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Laplace transforms of order statistics of Erlang random variables.

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LAPLACE TRANSFORMS OF ORDER STATISTICS OF ERLANG RANDOM VARIABLES

by

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Abstract

In this thesis, we present several methods which may be used to calculate the Laplace transform of order statistics of Erlang random variables. These methods are based on a probabilistic interpretation of the Laplace transform. A Markov chain analysis is included. Special cases and generalizations are discussed.

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TABLE OF CONTENTS

1. Introduction

Laplace transforms have a nice probabilistic interpretation (van Dantzig; Roy; Kleinrock). In fact, if X is a continuous, non-negative random variable with some probability density function (p.d.f.) $f(x)$, and Z is a continuous random variable, independent of X, with p.d.f. $g(z) = se^{-sz}$ $(s > 0, z \geq 0)$, then the Laplace transform of $f(x)$ is

$$L(s) = \int_0^\infty e^{-sx} f(x)dx = P(X < Z), s > 0. \tag{1.1}$$

More generally, if any non-negative random variable X has the cumulative distribution function (c.d.f.) $F(x)$ and Z is defined as before, then the Laplace Stieltjes transform of $F(x)$ is

$$L(s) = \int_0^\infty e^{-sx} dF(x), s > 0. \tag{1.2}$$

Thus, $L(s)$ is the probability that the value taken by X is less than the value taken by Z. This interpretation of the Laplace transform can be quite useful if we are interested in the probability that a given random event will occur prior to some catastrophe, where the time until the catastrophe occurs is exponentially distributed. Intuitively, we can think of this situation as a race of fixed distance, where X and Z are the race completion times of two racers. Thus, $L(s)$ is simply the probability that racer "X" wins the race. It can be easily verified (see Properties 3.1 and 3.3 in chapter 3) that if X is exponentially distributed with parameter λ $(X \sim exp(\lambda))$, then

$$L(s) = \frac{\lambda}{\lambda + s}. \tag{1.3}$$

Through the use of this interpretation, one can avoid extensive calculation by replacing integration with counting procedures.

Laplace transforms are useful, among other reasons, in finding the moments of a random variable X. The moment generating function of X, $M_X(t)$, can be manipulated in the following manner.

$$M_X(t) = E(e^{tX}) = E(e^{-(-t)X}) = L_X(-t). \tag{1.4}$$

Hence, $L(s)$ can be used to obtain $M(t)$, and both $L(s)$ and $M(t)$ can be used to obtain moments of X.

We note that a Laplace transform is also an expected value of a random variable. For instance, consider a random variable $Z \sim exp(s)$. Further, consider any non-negative random variable X. Now define the random variable Y as follows.

$$Y = \begin{cases} 0 \text{ if } X \geq Z \\ 1 \text{ if } X < Z \end{cases}$$

Hence

$$\begin{aligned} E(Y) &= 0P(Y = 0) + 1P(Y = 1) \\ &= 0P(X \geq Z) + 1P(X < Z) \\ &= P(X < Z) \\ &= L(s) \\ &= E(e^{-sX}). \end{aligned} \tag{1.5}$$

Order statistics are an important topic of study (see David) as they have a wide variety of applications. One such application is within the field of reliability prediction. It is of great advantage to be able to predict the reliability of a commercial or industrial manufacturing process. In particular, one may wish to predict the probability of a quota being filled prior to an incoming demand, or of a process remaining within production tolerance levels. "A k-out-of-n structure functions iff at least k out of n components function." (ref: Barlow and Proschan). Thus, the structure fails when the k-th order statistic occurs.

An application of order statistics is in queueing theory. Suppose two customers, A and B, arrive at a queueing system with two servers, where each server has a line containing at least one customer. A and B decide to wait in different lines so the first of them to reach a server can obtain service for both. In this case, we are interested in the smallest order statistic. If the service for each customer is exponential, then the waiting time of A and B is the minimum of two Erlang random variables. By finding the probability that this minimum is less than some (artificial) catastrophe variable Z, $Z \sim exp(s)$, we are finding the Laplace transform of the minimum.

Erlang random variables are a special case of Gamma random variables, and can be considered to be the sum of a number of exponential random variables. One important application of Erlang random variables is within queueing theory since they lend themselves nicely to applications where a single process can be modeled as a sequence of individual exponential processes.

Work on the Laplace transforms of order statistics of exponentially distributed random variables has been completed by Roy (1997). In this thesis, we shall concentrate on Laplace transforms of order statistics of Erlang random variables. Calculation of these Laplace transforms shall be completed using several methods.

2. Direct Calculation

One method by which the Laplace transform may be calculated is by direct application of the definition of a Laplace transform. Before we proceed further, let us state more formally a definition which was referred to in chapter 1.

Definition 2.1. *If $F(x)$ is the c.d.f. of a continuous, non-negative random variable X, then $L(s) = \int_0^\infty e^{-sx} dF(x)$ is the Laplace Stieltjes transform of $F(x)$. If $f(x)$ is the p.d.f. of that same random variable, then $L(s) = \int_0^\infty e^{-sx} f(x)dx$ is the Laplace transform of $f(x)$.*

We shall first consider two independent and identically distributed (i.i.d.) two-stage Erlang random variables. That is, consider two Erlang random variables Y_1 and Y_2, where

$$Y_1 = X_{11} + X_{12}$$

$$Y_2 = X_{21} + X_{22},$$

and where the X_{ij}'s are i.i.d. exponential random variables with parameter λ for $i, j = 1, 2$. Define Z to be a random variable which is exponentially distributed with parameter s.

Suppose we wish to determine the Laplace transform for $max(Y_1, Y_2)$. Since each Y_i $(i = 1, 2)$ is the sum of the same two i.i.d. exponential random variables, the common p.d.f. of the Y_i's for $i = 1, 2$ is

$$f(y) = \lambda^2 y e^{-\lambda y}. \tag{2.1}$$

Hence, using integration by parts, the common c.d.f. of the Y_i's for $i = 1, 2$ is

$$F(y) = P(Y \leq y) = -\lambda y e^{-\lambda y} - e^{-\lambda y} + 1. \tag{2.2}$$

We shall use the standard formula for the p.d.f. of order statistics of i.i.d. random variables, as given by Hogg and Craig, with some slight notational deviation. Let

$Y_1, Y_2, \ldots, Y_n$ denote a set of n i.i.d. random variables, for some $n \in \mathbf{N}$. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ denote the order statistics of these n random variables. Then the p.d.f. of the k-th order statistic, $W = Y_{(k)}$, is given by

$$g_{(k)}(w) = \frac{n!}{(k-1)!(n-k)!}[F(w)]^{k-1}[1-F(w)]^{n-k}f(w) \qquad (2.3)$$

where, for our purposes, $w \geq 0$. Otherwise, $g_{(k)}(w) = 0$.

