1985

TIME-LIKE AND SPACE-LIKE CURVES IN FRENET-SERRET FORMALISMS.

HIDEO. ICHIMURA
University of Windsor

0-315-20739-6

National Library of Canada | Bibliothèque nationale du Canada | CANADIAN THESES ON MICROFICHE | *THÈSES CANADIENNES SUR MICROFICHE*

NAME OF AUTHOR/*NOM DE L'AUTEUR* Hideo.Ichimura

TITLE OF THESIS/*TITRE DE LA THÈSE* Time-like and soace-like curves in Frenet-Serret formalisms.

UNIVERSITY/*UNIVERSITÉ* University of Windsor, Windsor,Ontario

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* Ph.D.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE GRADE* Fall, 1985

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* Prof. G. Szamosi

DATED/*DATÉ* SEPTEMBER 3, 1985 SIGNED/*SIGNÉ* Hideo Ichimura

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ*

NL-91 (3-74)

TIME-LIKE AND SPACE-LIKE CURVES IN FRENET-SERRET FORMALISMS

by

Hideo Ichimura

A Dissertation
submitted to the Faculty of Graduate Studies
through the Department of Physics
in Partial Fullfillment of
the Requirements for the Degree
of Doctor of Philosophy at
the University of Windsor

Windsor, Ontario, Canada

1985

838015

ABSTRACT

TIME-LIKE AND SPACE-LIKE CURVES IN FRENET-SERRET FORMALISMS

by

Hideo Ichimura

The Frenet-Serret formalism for both time-like and space-like curves is studied. The Frenet-Serret vectors and the Frenet-Serret coefficients in both three and four dimensions are expressed in terms of "world" quantities. The conversion from the three dimensional Frenet-Serret formalism to the four dimensional Frenet-Serret formalism and vice versa is described. Indicators of the four dimensional Frenet-Serret vector are investigated. The Frenet-Serret equations in both three and four dimensions are solved for constant Frenet-Serret coefficients with arbitrary initial conditions.

Nulltetrads, spinors, bispinors, spinor adjoint and bispinor adjoint are then defined. The Frenet-Serret equations for the nulltetrads, the spinors, the bispinors and the bispinor adjoints are introduced. Darboux bivector forms of the Frenet-Serret equations for the orthonormal tetrads and the nulltetrads are derived. Darboux bispinor forms of the Frenet-Serret equations for the spinors and the bispinor adjoints are derived. Solutions for the nulltetrads and the Darboux bivectors and the Darboux bispinors are discussed. Motion of a point charge in electromagnetic field and motion of a freely spinning particle are briefly discussed.

Most of the foregoing are duplicated for the skew-symmetrized descriptions. Examples of the above analysis are given for the time-like curve and the space-like curve with indicators ($\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$). The former is picked in order to compare with the existing literature and the latter is chosen as the simplest case of space-like curves.

ACKNOWLEDGEMENTS

This study was supervised by Prof.G.Szamosi. Dr.R.D.Kent was initially my personal tutor on the subject. Dr.E.Honig and Prof.N.D.Lane offered me discussions during the course of this study.

I also note the friendship and encouragement rendered by Profs.E.N.Glass, M.Schlesinger, H.Yamauchi, F.Holuj, Drs.J.Huschilt, H.Ogata, M.Czajkowski, N.E.Hedgecock, Mr.J.Demsky, Miss J.Amos, Mrs.P.Parungo, Mrs.B.Ridsdale and Mrs.M.Holmes.

TABLE OF CONTENTS

INTRODUCTION

As one of the most basic and important subjects of physics, the one particle equation of motion has historically been an object of intensive study. Classical mechanics treats it as a vector (or tensor) equation whereas the quantum mechanics treats it as a spinorial equation. Of these two, the latter is considered more fundamental than the former since it is possible to express any vector (or tensor) uniquely in terms of spinors but the converse is not true.

Therefore it is only natural for us to wonder how the classical equation of motion could be converted to the quantum mechanical spinor equation of motion. The use of Frenet-Serret equation to describe the relativistic motion of a point particle was pioneered by Synge(1937). Gursey(1957) then followed with the spinor approach to the geometry of time-like trajectories. Kent and Szamosi(1981) extended the above to the curved space-time.

The purpose of this study is two-fold:

(1) Systematic and thorough treatment of the Frenet-Serret equations.

(2) Generalization of (1) to include space-like curves.

In Chapter 1(i), we introduce the three and the four dimensional Frenet-Serret equations. The latter is first rewritten in terms of the skew-symmetrized tetrads denoted by a wavy bar (∼). Then both are further transformed to the reduced tetrads which are noted by a reverse wavy bar (∽). Finally the three and the four dimensional quantities (the Frenet-Serret vectors and the Frenet-Serret coefficients) are expressed in terms of each other using the fact that the coordinate of a particle in three dimensions is a spatial part of space-time coordinates of the same particle in four dimension. This part of the idea is an extension of a common practice in differential geometry (Adachi 1976).

In Chapter 1(ii), first we consider time-like and space-like motion with constant speed along a helix in four dimensional space-time and derive the Frenet-Serret vectors and the Frenet-Serret coefficients and the indicators. (Indicators indicate whether the vectors they refer to are time-like or space-like and are -1 for the former and +1 for the latter (Synge and Schild 1949)) We also derive the same for circular motion by setting the pitch to be zero in the above analysis. We then conduct the same study on the motion of a particle with constant speed in four dimensional space-time. We finally extend the above to the cases with not necessarily constant speed. We then express the three dimensional Frenet-Serret vectors and the coefficients in terms of velocity.

In Chapter 1(iii), the solution of the three and four dimensional Frenet-Serret equation with constant Frenet-Serret coefficients are presented.

In Chapter 2(i), the four dimensional Frenet-Serret equations are first re-expressed in terms of the nulltetrad. Then they are-expressed in terms of the spinors and finally in terms of the bispinors and their adjoints. Darboux bivector and Darboux bispinor forms are then introduced.

In Chapter 2(ii), the same treatment is applied on Chapter 2(i) as in Chapter 1(iii).

In Chapter 3, as applications, we describe the motion of a point charge in electromagnetic field and also of a freely spinning particle.

The chapters 2(i) & 3 mostly parallel the earlier treatise by Kent(1978) and many of the notations, conventions and definitions therein are not discussed here again. Among them, for example, are

$$I_a{}^b = \begin{bmatrix} \delta^A{}_B & 0 \\ 0 & \delta_{\dot{A}}{}^{\dot{B}} \end{bmatrix}, \tag{1}$$

$$\gamma^{(5)}{}_a{}^b = \begin{bmatrix} \delta^A{}_B & 0 \\ 0 & -\delta_{\dot{A}}{}^{\dot{B}} \end{bmatrix}, \tag{2}$$

and

$$\epsilon^{\mu\nu\alpha\beta} e_{\mu(0)} e_{\nu(1)} e_{\alpha(2)} e_{\beta(3)} = -1 \tag{3}$$

CHAPTER 1

FRENET-SERRET EQUATIONS

(i) Frenet-Serret Equations in Three and Four Dimensions

We consider a three-dimensional space and an four-dimensional space-time, a point on which is denoted by

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1-1-1}$$

and

$$\mathbf{Y} = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \tag{1-1-2}$$

respectively, or rather X^{λ} ($\lambda = 1,2,3$) and Y^{μ} ($\mu = 0,1,2,3$). At each point we can construct a frame of three and four linearly independent vectors which we call an orthonormal triad, denoted by

$$\boldsymbol{\varepsilon}^{\lambda} = \begin{bmatrix} \varepsilon^{\lambda}_{(1)} \\ \varepsilon^{\lambda}_{(2)} \\ \varepsilon^{\lambda}_{(3)} \end{bmatrix} \tag{1-1-3}$$

or $\mathcal{E}^{\lambda}_{(j)}$ ($j = 1,2,3$), and an orthonormal tetrad, denoted by

$$e^{\mu} = \begin{bmatrix} e^{\mu}_{(0)} \\ e^{\mu}_{(1)} \\ e^{\mu}_{(2)} \\ e^{\mu}_{(3)} \end{bmatrix} \qquad (1\text{-}1\text{-}4)$$

or $e^{\mu}_{(a)}$ ($a = 0,1,2,3$), respectively.

These triad and tetrad are locked onto the curve by demanding

$$\mathcal{E}^{\lambda}_{(1)} = \frac{dX^{\lambda}}{ds} \qquad (1\text{-}1\text{-}5)$$

and

$$e^{\mu}_{(0)} = \frac{dY^{\mu}}{ds} , \qquad (1\text{-}1\text{-}6)$$

whereupon we write the Frenet-Serret equations as

$$\frac{d}{ds}\mathcal{E}^{\lambda} = \underline{\underline{h}} \cdot \mathcal{E}^{\lambda} \qquad (1\text{-}1\text{-}7)$$

where

$$\underline{\underline{h}} = \begin{bmatrix} 0 & h_1 & 0 \\ -h_1 & 0 & h_2 \\ 0 & -h_2 & 0 \end{bmatrix} \tag{1-1-8}$$

and

$$(d\mathit{s})^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{1-1-9}$$

and

$$\frac{d}{ds}\mathbf{e}^{\mu} = \underline{\underline{k}} \cdot \mathbf{e}^{\mu} \tag{1-1-10}$$

where

$$\mathbf{e}^{\mu} = \begin{bmatrix} e^{\mu}_{(0)} \\ e^{\mu}_{(1)} \\ e^{\mu}_{(2)} \\ e^{\mu}_{(3)} \end{bmatrix} , \tag{1-1-11}$$

$$\underline{\underline{k}} = \begin{bmatrix} 0 & k_0 & 0 & 0 \\ -\varepsilon_0\varepsilon_1 k_0 & 0 & k_1 & 0 \\ 0 & -\varepsilon_1\varepsilon_2 k_1 & 0 & k_2 \\ 0 & 0 & -\varepsilon_2\varepsilon_3 k_2 & 0 \end{bmatrix} \tag{1-1-12}$$

and

$$(ds)^2 = |\,(dt)^2 - (d\mathit{s})^2\,| \tag{1-1-13}$$

namely

$$(ds)^2 = \begin{cases} (dt)^2 - (d\sigma)^2 \text{ for time-like curves} & (1\text{-}1\text{-}14) \\ (d\sigma)^2 - (dt)^2 \text{ for space-like curves} & (1\text{-}1\text{-}15) \end{cases}$$

and σ is the arc-length, $\underline{\underline{\mathcal{E}}}$ the Frenet-Serret vectors and $\underline{\underline{h}}$ the Frenet-Serret vectors for three-dimensional curves and s is the arc-length, e, ϵ's the Frenet-Serret vectors, indicators, and $\underline{\underline{k}}$ the Frenet-Serret coefficients for four-dimensional curves. The coefficients $\underline{\underline{h}}$ are related to the orthonormal triads $\mathcal{E}$ by

$$h_1^2 = \frac{d\mathcal{E}_{(1)}}{d\sigma} \cdot \frac{d\mathcal{E}_{(1)}}{d\sigma} \qquad (1\text{-}1\text{-}16)$$

$$h_1^2 + h_2^2 = \frac{d\mathcal{E}_{(2)}}{d\sigma} \cdot \frac{d\mathcal{E}_{(2)}}{d\sigma} \qquad (1\text{-}1\text{-}17)$$

and

$$h_2^2 = \frac{d\mathcal{E}_{(3)}}{d\sigma} \cdot \frac{d\mathcal{E}_{(3)}}{d\sigma} \qquad (1\text{-}1\text{-}18)$$

where in terms of unit normals

$$\mathcal{E}_{(a)} = \hat{x}\, \mathcal{E}^1_{(a)} + \hat{y}\, \mathcal{E}^2_{(a)} + \hat{z}\, \mathcal{E}^3_{(a)} \quad , \qquad (1\text{-}1\text{-}19)$$

and we define

$$\mathcal{E} = \begin{bmatrix} \mathcal{E}_{(1)} \\ \mathcal{E}_{(2)} \\ \mathcal{E}_{(3)} \end{bmatrix} \quad . \qquad (1\text{-}1\text{-}20)$$

Likewise the coefficients $\underline{\underline{k}}$ are related to the orthonormal tetrads $\mathbf{e}$ by

$$\varepsilon_1 k_0^2 = \frac{d\mathbf{e}_{(0)}}{ds} \cdot \frac{d\mathbf{e}_{(0)}}{ds} , \tag{1-1-21}$$

$$\varepsilon_0 k_0^2 + \varepsilon_2 k_1^2 = \frac{d\mathbf{e}_{(1)}}{ds} \cdot \frac{d\mathbf{e}_{(1)}}{ds} , \tag{1-1-22}$$

$$\varepsilon_1 k_1^2 + \varepsilon_3 k_1^2 = \frac{d\mathbf{e}_{(2)}}{ds} \cdot \frac{d\mathbf{e}_{(2)}}{ds} , \tag{1-1-23}$$

and

$$\varepsilon_2 k_2^2 = \frac{d\mathbf{e}_{(3)}}{ds} \cdot \frac{d\mathbf{e}_{(3)}}{ds} , \tag{1-1-24}$$

where in terms of unit normals

$$\mathbf{e}_{(a)} = \hat{t}\, e^0_{(a)} + \hat{x}\, e^1_{(a)} + \hat{y}\, e^2_{(a)} + \hat{z}\, e^3_{(a)} \tag{1-1-25}$$

and we define

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_{(0)} \\ \mathbf{e}_{(1)} \\ \mathbf{e}_{(2)} \\ \mathbf{e}_{(3)} \end{bmatrix} . \tag{1-1-26}$$

In terms of $\mathfrak{E}$ and $\mathbf{e}$, the Frenet-Serret equations written as in Eqs. (1-1-7) and (1-1-10) are now re-expressed as

$$\frac{d}{ds} \mathfrak{E} = \underline{\underline{h}} \cdot \mathfrak{E} \tag{1-1-27}$$

and

$$\frac{d}{ds}\,\mathbf{e} = \underline{\underline{k}} \cdot \mathbf{e} \qquad . \qquad (1\text{-}1\text{-}28)$$

Since motion of a particle can also be expressed in three dimensional and four dimensional forms of the Frenet-Serret equations, they are related, which we are now to investigate.

Let us first use the indicator matrix

$$\underline{\underline{\zeta}} = \begin{bmatrix} \sqrt{\varepsilon}_0 & 0 & 0 & 0 \\ 0 & \sqrt{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & \sqrt{\varepsilon}_2 & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon}_3 \end{bmatrix} \qquad (1\text{-}1\text{-}29)$$

to skew-symmetrize Eq. (1-1-10) as

$$\frac{d}{ds}\,\breve{\mathbf{e}} = \breve{\underline{\underline{k}}} \cdot \breve{\mathbf{e}} \qquad , \qquad (1\text{-}1\text{-}30)$$

where

$$\breve{\mathbf{e}} = \underline{\underline{\zeta}} \cdot \mathbf{e}$$

$$= \begin{bmatrix} \sqrt{\varepsilon}_0 & 0 & 0 & 0 \\ 0 & \sqrt{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & \sqrt{\varepsilon}_2 & 0 \\ 0 & 0 & 0 & \sqrt{\varepsilon}_3 \end{bmatrix} \cdot \begin{bmatrix} e_{(0)} \\ e_{(1)} \\ e_{(2)} \\ e_{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\varepsilon_0}\, e_{(0)} \\ \sqrt{\varepsilon_1}\, e_{(1)} \\ \sqrt{\varepsilon_2}\, e_{(2)} \\ \sqrt{\varepsilon_3}\, e_{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{e}_0 \\ \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{bmatrix} , \qquad (1\text{-}1\text{-}31)$$

and

$$\tilde{\underline{\underline{k}}} = \underline{\underline{\xi}} \cdot \underline{\underline{k}} \cdot \underline{\underline{\xi}}^{-1}$$

$$= \begin{bmatrix} 0 & \frac{\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_1}} k_0 & 0 & 0 \\ -\frac{\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_1}} k_0 & 0 & \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} k_1 & 0 \\ 0 & -\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} k_1 & 0 & \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_3}} k_2 \\ 0 & 0 & -\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_3}} k_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \breve{k}_0 & 0 & 0 \\ -\breve{k}_0 & 0 & \breve{k}_1 & 0 \\ 0 & -\breve{k}_1 & 0 & \breve{k}_2 \\ 0 & 0 & -\breve{k}_2 & 0 \end{bmatrix} . \tag{1-1-32}$$

We then rewrite Eqs. (1-1-7) and (1-1-30) by the transformation (reduced description)

$$\tilde{\mathcal{E}} = v\,\mathcal{E} \tag{1-1-33}$$

where $\vec{v}$ is the 3 velocity, i.e. (the dash indicates temporal differentiation)

$$\mathcal{E}' = \frac{1}{v}\tilde{\mathcal{E}}' + \left(\frac{1}{v}\right)'\tilde{\mathcal{E}} \tag{1-1-34}$$

and, with c = 1, by

$$\tilde{e} = \sqrt{|1-v^2|}\;\breve{e} \tag{1-1-35}$$

i.e.

$$\breve{e}' = \frac{1}{\sqrt{|1-v^2|}}\tilde{e}' + \left(\frac{1}{\sqrt{|1-v^2|}}\right)'\tilde{e} \tag{1-1-36}$$

respectively to

$$\tilde{\mathcal{E}}' = \underline{\underline{\tilde{h}}} \cdot \tilde{\mathcal{E}} \tag{1-1-37}$$

and

$$\tilde{e}' = \tilde{\underline{\underline{k}}} \cdot \tilde{e} \quad , \tag{1-1-38}$$

where

$$\tilde{\mathfrak{E}} = \begin{bmatrix} \tilde{\mathfrak{E}}_{(1)} \\ \tilde{\mathfrak{E}}_{(2)} \\ \tilde{\mathfrak{E}}_{(3)} \end{bmatrix}$$

$$= \begin{bmatrix} v\,\mathfrak{E}_{(1)} \\ v\,\mathfrak{E}_{(2)} \\ v\,\mathfrak{E}_{(3)} \end{bmatrix} \quad , \tag{1-1-39}$$

$$\tilde{\underline{\underline{h}}} = \begin{bmatrix} U & v\,h_1 & 0 \\ -v\,h_1 & U & v\,h_2 \\ 0 & -v\,h_2 & U \end{bmatrix}$$

$$= \begin{bmatrix} U & \tilde{h}_1 & 0 \\ -\tilde{h}_1 & U & \tilde{h}_2 \\ 0 & -\tilde{h}_2 & U \end{bmatrix} \quad , \tag{1-1-40}$$

$$\tilde{\mathbf{e}} = \begin{bmatrix} \tilde{\mathbf{e}}_{(0)} \\ \tilde{\mathbf{e}}_{(1)} \\ \tilde{\mathbf{e}}_{(2)} \\ \tilde{\mathbf{e}}_{(3)} \end{bmatrix}$$

$$= \sqrt{|1 - v^2|}\ \overset{\backsim}{\mathbf{e}}$$

$$= \sqrt{|1 - v^2|} \begin{bmatrix} \overset{\backsim}{\mathbf{e}}_{(0)} \\ \overset{\backsim}{\mathbf{e}}_{(1)} \\ \overset{\backsim}{\mathbf{e}}_{(2)} \\ \overset{\backsim}{\mathbf{e}}_{(3)} \end{bmatrix}$$

$$= \sqrt{|1 - v^2|}\ \underset{\sim}{\zeta} \cdot \mathbf{e}$$

$$= \sqrt{|1 - v^2|} \begin{bmatrix} \sqrt{\epsilon}_0 & \mathbf{e}_{(0)} \\ \sqrt{\epsilon}_1 & \mathbf{e}_{(1)} \\ \sqrt{\epsilon}_2 & \mathbf{e}_{(2)} \\ \sqrt{\epsilon}_3 & \mathbf{e}_{(3)} \end{bmatrix} , \qquad (1\text{-}1\text{-}41)$$

$$
\begin{aligned}
\underset{=}{\tilde{k}} &= v\,\underset{=}{I} + \sqrt{|1 - v^2|}\;\underset{=}{\overset{\backsim}{k}} \\
&= v\,\underset{=}{I} + \sqrt{|1 - v^2|}\;\underset{\approx}{\zeta} \cdot \underset{=}{k} \cdot \underset{\approx}{\zeta}^{-1} \\
&= \begin{bmatrix} v & \sqrt{|1-v^2|}\overset{\backsim}{k}_0 & 0 & 0 \\ -\sqrt{|1-v^2|}\overset{\backsim}{k}_0 & v & \sqrt{|1-v^2|}\overset{\backsim}{k}_1 & 0 \\ 0 & -\sqrt{|1-v^2|}\overset{\backsim}{k}_1 & v & \sqrt{|1-v^2|}\overset{\backsim}{k}_2 \\ 0 & 0 & -\sqrt{|1-v^2|}\overset{\backsim}{k}_2 & v \end{bmatrix} \\
&= \begin{bmatrix} v & \tilde{k}_0 & 0 & 0 \\ -\tilde{k}_0 & v & \tilde{k}_1 & 0 \\ 0 & -\tilde{k}_1 & v & \tilde{k}_2 \\ 0 & 0 & -\tilde{k}_2 & v \end{bmatrix} , \qquad (1\text{-}1\text{-}42)
\end{aligned}
$$

$$
\begin{aligned}
U &= -v\left(\frac{1}{v}\right)' \\
&= \frac{v'}{v} \qquad (1\text{-}1\text{-}43)
\end{aligned}
$$

and

$$
\begin{aligned}
V &= -\sqrt{|1 - v^2|}\left(\frac{1}{\sqrt{|1 - v^2|}}\right)' \\
&= \frac{vv'}{v^2 - 1} . \qquad (1\text{-}1\text{-}44)
\end{aligned}
$$

$\mathfrak{X}$ the coordinate of a particle in three dimensional space and $\mathfrak{Y}$ in four dimensional space time are related to each other as

$$\mathfrak{Y} = \begin{bmatrix} t \\ \mathfrak{X} \end{bmatrix} . \qquad (1\text{-}1\text{-}45)$$

If we utilize Eq. (1-1-45) together with Eqs. (1-1-5), (1-1-6), (1-1-33) and (1-1-35), we obtain

$$\sqrt{\epsilon_0}\,\tilde{e}_{(0)} = \begin{bmatrix} 1 \\ \tilde{\mathfrak{E}}_{(1)} \end{bmatrix} , \qquad (1\text{-}1\text{-}46)$$

$$\sqrt{\epsilon_0}\,\tilde{e}'_{(0)} = \begin{bmatrix} 0 \\ \tilde{\mathfrak{E}}'_{(1)} \end{bmatrix} , \qquad (1\text{-}1\text{-}47)$$

$$\sqrt{\epsilon_0}\,\tilde{e}''_{(0)} = \begin{bmatrix} 0 \\ \tilde{\mathfrak{E}}''_{(1)} \end{bmatrix} , \qquad (1\text{-}1\text{-}48)$$

$$\sqrt{\epsilon_0}\,\tilde{e}'''_{(0)} = \begin{bmatrix} 0 \\ \tilde{\mathfrak{E}}'''_{(1)} \end{bmatrix} , \qquad (1\text{-}1\text{-}49)$$

and so on.

Then we obtain from Eqs. (1-1-46) ~ (1-1-49) various scalar identities such as

$$\tilde{e}'_{(0)} \cdot \tilde{e}'_{(0)} = \tilde{\mathfrak{E}}'_{(1)} \cdot \tilde{\mathfrak{E}}'_{(1)} \qquad (1\text{-}1\text{-}50)$$

$$\tilde{\mathfrak{e}}''_{(0)} \cdot \tilde{\mathfrak{e}}''_{(0)} = \tilde{\mathfrak{E}}''_{(1)} \cdot \tilde{\mathfrak{E}}''_{(1)} \qquad (1\text{-}1\text{-}51)$$

$$\tilde{\mathfrak{e}}'''_{(0)} \cdot \tilde{\mathfrak{e}}'''_{(0)} = \tilde{\mathfrak{E}}'''_{(1)} \cdot \tilde{\mathfrak{E}}'''_{(1)} \qquad (1\text{-}1\text{-}52)$$

$$\tilde{\mathfrak{e}}_{(0)} \cdot \tilde{\mathfrak{e}}'_{(0)} = \tilde{\mathfrak{E}}_{(1)} \cdot \tilde{\mathfrak{E}}'_{(1)} \qquad (1\text{-}1\text{-}53)$$

$$\tilde{\mathfrak{e}}_{(0)} \cdot \tilde{\mathfrak{e}}''_{(0)} = \tilde{\mathfrak{E}}_{(1)} \cdot \tilde{\mathfrak{E}}''_{(1)} \qquad (1\text{-}1\text{-}54)$$

$$\tilde{\mathfrak{e}}_{(0)} \cdot \tilde{\mathfrak{e}}'''_{(0)} = \tilde{\mathfrak{E}}_{(1)} \cdot \tilde{\mathfrak{E}}'''_{(1)} \qquad (1\text{-}1\text{-}55)$$

$$\tilde{\mathfrak{e}}'_{(0)} \cdot \tilde{\mathfrak{e}}''_{(0)} = \tilde{\mathfrak{E}}'_{(1)} \cdot \tilde{\mathfrak{E}}''_{(1)} \qquad (1\text{-}1\text{-}56)$$

$$\tilde{\mathfrak{e}}'_{(0)} \cdot \tilde{\mathfrak{e}}'''_{(0)} = \tilde{\mathfrak{E}}'_{(1)} \cdot \tilde{\mathfrak{E}}'''_{(1)} \qquad (1\text{-}1\text{-}57)$$

$$\tilde{\mathfrak{e}}''_{(0)} \cdot \tilde{\mathfrak{e}}'''_{(0)} = \tilde{\mathfrak{E}}''_{(1)} \cdot \tilde{\mathfrak{E}}'''_{(1)} \qquad (1\text{-}1\text{-}58)$$

and so on.

