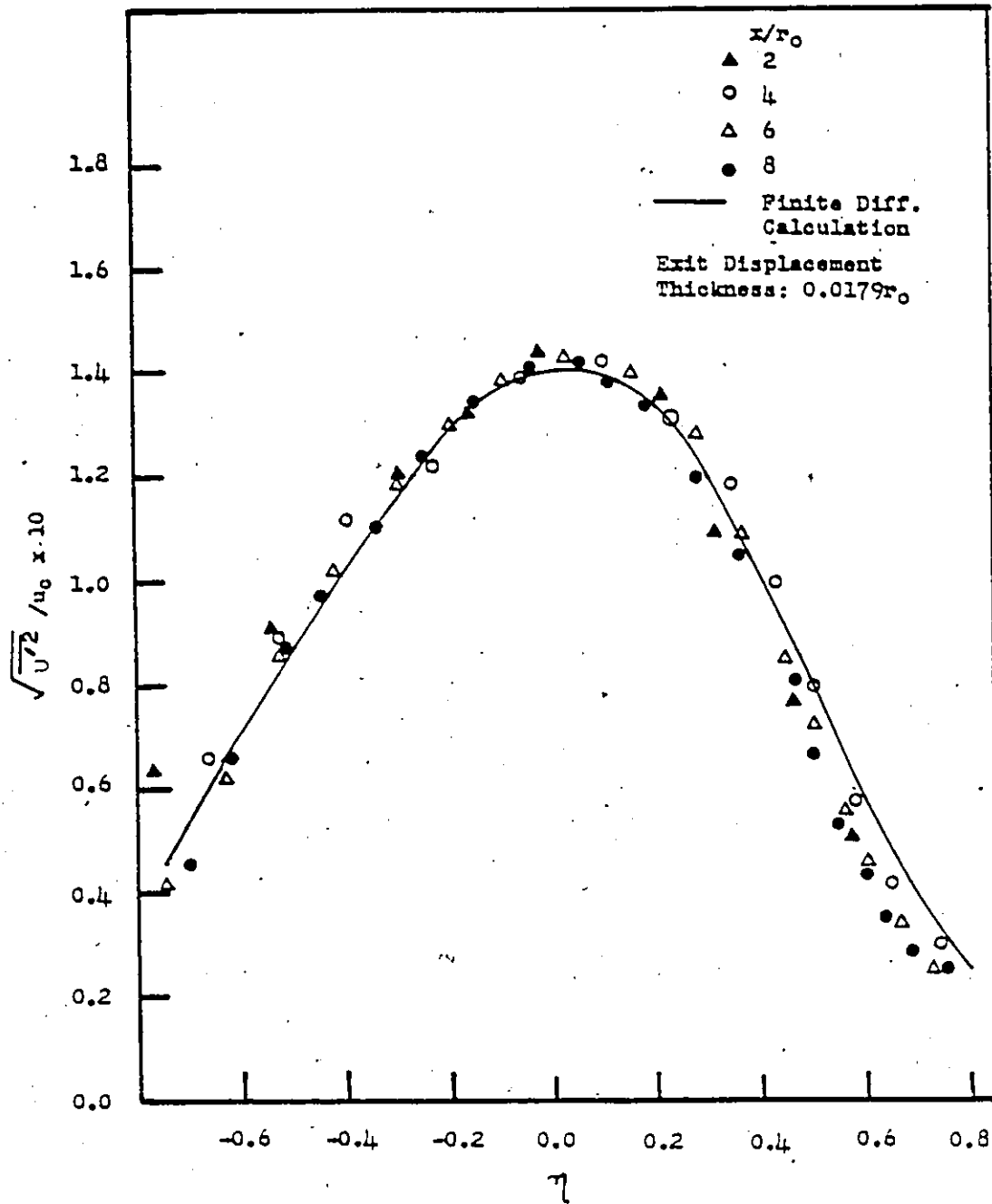
Fig.5.38

Calculated and Experimental Turbulent Radial
Intensity Distributions, Intermediate Boundary
Layer.
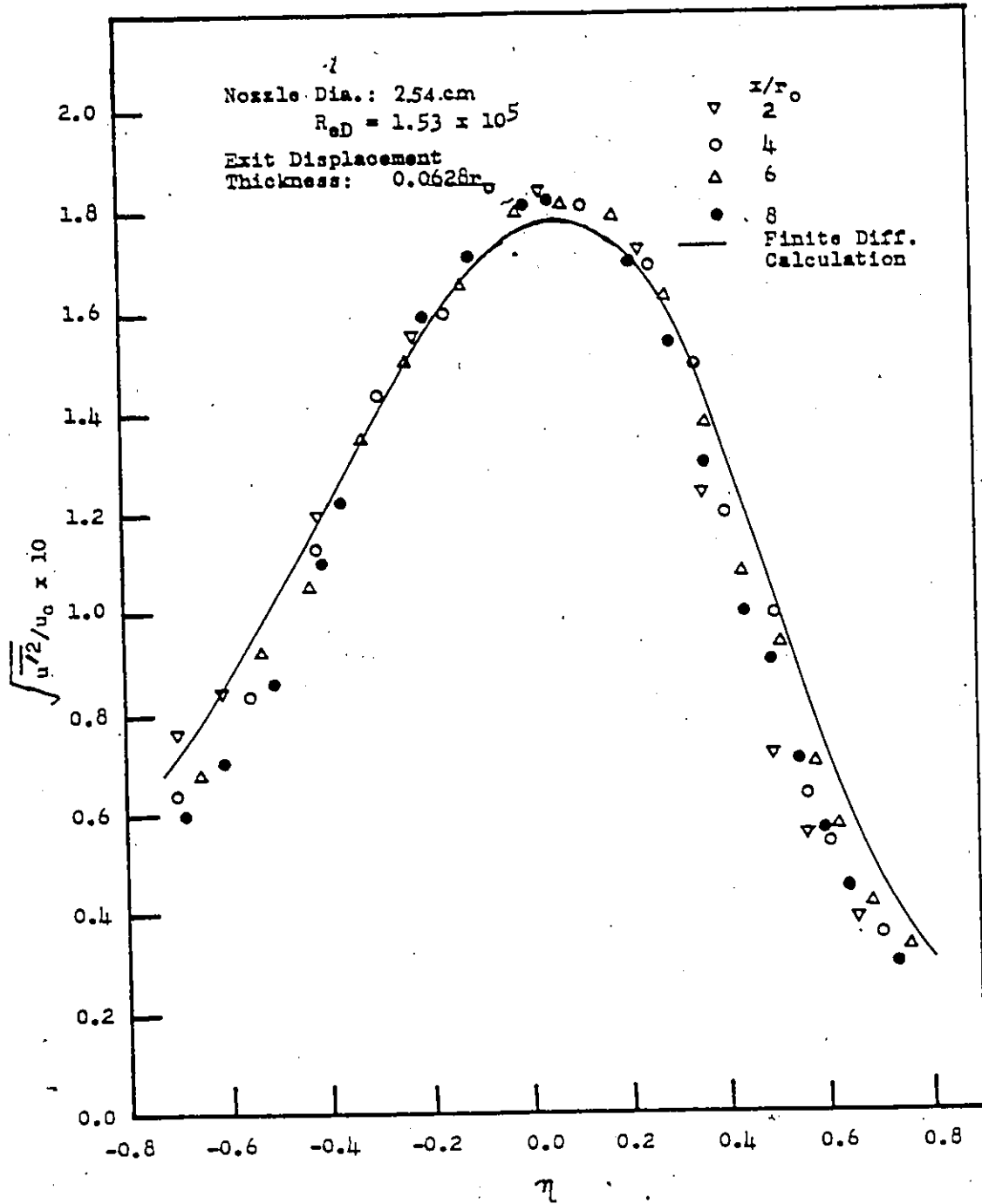
Fig.5.39

Calculated and Experimental Turbulent Axial

Intensity Distributions, Thick Boundary Layer.
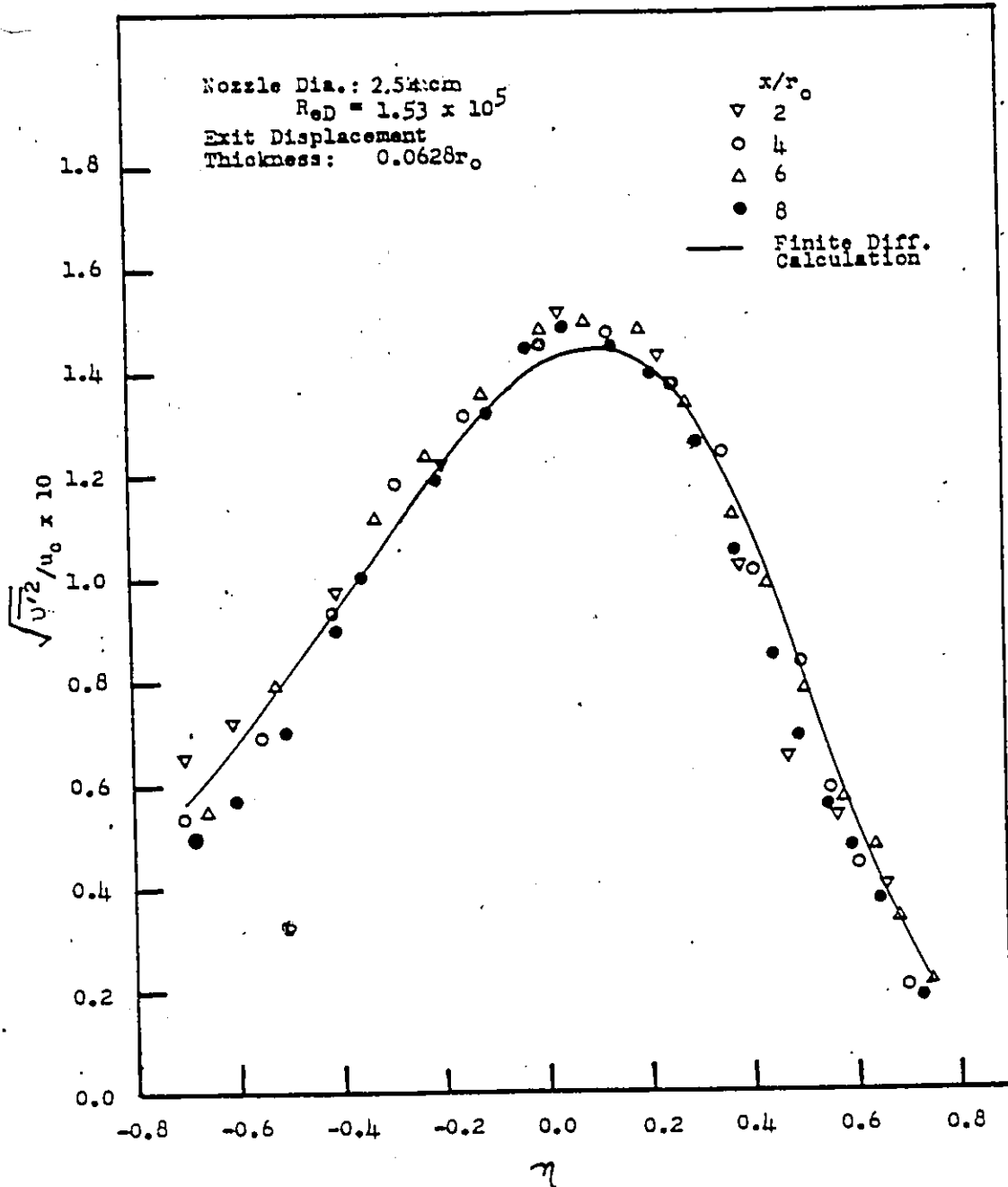
Fig.5.40

Calculated and Experimental Turbulent Radial

Intensity Distributions, Thick Boundary Layer.
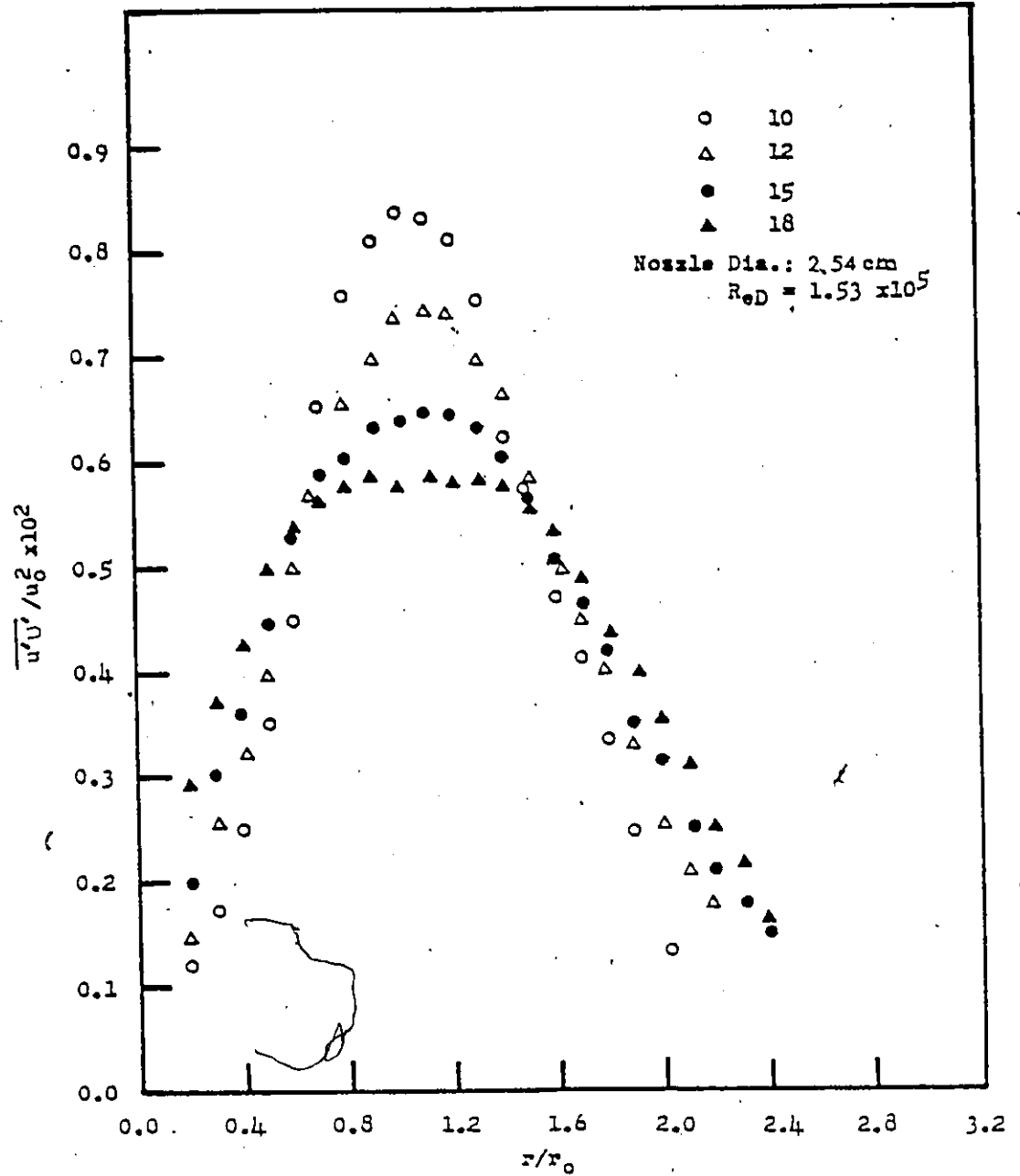
Fig.5.41

Experimental Turbulent Shear Stress Distributions
in the Transition Region.
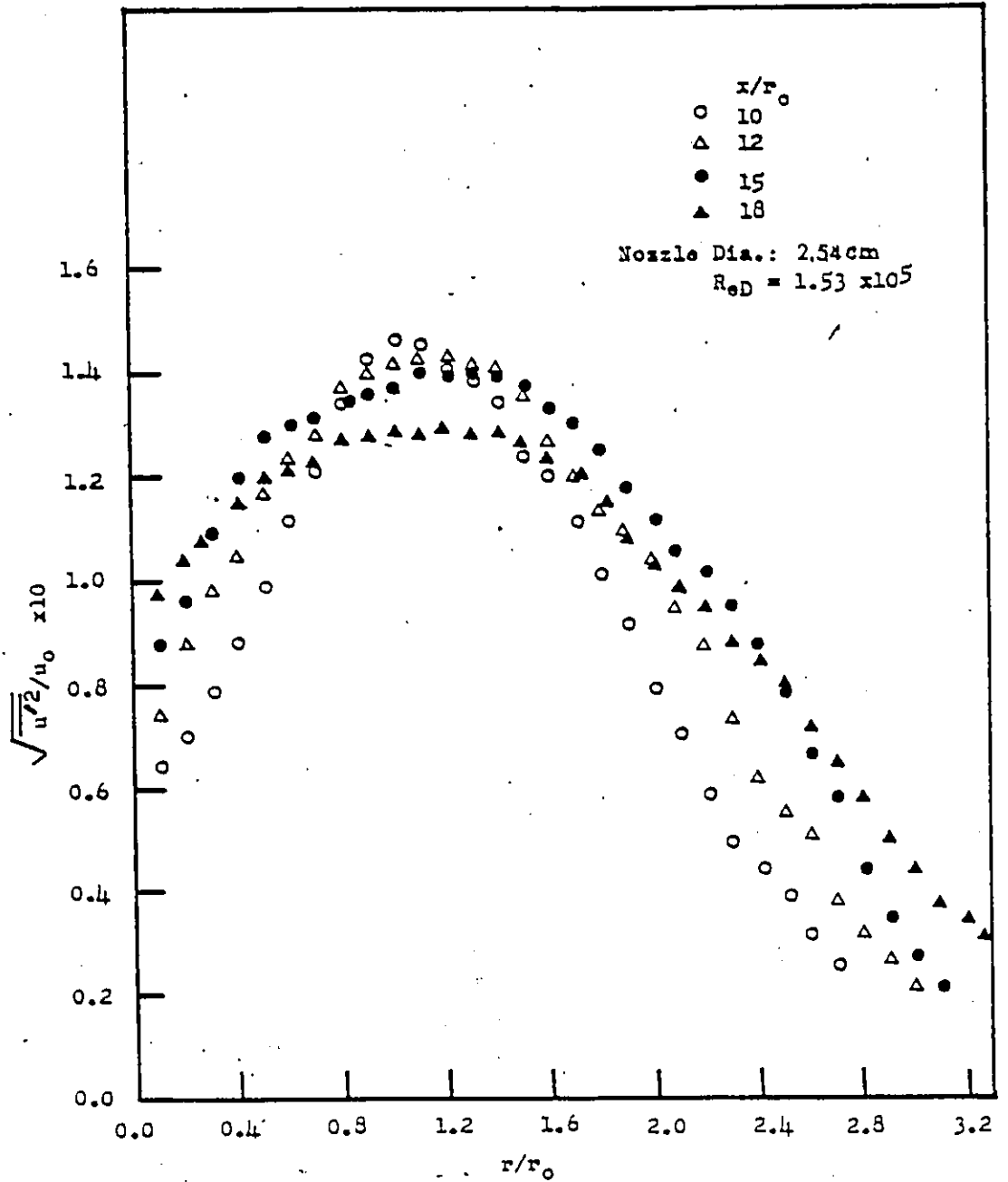
Fig.5.42

Experimental Turbulent x-Intensity Distributions
in the   Transition Region.
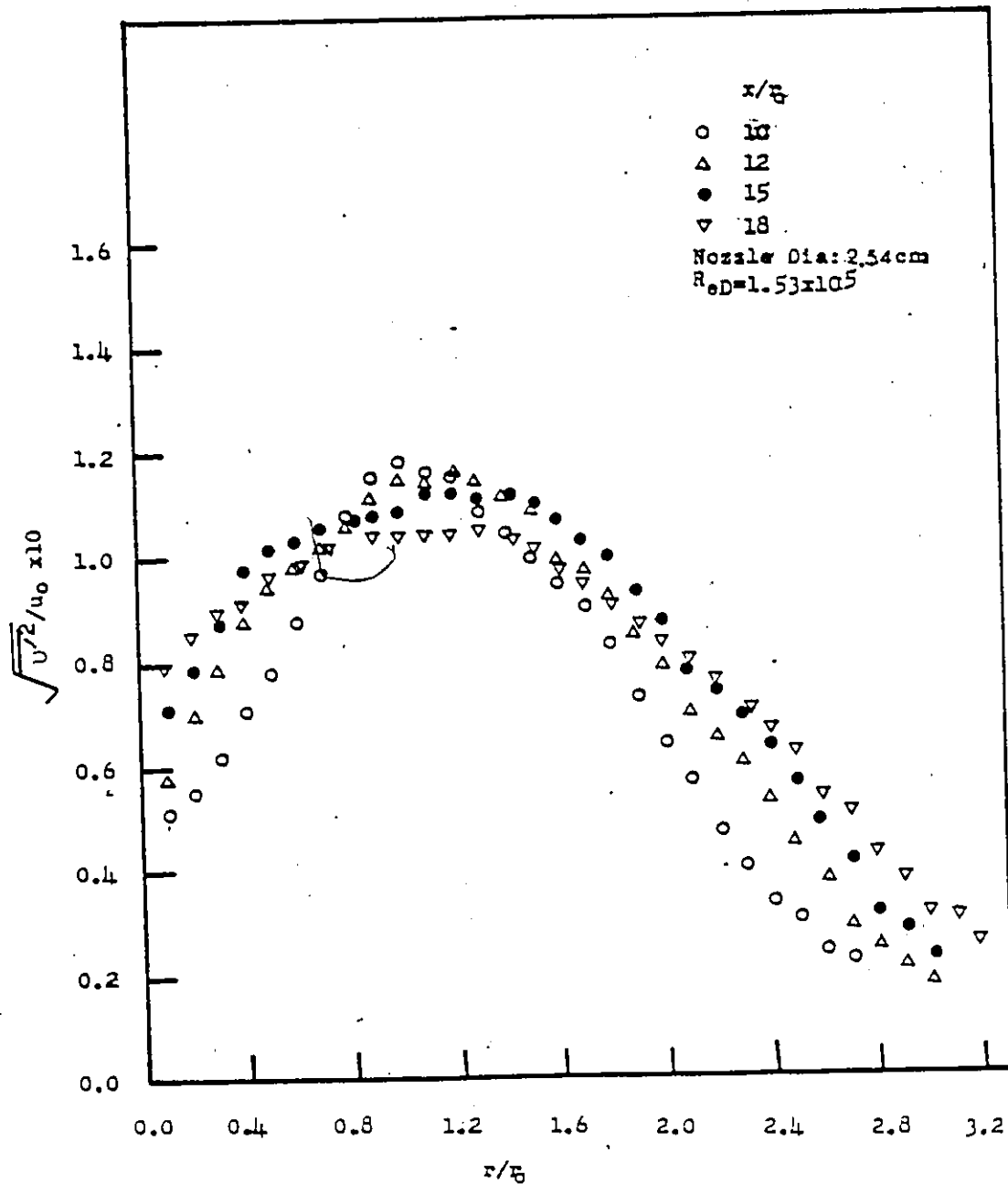
Fig.5.43

Experimental Turbulent r-Intensity Distributions
in the Transition Region.
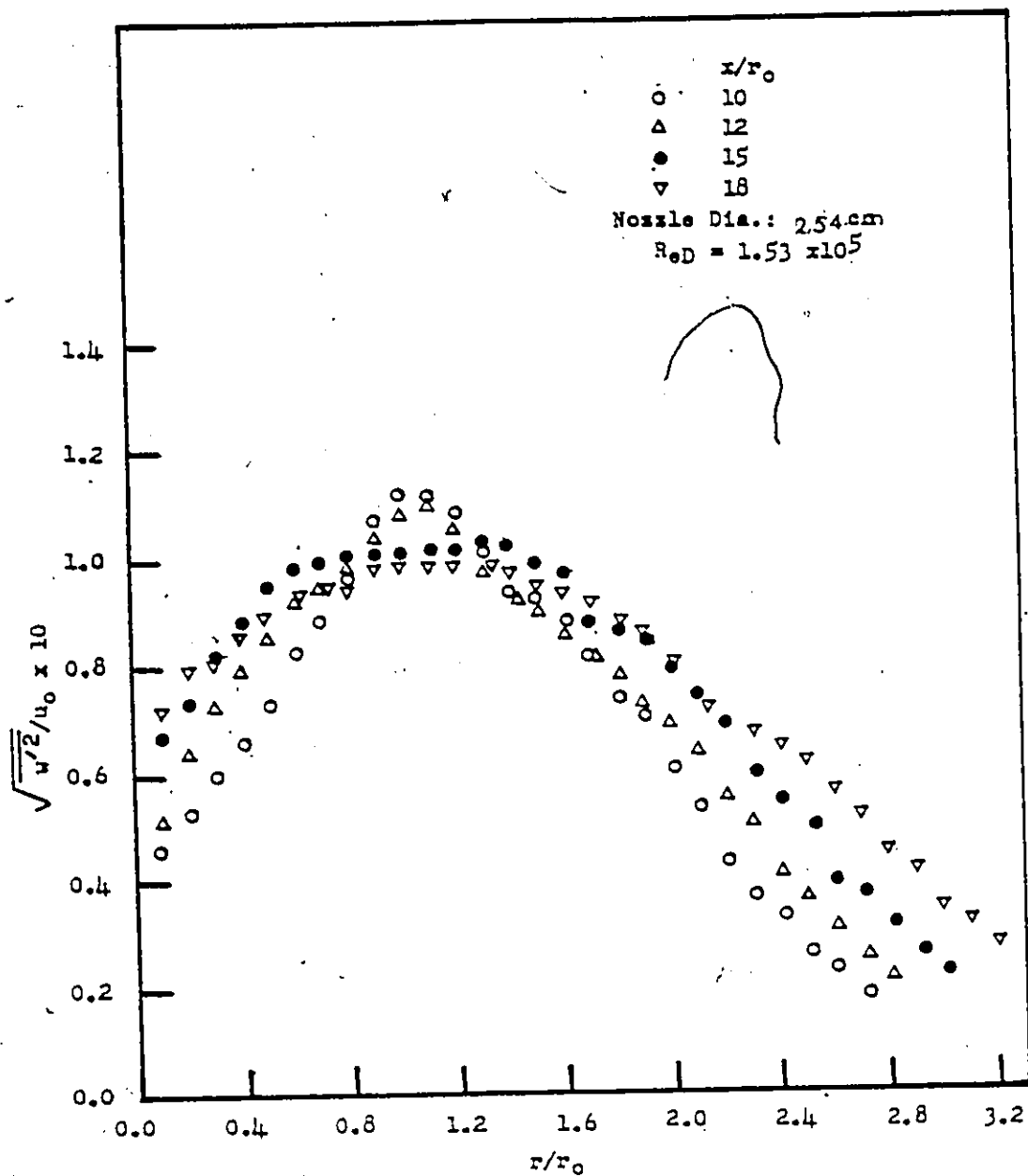
Fig.5.44

Experimental Turbulent s-Intensity Distributions

in the Transition Region.