Thus, the p.d.f. of the second order statistic, $Y_{(2)} = max(Y_1, Y_2)$, shall be represented notationally by $g_{(2)}(w)$ and, using (2.3), is derived as follows.

$$\begin{aligned} g_{(2)}(w) &= \frac{2!}{1!0!}F(w)f(w) \; (by (2.3)) \\ &= 2(-\lambda w e^{-\lambda w} - e^{-\lambda w} + 1)\lambda^2 w e^{-\lambda w} \\ &= -2\lambda^3 w^2 e^{-2\lambda w} - 2\lambda^2 w e^{-2\lambda w} + 2\lambda^2 w e^{-\lambda w}. \end{aligned} \qquad (2.4)$$

The following property will prove useful in the calculations to come.

Property 2.1.

$$\int_0^\infty w^m e^{-nw} dw = \frac{m!}{n^{m+1}}, m = 0, 1, \ldots, n \geq 0.$$

Proof. This follows directly from the definition of the Gamma function.■

Returning to the task at hand, we wish to calculate $L_{(2)}(s)$, where $L_{(2)}(s)$ is the Laplace transform of $g_{(2)}(w)$. Now, directly applying the definition of a Laplace transform to (2.4) and using Property 2.1, we obtain

$$\begin{aligned} L_{(2)}(s) &= \int_0^\infty e^{-sw} g_{(2)}(w) dw \\ &= -2\lambda^3 \int_0^\infty w^2 e^{-(2\lambda+s)w} dw - 2\lambda^2 \int_0^\infty w e^{-(2\lambda+s)w} dw \\ &\quad + 2\lambda^2 \int_0^\infty w e^{-(\lambda+s)w} dw \\ &= -2\lambda^3 \frac{2}{(2\lambda+s)^3} - 2\lambda^2 \frac{1}{(2\lambda+s)^2} + 2\lambda^2 \frac{1}{(\lambda+s)^2} \\ &= -\frac{4\lambda^3}{(2\lambda+s)^3} - \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{2\lambda^2}{(\lambda+s)^2}. \end{aligned} \qquad (2.5)$$

Suppose we also wish to determine $L_{(1)}(s)$. A similar set of calculations shows that

$$L_{(1)}(s) = \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3}. \tag{2.6}$$

The preceeding demonstration was for the simple case where we have only two i.i.d. two-stage Erlang random variables (and hence only two order statistics). Of course, we can consider more complex cases. Let us consider a situation in which we have five i.i.d. three-stage Erlang random variables. That is,

$$Y_i = X_{i1} + X_{i2} + X_{i3}, i = 1,2,3,4,5,$$

where the X_{ij}'s are i.i.d. exponential random variables with parameter λ for $i = 1,2,3,4,5$ and $j = 1,2,3$. Now suppose we are interested in calculating $L_{(2)}(s)$ in this case. We have the common p.d.f.

$$f(y) = \frac{1}{2}\lambda^3 y^2 e^{-\lambda y} \tag{2.7}$$

and the common c.d.f.

$$F(y) = -\frac{1}{2}\lambda^2 y^2 e^{-\lambda y} - \lambda y e^{-\lambda y} - e^{-\lambda y} + 1 \tag{2.8}$$

for the Y_i's, $i = 1,2,3,4,5$. Again, we use the standard formula (2.3) to obtain the p.d.f. for the second order statistic $Y_{(2)}$.

$$\begin{aligned} g_{(2)}(w) &= \frac{5!}{1!3!}F(w)(1-F(w))^3 f(w) \\ &= 20 f(w)F(w)(1-F(w))^3. \end{aligned} \tag{2.9}$$

Using (2.7), (2.8), and a little algebraic manipulation, we obtain

$$f(w)F(w) = -\frac{1}{4}\lambda^5 w^4 e^{-2\lambda w} - \frac{1}{2}\lambda^4 w^3 e^{-2\lambda w} - \frac{1}{2}\lambda^3 w^2 e^{-2\lambda w} + \frac{1}{2}\lambda^3 w^2 e^{-\lambda w} \tag{2.10}$$

and

$$
\begin{aligned}
(1-F(w))^3 &= (1-F(w))^2(1-F(w)) \\
&= \left(\frac{1}{4}\lambda^4 w^4 e^{-2\lambda w} + \lambda^3 w^3 e^{-2\lambda w} + 2\lambda^2 w^2 e^{-2\lambda w} + 2\lambda w e^{-2\lambda w} + e^{-2\lambda w}\right) \\
&\quad \times \left(\frac{1}{2}\lambda^2 w^2 e^{-\lambda w} + \lambda w e^{-\lambda w} + e^{-\lambda w}\right) \\
&= \frac{1}{8}\lambda^6 w^6 e^{-3\lambda w} + \frac{1}{4}\lambda^5 w^5 e^{-3\lambda w} + \frac{1}{4}\lambda^4 w^4 e^{-3\lambda w} + \frac{1}{2}\lambda^5 w^5 e^{-3\lambda w} \\
&\quad + \lambda^4 w^4 e^{-3\lambda w} + \lambda^3 w^3 e^{-3\lambda w} + \lambda^4 w^4 e^{-3\lambda w} + 2\lambda^3 w^3 e^{-3\lambda w} \\
&\quad + 2\lambda^2 w^2 e^{-3\lambda w} + \lambda^3 w^3 e^{-3\lambda w} + 2\lambda^2 w^2 e^{-3\lambda w} + 2\lambda w e^{-3\lambda w} \\
&\quad + \frac{1}{2}\lambda^2 w^2 e^{-3\lambda w} + \lambda w e^{-3\lambda w} + e^{-3\lambda w} \\
&= \frac{1}{8}\lambda^6 w^6 e^{-3\lambda w} + \frac{3}{4}\lambda^5 w^5 e^{-3\lambda w} + \frac{9}{4}\lambda^4 w^4 e^{-3\lambda w} + 4\lambda^3 w^3 e^{-3\lambda w} \\
&\quad + \frac{9}{2}\lambda^2 w^2 + 3\lambda w e^{-3\lambda w} + e^{-3\lambda w}. \qquad (2.11)
\end{aligned}
$$

Using (2.10) and (2.11) we obtain the p.d.f. of $Y_{(2)}$ as follows.