We are now to express $\tilde{\mathfrak{E}}'_{(1)}$, $\tilde{\mathfrak{E}}''_{(1)}$ and $\tilde{\mathfrak{E}}'''_{(1)}$ in terms of $\tilde{\mathfrak{E}}_{(1)}$, $\tilde{\mathfrak{E}}_{(2)}$ and $\tilde{\mathfrak{E}}_{(3)}$ and $\tilde{\mathfrak{e}}'_{(0)}$, $\tilde{\mathfrak{e}}''_{(0)}$ and $\tilde{\mathfrak{e}}'''_{(0)}$ in terms of $\tilde{\mathfrak{e}}_{(0)}$, $\tilde{\mathfrak{e}}_{(1)}$, $\tilde{\mathfrak{e}}_{(2)}$ and $\tilde{\mathfrak{e}}_{(3)}$ as

$$\tilde{\mathfrak{E}}'_{(1)} = v\,\tilde{\mathfrak{E}}_{(1)} + h_1\,\tilde{\mathfrak{E}}_{(2)} \quad , \qquad (1\text{-}1\text{-}59)$$

$$
\begin{aligned}
\tilde{\mathfrak{E}}''_{(1)} &= U\tilde{\mathfrak{E}}'_{(1)} + U'\tilde{\mathfrak{E}}_{(1)} + \tilde{h}_1\tilde{\mathfrak{E}}'_{(2)} + \tilde{h}_1'\tilde{\mathfrak{E}}_{(2)} \\
&= U(\tilde{\mathfrak{E}}_{(1)} + \tilde{h}_1\tilde{\mathfrak{E}}_{(2)}) + \tilde{h}_1(-\tilde{h}_1\tilde{\mathfrak{E}}_{(1)} + U\tilde{\mathfrak{E}}_{(2)} + \tilde{h}_2\tilde{\mathfrak{E}}_{(3)}) + U'\tilde{\mathfrak{E}}_{(1)} + \tilde{h}_1'\tilde{\mathfrak{E}}_{(2)} \\
&= (U^2 - \tilde{h}_1^2 + U')\tilde{\mathfrak{E}}_{(1)} + (2U\tilde{h}_1 + \tilde{h}_1')\tilde{\mathfrak{E}}_{(2)} + \tilde{h}_1\tilde{h}_2\tilde{\mathfrak{E}}_{(3)} \ , \qquad (1\text{-}1\text{-}60)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathfrak{E}}'''_{(1)} &= (U^2\tilde{h}_1^2 + U')\tilde{\mathfrak{E}}'_{(1)} + (2U\tilde{h}_1 + \tilde{h}_1')\tilde{\mathfrak{E}}'_{(2)} + \tilde{h}_1\tilde{h}_2\tilde{\mathfrak{E}}'_{(3)} + (U^2 - \tilde{h}_1^2 + U')'\tilde{\mathfrak{E}}_{(1)} + (2U\tilde{h}_1 + \tilde{h}_1')'\tilde{\mathfrak{E}}_{(2)} + (\tilde{h}_1\tilde{h}_2)'\tilde{\mathfrak{E}}_{(3)} \\
&= (U^2 - \tilde{h}_1^2 + U')(U\tilde{\mathfrak{E}}_{(1)} + \tilde{h}_1\tilde{\mathfrak{E}}_{(2)}) + (2U\tilde{h}_1 + \tilde{h}_1')(-\tilde{h}_1\tilde{\mathfrak{E}}_{(1)} + U\tilde{\mathfrak{E}}_{(2)} + \tilde{h}_2\tilde{\mathfrak{E}}_{(3)}) + \tilde{h}_1\tilde{h}_2(-\tilde{h}_2\tilde{\mathfrak{E}}_{(2)} + U\tilde{\mathfrak{E}}_{(3)}) \\
&\quad + (2UU' - 2\tilde{h}_1\tilde{h}_1' + U'')\tilde{\mathfrak{E}}_{(1)} + (2U'\tilde{h}_1 + 2U\tilde{h}_1' + \tilde{h}_1'')\tilde{\mathfrak{E}}_{(2)} + (\tilde{h}_1'\tilde{h}_2 + \tilde{h}_1\tilde{h}_2')\tilde{\mathfrak{E}}_{(3)} \\
&= (U^3 + U'' + 3UU' - 3U\tilde{h}_1^2 - 3\tilde{h}_1\tilde{h}_1')\tilde{\mathfrak{E}}_{(1)} + (3U^2\tilde{h}_1 + 3U'\tilde{h}_1 + 3U\tilde{h}_1' - \tilde{h}_1^3 - \tilde{h}_1\tilde{h}_2^2 + \tilde{h}_1'')\tilde{\mathfrak{E}}_{(2)} + (3U\tilde{h}_1\tilde{h}_2 + 2\tilde{h}_1'\tilde{h}_2 + \tilde{h}_1\tilde{h}_2')\tilde{\mathfrak{E}}_{(3)} \qquad (1\text{-}1\text{-}61)
\end{aligned}
$$

$$
\tilde{\mathfrak{C}}'_{(0)} = V\tilde{\mathfrak{C}}_{(0)} + \tilde{k}_0\tilde{\mathfrak{C}}_{(1)} \ , \qquad (1\text{-}1\text{-}62)
$$

$$
\begin{aligned}
\tilde{\mathfrak{C}}''_{(0)} &= V\tilde{\mathfrak{C}}'_{(0)} + \tilde{k}_0\tilde{\mathfrak{C}}'_{(1)} + V'\tilde{\mathfrak{C}}_{(0)} + \tilde{k}_0'\tilde{\mathfrak{C}}_{(1)} \\
&= V(V\tilde{\mathfrak{C}}_{(0)} + \tilde{k}_0\tilde{\mathfrak{C}}_{(1)}) + \tilde{k}_0(-\tilde{k}_0\tilde{\mathfrak{C}}_{(0)} + V\tilde{\mathfrak{C}}_{(1)} + \tilde{k}_1\tilde{\mathfrak{C}}_{(2)}) + V'\tilde{\mathfrak{C}}_{(0)} + \tilde{k}_0'\tilde{\mathfrak{C}}_{(1)} \\
&= (V^2 - \tilde{k}_0^2 + V')\tilde{\mathfrak{C}}_{(0)} + (2V\tilde{k}_0 + \tilde{k}_0')\tilde{\mathfrak{C}}_{(1)} + \tilde{k}_0\tilde{k}_1\tilde{\mathfrak{C}}_{(2)} \ , \qquad (1\text{-}1\text{-}63)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\mathfrak{C}}'''_{(0)} &= (V^2 - \tilde{k}_0^2 + V')\tilde{\mathfrak{C}}'_{(0)} + (2V\tilde{k}_0 + \tilde{k}_0')\tilde{\mathfrak{C}}'_{(1)} + \tilde{k}_0\tilde{k}_1\tilde{\mathfrak{C}}'_{(2)} + (V^2 - \tilde{k}_0^2 + V')'\tilde{\mathfrak{C}}_{(0)} + (2V\tilde{k}_0 + \tilde{k}_0')'\tilde{\mathfrak{C}}_{(1)} + (\tilde{k}_0\tilde{k}_1)'\tilde{\mathfrak{C}}_{(2)} \\
&= (V^2 - \tilde{k}_0^2 + V')(V\tilde{\mathfrak{C}}_{(0)} + \tilde{k}_0\tilde{\mathfrak{C}}_{(1)}) + (2V\tilde{k}_0 + \tilde{k}_0')(-\tilde{k}_0\tilde{\mathfrak{C}}_{(0)} + V\tilde{\mathfrak{C}}_{(1)} + \tilde{k}_1\tilde{\mathfrak{C}}_{(2)}) + \tilde{k}_0\tilde{k}_1(-\tilde{k}_1\tilde{\mathfrak{C}}_{(1)} + V\tilde{\mathfrak{C}}_{(2)} + \tilde{k}_2\tilde{\mathfrak{C}}_{(3)}) \\
&\quad + (2VV' - 2\tilde{k}_0\tilde{k}_0' + V'')\tilde{\mathfrak{C}}_{(0)} + (2V'\tilde{k}_0 + 2V\tilde{k}_0' + \tilde{k}_0'')\tilde{\mathfrak{C}}_{(1)} + (\tilde{k}_0'\tilde{k}_1 + \tilde{k}_0\tilde{k}_1')\tilde{\mathfrak{C}}_{(2)} \\
&= (V^3 + V'' + 3VV' - 3V\tilde{k}_0^2 - 3\tilde{k}_0\tilde{k}_0')\tilde{\mathfrak{C}}_{(0)} + (3V^2\tilde{k}_0 + 3V'\tilde{k}_0 + 3V\tilde{k}_0' - \tilde{k}_0^3 - \tilde{k}_0\tilde{k}_1^2 + \tilde{k}_0'')\tilde{\mathfrak{C}}_{(1)} \\
&\quad + (3V\tilde{k}_0\tilde{k}_1 + 2\tilde{k}_0'\tilde{k}_1 + \tilde{k}_0\tilde{k}_1')\tilde{\mathfrak{C}}_{(2)} + \tilde{k}_0\tilde{k}_1\tilde{k}_2\tilde{\mathfrak{C}}_{(3)} \ . \qquad (1\text{-}1\text{-}64)
\end{aligned}
$$

We then rewrite Eqs. (1-1-50) ~ (1-1-52) using Eqs. (1-1-59) ~ (1-1-64) as

$$(U^2+\tilde{h}_1^2)v^2=(V^2+\tilde{k}_0^2)|1-v^2| , \tag{1-1-65}$$

$$\left[(U^2-\tilde{h}_1^2+U')^2+(2U\tilde{h}_1+\tilde{h}_1')^2+\tilde{h}_1^2\tilde{h}_2^2\right]v^2$$
$$=\left[(V^2-\tilde{k}_0^2+V')^2+(2V\tilde{k}_0+\tilde{k}_0')^2+\tilde{k}_0^2\tilde{k}_1^2\right]|1-v^2| \tag{1-1-66}$$

and

$$\left[(U^3+U''+3UU'-3U\tilde{h}_1^2-3\tilde{h}_1\tilde{h}_1')^2+(3U^2\tilde{h}_1+3U'\tilde{h}_1+3U\tilde{h}_1'-\tilde{h}_1^3-\tilde{h}_1\tilde{h}_2^2+\tilde{h}_1'')^2\right.$$
$$\left.+(3U\tilde{h}_1\tilde{h}_2+2\tilde{h}_1'\tilde{h}_2+\tilde{h}_1\tilde{h}_2')^2\right]v^2=\left[(V^3+V''+3VV'-3V\tilde{k}_0^2-3\tilde{k}_0\tilde{k}_0')^2\right.$$
$$\left.+(3V^2\tilde{k}_0+3V'\tilde{k}_0+3V\tilde{k}_0'-\tilde{k}_0^3-\tilde{k}_0\tilde{k}_1^2+\tilde{k}_1'')^2+(3V\tilde{k}_0\tilde{k}_1+2\tilde{k}_0'\tilde{k}_1+\tilde{k}_0\tilde{k}_1')^2+\tilde{k}_0^2\tilde{k}_1^2\tilde{k}_2^2\right]|1-v^2| . \tag{1-1-67}$$

Thus knowing $\vec{v}$ the three-dimensional velocity of a particle, the Frenet-Serret coefficients in three dimension can be converted to those in four dimension and vice versa using Eqs. (1-1-65) ~ (1-1-67). The other identities Eqs. (1-1-53) and so on yield only redundant information. For example, Eq. (1-1-50) gives $\epsilon_0 = -1$ and $\epsilon_0 = 1$ for time-like and space-like vectors, respectively.

We now try to formulate the conversion of the Frenet-Serret vectors.

First from Eqs. (1-1-5), (1-1-6), (1-1-31), (1-1-33), (1-1-35) and (1-1-45), we obtain

$$\sqrt{\epsilon_0}\,\widetilde{\mathfrak{E}}_{(0)} = \begin{bmatrix} 1 \\ \widetilde{\mathfrak{E}}_{(1)} \end{bmatrix} \qquad (1\text{-}1\text{-}68)$$

We then derive by the use of Eqs. (1-1-37), (1-1-38) and (1-1-68),

$$\sqrt{\epsilon_0}\,\widetilde{\mathfrak{E}}_{(1)} = \begin{bmatrix} -\dfrac{V}{\widetilde{R}_0} \\ \dfrac{U-V}{\widetilde{R}_0}\widetilde{\mathfrak{E}}_{(1)} + \dfrac{\widetilde{h}_1}{\widetilde{R}_0}\widetilde{\mathfrak{E}}_{(2)} \end{bmatrix}, \qquad (1\text{-}1\text{-}69)$$

$$\sqrt{\epsilon_0}\,\widetilde{\mathfrak{E}}_{(2)} = \begin{bmatrix} -\dfrac{1}{\widetilde{R}_1}\left(\dfrac{V}{\widetilde{R}_0}\right)' + \dfrac{\widetilde{R}_0}{\widetilde{R}_1} + \dfrac{V^2}{\widetilde{R}_0\widetilde{R}_1} \\ \left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{U-V}{\widetilde{R}_0}\right)' + \dfrac{(U-V)^2}{\widetilde{R}_0\widetilde{R}_1} - \dfrac{\widetilde{h}_1^2}{\widetilde{R}_0\widetilde{R}_1} + \dfrac{\widetilde{R}_0}{\widetilde{R}_1}\right]\widetilde{\mathfrak{E}}_{(1)} + \left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{\widetilde{h}_1}{\widetilde{R}_0}\right)' + \dfrac{2\widetilde{h}_1}{\widetilde{R}_0}\left(\dfrac{U-V}{\widetilde{R}_0}\right)\right]\widetilde{\mathfrak{E}}_{(2)} + \dfrac{\widetilde{h}_1\widetilde{h}_2}{\widetilde{R}_0\widetilde{R}_1}\widetilde{\mathfrak{E}}_{(3)} \end{bmatrix}, \qquad (1\text{-}1\text{-}70)$$

and

$$\sqrt{\epsilon_0}\,\widetilde{\mathfrak{E}}_{(3)} = \left[\begin{array}{l} -\dfrac{\widetilde{R}_1 V}{\widetilde{R}_0\widetilde{R}_2} + \dfrac{1}{\widetilde{R}_2}\left[-\dfrac{1}{\widetilde{R}_1}\left(\dfrac{V}{\widetilde{R}_0}\right)' + \dfrac{\widetilde{R}_0}{\widetilde{R}_1} + \dfrac{V^2}{\widetilde{R}_0\widetilde{R}_1}\right]' - \dfrac{V}{\widetilde{R}_2}\left[-\dfrac{1}{\widetilde{R}_1}\left(\dfrac{V}{\widetilde{R}_0}\right)' + \dfrac{\widetilde{R}_0}{\widetilde{R}_1} + \dfrac{V^2}{\widetilde{R}_0\widetilde{R}_1}\right] \\ \left\{\dfrac{\widetilde{R}_1 U}{\widetilde{R}_2} + \dfrac{1}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{U-V}{\widetilde{R}_0}\right)' + \dfrac{(U-V)^2}{\widetilde{R}_0\widetilde{R}_1} - \dfrac{\widetilde{h}_1^2}{\widetilde{R}_0\widetilde{R}_1} + \dfrac{\widetilde{R}_0}{\widetilde{R}_1}\right]' + \dfrac{U-V}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{U-V}{\widetilde{R}_0}\right)' + \dfrac{(U-V)^2}{\widetilde{R}_0\widetilde{R}_1} - \dfrac{\widetilde{h}_1^2}{\widetilde{R}_0\widetilde{R}_1} + \dfrac{\widetilde{R}_0}{\widetilde{R}_1}\right]\right. \\ \left. -\dfrac{\widetilde{h}_1}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{\widetilde{h}_1}{\widetilde{R}_0}\right)' + \dfrac{2\widetilde{h}_1}{\widetilde{R}_0}\left(\dfrac{U-V}{\widetilde{R}_0}\right)\right]\right\}\widetilde{\mathfrak{E}}_{(1)} + \left\{\dfrac{\widetilde{h}_1\widetilde{R}_1}{\widetilde{R}_2} + \dfrac{1}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{\widetilde{h}_1}{\widetilde{R}_0}\right)' + \dfrac{2\widetilde{h}_1}{\widetilde{R}_0}\left(\dfrac{U-V}{\widetilde{R}_0}\right)\right]' + \dfrac{U-V}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{\widetilde{h}_1}{\widetilde{R}_0}\right)' + \dfrac{2\widetilde{h}_1}{\widetilde{R}_0}\left(\dfrac{U-V}{\widetilde{R}_0}\right)\right]\right. \\ \left. +\dfrac{\widetilde{h}_1}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{U-V}{\widetilde{R}_0}\right)' + \dfrac{(U-V)^2}{\widetilde{R}_0\widetilde{R}_1} - \dfrac{\widetilde{h}_1^2}{\widetilde{R}_0\widetilde{R}_1} + \dfrac{\widetilde{R}_0}{\widetilde{R}_1}\right]\right\}\widetilde{\mathfrak{E}}_{(2)} + \left\{\dfrac{1}{\widetilde{R}_2}\left(\dfrac{\widetilde{h}_1\widetilde{h}_2}{\widetilde{R}_0\widetilde{R}_1}\right)' + \dfrac{U-V}{\widetilde{R}_2}\left(\dfrac{\widetilde{h}_1\widetilde{h}_2}{\widetilde{R}_0\widetilde{R}_1}\right) + \dfrac{\widetilde{h}_2}{\widetilde{R}_2}\left[\dfrac{1}{\widetilde{R}_1}\left(\dfrac{\widetilde{h}_1}{\widetilde{R}_0}\right)' + \dfrac{2\widetilde{h}_1}{\widetilde{R}_0}\left(\dfrac{U-V}{\widetilde{R}_0}\right)\right]\right\}\widetilde{\mathfrak{E}}_{(3)} \end{array}\right].$$

(1-1-71)

Thus using Eqs. (1-1-68) ~ (1-1-71), we can convert the Frenet-Serret vectors in three dimension into those in four dimension and vice versa.

(ii) Time-Like and Space-Like Frenet-Serret Vectors

When we talk about the four dimensional space-time, we seem to be at liberty of choosing the indicator +1 and -1 to be for space-like and time-like curves, respectively or vice versa. However, we learned in the foregoing section that, if we are to relate it to the three dimensional space, it is not the case and the indicator +1 and -1 are inevitably for space-like vectors and time-like vectors, respectively. We naturally would like to know which of the vectors $\mathbf{e}_{(0)}$, $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ are time-like and the rest space-like.

Let us consider the constant velocity motion along the helix, which can, without loss of generality, be represented by

$$\mathbf{Y} = \begin{bmatrix} t \\ a \cos wt \\ a \sin wt \\ bt \end{bmatrix} , \qquad (1\text{-}2\text{-}1)$$

where a, b and w are constants, which is the solution of the set of equation of motion in Cartesian coordinate

$$\begin{cases} x'' + w^2 x = 0 & (1\text{-}2\text{-}2) \\ y'' + w^2 y = 0 & (1\text{-}2\text{-}3) \\ z' = b & (1\text{-}2\text{-}4) \end{cases}$$

under the initial condition ,

$$\begin{cases} x(0) = a & (1\text{-}2\text{-}5) \\ \dot{x}(0) = 0 & (1\text{-}2\text{-}6) \\ y(\frac{\pi}{2}) = a & (1\text{-}2\text{-}7) \\ \dot{y}(\frac{\pi}{2}) = 0 & (1\text{-}2\text{-}8) \\ z(0) = 0 \quad , & (1\text{-}2\text{-}9) \end{cases}$$

where the dot indicates the differentiation with respect to t.

In terms of the four dimensional Frenet-Serret description, first we find by the use of Eqs. (1-1-6) that

$$\begin{aligned} e_{(0)} &= \frac{d\Upsilon}{ds} \\ &= \frac{1}{\sqrt{|1 - v^2|}}\,\Upsilon' \\ &= \frac{1}{\sqrt{|1 - v^2|}} \begin{bmatrix} 1 \\ -a\ w \sin wt \\ a\ w \cos wt \\ b \end{bmatrix} , \end{aligned} \qquad (1\text{-}2\text{-}10)$$

and

$$\begin{aligned} \epsilon_0 &= e^{T}_{(0)} \cdot e_{(0)} \\ &= \frac{1}{\sqrt{|1-v^2|}} [-1, -aw \sin\theta, aw \cos\theta, b] . \frac{1}{\sqrt{|1-v^2|}} \begin{bmatrix} 1 \\ -aw \sin\theta \\ aw \cos\theta \\ b \end{bmatrix} \end{aligned}$$

(where $\theta = wt$)

$$= \frac{a^2w^2 + b^2 - 1}{|1 - v^2|}$$

$$= \frac{v^2 - 1}{|1 - v^2|}$$

$$= \mp 1 \qquad \left(\begin{matrix}\text{time-like}\\ \text{space-like}\end{matrix} \text{ curves}\right) \tag{1-2-11}$$

Then we use one of the Frenet-Serret equations to find

$$\mathbf{e}_{(1)} = \frac{1}{k_o} \frac{d\mathbf{e}_{(0)}}{ds}$$

$$= \frac{1}{k_o} \frac{1}{\sqrt{|1-v^2|}} \frac{d}{dt} \left\{ \frac{1}{\sqrt{|1-v^2|}} \begin{bmatrix} 1 \\ -aw \sin\theta \\ aw \cos\theta \\ b \end{bmatrix} \right\}$$

$$= - \frac{aw^2}{k_o |1 - v^2|} \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \tag{1-2-12}$$

and thus

$$-\epsilon_1 = \mathbf{e}_{(1)}^{T} \cdot \mathbf{e}_{(1)}$$

$$= \frac{a^2w^4}{k_o^2 |1 - v^2|^2} \tag{1-2-13}$$

Namely,

$$\epsilon_1 = 1 \tag{1-2-14}$$

and

$$k_0^2 = \frac{a^2 w^4}{|1 - v^2|^2} \quad . \tag{1-2-15}$$

Similarly we find

$$e_{(2)} = \frac{1}{k_1} \frac{de_{(1)}}{ds} + \varepsilon_0 \varepsilon_1 \frac{k_0}{k_1} e_{(0)}$$

$$= - \frac{aw^3}{k_0 k_1 |1-v^2|^{3/2}} \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} + \varepsilon_0 \varepsilon_1 \frac{k_0}{k_1} \frac{1}{\sqrt{|1-v^2|}} \begin{bmatrix} 1 \\ -aw \sin\theta \\ aw \cos\theta \\ b \end{bmatrix} \tag{1-2-16}$$

and thus

$$\varepsilon_2 = e^T_{(2)} \cdot e_{(2)}$$

$$= \frac{a^2 w^6}{k_0^2 k_1^2 |1-v^2|^3} + \frac{k_0^2}{k_1^2} \frac{v^2-1}{|1-v^2|} - 2\varepsilon_0 \varepsilon_1 \frac{a^2 w^4}{k_1^2 |1-v^2|^2}$$

$$= \frac{1}{k_1^2} \left(\frac{a^2 w^6}{k_0^2 |1-v^2|^3} + k_0^2 \frac{v^2-1}{|1-v^2|} - 2\varepsilon_0 \varepsilon_1 \frac{a^2 w^4}{|1-v^2|^2} \right)$$

$$= \frac{1}{k_1^2} \left(\frac{a^2 w^6}{|1-v^2|^3} \frac{|1-v^2|^2}{a^2 w^4} + \frac{v^2-1}{|1-v^2|} \frac{a^2 w^4}{|1-v^2|^2} - 2\varepsilon_0 \varepsilon_1 \frac{a^2 w^4}{|1-v^2|^2} \right)$$

$$= \frac{1}{k_1^2} w^2 \left(\frac{1}{|1-v^2|} \mp \frac{a^2 w^2}{|1-v^2|^2} \pm 2 \frac{a^2 w^2}{|1-v^2|^2} \right)$$

$$= \frac{1}{k_1^2} w^2 \left(\frac{1}{|1-v^2|} \pm \frac{a^2 w^2}{|1-v^2|^2} \right) \tag{1-2-17}$$

$$= \frac{1}{k_1^2} w^2 \frac{|1-v^2| \pm a^2w^2}{|1-v^2|^2}$$

$$= \pm \frac{1}{k_1^2} w^2 \frac{1-b^2}{|1-v^2|^2} \quad \cdot \quad \left(\begin{matrix}\text{time-like} \\ \text{space-like}\end{matrix} \text{ curves}\right) \qquad (1\text{-}2\text{-}18)$$

Namely

$$\epsilon_2 = \pm 1 \quad \left(\begin{matrix}\text{time-like, space like } (1>b>-1) \\ \text{space-like } (b>1,\ b<-1)\end{matrix} \text{ curves}\right) \qquad (1\text{-}2\text{-}19)$$

and

$$k_1^2 = w^2 \frac{|1-b^2|}{|1-v^2|^2} \quad \cdot \qquad (1\text{-}2\text{-}20)$$

Finally we find

$$\mathbf{e}_{(3)} = \frac{1}{k_2}\frac{d\mathbf{e}_{(2)}}{ds} + \epsilon_1 \epsilon_2 \frac{k_1}{k_2}\mathbf{e}_{(1)}$$

$$= \frac{1}{k_2}\left(\frac{aw^4}{k_0 k_1|1-v^2|^2} - \epsilon_0 \epsilon_1 \frac{k_0}{k_1}\frac{aw^2}{|1-v^2|}\right)\begin{bmatrix}0\\ \cos\theta\\ \sin\theta\\ 0\end{bmatrix} + \epsilon_1 \epsilon_2 \frac{k_1}{k_2}\left(\frac{-aw^2}{k_0|1-v^2|}\right)\begin{bmatrix}0\\ \cos\theta\\ \sin\theta\\ 0\end{bmatrix}$$

$$= \frac{1}{k_2}\left[\frac{aw^4}{k_0 k_1|1-v^2|^2} - \left(\epsilon_0 \epsilon_1 \frac{k_0}{k_1} + \epsilon_1 \epsilon_2 \frac{k_1}{k_0}\right)\frac{aw^2}{|1-v^2|}\right]\begin{bmatrix}0\\ \cos\theta\\ \sin\theta\\ 0\end{bmatrix} \cdot \qquad (1\text{-}2\text{-}21)$$

and thus

$$\epsilon_{(3)} = e^{(3)} \cdot e_{(3)}$$

$$1 = \frac{1}{R_2^2}\left[\frac{aw^4}{R_0 R_1 |1-v^2|^2} - \left(\epsilon_1 \epsilon_2 \frac{R_0}{R_1} + \epsilon_1 \epsilon_2 \frac{R_1}{R_0}\right)\frac{aw^2}{|1-v^2|}\right]^2 , \qquad (1\text{-}2\text{-}22)$$

Namely

$$\epsilon_3 = 1 \qquad (1\text{-}2\text{-}23)$$

and

$$R_2^2 = \left[\frac{aw^4}{|1-v^2|^2}\,\frac{|1-v^2|}{aw^2}\,\frac{|1-v^2|}{w\sqrt{|1-b^2|}} - \left(\epsilon_0 \epsilon_1 \frac{aw^2}{w\sqrt{|1-b^2|}} + \epsilon_1 \epsilon_2 \frac{w\sqrt{|1-b^2|}}{aw^2}\right)\frac{aw^2}{|1-v^2|}\right]^2$$

$$= \left[\frac{w}{\sqrt{|1-b^2|}} - \left(\epsilon_0 \epsilon_1 \frac{aw}{\sqrt{|1-b^2|}} + \epsilon_1 \epsilon_2 \frac{\sqrt{|1-b^2|}}{aw}\right)\frac{aw^2}{|1-v^2|}\right]^2$$

$$= w^2\left[\frac{1}{\sqrt{|1-b^2|}} - \left(\epsilon_0 \frac{aw}{\sqrt{|1-b^2|}} + \epsilon_2 \frac{\sqrt{|1-b^2|}}{aw}\right)\frac{aw}{|1-v^2|}\right]^2 . \qquad (1\text{-}2\text{-}24)$$

To recapitulate, for the time-like curves

$$\epsilon_0 = -1 \quad , \qquad (1\text{-}2\text{-}25)$$

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 1 \qquad (1\text{-}2\text{-}26)$$

and

$$\left\{\begin{aligned} k_0 &= \frac{aw^2}{1 - v^2} && (1\text{-}2\text{-}27) \\ k_1 &= w\,\frac{\sqrt{1 - b^2}}{1 - v^2} && (1\text{-}2\text{-}28) \\ k_2 &= w\left[\frac{1}{\sqrt{1-b^2}} + \left(\frac{aw}{\sqrt{1-b^2}} - \frac{\sqrt{1-b^2}}{aw}\right)\frac{aw}{1-v^2}\right] && (1\text{-}2\text{-}29) \end{aligned}\right.$$

and for the space-like curves with $-1 < b < 1$

$$\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1 \qquad (1\text{-}2\text{-}30)$$

$$\varepsilon_2 = -1 \qquad (1\text{-}2\text{-}31)$$

and

$$k_0 = \frac{aw^2}{v^2 - 1} \qquad (1\text{-}2\text{-}32)$$

$$k_1 = w\,\frac{\sqrt{1 - b^2}}{v^2 - 1} \qquad (1\text{-}2\text{-}33)$$

$$k_2 = w\left[\frac{1}{\sqrt{1-b^2}} - \left(\frac{aw}{\sqrt{1-b^2}} - \frac{\sqrt{1-b^2}}{aw}\right)\frac{aw}{v^2-1}\right] \qquad (1\text{-}2\text{-}34)$$

and for the space-like curves with $b > 1$ or $b < -1$

$$\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 1 \qquad (1\text{-}2\text{-}35)$$

and

$$k_0 = \frac{aw^2}{v^2 - 1} \qquad (1\text{-}2\text{-}36)$$

$$k_1 = w\,\frac{\sqrt{b^2 - 1}}{v^2 - 1} \qquad (1\text{-}2\text{-}37)$$

$$k_2 = 0. \qquad (1\text{-}2\text{-}38)$$

We here note that, if we set $b = 0$ in the foregoing analysis we obtain the result for the circular motion with constant velocity which is for the time-like curves

$$\varepsilon_0 = -1 \tag{1-2-39}$$

$$\varepsilon_1 = \varepsilon_2 = 1 \tag{1-2-40}$$

and

$$k_0 = \frac{vw}{1 - v^2} \tag{1-2-41}$$

$$k_1 = \frac{w}{1 - v^2}$$

$$= \frac{k_0}{v} \tag{1-2-42}$$

and for the space-like curves

$$\varepsilon_0 = \varepsilon_1 = 1 \tag{1-2-43}$$

$$\varepsilon_2 = -1 \tag{1-2-44}$$

and

$$k_0 = \frac{vw}{v^2 - 1} \tag{1-2-45}$$

$$k_1 = \frac{w}{v^2 - 1}$$

$$= \frac{k_0}{v} \tag{1-2-46}$$

In the above, we observe that, because k_2 being zero, $e_{(3)}$ is decoupled and this seems to be a consequence of Ψ being essentially three dimensional.