APPENDICES

APPENDIX A

DERIVATION OF EQUATIONS

A.1 Differential Equations

Turbulent flow is governed by the continuity equation and the Navier-Stokes equations. In cylindrical co-ordinates the mass conservation equation for incompressible flow field is:

$$\frac{\partial u^o}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r\upsilon^o) + \frac{1}{r} \frac{\partial w^o}{\partial \theta} = 0 \qquad (A.1)$$

The co-ordinate system is given in Figure 1.1 and u^o, υ^o and w^o are instantaneous velocity components in x, r and θ directions respectively. Splitting instantaneous variables into mean and fluctuating components:

$$\left. \begin{array}{l} u^o = u + u' \\[6pt] \upsilon^o = \upsilon + \upsilon' \\[6pt] w^o = w' \\[6pt] p^o = p + p' \end{array} \right\} \qquad (A.2)$$

where u and υ are mean components and u', υ' and w' are the fluctuating components of the instantaneous velocity. The instantaneous pressure p^o has a mean and a fluctuating component, p and p', respectively.

Introducing Equation (A.2) in Equation (A.1):

$$\frac{\partial}{\partial x} (u+u') + \frac{1}{r} \frac{\partial}{\partial r} [r(\upsilon+\upsilon')] + \frac{1}{r} \frac{\partial w'}{\partial \theta} = 0 \qquad (A.3)$$

taking a time average,

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (\upsilon r) = 0 \qquad (A.4)$$

Subtracting Equation (A.4) from Equation (A.3),

$$\frac{\partial u'}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r\upsilon') + \frac{1}{r} \frac{\partial w'}{\partial \theta} = 0 \qquad (A.5)$$

Writing the Navier-Stokes equations in cylindrical co-ordinates for incompressible flow in absence of body forces [89],

x-component:
$$\rho \frac{Du^{O}}{Dt} = -\frac{\partial p^{O}}{\partial x} - \left[\frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rx}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{xx}}{\partial x} \right]$$

r-component:
$$\rho(\frac{D\upsilon^{O}}{Dt} - \frac{w^{O2}}{r}) = -\frac{\partial p^{O}}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} \right.$$

$$\left. - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rx}}{\partial x} \right]$$

θ-component:
$$\rho(\frac{Dw^{O}}{Dt} + \frac{\upsilon^{O}w^{O}}{r}) = -\frac{1}{r} \frac{\partial p^{O}}{\partial \theta} - \left[\frac{1}{r^{2}} \frac{\partial}{\partial r}(r^{2} \tau_{r\theta}) + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} \right.$$

$$\left. + \frac{\partial \tau_{\theta x}}{\partial x} \right]$$

$$(A.6)$$

All τ's are stresses on fluid elements and their expressions for incompressible flow with constant viscosity, μ, are:

$$\tau_{xx} = -2\mu \frac{\partial u^{O}}{\partial x} \; ; \quad \tau_{rr} = -2\mu \frac{\partial u^{O}}{\partial r} \; ; \quad \tau_{\theta\theta} = -\mu \left[\frac{2}{r} \frac{\partial w^{O}}{\partial \theta} + \frac{\upsilon^{O}}{r} \right]$$

$$\tau_{xr} = \tau_{rx} = -\mu \left[\frac{\partial u^{O}}{\partial r} + \frac{\partial \upsilon^{O}}{\partial x} \right] \; ; \quad \tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r}(\frac{w^{O}}{r}) + \frac{1}{r} \frac{\partial \upsilon^{O}}{\partial \theta} \right]$$

$$\text{and} \quad \tau_{\theta z} = \tau_{z\theta} = -\mu \left[\frac{\partial w^{O}}{\partial x} + \frac{1}{r} \frac{\partial u^{O}}{\partial \theta} \right] \qquad (A.7)$$

Using Equation (A.7) in Equation (A.6):

$$\rho \frac{Du^0}{Dt} = -\frac{\partial p^0}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u^0}{\partial \theta^2} + \frac{\partial^2 u^0}{\partial x^2}\right] \quad (A.8)$$

$$\rho\left(\frac{Dv^0}{Dt} - \frac{w^{0_2}}{r}\right) = -\frac{\partial p^0}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv^0)\right) + \frac{1}{r^2}\frac{\partial^2 v^0}{\partial \theta^2}\right.$$
$$\left. - \frac{2}{r^2}\frac{\partial w^0}{\partial \theta} + \frac{\partial^2 v^0}{\partial x^2}\right] \quad (A.9)$$

$$\rho\left(\frac{Dw^0}{Dt} + \frac{v^0 w^0}{r}\right) = -\frac{1}{r}\frac{\partial p^0}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rw^0)\right) + \frac{1}{r^2}\frac{\partial^2 w^0}{\partial \theta^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v^0}{\partial \theta} + \frac{\partial^2 w^0}{\partial x^2}\right]$$
$$(A.10)$$

Using Equation (A.2) in Equation (A.8)

$$\frac{\partial(u+u')}{\partial t} + (u+u')\frac{\partial}{\partial x}(u+u') + (v+v')\frac{\partial}{\partial r}(u+u') + \frac{w'}{r}\frac{\partial}{\partial \theta}(u+u')$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x}(p+p') + \nu\left[\frac{\partial^2}{\partial r^2}(u+u') + \frac{1}{r}\frac{\partial}{\partial r}(u+u')\right.$$

$$\left. + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(u+u') + \frac{\partial^2}{\partial x^2}(u+u')\right] \quad (A.11)$$

Time Averaging Equation (A.11)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} + v\frac{\partial u}{\partial r} + \overline{v'\frac{\partial u'}{\partial r}} + \overline{\frac{w'}{r}\frac{\partial u'}{\partial \theta}}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right]$$
$$(A.12)$$

Multiplying Equation (A.5) by u', and time averaging:

$$\overline{u' \frac{\partial u'}{\partial x}} + \overline{\frac{1}{r} \frac{\partial}{\partial r} (ru'\upsilon')} - \overline{\upsilon' \frac{\partial u'}{\partial r}} + \overline{\frac{1}{r} \frac{\partial}{\partial \theta} (w'u')} - \overline{\frac{w'}{r} \frac{\partial u'}{\partial \theta}} = 0$$

(A.13)

Adding Equation (A.13) and Equation (A.12), and rearranging:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \upsilon \frac{\partial u}{\partial r} = - \frac{\partial}{\partial x} (\frac{p}{\rho} + \overline{u'^2}) - \frac{1}{r} \frac{\partial}{\partial r} (r \overline{u'\upsilon'}) - \frac{1}{r} \frac{\partial}{\partial \theta} (\overline{w'\upsilon'})$$

$$+ \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2} \right] \qquad (A.14)$$

$\overline{u'^2}$, $\overline{u'\upsilon'}$ and $\overline{w'u'}$ are the Reynolds stresses. Miller and Comings [19] experimentally showed that $\overline{u'^2}$ and p/ρ are approximately equal and opposite for jet flow, i.e., $(p/\rho + \overline{u'^2}) \simeq 0$. For axisymmetry, $\frac{\partial}{\partial \theta} (\overline{w'u'}) = 0$, and $\frac{\partial^2 u}{\partial \theta^2} = 0$. Applying the boundary layer approximations, $\frac{\partial^2 u}{\partial x^2}$ has been neglected. For steady flow, Equation (A.14) becomes:

$$u \frac{\partial u}{\partial x} + \upsilon \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left[r(\nu \frac{\partial u}{\partial r} - \overline{u'\upsilon'}) \right],$$

or, $\qquad u \frac{\partial u}{\partial x} + \upsilon \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\tau}{\rho} \right] \qquad (A.15)$

where $\qquad \frac{\tau}{\rho} = \nu \frac{\partial u}{\partial r} - \overline{u'\upsilon'} \qquad (A.16)$

Subtracting Equation (A.12) from Equation (A.11):

$$\frac{\partial u'}{\partial t} + u\,\frac{\partial u'}{\partial x} + u'\,\frac{\partial u}{\partial x} + u'\,\frac{\partial u'}{\partial x} - \overline{u'\,\frac{\partial u'}{\partial x}} + \upsilon\,\frac{\partial u'}{\partial r} + \upsilon'\,\frac{\partial u}{\partial r} + \upsilon'\,\frac{\partial u'}{\partial r}$$

$$- \overline{\upsilon'\,\frac{\partial u'}{\partial r}} + \frac{w'}{r}\,\frac{\partial u}{\partial \theta} + \frac{w'}{r}\,\frac{\partial u'}{\partial \theta} - \overline{\frac{w'}{r}\,\frac{\partial u'}{\partial \theta}} = \frac{-1}{\rho}\,\frac{\partial p'}{\partial x}$$

$$+ \nu\left[\frac{\partial^2 u'}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u'}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u'}{\partial \theta^2} + \frac{\partial^2 u'}{\partial x^2}\right]$$

Multiplying by u', time averaging and rearranging:

$$\frac{\partial}{\partial t}\left(\frac{\overline{u'^2}}{2}\right) + u\,\frac{\partial}{\partial x}\left(\frac{\overline{u'^2}}{2}\right) + \upsilon\,\frac{\partial}{\partial r}\left(\frac{\overline{u'^2}}{2}\right) + \overline{u'^2}\,\frac{\partial u}{\partial x} + \overline{u'\upsilon'}\,\frac{\partial u}{\partial r}$$

$$+ \frac{\overline{u'w'}}{r}\,\frac{\partial u}{\partial \theta} + \overline{u'^2\,\frac{\partial u'}{\partial x}} + \overline{u'\upsilon'\,\frac{\partial u'}{\partial r}} + \frac{\overline{u'w'}}{r}\,\frac{\partial u'}{\partial \theta}$$

$$= \frac{-1}{\rho}\,\frac{\partial p'}{\partial x} + \nu\left[\overline{u'^2\frac{\partial^2 u'}{\partial r^2}} + \frac{1}{r}\,\overline{u'\,\frac{\partial u'}{\partial r}} + \frac{\overline{u'}}{r^2}\,\frac{\partial^2 u'}{\partial \theta^2} + \overline{u'\,\frac{\partial^2 u'}{\partial x^2}}\right]$$

$$\text{(A.17)}$$

Multiplying Equation (A.5) by u'^2, time averaging and arranging:

$$\overline{u'^2\,\frac{\partial u'}{\partial x}} + \overline{\frac{u'}{r}\,\frac{\partial}{\partial r}\,(ru'\upsilon')} - \overline{u'\upsilon'\,\frac{\partial u'}{\partial r}}$$

$$+ \overline{\frac{u'}{r}\,\frac{\partial}{\partial \theta}\,(u'w')} - \overline{\frac{w'u'}{r}\,\frac{\partial u'}{\partial \theta}} = 0 \qquad \text{(A.18)}$$

Adding Equation (A.18) and Equation (A.17) and rearranging:

$$\frac{\partial}{\partial t}\left(\frac{\overline{u'^2}}{2}\right) + u\frac{\partial}{\partial x}\left(\frac{\overline{u'^2}}{2}\right) + \upsilon\frac{\partial}{\partial r}\left(\frac{\overline{u'^2}}{2}\right) = -\left[\overline{u'^2}\frac{\partial u}{\partial x} + \overline{u'\upsilon'}\frac{\partial u}{\partial r}\right.$$

$$\left.+ \frac{\overline{u'w'}}{r}\frac{\partial u}{\partial \theta}\right] - \overline{u'\frac{\partial}{\partial x}\left(\frac{p'}{\rho} + \frac{u'^2}{2}\right)} - \overline{\upsilon'\frac{\partial}{\partial r}\left(\frac{u'^2}{2}\right)} - \overline{\frac{w'}{r}\frac{\partial}{\partial \theta}\left(\frac{u'^2}{2}\right)}$$

$$- \overline{u'^2\left\{\frac{\partial u'}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\upsilon') + \frac{1}{r}\frac{\partial w'}{\partial \theta}\right\}} + \nu\left[\frac{\partial^2}{\partial r^2}\left(\frac{\overline{u'^2}}{2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\overline{u'^2}}{2}\right)\right.$$

$$\left.+ \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\overline{u'^2}}{2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\overline{u'^2}}{2}\right) - \overline{\left(\frac{\partial u'}{\partial r}\right)^2} - \overline{\left(\frac{1}{r}\frac{\partial u'}{\partial \theta}\right)^2} - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right]$$

(A.19)

Using continuity Equation (A.5), and rearranging:

$$\frac{D}{Dt}\left(\frac{\overline{u'^2}}{2}\right) = -\left[\overline{u'^2}\frac{\partial u}{\partial x} + \overline{u'\upsilon'}\frac{\partial u}{\partial r} + \frac{\overline{u'w'}}{r}\frac{\partial u}{\partial \theta}\right] - \overline{u'\frac{\partial}{\partial x}\left(\frac{p'}{\rho} + \frac{u'^2}{2}\right)}$$

$$- \overline{\upsilon'\frac{\partial}{\partial r}\left(\frac{u'^2}{2}\right)} - \overline{\frac{w'}{r}\frac{\partial}{\partial \theta}\left(\frac{u'^2}{2}\right)} + \nu\left[\frac{\partial^2}{\partial r^2}\left(\frac{\overline{u'^2}}{2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\overline{u'^2}}{2}\right)\right.$$

$$\left.+ \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\overline{u'^2}}{2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\overline{u'^2}}{2}\right)\right] - \nu\left[\overline{\left(\frac{\partial u'}{\partial r}\right)^2} + \overline{\left(\frac{1}{r}\frac{\partial u'}{\partial \theta}\right)^2} + \overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right]$$

(A.20)

Similar mathematical operations in Equations (A.9) and (A.10) yield:

$$\frac{D}{Dt}\left(\frac{\overline{\upsilon'^2}}{2}\right) = -\left[\overline{u'\upsilon'}\,\frac{\partial \upsilon}{\partial x} + \overline{\upsilon'^2}\,\frac{\partial \upsilon}{\partial r} + \frac{\overline{w'\upsilon'}}{r}\,\frac{\partial \upsilon}{\partial \theta}\right] - \overline{\upsilon'\,\frac{\partial}{\partial r}\left(\frac{p'}{\rho} + \frac{\upsilon'^2}{2}\right)}$$

$$+ \frac{\overline{w'^2\upsilon'}}{r} - \overline{u'\,\frac{\partial}{\partial x}\left(\frac{\upsilon'^2}{2}\right)} - \overline{\frac{w'}{r}\,\frac{\partial}{\partial \theta}\left(\frac{\upsilon'^2}{2}\right)} + \nu\left[\frac{\partial^2}{\partial r^2}\left(\frac{\overline{\upsilon'^2}}{2}\right) + \frac{1}{r}\,\frac{\partial}{\partial r}\left(\frac{\overline{\upsilon'^2}}{2}\right)\right.$$

$$+ \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\left(\frac{\overline{\upsilon'^2}}{2}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\overline{\upsilon'^2}}{2}\right)\right] - \nu\left[\overline{\left(\frac{\partial \upsilon'}{\partial r}\right)^2} + \overline{\left(\frac{1}{r}\,\frac{\partial \upsilon'}{\partial \theta}\right)^2} + \overline{\left(\frac{\partial \upsilon'}{\partial x}\right)^2}\right.$$

$$+ \frac{\overline{\upsilon'^2}}{r^2} + \overline{\frac{2\upsilon'}{r^2}\,\frac{\partial w'}{\partial \theta}}\right] \qquad\qquad (A.21)$$

and,

$$\frac{D}{Dt}\left(\frac{\overline{w'^2}}{2}\right) = -\,\overline{w'^2}\,\frac{\upsilon}{r} - \overline{\frac{w'}{r}\,\frac{\partial}{\partial \theta}\left(\frac{p'}{\rho} + \frac{w'^2}{2}\right)} - \frac{\overline{\upsilon'w'^2}}{r} - \overline{u'\,\frac{\partial}{\partial x}\left(\frac{w'^2}{2}\right)}$$

$$- \overline{\upsilon'\,\frac{\partial}{\partial r}\left(\frac{w'^2}{2}\right)} + \nu\left[\frac{\partial^2}{\partial r^2}\left(\frac{\overline{w'^2}}{2}\right) + \frac{1}{r}\,\frac{\partial}{\partial r}\left(\frac{\overline{w'^2}}{2}\right) + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}\left(\frac{\overline{w'^2}}{2}\right)\right.$$

$$+ \frac{\partial^2}{\partial x^2}\left(\frac{\overline{w'^2}}{2}\right)\right] - \nu\left[\overline{\left(\frac{\partial w'}{\partial r}\right)^2} + \overline{\left(\frac{1}{r}\,\frac{\partial w'}{\partial \theta}\right)^2} + \overline{\left(\frac{\partial w'}{\partial x}\right)^2} + \frac{\overline{w'^2}}{r} - \overline{\frac{2w'}{r^2}\,\frac{\partial \upsilon'}{\partial \theta}}\right]$$

$$(A.22)$$

Equations (A.20), (A.21) and (A.22) are Reynolds normal stress transport differential equations. The turbulent kinetic energy equation is formed by adding Equations (A.20), (A.21) and (A.22). For steady flow, the turbulent kinetic energy equation is:

$$u \frac{\partial k}{\partial x} + \upsilon \frac{\partial k}{\partial r} = -\left[\overline{u'^2} \frac{\partial u}{\partial x} + \overline{u'\upsilon'} \left(\frac{\partial u}{\partial r} + \frac{\partial \upsilon}{\partial x}\right) + \frac{\overline{u'w'}}{r} \frac{\partial u}{\partial \theta}\right.$$

$$+ \frac{\overline{w'\upsilon'}}{r} \frac{\partial \upsilon}{\partial \theta} + \overline{w'^2} \left.\frac{\upsilon}{r}\right] - \upsilon\left[\left(\overline{\frac{\partial u'}{\partial x}}\right)^2 + \left(\overline{\frac{\partial \upsilon'}{\partial x}}\right)^2 + \left(\overline{\frac{\partial w'}{\partial x}}\right)^2 + \left(\overline{\frac{\partial u'}{\partial r}}\right)^2\right.$$

$$+ \left(\overline{\frac{\partial \upsilon'}{\partial r}}\right)^2 + \left(\overline{\frac{\partial w'}{\partial r}}\right)^2 + \left(\overline{\frac{1}{r}\frac{\partial u'}{\partial \theta}}\right)^2 + \left(\overline{\frac{1}{r}\frac{\partial \upsilon'}{\partial \theta} - \frac{w'}{r}}\right)^2 + \left.\left(\overline{\frac{1}{r}\frac{\partial w'}{\partial \theta} + \frac{\upsilon'}{r}}\right)^2\right]$$

$$-\left[\frac{\partial}{\partial x} \overline{\{u'(\frac{p'}{\rho} + k)\}} + \frac{1}{r}\frac{\partial}{\partial r} \overline{\{\upsilon'r.(\frac{p'}{\rho} + k)\}} + \frac{1}{r}\frac{\partial}{\partial \theta} \overline{\{w'(\frac{p'}{\rho} + k)\}}\right.$$

$$\left.- \upsilon \left\{\frac{\partial^2 k}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r \frac{\partial k}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 k}{\partial \theta^2}\right\}\right] \hspace{2cm} \text{(A.23)}$$

where $k = (\overline{u'^2} + \overline{\upsilon'^2} + \overline{w'^2})/2$.

A.2 Integral Equations

Differential form of the mass and momentum conservation equations for the mean flow are given by Equations (A.4) and (A.15). Integral equations may be formed by taking integrals of Equation (A.15) over a control volume and using mass conservation Equation (A.4).