$$
\begin{aligned}
g_{(2)}(w) &= 20 f(w) F(w) (1-F(w))^3 \\
&= -\frac{5}{8}\lambda^{11} w^{10} e^{-5\lambda w} - \frac{15}{4}\lambda^{10} w^9 e^{-5\lambda w} - \frac{45}{4}\lambda^9 w^8 e^{-5\lambda w} - 20\lambda^8 w^7 e^{-5\lambda w} \\
&\quad - \frac{45}{2}\lambda^7 w^6 e^{-5\lambda w} - 15\lambda^6 w^5 e^{-5\lambda w} - 5\lambda^5 w^4 e^{-5\lambda w} - \frac{5}{4}\lambda^{10} w^9 e^{-5\lambda w} \\
&\quad - \frac{15}{2}\lambda^9 w^8 e^{-5\lambda w} - \frac{45}{2}\lambda^8 w^7 e^{-5\lambda w} - 40\lambda^7 w^6 e^{-5\lambda w} - 45\lambda^6 w^5 e^{-5\lambda w} \\
&\quad - 30\lambda^5 w^4 e^{-5\lambda w} - 10\lambda^4 w^3 e^{-5\lambda w} - \frac{5}{4}\lambda^9 w^8 e^{-5\lambda w} - \frac{15}{2}\lambda^8 w^7 e^{-5\lambda w} \\
&\quad - \frac{45}{2}\lambda^7 w^6 e^{-5\lambda w} - 40\lambda^6 w^5 e^{-5\lambda w} - 45\lambda^5 w^4 e^{-5\lambda w} - 30\lambda^4 w^3 e^{-5\lambda w} \\
&\quad - 10\lambda^3 w^2 e^{-5\lambda w} + \frac{5}{4}\lambda^9 w^8 e^{-4\lambda w} + \frac{15}{2}\lambda^8 w^7 e^{-4\lambda w} + \frac{45}{2}\lambda^7 w^6 e^{-4\lambda w} \\
&\quad + 40\lambda^6 w^5 e^{-4\lambda w} + 45\lambda^5 w^4 e^{-4\lambda w} + 30\lambda^4 w^3 e^{-4\lambda w} + 10\lambda^3 w^2 e^{-4\lambda w}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{5}{8}\lambda^{11}w^{10}e^{-5\lambda w} - 5\lambda^{10}w^{9}e^{-5\lambda w} - 20\lambda^{9}w^{8}e^{-5\lambda w} - 50\lambda^{8}w^{7}e^{-5\lambda w} \\
&\quad - 85\lambda^{7}w^{6}e^{-5\lambda w} - 100\lambda^{6}w^{5}e^{-5\lambda w} - 80\lambda^{5}w^{4}e^{-5\lambda w} - 40\lambda^{4}w^{3}e^{-5\lambda w} \\
&\quad - 10\lambda^{3}w^{2}e^{-5\lambda w} + \frac{5}{4}\lambda^{9}w^{8}e^{-4\lambda w} + \frac{15}{2}\lambda^{8}w^{7}e^{-4\lambda w} + \frac{45}{2}\lambda^{7}w^{6}e^{-4\lambda w} \\
&\quad + 40\lambda^{6}w^{5}e^{-4\lambda w} + 45\lambda^{5}w^{4}e^{-4\lambda w} + 30\lambda^{4}w^{3}e^{-4\lambda w} + 10\lambda^{3}w^{2}e^{-4\lambda w}.
\end{aligned}
\tag{2.12}
$$

Applying Property 2.1 to result (2.12), and after some simplification, we obtain $L_2(s)$ as follows.

$$
\begin{aligned}
L_{(2)}(s) &= \int_0^\infty g_{(2)}(w)e^{-sw}dw \\
&= -2268000\left(\frac{\lambda}{5\lambda+s}\right)^{11} - 1814400\left(\frac{\lambda}{5\lambda+s}\right)^{10} - 806400\left(\frac{\lambda}{5\lambda+s}\right)^{9} \\
&\quad - 252000\left(\frac{\lambda}{5\lambda+s}\right)^{8} - 61200\left(\frac{\lambda}{5\lambda+s}\right)^{7} - 12000\left(\frac{\lambda}{5\lambda+s}\right)^{6} \\
&\quad - 1920\left(\frac{\lambda}{5\lambda+s}\right)^{5} - 240\left(\frac{\lambda}{5\lambda+s}\right)^{4} - 20\left(\frac{\lambda}{5\lambda+s}\right)^{3} \\
&\quad + 50400\left(\frac{\lambda}{4\lambda+s}\right)^{9} + 37800\left(\frac{\lambda}{4\lambda+s}\right)^{8} + 16200\left(\frac{\lambda}{4\lambda+s}\right)^{7} \\
&\quad + 4800\left(\frac{\lambda}{4\lambda+s}\right)^{6} + 1080\left(\frac{\lambda}{4\lambda+s}\right)^{5} + 180\left(\frac{\lambda}{4\lambda+s}\right)^{4} \\
&\quad + 20\left(\frac{\lambda}{4\lambda+s}\right)^{3}.
\end{aligned}
\tag{2.13}
$$

Thus, we see from results such as (2.5) and (2.13) that the larger the number of Erlang random variables or stages per Erlang variable involved in the problem, the more involved the calculations become. That is, for a particular $i \in \mathbf{N}$, the number of steps required to calculate $L_{(i)}(s)$ increases as the number of Erlang variables or the number of stages per Erlang variable increases.

In addition, calculations in the above examples are further simplified by the fact that the Erlang variables are i.i.d.. For Erlang variables which are not identically distributed, the calculations become more complicated. The next two chapters of this thesis will introduce methods which allow for fairly easy calculation in under either of these conditions. In addition, these new methods produce more aesthetically and intuitively pleasing results.

3. A Probabilistic and Combinatorial Approach

We shall begin this chapter with some needed properties (see Roy) followed by a simple demonstration of how probabilistic interpretation can aid in calculating Laplace transforms. Properties 3.1, 3.2, 3.3, 3.4, and 3.5 are well known and are stated here as a necessity for the following results. Generalizations and examples will be given throughout the chapter.

Property 3.1. *If X and Y are independent, continuous, non-negative random variables with respective p.d.f.'s $f(x)$ and $g(y) = se^{-sy}$, then $P(X < Y) = L_X(s)$.*

Proof. $P(X < Y) = \int_0^\infty \int_x^\infty se^{-sy} f(x)dydx = \int_0^\infty e^{-sx} f(x)dx = L_X(s)$. ∎

Property 3.2. *If $X_1, X_2, \ldots, X_k$ are independent and exponentially distributed random variables with respective parameters $\lambda_1, \lambda_2, \ldots \lambda_k$, where $k \in \mathbb{N}$, then $min(X_1, X_2, \ldots, X_k) \sim exp(\sum_{i=1}^k \lambda_i)$.*

Proof. Let W represent $min(X_1, X_2, \ldots, X_k)$. Then the cumulative distribution function of W is

$$\begin{aligned}
F_W(w) &= P(W \le w) \\
&= P(min(X_1, X_2, \ldots, X_k) \le w) \\
&= 1 - P(min(X_1, X_2, \ldots, X_k) > w) \\
&= 1 - P(X_1 > w, X_2 > w, \ldots, X_k > w) \\
&= 1 - \prod_{i=1}^{k} P(X_i > w) \quad \text{(by independence of the } X_i\text{'s)} \\
&= 1 - \prod_{i=1}^{k} e^{-\lambda_i w} \\
&= 1 - e^{-(\sum_{i=1}^k \lambda_i)w}
\end{aligned}$$

which is the c.d.f. of an exponential random variable with parameter $\sum_{i=1}^k \lambda_i$. ∎