Let us here comment on the definition of the Frenet-Serret vectors, their indicators and the Frenet-Serret coefficients. In determining k's and ϵ's, for example in Eqs.(1-2-25) through (1-2-38), if we allow k's to take complex values, we could have either +1 or -1 as ϵ's, arbitrarily. We also notice that k^2 is uniquely determined. Therefore we decide here to define all k's to be real positive so that the Frenet-Serret coefficients and their indicator follow uniquely.

Let us now generalize the foregoing analysis a little further. Let us study the motion of a particle with constant speed. From the Frenet-Serret equation we obtain

$$\mathbb{e}_{(0)} = \frac{1}{\sqrt{|1-v^2|}}\begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} \quad , \tag{1-2-47}$$

$$\mathbb{e}_{(1)} = \frac{1}{R_0}\frac{1}{|1-v^2|}\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} \quad , \tag{1-2-48}$$

$$\mathbb{e}_{(2)} = \frac{1}{R_0 R_1}\frac{1}{(|1-v^2|)^{3/2}}\begin{bmatrix} 0 \\ \vec{v}'' \end{bmatrix} + \epsilon_0\,\epsilon_1\,\frac{R_0}{R_1\sqrt{|1-v^2|}}\begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} \tag{1-2-49}$$

$$\mathbb{e}_{(3)} = \frac{1}{R_0 R_1 R_2}\frac{1}{(|1-v^2|)^2}\begin{bmatrix} 0 \\ \vec{v}''' \end{bmatrix} + \left(\epsilon_0\epsilon_1\frac{R_0}{R_1 R_2} + \epsilon_1\epsilon_2\frac{R_1}{R_0 R_2}\right)\frac{1}{|1-v^2|}\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} \quad . \tag{1-2-50}$$

From Eqs. (1-2-47) through (1-2-50) we obtain

$$\begin{aligned} \epsilon_0 &= \mathbb{e}^T_{(0)}\mathbb{e}_{(0)} \\ &= \frac{1}{|1-v^2|}[-1, \vec{v}]\cdot\begin{bmatrix} 1 \\ \vec{v} \end{bmatrix} \\ &= \frac{-1+v^2}{|1-v^2|} \quad , \end{aligned} \tag{1-2-51}$$

$$\begin{aligned} \epsilon_1 &= \mathbb{e}^T_{(1)}\mathbb{e}_{(1)} \\ &= \frac{1}{R_0^2}\frac{1}{|1-v^2|}|\vec{v}'|^2 \quad , \end{aligned} \tag{1-2-52}$$

$$\begin{aligned} \epsilon_2 &= \mathbb{e}^T_{(2)}\mathbb{e}_{(2)} \\ &= \frac{1}{R_0^2 R_1^2(|1-v^2|)^3}|\vec{v}''|^2 - 2\epsilon_0\epsilon_1\frac{1}{R_1^2(|1-v^2|)^2}\vec{v}\cdot\vec{v}'' - \frac{R_0^2(-1+v^2)}{R_1^2(|1-v^2|)} \\ &= \frac{1}{R_1^2}\left[\frac{1}{R_0^2(|1-v^2|)^3}|\vec{v}''|^2 + 2\epsilon_0\epsilon_1\frac{1}{(|1-v^2|)^2}\vec{v}\cdot\vec{v}'' + R_0^2\frac{(-1+v^2)}{(|1-v^2|)}\right] \end{aligned} \tag{1-2-53}$$

$$\epsilon_3 = \rho^T_{(3)} \omega_{(3)}$$
$$= \frac{1}{R_2^2} \left| \frac{1}{R_0^2 R_1^2 (|1-v^2|)^2} \vec{v}''' + \left(\epsilon_0 \epsilon_1 \frac{R_0}{R_1} + \epsilon_1 \epsilon_2 \frac{R_1}{R_0} \right) \frac{1}{|1-v^2|} \vec{v}' \right|^2 \quad . \quad (1\text{-}2\text{-}54)$$

From Eqs. (1-2-51), we find

$$\epsilon_0 = \begin{matrix} -1 \\ +1 \end{matrix} \left(\begin{matrix} \text{time-like} \\ \text{space-like} \end{matrix} \text{ curves} \right) \quad . \quad (1\text{-}2\text{-}55)$$

From Eq. (1-2-52), we find

$$\epsilon_1 = 1 \quad . \quad (1\text{-}2\text{-}56)$$

and

$$k_0^2 = \frac{|\vec{v}'|^2}{|1 - v^2|^2} \quad . \quad (1\text{-}2\text{-}57)$$

From Eq. (1-2-53), we find

$$\epsilon_2 = 1 \quad (1\text{-}2\text{-}58)$$

and

$$k_1^2 = \frac{1}{1 - v^2} \left[\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{(v)^2}{1 - v^2} \right] \quad (1\text{-}2\text{-}59)$$

for time-like curves and

$$\epsilon_2 = \begin{matrix} +1 \\ -1 \end{matrix} \quad (1\text{-}2\text{-}60)$$

depending upon

$$\frac{|\vec{v}''|^2}{|\vec{v}'|^2} \gtrless \frac{|\vec{v}'|^2}{v^2 - 1} \qquad (1\text{-}2\text{-}61)$$

and

$$k_1^2 = \frac{1}{v^2 - 1} \left| \frac{|\vec{v}''|^2}{|\vec{v}'|^2} - \frac{(v)^2}{v^2 - 1} \right| \qquad (1\text{-}2\text{-}62)$$

for space-like curves.

From Eq. (1-2-53), we find

$$\varepsilon_3 = 1 \qquad (1\text{-}2\text{-}63)$$

and

$$k_2^2 = \left| \frac{1-v^2}{|\vec{v}'|^2} \frac{1}{\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}} \vec{v}''' - \left[\frac{|\vec{v}'|}{\sqrt{1-v^2}\sqrt{\left|\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}\right|}} \pm \frac{\sqrt{1-v^2}\sqrt{\left|\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}\right|}}{|\vec{v}'|} \right] \frac{1}{1-v^2}\vec{v}' \right|^2 . \qquad (1\text{-}2\text{-}64)$$

for time-like curves

and

$$k_2^2 = \left| \frac{v^2-1}{|\vec{v}'|^2} \frac{1}{\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}} \vec{v}'' + \left[\frac{|\vec{v}'|}{\sqrt{v^2-1}\sqrt{\left|\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}\right|}} \pm \frac{\sqrt{v^2-1}\sqrt{\left|\frac{|\vec{v}''|^2}{|\vec{v}'|^2} + \frac{|\vec{v}'|^2}{1-v^2}\right|}}{|\vec{v}'|} \right] \frac{1}{v^2-1}\vec{v}' \right|^2 \qquad (1\text{-}2\text{-}65)$$

for space-like curves

again double sign depending on the inequality Eq. (1-2-61).

As a next exercise, let us extend the foregoing to cases with not necessarily constant velocities. The Frenet-Serret vectors are expressed as

$$e_{(0)}=\frac{1}{\sqrt{|1-v^2|}}\begin{bmatrix}1\\ \vec{v}\end{bmatrix}, \tag{1-2-66}$$

$$e_{(1)}=\frac{1}{k_0|1-v^2|}\begin{bmatrix}0\\ \vec{v}'\end{bmatrix} \quad \frac{1}{2k_0}\left(\frac{1}{|1-v^2|}\right)'\begin{bmatrix}1\\ \vec{v}\end{bmatrix}, \tag{1-2-67}$$

$$e_{(2)}=\frac{1}{k_0k_1(|1-v^2|)^{3/2}}\begin{bmatrix}0\\ \vec{v}''\end{bmatrix}+\frac{1}{k_1\sqrt{|1-v^2|}}\left[\left(\frac{1}{k_0|1-v^2|}\right)'+\frac{1}{2}\left(\frac{1}{|1-v^2|}\right)'\right]\begin{bmatrix}0\\ \vec{v}'\end{bmatrix}$$

$$+\left[\frac{1}{k_1\sqrt{|1-v^2|}}\left(\frac{1}{|1-v^2|}\right)'+\epsilon_0\epsilon_1\frac{k_0}{k_1}\frac{1}{\sqrt{|1-v^2|}}\right]\begin{bmatrix}1\\ \vec{v}\end{bmatrix}, \tag{1-2-68}$$

$$e_{(3)}=\frac{1}{k_0k_1k_2(|1-v^2|)^2}\begin{bmatrix}0\\ \vec{v}'''\end{bmatrix}+\left\{\frac{1}{k_2\sqrt{|1-v^2|}}\left[\frac{1}{k_0k_1(|1-v^2|)^{3/2}}\right]'+\frac{1}{k_0k_1|1-v^2|}\left[\left(\frac{1}{k_0|1-v^2|}\right)'+\frac{1}{2}\left(\frac{1}{|1-v^2|}\right)'\right]\right\}\begin{bmatrix}0\\ \vec{v}''\end{bmatrix}$$

$$+\left(\frac{1}{k_2\sqrt{|1-v^2|}}\left\{\frac{1}{k_1\sqrt{|1-v^2|}}\left[\left(\frac{1}{k_0|1-v^2|}\right)'+\frac{1}{2}\left(\frac{1}{|1-v^2|}\right)'\right]\right\}'+\frac{1}{k_2\sqrt{|1-v^2|}}\left[\frac{1}{2k_1\sqrt{|1-v^2|}}\left(\frac{1}{|1-v^2|}\right)''+\epsilon_0\epsilon_1\frac{k_0}{k_1\sqrt{|1-v^2|}}\right]\right.$$

$$\left.+\epsilon_1\epsilon_2\frac{k_1}{k_0k_2|1-v^2|}\right)\begin{bmatrix}0\\ \vec{v}'\end{bmatrix}+\left\{\frac{1}{k_2\sqrt{|1-v^2|}}\left[\frac{1}{2k_1\sqrt{|1-v^2|}}\left(\frac{1}{|1-v^2|}\right)''+\epsilon_0\epsilon_1\frac{k_0}{k_1\sqrt{|1-v^2|}}\right]'+\epsilon_1\epsilon_2\frac{k_1}{2k_2}\left(\frac{1}{|1-v^2|}\right)'\right\}\begin{bmatrix}1\\ \vec{v}\end{bmatrix}.$$

(1-2-69)

The coefficients k's will be calculated shortly. However, in order to study the indicator of the Frenet-Serret vectors, it is convenient to recast Eqs. (1-2-66) through (1-2-69) to

$$\begin{bmatrix}1\\ \vec{v}\end{bmatrix}=\sqrt{|1-v^2|}\,e_{(0)}, \tag{1-2-70}$$

$$\begin{bmatrix}0\\ \vec{v}'\end{bmatrix}=k_0|1-v^2|\,e_{(1)}+\left(\sqrt{|1-v^2|}\right)'e_{(0)}, \tag{1-2-71}$$

$$\begin{bmatrix}0\\ \vec{v}''\end{bmatrix}=\left[\left(\sqrt{|1-v^2|}\right)''-\epsilon_0\epsilon_1k_0^2(|1-v^2|)^{3/2}\right]e_{(0)}+\left[(|1-v^2|k_0)'+\frac{1}{2}(|1-v^2|)'k_0\right]e_{(1)}+k_0k_1(|1-v^2|)^{3/2}e_{(2)},$$

(1-2-72)

$$\left[\frac{\varrho}{v'''}\right]=\left\{\left[\left(\sqrt{|1-v^2|}\right)''-\epsilon_{(0)}\epsilon_{(1)}k_0^2\left(|1-v^2|\right)^{3/2}\right]'-\epsilon_0\epsilon_1 k_0\sqrt{|1-v^2|}\left[\left(|1-v^2|k_0\right)'+\frac{1}{2}\left(|1-v^2|\right)'k_0\right]\right\}e_{(0)}$$

$$+\left\{\left[\left(|1-v^2|k_0\right)'+\frac{1}{2}\left(|1-v^2|\right)'k_0\right]'+k_0\sqrt{|1-v^2|}\left[\left(\sqrt{|1-v^2|}\right)''-\epsilon_0\epsilon_1 k_0^2\left(|1-v^2|\right)^{3/2}\right]-\epsilon_1\epsilon_2 k_1\sqrt{|1-v^2|}\left[\left(|1-v^2|\right)^{3/2}k_0k_1\right]\right\}e_{(1)}$$

$$+\left\{\left[\left(|1-v^2|\right)^{3/2}k_0k_1\right]'+k_1\sqrt{|1-v^2|}\left[\left(|1-v^2|k_0\right)'+\frac{1}{2}\left(|1-v^2|\right)'k_0\right]\right\}e_{(2)}$$

$$+k_2\sqrt{|1-v^2|}\left[\left(|1-v^2|\right)^{3/2}k_0k_1\right]e_{(3)} \qquad (1\text{-}2\text{-}73)$$

so that the orthogonality of the vectors can be taken advantage of. This orthogonality could be examined through the use of Eqs. (1-2-66) through (1-2-69). From Eq. (1-2-70), we obtain

$$\begin{aligned}\epsilon_0 &= e_{(0)}^T\, e_{(0)} \\ &= \frac{-1+v^2}{|1-v^2|} \\ &= \mp 1 \cdot \left(\begin{matrix}\text{time-like}\\ \text{space-like}\end{matrix}\text{ curves}\right).\end{aligned} \qquad (1\text{-}2\text{-}74)$$

From Eq. (1-2-71), we obtain

$$k_0 = \sqrt{\left|\frac{(v)^2}{(1-v^2)^2}+\frac{(\vec{v}.\vec{v})^2}{(1-v^2)^3}\right|} . \qquad (1\text{-}2\text{-}75)$$

Also we obtain

$$\epsilon_1 = 1 \qquad (1\text{-}2\text{-}76)$$

for all time-like curves and space-like curves with such $\vec{v}$ as

$$(\vec{v} \times \vec{v}') \cdot (\vec{v} \times \vec{v}') > \vec{v}' \cdot \vec{v}' \qquad (1\text{-}2\text{-}77)$$

and

$$\varepsilon_1 = -1 \qquad (1\text{-}2\text{-}78)$$

for space like curves with

$$(\vec{v} \times \vec{v}') \cdot (\vec{v} \times \vec{v}') < \vec{v}' \cdot \vec{v}' \quad . \qquad (1\text{-}2\text{-}79)$$

From Eq. (1-2-72), we obtain

$$k_1 = \frac{1}{k_0}\sqrt{\left| \frac{|\vec{v}'''|^2 - \left[\left(\sqrt{|1-v^2|}\right)'' - \epsilon_0 \epsilon_1 k_0^2 \left(|1-v^2|\right)^{3/2}\right]^2 \epsilon_0 - \left[\left(|1-v^2| k_0\right)' + \frac{1}{2}\left(|1-v^2|\right)' k_0\right] \epsilon_1}{\left(|1-v^2|\right)^3} \right|} \quad . \qquad (1\text{-}2\text{-}80)$$

Also

$$\varepsilon_2 = 1 \qquad (1\text{-}2\text{-}81)$$

or

$$\varepsilon_2 = -1 \qquad (1\text{-}2\text{-}82)$$

depending if the quantity inside the outermost absolute sign in Eq. (1-2-80) is positive or negative. From Eq. (1-2-73), we obtain

$$k_2 = \frac{1}{k_0 k_1 \left(|1-v^2|\right)^2}\sqrt{\Bigg(|\vec{v}'''|^2 - \left\{\left[\left(\sqrt{|1-v^2|}\right)'' - \epsilon_0 \epsilon_1 k_0^2 \left(|1-v^2|\right)^{3/2}\right]' - \epsilon_0 \epsilon_1 k_0 \sqrt{|1-v^2|}\left[\left(|1-v^2| k_0\right)' + \frac{1}{2}\left(|1-v^2|\right)' k_0\right]\right\}^2 \epsilon_0}$$
$$\overline{- \left\{\left[\left(|1-v^2| k_0\right)' + \frac{1}{2}\left(|1-v^2|\right)' k_0\right]' + k_0 \sqrt{|1-v^2|}\left[\left(\sqrt{|1-v^2|}\right)'' - \epsilon_0 \epsilon_1 k_0^2 \left(|1-v^2|\right)^{3/2}\right] - \epsilon_1 \epsilon_2 k_1 \sqrt{|1-v^2|}\left[\left(|1-v^2|\right)^{3/2} k_0 k_1\right]\right\} \epsilon_1}$$
$$\overline{- \left\{\left[\left(|1-v^2|\right)^{3/2} k_0 k_1\right]' + k_1 \sqrt{|1-v^2|}\left[\left(|1-v^2| k_0\right)' + \frac{1}{2}\left(|1-v^2|\right)' k_0\right]\right\} \epsilon_3 \Bigg)} \quad . \qquad (1\text{-}2\text{-}83)$$

Also

$$\varepsilon_3 = 1 \qquad (1\text{-}2\text{-}84)$$

or

$$\varepsilon_3 = -1 \qquad (1\text{-}2\text{-}85)$$

depending if the quantity inside the outermost absolute sign in Eq. (1-2-83) is positive or negative.

Let us comment on the determination of ε_0 and ε_1 as described in Eqs. (1-2-74) through (1-2-79). We specifically compare them to the cases with constant velocities as described in Eqs. (1-2-55) through (1-2-57). As we can see from Eqs. (1-2-74) and (1-2-55), it is the same for ε_0. Whereas for ε_1, it is a different story. Of course, if we set $\vec{v}.\vec{v} = 0$ in Eq. (1-2-75), it gives Eq. (1-2-57) and thereby Eq. (1-2-56).

In Eqs. (1-2-66) through (1-2-85), we have listed all the expressions by which the Frenet-Serret vectors, the Frenet-Serret coefficients and the indicators can be calculated. It is rather cumbersome if we express each one of them in terms of $\vec{v}$ alone.

For completeness, let us express the three dimensional Frenet-Serret vectors and coefficients in terms of $\vec{v}$. We first by the use of Eqs. (1-1-5) and (1-1-9) find

$$\begin{aligned}\mathcal{E}_{(1)} &= \frac{1}{|\vec{v}|}\vec{v} \\ &= |\hat{\vec{v}}| \quad .\end{aligned} \tag{1-2-86}$$

We then, by the use of Eq. (1-1-16) obtain

$$\begin{aligned}k_1 &= \sqrt{\frac{d\mathcal{E}_1}{ds}\cdot\frac{d\mathcal{E}_1}{ds}} \\ &= \left|\frac{d\mathcal{E}_1}{ds}\right| \\ &= \frac{\sqrt{\left||\vec{v}|^2|\vec{v}'|^2-(\vec{v}\cdot\vec{v}')^2\right|}}{|\vec{v}|^3} \\ &= \frac{|\vec{v}\times\vec{v}'|}{|\vec{v}|^3} \quad .\end{aligned} \tag{1-2-87}$$

The use of first of Eq. (1-1-7) yields

$$\begin{aligned}\mathcal{E}_{(2)} &= \frac{1}{k_1}\frac{d\mathcal{E}_{(1)}}{ds} \\ &= \frac{1}{\left|\frac{d\mathcal{E}_{(1)}}{ds}\right|}\frac{d\mathcal{E}_{(1)}}{ds} \\ &= \widehat{\frac{d\mathcal{E}_{(1)}}{ds}} \\ &= \frac{|\vec{v}|^2\vec{v}'-(\vec{v}\cdot\vec{v}')\vec{v}}{\left||\vec{v}|^2\vec{v}'-(\vec{v}\cdot\vec{v}')\vec{v}\right|} \\ &= \frac{|\vec{v}|\vec{v}'-\frac{(\vec{v}\cdot\vec{v}')}{|\vec{v}|}\vec{v}}{|\vec{v}\times\vec{v}'|} \quad .\end{aligned} \tag{1-2-88}$$

The use of Eqs. (1-1-16) and (1-1-17) yields

$$
\begin{aligned}
h_2 &= \sqrt{\frac{d\mathcal{E}_{(2)}}{ds}\cdot\frac{d\mathcal{E}_{(2)}}{ds} - \frac{d\mathcal{E}_{(1)}}{ds}\cdot\frac{d\mathcal{E}_{(1)}}{ds}} \\
&= \sqrt{\frac{1}{\left|\frac{d\mathcal{E}_{(1)}}{ds}\right|^2}\left[\frac{d^2\mathcal{E}_{(1)}}{ds^2}\cdot\frac{d^2\mathcal{E}_{(1)}}{ds^2} - \left(\widehat{\frac{d\mathcal{E}_{(1)}}{ds}}\cdot\frac{d^2\mathcal{E}_{(1)}}{ds^2}\right)^2\right] - \frac{d\mathcal{E}_{(1)}}{ds}\cdot\frac{d\mathcal{E}_{(1)}}{ds}} \\
&= \sqrt{\frac{\left\{\left[(\vec{v}\cdot\vec{v}\,')+|\vec{v}|^2\right]^2|\vec{v}\,'|^2 - 2\left[(\vec{v}\cdot\vec{v}\,')+|\vec{v}|^2\right]\left[|\vec{v}\,'|^2+(\vec{v}\cdot\vec{v}\,'')\right](\vec{v}\cdot\vec{v}\,') + \left[|\vec{v}\,'|^2+(\vec{v}\cdot\vec{v}\,'')\right]^2|\vec{v}|^2\right\}}{\left||\vec{v}|^2\vec{v}\,' - (\vec{v}\cdot\vec{v}\,')\vec{v}\right|^2} - \frac{\left||\vec{v}|^2\vec{v}\,' - (\vec{v}\cdot\vec{v}\,')\vec{v}\right|^2}{|\vec{v}|^8}}
\end{aligned}
\tag{1-2-89}
$$