A:2.1 Momentum Integral Equation

Multiplying Equation (A.4) by u and rearranging,

$$u \frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(ru\upsilon) - \upsilon \frac{\partial u}{\partial r} = 0 \hspace{2cm} \text{(A.24)}$$

Adding Equation (A.24) and Equation (A.15) and arranging,

$$\frac{\partial}{\partial x}(ru^2) + \frac{\partial}{\partial r}(ru\upsilon) = \frac{\partial}{\partial r}(r \tau/\rho) \hspace{2cm} \text{(A.25)}$$

For axisymmetric jet υ and τ are equal to zero at r=0 and ∞. Integrating Equation (A.25) over a control volume with r=0 to ∞,

$$\frac{\partial}{\partial x} \int_0^\infty ru^2 \, \partial r = 0 \qquad (A.26)$$

A.2.2 Mean Energy Integral Equation

Multiplying Equation (A.15) by u,

$$u^2 \frac{\partial u}{\partial x} + u\upsilon \frac{\partial u}{\partial r} = \frac{u}{r} \frac{\partial}{\partial r} (r \, \tau/\rho) \qquad (A.27)$$

Multiplying Equation (A.4) by u^2, and arranging,

$$\frac{u^2}{2} \frac{\partial u}{\partial x} + \frac{1}{2r} \frac{\partial}{\partial r} (u^2 \upsilon r) - u\upsilon \frac{\partial u}{\partial r} = 0 \qquad (A.28)$$

Adding Equation (A.27) and Equation (A.28) and arranging,

$$\frac{1}{2} \frac{\partial}{\partial x} (ru^3) + \frac{1}{2} \frac{\partial}{\partial r} (u^2 \upsilon r) = \frac{\partial}{\partial r} (ur \, \tau/\rho) - \tau/\rho (r \frac{\partial u}{\partial r}) \qquad (A.29)$$

For axisymmetric jets, υ and τ are equal to zero at r=0 and ∞. Integrating Equation (A.29) over a control volume in the limit r=0 to ∞,

$$\frac{1}{2} \frac{\partial}{\partial x} \int_0^\infty ru^3 \partial r = - \int_0^\infty (r \frac{\partial u}{\partial r}) \, \tau/\rho \, \partial r \qquad (A.30)$$

Using Equation (A.16) for τ/ρ in Equation (A.30),

$$\frac{1}{2} \frac{\partial}{\partial x} \int_0^\infty ru^3 \partial r = - \int_0^\infty (r \frac{\partial u}{\partial r}) (\nu \frac{\partial u}{\partial r} - \overline{u'\upsilon'}) \partial r \qquad (A.31)$$

$-\rho \, \overline{u'\upsilon'}$ is turbulent shear stress and $\rho\nu \frac{\partial u}{\partial r}$ represents shear stress due to molecular motion. Using $\sqrt{k}$ and L^1 as the velocity and length scales of the turbulence, the shear stress
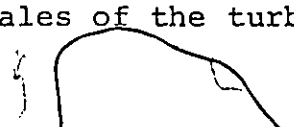
magnitude may be represented by:

$$\overline{u'v'} \sim \sqrt{k}\sqrt{k}$$

or

$$\overline{u'v'} \sim k \qquad (A.32)$$

$$\nu\frac{\partial u}{\partial r} \sim \nu\frac{\sqrt{k}}{L^1}$$

or

$$\nu\frac{\partial u}{\partial r} \sim \frac{k}{(\sqrt{k}L^1)/\nu} \qquad (A.33)$$

Comparing Equation (A.32) and Equation (A.33), $\overline{u'v'}$ is
(kL^1/ν) times greater than $\nu(\partial u/\partial r)$. For high turbulence
Reynolds numbers, ($\sqrt{k}L/\nu$), the viscous-shear stress is negli-
gible. Expressing turbulent shear stress $-\rho\,\overline{u'v'}$ by Prandtl's
mixing length model:

$$-\rho\,\overline{u'v'} = \rho\nu_t\frac{\partial u}{\partial r} \qquad (A.34)$$

where,

$$\nu_t = L^2\left|\frac{\partial u}{\partial r}\right| \qquad (A.35)$$

Neglecting molecular shear stress and using Equation (A.34) in
Equation (A.31), the energy integral equation becomes:

$$\frac{1}{2}\frac{\partial}{\partial x}\int_0^\infty u^3 r\,\partial r = -\int_0^\infty \nu_t\, r\,\left(\frac{\partial u}{\partial r}\right)^2 \partial r \qquad (A.36)$$

APPENDIX B

CALCULATION BY THE INTEGRAL METHOD

B.1 Integral Calculations

The empirical velocity profile obtained by fitting a curve to the experimental measurements is:

$$\frac{u}{u_c} = 1.0 \qquad\qquad ; \qquad\qquad \eta \leqslant -0.736$$

$$\frac{u}{u_c} = 0.5 - \eta + 0.05\eta^2 + 0.66\eta^3 \quad ; \quad -.736 \leqslant \eta \leqslant 0.5$$

$$\frac{u}{u_c} = 0.095\left[1.0 - \tanh 4.79(\eta - .5)\right] ; \qquad \eta \geqslant 0.5$$

$$(B.1)$$

The momentum integral equation is obtained using the boundary layer approximation:

$$\frac{\partial}{\partial x} \int_0^\infty u^2 \, r\partial r = 0$$

where x is axial coordinate and r is radial coordinate as shown in Figure 1.1.

The momentum equation can be written as:

$$\frac{\partial}{\partial x}\left[\int_0^{r_1} u^2 \, r\partial r + \int_{r_1}^\infty u^2 \, r\partial r\right] = 0$$

In the potential core $0 \leqslant r \leqslant r_1$, $u = u_c = $ constant, so the above equation becomes:

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{r_1}^\infty \left(\frac{u}{u_c}\right)^2 r\partial r\right] = 0 \qquad\qquad (B.2)$$

A new variable, η, is defined as:

$$\eta = \frac{r - r_{\frac{1}{2}}}{b}$$

where $r_{\frac{1}{2}}$ is the radial distance for $u/u_c = 0.5$, and

$$b/r_0 = \left[\frac{\partial r/r_0}{\partial u/u_c}\right]_{r=r_{\frac{1}{2}}} ,$$

Transforming Equation (B.2) into variable η:

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{\eta_1}^{\eta_2} (r_{\frac{1}{2}} + \eta b) b (u/u_c)^2 \partial\eta\right] = 0 \qquad (B.3)$$

where,

$$\eta_1 = \frac{r_1 - r_{\frac{1}{2}}}{b} \text{ and } \eta_2 = \frac{r_2 - r_{\frac{1}{2}}}{b} .$$

The energy equation in integral form is also obtained by using boundary layer approximation; it is:

$$\frac{1}{2}\frac{\partial}{\partial x}\int_0^\infty ru^3\partial r - L^2\int_0^\infty r\left(\frac{\partial u}{\partial r}\right)^3\partial r = 0$$

where L is Prandtl's mixing length,

$$\frac{1}{2}\frac{\partial}{\partial x}\left[\int_0^{r_1} ru^3\partial r + \int_{r_1}^{r_2} ru^3\partial r\right]$$

or,

$$- L^2\left[\int_0^{r_1} r\left(\frac{\partial u}{\partial r}\right)^3\partial r + \int_{r_1}^{r_2} r\left(\frac{\partial u}{\partial r}\right)^3\partial r\right] = 0$$

In core $0 \leq r \leq r_1$, $u = u_c$ and $\partial u/\partial r = 0$, so the above equation becomes:

$$\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{r_1}^{r_2} r(u/u_c)^3\partial r\right] - L^2\int_{r_1}^{r_2} r\left(\frac{\partial u/u_c}{\partial r}\right)^3\partial r = 0$$

Transforming into the variable, $\eta = \dfrac{r-r_{\frac{1}{2}}}{b}$:

$$\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{\eta_1}^{\eta_2}(r_{\frac{1}{2}}b+\eta b^2)(u/u_c)^2\partial\eta\right]$$

$$\dot{} \; (\frac{L}{b})^2\int_{\eta_1}^{\eta_2}(r_{\frac{1}{2}}+\eta b)\left(\frac{\partial(u/u_c)}{\partial\eta}\right)^3\partial\eta = 0 \quad (B.4)$$

$\eta_1 = -0.736$ which represents the edge of the core, and η_2 is chosen to be 0.8 where $u/u_c = 0.01$. This choice is made only to have easy access to the integration of higher powers of the product of η and tanh $4.79(\eta-\eta_o)$.

B.2 Momentum Integral Equation

Using velocity profile (B.1) in Equation (B.3) and using the limits for η,

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{-.736}^{0.5}(r_{\frac{1}{2}}b+\eta b^2)(0.5 - \eta + 0.5\eta^2 + .66\eta^3)^2\partial\eta\right.$$

$$\left. + \int_{0.5}^{0.8}(r_{\frac{1}{2}}b+\eta b^2)\{.095(1.0 - \tanh 4.79(\eta-0.5))\}^2\partial\eta\right] = 0$$

or,

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + r_{\frac{1}{2}}b\underbrace{\int_{-.736}^{0.5}(.5 - \eta + 0.5\eta^2 + .66\eta^3)^2\partial\eta}_{\text{1st Integral}}\right.$$

$$+ b^2\underbrace{\int_{-.736}^{0.5}\eta(.5 - \eta + .05\eta^2 + .66\eta^3)^2\partial\eta}_{\text{2nd Integral}}$$

$$+ r_{\frac{1}{2}}b\underbrace{\int_{0.5}^{0.8}\{.095(1 - \tanh 4.79(\eta-.5))\}^2\partial\eta}_{\text{3rd Integral}}$$

$$\left.+ b^2\underbrace{\int_{0.5}^{0.8}\eta\{.095(1 - \tanh 4.79(\eta-.5))\}^2\partial\eta}_{\text{4th Integral}}\right] = 0 \quad (B.5)$$

B.2.1 Calculation of Integrals of Equation (B.5)

$$\text{1st Integral} = \int_{-.736}^{0.5} (.5-\eta+.05\eta^2+.66\eta^3)\,\partial\eta$$

$$= \int_{-.736}^{0.5} \Big[.25-\eta+1.05\eta^2+.56\eta^3-1.3175\eta^4 \\ + .066\eta^5+.4356\eta^6\Big]\,\partial\eta$$

$$= 0.54685577$$

$$\text{2nd Integral} = \int_{-.736}^{0.5} \eta(0.5-\eta+0.05\eta^2+.66\eta^3)\,\partial\eta$$

$$= \int_{-.736}^{0.5} \Big[0.25\eta-\eta^2+1.05\eta^3+.56\eta^4+1.37175\eta^5 \\ + .066\eta^6+.4356\eta^7\Big]\,\partial\eta$$

$$= -0.21578334$$

$$\text{3rd Integral} = \int_{0.5}^{0.8} \Big[.095(1-\tanh 4.79(\eta-.5))\Big]^2\,\partial\eta$$

$$= (.095)^2 \int_{0.5}^{0.8} \Big[1-2\,\tanh 4.79(\eta-.5) \\ + \tanh^2 4.79(\eta-.5)\Big]\,\partial\eta$$

$$= (.095)^2 \int_{0.5}^{0.8} \Big[2-2\,\tanh 4.79(\eta-.5) \\ - \text{sech}^2 4.79(\eta-.5)\Big]\,\partial\eta$$

$$= (.095)^2 \Big[2\eta- \frac{2}{4.79}\,\ell n\,\cosh 4.79(\eta-.5) \\ - \frac{1}{4.79}\,\tanh 4.79(\eta-.5)\Big]_{.5}^{.8}$$

$$= 0.00072225$$

$$\text{4th Integral} = \int_{0.5}^{0.8} \eta \left[.095(1-\tanh 4.79(\eta-.5))\right]^2 \partial\eta$$

$$= (.095)^2 \int_{0.5}^{0.8} \left[\eta - 2\eta \tanh 4.79(\eta-.5)\right.$$

$$\left. + \eta\tanh^2 4.79(\eta-.5)\right]\partial\eta$$

$$= (.095)^2\left[\frac{\eta^2}{2} - 2\int\eta\tanh 4.79(\eta-.5) \partial\eta\right.$$

$$\left. + \int(\eta-\eta\text{sech}^2 4.79(\eta-.5))\partial\eta\right]_{0.5}^{0.8}$$

$$= (.095)^2\left[\eta^2 - 2\int\eta\tanh 4.79(\eta-.5)\partial\eta\right.$$

$$\left. - \int\eta\text{sech}^2 4.79(\eta-.5)\partial\eta\right]_{0.5}^{0.8}$$

$$= (.095)^2\left[\eta^2 - 2\int\eta\tanh 4.79(\eta-.5)\partial\eta\right.$$

$$- \frac{\eta}{4.79} \tanh 4.79(\eta-.5)$$

$$\left. + \frac{1}{4.79}\int\tanh 4.79(\eta-.5)\partial\eta\right]_{0.5}^{0.8}$$

$$= (.095)^2\left[\eta^2 - 2\int\eta\tanh 4.79(\eta-.5)\partial\eta\right.$$

$$- \frac{\eta}{4.79} \tanh 4.79(\eta-.5)$$

$$\left. + (\frac{1}{4.79})^2 \ln\cosh 4.79(\eta-.5)\right]_{0.5}^{0.8}$$

$$= (.095)^2\left[\eta^2 - \frac{2}{4.79}\int 4.79(\eta-.5)\tanh 4.79(\eta-.5)\partial\eta\right.$$

$$- 2\cdot 0.5\int\tanh 4.79(\eta-.5)\partial\eta$$

$$- \frac{\eta}{4.79} \tanh 4.79(\eta-.5)$$

$$\left. + (\frac{1}{4.79})^2 \ln\cosh 4.79(\eta-.5)\right]_{0.5}^{0.8}$$

4th Integral $= (.095)^2 \left[\eta^2 - \dfrac{2}{(4.79)^2} \int \{ (4.79(\eta-.5))^2 - \dfrac{(4.79(\eta-.5))^4}{3} \right.$

$\qquad + \dfrac{2}{15}(4.79(\eta-.5))^6 - \dfrac{17}{315}(4.79(\eta-.5))^8$

$\qquad + \dfrac{62}{2835}(4.79(\eta-.5))^{10} - \ldots \} \partial \{4.79(\eta-\eta_o)\}$

$\qquad - \dfrac{1}{4.79}\ell n \cosh 4.79(\eta-.5) - \dfrac{\eta}{4.79}\tanh 4.79(\eta-.5)$

$\qquad \left. + (\dfrac{1}{4.79})^2 \ell n \cosh 4.79(\eta-.5) \right]_{0.5}^{0.8}$

$= (.095)^2 \left[\eta^2 - \dfrac{2}{(4.79)^2} \{ \dfrac{(4.79(\eta-.5))^3}{3} - \dfrac{(4.79(\eta-.5))^5}{15} \right.$

$\qquad + \dfrac{2}{105}(4.79(\eta-.5))^7 - \dfrac{17}{2835}(4.79(\eta-.5))^9$

$\qquad + \dfrac{62}{31185}(4.79(\eta-.5))^{11} - \ldots \}$

$\qquad - \dfrac{1}{4.79}\ell n \cosh 4.79(\eta-.5)$

$\qquad - \dfrac{\eta}{4.79} \tanh 4.79(\eta-.5)$

$\qquad \left. + \dfrac{1}{(4.79)^2}\ell n \cosh 4.79(\eta-.5) \right]_{0.5}^{0.8}$

$= 0.00037526$

Substituting the values of the integrals into Equation (B.5):

$$\frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + 0.547578\ r_{\frac{1}{2}}b - 0.215408b^2 \right] = 0 \qquad \text{(B.6)}$$

This is a first order differential equation with starting condition at x=0 is $r_1=r_o$ and b=0; integrating:

$$\frac{r_1^2}{2} + .547578\ r_{\frac{1}{2}}b - .215408\ b^2 = \frac{r_o^2}{2} \qquad \text{(B.7)}$$

Since,

$$\eta_1 = -.736 = \frac{r_1 - r_{\frac{1}{2}}}{b}$$

$$r_1 = r_{\frac{1}{2}} - 0.736b \qquad\qquad (B.8)$$

Substituting (B.8) into Equation (B.7):

$$\frac{(r_{\frac{1}{2}} - .736b)^2}{2} + .547578\ r_{\frac{1}{2}}b - .215408b^2 = \frac{r_o^2}{2}$$

or

$$r_{\frac{1}{2}}^2 - 0.37684397\ r_{\frac{1}{2}}b + 0.05544\ b^2 = \frac{r_o^2}{2}$$

or

$$r_{\frac{1}{2}} = 0.18842198b \pm \sqrt{r_o^2 - .075377b^2}$$

Since at b=0, $r_{\frac{1}{2}}$ can never be negative, the positive sign is included with the following relationship:

$$\frac{r_{\frac{1}{2}}}{r_o} = 0.18842198\ b/r_o + \sqrt{1.0 - 0.075377(b/r_o)^2} \quad (B.9)$$

B.3 Energy Integral Equations

Using velocity profile (B.1) in Equation (B.4), we get:

$$\frac{1}{2} \frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + \int_{-.736}^{0.5} (r_{\frac{1}{2}} + \eta b^2)(0.5 - \eta + 0.5\eta^2 + .66\eta^3)^3 \partial\eta \right.$$

$$+ \int_{0.5}^{0.8} (r_{\frac{1}{2}}b + \eta b)\{.095(1 - \tanh 4.79(\eta-.5))\}^3 \partial\eta \Big]$$

$$- \left(\frac{L}{b}\right)^2 \left[\int_{-.736}^{0.5} (r_{\frac{1}{2}} + \eta b)(-1 + 0.1\eta + 1.98\eta^2)^3 \partial\eta \right.$$

$$+ \int_{0.5}^{0.8} (r_{\frac{1}{2}} + \eta b)\{-0.455\operatorname{sech}^2 4.79(\eta - 0.5)\}^3 \partial\eta \Big] = 0$$

or,

$$\frac{1}{2} \frac{\partial}{\partial x}\left[\frac{r_1^2}{2} + r_{\frac{1}{2}}b \underbrace{\int_{-.736}^{0.5} (0.5 - \eta + 0.05\eta^2 + 0.66\eta^3)^3 \partial\eta}_{\text{1st Integral}} \right.$$

$$+ b^2 \underbrace{\int_{-.736}^{0.5} (0.5 - \eta + 0.05\eta^2 + .66\eta^3)^3 \partial\eta}_{\text{2nd Integral}}$$

$$+ r_{\frac{1}{2}}b(0.095)^3 \underbrace{\int_{0.5}^{0.8} \{1 - \tanh 4.79(\eta-.5)\}^3 \partial\eta}_{\text{3rd Integral}}$$

$$+ b^2 (.095)^3 \underbrace{\int_{0.5}^{0.8} \eta\{1 - \tanh 4.79(\eta-.5)\}^3 \partial\eta}_{\text{4th Integral}}$$

$$- \left(\frac{L}{b}\right)^2 \left[r_{\frac{1}{2}} \underbrace{\int_{-.736}^{0.5} (-1 + 0.1\eta + 1.98\eta^2) \, \partial\eta}_{\text{5th Integral}} \right.$$

$$+ b \underbrace{\int_{-.736}^{0.5} \eta(-1 + 0.1\eta + 1.98\eta^2)^3 \, \partial\eta}_{\text{6th Integral}}$$

$$- r_{\frac{1}{2}}(0.455)^3 \underbrace{\int_{0.5}^{0.8} \{\text{sech}^2 \, 4.79(\eta-.5)\}^3 \partial\eta}_{\text{7th Integral}}$$

$$- b(0.455)^3 \underbrace{\int_{0.5}^{0.8} \eta\{\text{sech}^2 \, 4.79(\eta-.5)\}^3 \partial\eta}_{\text{8th Integral}} \ = \ 0$$

$$(B.10)$$

B.3.1   Calculation of Integrals in Equation (B.10)

$$\text{1st Integral} = \int_{-.736}^{0.5} (0.5-\eta+.05\eta^2+.66\eta^3)^3 \, \partial\eta$$

$$= \int_{-.736}^{0.5} \{.125-.75\eta+1.5375\eta^2-0.655\eta^3$$

$$- 1.82623\eta^4+2.0715\eta^5+0.455525\eta^6$$

$$- 1.30165\eta^7+0.06534\eta^8+0.287496\eta^9\} \partial\eta$$

$$= 0.45097066$$

$$\text{2nd Integral} = \int_{-.736}^{0.5} \eta(0.5-\eta+.05\eta^2+0.66\eta^3) \, \partial\eta$$

$$= \int_{-.736}^{0.5} \{0.125\eta-0.75\eta^2+1.5375\eta^3-0.655\eta^4$$

$$- 1.82625\eta^5+2.0715\eta^6+0.455525\eta^7$$

$$- 1.30185\eta^8+0.06534\eta^9+0.287496\eta^{10}\} \partial\eta$$

2nd Integral = -0.20326464

$$\text{3rd Integral} = (0.095)^3 \int_{0.5}^{0.8} \{1-\tanh\eta\,4.79(\eta-.5)\}^3 \partial\eta$$

$$= (0.095)^3 \int_{0.5}^{0.8} \Big[1-3\tanh\,4.79(\eta-.5)$$

$$+ 3\tanh\,4.79(\eta-.5)-\tanh^3 4.79(\eta-.5)\Big]\partial\eta$$

$$= (0.095)^3 \int_{0.5}^{0.8} \Big[4-4\tanh\,4.79(\eta-.5)$$

$$- 3\,\text{sech}^2\,4.79(\eta-.5)$$

$$+ \tanh\,4.79(\eta-.5)\,\text{sech}^2\,4.79(\eta-.5)\Big]\,\partial\eta$$

$$= (.095)^3 \Big[4\eta - \frac{4}{4.79}\ell n\,\cosh\,4.79(\eta-.5)$$

$$- \frac{3}{4.79}\tanh\,4.79(\eta-.5)$$

$$+ \frac{1}{4.79}\int \tanh\,4.79(\eta-.5)\,\partial\{\tanh\,4.79(\eta-.5)\}\Big]_{0.5}^{0.8}$$

$$= (.095)^3 \Big[4\eta - \frac{4}{4.79}\ell\,n\cosh\,4.79(\eta-.5)$$

$$- \frac{3}{4.79}\tanh\,4.79(\eta-.5)$$

$$+ \frac{1}{4.79}\,\frac{\tanh^2 4.79(\eta-.5)}{2}\Big]_{0.5}^{0.8}$$

$$= 0.00004875$$

$$\text{4th Integral} = (.095)^3 \int_{0.5}^{0.8} \eta\{1-\tanh\,4.79(\eta-.5)\}^3 \partial\eta$$

4th Integral $= (.095)^3 \int_{0.5}^{0.8} \Big[4\eta - 4\eta\tanh 4.79(\eta-.5)$

$- 3\eta\operatorname{sech}^2 4.79(\eta-.5)$

$+ \eta\tanh 4.79(\eta-.5)\operatorname{sech}^2 4.79(\eta-.5) \Big] \partial\eta$

$= (.095)^3 \Big[2\eta^2 - 4\int \eta\tanh 4.79(\eta-.5)\partial\eta$

$- 3\int \eta\operatorname{sech}^2 4.79(\eta-.5)\partial\eta$

$+ \int \eta\tanh 4.79(\eta-.5)*\operatorname{sech}^2 4.79(\eta-.5)\partial\eta \Big]_{0.5}^{0.8}$

$= (.095)^3 \Big[2\eta^2 - \dfrac{4}{4.79}\int \{ 4.79(\eta-.5)\tanh 4.79(\eta-.5)$

$+ 4.79*0.5\tanh 4.79(\eta-.5) \}\partial\eta$

$- \dfrac{3\eta}{4.79} * \tan 4.79(\eta-.5)$

$+ \dfrac{3}{(4.79)^2}\ell n \cosh 4.79(\eta-.5)$

$+ \dfrac{\eta}{4.79}\dfrac{\tanh^2 4.79(\eta-.5)}{2}$

$- \dfrac{1}{9.58}\tanh^2 4.79(\eta-.5)\partial\eta \Big]_{0.5}^{0.8}$

$= (.095)^3 \Big[2\eta^2 - \dfrac{4}{4.79}\int \{ \dfrac{(4.79(\eta-.5))^2}{1} - \dfrac{(4.79(\eta-.5))^4}{3}$

$+ \dfrac{2}{15}(4.79(\eta-.5))^6 - \dfrac{17}{315}(4.79(\eta-.5))^8$

$+ \dfrac{62}{2835}(4.79(\eta-.5))^{10} - \ldots\ldots \}\partial\eta$

$- \dfrac{2}{4.79}\ell n \cosh 4.79(\eta-.5) - \dfrac{3\eta}{4.79}\tanh 4.79(\eta-.5)$

$$+ \frac{3}{(4.79)^2} \ell n \cosh 4.79(\eta-.5)$$

$$+ \frac{\eta}{4.79} \frac{\tanh^2 4.79(\eta-.5)}{2}$$

$$\left. - \frac{1}{9.58} \int \{1-\mathrm{sech}^2 4.79(\eta-.5)\} \partial \eta \right]_{0.5}^{0.8}$$

$$= (.095)^3 \left[2\eta^2 - \frac{4}{(4.79)^2} \left\{ \frac{(4.79(\eta-.5))^3}{3} \right. \right.$$

$$- \frac{(4.79(\eta-.5))^5}{15} + \frac{2(4.79(\eta-.5)^7}{105} - \frac{17(4.79(\eta-.5))^9}{315*9}$$

$$\left. + \frac{62(4.79(\eta-.5))^{11}}{2835*11} - \ldots \right\}$$

$$- \frac{2}{4.79} \ell n \cosh 4.79(\eta-.5) - \frac{3\eta}{4.79} \tanh 4.79(\eta-.5)$$

$$+ \frac{3}{(4.79)^2} \ell n \cosh 4.79(\eta-.5) + \frac{\eta}{4.79} \frac{\tanh^2 4.79(\eta-.5)}{2}$$

$$\left. - \frac{\eta}{9.58} + \frac{1}{9.58*4.79} \tanh 4.79(\eta-.5) \right]_{0.5}^{0.8}$$