Property 3.3. *If X_1 and X_2 are independent and exponentially distributed random variables with respective parameters λ_1 and λ_2, then $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$.*

Proof.

$$\begin{aligned} P(X_1 < X_2) &= \int_0^\infty \int_{x_1}^\infty \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_2 dx_1 \\ &= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x_1} dx_1 \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}. \blacksquare \end{aligned}$$

Property 3.4. *If $X_1, X_2, \ldots, X_k$ are independent and exponentially distributed random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$, where $k \in \mathbb{N}$, then for each $j \in \mathbb{N}$ with $j \leq k$,*

$$\begin{aligned} &P(X_j < X_1, X_j < X_2, \ldots, X_j < X_{j-1}, X_j < X_{j+1}, \ldots, X_j < X_k)) \\ &= P(X_j = min(X_1, X_2, \ldots, X_k)) \\ &= \frac{\lambda_j}{\sum_{i=1}^k \lambda_i}. \end{aligned}$$

Proof. We first recognize that

$$P(X_j = min(X_1, X_2, \ldots, X_k)) = P(X_j < min(X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k)).$$

Then, by Property 3.2, we have that $min(X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k))$ is exponentially distributed with parameter $\sum_{\substack{i=1 \\ i \neq j}}^k \lambda_i$. Hence, by Property 3.3,

$$\begin{aligned} P(X_j < min(X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k)) &= \frac{\lambda_j}{\lambda_j + \sum_{\substack{i=1 \\ i \neq j}}^k \lambda_i} \\ &= \frac{\lambda_j}{\sum_{i=1}^k \lambda_i}. \blacksquare \end{aligned}$$

Property 3.5. *If Y is an exponentially distributed random variable, and s and t are non-negative real numbers, then*

$$P(Y > s + t | Y > s) = P(Y > t).$$

This is known as the "memoryless property" of the exponential distribution.

Proof. Suppose $Y \sim exp(\lambda)$. Then,

$$\begin{aligned} P(Y > s + t | Y > s) &= \frac{P(Y > s + t)}{P(Y > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(Y > t). \blacksquare \end{aligned}$$

We are now ready to consider an example. As at the beginning of chapter 2, define two Erlang random variables $Y_1 = X_{11} + X_{12}$ and $Y_2 = X_{21} + X_{22}$, where the $X_{ij} \sim exp(\lambda)$ for $i, j = 1, 2$. Again, define Z to be a random variable which is exponentially distributed with parameter s.

Again, suppose we wish to determine $L_{(2)}(s)$, the Laplace transform of the second order statistic. By Property 3.1, this is equivalent to calculating $P(Y_{(2)} < Z)$. That is the probability that both Y_1 and Y_2 are less than Z. There are $\binom{4}{2} = 6$ cases where this is possible, as follows. Note that the order from left to right represents first place through fifth place, respectively.

(1) $X_{11}, X_{12}, X_{21}, X_{22}, Z$

(2) $X_{21}, X_{22}, X_{11}, X_{12}, Z$

(3) $X_{11}, X_{21}, X_{12}, X_{22}, Z$

(4) $X_{21}, X_{11}, X_{22}, X_{12}, Z$

(5) $X_{11}, X_{21}, X_{22}, X_{12}, Z$

(6) $X_{21}, X_{11}, X_{12}, X_{22}, Z$

Thus, $P(Y_{(2)} < Z)$ is the sum of the probabilities of these six disjoint cases.

We now calculate the probability of each of these cases. We start by considering a race between three participants; X_{11}, X_{21}, and Z. A racer will be considered to have finished racing once it has defeated all participants against whom it is competing. In a way, we can consider this situation to be an exponential relay race with three teams. Team Y_1 has two successive members, X_{11} and X_{12}. Similarly, team Y_2 has two members. Team Z has one member. For team Y_1, once X_{11} finishes racing, X_{12} begins to race. Once X_{11} and X_{12} have both completed the race, team Y_1 is done racing. Analogous comments can be made for teams Y_2 and Z.

The probability of case (1) above can be derived as follows. We see that X_{11} is the first to complete the race. By Property 3.4, the probability that X_{11} wins against X_{21} and Z is

$$P(X_{11} = min(X_{11}, X_{21}, Z)) = P(X_{11} < min(X_{21}, Z)) = \frac{\lambda}{2\lambda + s}. \tag{3.1}$$

Once X_{11} finishes the race, X_{12} must begin to race. By Property 3.5, X_{21} and Z start the race from the beginning against X_{12}. That is, by the "memoryless property", it does not matter how much time has already passed during the race; it is as if X_{21} and Z had never started racing at all. In case (1), X_{12} is the next to finish. By Property 3.4, X_{12} does so with probability

$$P(X_{12} = min(X_{12}, X_{21}, Z)) = P(X_{12} < min(X_{21}, Z)) = \frac{\lambda}{2\lambda + s}. \tag{3.2}$$

Now team Y_1 has no participants remaining. Again, by the memoryless property of the exponential distribution, X_{21} and Z restart the race. In case (1), X_{21} must win against Z and does so with probability

$$P(X_{21} = min(X_{21}, Z)) = P(X_{21} < Z) = \frac{\lambda}{\lambda + s}. \tag{3.3}$$

Since X_{21} has completed the race, X_{22} must begin to race. X_{22} wins against Z with probability

$$P(X_{22} = min(X_{22}, Z)) = P(X_{22} < Z) = \frac{\lambda}{\lambda + s}. \tag{3.4}$$

Thus, the overall probability of case (1) is obtained by multiplying equations (3.1) through (3.4). That is,

$$\begin{aligned} P(\text{case (1)}) &= P(X_{11} = min(X_{11}, X_{21}, Z)) \times P(X_{12} = min(X_{12}, X_{21}, Z)) \\ &\quad \times P(X_{21} = min(X_{21}, Z)) \times P(X_{22} = min(X_{22}, Z)) \\ &= P(X_{11} < min(X_{21}, Z)) \times P(X_{12} < min(X_{21}, Z)) \\ &\quad \times P(X_{21} < Z) \times P(X_{22} < Z) \\ &= \frac{\lambda^4}{(2\lambda + s)^2(\lambda + s)^2}. \end{aligned} \tag{3.5}$$

The probabilities in (3.5) are actually conditional probabilities but the conditioning notation has been suppressed by application of the memoryless property.

A symmetrical argument, reversing all values of i, shows the overall probability of case (2) to also be the value given in result (3.5).