Finally the use of the second of Eq. (1-1-7) gives

$$
\begin{aligned}
\mathcal{E}_{(3)} &= \frac{1}{h_2}\frac{d\mathcal{E}_{(2)}}{ds} + \frac{h_1}{h_2}\mathcal{E}_{(1)} \\
&= \frac{\frac{1}{\left|\frac{d\mathcal{E}_{(1)}}{ds}\right|}\frac{d^2\mathcal{E}_{(1)}}{ds^2} - \frac{1}{\left|\frac{d\mathcal{E}_{(1)}}{ds}\right|}\left(\widehat{\frac{d\mathcal{E}_{(1)}}{ds}}\cdot\frac{d^2\mathcal{E}_{(1)}}{ds^2}\right)\widehat{\frac{d\mathcal{E}_{(1)}}{ds}} + \left|\frac{d\mathcal{E}_{(1)}}{ds}\right|\mathcal{E}_{(1)}}{\sqrt{\frac{1}{\left|\frac{d\mathcal{E}_{(1)}}{ds}\right|^2}\left[\frac{d^2\mathcal{E}_{(1)}}{ds^2}\cdot\frac{d^2\mathcal{E}_{(1)}}{ds^2} - \left(\widehat{\frac{d\mathcal{E}_{(1)}}{ds}}\cdot\frac{d^2\mathcal{E}_{(1)}}{ds^2}\right)^2\right] - \frac{d\mathcal{E}_{(1)}}{ds}\cdot\frac{d\mathcal{E}_{(1)}}{ds}}} \\
&= \frac{\frac{\left[(\vec{v}\cdot\vec{v}\,')+|\vec{v}|^2\right]^2\vec{v}\,' - \left[|\vec{v}\,'|^2+(\vec{v}\cdot\vec{v}\,'')\right]^2\vec{v}}{\left||\vec{v}|^2\vec{v}\,' - (\vec{v}\cdot\vec{v}\,')\vec{v}\right|} - \frac{\sqrt{\left||\vec{v}|^2|\vec{v}\,'|^2 - (\vec{v}\cdot\vec{v}\,')^2\right|}}{|\vec{v}|^4}\vec{v}}{\sqrt{\frac{\left\{\left[(\vec{v}\cdot\vec{v}\,')+|\vec{v}|^2\right]^2|\vec{v}\,'|^2 - 2\left[(\vec{v}\cdot\vec{v}\,')+|\vec{v}|^2\right]\left[|\vec{v}\,'|^2+(\vec{v}\cdot\vec{v}\,'')\right](\vec{v}\cdot\vec{v}\,') + \left[|\vec{v}\,'|^2+(\vec{v}\cdot\vec{v}\,'')\right]^2|\vec{v}|^2\right\}}{\left||\vec{v}|^2\vec{v}\,' - (\vec{v}\cdot\vec{v}\,')\vec{v}\right|^2} - \frac{\left||\vec{v}|^2\vec{v}\,' - (\vec{v}\cdot\vec{v}\,')\vec{v}\right|^2}{|\vec{v}|^8}}}
\end{aligned}
\tag{1-2-90}
$$

(iii) SOLUTIONS

We are now to solve the Frenet-Serret equations in three and four dimension assuming that all the Frenet-Serret coefficients are constant. Let us start with the Frenet-Serret equations in three dimensions.

$$\frac{d}{ds}\mathcal{E} = \underline{\underline{h}} \cdot \mathcal{E} \quad . \qquad (1\text{-}1\text{-}7)$$

Characteristic roots λ's can be found by solving

$$\begin{vmatrix} -\lambda & h_1 & 0 \\ -h_1 & -\lambda & h_2 \\ 0 & -h_2 & -\lambda \end{vmatrix} = -\lambda(\lambda^2+h_1{}^2+h_2{}^2) = 0 \qquad (1\text{-}3\text{-}1)$$

to be

$$\lambda_1 = 0, \quad \lambda_2 = i\sqrt{h_1{}^2+h_2{}^2}, \quad \lambda_3 = -i\sqrt{h_1{}^2+h_2{}^2} \ . \qquad (1\text{-}3\text{-}2)$$

General solutions can be written as

$$\begin{bmatrix} \mathcal{E}_{(1)} \\ \mathcal{E}_{(2)} \\ \mathcal{E}_{(3)} \end{bmatrix} = \begin{bmatrix} C_{11}C_{12}C_{13} \\ C_{21}C_{22}C_{23} \\ C_{31}C_{32}C_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ e^{i\sqrt{h_1{}^2+h_2{}^2}\, s} \\ e^{-i\sqrt{h_1{}^2+h_2{}^2}\, s} \end{bmatrix} , \qquad (1\text{-}3\text{-}3)$$

where C's are integration constants and can be expressed in terms of initial values as

$$\begin{bmatrix} 1 & 1 & 1 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1^{\,2} & \lambda_2^{\,2} & \lambda_3^{\,2} \end{bmatrix} \cdot \begin{bmatrix} C_{11}C_{21}C_{31} \\ C_{12}C_{22}C_{32} \\ C_{13}C_{23}C_{33} \end{bmatrix} = \begin{bmatrix} \mathfrak{E}_{(1)}(0) & \mathfrak{E}_{(2)}(0) & \mathfrak{E}_{(3)}(0) \\ \dfrac{d\mathfrak{E}_{(1)}}{ds}(0) & \dfrac{d\mathfrak{E}_{(2)}}{ds}(0) & \dfrac{d\mathfrak{E}_{(3)}}{ds}(0) \\ \dfrac{d^2\mathfrak{E}_{(1)}}{ds^2}(0) & \dfrac{d^2\mathfrak{E}_{(2)}}{ds^2}(0) & \dfrac{d^2\mathfrak{E}_{(3)}}{ds^2}(0) \end{bmatrix} \tag{1-3-4}$$

or

$$\underline{\underline{\lambda}} \cdot \underline{\underline{C}} = \begin{bmatrix} \mathfrak{E}^+(0) \\ \dfrac{d}{ds}\,\mathfrak{E}^+(0) \\ \dfrac{d^2}{ds^2}\,\mathfrak{E}^+(0) \end{bmatrix}$$

$$= \mathfrak{E}^+(0) \cdot \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{h}}^+ \\ \underline{\underline{h}}^+ \cdot \underline{\underline{h}}^+ \end{bmatrix} \tag{1-3-5}$$

where

$$\underline{\underline{\lambda}} = \begin{bmatrix} 1 & 1 & 1 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1^{\,2} & \lambda_2^{\,2} & \lambda_3^{\,2} \end{bmatrix} \tag{1-3-6}$$

and

$$\underline{\underline{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad (1\text{-}3\text{-}7)$$

C's now can be solved as

$$\underline{\underline{C}} = \underline{\underline{\lambda}}^{-1} \cdot \begin{bmatrix} \xi^{+}(0) \\ \frac{d}{ds}\xi^{+}(0) \\ \frac{d^2}{ds^2}\xi^{+}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{(1)}(0) & \xi_{(2)}(0) & \xi_{(3)}(0) \\ \frac{\lambda}{2\sqrt{h_1^2+h_2^2}}\left[\frac{d\xi_{(1)}}{ds}(0) - \frac{d^2\xi_{(1)}}{ds^2}(0)\right] & \frac{\lambda}{2\sqrt{h_1^2+h_2^2}}\left[\frac{d\xi_{(2)}}{ds}(0) - \frac{d^2\xi_{(2)}}{ds^2}(0)\right] & \frac{\lambda}{2\sqrt{h_1^2+h_2^2}}\left[\frac{d\xi_{(3)}(0)}{ds} - \frac{d^2\xi_{(3)}}{ds^2}(0)\right] \\ \frac{2\xi_{(1)}(0) - \frac{d\xi_{(1)}}{ds}(0) - \frac{d^2\xi_{(1)}}{ds^2}(0)}{2(h_1^2+h_2^2)} & \frac{2\xi_{(2)}(0) - \frac{d\xi_{(2)}}{ds}(0) - \frac{d^2\xi_{(2)}}{ds^2}(0)}{2(h_1^2+h_2^2)} & \frac{2\xi_{(3)}(0) - \frac{d\xi_{(3)}(0)}{ds} - \frac{d^2\xi_{(3)}}{ds^2}(0)}{2(h_1^2+h_2^2)} \end{bmatrix}, \qquad (1\text{-}3\text{-}8)$$

or it can also be solved as

$$\underline{\underline{C}} = \underline{\underline{\lambda}}^{-1} \cdot \xi^{+}(0) \cdot \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{h}}^{+} \\ \underline{\underline{h}}^{+} \cdot \underline{\underline{h}}^{+} \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\lambda}{2\sqrt{k_1^2+k_2^2}} & -\frac{\lambda}{2\sqrt{k_1^2+k_2^2}} \\ \frac{1}{k_1^2+k_2^2} & -\frac{1}{2(k_1^2+k_2^2)} & -\frac{1}{2(k_1^2+k_2^2)} \end{bmatrix} \cdot \left[\mathcal{E}_{(1)}(0), \mathcal{E}_{(2)}(0), \mathcal{E}_{(3)}(0) \right] \cdot \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -k_1 & 0 \\ k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix} \\ \begin{bmatrix} -k_1^2 & 0 & k_1 k_2 \\ 0 & -(k_1^2+k_2^2) & 0 \\ k_1 k_2 & 0 & -k_2^2 \end{bmatrix} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathcal{E}_{(1)}(0) & \mathcal{E}_{(2)}(0) & \mathcal{E}_{(3)}(0) \\ \frac{\lambda\left[k_1^2\mathcal{E}_{(1)}(0)+k_1\mathcal{E}_{(2)}(0)-k_1k_2\mathcal{E}_{(3)}(0)\right]}{2\sqrt{k_1^2+k_2^2}} & \frac{\lambda\left[-k_1\mathcal{E}_{(1)}(0)+(k_1^2+k_2^2)\mathcal{E}_{(2)}(0)+k_2\mathcal{E}_{(3)}(0)\right]}{2\sqrt{k_1^2+k_2^2}} & \frac{\lambda\left[-k_1k_2\mathcal{E}_{(1)}(0)-k_2\mathcal{E}_{(2)}(0)+k_2^2\mathcal{E}_{(3)}(0)\right]}{2\sqrt{k_1^2+k_2^2}} \\ \frac{(k_1^2+2)\mathcal{E}_{(1)}(0)-k_1\mathcal{E}_{(2)}(0)-k_1k_2\mathcal{E}_{(3)}(0)}{2(k_1^2+k_2^2)} & \frac{-k_1\mathcal{E}_{(1)}(0)+(k_1^2+k_2^2+2)\mathcal{E}_{(2)}(0)-k_2\mathcal{E}_{(3)}(0)}{2(k_1^2+k_2^2)} & \frac{-k_1k_2\mathcal{E}_{(1)}(0)+k_2\mathcal{E}_{(2)}(0)+(k_2^2+2)\mathcal{E}_{(3)}(0)}{2(k_1^2+k_2^2)} \end{bmatrix} \cdot \quad (1\text{-}3\text{-}9)
$$

Let us now proceed to the case of the Frenet-Serret equations in four dimension as

$$
\frac{d}{ds}\tilde{e} = \underline{\underline{\tilde{k}}} \cdot \tilde{e} \qquad (1\text{-}1\text{-}20)
$$

Characteristic roots λ's can be found by solving

$$
\begin{vmatrix} -\lambda & \tilde{k} & 0 & 0 \\ -\tilde{k} & -\lambda & \tilde{\tau}_1 & 0 \\ 0 & -\tilde{\tau}_1 & -\lambda & \tilde{\tau}_2 \\ 0 & 0 & -\tilde{\tau}_2 & -\lambda \end{vmatrix} = \lambda^4 + a^2\lambda^2 + \Omega^2 = 0 \qquad (1\text{-}3\text{-}10)
$$

as

$$\lambda = \pm\sqrt{\frac{-a^2 + \sqrt{a^4 - 4\Omega^2}}{2}} \qquad (1\text{-}3\text{-}11)$$

or

$$\lambda_1 = \sqrt{\frac{-a^2 + \sqrt{a^4 - 4\Omega^2}}{2}}, \quad \lambda_2 = \sqrt{\frac{-a^2 - \sqrt{a^4 - 4\Omega^2}}{2}}$$

$$\lambda_3 = -\sqrt{\frac{-a^2 + \sqrt{a^4 - 4\Omega^2}}{2}}, \quad \lambda_4 = -\sqrt{\frac{-a^2 - \sqrt{a^4 - 4\Omega^2}}{2}}$$

$$(1\text{-}3\text{-}12)$$

where

$$\begin{aligned} a^2 &= \tilde{k}^2 + \tilde{\tau}_1{}^2 + \tilde{\tau}_2{}^2 \\ &= \varepsilon_0 \varepsilon_1 k^2 + \varepsilon_1 \varepsilon_2 \tau_1{}^2 + \varepsilon_2 \varepsilon_3 \tau_2{}^2 \end{aligned}$$

$$(1\text{-}3\text{-}13)$$

and

$$\begin{aligned} \Omega^2 &= \tilde{k}^2 . \tilde{\tau}_2{}^2 \\ &= \varepsilon_0 \varepsilon_1 \varepsilon_2 \varepsilon_3 k^2 \tau_2{}^2 \quad . \end{aligned} \qquad (1\text{-}3\text{-}14)$$

We first note that

$$\begin{aligned} a^4 - 4\Omega^2 &= (\tilde{k}^2 + \tilde{\tau}_1{}^2 + \tilde{\tau}_2{}^2)^2 - 4\tilde{k}^2\tilde{\tau}_2{}^2 \\ &= [(\tilde{k} + \tilde{\tau}_2)^2 + \tilde{\tau}_1{}^2][(\tilde{k} - \tilde{\tau}_2)^2 + \tilde{\tau}_1{}^2] > 0 \quad . \end{aligned}$$

$$(1\text{-}3\text{-}15)$$

In terms of the characteristic roots described above, we can write the general solution as

$$\begin{bmatrix} e^{\mu}_{(0)} \\ e^{\mu}_{(1)} \\ e^{\mu}_{(2)} \\ e^{\mu}_{(3)} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} & C_{02} & C_{03} \\ C_{10} & C_{11} & C_{12} & C_{13} \\ C_{20} & C_{21} & C_{22} & C_{23} \\ C_{30} & C_{31} & C_{32} & C_{33} \end{bmatrix} \cdot \begin{bmatrix} e^{-\lambda_1 S} \\ e^{-\lambda_2 S} \\ e^{-\lambda_3 S} \\ e^{-\lambda_4 S} \end{bmatrix}, \quad (1\text{-}3\text{-}16)$$

where C's, integration constants, can be deduced from

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 \\ \lambda_1^{\,2} & \lambda_2^{\,2} & \lambda_3^{\,2} & \lambda_4^{\,2} \\ -\lambda_1^{\,3} & -\lambda_2^{\,3} & -\lambda_3^{\,3} & -\lambda_4^{\,3} \end{bmatrix} \cdot \begin{bmatrix} C_{00} & C_{01} & C_{02} & C_{03} \\ C_{10} & C_{11} & C_{12} & C_{13} \\ C_{20} & C_{21} & C_{22} & C_{23} \\ C_{30} & C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} e^{\mu}_{(0)}(0) & e^{\mu}_{(1)}(0) & e^{\mu}_{(2)}(0) & e^{\mu}_{(3)}(0) \\ \frac{de^{\mu}_{(0)}}{ds}(0) & \frac{de^{\mu}_{(1)}}{ds}(0) & \frac{de^{\mu}_{(2)}}{ds}(0) & \frac{de^{\mu}_{(3)}}{ds}(0) \\ \frac{d^2e^{\mu}_{(0)}}{ds^2}(0) & \frac{d^2e^{\mu}_{(1)}}{ds^2}(0) & \frac{d^2e^{\mu}_{(2)}}{ds^2}(0) & \frac{d^2e^{\mu}_{(3)}}{ds^2}(0) \\ \frac{d^3e^{\mu}_{(0)}}{ds^3}(0) & \frac{d^3e^{\mu}_{(1)}}{ds^3}(0) & \frac{d^3e^{\mu}_{(2)}}{ds^3}(0) & \frac{d^3e^{\mu}_{(3)}}{ds^3}(0) \end{bmatrix}$$

$$(1\text{-}3\text{-}17)$$

or

$$\underline{\underline{\lambda}} \cdot \underline{\underline{C}} = \begin{bmatrix} e^{\mu+}(0) \\ \frac{de^{\mu+}}{ds}(0) \\ \frac{d^2e^{\mu+}}{ds^2}(0) \\ \frac{d^3e^{\mu+}}{ds^3}(0) \end{bmatrix}$$

$$= (e^{\mu})^{+}(0) \cdot \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{k}}^{+} \\ \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \\ \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \end{bmatrix} \qquad (1\text{-}3\text{-}18)$$

where

$$\underline{\underline{\lambda}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 \\ \lambda_1^{2} & \lambda_2^{2} & \lambda_3^{2} & \lambda_4^{2} \\ -\lambda_1^{2} & -\lambda_2^{3} & -\lambda_3^{3} & -\lambda_4^{3} \end{bmatrix} \qquad (1\text{-}3\text{-}19)$$

and

$$\underline{\underline{C}} = \begin{bmatrix} C_{00} & C_{01} & C_{02} & C_{03} \\ C_{10} & C_{11} & C_{12} & C_{13} \\ C_{20} & C_{21} & C_{22} & C_{23} \\ C_{30} & C_{31} & C_{32} & C_{33} \end{bmatrix} \qquad (1\text{-}3\text{-}20)$$

Now C's can be solved as

$$\underline{\underline{C}} = \underline{\underline{\lambda}}^{-1} \cdot \begin{bmatrix} (e^{\mu})^{+}(0) \\ \frac{d}{ds}(e^{\mu})^{+}(0) \\ \frac{d^2}{ds^2}(e^{\mu})^{+}(0) \\ \frac{d^3}{ds^3}(e^{\mu})^{+}(0) \end{bmatrix} \qquad (1\text{-}3\text{-}21)$$

or as

$$\underline{\underline{C}} = \underline{\underline{\lambda}}^{-1} \cdot \underline{\underline{(e^{\mu})}}^{+}(0) \cdot \begin{bmatrix} \underline{\underline{I}} \\ \underline{\underline{k}}^{+} \\ \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \\ \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \cdot \underline{\underline{k}}^{+} \end{bmatrix} \quad . \qquad (1\text{-}3\text{-}22)$$

We will not here complete Eq. (1-3-22) in a manner of Eq. (1-3-9) since it is only cumbersome although straightforward.

CHAPTER 2

SPINORS

(i) Frenet-Serret Equations in Spinor Forms

Let us now introduce null tetrads in general by

$$\begin{matrix} \ell^{\mu} \\ \bar{\ell}^{\mu} \end{matrix} = \frac{1}{\sqrt{2}} [\, e^{\mu}_{(a)} \pm i^{\delta_{\varepsilon_a,\varepsilon_b}}\, e^{\mu}_{(b)} \,]$$

$$= \frac{1}{\sqrt{2}} [\, e^{\mu}_{(a)} \pm \sqrt{-\varepsilon_a \varepsilon_b}\, e^{\mu}_{(b)} \,] \qquad (2\text{-}1\text{-}1)$$

which insures

$$\ell^{\mu} \bar{\ell}^{\mu} = \frac{1}{2} (\varepsilon_a + \varepsilon_a \overline{\varepsilon_b} \varepsilon_b)$$

$$= \varepsilon_a \quad . \qquad (2\text{-}1\text{-}2)$$

Specifically for our case, we define

$$\begin{matrix} \ell^{\mu}_{1} \\ \bar{\ell}^{\mu}_{1} \end{matrix} = \frac{1}{\sqrt{2}} [\, e^{\mu}_{(1)} \pm \sqrt{-\varepsilon_0 \varepsilon_3}\, e^{\mu}_{(3)} \,] \qquad (2\text{-}1\text{-}3)$$

and

$$\begin{matrix} \ell^{\mu}_{2} \\ \bar{\ell}^{\mu}_{2} \end{matrix} = \frac{1}{\sqrt{2}} [\, e^{\mu}_{(1)} \pm \sqrt{-\varepsilon_1 \varepsilon_2}\, e^{\mu}_{(2)} \,] \qquad (2\text{-}1\text{-}4)$$

namely

$$\boldsymbol{\ell}^{\mu} = \underline{\underline{T}} \cdot \mathbf{e}^{\mu} \qquad (2\text{-}1\text{-}5)$$

where

$$\ell^{\mu} = \begin{bmatrix} \ell_1^{\mu} \\ \bar{\ell}_1^{\mu} \\ \ell_2^{\mu} \\ \bar{\ell}_2^{\mu} \end{bmatrix} \qquad (2\text{-}1\text{-}6)$$

and

$$\underline{\underline{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & \sqrt{-\varepsilon_0\varepsilon_3} \\ 1 & 0 & 0 & -\sqrt{-\varepsilon_0\varepsilon_3} \\ 0 & 1 & \sqrt{-\varepsilon_1\varepsilon_2} & 0 \\ 0 & 1 & -\sqrt{-\varepsilon_1\varepsilon_2} & 0 \end{bmatrix} \qquad (2\text{-}1\text{-}7)$$

We recognize for time-like curves Eq.(2-1-5) becomes

$$\ell^{\mu} = \begin{bmatrix} \ell^{\mu} \\ n^{\mu} \\ m^{\mu} \\ \bar{m}^{\mu} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\mu}_{(0)} + e^{\mu}_{(3)} \\ e^{\mu}_{(0)} - e^{\mu}_{(3)} \\ e^{\mu}_{(1)} + ie^{\mu}_{(2)} \\ e^{\mu}_{(1)} - ie^{\mu}_{(2)} \end{bmatrix} \qquad (2\text{-}1\text{-}8)$$

We note here that the definition of nulltetrads by Eq. (2-1-1) is quite arbitrary. We could have just as well defined them as

$$\begin{matrix} \ell^{\mu} \\ \bar{\ell}^{\mu} \end{matrix} = \frac{1}{\sqrt{2}} [\, e^{\mu}_{(b)} \pm \sqrt{-\varepsilon_b \varepsilon_a}\; e^{\mu}_{(a)} \,] \qquad (2\text{-}1\text{-}9)$$

which would be different from Eq. (2-1-1) when $\varepsilon_a \neq \varepsilon_b$. We also note that there are arbitrariness involved in Eqs. (2-1-3) and (2-1-4). We could have picked different combinations just as well.

We can also define the null tetrads for $\tilde{e}$ as

$$\begin{matrix} \tilde{\ell}^{\mu} \\ \tilde{\bar{\ell}}^{\mu} \end{matrix} = \frac{1}{\sqrt{2}} [\, \tilde{e}^{\mu}_{(a)} \pm \sqrt{-\varepsilon_a \varepsilon_b}\; \tilde{e}^{\mu}_{(b)} \qquad (2\text{-}1\text{-}10)$$

or

$$\tilde{\ell}^{\mu} = \underline{\underline{T}} \cdot \tilde{e}^{\mu} \qquad (2\text{-}1\text{-}11)$$

which insures

$$\tilde{\ell}^{\mu} \tilde{\bar{\ell}}^{\mu} = 1 \qquad (2\text{-}1\text{-}12)$$

as compared to Eqs. (2-1-1), (2-1-5) and (2-1-2). Using Eqs. (2-1-5) and (2-1-11), the Frenet-Serret equations Eqs. (1-1-10) and (1-1-30) are transformed to

$$\frac{d}{ds} \ell^{\mu} = \underline{\underline{L}} \cdot \ell^{\mu} \qquad (2\text{-}1\text{-}13)$$

and

$$\frac{d}{ds}\tilde{k}^{\mu} = \underline{\underline{L}} \cdot \tilde{k}^{\mu} \tag{2-1-14}$$

where

$$\underline{\underline{T}}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{-\epsilon_1\epsilon_2} & -1/\sqrt{-\epsilon_1\epsilon_2} \\ 1/\sqrt{-\epsilon_0\epsilon_3} & -1/\sqrt{-\epsilon_0\epsilon_3} & 0 & 0 \end{bmatrix} \tag{2-1-15}$$

$$\underline{\underline{L}} = \underline{\underline{T}} \cdot \underline{\underline{k}} \cdot \underline{\underline{T}}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & k_0 - (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 k_2 & k_0 + (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 k_2 \\ 0 & 0 & k_0 + (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 k_2 & k_0 - (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 k_2 \\ -\epsilon_0\epsilon_1 k_0 + (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})k_2 & -\epsilon_0\epsilon_1 k_0 - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})k_2 & (2/\sqrt{-\epsilon_1\epsilon_2})k_1 & 0 \\ -\epsilon_0\epsilon_1 k_0 - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})k_2 & -\epsilon_0\epsilon_1 k_0 + (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})k_2 & 0 & (2/\sqrt{-\epsilon_1\epsilon_2})k_1 \end{bmatrix} \tag{2-1-16}$$

and

$$\tilde{\underline{\underline{L}}} = \underline{\underline{T}} \cdot \tilde{\underline{\underline{K}}} \cdot \underline{\underline{T}}^{-1}$$

$$= \underline{\underline{T}} \cdot \underline{\underline{\zeta}} \cdot \underline{\underline{k}} \;\; \underline{\underline{\zeta}}^{-1} \cdot \underline{\underline{T}}^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & \tilde{k}_0 - (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{k}_2 & \tilde{k}_0 + (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{k}_2 \\ 0 & 0 & \tilde{k}_0 + (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{k}_2 & \tilde{k}_0 - (\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{k}_2 \\ -\tilde{k}_0 + (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\tilde{k}_2 & -\tilde{k}_0 - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\tilde{k}_2 & \{(1/\sqrt{-\epsilon_1\epsilon_2}) - \sqrt{-\epsilon_1\epsilon_2}\}\tilde{k}_1 & 0 \\ -\tilde{k}_0 - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\tilde{k}_2 & -\tilde{k}_0 + (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\tilde{k}_2 & 0 & \{-(1/\sqrt{-\epsilon_1\epsilon_2}) + \sqrt{-\epsilon_1\epsilon_2}\}\tilde{k}_1 \end{bmatrix} \tag{2-1-17}$$

For example $\underline{\underline{L}}$ is calculated for time-like curves as

$$\underline{\underline{L}} = \frac{1}{2}\begin{bmatrix} 0 & 0 & k_0+ik_2 & k_0-ik_2 \\ 0 & 0 & k_0-ik_2 & k_0+ik_2 \\ k_0+ik_2 & k_0-ik_2 & -i2k_1 & 0 \\ k_0-ik_2 & k_0+ik_2 & 0 & i2k_1 \end{bmatrix} \quad (2\text{-}1\text{-}18)$$

and for space-like curves with indicators $\varepsilon_2 = -1$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1$ as

$$\underline{\underline{L}} = \frac{1}{2}\begin{bmatrix} 0 & 0 & k_0+ik_2 & k_0-ik_2 \\ 0 & 0 & k_0-ik_2 & k_0+ik_2 \\ -k_0-ik_2 & -k_0+ik_2 & 2k_1 & 0 \\ -k_0+ik_2 & -k_0-ik_2 & 0 & 2k_1 \end{bmatrix} \quad (2\text{-}1\text{-}19)$$