4th Integral = 0.00002021

5th Integral = $\int_{-.736}^{0.5} \{-1+0.1\eta+1.98\eta^2\}^3 \partial \eta$

$$= \int_{-.736}^{0.5} \{-1+0.3\eta+5.91\eta^2-1.187\eta^3-11.7018\eta^4$$

$$+1.17604\eta^5+7.762392\eta^6\} \partial \eta$$

$$= -0.64783662$$

6th Integral = $\int_{-.736}^{0.5} \eta\{-1+0.1\eta+1.98\eta^2\}^2\,\partial\eta$

$= \int_{-.736}^{0.5}\{-\eta+0.3\eta^2+5.91\eta^3-1.187\eta^4-11.7018\eta^5$

$\qquad\qquad + 1.17604\eta^6+7.762392\eta^7\}\partial\eta$

$= 0.01906390$

7th Integral $= (0.455)^3 \int_{0.5}^{0.8} \text{sech}^6 4.79(\eta-.5)\,\partial\eta$

$= \dfrac{(0.455)^3}{4.79} \int_{0.5}^{0.8}\{1-\tanh^2 4.79(\eta-.5)\}^2$

$\qquad\qquad *\delta\{\tanh 4.79(\eta-.5)$

$= \dfrac{(0.455)^3}{4.79} \int_{0.5}^{0.8}\Big[1-2\tanh^2 4.79(\eta-.5)$

$\qquad\qquad + \tanh^4 4.79(\eta-.5)\Big]\,\partial\tanh 4.79(\eta-.5)$

$= \dfrac{(0.455)^3}{4.79}\Big[\tanh 4.79(\eta-.5)-\dfrac{2}{3}\tanh 4.79(\eta-.5)$

$\qquad\qquad + \dfrac{1}{5}\tanh^5 4.79(\eta-.5)\Big]_{0.5}^{0.8}$

$= 0.01045859$

8th Integral $= (0.455)^3 \int_{0.5}^{0.8} \eta\,\text{sech}^6 4.79(\eta-.5)\,\partial\eta$

$= (0.455)^3\Big[\eta\int\text{sech}^6 4.79(\eta-.5)\,\partial\eta$

$\qquad\qquad -\int\{\int\text{sech}^6 4.79(\eta-.5)\,\partial\eta\}\partial\eta\Big]_{0.5}^{0.8}$

Using 7th Integral,

8th Integral $= \dfrac{(0.455)^3}{4.79}\Big[\eta\{\tanh 4.79(\eta-.5)-\dfrac{2}{3}\tanh^3 4.79(\eta-.5)$

$$+ \dfrac{1}{5}\tanh^5 4.79(\eta-.5)\}-\int\{\tanh 4.79(\eta-.5)$$

$$- \dfrac{2}{3}\tanh^3 4.79(\eta-.5)+\dfrac{1}{5}\tanh^5 4.79(\eta-.5)\}\partial\eta\Big]_{0.5}^{0.8}$$

$$= \dfrac{(0.455)^3}{4.79}\Big[\eta\{\tanh 4.79(\eta-.5)-\dfrac{2}{3}\tanh^3 4.79(\eta-.5)$$

$$+ \dfrac{1}{5}\tanh^5 4.79(\eta-.5)\}-\int\{\dfrac{8}{15}\tanh 4.79(\eta-.5)$$

$$+ \dfrac{7}{5}\tanh 4.79(\eta-.5)\operatorname{sech}^2 4.79(\eta-.5)$$

$$- \dfrac{1}{5}\tanh^3 4.79(\eta-.5)\operatorname{sech}^2 4.79(\eta-.5)\}\partial\eta\Big]_{0.5}^{0.8}$$

$$= \dfrac{(0.455)^3}{4.79}\Big[\eta\{\tanh 4.79(\eta-.5)-\dfrac{2}{3}\tanh^3 4.79(\eta-.5)$$

$$+ \dfrac{1}{5}\tanh^5 4.79(\eta-.5)\}-\dfrac{1}{4.79}\{\dfrac{8}{15}\ell n\cosh 4.79(\eta-.5)$$

$$+ \dfrac{7}{30}\tanh^2 4.79(\eta-.5)-\dfrac{1}{20}\tanh^4 4.79(\eta-.5)\}\Big]_{0.5}^{0.8}$$

$$= 0.00598443$$

Substituting the values of integrals in Equation (B.10):

$$\dfrac{1}{2}\dfrac{\partial}{\partial x}\Big[\dfrac{r_1^2}{2} + 0.45101941 r_{\frac{1}{2}}b - 0.20326464b^2\Big] - \Big(\dfrac{L}{b}\Big)^2$$

$$* \Big[-0.65829521 r_{\frac{1}{2}} + 0.01311954b\Big] = 0$$

or,

$$\frac{1}{2}\frac{\partial}{\partial x}\left[\{\frac{r_1^2}{2} + 0.547578r_{\frac{1}{2}}b - 0.215408b^2\} - 0.9655859r_{\frac{1}{2}}b\right.$$

$$\left. + 0.01214336b^2\right] - \frac{L^2}{b^2}\left[-0.65829521r_{\frac{1}{2}} + 0.01311954b\right]$$

$$= 0$$

Using Equation (B.6):

$$\frac{1}{2}\frac{\partial}{\partial x}\left[0.09655859r_{\frac{1}{2}}b - 0.01214336b^2\right.$$

$$= + \frac{L^2}{b^2}\left[0.65829521r_{\frac{1}{2}} - 0.01311954b\right]$$

Substituting the expression for $r_{\frac{1}{2}}$ from Equation (B.9):

$$\frac{1}{2}\frac{\partial}{\partial x}\left[0.09655859b\{0.18842198b + \sqrt{r_o^2 - 0.075377b^2}\}\right.$$

$$\left. - 0.01214336b^2\right] = (\frac{L}{b})^2\left[0.65829521\ 0.18842198b\right.$$

$$\left. + \sqrt{r_o^2 - 0.075377b^2}\} - 0.01311954b\right]$$

or

$$\frac{1}{2}\frac{\partial}{\partial x}\left[.09655859b\sqrt{r_o^2 - 0.075377b^2} + 0.0060504b^2\right]$$

$$= \frac{L^2}{b^2}\left[0.65829521\sqrt{r_o^2 - 0.075377b} + 0.11019775b\right]$$

Differentiating,

$$\frac{1}{2}\frac{\partial b}{\partial x}\left[0.09655859\sqrt{r_o^2 - 0.075377b^2}\right.$$

$$\left. - \frac{.09655859*0.075377b^2}{\sqrt{r_o^2 - .075377b^2}} + 0.121008b\right]$$

$$= \left(\frac{L}{b}\right)^2\left[0.65829521\sqrt{r_o^2 - 0.075377b^2} + 0.11091775b\right]$$

or,

$$\frac{1}{2}\frac{\partial b}{\partial x}\left[.09655859(r_o^2 - 0.075377b^2) - .09655859*0.075377b^2\right.$$

$$\left. + 0.0121008b\sqrt{r_o^2 - 0.075377b^2}\right]$$

$$= \frac{L^2}{b^2}\left[0.65829521(r_o^2 - 0.075377b^2)\right.$$

$$\left. + 0.11091775b\sqrt{r_o^2 - 0.075377b^2}\right]$$

or,

$$\frac{\partial b}{\partial x} = 2\left(\frac{L}{b}\right)^2\left[0.6582952r_o^2 - 0.04962032b^2 + \right.$$

$$\left. + 0.11091775b\sqrt{r_o^2 - 0.075377b^2}\right] /$$

$$\left[0.09655859r_o^2 - 0.01455659b^2\right.$$

$$\left. + 0.0121008\sqrt{r_o^2 - 0.075377b^2}\right]$$

(B.11)

APPENDIX C

HOT-WIRE EQUATIONS AND CALIBRATION

C.1 Hot-Wire Equations

C.1.1 Without A Linearizer

For a hot-wire anemometer without linearizer,

$$E^2 = A + BU_E^C \tag{C.1}$$

where E is the output voltage, U_E is the effective cooling velocity and A,B,C are constants. The effective cooling velocity is given by the following empirical equation [22]:

$$U_E^2 = U_N^2 + k_2^2 U_T^2 \tag{C.2}$$

where U_N is the velocity component normal to the wire and U_T is the tangential velocity component. Webster [90] and Champagne [91] determined k_2 to be 0.20, with no dependence on the wire ℓ/d ratio, and only a weak dependence on velocity. Considering the wire in x-y plane; Figure C.1.

$$U_N^2 = \{(u+u')\cos\alpha + \upsilon'\sin\alpha\}^2 + w'^2$$

and

$$U_T^2 = \{(u+u')\sin\alpha - \upsilon'\cos\alpha\}^2 \tag{C.3}$$

Assuming weak turbulence, so that:

$$u \gg u', \quad u \gg \upsilon', \quad u \gg w'$$

Neglecting terms containing quadratic and higher powers of fluctuating velocities, Equation (C.3) becomes:

$$U_N = u\cos\alpha\left[(1 + \frac{u'}{u}) + \frac{\upsilon'}{u}\tan\alpha\right]$$

$$U_T = u\cos\alpha\left[(1 + \frac{u'}{u})\tan\alpha - \frac{\upsilon'}{u}\right] \tag{C.4}$$

Using Equation (C.4) in Equation (C.2):

$$U_E^2 = u^2 \cos^2\alpha \left[1 + 2\frac{u'}{u} + \frac{u'^2}{u^2} + 2\tan\alpha(\frac{\upsilon'}{u} + \frac{u'\upsilon'}{u^2}) \right.$$

$$+ \frac{\upsilon'^2}{u^2}\tan^2\alpha + k_2^2\{\frac{\upsilon'^2}{u^2} - 2\tan\alpha(\frac{\upsilon'}{u} + \frac{u'\upsilon'}{u^2})$$

$$\left. + \tan^2\alpha(1 + 2\frac{u'}{u} + \frac{u'^2}{u^2})\} \right]$$

or

$$U_E^c = u^c \cos^c\alpha \left[1 + 2\frac{u'}{u} + \frac{u'^2}{u^2} + 2\tan\alpha(\frac{\upsilon'}{u} + \frac{u'\upsilon'}{u^2}) \right.$$

$$+ \frac{\upsilon'^2}{u^2}\tan^2\alpha + k_2^2\{\frac{\upsilon'^2}{u^2} - 2\tan\alpha(\frac{\upsilon'}{u} + \frac{u'\upsilon'}{u^2})$$

$$\left. + \tan^2\alpha(1 + 2\frac{u'}{u} + \frac{u'^2}{u^2})\} \right]^{c/2} \tag{C.5}$$

Expanding Equation (C.5) in power series and retaining the first power terms of fluctuating components:

$$U_E^c = s_1 u^c \cos^c\alpha \left[1 + c\frac{u'}{u} + c\frac{s_1}{s_2}\frac{\upsilon'}{u} \right] \tag{C.6}$$

where,

$$s_1 = 1 + \frac{c}{2}k_2^2\tan^2\alpha + \frac{c(c-2)}{8}k_2^4\tan^4\alpha$$

$$s_2 = \tan\alpha \left[1 - k_2^2(1 - \frac{c-2}{2}\tan^2\alpha) + \frac{k_2^4}{2}\tan^2\alpha \right.$$

$$\left. * (2 - c + \frac{(c-2)(c^2-4)}{4}\tan^2\alpha) \right] \tag{C.7}$$

The hot-wire anemometer output voltage, E, has mean and fluctuating components:

$$E = \bar{E} + e \tag{C.8}$$

Using Equations (C.6) and (C.8) in Equation (C.1) and assuming $\bar{E} \gg e$:

$$\bar{E}^2 + 2\bar{E}e = A + Bs_1u^c\cos^c\alpha\left[1 + c\,\frac{u'}{u} + c\,\frac{s_2}{s_1}\,\frac{u'}{u}\right] \quad (C.9)$$

Taking average of Equation (C.9):

$$\bar{E}^2 = A + (Bs_1\cos^c\alpha)u^c \quad (C.10)$$

$(Bs_1\cos^c\alpha)$ is the slope in the plane $\bar{E}^2$ vs. u^c. Using experimental values of $\bar{E}$ and u, a series of curves can be drawn in the plane with different values of c. The best fit curve designates the best value of c, and it is represented by calibration curves in Figures C.4 and C.5 for normal and slanting wires respectively.

To linearize a signal it is necessary to set the value of c on the linearizer. The determined value of c does not always give the best linearization, but it can be adjusted by trial.

C.1.2 With a Linearizer

For a hot-wire anemometer with a linearizer:

$$E_L = B_LU_E \quad (C.11)$$

where E_L is the linearizer output and B_L is a constant. The expression for cooling velocity, U_E for this case is obtained by using c=1 in Equation (C.6):

$$U_E = s_1u\cos\alpha\left[1 + \frac{u'}{u} + \frac{s_1}{s_2}\,\frac{u'}{u}\right] \quad (C.12)$$

For c=1, s_1 and s_2 from Equation (C.7):

$$s_1 = 1 + \frac{1}{2}k_2^2 \tan^2\alpha - \frac{1}{8}k^4 \tan^4\alpha$$

$$s_2 = \tan\alpha\left[1 - k_2^2(1 + \frac{1}{2}\tan^2\alpha) + \frac{k_2^4}{2} \tan^4\alpha(1 + \frac{3}{4}\tan^2\alpha)\right] \quad \text{(C.13)}$$

The output voltage, E_L, has two components:

$$E_L = \bar{E}_L + e_L \quad \text{(C.14)}$$

Using Equations (C.14) and (C.12) in Equation (C.11):

$$\bar{E}_L + e_L = B_L s_1 u\cos\alpha\left[1 + \frac{u'}{u} + \frac{s_2}{s_1} \frac{v'}{u}\right] \quad \text{(C.15)}$$

Taking mean of Equation (C.15):

$$\bar{E}_L = (B_L s_1 \cos\alpha) u \quad \text{(C.16)}$$

$$\bar{E}_L = (\text{slope}) u \quad \text{(C.17)}$$

where,

$$(\text{slope}) = B_L s_1 \cos\alpha$$

For normal wire, $\alpha=0$ and $s_1=1$, so, the slope, n_1 for normal wire becomes, $n_1=B_L$, and the slope, n_2 for slanting wire is; $n_2=B_L s_1 \cos\alpha$. The best values n_1 and n_2 for normal and slanting wires are determined by calibration presented later:

Subtracting Equation (C.16) from Equation (C.15):

$$e_L = (B_L s_1 \cos\alpha)\left[u' + \frac{s_2}{s_1} v'\right] \quad \text{(C.18)}$$

If a wire is located normal to the flow in Position I, and a single slanting wire is placed alternately in Positions II and III, as shown in Figure C.2, Equation (C.18) can be written as:

$$e_{L1} = n_1 u_1 \text{, since } s_2 = 0$$

$$e_{L2} = n_2(u' + \frac{s_2}{s_1} \upsilon')$$

$$e_{L3} = n_2(u' - \frac{s_2}{s_1} \upsilon')$$

$$\text{(C.19)}$$

If Position III is obtained by rotating the wire 180° about the axis of the wire support, then $n_2 = n_3$.

Equation (C.19) can be written as:

$$\overline{u'^2} = \overline{e_{L1}^2}/n_1^2$$

$$\overline{\upsilon'^2} = (s_1/s_2)^2 \left[(\overline{e_{L2}^2} + \overline{e_{L3}^2})/2n_2^2 - \overline{u'^2} \right] \qquad \text{(C.20)}$$

$$\overline{u'\upsilon'} = s_1/s_2 \ (\overline{e_{L2}^2} - \overline{e_{L3}^2})/4n_2^2$$

The values of $\overline{e_{L1}^2}$, $\overline{e_{L2}^2}$ and $\overline{e_{L3}^2}$ were obtained by squaring the voltage which were read from the RMS meter to the hot-wire anemometer, and with the wires in Positions I, II and III. The constant (s_1/s_2) was determined by calibration in a developed pipe flow as in ref. [3]. As a check, s_1 and s_2 were also calculated from Equation (C.13) knowing α from measurement and $k_2 = 0.2$ for wires ($\ell/d = 200$) used here.

C.2 Calibration

C.2.1 Determination of c

The hot-wire Equation (C.10) for an anemometer without a linearizer contains three empirical constants. For calibration purpose $\overline{E}^2$ and u were measured by increasing the main flow, in steps, to give the nozzle exit velocities

from 2.5 m/s to 100 m/s. The measured values of $\bar{E}^2$ and u^c were plotted by least square principle varying the values of c between 0.4 and 0.5. The least RMS deviation designates the best value of c, and typical plottings for normal and slanting wires are shown in Figures $\overline{C.4}$ and C.5, respectively.

C.2.2 Anemometer with a Linearizer

The hot-wire anemometer signal is nonlinear. At higher degrees of turbulence, the distortion caused by the curvature of the calibration curve becomes noticeable; also, measurements at varying degrees of mean flow velocity are rendered difficult by consequent sensitivity variations. This problem is overcome by linearizing the signal through an electronic circuit. The hot-wire Equation (C.17) for this case contains one empirical constant.

For calibrating the hot-wire anemometer, the hot-wire and the pitot tube were placed side by side in a core of uniform velocity. A suitable value of the exponent, c, is necessary to set to the linearizer to produce a linearized signal. The best value of c obtained by curve fitting principle described in Section C.2.1 does not always produce the best linearization. By trial, c was adjusted to the linearizer to achieve the best linear output. The values of c for the best linear output and that obtained in Section C.2.1 for a number of wires (5μ, $\ell/d = 200$) are shown in Table C.1. Adjusting the value of c to the linearizer, $\bar{E}$ and u were measured. Typical calibration curves for both normal and slanting wire are shown in Figure C.6. The calibration was

TABLE C.1: Best Values of Exponent c in Hot-Wire Equation

Wire	Velocity Range (m/s)	Points of Calibration	Best Fit Without Linearizer		Best Fit With Linearizer	
			c	RMS Dev.	c	RMS Dev.
N-1	1.5-100	12	0.46	0.128	0.47	0.078
N-2	1.5-100	12	0.44	0.095	0.48	0.101
N-3	1.5-100	12	0.40	0.132	0.48	0.121
N-4	1.5-100	12	0.45	0.115	0.46	0.098
S-1	1.5-100	12	0.49	0.085	0.50	0.131
S-1	1.5-100	12	0.50	0.068	0.48	0.110
S-2	1.5-100	12	0.50	0.099	0.50	0.115

checked by measuring shear stress and turbulent intensities in a fully developed pipe flow and comparing these with the measurements of Tucker [3] and Laufer [92] as shown in Figures C.8 and C.9.

A straight commercial aluminum pipe of 10.8 cm inside diameter and 15.24 m long was connected to a wooden settling chamber of length 1.65 m and cross-section 0.91 m x 0.91 m square. The settling chamber, having flow straighteners and screens inside, received air from a blower through a flexible pipe. The turbulent intensities and shear stress were measured by a hot-wire anemometer (type 55M10) with a linearizer (type 55D10) at the exit of the pipe. The inside surface of the pipe was polished by glasswool. The hot-wire probe was traversed over a scale of precision 0.0254 mm.

The low velocity calibration was performed following the procedure given in the instruction manual for the linearizer type 55D10. A typical low velocity calibration curve is shown in Figure C.7. The calibration of the pitot tube with the micromanometer is shown in Figure C.3.
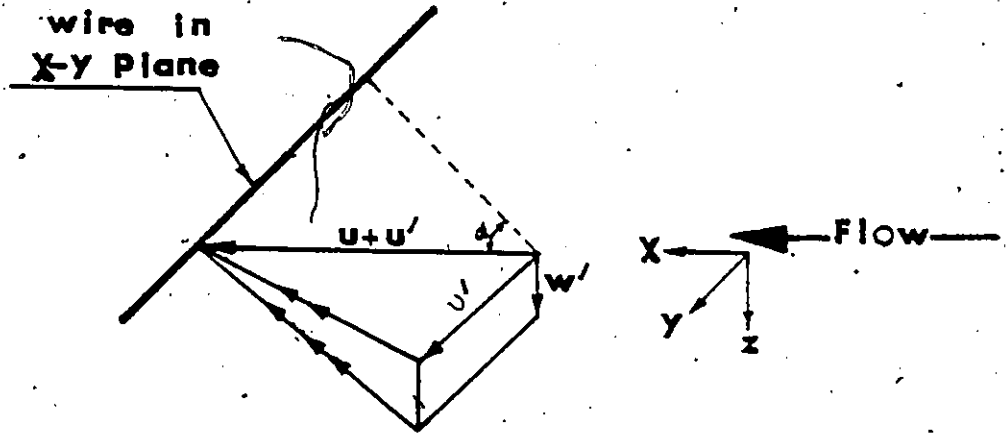
Fig.C.1

Wire Configuration in 3-Dimensional Velocity Field.