Similar arguments show the overall probabilities of cases (3) through (6) to each be

$$\left(\frac{\lambda}{2\lambda + s}\right)^3 \left(\frac{\lambda}{\lambda + s}\right). \tag{3.6}$$

By (3.5), (3.6) and Property 3.1,

$$L_{(2)}(s) = \frac{2\lambda^4}{(2\lambda + s)^2(\lambda + s)^2} + \frac{4\lambda^4}{(2\lambda + s)^3(\lambda + s)}. \tag{3.7}$$

Notation. Let $Y_1, Y_2, \ldots, Y_k$ represent k arbitrarily distributed Erlang random variables. The notation

$$(r_1\ r_2 \ldots\ r_k)$$

shall represent the state where $Y_1, Y_2, \ldots, Y_k$ have $r_1, r_2, \ldots, r_k$ respective stages remaining.

Diagram 3.1.

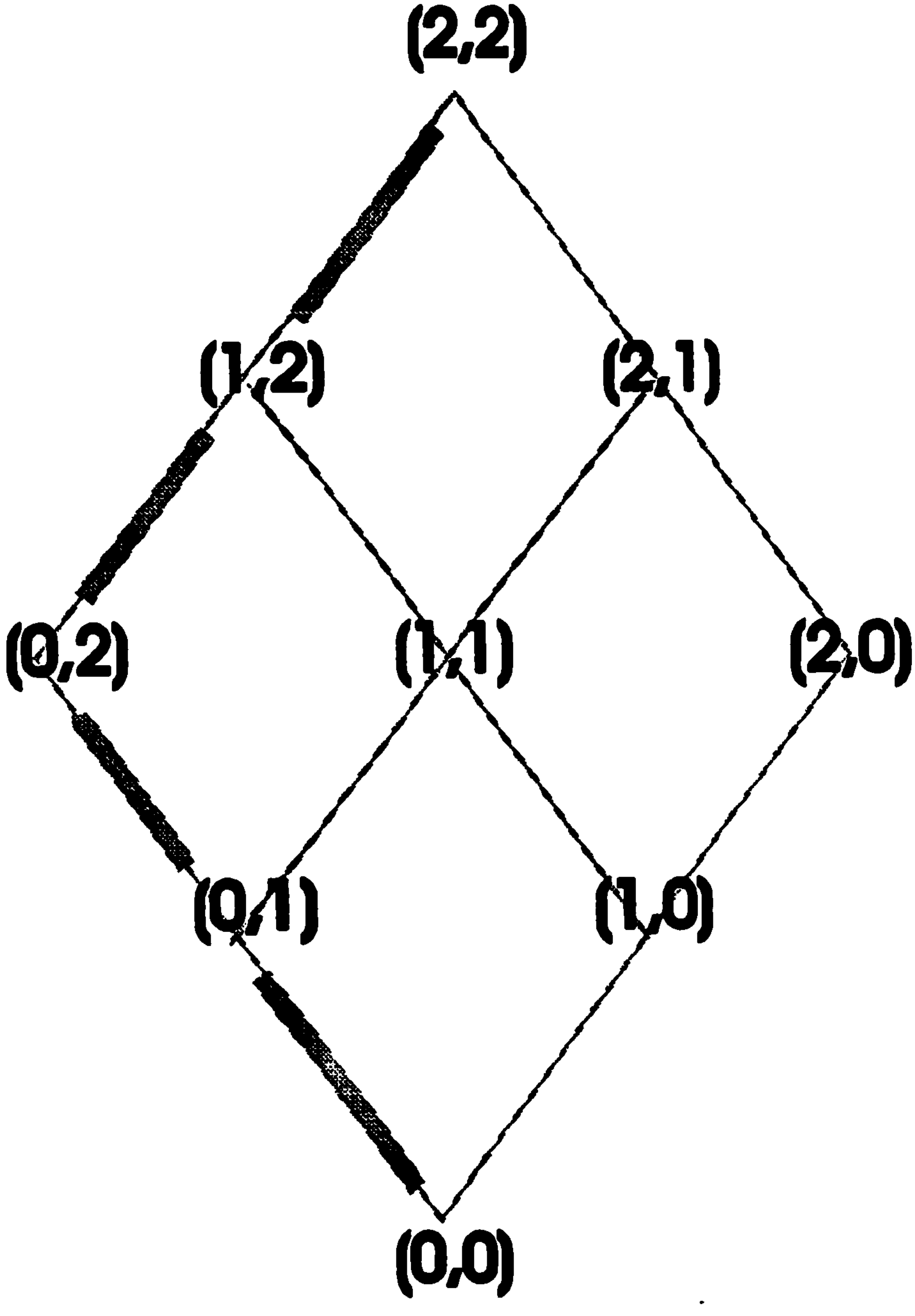


Diagram 3.1 gives a visual representation of all possible outcomes of the race where $Y_{(2)} < Z$. Under our analogy, the notation $(r_1\ r_2)$ represents the state where teams Y_1 and Y_2 have r_1 and r_2 members, respectively, remaining to finish the race. It shall be implicitly understood that the one member of team Z does not finish

racing until all other participants have finished the race. Each path from the state (2 2) to the state (0 0) represents one of the six possible outcomes listed above. The path representing case (1) is highlighted in Diagram 3.1.

Comparing (2.5) to (3.7), we see that the two results appear nothing alike. However, algebraic manipulation quickly reveals that the two results are equivalent. However, result (3.7) is more intuitive and more aesthetically pleasing than result (2.5).

Now suppose we wish to determine $L_{(1)}(s)$. By Property 3.1, this is equivalent to calculating $P(Y_{(1)} < Z)$. That is, the probability that at least one of Y_1 and Y_2 is less than or equal to Z. There are $\binom{2}{1}\left[\binom{2}{2}+\binom{3}{2}\binom{1}{1}\right]=6$ cases where this is possible, as follows.

(1) X_{11}, X_{12}

(2) X_{21}, X_{22}

(3) X_{11}, X_{21}, X_{12}

(4) X_{21}, X_{11}, X_{22}

(5) X_{11}, X_{21}, X_{22}

(6) X_{21}, X_{11}, X_{12}

Thus, $P(Y_{(1)} < Z)$ is the sum of the probabilities of these six disjoint cases. Note that it makes no difference which racers finish in which order so long as at least one team defeats team Z.