We now introduce two spinors ξ^A and η^A which satisfy the "normalization condition"

$$\xi_A \, \eta^A = 1 \quad . \quad (2\text{-}1\text{-}20)$$

We note a unique correspondence between these spinors and the orthonormal tetrads as

$$\mathbf{e}^{\mu} \frac{1}{\sqrt{2}} \sigma_{\mu}^{A\dot{B}} = \begin{bmatrix} \xi^A & \xi^B \\ \eta^A & \eta^B \\ \xi^A & \xi^B \\ \eta^A & \eta^B \end{bmatrix}$$

$$= \underline{\underline{\Sigma}}^{-1} \cdot \frac{d}{ds} \boldsymbol{\xi} \quad , \tag{2-1-21}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \xi^{A} \\ \eta^{A} \\ \xi^{\dot{B}} \\ \eta^{\dot{B}} \end{bmatrix} \tag{2-1-22}$$

and

$$\underline{\underline{\Sigma}} = \frac{1}{2} \begin{bmatrix} -\eta_{\dot{B}} & 0 & \xi_{\dot{B}} & 0 \\ 0 & \xi_{\dot{B}} & 0 & -\eta_{\dot{B}} \\ -\eta_{A} & 0 & 0 & \xi_{A} \\ 0 & \xi_{A} & -\eta_{A} & 0 \end{bmatrix} \tag{2-1-23}$$

We thus operate $\frac{1}{\sqrt{2}} \sigma_{\mu}{}^{A\dot{B}}$ on both sides of Eqs. (2-1-13) and (2-1-14) and then pre-multiply both sizes by $\underline{\underline{\Sigma}}$ to obtain

$$\frac{d}{ds} \boldsymbol{\xi} = \underline{\underline{\Gamma}} \cdot \boldsymbol{\xi} \tag{2-1-24}$$

and

$$\frac{d}{ds} \tilde{\boldsymbol{\xi}} = \underline{\underline{\tilde{\Gamma}}} \cdot \tilde{\boldsymbol{\xi}} \quad , \tag{2-1-25}$$

where

$$\tilde{\boldsymbol{\xi}} = \begin{bmatrix} \tilde{\xi}^{A} \\ \tilde{\eta}^{A} \\ \tilde{\xi}^{\dot{B}} \\ \tilde{\eta}^{\dot{B}} \end{bmatrix} \tag{2-1-26}$$

$$\underline{\underline{\Gamma}} = \underline{\underline{\Sigma}} \cdot \underline{\underline{L}} \cdot \underline{\underline{\Sigma}}^{-1} = \frac{1}{2}\begin{bmatrix} -\eta_{\dot B} & 0 & \xi_{\dot B} & 0 \\ 0 & \xi_{\dot B} & 0 & -\eta_{\dot B} \\ -\eta_A & 0 & 0 & \xi_A \\ 0 & \xi_A & -\eta_A & 0 \end{bmatrix} \cdot \underline{\underline{L}} \cdot \begin{bmatrix} \xi^{\dot B} & 0 & \xi^A & 0 \\ 0 & \eta^{\dot B} & 0 & \eta^A \\ \eta^{\dot B} & 0 & 0 & \xi^A \\ 0 & \xi^{\dot B} & \eta^A & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & \Gamma_{43} & \Gamma_{44} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} L_{11}+L_{33} & L_{32}+L_{14} & 0 & 0 \\ L_{41}+L_{23} & L_{22}+L_{44} & 0 & 0 \\ 0 & 0 & L_{11}+L_{44} & L_{42}+L_{13} \\ 0 & 0 & L_{31}+L_{24} & L_{22}+L_{33} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (2/\sqrt{-\epsilon_1\epsilon_2})R_1 & (1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2 & 0 & 0 \\ (1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2 & (2/\sqrt{-\epsilon_1\epsilon_2})R_1 & 0 & 0 \\ 0 & 0 & (-2/\sqrt{-\epsilon_1\epsilon_2})R_1 & (1-\epsilon_0\epsilon_1)R_0-[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_1 \\ 0 & 0 & (1-\epsilon_0\epsilon_1)R_0-[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_1 & (-2/\sqrt{-\epsilon_1\epsilon_2})R_1 \end{bmatrix} \quad (2\text{-}1\text{-}27)$$

likewise

$$\tilde{\underline{\underline{\Gamma}}} = \underline{\underline{\Sigma}} \cdot \tilde{\underline{\underline{L}}} \cdot \underline{\underline{\Sigma}}^{-1}$$

$$= \frac{1}{4}\begin{bmatrix} [(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde R_1 & [(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]\tilde R_2 & 0 & 0 \\ [(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]\tilde R_2 & [(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde R_1 & 0 & 0 \\ 0 & 0 & -[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde R_1 & -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})]\tilde R_2 \\ 0 & 0 & -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]\tilde R_2 & -[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde R_1 \end{bmatrix} \quad (2\text{-}1\text{-}28)$$

and also note that ε is introduced via

$$\overset{\urcorner}{\varepsilon}{}^{\mu} \frac{1}{\sqrt{2}} \sigma_{\mu}{}^{A\dot{B}} = \begin{bmatrix} \overset{\urcorner}{\xi}{}^{A} \overset{\urcorner}{\xi}{}^{\dot{B}} \\ \overset{\backsim\!\urcorner}{\eta}{}^{A} \overset{\backsim\!\urcorner}{\eta}{}^{\dot{B}} \\ \overset{\backsim\!\urcorner}{\xi}{}^{A} \overset{\backsim\!\urcorner}{\xi}{}^{\dot{B}} \\ \overset{\backsim\!\urcorner}{\eta}{}^{A} \overset{\backsim\!\urcorner}{\eta}{}^{\dot{B}} \end{bmatrix} . \qquad (2\text{-}1\text{-}29)$$

For example P is calculated for time-like curves as

$$\underline{\underline{\Gamma}} = \frac{1}{2} \begin{bmatrix} -ik_1 & k_0-ik_2 & 0 & 0 \\ k_0-ik_2 & ik_1 & 0 & 0 \\ 0 & 0 & ik_1 & k_0+ik_2 \\ 0 & 0 & k^0+ik_2 & -ik_1 \end{bmatrix} \qquad (2\text{-}1\text{-}30)$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$ as

$$\underline{\underline{\Gamma}} = \frac{1}{2} \begin{bmatrix} k_1 & ik_2 & 0 & 0 \\ ik_2 & -k_1 & 0 & 0 \\ 0 & 0 & -k_1 & ik_2 \\ 0 & 0 & ik_2 & k_1 \end{bmatrix} . \qquad (2\text{-}1\text{-}31)$$

We also note that we kept L_{11}, L_{22} in general in Eq. (2-1-27) although they are zero in Eq. (2-1-16) since the latter is due to our choice of way to construct the null tetrads (See Eq. (2-1-3) and (2-1-4)). Eq. (2-1-30) agrees with the previously available result. (See Kent p. 28.)

Eqs. (2-1-24) and (2-1-25) can be re-expressed in terms of bispinors using

$$\Psi = \frac{1}{2}\ \underline{\underline{\text{Ⓗ}}} \cdot \xi \qquad (2\text{-}1\text{-}32)$$

and

$$\tilde{\Psi} = \frac{1}{2}\ \underline{\underline{\text{Ⓗ}}} \cdot \xi \quad , \qquad (2\text{-}1\text{-}33)$$

respectively, where

$$\underline{\underline{\text{Ⓗ}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{\dot{A}\dot{B}} & 0 \\ -\delta^{B}{}_{A} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_{\dot{B}\dot{C}}\delta^{\dot{C}}{}_{\dot{B}-} \end{bmatrix} \qquad (2\text{-}1\text{-}33)$$

$$\Psi = \frac{1}{2} \begin{bmatrix} \eta^{A} \\ \xi_{\dot{A}} \\ \xi^{B} \\ \eta_{\dot{B}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_{a} \\ \chi_{b} \end{bmatrix} , \qquad (2\text{-}1\text{-}34)$$

$$\tilde{\Psi} = \frac{1}{2} \begin{bmatrix} \eta^{A} \\ \xi_{\dot{A}} \\ \xi^{B} \\ \eta_{\dot{B}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\varphi}_a \\ \tilde{\chi}_b \end{bmatrix} , \qquad (2\text{-}1\text{-}35)$$

$$\varphi_a = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta^A \\ \xi_{\dot{A}} \end{bmatrix} , \qquad (2\text{-}1\text{-}36)$$

$$\chi_b = \frac{1}{\sqrt{2}} \begin{bmatrix} \xi^B \\ \eta_{\dot{B}} \end{bmatrix} , \qquad (2\text{-}1\text{-}37)$$

$$\tilde{\varphi}_a = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\eta}^A \\ \tilde{\xi}_{\dot{A}} \end{bmatrix} , \qquad (2\text{-}1\text{-}38)$$

$$\tilde{\chi}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\xi}^B \\ \tilde{\eta}_{\dot{B}} \end{bmatrix} , \qquad (2\text{-}1\text{-}39)$$

as

$$\frac{d}{ds} \Psi = \underline{\underline{\Phi}} \cdot \Psi \qquad (2\text{-}1\text{-}40)$$

and

$$\frac{d}{ds} \tilde{\Psi} = \underline{\underline{\Phi}} \cdot \Psi \qquad (2\text{-}1\text{-}41)$$

respectively, where

$$\underline{\underline{\Phi}} = \underline{\underline{H}} \cdot \underline{\underline{\Gamma}} \cdot \underline{\underline{H}}^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_{\dot{A}\dot{B}} & 0 \\ -\delta^B{}_A & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{\dot{B}\dot{C}}\delta^{\dot{C}}{}_{\dot{B}} \end{bmatrix} \cdot \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & \Gamma_{43} & \Gamma_{44} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -\delta^A{}_B & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\epsilon^{\dot{B}\dot{A}} & 0 & 0 \\ 0 & 0 & 0 & -\delta^{\dot{B}}{}_{\dot{C}}\epsilon^{\dot{C}\dot{B}} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_{22} & 0 & -\Gamma_{21}\delta^{A}{}_{B} & 0 \\ 0 & \Gamma_{33} & 0 & -\Gamma_{34}\delta_{\dot{A}}{}^{\dot{B}} \\ -\Gamma_{12}\delta^{B}{}_{A} & 0 & \Gamma_{11} & 0 \\ 0 & -\Gamma_{43}\delta_{\dot{B}}{}^{\dot{A}} & 0 & \Gamma_{44} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(2/\sqrt{-\epsilon_1\epsilon_2})R_1 & 0 & -\{(1-\epsilon_0\epsilon_1)+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}\delta^{A}{}_{B} & 0 \\ 0 & -(2/\sqrt{-\epsilon_1\epsilon_2})R_1 & 0 & -\{(1-\epsilon_0\epsilon_1)R_0-[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}\delta_{\dot{A}}{}^{\dot{B}} \\ -\{(1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}\delta^{B}{}_{A} & 0 & (2/\sqrt{-\epsilon_1\epsilon_2})R_1 & 0 \\ 0 & -\{(1-\epsilon_0\epsilon_1)R_0-[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}\delta_{\dot{B}}{}^{\dot{A}} & 0 & (2/\sqrt{-\epsilon_1\epsilon_2})R_1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(2/\sqrt{-\epsilon_1\epsilon_2})R_1 & -(1-\epsilon_0\epsilon_1)R_0 I_a{}^{b} - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\gamma_a^{(5)b} \\ -(1-\epsilon_0\epsilon_1)R_0 I_b{}^{a} - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\gamma_b^{(5)a} & (2/\sqrt{-\epsilon_1\epsilon_2})R_1 \end{bmatrix}, \quad (2\text{-}1\text{-}42)$$

and

$$\underline{\underline{\tilde{\Phi}}} = \underline{\underline{H}} \cdot \underline{\underline{\tilde{\Gamma}}} \cdot \underline{\underline{H}}^{-1}$$

$$= \frac{1}{4}\begin{bmatrix} -[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde{R}_1 & -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]\tilde{R}_2\gamma_a^{(5)b} \\ -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]\tilde{R}_2\gamma_b^{(5)a} & \{(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]\tilde{R}_1 \end{bmatrix}. \quad (2\text{-}1\text{-}43)$$

For example, $\underline{\underline{\Phi}}$ is calculated for time-like curve as

$$\underline{\underline{\Phi}} = \frac{1}{2}\begin{bmatrix} ik_1 & -k_0 I_a{}^{b} + ik_2\gamma_a^{(5)b} \\ -k_0 I_b{}^{a} + ik_2\gamma_b^{(5)a} & -ik_1 \end{bmatrix} \quad (2\text{-}1\text{-}44)$$

and for space-like curves with indicators $\epsilon_2 = -1$, $\epsilon_0 = \epsilon_1 = \epsilon_3 = 1$

as

$$\underline{\underline{\Phi}} = \frac{1}{2}\begin{bmatrix} -k_1 & ik_2\gamma_a^{(5)b} \\ ik_2\gamma_b^{(5)a} & k_1 \end{bmatrix} \quad (2\text{-}1\text{-}45)$$

We now define the spinor adjoint operation indicated by 'x' as

$$(\varphi_a)^x = (\varphi_a)^+ \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \; [\xi_A, \eta^{\dot{A}}]$$

$$= \chi^a \quad , \qquad (2\text{-}1\text{-}46)$$

$$(\chi_b)^x = (\chi_b)^+ \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \; [\eta_B, \xi^{\dot{B}}]$$

$$= \varphi^b \quad , \qquad (2\text{-}1\text{-}47)$$

$$(\breve{\varphi}_a)^x = \frac{1}{\sqrt{2}} \; [\breve{\xi}_A, \breve{\eta}^{\dot{A}}]$$

$$= \breve{\chi}^a \quad , \qquad (2\text{-}1\text{-}48)$$

and

$$(\breve{\chi}_b)^x = \frac{1}{\sqrt{2}} \; [\breve{\eta}_B, \breve{\xi}^{\dot{B}}]$$

$$= \varphi^b \quad . \qquad (2\text{-}1\text{-}49)$$

We also define the bispinor adjoint operation indicated by '+' as

$$\bar{\chi} = (\Psi)^+$$

$$= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_a \\ \chi_b \end{bmatrix}\right)^+$$

$$= \frac{1}{\sqrt{2}} \; [\varphi_a{}^x, \chi_b{}^x]$$

$$= \frac{1}{\sqrt{2}} \; [\chi^a, \varphi^b] \qquad (2\text{-}1\text{-}48)$$

and similarly

$$\breve{\chi} = (\breve{\psi})^{+}$$

$$= \frac{1}{\sqrt{2}} [\breve{\varphi}_a{}^{x}, \breve{\chi}_b{}^{x}]$$

$$= \frac{1}{\sqrt{2}} [\breve{\chi}^{a}, \breve{\varphi}^{b}] \quad . \qquad (2\text{-}1\text{-}49)$$

Bispinor adjoint operation can also be expressed as a four component operation as

$$\chi = (\psi)^{\oplus}$$

$$= \psi^{+} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \psi^{T} \cdot \begin{pmatrix} 0 & 0 & \varepsilon_{AB} & 0 \\ 0 & 0 & 0 & \varepsilon^{AB} \\ \varepsilon_{BA} & 0 & 0 & 0 \\ 0 & \varepsilon^{BA} & 0 & 0 \end{pmatrix} \qquad (2\text{-}1\text{-}50)$$

Spinor adjoint operation on two by two matrix and the bispinor adjoint operation on four by four matrix are expressed as

$$(\underline{\underline{A}})^{x} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{x}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \underline{\underline{A}}^{+} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} A_{11}^{*} & A_{21}^{*} \\ A_{12}^{*} & A_{22}^{*} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_{22}^{*} & A_{12}^{*} \\ A_{21}^{*} & A_{11}^{*} \end{bmatrix} \qquad (2\text{-}1\text{-}51)$$

and

$$(\underline{B})^{\oplus} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{\oplus}$$

$$= \begin{bmatrix} B_{11}^{x} & B_{21}^{x} \\ B_{12}^{x} & B_{22}^{x} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \underline{\underline{B}}^{+} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (2\text{-}1\text{-}52)$$

By taking bispinor adjoint, we can rewrite Eqs. (2-1-40) and (2-1-41) as

$$\frac{d}{dS} \dot{\chi} = \dot{\chi} \cdot \underline{\underline{\Phi}}^{\oplus} \qquad (2\text{-}1\text{-}53)$$

and

$$\frac{d}{ds}\tilde{X} = \tilde{X} \cdot \underline{\underline{\Phi}}^{\oplus} \quad , \tag{2-1-54}$$

where

$$\underline{\underline{\Phi}}^{\oplus} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \underline{\underline{\Phi}}^{+} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Gamma^*_{22} & 0 & -\Gamma^*_{43}\delta^A{}_B & 0 \\ 0 & \Gamma^*_{33} & 0 & -\Gamma^*_{12}\delta^{\dot{B}}{}_{\dot{A}} \\ -\Gamma^*_{21}\delta^{\dot{A}}{}_{\dot{B}} & 0 & \Gamma^*_{11} & 0 \\ 0 & -\Gamma^*_{34}\delta^B{}_A & 0 & \Gamma^*_{44} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Phi^{\times}_{11} & \Phi^{\times}_{21} \\ \Phi^{\times}_{12} & \Phi^{\times}_{22} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 & 0 & -\{(1-\epsilon_0\epsilon_1)R_0 - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*\delta^A{}_B & 0 \\ 0 & -(2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 & 0 & -\{(1-\epsilon_0\epsilon_1)R_0 + [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*\delta^{\dot{B}}{}_{\dot{A}} \\ -\{(1-\epsilon_0\epsilon_1)R_0 - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*\delta^B{}_A & 0 & (2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 & 0 \\ 0 & -\{(1-\epsilon_0\epsilon_1)R_0 + [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*\delta^{\dot{A}}{}_{\dot{B}} & 0 & (2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -(2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 & -\{(1-\epsilon_0\epsilon_1)R_0 I_a{}^{a} - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]^* R_2 \gamma^{(5)}_a{}^{a}\} \\ -\{(1-\epsilon_0\epsilon_1)R_0 I_a{}^{a} - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]^* R_2 \gamma^{(5)}_a{}^{a}\} & (2/\sqrt{-\epsilon_1\epsilon_2})^* R_1 \end{bmatrix} \tag{2-1-55}$$

and similarly

$$\underline{\underline{\tilde{\Phi}}}^{\oplus} = \frac{1}{4}\begin{bmatrix} -[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]^*\tilde{k}_1^* & -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]^*\tilde{k}_2^*\gamma_a^{(5)b} \\ -[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]^*\tilde{k}_2^*\gamma_b^{(5)a} & [(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]^*\tilde{k}_1^* \end{bmatrix}. \quad (2\text{-}1\text{-}56)$$

For example Φ^{+} is calculated for time-like curve as

$$\underline{\underline{\Phi}}^{\oplus} = \frac{1}{2}\begin{bmatrix} -ik_1 & -k_0 I_a{}^b + ik_2\gamma_a^{(5)b} \\ -k_0 I_b{}^a + ik^2\gamma_b^{(5)a} & ik_1 \end{bmatrix} \quad (2\text{-}1\text{-}57)$$

and for space-like curve s with indicators $\varepsilon_2 = -1$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1$ as

$$\underline{\underline{\Phi}}^{\oplus} = \frac{1}{2}\begin{bmatrix} -k_1 & -ik_2\gamma_a^{(5)b} \\ -ik_2\gamma_b^{(5)a} & k_1 \end{bmatrix} . \quad (2\text{-}1\text{-}58)$$

We are now to proceed to the Darboux bivector description of Eqs. (1-1-10) and (1-1-30). We first rewrite Eq. (1-1-10) as

$$\begin{aligned} \frac{d}{ds}\,e^{\mu} &= \underline{\underline{k}} \cdot e^{\mu} \\ &= \underline{\underline{e}}^{\nu} \cdot \mathbb{D}^{\mu}{}_{\nu} \end{aligned} \quad (2\text{-}1\text{-}59)$$

where

$$\underline{\underline{e}}^{\nu} = \begin{bmatrix} e^{\nu}_{(0)} & 0 & 0 & 0 \\ 0 & e^{\nu}_{(1)} & 0 & 0 \\ 0 & 0 & e^{\nu}_{(2)} & 0 \\ 0 & 0 & 0 & e^{\nu}_{(3)} \end{bmatrix} . \quad (2\text{-}1\text{-}60)$$

Then $\mathbb{D}^{\mu}{}_{\nu}$ is calculated as

$$D^{\mu}{}_{\nu} = \underline{\underline{\varepsilon}} \cdot \underline{\underline{e}}_{\nu} \cdot \underline{\underline{k}} \cdot \mathbb{e}^{\mu}$$

$$= \begin{bmatrix} k_0\varepsilon_0 e^{\mu}_{(1)} e_{\nu(0)} \\ -k_0\varepsilon_0 e^{\mu}_{(0)} e_{\nu(1)} + k_1\varepsilon_1 e^{\mu}_{(2)} e_{\nu(1)} \\ -k_1\varepsilon_1 e^{\mu}_{(1)} e_{\nu(2)} + k_2\varepsilon_2 e^{\mu}_{(3)} e_{\nu(2)} \\ -k_2\varepsilon_2 e^{\mu}_{(2)} e_{\nu(3)} \end{bmatrix} . \quad (2\text{-}1\text{-}61)$$

$\underline{\underline{e}}^{\nu} \cdot \mathbb{D}^{\mu}{}_{\nu}$ can equally well be expressed as $e^{\nu}_{ij}\,\mathbb{i}\mathbb{j}\cdot(D^{\mu}{}_{\nu})_k\mathbb{k}$. In the present case, because of the nature of $\mathbb{e}^{\mu}$, $\mathbb{i}$, $\mathbb{j}$ and $\mathbb{k}$ become redundant. Thus we rewrite Eq. (2-1-61) as

$$D^{\mu\nu} = -\varepsilon_0 k_0\, E^{\mu\nu}_{(01)} - \varepsilon_1\tau_1 E^{\mu\nu}_{(12)} - \varepsilon_2\tau_2 E^{\mu\nu}_{(23)} \quad . \qquad (2\text{-}1\text{-}62)$$

Similarly Eq. (1-1-30) is rewritten as

$$\frac{d}{ds}\overset{\frown}{\mathbb{e}} = \underline{\underline{\overset{\frown}{k}}} \cdot \overset{\frown}{\mathbb{e}}^{\mu}$$

$$= \overset{\frown}{e}^{\nu} \cdot \overset{\frown}{\mathbb{D}}^{\mu}{}_{\nu} \qquad (2\text{-}1\text{-}63)$$

where

$$\overset{\frown}{D}^{\mu\nu} = -\overset{\frown}{k}_0 \overset{\frown}{E}^{\mu\nu}_{(01)} - \overset{\frown}{k}_1 \overset{\frown}{E}^{\mu\nu}_{(12)} - \overset{\frown}{k}_2 \overset{\frown}{E}^{\mu\nu}_{(23)} \qquad (2\text{-}1\text{-}64)$$

and

$$\overset{\frown}{E}^{\mu\nu}_{(ij)} = \overset{\frown}{e}^{\mu}_{(i)}\, \overset{\frown}{e}^{\nu}_{(j)} - \overset{\frown}{e}^{\mu}_{(j)}\, \overset{\frown}{e}^{\nu}_{(i)} \quad . \qquad (2\text{-}1\text{-}65)$$

For example, $D^{\mu\nu}$ is calculated from Eq. (2-1-62) for time-like curve as

$$D^{\mu\nu} = k_0 E^{\mu\nu}_{(01)} - \tau_1 E^{\mu\nu}_{(12)} - \tau_2 E^{\mu\nu}_{(23)} \qquad (2\text{-}1\text{-}66)$$

and for space-like curves with indicators $\epsilon_2 = -1$, $\epsilon_0 = \epsilon_1 = \epsilon_3 = 1$ as

$$D^{\mu\nu} = -k_0 E^{\mu\nu}_{(01)} - \tau_1 E^{\mu\nu}_{(12)} + \tau_2 E^{\mu\nu}_{(23)} \ . \qquad (2\text{-}1\text{-}67)$$

We then rewrite Eqs. (2-1-13) and (2-1-14) as

$$\begin{aligned} \frac{d}{ds}\ell^{\mu} &= \underline{\underline{L}} \cdot \ell^{\mu} \\ &= \underline{\underline{\ell}}^{\nu} \cdot \triangle^{\mu}{}_{\nu} \end{aligned} \qquad (2\text{-}1\text{-}68)$$

and

$$\begin{aligned} \frac{d}{ds}\breve{\ell}^{\mu} &= \breve{\underline{\underline{L}}} \cdot \breve{\ell}^{\mu} \\ &= \breve{\underline{\underline{\ell}}}^{\nu} \cdot \breve{\triangle}^{\mu}{}_{\nu} \end{aligned} \qquad (2\text{-}1\text{-}69)$$

where

$$\underline{\underline{\ell}}^{\nu} = \begin{pmatrix} \ell^{\nu}_{1} & 0 & 0 & 0 \\ 0 & \bar{\ell}^{\nu}_{1} & 0 & 0 \\ 0 & 0 & \ell^{\nu}_{2} & 0 \\ 0 & 0 & 0 & \bar{\ell}^{\nu}_{2} \end{pmatrix} \qquad (2\text{-}1\text{-}70)$$

and

$$\underline{\underline{\ell}}^{\nu} = \begin{bmatrix} \ell_1^{\nu} & 0 & 0 & 0 \\ 0 & \ell_1^{\nu} & 0 & 0 \\ 0 & 0 & \ell_2^{\nu} & 0 \\ 0 & 0 & 0 & \ell_2^{\nu} \end{bmatrix} . \qquad (2\text{-}1\text{-}71)$$