Normal Wire
Position I

Slanting Wire
Position II

Slanting Wire
Position III
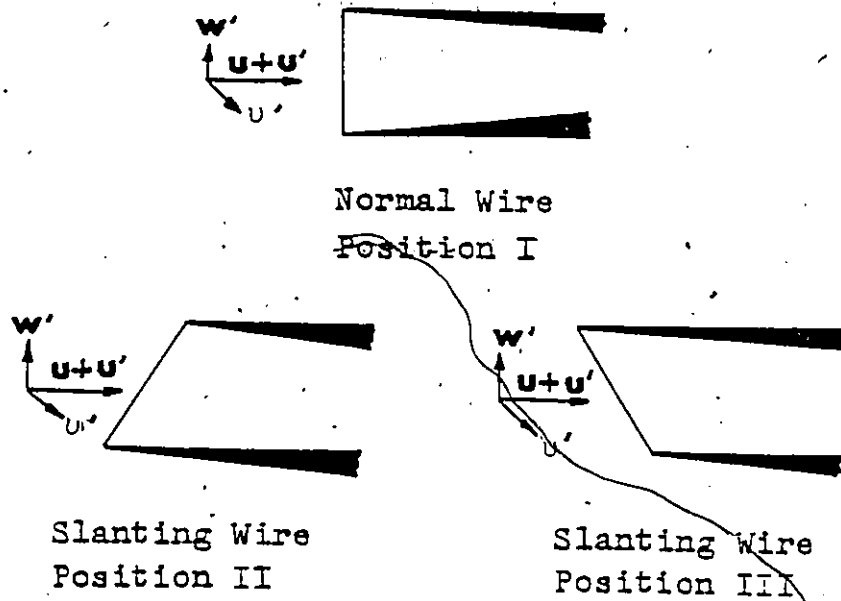
Fig.C.2

Hot-Wire Configuration for Measurements of
$\overline{u'^2}$, $\overline{v'^2}$ and $\overline{u'v'}$.
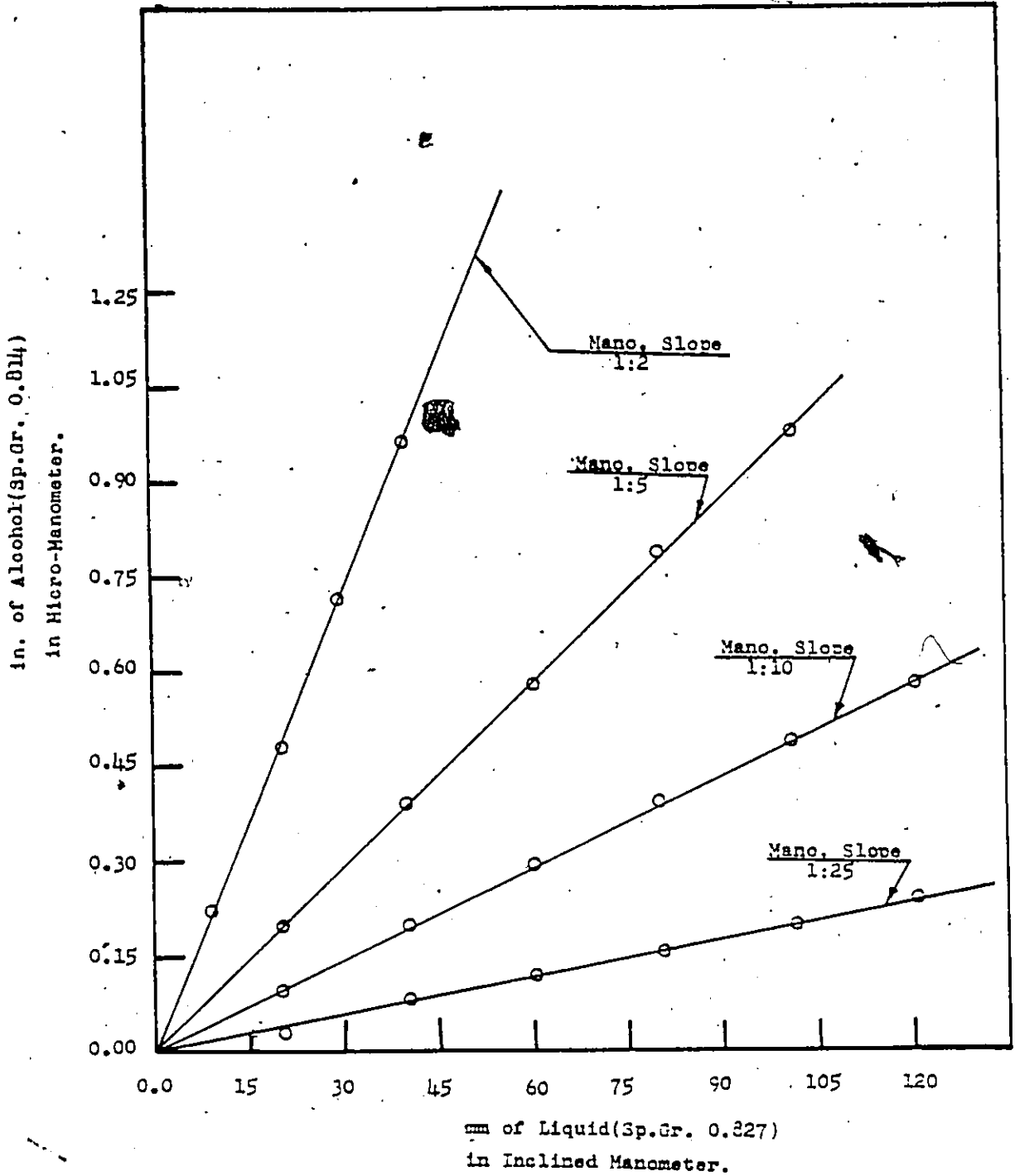
Fig.C.3

Calibration of a Pitot Tube Against a Micro-Manometer.
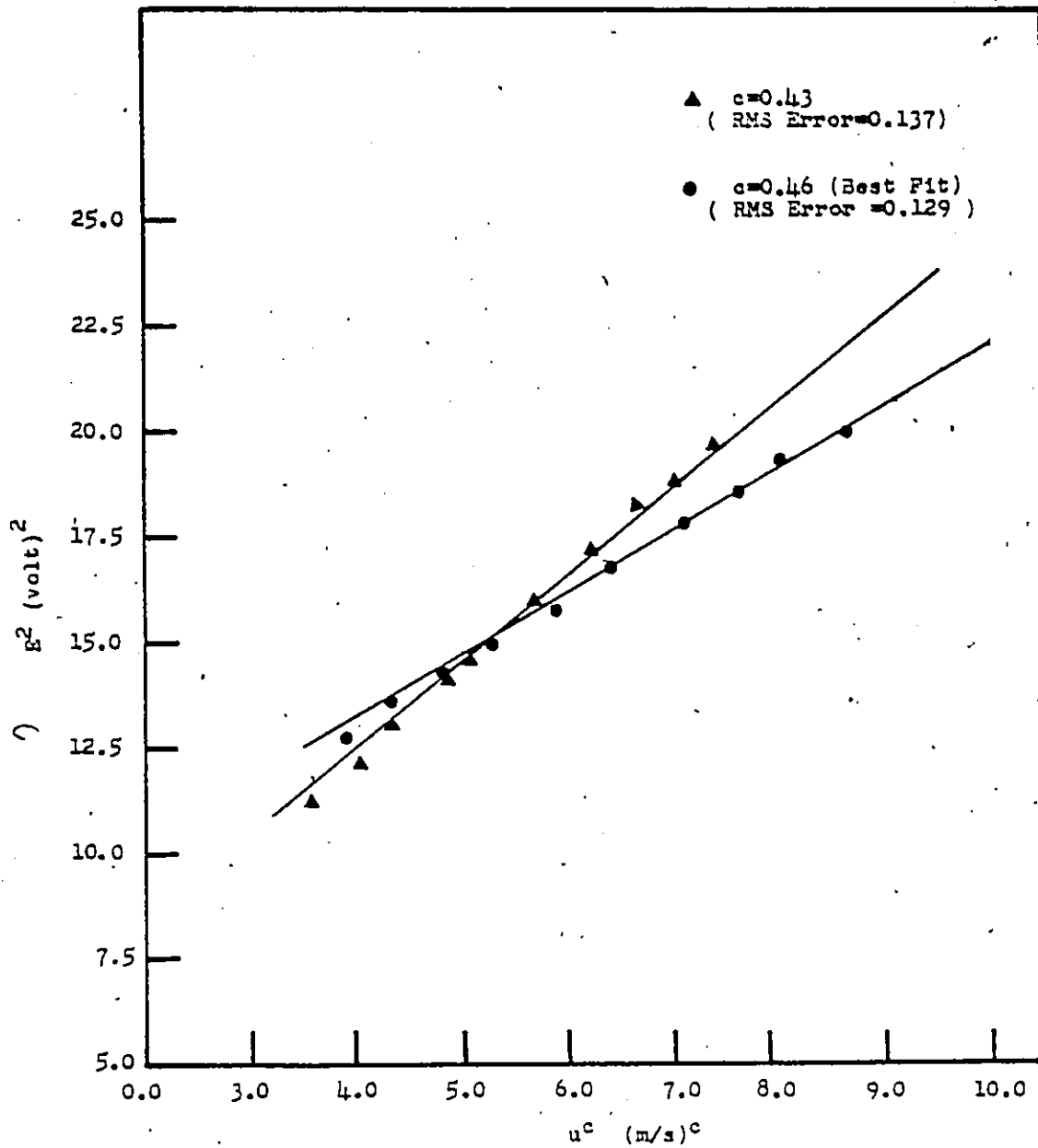
Fig.C.4

A Typical Calibration Curve for a Normal Wire
Without a Linearizer.
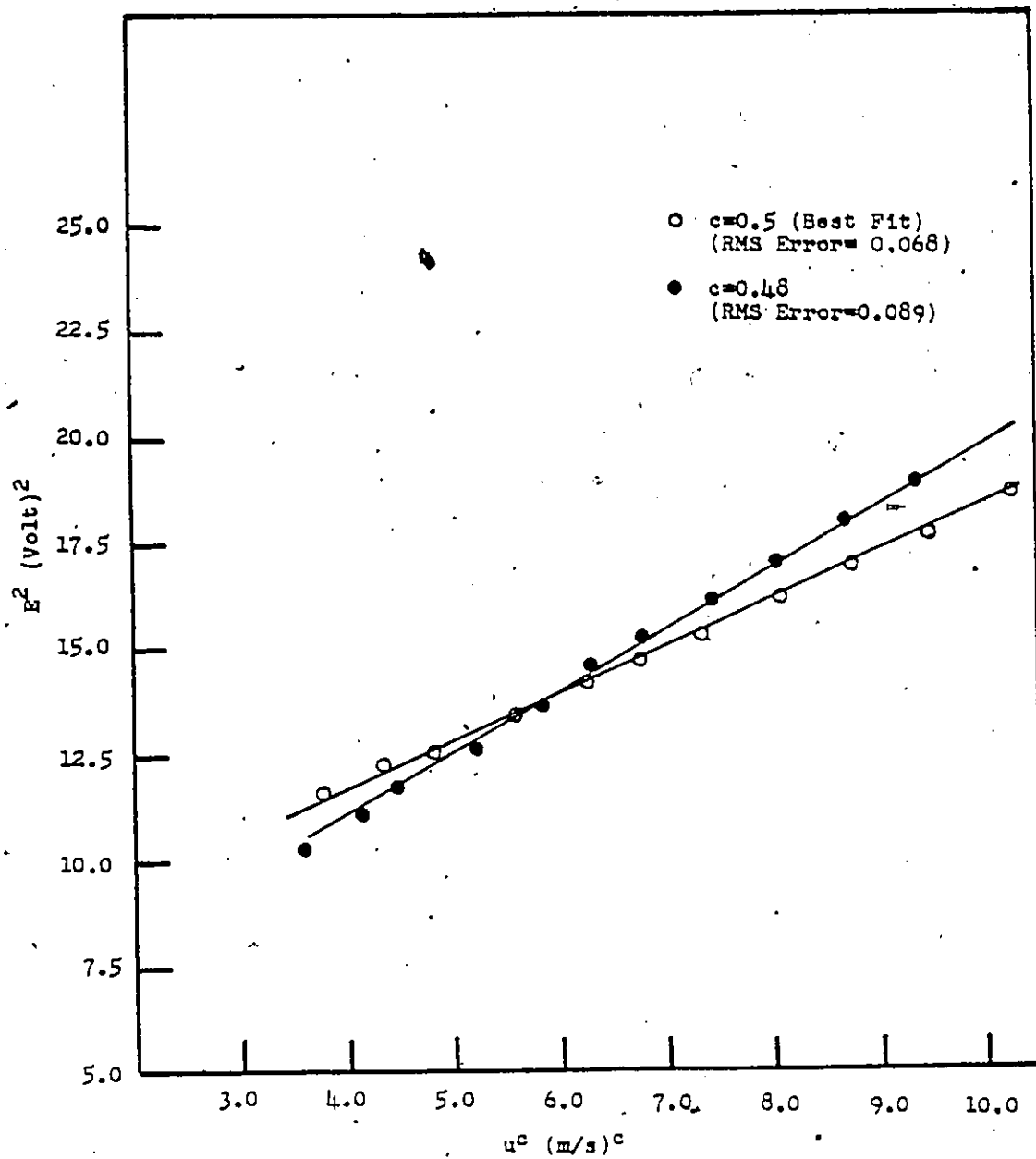
Fig.C.5

Typical Calibration Curves for a Slanting Wire Without
a Linearizer.
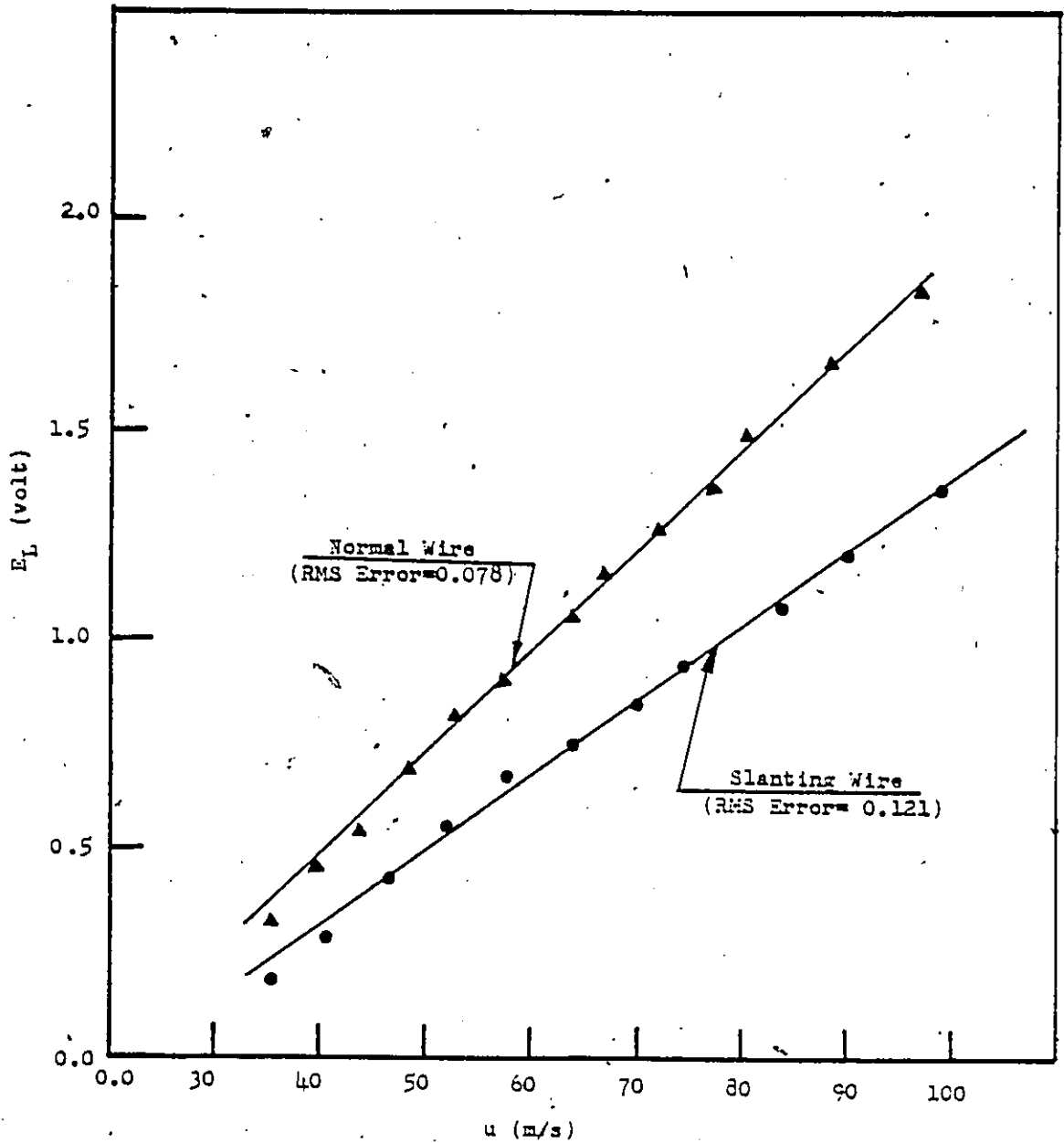
Fig.C.6

Typical Calibration Curves for Normal and Slanting

Wires with a Linearizer.
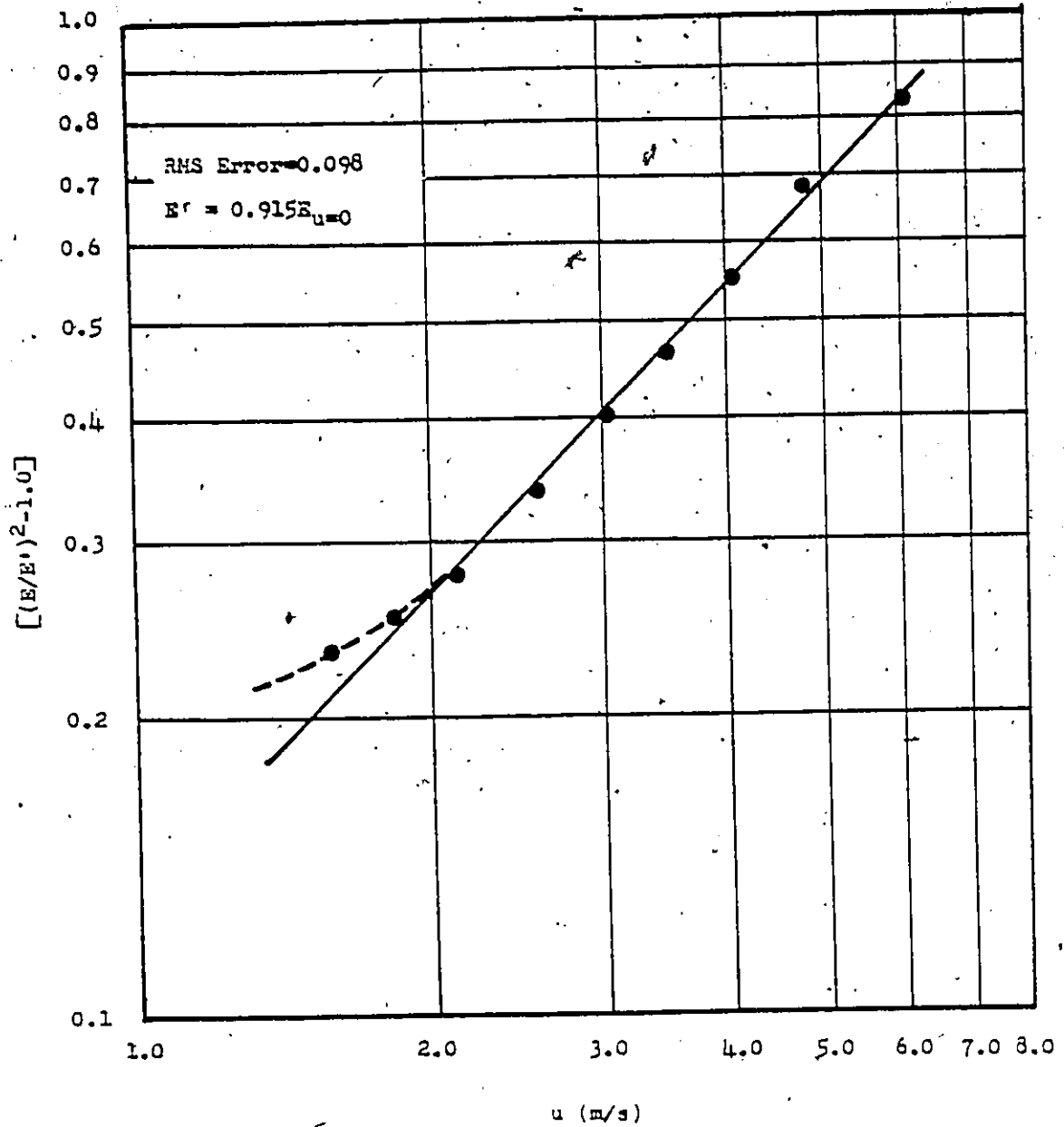
Fig.C.7

Typical Calibration Curve for Normal Wire at
Low Velocity.
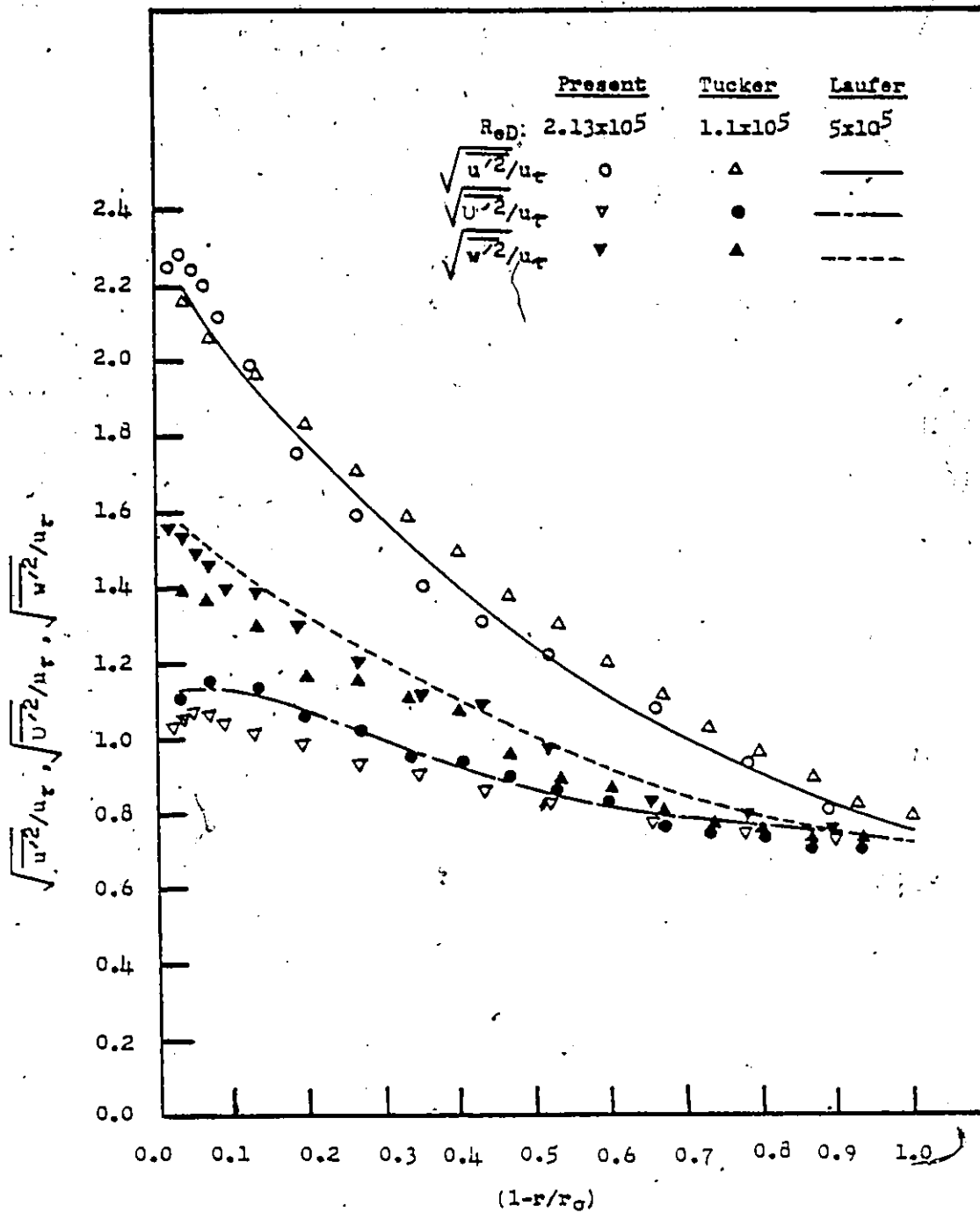
Fig.C.8

Distributions of Turbulent Intensities in a Fully
Developed Pipe Flow.
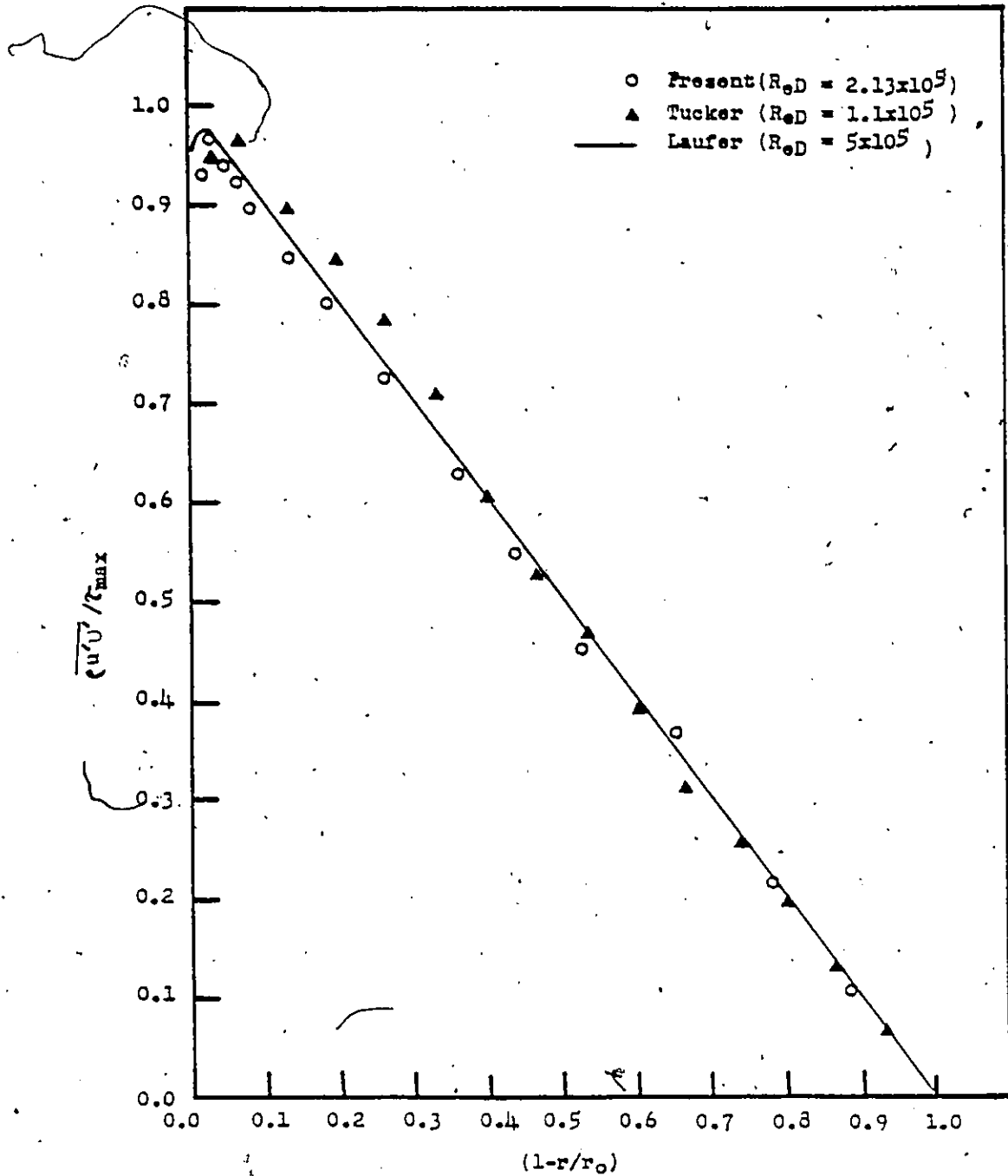
Fig.0.9

Reynold's Shear Stress Distributions in a Fully

Developed Pipe Flow.