We calculate the probability of each of these cases in the same manner as before. The probability of case (1) equals the probability that X_{11} finishes first multiplied by the probability that X_{12} finishes second. That is,

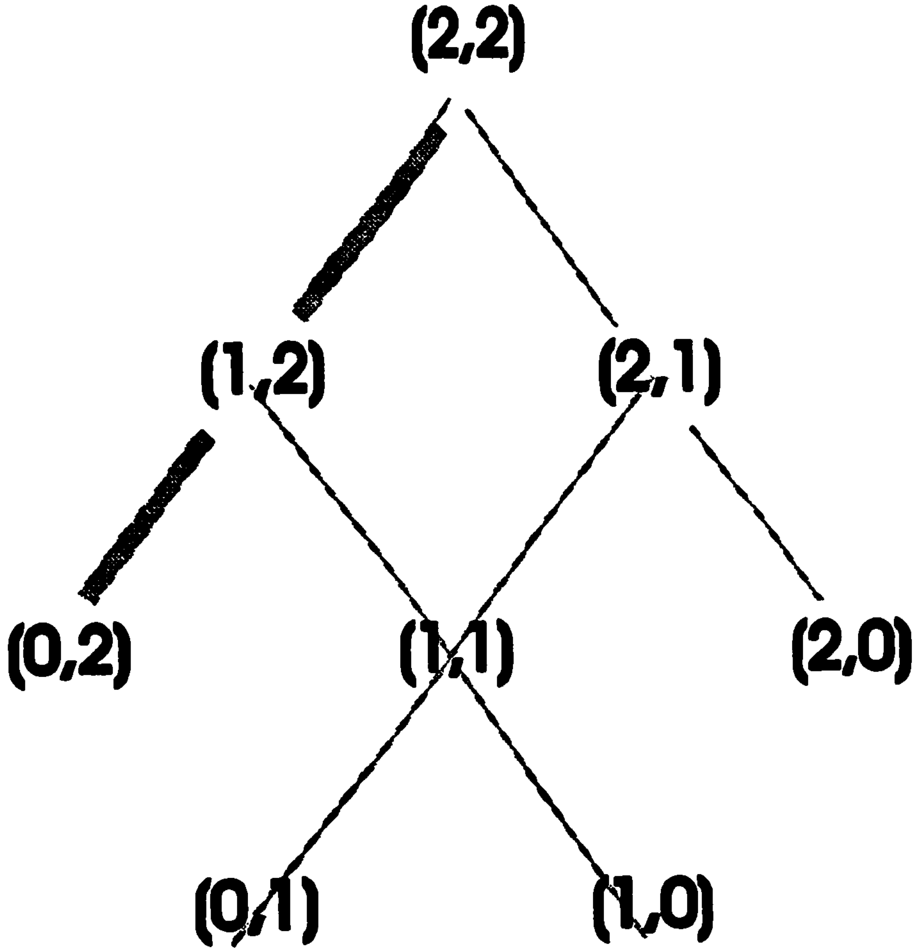
$$P(X_{11} = min(X_{11}, X_{21}, Z))P(X_{12} = min(X_{12}, X_{21}, Z)) = \left(\frac{\lambda}{2\lambda+s}\right)\left(\frac{\lambda}{2\lambda+s}\right) = \left(\frac{\lambda}{2\lambda+s}\right)^2. \tag{3.8}$$

A symmetrical argument, reversing all values of i, shows the overall probability of case (2) to also be the value given in result (3.8).

Similar arguments show the overall probabilities of cases (3) through (6) to each be

$$\left(\frac{\lambda}{2\lambda+s}\right)^3. \tag{3.9}$$

Diagram 3.2.



By (3.8), (3.9) and Property 3.1,

$$L_{(1)}(s) = \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3}. \tag{3.10}$$

Diagram 3.2 gives a visual representation of all possible outcomes of the race where $Y_{(1)} < Z$. It shall be implicitly understood that the one member of team Z does not finish racing until at least one of the other teams has completed the race. Each path from the state $(2\ 2)$ to a state of the form $(i\ 0)$ or $(0\ j)$ $(i, j = 1, 2)$ represents one of the six possible outcomes listed above. The path representing case (1) is highlighted in Diagram 3.2.

Comparing (2.6) to (3.10), we see that the two results are identical. Thus, at least for the first order statistic, the probabilistic approach does not improve the appearance of our end result.

We now generalize the preceding results to the case where we have two arbitrarily distributed Erlang random variables.

Property 3.6. *Let Y_1 and Y_2 represent two Erlang random variables with*

$$Y_i = \sum_{j=1}^{n_i} X_{ij}, n_i \in \mathbb{N}, i = 1, 2$$

and where $X_{ij} \sim exp(\lambda_i), i = 1, 2, j = 1, 2, \ldots, n_i$. Then,

$$L_{(1)}(s) = \sum_{r_1=1}^{n_1} \binom{n_1+n_2-r_1-1}{n_2-1} \frac{\lambda_1^{n_1-r_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_1}} + \sum_{r_2=1}^{n_2} \binom{n_1+n_2-r_2-1}{n_1-1} \frac{\lambda_1^{n_1}\lambda_2^{n_2-r_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_2}}$$

and

$$L_{(2)}(s) = \sum_{r_1=1}^{n_1} \binom{n_1+n_2-r_1-1}{n_2-1} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_1}(\lambda_1+s)^{r_1}} + \sum_{r_2=1}^{n_2} \binom{n_1+n_2-r_2-1}{n_1-1} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_2}(\lambda_2+s)^{r_2}}.$$

Proof. We begin in the state $(n_1\ n_2)$. If we decrease the i-th component by a (i.e. complete a stages of the i-th Erlang random variable), we shall say that we have moved a steps in the direction D_i, $i = 1, 2$. When we say we have reached a state, we shall mean that the state has been reached prior to the catastrophe Z, $Z \sim exp(s)$. For example, suppose we reach the state $(r_1\ r_2)$. Then we have moved a total of $n_1 - r_1$ steps in direction D_1 and $n_2 - r_2$ steps in direction D_2 prior to catastrophe. Also, when we say we have reached a state where the i-th component is zero $(i = 1, 2)$, we shall mean that this is the first state in which that component has been zero unless it was stated to be zero in a previous state.

To calculate $L_{(1)}(s) = P(Y_{(1)} < Z)$, we want to ensure that at least one of Y_1 and Y_2 has occurred prior to Z. That is, one of the two components of the state reaches zero prior to catastrophe. Now,

$$
\begin{aligned}
P(Y_{(1)} < Z) &= P(\text{at least one of } Y_1, Y_2 \text{ is less than } Z) \\
&= P(Y_1 = min(Y_1, Y_2, Z) \text{ or } Y_2 = min(Y_1, Y_2, Z)) \\
&= P(Y_1 = min(Y_1, Y_2, Z)) + P(Y_2 = min(Y_1, Y_2, Z)) \\
&\qquad \text{(almost surely disjoint events)} \\
&= P(\text{we reach } (0\ r_2), r_2 = 1, 2, \ldots, n_2) \\
&\qquad + P(\text{we reach } (r_1\ 0), r_1 = 1, 2, \ldots, n_1) \\
&= \sum_{r_1=1}^{n_1} P(\text{we reach } (r_1\ 0)) + \sum_{r_2=1}^{n_2} P(\text{we reach } (0\ r_2)).
\end{aligned} \tag{3.11}
$$