Then $\Delta^{\mu}{}_{\nu}$ and $\tilde{\Delta}^{\mu}{}_{\nu}$ are calculated as

$$\Delta^{\mu}{}_{\nu} = \underline{\underline{\bar{\ell}}}^{\nu} \cdot \underline{\underline{L}} \cdot \ell^{\mu}$$

$$= \begin{bmatrix} \epsilon_0 \bar{\ell}_1^{\nu}(L_{11}\ell_1^{\mu} + L_{12}\bar{\ell}_1^{\mu} + L_{13}\ell_2^{\mu} + L_{14}\bar{\ell}_2^{\mu}) \\ \epsilon_0 \ell_1^{\nu}(L_{21}\ell_1^{\mu} + L_{22}\bar{\ell}_1^{\mu} + L_{23}\ell_2^{\mu} + L_{24}\bar{\ell}_2^{\mu}) \\ \epsilon_1 \bar{\ell}_2^{\nu}(L_{31}\ell_1^{\mu} + L_{32}\bar{\ell}_1^{\mu} + L_{33}\ell_2^{\mu} + L_{34}\bar{\ell}_2^{\mu}) \\ \epsilon_1 \ell_2^{\nu}(L_{41}\ell_1^{\mu} + L_{42}\bar{\ell}_1^{\mu} + L_{43}\ell_2^{\mu} + L_{44}\bar{\ell}_2^{\mu}) \end{bmatrix}$$

$$= \begin{bmatrix} \epsilon_0 \bar{\ell}_1^{\nu}\left\{\left[R_0 - \left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\epsilon_2\epsilon_3 R_2\right]\ell_2^{\mu} + \left[R_0 + \left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\epsilon_2\epsilon_3 R_2\right]\bar{\ell}_2^{\mu}\right\} \\ \epsilon_0 \ell_1^{\nu}\left\{\left[R_0 + \left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\epsilon_2\epsilon_3 R_2\right]\ell_2^{\mu} + \left[R_0 - \left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\epsilon_2\epsilon_3 R_2\right]\bar{\ell}_2^{\mu}\right\} \\ \epsilon_1 \bar{\ell}_2^{\nu}\left\{\left[-\epsilon_0\epsilon_1 R_0 + \left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)R_2\right]\ell_1^{\mu} + \left[-\epsilon_0\epsilon_1 R_0 - \left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)R_2\right]\bar{\ell}_1^{\mu} + \left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right) - \sqrt{-\epsilon_1\epsilon_2}\,\epsilon_1\epsilon_3\right]R_1\ell_2^{\mu} - \left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right) + \sqrt{-\epsilon_1\epsilon_2}\,\epsilon_1\epsilon_2\right]R_1\bar{\ell}_2^{\mu}\right\} \\ \epsilon_1 \ell_2^{\nu}\left\{\left[-\epsilon_0\epsilon_1 R_0 - \left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)R_2\right]\ell_1^{\mu} + \left[-\epsilon_0\epsilon_1 R_0 + \left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)R_2\right]\bar{\ell}_1^{\mu} + \left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right) - \sqrt{-\epsilon_1\epsilon_2}\,\epsilon_1\epsilon_3\right]R_1\ell_2^{\mu} - \left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right) - \sqrt{-\epsilon_1\epsilon_2}\,\epsilon_1\epsilon_2\right]R_1\bar{\ell}_2^{\mu}\right\} \end{bmatrix}$$

(2-1-72)

and

$$\tilde{\Delta}^{\mu}{}_{\nu} = \underline{\underline{\tilde{\ell}}}^{\nu} \cdot \underline{\underline{L}} \cdot \tilde{\ell}^{\mu}$$

$$= \begin{bmatrix} \bar{\tilde{\ell}}{}_1^{\nu}(\tilde{L}_{11}\tilde{\ell}_1^{\mu}+\tilde{L}_{12}\bar{\tilde{\ell}}{}_1^{\mu}+\tilde{L}_{13}\tilde{\ell}_2^{\mu}+\tilde{L}_{14}\bar{\tilde{\ell}}{}_2^{\mu}) \\ \tilde{\ell}_1^{\nu}(\tilde{L}_{21}\tilde{\ell}_1^{\mu}+\tilde{L}_{22}\bar{\tilde{\ell}}{}_1^{\mu}+\tilde{L}_{23}\tilde{\ell}_2^{\mu}+\tilde{L}_{24}\bar{\tilde{\ell}}{}_2^{\mu}) \\ \bar{\tilde{\ell}}{}_2^{\nu}(\tilde{L}_{31}\tilde{\ell}_1^{\mu}+\tilde{L}_{32}\bar{\tilde{\ell}}{}_1^{\mu}+\tilde{L}_{33}\tilde{\ell}_2^{\mu}+\tilde{L}_{34}\bar{\tilde{\ell}}{}_2^{\mu}) \\ \tilde{\ell}_2^{\nu}(\tilde{L}_{41}\tilde{\ell}_1^{\mu}+\tilde{L}_{42}\bar{\tilde{\ell}}{}_1^{\mu}+\tilde{L}_{43}\tilde{\ell}_2^{\mu}+\tilde{L}_{44}\bar{\tilde{\ell}}{}_2^{\mu}) \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\tilde{\ell}}{}_1^{\nu}\left\{\left[\tilde{R}_0-\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\tilde{R}_2\right]\tilde{\ell}_2^{\mu}+\left[\tilde{R}_0+\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\tilde{R}_2\right]\bar{\tilde{\ell}}{}_2^{\mu}\right\} \\ \tilde{\ell}_1^{\nu}\left\{\left[\tilde{R}_0+\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\tilde{R}_2\right]\tilde{\ell}_2^{\mu}+\left[\tilde{R}_0-\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)\tilde{R}_2\right]\bar{\tilde{\ell}}{}_2^{\mu}\right\} \\ \bar{\tilde{\ell}}{}_2^{\nu}\left\{\left[-\tilde{R}_0+\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\tilde{R}_2\right]\tilde{\ell}_1^{\mu}+\left[-\tilde{R}_0-\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\tilde{R}_2\right]\bar{\tilde{\ell}}{}_1^{\mu}+\left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right)-\sqrt{-\epsilon_1\epsilon_2}\right]\tilde{R}_1\tilde{\ell}_2^{\mu}\right\} \\ \tilde{\ell}_2^{\nu}\left\{\left[-\tilde{R}_0-\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\tilde{R}_2\right]\tilde{\ell}_1^{\mu}+\left[-\tilde{R}_0+\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\tilde{R}_2\right]\bar{\tilde{\ell}}{}_1^{\mu}-\left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right)-\sqrt{-\epsilon_1\epsilon_2}\right]\tilde{R}_1\bar{\tilde{\ell}}{}_2^{\mu}\right\} \end{bmatrix}$$

where

$$\underline{\underline{\bar{\ell}}}^{\nu} = \begin{bmatrix} \epsilon_0\bar{\ell}_1^{\mu} & 0 & 0 & 0 \\ 0 & \epsilon_0\ell_1^{\mu} & 0 & 0 \\ 0 & 0 & \epsilon_1\bar{\ell}_2^{\mu} & 0 \\ 0 & 0 & 0 & \epsilon_1\ell_2^{\mu} \end{bmatrix} \qquad (2\text{-}1\text{-}74)$$

and

$$\underline{\underline{\tilde{\ell}}}^{\nu} = \begin{bmatrix} \bar{\tilde{\ell}}{}_1^{\nu} & 0 & 0 & 0 \\ 0 & \tilde{\ell}_1^{\nu} & 0 & 0 \\ 0 & 0 & \bar{\tilde{\ell}}{}_2^{\nu} & 0 \\ 0 & 0 & 0 & \tilde{\ell}_2^{\nu} \end{bmatrix} \qquad (2\text{-}1\text{-}75)$$

In a similar manner to $D^{\mu\nu}$, $\Delta^{\mu\nu}$ and $\tilde{\Delta}^{\mu\nu}$ can be re-expressed as

$$\Delta^{\mu\nu}=\epsilon_0(\bar{l}_1^\nu l_2^\mu+l_1^\nu \bar{l}_2^\mu)\left[R_0-(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 R_2\right]+\epsilon_0(\bar{l}_1^\nu \bar{l}_2^\mu+l_1^\nu l_2^\mu)\left[-R_0+(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\epsilon_2\epsilon_3 R_2\right]$$
$$+\epsilon_1(\bar{l}_2^\nu l_1^\mu+l_2^\nu \bar{l}_1^\mu)\left[-\epsilon_0\epsilon_1 R_0+(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})R_2\right]+\epsilon_1(\bar{l}_2^\nu \bar{l}_1^\mu+l_2^\nu l_1^\mu)\left[-\epsilon_0\epsilon_1 R_0-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})R_2\right]$$
$$+\epsilon_1(\bar{l}_2^\nu l_2^\mu-l_2^\nu \bar{l}_2^\mu)(2/\sqrt{-\epsilon_1\epsilon_2})R_1 \quad . \qquad (2\text{-}1\text{-}76)$$

$$\tilde{\Delta}^{\mu\nu}=(\tilde{\bar{l}}_1^\nu \tilde{l}_2^\mu+\tilde{l}_1^\nu \tilde{\bar{l}}_2^\mu)\left[\tilde{R}_0-(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{R}_2\right]+(\tilde{\bar{l}}_1^\nu \tilde{\bar{l}}_2^\mu+\tilde{l}_1^\nu \tilde{l}_2^\mu)\left[-\tilde{R}_0+(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{R}_2\right]$$
$$+(\tilde{\bar{l}}_2^\nu \tilde{l}_1^\mu+\tilde{l}_2^\nu \tilde{\bar{l}}_1^\mu)\left[-\tilde{R}_0+(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\tilde{R}_2\right]+(\tilde{\bar{l}}_2^\nu \tilde{\bar{l}}_1^\mu+\tilde{l}_2^\nu \tilde{l}_1^\mu)\left[-\tilde{R}_0-(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})\tilde{R}_2\right]$$
$$+(\bar{l}_2^\nu l_2^\mu-l_2^\nu \bar{l}_2^\mu)\left[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}\right]\tilde{R}_1 \quad . \qquad (2\text{-}1\text{-}77)$$

For example $\Delta^{\mu\nu}$ can be calculated for time-like curve as

$$\Delta^{\mu\nu}=-(n^\nu m^\mu+l^\nu \bar{m}^\mu)(R_0+iR_2)-(n^\nu \bar{m}^\mu+l^\nu m^\mu)(R_0-iR_2)$$
$$+(\bar{m}^\nu l^\mu+m^\nu n^\mu)(R_0+iR_2)+(\bar{m}^\nu n^\mu+m^\nu l^\mu)(R_0-iR_2)$$
$$+(\bar{m}^\nu m^\mu-m^\nu \bar{m}^\mu)(-i2R_1) \qquad (2\text{-}1\text{-}78)$$

and for space-like curves with indicators $\epsilon_2 \doteq -1$, $\epsilon_0 = \epsilon_1 = \epsilon_3 = 1$ as

$$\Delta^{\mu\nu}=(\bar{l}_1^\nu l_2^\mu+l_1^\nu \bar{l}_2^\mu)(R_0+iR_2)+(\bar{l}_1^\nu \bar{l}_2^\mu+l_1^\nu l_2^\mu)(R_0-iR_2)$$
$$+(\bar{l}_2^\nu l_1^\mu+l_2^\nu \bar{l}_1^\mu)(R_0+iR_2)+(\bar{l}_2^\nu \bar{l}_1^\mu+l_2^\nu l_1^\mu)(R_0-iR_2)$$
$$+(\bar{l}_2^\nu l_2^\mu-l_2^\nu \bar{l}_2^\mu)(2R_1) \quad . \qquad (2\text{-}1\text{-}79)$$

We now proceed to the rewriting of Eqs. (2-1-24) and (2-1-25) as

$$\frac{d}{ds}\,\mathfrak{x}^A = \underline{\underline{\Gamma}} \cdot \mathfrak{x}^A$$

$$= \underline{\underline{\xi}}^C \cdot \mathbb{D}^A{}_C \qquad (2\text{-}1\text{-}80)$$

and

$$\frac{d}{ds}\,\breve{\mathfrak{x}}^A = \underline{\underline{\breve{\xi}}}^C \cdot \breve{\mathbb{D}}^A{}_C \;, \qquad (2\text{-}1\text{-}81)$$

where

$$\mathfrak{x}^A = \mathfrak{x}$$

$$= \frac{1}{2}\begin{bmatrix} \xi^A \\ \eta^A \\ \xi^{\dot{B}} \\ \eta^{\dot{B}} \end{bmatrix} \qquad (2\text{-}1\text{-}82)$$

$$\underline{\underline{\xi}}^C = \frac{1}{2}\begin{bmatrix} \xi^C & 0 & 0 & 0 \\ 0 & \eta^C & 0 & 0 \\ 0 & 0 & \xi^{\dot{D}} & 0 \\ 0 & 0 & 0 & \eta^{\dot{D}} \end{bmatrix} \;, \qquad (2\text{-}1\text{-}83)$$

$$\breve{\mathfrak{x}}^A = \breve{\mathfrak{x}}$$

$$= \frac{1}{2}\begin{bmatrix} \breve{\xi}^A \\ \breve{\eta}^A \\ \breve{\xi}^{\dot{B}} \\ \breve{\eta}^{\dot{B}} \end{bmatrix} \qquad (2\text{-}1\text{-}84)$$

and

$$\underline{\underline{\tilde{\xi}^{C}}} = \frac{1}{2}\begin{bmatrix} \tilde{\xi}^{C} & 0 & 0 & 0 \\ 0 & \tilde{\eta}^{C} & 0 & 0 \\ 0 & 0 & \tilde{\xi}^{\dot{D}} & 0 \\ 0 & 0 & 0 & \tilde{\eta}^{\dot{D}} \end{bmatrix} . \qquad (2\text{-}1\text{-}85)$$

Then noting $\Gamma_{11} = \Gamma_{44}$, $\Gamma_{22} = \Gamma_{33}$, $\Gamma_{11} = -\Gamma_{22}$, $\Gamma_{12} = \Gamma_{21}$ and $\Gamma_{34} = \Gamma_{43}$, $\mathbb{D}^{A}{}_{C}$ is calculated as

$$\begin{aligned} \mathbb{D}^{A}{}_{C} &= \underline{\underline{\eta_{C}}} \cdot \underline{\underline{\Gamma}} \cdot \xi^{A} \\ &= \frac{1}{4}\begin{bmatrix} -(\Gamma_{11}\xi^{A} + \Gamma_{12}\eta^{A})\ \eta_{C} \\ (\Gamma_{12}\xi^{A} - \Gamma_{11}\eta^{A})\ \xi_{C} \\ (\Gamma_{11}\xi^{B} + \Gamma_{12}\eta^{B})\ \eta_{D} \\ -(\Gamma_{12}\xi^{B} - \Gamma_{11}\eta^{B})\ \xi_{D} \end{bmatrix} , \end{aligned} \qquad (2\text{-}1\text{-}86)$$

where

$$\underline{\underline{\eta_{C}}} = \frac{1}{2}\begin{bmatrix} -\eta_{C} & 0 & 0 & 0 \\ 0 & \xi_{C} & 0 & 0 \\ 0 & 0 & -\eta_{\dot{D}} & 0 \\ 0 & 0 & 0 & \xi_{\dot{D}} \end{bmatrix} . \qquad (2\text{-}1\text{-}87)$$

Likewise, noting $\tilde{\Gamma}_{11} = \tilde{\Gamma}_{44}$, $\tilde{\Gamma}_{22} = \tilde{\Gamma}_{33}$, $\tilde{\Gamma}_{11} = -\tilde{\Gamma}_{22}$, $\tilde{\Gamma}_{12} = \tilde{\Gamma}_{21}$ and

$\overset{\rightharpoonup}{\Gamma}_{34} = \overset{\rightharpoonup}{\Gamma}_{43} = -\overset{\rightharpoonup}{\Gamma}_{12} = -\overset{\rightharpoonup}{\Gamma}_{21}$, $\mathbb{D}^{A}{}_{C}$ is calculated as

$$\overset{\rightharpoonup}{\mathbb{D}}^{A}{}_{C} = \underline{\underline{\overset{\rightharpoonup}{\eta}_{C}}} \cdot \underline{\underline{\overset{\rightharpoonup}{\Gamma}}} \cdot \overset{\rightharpoonup}{\xi}^{A}$$

$$= \frac{1}{4} \begin{pmatrix} -(\overset{\rightharpoonup}{\Gamma}_{11}\overset{\rightharpoonup}{\xi}^{A} + \overset{\rightharpoonup}{\Gamma}_{12}\overset{\rightharpoonup}{\eta}^{A})\,\overset{\rightharpoonup}{\eta}_{C} \\ (\overset{\rightharpoonup}{\Gamma}_{12}\overset{\rightharpoonup}{\xi}^{A} - \overset{\rightharpoonup}{\Gamma}_{11}\overset{\rightharpoonup}{\eta}^{A})\,\overset{\rightharpoonup}{\xi}_{C} \\ (\overset{\rightharpoonup}{\Gamma}_{11}\overset{\rightharpoonup}{\xi}^{\dot{B}} + \overset{\rightharpoonup}{\Gamma}_{12}\overset{\rightharpoonup}{\eta}^{\dot{B}})\,\overset{\rightharpoonup}{\eta}_{\dot{D}} \\ -(\overset{\rightharpoonup}{\Gamma}_{12}\overset{\rightharpoonup}{\xi}^{\dot{B}} - \overset{\rightharpoonup}{\Gamma}_{11}\overset{\rightharpoonup}{\eta}^{\dot{B}})\,\overset{\rightharpoonup}{\xi}_{\dot{D}} \end{pmatrix}, \qquad (2\text{-}1\text{-}88)$$

where

$$\underline{\underline{\overset{\rightharpoonup}{\eta}_{C}}} = \frac{1}{2} \begin{pmatrix} -\overset{\rightharpoonup}{\eta}_{C} & 0 & 0 & 0 \\ 0 & \overset{\rightharpoonup}{\xi}_{C} & 0 & 0 \\ 0 & 0 & -\overset{\rightharpoonup}{\eta}_{D} & 0 \\ 0 & 0 & 0 & \overset{\rightharpoonup}{\xi}_{D} \end{pmatrix}. \qquad (2\text{-}1\text{-}89)$$

Similar consideration as given to $\mathbb{D}^{\mu\nu}$ in Eqs. (2-1-62) and (2-1-64) reduces Eqs. (2-1-86) and (2-1-88) to

$$\mathbb{D}^{A}{}_{C} = \frac{1}{4}\Big[-\Gamma_{11}(\xi^{A}\eta_{C} + \eta^{A}\xi_{C} - \xi^{\dot{B}}\eta_{\dot{D}} - \eta^{\dot{B}}\xi_{\dot{D}}) - \Gamma_{12}(\eta^{A}\eta_{C} - \xi^{A}\xi_{C}) - \Gamma_{34}(\eta^{\dot{B}}\eta_{\dot{D}} - \xi^{\dot{B}}\xi_{\dot{D}})\Big]$$

$$= \frac{1}{4}\Big(-(2/\sqrt{-\epsilon_1\epsilon_2})R_1(\xi^{A}\eta_{C} + \eta^{A}\xi_{C} - \xi^{\dot{B}}\eta_{\dot{D}} - \eta^{\dot{B}}\xi_{\dot{D}}) - \Big\{(1-\epsilon_0\epsilon_2)R_0 + \Big[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})\Big]R_2\Big\}(\eta^{A}\eta_{C} - \xi^{A}\xi_{C})$$

$$- \Big\{(1-\epsilon_0\epsilon_1)R_0 - \Big[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})\Big]R_2\Big\}(\eta^{\dot{B}}\eta_{\dot{D}} - \xi^{\dot{B}}\xi_{\dot{D}})\Big) \qquad (2\text{-}1\text{-}90)$$

and

$$\tilde{\mathbb{D}}^{A}{}_{C}=\frac{1}{4}\Big\{-\big(2/\sqrt{-\epsilon_1\epsilon_2}\big)\tilde{k}_1\big(\tilde{\xi}^{A}\tilde{\eta}_{C}+\tilde{\eta}^{A}\tilde{\xi}_{C}-\tilde{\xi}^{\dot{B}}\tilde{\eta}_{\dot{D}}-\tilde{\eta}^{\dot{B}}\tilde{\xi}_{\dot{D}}\big)-\Big[\big(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\big)-\big(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\big)\Big]\tilde{k}_2\big(\tilde{\eta}^{A}\tilde{\eta}_{C}-\tilde{\xi}^{A}\tilde{\xi}_{C}-\tilde{\eta}^{\dot{B}}\tilde{\eta}_{\dot{D}}+\tilde{\xi}^{\dot{B}}\tilde{\xi}_{\dot{D}}\big)\Big] \,. \qquad (2\text{-}1\text{-}91)$$

For example $\mathbb{D}^{A}{}_{C}$ is calculated for time-like curve as

$$\mathbb{D}^{A}{}_{C}=\frac{1}{2}\Big[ik_1\big(\xi^{A}\eta_{C}+\eta^{A}\xi_{C}-\xi^{\dot{B}}\eta_{\dot{D}}-\eta^{\dot{B}}\xi_{\dot{D}}\big)-\big(k_1-ik_2\big)\big(\eta^{A}\eta_{C}-\xi^{A}\xi_{C}\big)-\big(k_1+ik_2\big)\big(\eta^{\dot{B}}\eta_{\dot{D}}-\xi^{\dot{B}}\xi_{\dot{D}}\big)\Big] \,, \qquad (2\text{-}1\text{-}92)$$

and for space-like curves with indicators $\epsilon_2 = -1$, $\epsilon_0 = \epsilon_1 = \epsilon_3 = 1$

as

$$\mathbb{D}^{A}{}_{C}=\frac{1}{2}\Big[k_1\big(\xi^{A}\eta_{C}+\eta^{A}\xi_{C}-\xi^{\dot{B}}\eta_{\dot{D}}-\eta^{\dot{B}}\xi_{\dot{D}}\big)+ik_2\big(\eta^{A}\eta_{C}-\xi^{A}\xi_{C}\big)-ik_2\big(\eta^{\dot{B}}\eta_{\dot{D}}-\cancel{\xi^{\dot{B}}\xi_{\dot{D}}}\big)\Big] \,. \qquad (2\text{-}1\text{-}93)$$

We then rewrite Eqs. (2-1-40) and (2-1-41) as

$$\frac{d}{ds}\Psi^{A} = \underline{\underline{\Phi}}\cdot\Psi^{A}$$

$$= \underline{\underline{\varphi}}^{C}\cdot\mathbb{Q}^{A}{}_{C} \,, \qquad (2\text{-}1\text{-}94)$$

and

$$\frac{d}{ds}\tilde{\Psi}^{A} = \underline{\underline{\tilde{\Phi}}}\cdot\tilde{\Psi}^{A}$$

$$= \underline{\underline{\tilde{\varphi}}}^{C}\cdot\tilde{\mathbb{Q}}^{A}{}_{C} \,, \qquad (2\text{-}1\text{-}95)$$

where

$$\Psi^A = \Psi$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_a \\ \chi_b \end{bmatrix}$$

$$= \frac{1}{2} \cdot \begin{bmatrix} \eta^A \\ \xi_{\dot{A}} \\ \xi^B \\ \eta_{\dot{B}} \end{bmatrix} , \tag{2-1-96}$$

$$\tilde{\Psi}^A = \tilde{\Psi}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\varphi}_a \\ \tilde{\chi}_b \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \tilde{\eta}^A \\ \tilde{\xi}_{\dot{A}} \\ \tilde{\xi}^B \\ \tilde{\eta}_{\dot{B}} \end{bmatrix} , \tag{2-1-97}$$

$$\underline{\underline{\varphi}}^c = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_c & 0 \\ 0 & \chi_d \end{bmatrix} \tag{2-1-98}$$

and

$$\underline{\underline{\tilde{\varphi}}}^c = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{\varphi}_c & 0 \\ 0 & \tilde{\chi}_d \end{bmatrix} . \tag{2-1-99}$$

Then $\mathcal{Q}^{A}{}_{C}$ and $\overleftarrow{\mathcal{Q}}^{A}{}_{C}$ are calculated as

$$
\begin{aligned}
\mathcal{Q}^{A}{}_{c} &= \underline{\underline{\chi}}_c \cdot \underline{\underline{\Phi}} \cdot \psi^{A} \\
&= \underline{\underline{\chi}}_c \cdot \underline{\underline{\Theta}} \cdot \underline{\underline{\Gamma}} \cdot \underline{\underline{\Theta}}^{-1} \cdot \psi^{A} \\
&= \underline{\underline{\chi}}_c \cdot \underline{\underline{\Theta}} \cdot \underline{\underline{\Sigma}} \cdot \underline{\underline{L}} \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{\underline{\Theta}}^{-1} \cdot \psi^{A} \\
&= \underline{\underline{\chi}}_c \cdot \underline{\underline{\Theta}} \cdot \underline{\underline{\Sigma}} \cdot \underline{\underline{T}} \cdot \underline{\underline{R}} \cdot \underline{\underline{T}}^{-1} \cdot \underline{\underline{\Sigma}}^{-1} \cdot \underline{\underline{\Theta}}^{-1} \cdot \psi^{A} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} \chi^{c} & 0 \\ 0 & -\psi^{d} \end{bmatrix} \cdot \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_a \\ \chi_b \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \Phi_{11} \psi_a \chi^{c} + \Phi_{12} \chi_b \chi^{c} \\ -\Phi_{21} \psi_a \psi^{d} - \Phi_{22} \chi_b \psi^{d} \end{bmatrix} \\
&= \frac{1}{8} \begin{bmatrix} -(2/\sqrt{-\epsilon_1\epsilon_2}) R_1 \psi_a \chi^{c} - \{(1-\epsilon_0\epsilon_1) R_0 I_a{}^{b} + [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})] R_2 \gamma^{(5)}{}_a{}^{b}\} \chi_b \chi^{c} \\ \{(1-\epsilon_0\epsilon_1) R_0 I_b{}^{a} + [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})] R_2 \gamma^{(5)}{}_b{}^{a}\} \psi_a \psi^{d} - (2/\sqrt{-\epsilon_1\epsilon_2}) R_1 \chi_b \psi^{d} \end{bmatrix}
\end{aligned}
\qquad (2\text{-}1\text{-}100)
$$

and

$$
\begin{aligned}
\overleftarrow{\mathcal{Q}}^{A}{}_{c} &= \overleftarrow{\underline{\underline{\chi}}}_c \cdot \overleftarrow{\underline{\underline{\Phi}}} \cdot \overleftarrow{\psi}^{A} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} \overleftarrow{\chi}_c & 0 \\ 0 & -\overleftarrow{\psi}^{d} \end{bmatrix} \cdot \begin{bmatrix} \overleftarrow{\Phi}_{11} & \overleftarrow{\Phi}_{12} \\ \overleftarrow{\Phi}_{21} & \overleftarrow{\Phi}_{22} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} \overleftarrow{\psi}_a \\ \overleftarrow{\chi}_b \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \overleftarrow{\Phi}_{11} \overleftarrow{\psi}_a \overleftarrow{\chi}^{c} + \overleftarrow{\Phi}_{12} \overleftarrow{\chi}_b \overleftarrow{\chi}^{c} \\ -\overleftarrow{\Phi}_{12} \overleftarrow{\psi}_a \overleftarrow{\psi}^{d} - \overleftarrow{\Phi}_{22} \overleftarrow{\chi}_b \overleftarrow{\psi}^{d} \end{bmatrix} \\
&= \frac{1}{8} \begin{bmatrix} -[(1/\sqrt{-\epsilon_1\epsilon_2}) - \sqrt{-\epsilon_1\epsilon_2}] \overleftarrow{R}_1 \overleftarrow{\psi}_a \overleftarrow{\chi}^{c} - [(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}) - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})] \overleftarrow{R}_2 \gamma^{(5)}{}_a{}^{b} \overleftarrow{\chi}_b \overleftarrow{\chi}^{c} \\ [(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}) - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})] \overleftarrow{R}_2 \gamma^{(5)}{}_b{}^{a} \overleftarrow{\psi}_a \overleftarrow{\psi}^{d} - [(1/\sqrt{-\epsilon_1\epsilon_2}) - \sqrt{-\epsilon_1\epsilon_2}] \overleftarrow{R}_1 \overleftarrow{\chi}_b \overleftarrow{\psi}^{d} \end{bmatrix}.
\end{aligned}
\qquad (2\text{-}1\text{-}101)
$$