APPENDIX D

THE FINITE-DIFFERENCE FORMULATION

D.1 General

A standard explicit finite-difference technique requires very small streamwise steps [93] to satisfy the stability criterion. The Dufort-Frankel [94] method was found to be stable for the varying step size that was used here. The standard explicit scheme was used as a starting method for the Dufort-Frankel procedure which requires information from the two previous streamwise stations.

The finite-difference problem domain is usually established by letting ΔX and ΔR be small increments of the coordinates X and R and the considering of all the variables as existing on the finite set of points $X=i\Delta X$, $R=j\Delta R$ (valid for fixed ΔX, ΔR) where i and j are integers. Here the finite-difference equations will be written in a form that will be applicable for uneven grid spacing with x and r directions. For this purpose, the notation $\Delta R_+ = (R_{j+1} - R_j)$, $\Delta R_- = (R_j - R_{j-1})$, $\Delta X_+ = (X_{i+1} - X_i)$ and $\Delta X_- = (X_i - X_{i-1})$ will be introduced. The dependent variables are expanded in Taylor series [95].

The basic variables are made non-dimensional by using the following transformation.

$$X = \frac{x u_o}{\nu}, \quad R = \frac{r u_o}{\nu}, \quad L^* = \frac{L u_o}{\nu}, \quad N = \frac{\nu + \nu_t}{\nu},$$

$$U = \frac{u}{u_o}, \quad V = \frac{\upsilon}{u_o}, \quad K^* = \frac{k}{u_o^2}$$

$$(D.1)$$

Replacing shear stress, τ/ρ by $(\nu+\nu_t)\partial u/\partial r$ and introducing the above transformation (D.1) in the continuity Equation (3.1), the momentum Equation (3.2) and the turbulent kinetic energy Equation (3.13):

the continuity equation is:

$$\frac{\partial(UR)}{\partial X} + \frac{\partial(VR)}{\partial R} = 0 \qquad (D.2)$$

the momentum equation is:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial R} = \frac{1}{R}\frac{\partial}{\partial R}(RN\frac{\partial U}{\partial R}) \qquad (D.3)$$

and the turbulent kinetic energy equation is:

$$U\frac{\partial K^*}{\partial X} + V\frac{\partial K^*}{\partial R} = P - \varepsilon + \frac{1}{R}\frac{\partial}{\partial R}\left[k_q L^* K^* R\frac{\partial K^*}{\partial R}\right] \qquad (D.4)$$

In all calculations $L^1 = L$.

D.2 Finite Difference Equations for Dufort-Frankel Method

D.2.1 Continuity Equation

Taylor's expansions about half a grid in r-direction and one grid in x-direction.

$$U(i+1,j+1) = U(i,j+\tfrac{1}{2}) + \Delta X_+ U_x + \frac{\Delta R_+}{2} U_r + \tfrac{1}{2}\{(\Delta X_+)^2 U_{xx}$$

$$+ \Delta X_+ \Delta R_+ U_{xr} + (\frac{\Delta R_+}{2})^2 U_{rr}\} + \bar{0}(\Delta^3) \qquad (D.5)$$

$$U(i-1,j+1) = U(i,j+\tfrac{1}{2}) - \Delta X_- U_x + \frac{\Delta R_+}{2} U_r + \tfrac{1}{2}\{(\Delta X_-)^2 U_{xx}$$

$$- \Delta X_- \Delta R_+ U_{xr} + (\frac{\Delta R_+}{2})^2 U_{rr}\} + 0(\Delta^3) \qquad (D.6)$$

$$U(i+1,j) = U(i,j+\tfrac{1}{2}) + \Delta X_+ U_x - \frac{\Delta R_-}{2} U_r + \tfrac{1}{2}\{(\Delta X_+)^2 U_{xx}$$

$$- \Delta X_+ \Delta R_- U_{xr} + (\frac{\Delta R_-}{2})^2 U_{rr}\} + 0(\Delta^3) \qquad (D.7)$$

$$U(i-1,j) = U(i,j+\tfrac{1}{2}) - \Delta X_- U_x - \frac{\Delta R_-}{2} U_r + \tfrac{1}{2}\{(\Delta X_-)^2 U_{xx}$$

$$+ \Delta X_- \Delta R_- U_{xr} + (\frac{\Delta R_+}{2})^2 U_{rr}\} + 0(\Delta^3) \qquad (D.8)$$

Subtracting (D.8) from (D.5), (D.7) from (D.6), and adding,

$$(\frac{\partial U}{\partial X})_{i,j+\frac{1}{2}} = \frac{U(i+1,j+1)+U(i+1,j)-U(i-1,j+1)-U(i-1,j)}{2(\Delta X_+ + \Delta X_-)} + 0(\Delta)$$

$$(D.9)$$

Subtracting expansions for V as in Equation (D.5) and Equation (D.7):

$$(\frac{\partial V}{\partial R})_{i,j+\frac{1}{2}} = \frac{V(i+1,j+1)-V(i+1,j)}{(\Delta R_+ + \Delta R_-)/2} \qquad (D.10)$$

Using approximations (D.9) and (D.10) in the continuity Equation (D.1), the finite difference equation becomes:

$$\frac{R(j)+R(j+1)}{2} \frac{U(i+1,j+1)+U(i+1,j)-U(i-1,j+1)-U(i-1,j)}{2(\Delta X_+ + \Delta X_-)}$$

$$+ \frac{R(j+1)V(i+1,j+1)-R(j)V(i+1,j)}{(\Delta R_+ + \Delta R_-)/2} = 0 \qquad (D.11)$$

D.2.2 Momentum Equation

Taylor's expansion of U about one grid in r-direction,

$$U(i,j+1) = U(i,j) + \Delta R_+ U_r + \frac{(\Delta R_+)^2}{2} U_{rr} + 0(\Delta^3) \qquad (D.12)$$

$$U(i,j-1) = U(i,j) - \Delta R_- U_r + \frac{(\Delta R_-)^2}{2} U_{rr} + 0(\Delta^3) \qquad (D.13)$$

Subtracting Equation (D.13) from Equation (D.12),

$$\left(\frac{\partial U}{\partial R}\right)_{i,j} = \frac{U(i,j+1)-U(i,j-1)}{\Delta R_+ + \Delta R_-} + 0(\Delta^2) \qquad (D.14)$$

Similarly, expanding U about one grid distance in x-direction it is obtained,

$$\left(\frac{\partial U}{\partial X}\right)_{i,j} = \frac{U(i+1,j)-U(i-1,j)}{(\Delta X_+ + \Delta X_-)} + 0(\Delta^2) \qquad (D.15)$$

Taylor's expansion of U about half a grid spacing in r-direction:

$$U(i,j+\tfrac{1}{2}) = U(i,j) + \frac{\Delta R_+}{2} U_r + \frac{1}{2} \left(\frac{\Delta R_+}{2}\right)^2 U_{rr} + 0(\Delta^3)$$
$$(D.16)$$

$$U(i,j-\tfrac{1}{2}) = U(i,j) - \frac{\Delta R_-}{2} U_r + \frac{1}{2} \left(\frac{\Delta R_-}{2}\right)^2 U_{rr} + 0(\Delta^3)$$
$$(D.17)$$

Subtracting Equation (D.17) from Equation (D.16),

$$\left(\frac{\partial U}{\partial R}\right)_{i,j} = \frac{2}{\Delta R_+ + \Delta R_-} \left[U(i,j+\tfrac{1}{2}) - U(i,j-\tfrac{1}{2}) \right] \qquad (D.18)$$

Similarly, it is written:

$$\left[\frac{\partial}{\partial R}\{RN \frac{\partial U}{\partial R}\}\right]_{i,j} = \frac{2}{\Delta R_+ + \Delta R_-} \left[(RN \frac{\partial U}{\partial R})_{i,j+\tfrac{1}{2}} - (RN \frac{\partial U}{\partial R})_{i,j-\tfrac{1}{2}} \right]$$

or,

$$\left[\frac{\partial}{\partial R}\{RN \frac{\partial U}{\partial R}\}\right]_{i,j} = \frac{2}{\Delta R_+ + \Delta R_-} \left[\frac{R(j)+R(j+1)}{2} \cdot \frac{N(i,j)+N(i,j+1)}{2} \right.$$

$$* \left(\frac{\partial U}{\partial R}\right)_{i,j-\tfrac{1}{2}} - \frac{R(j)+R(j-1)}{2}$$

$$\left. * \frac{N(i,j)+N(i,j-1)}{2} \left(\frac{\partial U}{\partial R}\right)_{i,j-\tfrac{1}{2}} \right] \qquad (D.19)$$

Using expansions similar to Equations (D.16) and (D.17), it is written:

$$\left(\frac{\partial U}{\partial R}\right)_{i,j+\frac{1}{2}} = \frac{U(i,j+1)-U(i,j)}{\Delta R_+} + 0(\Delta^2)$$

(D.20)

$$\left(\frac{\partial U}{\partial R}\right)_{i,j-\frac{1}{2}} = \frac{U(i,j)-U(i,j-1)}{\Delta R_-} + 0(\Delta^2)$$

Writing the following expansions for U:

$$U(i+1,j) = U(i,j) + \Delta X_+ U_x + \frac{\Delta X_+^2}{2} U_{xx} + 0(\Delta^3)$$
(D.21)

$$U(i-1,j) = U(i,j) - \Delta X_- U_x + \frac{\Delta X_-^2}{2} U_{xx} + 0(\Delta^3)$$
(D.22)

Adding Equation (D.21) and Equation (D.22):

$$U(i,j) = 0.5\left[U(i+1,j) + U(i-1,j)\right] - \frac{\Delta X_+^2 + \Delta X_-^2}{2} U_{xx} + 0(\Delta^4)$$

(D.23)

The differential equation is for a thin shear layer where U_{xx} is negligibly small compared to U_{rr}. Using Equations (D.23) and (D.20) in Equation (D.19):

$$\frac{\partial}{\partial R}\left\{RN \frac{\partial U}{\partial R}\right\}_{i,j} = \frac{2}{\Delta R_+ + \Delta R_-}\left[\frac{R(j)+R(J+1)}{2} \cdot \frac{N(i,j)+N(i,j+1)}{2}\right.$$

$$* \frac{U(i,j+1)-0.5\{U(i+1,j)+U(i-1,j)\}}{\Delta R_+}$$

$$- \frac{R(j)+R(j-1)}{2} \cdot \frac{N(i,j)+N(i,j+1)}{2}$$

$$\left.* \frac{0.5\{U(i+1,j)+U(i-1,j)-U(i,j-1)\}}{\Delta R_-}\right] + 0(\Delta)$$

(D.24)

Using Equations (D.24), (D.15) and (D.14) in Equation (D.3),
the finite-difference equation for the momentum equation is:

$$U(i,j) \frac{U(i+1,j)-U(i-1,j)}{\Delta X_+ + \Delta X_-} + V(i,j) \frac{U(i,j+1)-U(i,j-1)}{\Delta R_+ + \Delta R_-}$$

$$= \frac{1}{R(j)} \frac{2}{\Delta R_+ + \Delta R_-} \left[\frac{R(j)+R(j+1)}{2} \cdot \frac{N(i,j)+N(i,j+1)}{2} \right.$$

$$* \frac{U(i,j+1)-0.5\{U(i+1,j)-U(i-1,j)\}}{\Delta R_+} - \frac{R(j)+R(j-1)}{\Delta 2}$$

$$\left. * \frac{N(i,j)+N(i,j-1)}{2} \cdot \frac{0.5\{U(i+1,j)+U(i-1,j)\}-U(i,j-1)}{\Delta R_-} \right]$$

$$(D.25)$$

D.2.3 Turbulent Energy Equation

Using the similar approximation as for the momentum
equation, the finite-difference equation for the turbulent
energy equation (D.4) is:

$$U(i,j) \frac{K^*(i+1,j)-K^*(i-1,j)}{\Delta X_+ + \Delta X_-} + V(i,j) \frac{K^*(i,j+1)-K^*(i,j-1)}{\Delta R_+ + \Delta R_-}$$

$$= P(i,j) - \epsilon(i,j) + \frac{k_q}{R(j)} \frac{2L^*(i)}{\Delta R_+ + \Delta R_-}$$

$$* \left[\frac{R(j)+R(j+1)}{2} \cdot \frac{K^*(i,j)+K^*(i,j+1)}{2} \right.$$

$$* \frac{K^*(i,J+1)-0.5\{K^*(i+1,j)+K^*(i-1,j)\}}{\Delta R_+}$$

$$- \frac{R(j)+R(j-1)}{2} \cdot \frac{K^*(i,j)+K^*(i,j-1)}{2}$$

$$\left. * \frac{0.5\{K^*(i+1,j)+K^*(i-1,j)\}-K^*(i,j-1)}{\Delta R_-} \right]$$

$$(D.26)$$

and,

$$P(i,j) \;.= \; a_1 TAU(i,j) \; . \; \frac{U(i,j+1)-U(i,j-1)}{\Delta R_+ + \Delta R_-} \qquad \text{(D.27)}$$

$$c(i,j) \quad = c_r \; \frac{\left[K^*(i,j)\right]^{3/2}}{L^*(i)} \qquad \text{(D.28)}$$

$$TAU(i,j) = L^2_*(i) \left| \frac{U(i,j+1)-U(i,j-1)}{\Delta R_+ + \Delta R_-} \right| \left(\frac{U(i,j+1)-U(i,j-1)}{\Delta R_+ + \Delta R_-}\right) \qquad \text{(D.29)}$$

D.3 Direct Explicit Scheme

The finite-difference equations for this scheme were used to start the Dufort-Frankel method. These equations can be derived by standard method [95].

the continuity equation is:

$$\frac{R(j+1)}{4\Delta X} \left[U(i+1,j)-U(i,j)+U(i+1,j+1)-u(i,j+1)\right]$$

$$+ \; \frac{1}{\Delta R} \left[R(j+1)V(i+1,j+1)-R(j)V(i+1,j)\right] \; = \; 0$$

$$\text{(D.30)}$$

the momentum equation is:

$$U(i,j) \; \frac{U(i+1,j)-U(i,j)}{\Delta X} \; + \; V(i,j) \; \frac{U(i,j)-U(i,j-1)}{\Delta R_-}$$

$$= \; \frac{1}{R(j)} \; \frac{2}{\Delta R_+ + \Delta R_-} \left[\frac{R(j)+R(j+1)}{2} \; . \; \frac{N(i,j)+N(i,j+1)}{2} \right.$$

$$* \; \frac{U(i,j+1)-U(i,j)}{\Delta R_+} \; - \; \frac{R(j)+R(j-1)}{2}$$

$$\left. * \; \frac{N(i,j)+N(i,j-1)}{2} \; . \; \frac{U(i,j)-U(i,j-1)}{\Delta R_-} \right] \qquad \text{(D.31)}$$

the turbulent energy equation is:

$$U(i,j) \frac{K^*(i+1,j)-K^*(i,j)}{\Delta X} + V(i,j) \frac{K^*(i,j)-K^*(i,j-1)}{\Delta R_-}$$

$$= P(i,j) - \varepsilon(i,j) + \frac{1}{R(j)} \frac{2k_o L^*(i)}{\Delta R_+ + \Delta R_-} \left[\frac{R(j)+R(j+1)}{2} \right.$$

$$* \frac{K^*(i,j)+K^*(i,j+1)}{2} \frac{K^*(i,j+1)-K^*(i,j)}{\Delta R_+} - \frac{R(j)+R(j-1)}{2}$$

$$* \frac{K^*(i,j)+K^*(i,j-1)}{2} \frac{K^*(i,j)-K^*(i,j-1)}{\Delta R_-} \right] \qquad (D.32)$$

where,

$$P(i,j) = a_1 TAU(i,j) \frac{U(i,j+1)-U(i,j)}{\Delta R_+} \qquad (D.33)$$

$$\varepsilon(i,j) = c_r \frac{\left[K^*(i,j)\right]^{3/2}}{L^*(i)} \qquad (D.34)$$

$$TAU(i,j) = L^{*2}(i)\left|\frac{U(i,j+1)-U(i,j)}{\Delta R_+}\right| \frac{U(i,j+1)-U(i,j)}{\Delta R_+}$$

$$(D.35)$$

TAU(i,j) is the shear stress, P(i,j) is the production of turbulent energy and $\varepsilon(i,j)$ is the dissipation of energy.

APPENDIX E

STABILITY ANALYSIS OF THE MOMENTUM EQUATION

The finite-difference solution should ensure:

1)   the solution being obtained is that corresponding to the partial differential equations, at least if the mesh were to be refined in a particular manner.

2)   due to the particular method of solution, round off errors or errors from any source are not amplified or allowed to grow in subsequent steps in the solution.

The first point is called the consistency condition [96] which can be studied by expanding the dependent variables in Taylor series expansions in a manner such that the difference between the partial differential equations and the finite difference representation can be observed [96,97]. This difference is known as truncation error of the equations, and if it vanishes in the limit as the mesh size is shrunk, the finite difference representation is said to be consistent.

The second point is called the stability condition. In dealing with stability and convergence, the ideas of Von Neumann [79] were used.

Let the error growth in U be δ, and according to Neumann [79], it was expressed in first hermonic by:

$$\delta = Ae^{\beta_1 R}e^{i\beta_2 X} \qquad (E.1)$$

With the error, the velocities change to:

$$\check{U}(i,j+1) \sim U(i,j+1) + \delta(i,j+1)$$

$$U(i,j-1) \sim U(i,j-1) - \delta(i,j-1)$$

$$U(i+1,j) \sim U(i+1,j) + \delta(i+1,j)$$

$$\text{(E.2)}$$

and

$$\delta(i,j+1) = Ae^{\beta_1(R+\Delta R)}e^{i\beta_2 X}$$

$$\delta(i,j-1) = Ae^{\beta_1(R-\Delta R)}e^{i\beta_2 X}$$

$$\text{(E.3)}$$

etcetera

Substituting Equation (E.2) in Equation (D.25) and then subtracting Equation (D.25) and using $\Delta R_+ = \Delta R_- = \Delta R$:

$$U(i,j)\,\frac{\delta(i+1,j)-\delta(i-1,j)}{\Delta X_+ + \Delta X_-} + V(i,j)\,\frac{\delta(i,j+1)-\delta(i,j-1)}{\Delta R}$$

$$= \frac{1}{2(\Delta R)^2}\left[\left(1 + \frac{\Delta R}{2R(j)}\right)(N(i,j)+N(i,j+1))\right.$$

$$* \quad \{\delta(i,j+1)-0.5(\delta(i+1,j)+\delta(i-1,j))\} - \left(1 - \frac{\Delta R}{2R(j)}\right)$$

$$* \quad \left.(N(i,j)+N(i,j-1))\{0.5(\delta(i+1,j)+\delta(i-1,j))-\delta(i,j-1)\}\right]$$

$$\text{(E.4)}$$

Substituting Equation (E.3) in Equation (E.4), using $\xi = e^{\beta_1 \Delta R}$ and rearranging:

$$\xi^2 + A_o\xi + B_o = 0 \qquad \text{(E.5)}$$

where,

$$A_o = \frac{-U(i,j)/(\Delta X_+ + \Delta X_-)}{\dfrac{V(i,j)}{2\Delta R} - \dfrac{1}{2(\Delta R)^2}\left(1 + \dfrac{\Delta R}{2R(j)}\right)(N(i,j)+N(i,j+1))}$$

$$B_O = \frac{V(i,j)/2\Delta R + \frac{1}{2(\Delta R)^2}(1 - \frac{\Delta R}{R(j)})(N(i,j)+N(i,j-1))}{\frac{V(i,j)}{2\Delta R} - \frac{1}{2(\Delta R)^2}(1 + \frac{\Delta R}{2R(j)})(N(i,j)+N(i,j+1))}$$

The roots of Equation (E.5) is:

$$\xi = -\frac{A_O}{2} \pm \sqrt{(\frac{A_O}{2})^2 - B_O} \qquad (E.6)$$

According to Von Neumann [79] stability condition,

$$|\xi| \leq 1 \qquad (E.7)$$

For real roots, inequality (E.7) are:

$$-\frac{A_O}{2} + (\frac{A_O}{2})^2 - B_O \leq 1 \qquad (E.8)$$

$$-\frac{A_O}{2} - (\frac{A_O}{2})^2 - B_O \geq -1 \qquad (E.9)$$

For $A_O > 0$, Equation (E.9), and for $A_O < 0$, Equation (E.8) yield:

$$(A_O - B_O) \leq 1 \qquad (E.10)$$

Using expressions for A_O and B_O in Equation (E.10) and re-arranging,

$$-\frac{U(i,j)}{\Delta X_+ + \Delta X_-} \leq \left[\frac{V(i,j)}{\Delta R} + \frac{1}{2\Delta R}(\frac{N(i,j-1)+N(i,j+1)}{\Delta R})(1 - \frac{\Delta R}{2R(j)})\right]$$

$$(E.11)$$

Since $U(i,j)/(X_+ + X_-)$ is always positive,

$$\frac{U(i,j)}{\Delta X_+ + \Delta X_-} \geq \left|\frac{V(i,j)}{\Delta R} + \frac{1}{2\Delta R}(\frac{N(i,j-1)+N(i,j+1)}{\Delta R})(1 - \frac{\Delta R}{2R(j)})\right|$$

or,

$$(\Delta X_+ + \Delta X_-) \leq \frac{U(i,j)\Delta R}{\left|V(i,j) + \frac{N(i,j-1)+N(i,j+1)}{2\Delta R}(1 - \frac{\Delta R}{2R(j)}\right|}$$

For complex roots,

$$\zeta = -\frac{A_o}{2} \pm i \sqrt{B_o - (A_o/2)^2}$$

and Inequality (E.7) becomes:

$$(\frac{A_o}{2})^2 + B_o - (\frac{A_o}{2})^2 \le 1$$

or,   (E.13)

$$B_o \le 1$$

The stability constraint given by Equation (E.12) determines the grid spacings in x-direction with uniform grid spacings in r-direction. As the radial velocity on the outerside is negative, the condition (E.13) satisfies automatically, but it is not satisfied in the inner side where instability did not show at all.