In order to reach the state $(r_1\ 0)$, we must first reach $(r_1\ 1)$ and then $(r_1\ 0)$. From $(n_1\ n_2)$ to $(r_1\ 0)$, there were a total of $n_1 - r_1$ steps in direction D_1 and n_2 steps in direction D_2. However, we must reserve one step in the direction D_2 until the very last step since that is when the zero component must appear for the first time. The total number of ways of rearranging $n_1 - r_1$ D_1's and $n_2 - 1$ D_2's is

$$\frac{(n_1 - r_1 + n_2 - 1)!}{(n_1 - r_1)!(n_2 - 1)!} = \binom{n_1 + n_2 - r_1 - 1}{n_2 - 1}. \tag{3.12}$$

The probability of reaching the state (r_1 0) along any of these paths can be obtained by the same reasoning used in the previous example. That is, among the three exponential variables X_{1j_1}, X_{2j_2} and Z, Property 3.4 yields

$$P(X_{1j_1} = min(X_{1j_1}, X_{2j_2}, Z)) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + s}$$

and

$$P(X_{2j_2} = min(X_{1j_1}, X_{2j_2}, Z)) = \frac{\lambda_2}{\lambda_1 + \lambda_2 + s},$$

for $j_1 = 1, 2, \ldots, n_1$ and $j_2 = 1, 2, \ldots, n_2$. Since we complete $n_1 - r_1$ stages of Y_1 and n_2 stages of Y_2, the probability of reaching (r_1 0) in a given way is

$$\left(\frac{\lambda_1}{\lambda_1 + \lambda_2 + s}\right)^{n_1 - r_1} \left(\frac{\lambda_2}{\lambda_1 + \lambda_2 + s}\right)^{n_2} = \frac{{\lambda_1}^{n_1 - r_1} {\lambda_2}^{n_2}}{(\lambda_1 + \lambda_2 + s)^{n_1 + n_2 - r_1}}. \tag{3.13}$$

We have a similar calculation for the probability of reaching state (0 r_2).

By (3.12) and (3.13), for fixed r_1, we obtain

$$P(\text{we reach } (r_1\ 0)) = \binom{n_1 + n_2 - r_1 - 1}{n_2 - 1} \frac{{\lambda_1}^{n_1 - r_1} {\lambda_2}^{n_2}}{(\lambda_1 + \lambda_2 + s)^{n_1 + n_2 - r_1}}. \tag{3.14}$$

Similarly, for fixed r_2, we obtain

$$P(\text{we reach } (0\ r_2)) = \binom{n_1 + n_2 - r_2 - 1}{n_1 - 1} \frac{{\lambda_1}^{n_1} {\lambda_2}^{n_2 - r_2}}{(\lambda_1 + \lambda_2 + s)^{n_1 + n_2 - r_2}}. \tag{3.15}$$

Applying results (3.14) and (3.15) to (3.11), the result for $L_{(1)}(s)$ follows.

To obtain $L_{(2)}(s)$=$P(Y_{(2)} < Z)$, we wish to ensure that both Y_1 and Y_2 have occurred prior to Z. Thus,

$$\begin{aligned}
P(Y_{(2)} < Z) &= P(\text{both } Y_1 \text{ and } Y_2 \text{ occur before } Z) \\
&= P(\text{we reach } (0\ 0)) \\
&= P(Y_1 < Y_2 < Z \text{ or } Y_2 < Y_1 < Z) \\
&= P(Y_1 < Y_2 < Z) + P(Y_2 < Y_1 < Z) \\
&\qquad \text{(almost surely disjoint events)} \\
&= P(\text{we reach } (0\ r_2) \text{ then } (0\ 0), r_2 = 1, 2, \ldots, n_2) \\
&\qquad + P(\text{we reach } (r_1\ 0) \text{ then } (0\ 0), r_1 = 1, 2, \ldots, n_1) \\
&= \sum_{r_2=1}^{n_2} P(\text{we reach } (0\ r_2) \text{ then } (0\ 0)) \\
&\qquad + \sum_{r_1=1}^{n_1} P(\text{we reach } (r_1\ 0) \text{ then } (0\ 0)) \\
&= \sum_{r_1=1}^{n_1} P(\text{we reach } (r_1\ 0)) P(\text{we reach } (0\ 0) \text{ from } (r_1\ 0)) \\
&\qquad + \sum_{r_2=1}^{n_2} P(\text{we reach } (0\ r_2)) P(\text{we reach } (0\ 0) \text{ from } (0\ r_2)).
\end{aligned} \tag{3.16}$$

Now $P(\text{we reach } (r_1\ 0))$, for fixed r_1, and $P(\text{we reach } (0\ r_2))$, for fixed r_2, are given by results (3.14) and (3.15) respectively. Also, starting at state $(r_1\ 0)$, there is only one way to reach state $(0\ 0)$; that is, we must complete the remaining r_1 stages of Y_1 prior to Z occurring. But

$$P(X_{1i} = min(X_{1i}, Z)) = \frac{\lambda_1}{\lambda_1 + s}$$

for $i = 1, 2, \ldots, n_1$ and so, for fixed r_1,

$$P(\text{reach } (0\ 0) \text{ from } (r_1\ 0)) = \left(\frac{\lambda_1}{\lambda_1 + s}\right)^{r_1}. \tag{3.17}$$

Similarly, for fixed r_2,

$$P(\text{reach } (0\ 0) \text{ from } (0\ r_2)) = \left(\frac{\lambda_2}{\lambda_2+s}\right)^{r_2}. \tag{3.18}$$

By results (3.14) and (3.17), for fixed r_1,

$$\begin{aligned} &P(\text{we reach } (r_1\ 0) \text{ then } (0\ 0)) \\ &= \binom{n_1+n_2-r_1-1}{n_2-1} \frac{\lambda_1^{n_1-r_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_1}} \cdot \left(\frac{\lambda_1}{\lambda_1+s}\right)^{r_1} \\ &= \binom{n_1+n_2-r_1-1}{n_2-1} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_1}(\lambda_1+s)^{r_1}}. \end{aligned} \tag{3.19}$$

Similarly, by results (3.15) and (3.18), for fixed r_2,

$$\begin{aligned} &P(\text{we reach } (0\ r_2) \text{ then } (0\ 0)) \\ &= \binom{n_1+n_2-r_2-1}{n_1-1} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{(\lambda_1+\lambda_2+s)^{n_1+n_2-r_2}(\lambda_2+s)^{r_2}}. \end{aligned} \tag{3.20}$$

Applying results (3.19) and (3.20) to (3.16), the result for $L_{(2)}(s)$ follows.■

Notation. When we say that a variable is Erlang with parameters (n, λ) we shall mean that it has n stages which are each exponential with rate parameter λ.

Example 3.1. Let Y_1 and Y_2 represent two Erlang random variables with respective parameters $(4,3)$ and $(2,2)$. Calculate $L_{(1)}(5)$ and $L_{(2)}(5)$.