Similar to Eqs. (2-1-90) and (2-1-91), from Eqs. (2-1-100) and (2-1-101), we obtain

$$\mathbb{Q}^{A}{}_{c}=\frac{1}{8}\left(-\left(2/\sqrt{-\epsilon_1\epsilon_2}\right)\mathcal{R}_1\left(\psi_a\chi^c+\chi_b\psi^d\right)-\left\{(1-\epsilon_0\epsilon_1)\mathcal{R}_0 I_a{}^{e}+\left[\epsilon_2(\epsilon_1-\epsilon_0)/\left(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2}\right)\right]\mathcal{R}_2\gamma_a^{(5)e}\right\}\left(\chi_b\chi^c-\psi_a\psi^d\right)\right) \tag{2-1-102}$$

and

$$\breve{\mathbb{Q}}^{A}{}_{c}=\frac{1}{8}\left\{-\left[\left(1/\sqrt{-\epsilon_1\epsilon_2}\right)-\sqrt{-\epsilon_1\epsilon_2}\right]\breve{\mathcal{R}}_1\left(\breve{\psi}_a\breve{\chi}^c+\breve{\chi}_b\breve{\psi}^d\right)-\left[\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)-\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\right]\breve{\mathcal{R}}_2\gamma_a^{(5)e}\left(\breve{\chi}_b\breve{\chi}^c-\breve{\psi}_a\breve{\psi}^d\right)\right\}. \tag{2-1-103}$$

For example, $\mathbb{Q}^{A}{}_{C}$ is calculated for time-like curves as

$$\mathbb{Q}^{A}{}_{c}=\frac{1}{4}\left[i\mathcal{R}_1\left(\psi_a\chi^c+\chi_b\psi^d\right)-\left(\mathcal{R}_0 I_a{}^{e}-i\mathcal{R}_2\gamma_a^{(5)e}\right)\left(\chi_b\chi^c-\psi_a\psi^d\right)\right] \tag{2-1-104}$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$ as

$$\mathbb{Q}^{A}{}_{c}=\frac{1}{4}\left[-\mathcal{R}_1\left(\psi_a\chi^c+\chi_b\psi^d\right)-i\mathcal{R}_2\gamma_a^{(5)e}\left(\chi_b\chi^c-\psi_a\psi^d\right)\right]. \tag{2-1-105}$$

Just as we rewrite Eqs. (2-1-40) and (2-1-41) as in Eqs. (2-1-94) and (2-1-95), we now rewrite Eqs. (2-1-53) and (2-1-54) as

$$\begin{aligned}\frac{d}{ds}\chi_A &= \chi_A\cdot\left(\underline{\underline{\Phi}}\right)^{\oplus}\\ &= \mathbb{Q}_A{}^{C}\cdot\underline{\underline{\chi}}_C\end{aligned} \tag{2-1-106}$$

where

$$\underline{\underline{\Phi}}^{\oplus}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\cdot\underline{\underline{\Phi}}^{+}\cdot\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$$=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\cdot\begin{bmatrix}\Gamma^*_{22}&0&-\Gamma^*_{12}\delta^{\dot B}{}_{\dot A}&0\\0&\Gamma^*_{33}&0&-\Gamma^*_{43}\delta^{A}{}_{B}\\-\Gamma^*_{12}\delta^{\dot A}{}_{\dot B}&0&\Gamma^*_{11}&0\\0&-\Gamma^*_{34}\delta^{B}{}_{A}&0&\Gamma^*_{44}\end{bmatrix}\cdot\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$$=\begin{bmatrix}\Gamma^*_{33}&0&-\Gamma^*_{43}\delta^{A}{}_{B}&0\\0&\Gamma^*_{22}&0&-\Gamma^*_{12}\delta^{\dot B}{}_{\dot A}\\-\Gamma^*_{34}\delta^{B}{}_{A}&0&\Gamma^*_{44}&0\\0&-\Gamma^*_{21}\delta^{\dot A}{}_{\dot B}&0&\Gamma^*_{11}\end{bmatrix}, \qquad (2\text{-}1\text{-}107)$$

and

$$d\mathbb{Q}_A{}^C=(d\mathbb{Q}_C{}^A)^{\oplus}$$
$$=(\psi^A)^{\oplus}\cdot(\underline{\underline{\Phi}})^{\oplus}\cdot(\underline{\underline{\chi}}_C)^{\oplus}$$
$$=\chi_A\cdot\underline{\underline{\Phi}}^{\oplus}\cdot\underline{\underline{\psi}}^C$$

$$=\tfrac{1}{2}\left[\xi_A,\eta^{\dot A},\eta_B,\xi^{\dot B}\right]\cdot\begin{bmatrix}\Gamma^*_{33}&0&-\Gamma^*_{43}\delta^{A}{}_{B}&0\\0&\Gamma^*_{22}&0&-\Gamma^*_{12}\delta^{\dot B}{}_{\dot A}\\-\Gamma^*_{34}\delta^{B}{}_{A}&0&\Gamma^*_{44}&0\\0&-\Gamma^*_{21}\delta^{\dot A}{}_{\dot B}&0&\Gamma^*_{11}\end{bmatrix}\cdot\tfrac{1}{2}\begin{bmatrix}\eta^C&0&0&0\\0&\xi_C&0&0\\0&0&\xi^D&0\\0&0&0&\eta_{\dot D}\end{bmatrix}$$

$$=\tfrac{1}{4}\left[\Gamma^*_{33}\xi_A\eta^C-\Gamma^*_{34}\delta^{B}{}_{A}\eta_B\eta^C,\ \Gamma^*_{22}\eta^{\dot A}\xi_C-\Gamma^*_{21}\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_C,\ -\Gamma^*_{43}\delta^{A}{}_{B}\xi_A\xi^D+\Gamma^*_{44}\eta_B\xi^D,\ -\Gamma^*_{12}\delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D}+\Gamma^*_{11}\xi^{\dot B}\eta_{\dot D}\right]$$

$$=\tfrac{1}{4}\left[-\Gamma^*_{11}\xi_A\eta^C-\Gamma^*_{34}\delta^{B}{}_{A}\eta_B\eta^C,\ -\Gamma^*_{11}\eta^{\dot A}\xi_C-\Gamma^*_{12}\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_C,\ -\Gamma^*_{34}\delta^{A}{}_{B}\xi_A\xi^D+\Gamma^*_{11}\eta_B\xi^D,\ -\Gamma^*_{12}\delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D}+\Gamma^*_{11}\xi^{\dot B}\eta_{\dot D}\right]$$

(2-1-108)

and

$$\frac{d}{ds}\vec{\chi}_A=d\vec{\mathbb{Q}}_A{}^C\cdot\vec{\underline{\underline{\chi}}}_C \qquad , \qquad (2\text{-}1\text{-}109)$$

where

$$\overleftarrow{\underline{\underline{\Phi}}}^{\oplus}=\begin{bmatrix} \overleftarrow{\Gamma}^*_{33} & 0 & -\overleftarrow{\Gamma}^*_{43}\delta^A{}_B & 0 \\ 0 & \overleftarrow{\Gamma}^*_{22} & 0 & -\overleftarrow{\Gamma}^*_{12}\delta^{\dot B}{}_{\dot A} \\ -\overleftarrow{\Gamma}^*_{34}\delta^B{}_A & 0 & \overleftarrow{\Gamma}^*_{44} & 0 \\ 0 & -\overleftarrow{\Gamma}^*_{21}\delta^{\dot A}{}_{\dot B} & 0 & \overleftarrow{\Gamma}^*_{11} \end{bmatrix} \tag{2-1-110}$$

and

$$d\overleftarrow{Q}_A{}^C=\frac{1}{4}\Big[-\overleftarrow{\Gamma}^*_{11}\overleftarrow{\xi}_A\overleftarrow{\eta}^C-\overleftarrow{\Gamma}^*_{34}\delta^B{}_A\overleftarrow{\eta}_B\overleftarrow{\eta}^C,-\overleftarrow{\Gamma}^*_{11}\overleftarrow{\eta}^{\dot A}\overleftarrow{\xi}_{\dot C}-\overleftarrow{\Gamma}^*_{12}\delta^{\dot A}{}_{\dot B}\overleftarrow{\xi}^{\dot B}\overleftarrow{\xi}_{\dot C},\ \overleftarrow{\Gamma}^*_{34}\delta^A{}_B\overleftarrow{\xi}_A\overleftarrow{\xi}^D+\overleftarrow{\Gamma}^*_{11}\overleftarrow{\eta}_B\overleftarrow{\xi}^D,\ \overleftarrow{\Gamma}^*_{12}\delta^{\dot B}{}_{\dot A}\overleftarrow{\eta}^{\dot A}\overleftarrow{\eta}_{\dot D}+\overleftarrow{\Gamma}^*_{11}\overleftarrow{\xi}^{\dot B}\overleftarrow{\eta}_{\dot D}\Big], \tag{2-1-111}$$

respectively. In a similar way as Eq. (2-1-60), we re-express Eqs. (2-1-108) and (2-1-111) as

$$\begin{aligned} dQ_A{}^C&=\frac{1}{4}\Big[-\Gamma^*_{11}(\xi_A\eta^C+\eta^{\dot A}\xi_{\dot C}-\eta_B\xi^D-\xi^{\dot B}\eta_{\dot D})-\Gamma^*_{34}(\delta^B{}_A\eta_B\eta^C+\delta^A{}_B\xi_A\xi^D)-\Gamma^*_{12}(\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_{\dot C}+\delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D})\Big]\\ &=\frac{1}{4}\Big[-(2/\sqrt{-\epsilon_1\epsilon_2})^*R_1(\xi_A\eta^C+\eta^{\dot A}\xi_{\dot C}-\eta_B\xi^D-\xi^{\dot B}\eta_{\dot D})-\{(1-\epsilon_0\epsilon_1)R_0-[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*(\delta^B{}_A\eta_B\eta^C+\delta^A{}_B\xi_A\xi^D)\\ &\quad-\{(1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2\}^*(\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_{\dot C}+\delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D})\Big] \end{aligned} \tag{2-1-112}$$

and

$$\begin{aligned} d\overleftarrow{Q}_A{}^C&=\frac{1}{4}\Big[-\overleftarrow{\Gamma}^*_{11}(\overleftarrow{\xi}_A\overleftarrow{\eta}^C+\overleftarrow{\eta}^{\dot A}\overleftarrow{\xi}_{\dot C}+\overleftarrow{\eta}_{\dot B}\overleftarrow{\xi}^{\dot D}+\overleftarrow{\xi}^{\dot B}\overleftarrow{\eta}_{\dot D})-\overleftarrow{\Gamma}^*_{34}(\delta^B{}_A\overleftarrow{\eta}_B\overleftarrow{\eta}^C-\delta^A{}_B\overleftarrow{\xi}_A\overleftarrow{\xi}^D)-\overleftarrow{\Gamma}^*_{12}(\delta^{\dot A}{}_{\dot B}\overleftarrow{\xi}^{\dot B}\overleftarrow{\xi}_{\dot C}-\delta^{\dot B}{}_{\dot A}\overleftarrow{\eta}^{\dot A}\overleftarrow{\eta}_{\dot D})\Big]\\ &=\frac{1}{4}\Big[-[(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}]^*\overleftarrow{R}^*_1(\overleftarrow{\xi}_A\overleftarrow{\eta}^C+\overleftarrow{\eta}^{\dot A}\overleftarrow{\xi}_{\dot C}+\overleftarrow{\eta}_{\dot B}\overleftarrow{\xi}^{\dot D}+\overleftarrow{\xi}^{\dot B}\overleftarrow{\eta}_{\dot D})\\ &\quad-[(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})]^*(\delta^B{}_A\overleftarrow{\eta}_B\overleftarrow{\eta}_C+\delta^A{}_B\overleftarrow{\xi}_A\overleftarrow{\xi}^D-\delta^{\dot A}{}_{\dot B}\overleftarrow{\xi}^{\dot B}\overleftarrow{\xi}_{\dot C}-\delta^{\dot B}{}_{\dot A}\overleftarrow{\eta}^{\dot A}\overleftarrow{\eta}_{\dot D})\Big]. \end{aligned} \tag{2-1-113}$$

For example, for time-like curve, $\mathcal{D}_A{}^C$ is calculated as

$$\mathcal{D}_A{}^C = \frac{1}{2}\Big[-i\kappa_1(\xi_A\eta^C + \eta^{\dot A}\xi_{\dot C} - \xi_B\xi^D - \xi^{\dot B}\eta_{\dot D}) - (\kappa_0 - i\kappa_2)(\delta^B{}_A\eta_B\eta^C + \delta^A{}_B\xi_A\xi^D) - (\kappa_0 + i\kappa_2)(\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_{\dot C} + \delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D})\Big] \tag{2-1-114}$$

and for space-like curves with indicators $\varepsilon_2 = -1$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1$, we obtain

$$\mathcal{D}_A{}^C = \frac{1}{2}\Big[\kappa_1(\xi_A\eta^C + \eta^{\dot A}\xi_{\dot C} - \eta_B\xi^D - \xi^{\dot B}\eta_{\dot D}) - (\kappa_0 - i\kappa_2)(\delta^B{}_A\eta_B\eta^C + \delta^A{}_B\xi_A\xi^D) - (\kappa_0 + i\kappa_2)(\delta^{\dot A}{}_{\dot B}\xi^{\dot B}\xi_{\dot C} + \delta^{\dot B}{}_{\dot A}\eta^{\dot A}\eta_{\dot D})\Big]; \tag{2-1-115}$$

We then rewrite Eqs. (2-1-24) and (2-1-25) as

$$\begin{aligned} \frac{d}{ds}\xi^A &= \underline{\underline{\Gamma}} \cdot \xi^A \\ &= \underline{\underline{\eta}}_{\dot C} \cdot \Pi^{\dot C A} \end{aligned} \tag{2-1-116}$$

and

$$\begin{aligned} \frac{d}{ds}\tilde{\xi}^A &= \underline{\underline{\tilde{\Gamma}}} \cdot \tilde{\xi}^A \\ &= \underline{\underline{\tilde{\eta}}}_{\dot C} \cdot \tilde{\Pi}^{\dot C A} , \end{aligned} \tag{2-1-117}$$

where

$$\underline{\underline{\eta}}_{\dot C} = \begin{bmatrix} \eta_{\dot C} & 0 & 0 & 0 \\ 0 & \xi_{\dot C} & 0 & 0 \\ 0 & 0 & \eta_D & 0 \\ 0 & 0 & 0 & \xi_D \end{bmatrix} , \tag{2-1-118}$$

$$\underline{\underline{\breve{\eta}}}_{\dot{C}} = \begin{bmatrix} \breve{\eta}_{\dot{C}} & 0 & 0 & 0 \\ 0 & \breve{\xi}_{\dot{C}} & 0 & 0 \\ 0 & 0 & \breve{\eta}_{D} & 0 \\ 0 & 0 & 0 & \breve{\xi}_{D} \end{bmatrix} , \qquad (2\text{-}1\text{-}119)$$

$$\Pi^{\dot{C}A} = \underline{\underline{\xi}}^{\dot{C}} \cdot \underline{\underline{\Gamma}} \cdot \underline{\xi}^{A}$$

$$= \frac{1}{2} \begin{bmatrix} -\xi^{\dot{C}} & 0 & 0 & 0 \\ 0 & \eta^{\dot{C}} & 0 & 0 \\ 0 & 0 & -\xi^{D} & 0 \\ 0 & 0 & 0 & \eta^{D} \end{bmatrix} \cdot \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & \Gamma_{43} & \Gamma_{44} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \xi^{A} \\ \eta^{A} \\ \xi^{\dot{B}} \\ \eta^{\dot{B}} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -\Gamma_{11}\xi^{A}\xi^{\dot{C}} - \Gamma_{12}\eta^{A}\xi^{\dot{C}} \\ \Gamma_{21}\xi^{A}\eta^{\dot{C}} + \Gamma_{22}\eta^{A}\eta^{\dot{C}} \\ -\Gamma_{33}\xi^{D}\xi^{\dot{B}} - \Gamma_{34}\xi^{D}\eta^{\dot{B}} \\ \Gamma_{43}\eta^{D}\xi^{\dot{B}} + \Gamma_{44}\eta^{D}\eta^{\dot{B}} \end{bmatrix} \qquad (2\text{-}1\text{-}120)$$

and

$$\breve{\Pi}^{\dot{C}A} = \underline{\underline{\breve{\xi}}}^{\dot{C}} \cdot \underline{\underline{\breve{\Gamma}}} \cdot \breve{\xi}^{A}$$

$$= \frac{1}{4} \begin{bmatrix} -\breve{\Gamma}_{11}\breve{\xi}^{A}\breve{\xi}^{\dot{C}} - \breve{\Gamma}_{12}\breve{\eta}^{A}\breve{\xi}^{\dot{C}} \\ \breve{\Gamma}_{21}\breve{\xi}^{A}\breve{\eta}^{\dot{C}} + \breve{\Gamma}_{22}\breve{\eta}^{A}\breve{\eta}^{\dot{C}} \\ -\breve{\Gamma}_{33}\breve{\xi}^{\dot{B}}\breve{\xi}^{D} - \breve{\Gamma}_{34}\breve{\xi}^{\dot{B}}\breve{\xi}^{D} \\ \breve{\Gamma}_{43}\breve{\xi}^{\dot{B}}\breve{\eta}^{D} + \breve{\Gamma}_{44}\breve{\eta}^{\dot{B}}\breve{\eta}^{D} \end{bmatrix} \qquad (2\text{-}1\text{-}121)$$

In a similar manner to Eq. (2-1-60), Eqs. (2-1-120) and (2-1-121) are rewritten as

$$\Pi^{\dot{C}A} = \frac{1}{4}\left[-\Gamma_{12}(\eta^{A}\xi^{\dot{C}} - \xi^{A}\eta^{\dot{C}}) - \Gamma_{34}(\eta^{\dot{B}}\xi^{D} - \xi^{\dot{B}}\eta^{D})\right]$$

$$= \frac{1}{4}\left[-\left\{(1-\epsilon_0\epsilon_1)R_0 + \left[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{1-\epsilon_0\epsilon_3}\sqrt{1-\epsilon_1\epsilon_2})\right]R_2\right\}^{*}(\eta^{A}\xi^{\dot{C}} - \xi^{A}\eta^{\dot{C}}) - \left\{(1-\epsilon_0\epsilon_1)R_0 - \left[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{1-\epsilon_0\epsilon_3}\sqrt{1-\epsilon_1\epsilon_2})\right]R_2\right\}(\eta^{\dot{B}}\xi^{D} - \xi^{\dot{B}}\eta^{D})\right]$$

(2-1-   )

and

$$\tilde{\overline{\Pi}}^{\dot{C}A}=\frac{1}{2}\left[-\tilde{\Gamma}_{12}\left(\tilde{\eta}^{A}\tilde{\xi}^{\dot{C}}-\tilde{\xi}^{A}\tilde{\eta}^{\dot{C}}\right)\right]$$

$$=\frac{1}{2}\left\{-\left[\left(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2}\right)-\left(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3}\right)\right]\tilde{k}_2\left(\tilde{\eta}^{A}\tilde{\xi}^{\dot{C}}-\tilde{\xi}^{C}\tilde{\eta}^{\dot{C}}\right)\right\}. \qquad (2\text{-}1\text{-}123)$$

For example, $\overline{\Pi}^{\dot{B}A}$ is calculated for time like curves as

$$\overline{\Pi}^{\dot{C}A}=\frac{1}{2}\left\{-(k_0-ik_2)(\eta^{A}\xi^{\dot{C}}-\xi^{A}\eta^{\dot{C}})-(k_0+ik_2)(\eta^{\dot{B}}\xi^{D}-\xi^{\dot{B}}\eta^{D})\right\} \qquad (2\text{-}1\text{-}124)$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3$ $=1$ as

$$\overline{\Pi}^{\dot{C}A}=ik_2\left(\eta^{A}\xi^{\dot{C}}-\xi^{A}\eta^{\dot{C}}-\eta^{\dot{B}}\xi^{D}+\xi^{\dot{B}}\eta^{D}\right). \qquad (2\text{-}1\text{-}125)$$

We then rewrite Eqs. (2-1-40) and (2-1-41) instead of Eqs. (2-1-94) and (2-1-95) as

$$\begin{aligned}\frac{d}{ds}\Psi^{A} &= \underline{\underline{\Phi}}\cdot\Psi^{A}\\ &= \underline{\underline{\chi}}^{c}\cdot\overline{\Pi}_{c}{}^{a}\end{aligned} \qquad (2\text{-}1\text{-}126)$$

and

$$\begin{aligned}\frac{d}{ds}\tilde{\Psi}^{A} &= \underline{\underline{\tilde{\Phi}}}\cdot\tilde{\Psi}^{A}\\ &= \underline{\underline{\tilde{\chi}}}^{c}\cdot\tilde{\overline{\Pi}}_{c}{}^{a}\end{aligned} \qquad (2\text{-}1\text{-}127)$$

where

$$\Pi_c{}^{a} = \underline{\underline{\varphi}}_c \cdot \underline{\underline{\Phi}} \cdot \psi^{A}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \psi^c & 0 \\ 0 & -\chi^d \end{bmatrix} \cdot \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} \psi_a \\ \chi_b \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \Phi_{11}\psi_a\psi^c + \Phi_{12}\chi_b\psi^c \\ -\Phi_{21}\psi_a\chi^d - \Phi_{22}\chi_b\chi^d \end{bmatrix} \tag{2-1-128}$$

and

$$\breve{\Pi}_c{}^{a} = \underline{\underline{\breve{\varphi}}}_c \cdot \underline{\underline{\breve{\Phi}}} \cdot \breve{\psi}^{A}$$

$$= \frac{1}{2}\begin{bmatrix} \breve{\Phi}_{11}\breve{\psi}_a\breve{\psi}^c + \breve{\Phi}_{12}\breve{\chi}_b\breve{\psi}^c \\ -\breve{\Phi}_{21}\breve{\psi}_a\breve{\chi}^d - \breve{\Phi}_{22}\breve{\chi}_b\breve{\chi}^d \end{bmatrix} \tag{2-1-129}$$

In a similar manner to Eq. (2-1-62), Eqs. (2-1-128) and (2-1-129) are rewritten as

$$\begin{aligned} \Pi_c{}^{a} &= \frac{1}{2}\left[\Phi_{11}\psi_a\psi^c + \Phi_{12}\chi_b\psi^c - \Phi_{21}\psi_a\chi^d - \Phi_{22}\chi_b\chi^d\right] \\ &= \frac{1}{2}\Big\{-(2/\sqrt{-\epsilon_1\epsilon_2})R_1(\psi_a\psi^c + \chi_b\chi^d) - (1-\epsilon_0\epsilon_1)R_0(I_a{}^{b}\chi_b\psi^c - I_b{}^{a}\psi_a\chi^d) \\ &\quad -\left[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})\right]R_2(\gamma^{(15)}_a{}^{b}\chi_b\psi^c - \gamma^{(15)}_b{}^{a}\psi_a\chi^d)\Big] \end{aligned} \tag{2-1-130}$$

and

$$\begin{aligned} \breve{\Pi}_c{}^{a} &= \frac{1}{2}\left[\breve{\Phi}_{11}\breve{\psi}_a\breve{\psi}^c + \breve{\Phi}_{12}\breve{\chi}_b\breve{\psi}^c - \breve{\Phi}_{21}\breve{\psi}_a\breve{\chi}^d - \breve{\Phi}_{22}\breve{\chi}_b\breve{\psi}^d\right] \\ &= \frac{1}{2}\Big\{-\left[(1/\sqrt{-\epsilon_1\epsilon_2}) - \sqrt{-\epsilon_1\epsilon_2}\right]\breve{R}_1(\breve{\psi}_a\breve{\psi}^c + \breve{\chi}_b\breve{\chi}^d) - \left[(\sqrt{-\epsilon_0\epsilon_2}/\sqrt{-\epsilon_1\epsilon_3}) - (\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\right]\breve{R}_2(\gamma^{(15)}_a{}^{b}\breve{\chi}_b\breve{\psi}^c - \gamma^{(15)}_b{}^{a}\breve{\psi}_a\breve{\chi}^d)\Big\}. \end{aligned} \tag{2-1-131}$$

For example $\widehat{\Pi}_c{}^a$ is calculated for time-like curves as

$$\widehat{\Pi}_c{}^a = i\mathcal{R}_1(\psi_a\psi^c + \chi_b\chi^d) - \mathcal{R}_0(I_a{}^a\chi_b\psi^c - I_a{}^a\psi_a\chi^d) + i\mathcal{R}_2(\gamma^{(5)e}_{a}{}^{a}\chi_b\psi^c - \gamma^{(5)a}_{e}\psi_a\chi^d) \tag{2-1-132}$$

and for space-like curves with indicators $\epsilon_2 = -1$, $\epsilon_0 = \epsilon_1 = \epsilon_2$ $= 1$ as

$$\widehat{\Pi}_c{}^a = -\mathcal{R}_1(\psi_a\psi^c + \chi_b\psi^d) - i\mathcal{R}_2(\gamma^{(5)e}_{a}\chi_b\psi^c - \gamma^{(5)a}_{e}\psi_a\chi^d) . \tag{2-1-133}$$