APPENDIX F

COMPUTER PROGRAMMES

```
C****************************************************************
C     THIS IS A PROGRAMME FOR CALCULATING MEAN VELOCITIES FOR AXISYMMETRIC
C     JETS IN STILL AIR FROM THE AXIAL MOMENTUM AND CONTINUITY EQUATIONS BY
C     FINITE DIFFERENCE METHOD OF DUFORT-FRANKEL TYPE.THIS SIMULATES MEAN
C     VELOCITIES FOR ANY ARBITRARY VELOCITY VELOCITY PROFILE AT THE NOZZLE
C     EXIT WHICH HAS TO BE SUPPLIED.FOR TURBULENT JETS, THE EDDY DIFFUSIVITY,
C     IS REPLACED BY AN EXPRESSION IN TERMS OF THE PRANDTL'S MIXING LENGTH AND
C     THE MEAN VELOCITY GRADIENT.
C****************************************************************
0001       REAL LNGP
0002       COMMON/AREA1/J(200),JP(200),JM(200),V(200),VP(200),VM(200),
          1JTEST,JE,JF,VAXIS,USTART,ALT,ALO,
0003       COMMON/AREA2/DELXP,DELXM,DELXT,DELY,DELTA,Y(200),L,S,NY,EV(200)
          1,FCONV,KCONV,KCONV,H,DELMIX,YCONV
0004       COMMON/AREA3/COEFP(200),COEFM(200),SMUP(200),SMUM(200)
0005       COMMON/AREA4/YL,AJ,DENS ,S)
0006       COMMON/AREA5/X,AJ,RAD,LT
0007       COMMON/AREA6/CONST1,CONST2,FRAC,DXFMIN
0008       COMMON/AREA7/YUHALF
0009       COMMON/AREA8/JCORE,JJ,JINFL
0010       READ(5,10) IS,D,IOUT,S,NYI
0011    10 FORMAT(4I3)
0012       WRITE(6,11) IS,D,IOUT,NS,NYI
0013    11 FORMAT(' ','INS=',I3,10X,'IOUT=',I2,10X,'NS=',I2,10X,'NYI=',I3)
0014       READ(5,12) CONST1,CONST2,FRAC,DXFMIN
0015    12 FORMAT(4F12.5)
0016       WRITE(6,13)CONST1,CONST2,FRAC,DXFMIN
0017    13 FORMAT(' ','CONST1=',F12.5,5X,'CONST2=',F12.5,5X,'FRAC=',F12.5,
          15X,'DXFMIN=',F12.5)
0018       READ(5,15) DENS,ANU,USTART,RAD,VAXIS,JE,JF
0019    15 FORMAT(7E10.5)
0020       WRITE(6,16) DENS,ANU,USTART,RAD,VAXIS,JE,JF
0021    16 FORMAT(' ','DENS=',F12.5,5X,'ANU=',F12.5,5X,'USTART=',F12.5,5X,
          1'RAD=',F12.5,5X,' ','VAXIS=',F12.5,5X,'JE=',F12.5,5X,'JF=',F12.5)
0022       READ(5,17) KCH,DXFMAX
0023    17 FORMAT(2F12.5)
0024       WRITE(6,18) KCH,DXFMAX
0025    18 FORMAT(' ','KCH=',F12.5,5X,'DXFMAX=',F12.5)
0026       USTART=300.0
0027    24 KCONV=1J/USTART
0028       SCONV=ANJ
0029       JCONV=USTART
0030       RAD=RAD/KCONV
0031       DELTY=RAD/NYI
0032       JE=JE/USTART
0033       JF=JF/KCONV
0034       USTART=USTART/JCONV
0035       I=0
0036       KKP=0
0037       X=0.0
0038       NYIY=NYI-1
0039       DO 25 J=1,NYI
0040       Y(J)=(J-1)*DELY
0041       J(J)=1.0
0042       V(J)=0.0
0043       EV(J)=1.0
C****************************************************************
C     A TYPICAL MEAN AXIAL VELOCITY DISTRIBUTION AT THE NOZZLE EXIT. MEAN
C     RADIAL VELOCITY AT THE NOZZLE EXIT IS ZERO.
C****************************************************************
0044    25 CONTINUE
0045       J(NNYI+1)=0.993
0046       J(NNYI+2)=0.971
0047       J(NNYI+3)=0.937
0048       J(NNYI+4)=0.91
0049       J(NNYI+5)=0.979
0050       J(NNYI+6)=0.843
0051       J(NNYI+7)=0.714
0052       J(NNYI+8)=0.724
0053       J(NNYI+9)=0.649
0054       J(NNYI+10)=0.25
0055       V(NNYI+11)=0.15
0056       V(NNYI+1)=0.0
0057       V(NNYI+2)=0.0
0058       V(NNYI+3)=0.0
0059       V(NNYI+4)=0.0
0060       J(NNYI+5)=0.0
0061       J(NNYI+6)=0.0
```

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```
FORTRAN IV G LEVEL  21              MAIN              DATE = 79173        11/13/7

3352          V(NNYI+7)=0.0
3303          V(NNYI+9)=0.0
3304          V(NNYI+3)=0.0
3305          V(NNYI+10)=0.0
3306          V(NNYI+11)=0.0
3307          EV(NNYI+2)=1.0
3364          EV(NNYI+3)=1.0
3369          EV(NNYI+4)=1.0
3370          EV(NNYI+5)=1.0
3371          EV(NNYI+6)=1.0
3372          EV(NNYI+7)=1.0
3373          EV(NNYI+9)=1.0
3374          EV(NNYI+3)=1.0
3375          EV(NNYI+10)=1.0
3377          EV(NNYI+11)=1.0
3375          V(NNYI=NNYI+4+J+4
3379          DO 43 J=NNYI+2,30
3380          Y(J)=(J-1)*DELY
3381          J(J)=UF
3382          V(J)=0.0
3383          EV(J)=1.0
3384    43 CONTINUE
3385          DELTA=RAD
3386          DELHIX=0.0
C*********************************************************************
C     THE SOLUTION STARTS BY CALCULATING THE INFORMATION AT THE SECOND STATION
C     BY SIMPLE EXPLICIT SCHEME WITH KNOWN INFORMATION AT THE FIRST STATION.
C     THE DUFORT-FRANKEL SCHEME STARTS CALCULATION WITH THE KNOWN INFORMATION
C     AT THE TWO PREVIOUS STATIONS.
C*********************************************************************
3387    54 WRITE(6,55)
3388    55 FORMAT(' ',15X,'SIMPLE EXPLICIT SCHEME')
3389          IY=NNYI
3390          N=1
3391          I=1
3392          LT=0
3393          GO TO 57
3394    56 I=I+1
3395          IF(I.GT.N)GO TO 65
3396          LT=1
3397    57 CALL SDELX
3391          X=X+DELXP
3399          CALL UVEL
3100          CALL VVEL
3101          DELXI=DELXP
3102          DO 50 J=1,230
3103          JA(J)=U(J)
3104          J(J)=UP(J)
3105          VA(J)=V(J)
3106    50 V(J)=VP(J)
3107          CALL LFVISC
3108          CALL OUTPUT
3109          GO TO 56
3110    65 WRITE(6,66)
3111    66 FORMAT('_',15X,'DUFORT-FRANKEL SCHEME')
3112          I=1
3113          K=1
3114          R=1
3115    51 DXF= DELX/DELMIX
3116          WRITE(6,52) DXF
3117    52 FORMAT(' ','STA-TION DXF=',E12.5)
3118    57 IF(J(1).LT.0.0.R.R=JSTART))GO TO 63
3119    54 DELXP=DXF*DELMIX
3120          GO TO 51
3121    63 IF(K.EQ.1)GO TO 64
3122          GO TO 61
3123    64 DELXP=DELXM
3124          DXF=DELXP/DELTA
3125          GO TO 77
3126    65 DELXP=DXF*DELTA
3127          GO TO 51
3128    77 DISTI=X/RAD
3129          WRITE(6,52) DISTI
3130    52 FORMAT(' ','PREDICTED STATION WEIGHT=',E14.5)
3131          J=K+1
3132    61 FMC1=0.0
3133          DO 125 J=2,IY
```

VERY POOR PRINT

```
1134          FJUBANS((V(J)/(C.)*DELY*U(J)))
1135          A.INC(=AAXY(F.INC,FJ)C1)
1136   123 CONTINUE
1137          JELX(I=1.)/J.C1-JELX?
1138          IF(JELX?.LE.)IG(?I)JU TU 13)
1139          JELXP=).5*JELX?I
1140   130 J.LXT=JELX(?+JELXV
1141          (AX+JELX?
1142          CALL JVEL
1143          CALL VVEL
1144          DJ 7) J=1,2))
1145          JR(J)=U(J)
1146          J(J)=UP(J)
1147          VR(J)=V(J)
1148    70 V(J)=VP(J)
1149          CALL CFVISC
1150          I=AI-NS
1151          II=(IJUT-N
1152          IF(IN.EQ.11)GU TC 75
1153          GU TJ 76
1154    75 CALL OUTPUT
1155    76 I=I+I
1156          IF(I.EO.1)0) JXFMAX=0.5*CXFMAX
1157          IF(I.EO.2)0)JXFMAX=0.75*CXFMAX
1158          JELX=A*JELXP
1159          JXF=INCR?*JXF
1160          IF(JXF.JE.JXFMAX)GU TJ 40
1161          GU TO 45
1162    40 WRITE(6,42) JXF
1163    42 FJRMAT(' ','JXF HAS REACHED THE VALUE=',G14.J)
1164          J(F=JXFMAX
1165          INCR=1.)
1166    45 IF(I.GT.IEND)GO TO 100
1167          GU TJ 67
1168   100 STOP
1169          END
```

```
0001          SUBROUTINE SDELX
C*****************************************************************************
C     USING THE STABILITY CONSTRAINTS DERIVED BY VON NEUMANN METHOD, THE AXIAL
C     UNEQUAL GRID SPACINGS ARE CALCULATED. THE RADIAL GRIDS ARE UNIFORM.
C*****************************************************************************
0002          COMMON/AREA1/U(2)),UP(2)),UR(2)),V(200),VP(2)),VR(2)),
     1 JTEST,JE,JF,VAXIS,JSTART,AL1,AL2
0003          COMMON/AREA2/DELXP,DELXM,DELT,DELY,JELTA,Y(2)),I,NS,NY,EV(2))
     1 UCONV,UCONV,XCONV,H,DCLMIX
0004          COMMON/AREA4/COEFP(2)),CCEFM(200),SMUP(200),SMUM(200)
0005          COMMON/AREA5/X,XO,RAD,LT
0006          COMMON/AREA6/CONST1,CONST2,FRAC,JXFMIN
0007    11 Y(LT)=0.0
0008          SMUP(1)=EV(1)+EV(2)
0009          DO 4 J=2,199
0010          Y(J)=(J-1)*DELY
0011          YJ=Y(J)
0012          YJP=YJ+DELY
0013          YJM=Y(J-1)
0014          CC1=A.0*U(J)+YJ*DELY
0015          SMUP(J)=EV(J)+EV(J+1)
0016          SMUM(J)=EV(J)+EV(J-1)
0017          IF(U(J).EQ.0.0)GO TJ 4
0018          COEFP(J)=(YJ+YJP)/(CC1*DELY)
0019          COEFM(J)=(YJ+YJM)/(CC1*DELY)
0020     4 CONTINUE
0021          IF(LT.EQ.0.0)GO TO 10
0022          GO TO 15
0023    10 DELXP=10.0*DELY
0024          GO TO 15
0025    15 CC3=0.0
0026          CC4=0.0
0027          DO 5 J=2,NY
0028          AL1=SMUP(J)
0029          AL2=SMUM(J)
0030          SPMAX=AMAX1(AL1,CC4)
0031          SMMAX=AMAX1(AL2,CC4)
0032          IF(CC1.EQ.0.0)GO TO 5
0033          CC2=(ABS(V(J))/((J)*DELY)) +COEFP(J)*SPMAX+COEFM(J)*SMMAX
0034          CC3=AMAX1(CC2,CC3)
0035     5 CONTINUE
0036    23 DELXP1=0.5/CC3
0037          IF(I.EQ.1)GO TO 7
0038          GO TO 6
0039     7 DELXP2=JXFMIN*DEL4IX
0040          IF(DELXP2.GT.)DELXP1)GO TO 4
0041          DELXP1=DELXP2
0042     9 IF(DELXP1.GT.(FRAC*DELXP))GO TO 14
0043     8 DELXP=DELXP1
0044    14 DELXP=FRAC*DELXP
0045    13 RETURN
0046          END
```

```
FORTRAN IV G LEVEL   21                 UVEL            DATE = 79179          11/10/96

0075                       VJ=V(J)
0076                       UH=-VJ*(UJP-UJM)/(2.0*DELY)
0077                       JI=(UJP-0.5*U*J)*UB*2.0
0078                       JG=(UJM-0.5*U*J)*UC*2.0
0079                       UP(J)=(UD+UH+UI+UG)/(UA+UB+UC)
0080                       UPJ=UP(J)
0081                       IF(UPJ.GE.UE)GO TO 125
0082                       UP(J)=UE
0083                 125 IF(K.GT.1)GU TO 40
0084                       GO TO 33
0085                  40 IF(K.GT.2)GO TO 41
0086                       UP(1)=(4.0*UP(2)-UP(3))/3.0
0087                  41 IF(4.GT.1)GO TO 33
0088                       UPMEAN=(UP(1)+UE)/2.0
0089                       IF(UPJ.LE.UPMEAN)GO TO 42
0090                       GO TO 33
0091                  42 JJ=J
0092                       JM=J-1
0093                       UPJJ=UP(JJ)
0094                       UPJM=UP(JM)
0095                       YJJ=(JJ-1)*DELY
0096                       YJM=(JM-1)*DELY
0097                       YUHALF=YJM+((YJJ-YJM)/(UPJJ-UPJM))*(UPMEAN-UPJM)
0098                       M=M+1
0099                  33 K=K+1
0100                       IF(K.LT.3)GO TO 21
0101                       IF(UP(1).LT.(0.99*USTART))GU TO 23
0102                  21 IF(L.GT.1)GO TO 22
0103                       IF(UPJ.LT.0.99*USTART)GO TO 24
0104                       GO TO 22
0105                  24 YCORE=(J-1)*DELY
0106                       JCORE=J
0107                       L=L+1
0108                       GO TO 22
0109                  23 YCORE=0.0
0110                       JCORE=1
0111                  22 IF(K.LT.4)GO TO 26
0112                       UTEST=(UP(1)-UPJ)/(UP(1)-UE)
0113                       DER(J-1)=ABS(UPJ-UP(J-2))
0114                       DER(J-2)=ABS(UP(J-1)-UP(J-3))
0115                       IF(IM.GT.1)GO TO 100
0116                       IF(DER(J-1).LT.DER(J-2))GO TO 105
0117                       GO TO 100
0118                 105 JINFL=J-2
0119                       IM=IM+1
0120                 100 IF(UTEST.GE.0.99)GO TO 27
0121                  26 CONTINUE
0122                  27 DELTA=(J-1)*DELY
0123                       DELMIX=DELTA-YCORE
0124                       NY=J
0125                       NNY=J+1
0126                       DO 28 J=NNY,200
0127                  28 UP(J)=UF
0128                       VP(1)=VAXIS
0129                  29 RETURN
0130                       END
```

```
FORTRAN IV G LEVEL   21                 VVEL            DATE = 79179          11/10/96

0001                       SUBROUTINE  VVEL
0002                       COMMON/AREA1/U(200),UP(200),UM(200),V(200),VP(200),VM(200),
                          1UTEST,UE,UF,VAXIS,USTART,AL1,AL2
0003                       COMMON/AREA2/DELXP,DELXM,DELXT,DELY,DELTA,Y(200),I,NS,NY,CV(200)
                          1,ECONV,UCONV,XCONV,M,DELMIX
0004                       IF(I.GT.NS)GO TO 20
0005                       DO 10 J=1,199
0006                       YJ=Y(J)
0007                       YJP=YJ+DELY
0008                       VP(J+1)=YJ*VP(J)/YJP-(DELY/(4.0*DELXP*YJP))*(YJP+YJ)*(UP(J+1)
                          1+UP(J)-U (J+1)-U (J))
0009                       IF(ABS(VP(J+1)).GT.0.6) VP(J+1)=VP(J)
0010                  10 CONTINUE
0011                       GO TO 25
0012                  20 DO 21 J=1,199
0013                       YJ=Y(J)
0014                       YJP=YJ+DELY
0015                       VP(J+1)=YJ*VP(J)/YJP-(DELY/(4.0*DELXT*YJP))*(YJP+YJ)*(UP(J+1)
                          1+UP(J)-UM(J+1)-UM(J))
0016                       IF(ABS(VP(J+1)).GT.0.6) VP(J+1)=VP(J)
0017                  21 CONTINUE
0018                  25 RETURN
0019                       END
```

VERY POOR PRINT

```
3301            SUBROUTINE  EFVISC
        C*****************************************************************
        C      PRANDTL'S MIXING LENGTH MODEL FOR EDDY DIFFUSIVITY IS USED IN THE INITIAL
        C      REGIONS. THE KADJI-PLETCHER'S MODEL AND THE SCH-TAUB MODEL ARE COMBINED
        C      TOGETHER FOR THE TRANSITION REGION.
        C*****************************************************************
3302            DIMENSION FUNC(200)
3303            COMMON/AREA1/U(200),UP(200),U (200),V(200),VP(200),VM(200),
               1UTEST,UE,UF,VAXIS,USTART,ALI,AL2
3304            COMMON/AREA2/DELXP,DELXM,DELAT,DELY,DELTA,Y(200),I,IS,NY,EV(200)
               1,ICONV,JCDIV,CONV,N,DELMIX,SH
3305            COMMON/AREA4/YL,ANG,DENS
3306            COMMON/AREA5/X,XO,RAD,LT
3307            COMMON/AREA6/CONST1,CONST2,FANC,DAPAIN,
3308            COMMON/AREA7/YUHALF
3309            COMMON/AREA9/JCORE,JJ,JINAL
3310            CONST1=0.13625
3311            CONST2=0.015
3312            YHALF=YUHALF
3313            JC=JCORE
3314            NYJ=NY
3315            J=1
3316          5 J=J+1
3317            IF(Y(J).LT.YJHALF)GO TO 5
3318            DUDP=(U(J+1)-U(J-1))/(2.0+0.9*DELY)
3319            S=1.0/ABS(DUDP)
3320            IF(X/RAD.GE..01)GO TO 10
3321         15 XL=CONST1*S
3322         10 DO 45 J=2,NYJ
3323            EV(J)=1.0+(XL*XL)*ABS((U(J+1)-U(J-1))/(2.0*DELY))
3324         45 CONTINUE
3325            EV(1)=1.0
3326            GO TO 200
3327        100 CONTINUE
3328            KKKP=KKKP+1
3329            NJ1=(NYJ-JC+1)/2
3330            NJ2=(NYJ+2-JC)/2
3331            IF(NJ1.EQ.NJ2)GO TO 120
3332            NJ=NYJ
3333            GO TO 125
3334        120 NJ=NYJ+1
3335        125 SUM=0.0
3336            CDEF=0.0*(DELY/3.0)
3337            JC1=JC+1
3338            JC2=JC+2
3339            FUNC(JC)=ABS(UE-U(JC))*Y(JC)
3340            FUNC(JC1)=ABS(UF-U(JC1))*Y(JC1)
3341            DO 130 J=JC2,NJ,2
3342            FUNC(J-1)=ABS(UE-U(J-1))*Y(J-1)
3343            FUNC(J)=ABS(UE-U(J))*Y(J)
3344            TERM=FUNC(J)+4.0*FUNC(J-1)+FUNC(J-2)
3345            SUM=SUM+TERM
3346        130 CONTINUE
3347            DINTEG=SUM*CDEF/RAD
3348            DINT1=DINTEG
3349            DO 33 JC=1,JCORE
3350         33 EV(JC)=CONST2*DINTEG
3351            DO 135 J=JC1,NYJ
3352            IF(ABS(V(J)).GE.UM(J))V(J)=VM(J-1)
3353            IF(ABS(V(J)).GE.U(J))V(J)=V(J-1)
3354            PA1=Y(J)/YA)
3355            IF(J(J).GT.J.1)GO TO 50
3356            DDJ=ABS(U(J-1)-U(J))
3357            IF(DDJ.GT.0.015)U(J)=U(J-1)-0.015
3358         50 CONTINUE
3359            IF(PA1.LT.1.0)CONST2=0.015
3360            IF(PA1.GE.1.0)CONST2=0.015
3361            TERM=(FUNC(J)+FUNC(J-1))*(DELY/2.0)
3362            DINTEG=DINTEG-TERM/RAD
3363            XINT=1.0
3364            IF(PA1.LE.1.0)GO TO 950
3365            DINTEG=DINT1
3366            IF((Y(J)/YHALF).GT.0.6)GO TO 925
3367            GO TO 950
3368        925 Z=(Y(J)/YHALF-0.5)**2.0
3369            XINT=(J.5)**2
3370        950 EV(J)=XINT*CONST2*DINTEG
3371            IF(PA1.LT.1.0)GO TO 135
```

```
3172            IF(EV(J).GT.EV(J-1))EV(J)=EV(J-1)
3173        135 CONTINUE
3174        200 RETURN
3175            END
```

```
FORTRAN IV G LEVEL   21              OUTPUT              DATE = 79174        11/13/76

0001            SUBROUTINE OUTPUT
0002            COMMON/AREA1/U(200),UP(200),UM(200),V(200),VP(200),VM(200),
             1 JTEST,UE,UF,VAXIS,USTART,ALI,XL2
0003            COMMON/AREA2/DELXP,DELXN,DELXT,DELY,DELTA,Y(200),I,JS,NY,EV(200)
             1 ,XCONV,UCONV,VCONV,N,DELVIX ,S3,YCLPE
0004            COMMON/AREA5/X,XJ,RAD,XL2
0005            COMMON/AREA7/YUHALF
0006            N=N+1
0007            NNY=NY+1
0008            N1J=NNY
0009            DIST=X*XCONV
0010            DDEL=DELTA/RAD
0011            DCLPE=(DELTA-DELVIX)/RAD
0012            XBAR=X/RAD
0013            SL3=S3 /RAD
0014            RUHALF=YUHALF/RAD
0015            UCL=(U(1)-UE)/(USTART-UE)
0016            DF1=DELY*XCONV*12.0
0017            DXP=DELXP*XCONV*12.0
0018            DXN=DELXN*XCONV*12.0
0019            WRITE(6,10)
0020    10 FORMAT(' ','*****************************************************')
0021            WRITE(6,15)I,DIST,DDEL,DCLPE
0022    15 FORMAT(' ',5X,'I=',I4,5X,'X_DISTANCE=',G12.5,2X,'HALF_WIDTH=',
             1G12.5,5X,'CLPE RADIUS=',G12.5)
0023            WRITE(6,16) XBAR,RUHALF,UCL
0024    16 FORMAT(' ',15X,'X/R0=',G12.5,3X,'HALF_RAD=',G12.5,3X,'CENTRE VELOC
             1ITY=',G12.5)
0025            WRITE(6,17) DF1,DXP,DXN,SL3
0026    17 FORMAT(' ',15X,'DF1=',G12.5,3X,'DXP=',G12.5,3X,'DXN=',G12.5,3X,'SL
             13=',G12.5)
0027            WRITE(6,20)
0028    20 FORMAT(' ',2X,'J',3X,'YBARU',7X,'UBARU',7X,'V_VEL',7X,'VISCOSITY',
             15X,'(4-RH)/B')
0029            DO 30 J=1,NNJ
0030            E/J=EV(J)*ECONV
0031            YJ=Y(J)
0032            YJA=YJ=YJ/RAD
0033            YETA=(YBARU-RUHALF)/SL3
0034            JJ=U(J)
0035            VJ=V(J)
0036            IF(J.EQ.1)GO TO 1
0037            DURA=(U(J+1)-U(J-1))/(2.0*DELY)
0038            SSTR=EV(J)*DUR
0039            DUR=DUR*RAD
0040            GO TO 2
0041    1  DUR=0.0
0042            SSTR=EV(J)*DUR
0043    2  WRITE(6,25) J,YBARU,JJ,VJ,E/J,YETA,SSTR
0044    25 FORMAT(' ',I3,7(2X,G12.5))
0045    30 CONTINUE
0046            WRITE(6,31)
0047    31 FORMAT(' ','*****************************************************')
0048    41 RETURN
0049            END
```