We then take the bispinor adjoint of Eqs. (2-1-126) and (2-1-127) or rewrite Eqs. (2-1-53) and (2-1-54) to obtain

$$\begin{aligned} \frac{d}{ds}\chi_A &= \frac{d}{ds}(\psi^A)^{\oplus} \\ &= \chi_A \cdot \underline{\underline{\Phi}}^{\oplus} \\ &= (\widehat{\Pi}^A{}_C)^{\oplus} \cdot (\underline{\underline{\psi}}^C)^{\oplus} \\ &= (\widehat{\Pi}^{\oplus})_A{}^C \cdot \underline{\underline{\chi}}_C \end{aligned} \tag{2-1-134}$$

and

$$\begin{aligned} (\widehat{\Pi}^{\oplus})_A{}^C &= \chi_A \cdot \underline{\underline{\Phi}}^{\oplus} \cdot \underline{\underline{\psi}}^c \\ &= \frac{1}{\sqrt{2}}[\chi^a, \psi^b] \cdot \begin{bmatrix} \Phi_{11}^{\oplus} & \Phi_{12}^{\oplus} \\ \Phi_{21}^{\oplus} & \Phi_{22}^{\oplus} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_c & 0 \\ 0 & \chi_d \end{bmatrix} \\ &= \frac{1}{2}\left[\Phi_{11}^{\oplus}\chi^a\psi_c + \Phi_{21}^{\oplus}\psi^b\psi_c ,\ \Phi_{12}^{\oplus}\chi^a\chi_d + \Phi_{22}^{\oplus}\psi^b\chi_d\right] \\ &= \frac{1}{2}\Big[-(2/\sqrt{-\epsilon_1\epsilon_2})^*\mathcal{R}_1\chi^a\psi_c - \{(1-\epsilon_0\epsilon_1)\mathcal{R}_0 I_a{}^a - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]^*\mathcal{R}_2\gamma^{(5)a}_{a}\}\psi^b\psi_c , \\ &\quad -\{(1-\epsilon_0\epsilon_1)\mathcal{R}_0 - [\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]^*\mathcal{R}_2\gamma^{(5)a}_{a}\}\chi^a\chi_d + (2/\sqrt{-\epsilon_1\epsilon_2})^*\mathcal{R}_1\psi^b\chi_d\Big] . \end{aligned} \tag{2-1-135}$$

and

$$(\overleftarrow{\Pi}^{\oplus})_A{}^C = \frac{1}{2}\Big[-\{(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}\}^* \overleftarrow{R}_1^* \overleftarrow{\chi}^a \overleftarrow{\psi}_c - \{(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\}^* \overleftarrow{R}_2^* \gamma^{(5)}{}_e{}^a \overleftarrow{\psi}^b \overleftarrow{\psi}_c$$

$$-\{(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\}^* \overleftarrow{R}_2^* \gamma^{(5)}{}_a{}^e \chi^a \chi_d + \{(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}\}^* \overleftarrow{R}_1^* \psi^b \chi_d \, .$$

(2-1-136)

In a similar manner to Eq.(2-1-62) , Eqs. (2-1-135) and (2-1-136) are re-expressed as

$$(\Pi^{\oplus})_A{}^C = \frac{1}{2}\Big[\Phi_{11}^{\oplus}\chi^a\psi_c + \Phi_{21}^{\oplus}\psi^b\psi_c + \Phi_{12}^{\oplus}\chi^a\chi_d + \Phi_{22}^{\oplus}\psi^b\chi_d\Big]$$

$$= \frac{1}{2}\Big\{-(2/\sqrt{-\epsilon_1\epsilon_2})^* R_1(\chi^a\psi_c - \psi^b\chi_d) - (1-\epsilon_0\epsilon_1)R_0(I_e{}^a\psi^b\psi_c + I_a{}^e\chi^a\chi_d)$$

$$+\{\epsilon_2 i(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})\}^* R_2(\gamma^{(5)}{}_e{}^a\psi^b\psi_c + \gamma^{(5)}{}_a{}^e\chi^a\chi_d)\Big\} \qquad (2\text{-}1\text{-}137)$$

and

$$(\overleftarrow{\Pi}^{\oplus})_A{}^C = \frac{1}{2}\Big[\overleftarrow{\Phi}_{11}^{\oplus}\overleftarrow{\chi}^a\overleftarrow{\psi}_c + \overleftarrow{\Phi}_{21}^{\oplus}\overleftarrow{\psi}^b\overleftarrow{\psi}_c + \overleftarrow{\Phi}_{12}^{\oplus}\overleftarrow{\chi}^a\overleftarrow{\chi}_d + \overleftarrow{\Phi}_{22}^{\oplus}\overleftarrow{\psi}^b\overleftarrow{\chi}_d\Big]$$

$$= \frac{1}{2}\Big\{-\{(1/\sqrt{-\epsilon_1\epsilon_2})-\sqrt{-\epsilon_1\epsilon_2}\}^* \overleftarrow{R}_1^*(\overleftarrow{\chi}^a\overleftarrow{\psi}_c - \overleftarrow{\psi}^b\overleftarrow{\chi}_d) - \{(\sqrt{-\epsilon_0\epsilon_3}/\sqrt{-\epsilon_1\epsilon_2})-(\sqrt{-\epsilon_1\epsilon_2}/\sqrt{-\epsilon_0\epsilon_3})\}^* \overleftarrow{R}_2^*(\gamma^{(5)}{}_e{}^a\overleftarrow{\psi}^b\overleftarrow{\psi}_c + \gamma^{(5)}{}_a{}^e\overleftarrow{\chi}^a\overleftarrow{\chi}_d)\Big\} \, .$$

(2-1-138)

For example $(\Pi^{\oplus})_A{}^C$ is calculated for time-like curve as

$$(\Pi^{\oplus})_A{}^C = -iR_1(\chi^a\psi_c - \psi^b\chi_d) - R_0(I_e{}^a\psi^b\psi_c + I_a{}^e\chi^a\chi_d) + iR_2(\gamma^{(5)}{}_e{}^a\psi^b\psi_c + \gamma^{(5)}{}_a{}^e\chi^a\chi_d)$$

(2-1-139)

and for space-like curves with indicators $\varepsilon_2 = -1$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1$ as

$$(\overset{\oplus}{\widetilde{\Pi}})_A{}^c = -k_1(\chi^a \psi_c - \psi^b \chi_d) + i k_2 (\gamma^{(5)}{}_a{}^a \psi^b \psi_c \pm \gamma^{(5)}{}_a{}^a \chi^a \chi_d) . \qquad (2\text{-}1\text{-}140)$$

(ii) SOLUTIONS

In Section 1-(iii) we solved a set of simultaneous linear ordinary differential equations of third order for ξ and of fourth order for φ assuming all the Frenet-Serret coefficients to be constant. The latter can be readily applied to the solution of the same for u and $\tilde{z}$. Moreover characteristic roots are the same since only linear transformation is involved. However, since ξ and $\tilde{\xi}$ are not introduced via linear transformation, characteristic roots are not the same. Also noteworthy is the fact that $\underline{\underline{\Gamma}}$ and $\underline{\underline{\tilde{\Gamma}}}$ are decoupled into a set of two by two matrices. Thus we write again

$$\frac{d}{ds}\xi = \underline{\underline{\Gamma}} \cdot \xi \qquad . \tag{2-1-24}$$

Characteristic roots can be found by solving

$$\begin{vmatrix} \Gamma_{11}-\lambda & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22}-\lambda & 0 & 0 \\ 0 & 0 & \Gamma_{33}-\lambda & \Gamma_{34} \\ 0 & 0 & \Gamma_{43} & \Gamma_{44}-\lambda \end{vmatrix}$$

$$= [\lambda^2-(\Gamma_{11}+\Gamma_{22})\lambda+\Gamma_{11}\Gamma_{22}-\Gamma_{12}\Gamma_{21}][\lambda^2-(\Gamma_{33}+\Gamma_{44})\lambda+\Gamma_{33}\Gamma_{44}-\Gamma_{34}\Gamma_{43}]$$

$$= 0 \qquad , \tag{2-2-1}$$

to be

$$\lambda_{1,2}=\frac{(\Gamma_{11}+\Gamma_{22})\pm\sqrt{(\Gamma_{11}+\Gamma_{22})^2+4(\Gamma_{12}\Gamma_{21}-\Gamma_{11}\Gamma_{22})}}{2}$$

$$=\pm\sqrt{\Gamma_{12}\Gamma_{21}-\Gamma_{11}\Gamma_{22}}$$

$$=\pm\sqrt{\frac{1}{8}(1-\epsilon_0\epsilon_1)\left\{k_0^2+\left[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2}\right]k_0k_2+\frac{1}{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}k_2^2\right\}-\frac{1}{4\epsilon_1\epsilon_2}k_1^2} \tag{2-2-2}$$

and

$$\lambda_{3,4}=\frac{(\Gamma_{33}+\Gamma_{44})\pm\sqrt{(\Gamma_{33}+\Gamma_{44})^2+4(\Gamma_{34}\Gamma_{43}-\Gamma_{33}\Gamma_{44})}}{2}$$

$$=\pm\sqrt{\Gamma_{34}\Gamma_{43}-\Gamma_{33}\Gamma_{44}}$$

$$=\pm\sqrt{\frac{1}{8}(1-\epsilon_0\epsilon_1)\left\{k_0^2-\left[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})\right]k_0k_2+\frac{1}{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}k_2^2\right\}-\frac{1}{4\epsilon_1\epsilon_2}k_1^2} \tag{2-2-3}$$

They are calculated to be

$$\lambda_{1,2}=\frac{1}{2}(k^2-\tau_1^2-\tau_2^2-i\,2k\tau_2) \tag{2-2-4}$$

and

$$\lambda_{3,4}=\frac{1}{2}(k^2-\tau_1^2-\tau_2^2+i\,2k\tau_2) \tag{2-2-5}$$

for time-like curve and

$$\lambda_{1,2,3,4} = \frac{k_1}{2} \tag{2-2-6}$$

for space-like curve with indicators $\epsilon_0=\epsilon_1=\epsilon_3=1,\ \epsilon_2=-1$.

General solution for ξ^A and η^A can be written as

$$\begin{bmatrix} \xi^A \\ \eta^A \end{bmatrix} = \begin{bmatrix} C_{11} C_{12} \\ C_{21} C_{22} \end{bmatrix} \cdot \begin{bmatrix} e^{\lambda_1 S} \\ e^{\lambda_2 S} \end{bmatrix} , \tag{2-2-7}$$

where C's are integration constants and can be expressed in terms of initial values as

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} C_{11} C_{12} \\ C_{21} C_{22} \end{bmatrix} = \begin{bmatrix} \xi^A(0) & \eta^A(0) \\ \frac{d\xi^A}{dS}(0) & \frac{d\eta^A}{dS}(0) \end{bmatrix}$$

$$= \begin{bmatrix} \xi^A(0) & \eta^A(0) \\ \Gamma_{11}\xi^A(0)+\Gamma_{12}\eta^A(0) & \Gamma_{21}\xi^A(0)+\Gamma_{22}\eta^A(0) \end{bmatrix} \tag{2-2-8}$$

C's are expressed as

$$C_{11} = \frac{(\Gamma_{11}-\lambda_2)\xi^A(0) + \Gamma_{12}\eta^A(0)}{\lambda_1 - \lambda_2}$$

$$= [\frac{\Gamma_{11}-\Gamma_{22}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}} + 1]\xi^A(0) + \frac{\Gamma_{12}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}}\eta^A(0)$$

$$=\left(\frac{(4/\sqrt{-\epsilon_1\epsilon_2})R_1}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}+1\right)\xi^A(0)$$

$$+\frac{(1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}\eta^A(0) \qquad (2\text{-}2\text{-}9)$$

$$c_{21} = \frac{(\Gamma_{11}-\lambda_1)\xi^A(0) + \Gamma_{12}\eta^A(0)}{\lambda_2 - \lambda_1}$$

$$= [-\frac{\Gamma_{11}-\Gamma_{22}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}} + 1]\xi^A(0) - \frac{\Gamma_{12}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}}\eta^A(0)$$

$$=\left(-\frac{(4/\sqrt{-\epsilon_1\epsilon_2})R_1}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}+1\right)\xi^A(0)$$

$$-\frac{(1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}\eta^A(0) \qquad (2\text{-}2\text{-}10)$$

$$c_{12} = \frac{\Gamma_{21}\xi^A(0) + (\Gamma_{22}-\lambda_2)\eta^A(0)}{\lambda_1 - \lambda_2}$$

$$= \frac{\Gamma_{21}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}}\xi^A(0) + [-\frac{\Gamma_{11}-\Gamma_{22}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}} + 1]\eta^A(0)$$

$$=\frac{(1-\epsilon_0\epsilon_1)R_0+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_2}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}\xi^A(0)$$

$$+\left(-\frac{(4/\sqrt{-\epsilon_1\epsilon_2})R_1}{\sqrt{-\frac{16R_1^2}{\epsilon_1\epsilon_2}+8(1-\epsilon_0\epsilon_1)\{R_0^2+[\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]R_0R_2+\epsilon_0\epsilon_1\epsilon_2\epsilon_3R_2^2\}}}+1\right)\eta^A(0) \qquad (2\text{-}2\text{-}11)$$

$$C_{22} = \frac{\Gamma_{21}\xi^A(0) + (\Gamma_{22}-\lambda_1)\eta^A(0)}{\lambda_2 - \lambda_1}$$

$$= \frac{\Gamma_{21}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}}\xi^A(0) + [\frac{\Gamma_{11}-\Gamma_{22}}{\sqrt{(\Gamma_{11}-\Gamma_{22})^2+4\Gamma_{12}\Gamma_{21}}} + 1]\eta^A(0)$$

$$= -\frac{(1-\epsilon_0\epsilon_1)\mathcal{R}_0 + \{\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]\mathcal{R}_2}{\sqrt{-\frac{16\mathcal{R}_1^2}{-\epsilon_1\epsilon_2} + 8(1-\epsilon_0\epsilon_1)\{\mathcal{R}_0^2 + \{\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]\mathcal{R}_0\mathcal{R}_2 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3\mathcal{R}_2^2\}}}\xi^A(0)$$

$$+\left(\frac{(4/\sqrt{-\epsilon_1\epsilon_2})\mathcal{R}_1}{\sqrt{-\frac{16\mathcal{R}_1^2}{-\epsilon_1\epsilon_2} + 8(1-\epsilon_0\epsilon_1)\{\mathcal{R}_0^2 + \{\epsilon_2(\epsilon_1-\epsilon_0)/(\sqrt{-\epsilon_0\epsilon_3}\sqrt{-\epsilon_1\epsilon_2})]\mathcal{R}_0\mathcal{R}_2 + \epsilon_0\epsilon_1\epsilon_2\epsilon_3\mathcal{R}_2^2\}}} + 1\right)\eta^A(0) \qquad (2\text{-}2\text{-}12)$$

For example C's are calculated for time-like curve as

$$C_{11} = [\frac{-i\tau_1}{\sqrt{(k-i\tau_2)^2-\tau_1^2}} + 1]\xi^A(0) + \frac{k-i\tau_2}{2\sqrt{(k-i\tau_2)^2-\tau_1^2}}\eta^A(0) \qquad (2\text{-}2\text{-}13)$$

$$C_{21} = [\frac{i\tau_1}{\sqrt{(k-i\tau_2)^2-\tau_1^2}} + 1]\xi^A(0) - \frac{k-i\tau_2}{2\sqrt{(k-i\tau_2)^2-\tau_1^2}}\eta^A(0) \qquad (2\text{-}2\text{-}14)$$

$$C_{12} = \frac{k-i\tau_2}{2\sqrt{(k-i\tau_2)^2-\tau_1^2}}\xi^A(0) + [\frac{i\tau_1}{\sqrt{(k-i\tau_2)^2-\tau_1^2}} + 1]\eta^A(0) \qquad (2\text{-}2\text{-}15)$$

$$C_{22} = -\frac{k - i\hat{\tau}_2}{2\sqrt{(k - i\hat{\tau}_2)^2 - \hat{\tau}_1^2}}\, \xi^A(0) + \left[\frac{-i\hat{\tau}_1}{\sqrt{(k - i\hat{\tau}_2)^2 - \hat{\tau}_1^2}} + 1\right] \eta^A(0) \tag{2-2-16}$$

and for space-like curves with indicators $\varepsilon_2 = -1$, $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 1$ as

$$C_{11} = 2\,\xi^A(0) \tag{2-2-17}$$

$$C_{21} = 0 \tag{2-2-18}$$

$$C_{12} = 0 \tag{2-2-19}$$

$$C_{22} = 2\,\eta^A(0) \tag{2-2-20}$$

CHAPTER 3

A POINT CHARGE AND A FREELY SPINNING PARTICLE

(i) We first describe the motion of a point particle with charge e and mass m in an electromagnetic field $F^{\mu\nu}$ by the Lorentz equation

$$\frac{d}{ds} u^{\mu} = \frac{e}{m} F^{\mu}{}_{\nu} u^{\nu} , \tag{3-1}$$

where u^{μ} is the 4 velocity of this particle.

Setting

$$u^{\mu} = e^{\mu}{}_{(0)} \tag{3-2}$$

and assuming the field to be static, we obtain

$$\frac{d}{ds} e^{\mu}{}_{(a)} = \frac{e}{m} F^{\mu}{}_{\nu} e^{\nu}{}_{(a)} . \tag{3-3}$$

Darboux bivector is related to the field by

$$D^{\mu\nu} = \frac{e}{m} F^{\mu\nu} . \tag{3-4}$$

Hence, using Eq.(2-1-62), we obtain the local electric, magnetic and 'Poynting' projections as

$$\begin{aligned} \frac{e}{m} E^{\mu} &= D^{\mu}{}_{\nu} e^{\nu}{}_{(0)} \\ &= a^{\mu} \\ &= k_0 e^{\mu}{}_{(1)} , \end{aligned} \tag{3-5}$$

$$\begin{aligned} \frac{e}{m} H^{\mu} &= -\hat{D}^{\mu}{}_{\nu} e^{\nu}{}_{(0)} \\ &= \Omega^{\mu} \\ &= \epsilon_0 \left(\epsilon_1 k_1 e^{\mu}{}_{(3)} + \epsilon_2 k_2 e^{\mu}{}_{(1)} \right) , \end{aligned} \tag{3-6}$$

and

$$\begin{aligned} \left(\frac{e}{m}\right)^2 R^{\mu} &= \left(\frac{e}{m}\right)^2 \epsilon^{\mu\nu\alpha\beta} E_{\nu} H_{\alpha} u_{\beta} \\ &= -\epsilon_0 \epsilon_1 k_0 k_1 e^{\mu}{}_{(2)} . \end{aligned} \tag{3-7}$$

They are calculated, for example, for time-like curves as

$$\frac{e}{m}E^\mu = K_0 e^\mu_{(1)} \quad , \tag{3-8}$$

$$\frac{e}{m}H^\mu = -(K_1 e^\mu_{(3)} + K_2 e^\mu_{(1)}) \quad , \tag{3-9}$$

and

$$\left(\frac{e}{m}\right)^2 R^\mu = K_0 K_1 e^\mu_{(2)} \tag{3-10}$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$ as

$$\frac{e}{m}E^\mu = K_0 e^\mu_{(1)} \quad , \tag{3-11}$$

$$\frac{e}{m}H^\mu = K_1 e^\mu_{(3)} - K_2 e^\mu_{(1)} \quad , \tag{3-12}$$

and

$$\left(\frac{e}{m}\right)^2 R^\mu = -K_0 K_1 e^\mu_{(2)} \quad . \tag{3-13}$$

Also from Eqs.(3-1-5), (3-1-6) & (3-1-7), we obtain

$$K_0^2 = \left(\frac{e}{m}\right)^2 \epsilon_1 E_\mu E^\mu \quad , \tag{3-14}$$

$$K_1^2 = \left(\frac{e}{m}\right)^2 \frac{\epsilon_2 R_\mu R^\mu}{\epsilon_1 E_\alpha E^\alpha} \quad , \tag{3-15}$$

and

$$K_2^2 = \left(\frac{e}{m}\right)^2 \frac{(E_\mu H^\mu)^2}{\epsilon_1 E_\alpha E^\alpha} \tag{3-16}$$

which are calculated for time-like curves as

$$K_0^2 = \left(\frac{e}{m}\right)^2 E_\mu E^\mu \quad , \tag{3-17}$$

$$K_1^2 = \left(\frac{e}{m}\right)^2 \frac{R_\mu R^\mu}{E_\alpha E^\alpha} \quad , \tag{3-18}$$

and

$$K_2^2 = \left(\frac{e}{m}\right)^2 \frac{(E_\mu H^\mu)^2}{E_\alpha E^\alpha} \tag{3-19}$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$ as

$$K_0^2=\left(\frac{e}{m}\right)^2 E_\mu E^\mu \quad ; \tag{3-20}$$

$$K_1^2=-\left(\frac{e}{m}\right)^2\frac{R_\mu R^\mu}{E_\alpha E^\alpha} , \tag{3-21}$$

and

$$K_2^2=\left(\frac{e}{m}\right)^2\frac{(E_\mu H^\mu)^2}{E_\alpha E^\alpha} . \tag{3-22}$$

Two field invariants are expressed as

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}F_{\mu\nu}=-\left(\epsilon_0\epsilon_1 K_0^2+\epsilon_1\epsilon_2 K_1^2+\epsilon_2\epsilon_3 K_2^2\right) \tag{3-23}$$

and

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}\hat{F}_{\mu\nu}=\epsilon_0\epsilon_2(\epsilon_0\epsilon_1-\epsilon_2\epsilon_3)K_0K_2 \tag{3-24}$$

which are calculated for time-like curves as

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}F_{\mu\nu}=K_0^2-K_1^2-K_2^2 \tag{3-25}$$

and

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}\hat{F}_{\mu\nu}=2K_0K_2 \tag{3-26}$$

and for space-like curves with indicators $\epsilon_2=-1$, $\epsilon_0=\epsilon_1=\epsilon_3=1$ as

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}F_{\mu\nu}=K_1^2+K_2^2-K_0^2 \tag{3-27}$$

and

$$\frac{1}{2}\left(\frac{e}{m}\right)^2 F^{\mu\nu}\hat{F}_{\mu\nu}=-2K_0K_2 . \tag{3-28}$$

(ii) We then describe the motion of a 'freely-spinning' point particle having spin S and total energy m by the Frenkel-Thomas equations

$$\frac{d}{ds}P^{\mu}=0 \quad , \tag{3-29}$$

and

$$\frac{d}{ds}S^{\mu\nu}=P^{\mu}U^{\nu}-U^{\mu}P^{\nu} \quad , \tag{3-30}$$

where P^{μ} is the energy-momentum 4-vector satisfying

$$m^2=P_{\mu}P^{\mu} \tag{3-31}$$

and $S^{\mu\nu}$ is the spin tensor. Then we write

$$P^{\mu}=Me^{\mu}_{(0)}-P_{(1)}e^{\mu}_{(1)}-P_{(2)}e^{\mu}_{(2)}-P_{(3)}e^{\mu}_{(3)} \quad , \tag{3-32}$$

and

$$S^{\mu\nu}=S_{(12)}E^{\mu\nu}_{(12)}+S_{(23)}E^{\mu\nu}_{(23)}+S_{(31)}E^{\mu\nu}_{(31)} . \tag{3-33}$$

Substituting Eqs.(3-32) & (3-33) into Eqs.(3-29) & (3-30), respectively and utilizing Eq.(1-1-12), we obtain

$$\frac{dM}{ds}=-\epsilon_0\epsilon_1 K_0 P_{(1)} \quad , \tag{3-34}$$

$$\frac{dP_{(1)}}{ds}=K_0 M+\epsilon_1\epsilon_2 K_1 P_{(2)} \quad , \tag{3-35}$$

$$\frac{dP_{(2)}}{ds}=-K_1 P_{(1)}+\epsilon_2\epsilon_3 K_2 P_{(3)} \quad , \tag{3-36}$$

$$\frac{dP_{(3)}}{ds}=-K_2 P_{(2)} \quad , \tag{3-37}$$

$$\frac{dS_{(12)}}{ds}=-\epsilon_2\epsilon_3 K_2 S_{(31)} \quad , \tag{3-38}$$

$$\frac{dS_{(23)}}{ds}=K_1 S_{(31)} \tag{3-39}$$

$$\frac{dS_{(31)}}{ds}=K_2 S_{(12)}-\epsilon_1\epsilon_2 K_1 S_{(23)} \tag{3-40}$$

$$P_{(1)}=0 \tag{3-41}$$

$$P_{(2)}=\epsilon_0\epsilon_1 K_0 S_{(12)} \tag{3-42}$$

and

$$P_{(3)}=-\epsilon_0\epsilon_1 K_0 S_{(31)} . \tag{3-43}$$

We can write Eqs.(3-34) (3-40) as

$$\frac{d}{ds}\mathbb{M} = \underline{\underline{K}} \cdot \mathbb{M} \quad , \tag{3-44}$$

where

$$\mathbb{M} = \begin{bmatrix} \underline{M} \\ P_{(1)} \\ P_{(2)} \\ P_{(3)} \\ S_{(12)} \\ S_{(23)} \\ S_{(31)} \end{bmatrix} \tag{3-45}$$

and

$$\underline{\underline{K}} = \begin{bmatrix} 0 & -\epsilon_0\epsilon_1 K_0 & 0 & 0 & 0 & 0 & 0 \\ K_0 & 0 & \epsilon_1\epsilon_2 K_1 & 0 & 0 & 0 & 0 \\ 0 & -K_1 & 0 & \epsilon_2\epsilon_3 K_2 & 0 & 0 & 0 \\ 0 & 0 & K_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2\epsilon_3 K_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_1 \\ 0 & 0 & 0 & 0 & K_2 & -\epsilon_1\epsilon_2 K_1 & 0 \end{bmatrix} \tag{3-46}$$

where the indicator-dependent symmetry is explicit.

BIBLIOGRAPHY

Adachi, C., "Outlines of Differential Geometry", BAIHUKAN PRESS, Tokyo(1976)

Gursey, F., Nuovo Cimento 5, 784(1957)

Kent, R.D. and Szamosi, G., Il Nuovo Cimento 64, 67(1981)

Kent, R.D., Ph.D.dissertation, University of Windsor, Windsor(1978)

Synge, J.L., Proc.London Math.Soc. 43, 376(1937)

Synge, J.L. and Schild, A., "Tensor Calculus", University of Toronto Press, Toronto(1949)

VITA AUCTORIS

HIDEO ICHIMURA was born in Tunghua, China on September 20, 1941. He received his B.S. & M.S. degrees from Shizuoka University, Japan in 1965 & 1967, respectively and Ph.D. degree from University of Pennsylvania, U.S.A. in 1972, all in electrical engineering. He is also a recipient of the 1969 William G. Tuller memorial award from the Parts, Materials and Packaging group of the Institute of Electrical and Electronics Engineers, U.S.A.