```
FORTRAN IV G LEVEL  21                    MAIN            DATE = 79179        13/06/43

      C**************************************************************************
      C     THIS PROGRAMME SIMULATES THE TURBULENT KINETIC ENERGY FROM A SINGLE
      C     TRANSPORT EQUATION FOR TURBULENT KINETIC ENERGY ASSOCIATED WITH AN
      C     EMPIRICAL EQUATION FOR LENGTH SCALE IN AN ALGEBRAIC FLOW.THE CALCULATION
      C     STARTS WITH A SIMPLE EXPLICIT FINITE DIFFERENCE METHOD AND FOLLOWS WITH
      C     THE EXPLICIT FINITE DIFFERENCE METHOD OF DUFORT FRANKEL TYPE.THIS
      C     PROGRAMME USES AN EMPIRICAL VELOCITY PROFILE FOR MEAN AXIAL VELOCITY.
      C     THE TURBULENT NORMAL STRESSES ARE PROPORTIONAL TO THE TURBULENT KINETIC
      C     ENERGY.ALL CALCULATION WAS PERFORMED FOR THE INITIAL REGION OF THE
      C     AXISYMMETRIC JETS.
      C     THIS PROGRAMME DETERMINES THE PRINCIPAL  CO-ORDINATE SYSTEM OF REYNOLD'S
      C     STRESSES.
      C     BESIDES THE PRANDTL'S MODEL OF SHEAR STRESS, THE ENERGY MODEL AND THE
      C     LAUNDER'S MODEL OF SHEAR STRESS ARE ALSO EXAMINED.
      C     A EMPIRICAL MODEL OF INTERMITTENCY WAS USED IN CALCULATION.
      C**************************************************************************
0001        DIMENSION X(13),XL(13),XA(13),B(13),P1(13),P2(13),R(40),XG(3,90),
           1R3(90),U(3,33),V(3,40),ERU3(3,90),PH2(3,33),TAU(3,33)
           1,J13S(3,90),J1S(3,90),C(3,30),S(3,90),JUF(3,9),E(3,9)
           1,TU(3,30),FV(3,30),TG(3,30),C1FT(3,90),GUK(3,90),H(13)
           1,JIFF(3,90),TRA(3,90),AS(13),PS(13),TAJH(3,90),RH1(13)
      READ1, DR1,X1,R0,C5,CK0,XY2
      READ 5,(RO(I,4),X=1,NYT)
0002    1 FORMAT(5F5.3)(2)
0003    5 FORMAT(16F5.3)
0004      STG=0.730
0005      SL=0.1025**2
0006      XK=0.10625
0007      SS=0.0
0008      DX=DR1/12.
0009      DX=DX1/12.
0010      DXP=DR
0011      DPN=DR
0012      DXP=DR
0013      DXN=DR
0014      FJ=R0/12.
0015      X(1)=0.01
0016      XA(1)=X(1)/R0
0017      B(1)=0.01
0018      RH(1)=0.138421*R0*H(1)+(1.0-0.07158050*B(1)**2)**0.5
0019      P1(1)=RH(1)-0.730*B(1)
0020      P2(1)=RH(1)+0.3*B(1)
0021      RH1(1)=RH(1)
0022      S1=X(1)-DXP
0023      JXL=0*0
0024      C*=C5
0025      IS=0
0026      I=0
0027    40 I=I+1
0028       IS=IS+1
0029       S0=0.
0030       X(I)=S1+DXP
0031       S1=X(I)
0032       XA(I)=X(I)/R0
0033       X(I+1)=X(I)+DXP
0034       XL(I)=0.1065*H(I)*R0
0035       B7=0.437
0036       B3=0.606
0037       B0=1.267
0038       P(I)=0.
0039       F1=2.0*SL*(0.15320860-0.04712130*B(I)**2+.09423667*B(I)*(1.0-
           10.07158050*B(I)**2)**0.5 )
0040       F2=0.0490073*-0.31363304*B(I)**2-.06532057*B(I)*(1.0-0.07158050
           1*B(I)**2)**0.5
0041       FF=F1/F2
           IF(F(1).GT.1)GO TO 45
           B(I+1)=B(I)+(DXP/R0)*FF
           GO TO 46
0042    45 B(I+1)=B(I-1)+2.0*(DXP/R0)*FF
0043    46 RH(I+1)=0.163421*H(I+1)+(1.0-0.07158050*B(I+1)**2)**0.5
0044       R1(I+1)=RH(I+1)-0.730*B(I+1)
0045       P2(I+1)=RH(I+1)-0.80*B(I+1)
0046       RH1(I+1)=RH(I+1)
0047       S1=FF
           S=H*0.138421*J*DXP*(0.07151*B*(I)*DX/(1.0-0.07158*B*(I)**2)*0.5)
           IF(B1(I).LE.0.0001)GO TO 30
0048    50 J=0
```

```
FORTRAN IV G LEVEL  21              MAIN            DATE = 79179        13/05/33

0057            60 J=J+1
0058               P(J+1)=R(J)+DR
0059               RH(J+1)=R(J+1)/R0
0060               IF(RH(J).LE.R1(I))GO TO 50
0061            70 E(I,J)=(RH(J)-RH(I))/X1(I)
0062               IF(E(I,J).GT.0.5)GO TO 75
0063               U(I,J)=0.5-E(I,J)+0.05*E(I,J)**2+0.0C*E(I,J)**3
0064               GUR(I,J)=-1.0+0.1)*E(I,J)+1.4*E(I,J)**2
0065               GUR(I,J)=-GUR(I,J)
0066               GO TO 79
0067            75 EKE=4.7)*(U(I,J)-0.5)
0068               U(I,J)=0.095*(1.7-(EXP(EKE)-EXP(-EKE))/(EXP(EKE)+EXP(-EKE)))
0069               T4=(EXP(EKE)-EXP(-EKE))/(EXP(EKE)+EXP(-EKE))
0070               GUR(I,J)=-0.095*4.7)*(1.0-T4**2)
0071               GUR(I,J)=-GUR(I,J)
0072            79 GUXR(I,J)=-((E(I,J)*G0X*GAMX)/R(I))*GUR(I,J)
0073               GUXR(I,J)=GUXR(I,J)*R(J)/R0
0074               S/=S/+GUXR(I,J)
0075               V(I,J)=S/*0.5/R(J)
0076               IF(V(I,J)-.)01)-5.0)80,00
0077            80 U(I,J)=1.
0078               E(I,J)=-0.735
0079               V(I,J)=0.
0080               GUR(I,J)=0.
0081               GO TO 60
0082            85 M2=J+5
0083               DO 84 M1=J,M2
0084               U(I,M1)=U(I,J-1)
0085               V(I,M1)=V(I,J-1)
0086            84 GUR(I,M1)=GUR(I,J-1)
0087            90 IF(I.GE.2)GO TO 100
0088               J=J+5
0089               DO 95 L=1,J
0090            95 U(I,L)=S0(I,L)**0.5
0091           100 J2=0
0092            95 J2=J2+1
0093               IF(U(I,J2).GT.0.5)GO TO 96
0094               DU1=U(I,J2)-0.5
0095               IF(ABS(DU1).GT.0.001)GO TO 97
0096               RH(I)=R(J2)
0097               GO TO 98
0098            97 RH(I)=(R(J2-1)+R(J2))/2.0
0099            98 IF(I.EQ.1)GO TO 49
0100               H=RH(I)/R0
0101               AA=1.0
0102               G2=1.0-0.15*(4H-R1(I))
0103               K=0
0104           105 K=K+1
0105               IF(I.GT.2)GO TO 106
0106               AL1=((P(K)-RH(I-1))/X(I-1))**2
0107               AL1=EXP(-AL1)
0108               G(I-1,K)=XL(I-1)
0109           106 AL1=((R(K)-RH(I))/X(I))**2
0110               AL1=EXP(-AL1)
0111               X0(I,K)=XL(I)
0112               IF(I.GT.2)GO TO 118
0113               0.SS(I-1,K)=S0(I-1,K)*Q(I-1,K)/XS(I-1,K)*C7
0114               TAU(I-1,K)=(XK*GUR(I-1,K))**2
0115               P.C0(I-1,K)=AA*TAU(I-1,K)*GUR(I-1,K)/(P(I-1)**4:*(1.50))
0116               J=J-1
0117           118 IF(K.LE.J)GO TO 110
0118               R(K+1)=R(K)+DH
0119               RH(K+1)=R(K+1)/R0
0120           110 IF(P(K)/RH(I)-96)15,25,25
0121            15 Z=(3H-P(K)/RH(I))**2.5*5.J*(1.0+EXP(-XA(I)/1.5))
0122               GO TO 31
0123            25 IF(R(K)/RH(I)-AA)120,120,20
0124            30 Z=(P(K)/RH(I)-AA)**2.5*3.5
0125            31 XINT(I,K)=0.5**2
0126               GO TO 130
0127           120 XINT(I,K)=1.
0128           130 CONTINUE
0129               IF(I.GT.2)GO TO 132
0130               XINT(I-1,K)=XINT(I,K)
0131               XINT(I-1,K+1)=XINT(I,K)
0132           132 IF(K.GT.J)GO TO 135
0133               IF(I.GT.2)GO TO 130
0134               AN=S0(I-1,K)-S0(I-1,K-1)
```

```
FORTRAN IV G LEVEL  21              MAIN              DATE = 79170           13/06/33

0135          A)=SQ(I-1,K+1)-SQ(I-1,K)
0136          A7=A+B
0137          IF(V(I-1,K).LT.0.)A7=A9
0138          C1=(R(K+1)+R(K))*(QJ(I-1,K+1)+QJ(I-1,K))*XG(I-1,K)*BK)/
              ((4.*R(K)*QJ(I-1,K)*DP*2)
0139          C2=(R(K-1)+R(K))*(QJ(I-1,K-1)+QJ(I-1,K))*XG(I-1,K)*BK)/
              ((4.*R(K)*QJ(I-1,K)*DP*N)
0140          SQ(I,K)=SQ(I-1,K)-(V(I-1,K)*DX*A7)/(QJ(I-1,K)*CR)+DX*(PROD(I-1,K)
              )-)(SS(I-1,K))/QJ(I-1,K)+2.*DX*(C1*(SQ(I-1,K+1)-SQ(I-1,K))-
              C2*(SQ(I-1,K)-SQ(I-1,K-1)))/(DRP+DRN)
0141          IF(SQ(I,K).LT.0.)SQ(I,K)=0.00001
0142          Q(I,K)=SQ(I,K)**.5
0143          IF(I.EQ.2)GO TO 155
0144  150     C1=(R(K+1)+R(K))*(XG(I-1,K+1)+XG(I-1,K))*(Q(I-1,K+1)+Q(I-1,K))/(5.
              )*DP)
0145          C2=(R(K-1)+R(K))*(XG(I-1,K-1)+XG(I-1,K))*(Q(I-1,K-1)+Q(I-1,K))/(5.
              )*DN)
0146          C3=QJ(I-1,K)/((DXP+DXN)+BK)*(C1+C2/(R(K)*(DRP+DRN)))
0147          P1=SQ(I-1,K+1)-SQ(I-1,K+1)
0148          D1=QJ(I-1,K)*SQ(I-2,K)/(DXP+DX.)-V(I-1,K)*P2
              /(DPP+DRN)+PRUD(I-1,K)-Q(SS(I-1,K)*2.+RXDN*(C1*(SQ(I-1,K+1)-
              )..5+SQ(I-2,K)-C2*(.5+SQ(I-2,K)-SQ(I-1,K-1)))/(R(K)*(DXN+DXP))
0149          SQ(I,K)=D1/C3
0150          IF(SQ(I,K).LT.0.0)SQ(I,K)=0.0001
0151          Q(I,K)=SQ(I,K)**.5
0152  155     TAU(I,K)=(XK*QJR(I,K))**2
0153          Q(SS(I,K)=SQ(I,K)*Q(I,K)/XG(I,K)*C7
0154          TU(I,K)=(2.0**7*SQ(I,K))**0.5
0155          TV(I,K)=(2.)**4*SQ(I,K))**0.5
0156          T.(I,K)=(2.)**09*SQ(I,K))**0.5
0157          PROD(I,K)=A*TAU(I,K)*QUR(I,K)/(B(I)*+0=1.50)
0158          IF(P9(K).LE.P2(I))GO TO 155
0159          R4(I)=RM(I)/R9
0160          TAU(I,1)=TAU(I,2)
0161          TU(I,1)=TU(I,2)
0162          TV(I,1)=TV(I,2)
0163          T.(I,1)=T.(I,2)
0164          Q(SS(I,1)=Q(SS(I,2)
0165          PROD(I,1)=PROD(I,2)
0166          SQ(I,1)=SQ(I,2)
0167          Q(I,1)=SQ(I,2)**0.5
0168          D=190  M=1,K
0169          Q(SS(I,M)=C3*SQ(I,M)*Q(I,M)*P9/XG(I,M)
0170  190     PROD(I,M)=-A*TAU(I,M)*QUR(I,M)/R(I)
0171          L1=K+5
0172          DO 230  L2=K,L1
0173          SQ(I,L2)=SQ(I,K)
0174          PRUD(I,L2)=PROD(I,K)
0175          Q(SS(I,L2)=Q(SS(I,K)
0176          E(I,L2)=E(I,K)
0177          TAU(I,L2)=TAU(I,K)
0178          XINT(I,L2)=XINT(I,K)
0179          XINT(I-1,L2)=XINT(I-1,K)
0180          TU(I,L2)=TU(I,K)
0181          TV(I,L2)=TV(I,K)
0182          T.(I,L2)=T.(I,K)
0183          XG(I,L2)=XG(I,K)
0184          Q(SS(I,L2)=Q(SS(I,K)
0185          PROD(I,L2)=PROD(I,K)
0186          QUR(I,L2)=QUR(I,K)
0187  230     Q(I,L2)=Q(I,K)
0188          IF(I.LT.3)GO TO 40
0189          I=I-1
0190          DO 190  M=1,K
0191          P1=(SQ(I+1,M)-SQ(I-1,M))/(DXP+DXN)
0192          IF((P1.LT.0.0).AND.(R9(M).LT.1.0))P1=-P1
0193          IF((P1.GT.0.0).AND.(R4(M).GT.1.0))GO TO 245
0194          GO TO 246
0195  245     IF((SQ(I,M).LE.SQ(I,M2)).AND.(V(I,M).GT.0.0))P1=-P1
0196  246     IF(M.EQ.1)P2=(SQ(I,M+1)-SQ(I,M))/DRP
0197          IF(M.GT.1)P2=(SQ(I,M+1)-SQ(I,M-1))/(DRP+DRN)
0198          IF(V(I,M).LT.0)P2=-P2
0199          QU(SQ(I,M)=PROD(I,M)*XINT(I,M)**2
0200          Q(I,M)=Q(I,M)*XINT(I,M)
0201          S.(I,M)=SQ(I,M)-XINT(I,M)**2
0202          Q(SS(I,M)=Q(SS(I,M)*XINT(I,M)**2
0203          TAU(I,M)=0.5*SQ(I,M)
0204          XG(I,M)=XG(I,M)/E)
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0205              TAU(I,4)=TAU(I,4)+XINT(I,4)**2
0206              TU(I,4)=TU(I,4)+XINT(I,4)
0207              TV(I,4)=TV(I,4)+XINT(I,4)
0208              TX(I,4)=TX(I,4)+XINT(I,4)
0209              TTA(I,4)=TU(I,4)+V(I,4)**2)*C)+XINT(I,4)**2
0210              IF(ABS(X4(I)-2.0).LE.0.001)GO TO 2
0211              IF(ABS(X4(I)-4.0).LE.0.001)GO TO 2
0212              IF(ABS(X4(I)-6.0).LE.0.001)GO TO 2
0213              IF(ABS(X4(I)-8.0).LE.0.001)GO TO 2
0214              GO TO 190
0215            2 AS(1)=TU(I,4)**2
0216              AS(2)=-TAU(I,4)
0217              AS(3)=TV(I,4)**2
0218              AS(4)=0.0
0219              AS(5)=0.0
0220              AS(6)=TX(I,4)**2
0221              MV=0
0222              N=0
0223              IF(ABS(X4(I)-4.0)-0.001) 6,6,7
0224            6 IF(ABS(AS(2)).LT.0.0004) AS(2)=0.000
0225            7 CALL SIMSEX(AS,75,MV,IV)
0226              AS(1)=AS(1)/S(I,4)**0.5
0227              AS(3)=AS(3)/S(I,4)**0.5
0228              AS(6)=AS(6)/S(I,4)**0.5
0229              SER=TAU(I,4)/SQ(I,4)
0230              IF(SER.GT.0.5) SER=0.5
0231              TAU=0.40+S(I,4)+XINT(I,4)
0232              TKE=SQ(I,4)
0233              TN0IS=(TKE**2)/DIS(I,4)
0234              TAUR=0.09+TN0IS+SQU+(I,4)/2(I)
0235              DO 5 (I)=1,9
0236            5 AS(I)=ARCOS(AS(I))*180.0/3.141593
0237              THETA=HS(I)*3.141593/180.0
0238              THETA=TAN(THETA)
0239              FIP=(TKE**0.5)*(0.02+0.45*THETA+0.07*THETA**2)
0240              PRINT 3, AS(I),AS(3),AS(6),X4(I),SER,TAU,  E45    ,TAUR1
0241              PRINT 4, (HS(I),I=1,9)
0242            3 FORMAT(' ','A(1)=',F9.5,6X,'A(2)=',F9.5,6X,'A(6)=',F9.5,
                 17X,'D/RJ=',F9.5,1X,'TAU/SQ=',F9.5,1X,'TAUR=',F9.5,1X,'THPE=',F9.5,
                 1X,'TAU=',F9.5)
                4 FORMAT(' ',3F9.2,2X,3F6.2,3X,3F5.2)
0243          190 DIFF(I,4)=-(TKA(I,4)+PRL(I,4)+DIS(I,4))
0244          200 PRINT 240
0245          240 FORMAT(' ',' ','J',3X,'U(I,J)',3X,'V(I,J)',3X,'TU(I,J)',3X,'SU(I,J)',3X,'DI(I,J)'
                 1,4X,'   X5(I,J)',7X,'TAUR(I,J)',3X,'  T=P ',4X,'TAU(I,J)')
0246              DO 225 L=1,K
0247              TKP=0.545*X5(I,L)*U(I,L)*SU4(I,L)/S(I)*XINT(I,L)**2
0248          225 PRINT 230, L,U(I,L),V(I,L),S(I,L),U(I,L), X5(I,L),TAUR(I,L),
                 1T=P,TAU(I,L)
0249          230 FORMAT(' ',I2,8(2X,E1=.7))
0250              PRINT 240,X4(I),'IM
0251          240 FORMAT(' ','AXIAL DISTANCE  X/5=',E1=.7,3X,'HALF RADIUS  R/7=',
                 1E1=.7)
0252              PRINT 250
0253          250 FORMAT(' ','************************************************************
                 1*******************************************************')
0254              DO 260 L=1,K
0255              S(I,L)=SQ(I,L)/(XINT(I,L)**2)
0256              TU(I,L)=TAU(I,L)/(XINT(I,L)**2)
0257              X5(I,L)=X5(I,L)*6)
0258          260 U(I,L)=U(I,L)/XINT(I,L)
0259              IF(I5.GE.21))XN=04
0260              IF(I5.GE.20))XP=0X
0261              I=I+1
0262              K=K+5-1
0263              DO 290  I=1,K5
0264              U(I-2,4)=U(I-1,4)
0265              V(I-2,4)=V(I-1,4)
0266              XINT(I-2,4)=XINT(I-1,4)
0267              TAU(I-2,4)=TAU(I-1,4)
0268              DISS(I-2,4)=DISS(I-1,4)
0269              TU(I-2,4)=TU(I-1,4)
0270              TV(I-2,4)=TV(I-1,4)
0271              TX(I-2,4)=TX(I-1,4)
0272              X5(I-2,4)=X5(I-1,4)
0273              0.5(I-2,4)=DIS(I-1,4)
0274              PRL(I-2,4)=PRL(I-1,4)
0275              PR0(I-2,4)=PR00(I-1,4)
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VERY POOR PRINT

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J277              GUR(I-2,4)=GUR(I-1,4)
J273              SG(I-2,4)=SG(I-1,4)
J279              O(I-2,4)=O(I-1,4)
J290        290   E(I-2,4)=E(I-1,4)
J231              DO 295 M=1,K3
J232              OU(I-1,4)=OU(I,4)
J293              V(I-1,4)=V(I,4)
J294              TAU(I-1,4)=TAU(I,4)
J295              XINT(I-1,4)=XINT(I,4)
J296              TJ(I-1,4)=TJ(I,4)
J297              TV(I-1,4)=TV(I,4)
J298              TU(I-1,4)=TU(I,4)
J299              OIS(I-1,4)=OIS(I,4)
J290              JU(I-1,4)=JOH(I,4)
J521              XU(I-1,4)=XU(I,4)
J542              OISS(I-1,4)=OISS(I,4)
J293              PWOO(I-1,4)=POOJ(I,4)
J294              OUR(I-1,4)=OUR(I,4)
J275              SC(I-1,4)=SC(I,4)
J296              O(I-1,4)=O(I,4)
J297        295   E(I-1,4)=E(I,4)
J293              X(I-2)=X(I-1)
J294              X(I-1)=X(I)
J300              XA(I-2)=XA(I-1)
J301              XA(I-1)=XA(I)
J302              XL(I-2)=XL(I-1)
J303              XL(I-1)=XL(I)
J304              PHI(I-2)=PHI(I-1)
J305              PHI(I-1)=PHI(I)
J306              PHI(I)=PHI(I+1)
J307              B(I-2)=B(I-1)
J308              B(I-1)=B(I)
J309              B(I)=B(I+1)
J310              PI(I-2)=PI(I-1)
J311              PI(I-1)=PI(I)
J312              PI(I)=PI(I+1)
J313              P2(I-2)=P2(I-1)
J314              P2(I-1)=P2(I)
J315              P2(I)=P2(I+1)
J315              PH(I-2)=PH(I-1)
J317              PH(I-1)=PH(I)
J318              PH(I)=PH(I+1)
J319              I=I-1
J320              DO 299 M=1,K
J321              TRA(I-1,4)=TRA(I,4)
J322        299   DIFF(I-1,4)=DIFF(I,4)
J323              GO TO 40
J324        300   STOP
J325              END
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VITA AUCTORIS

1948	Born at Hiran, Gopalganj, Bangladesh, on February 10.
1963	Completed Secondary School Certificate Education from Bagdha High School, Barisal, Bangladesh.
1965	Completed Higher Secondary Certificate Education from Jagannath College, Dacca, Bangladesh.
1969	Received the Degree of Bachelor of Science in Mechanical Engineering from University of Engineering and Technology, Dacca, Bangladesh.
1970	Joined the Department of Mechanical Engineering of University of Engineering and Technology, Dacca, Bangladesh, as a lecturer.
1975	Received the Degree of Master of Science in Mechanical Engineering from University of Engineering and Technology, Dacca, Bangladesh.
1975	Promoted to the Rank of Assistant Professor, University of Engineering and Technology, Dacca, Bangladesh.
1979	Currently a candidate for the Degree of Doctor of Philosophy in Mechanical Engineering at the University of Windsor, Windsor, Ontario, Canada.