Solution. We have that $n_1 = 4, n_2 = 2, \lambda_1 = 3, \lambda_2 = 2$ and $s = 5$. Then by Property 3.6 we have

$$\begin{aligned} L_{(1)}(5) &= \sum_{r_1=1}^{4} \binom{5-r_1}{1} \frac{3^{4-r_1}2^2}{10^{6-r_1}} + \sum_{r_2=1}^{2} \binom{5-r_2}{3} \frac{3^4 2^{2-r_2}}{10^{6-r_2}} \\ &= \binom{4}{1}\frac{3^3 2^2}{10^5} + \binom{3}{1}\frac{3^2 2^2}{10^4} + \binom{2}{1}\frac{3^1 2^2}{10^3} + \binom{1}{1}\frac{3^0 2^2}{10^2} \\ &\quad + \binom{4}{3}\frac{3^4 2^1}{10^5} + \binom{3}{3}\frac{3^4 2^0}{10^4} \\ &= \frac{432+1080+2400+4000+648+810}{10000} \\ &= 0.9370 \end{aligned}$$

and

$$
\begin{aligned}
L_{(2)}(5) &= \sum_{r_1=1}^{4}\binom{5-r_1}{1}\frac{3^4 2^2}{10^{6-r_1}8^{r_1}} + \sum_{r_2=1}^{2}\binom{5-r_2}{3}\frac{3^4 2^2}{10^{6-r_2}7^{r_2}} \\
&= \binom{4}{1}\frac{3^4 2^2}{10^5 8^1} + \binom{3}{1}\frac{3^4 2^2}{10^4 8^2} + \binom{2}{1}\frac{3^4 2^2}{10^3 8^3} + \binom{1}{1}\frac{3^4 2^2}{10^2 8^4} \\
&\quad + \binom{4}{3}\frac{3^4 2^2}{10^5 7^1} + \binom{3}{3}\frac{3^4 2^2}{10^4 7^2} \\
&= \frac{663552 + 622080 + 518400 + 324000}{409600000} + \frac{9072 + 3240}{4900000} \\
&\approxeq 0.0077. \blacksquare
\end{aligned}
$$

Corollary 3.1. *In Property 3.6, if $n_1 = n_2 = n$ and $\lambda_1 = \lambda_2 = \lambda$, then we obtain the more compact results*

$$
L_{(1)}(s) = 2\sum_{r=1}^{n}\binom{2n-r-1}{n-1}\left(\frac{\lambda}{2\lambda+s}\right)^{2n-r}
$$

and

$$
L_{(2)}(s) = 2\sum_{r=1}^{n}\binom{2n-r-1}{n-1}\left(\frac{\lambda}{2\lambda+s}\right)^{2n-r}\left(\frac{\lambda}{\lambda+s}\right)^{r}.
$$

We shall now verify that the formulas given in Property 3.6 give the same results as those obtained in the two variable example solved earlier. Setting $n_1 = n_2 = n = 2$ and $\lambda_1 = \lambda_2 = \lambda$, we use Corollary 3.1 to obtain

$$
\begin{aligned}
L_{(1)}(s) &= 2\sum_{r=1}^{2}\binom{4-r-1}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{4-r} \\
&= 2\left[\binom{2}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{3} + \binom{1}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{2}\right] \\
&= \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3}
\end{aligned}
$$

and

$$L_{(2)}(s) = 2\sum_{r=1}^{2}\binom{4-r-1}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{4-r}\left(\frac{\lambda}{\lambda+s}\right)^{r}$$
$$= 2\left[\binom{2}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{3}\left(\frac{\lambda}{\lambda+s}\right)+\binom{1}{1}\left(\frac{\lambda}{2\lambda+s}\right)^{2}\left(\frac{\lambda}{\lambda+s}\right)^{2}\right]$$
$$= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2}+\frac{4\lambda^4}{(2\lambda+s)^3(\lambda+s)}.$$

We see that these are results (3.10) and (3.7) respectively.

Example 3.2. Let Y_1 and Y_2 represent two Erlang random variables with common parameters $(3, 10)$. Calculate $L_{(1)}(20)$.

Solution. We have that $n = 3, \lambda = 10$ and $s = 20$. Then by Corollary 3.1 we have

$$L_{(1)}(20) = 2\sum_{r=1}^{3}\binom{5-r}{2}\left(\frac{10}{40}\right)^{6-r}$$
$$= 2\left[\binom{4}{2}\left(\frac{10}{40}\right)^{5}+\binom{3}{2}\left(\frac{10}{40}\right)^{4}+\binom{2}{2}\left(\frac{10}{40}\right)^{3}\right]$$
$$= \frac{12}{1024}+\frac{6}{256}+\frac{2}{64}$$
$$= \frac{17}{256}$$
$$\approxeq 0.0664. \blacksquare$$

We now generalize the preceding results to the case where we have an arbitrary number of arbitrarily distributed Erlang random variables.

Notation. Let $Y_1, Y_2, \ldots, Y_k$ represent k arbitrarily distributed Erlang random variables with $n_1, n_2, \ldots, n_k$ stages respectively. The notation

$$(r_1^{(m)}\ r_2^{(m)} \ldots\ r_k^{(m)})$$

shall represent the state where $Y_1, Y_2, \ldots, Y_k$ have $r_1, r_2, \ldots, r_k$ respective stages remaining just after m of the k Erlang variables have zero stages remaining; i.e. just after m of k state components have reached zero. Define $r_i^{(0)} = n_i$, $i = 1, 2, \ldots, k$.

In order to obtain the Laplace transform, $L_{(u)}(s)$, of the u-th order statistic, we will consider all paths from the initial state $(n_1\ n_2 \ldots n_k)$ to states where exactly u of the k components are zero. Let the positions of the zeros, as they appear in order, be $i_1, i_2, \ldots, i_u$.

For some $m \in \{1, 2, \ldots, k\}$, define notation for multinomial coefficients as follows. Let

$$\binom{C}{C_1, \ldots, C_{i_m}, \ldots, C_k} = \frac{C!}{C_1! \cdots C_{i_m}! \cdots C_k!},$$

where $C_i = r_i^{(m-1)} - r_i^{(m)}$ for $i \neq i_m, i = 1, 2, \ldots, C_{i_m} = r_{i_m}^{(m-1)} - 1$, and $C = \sum_{i=1}^{k} C_i$ and let

$$\binom{D}{D_m, D_{m+1}, \ldots, D_k} = \frac{D!}{D_{m+1}! \cdots D_k!},$$

where $D_i = r_i^{(m-1)} - r_i^{(m)}, i = m+1, \ldots, k, D_m = r_m^{(m-1)} - 1$, and $D = \sum_{i=m}^{k} D_i$.

Property 3.7. *Let $Y_1, Y_2, \ldots, Y_k$ represent k Erlang random variables with*

$$Y_i = \sum_{j=1}^{n_i} X_{ij}, n_i \in \mathbb{N}, i = 1, 2, \ldots, k$$

where $X_{ij} \sim exp(\lambda_i), i = 1, 2, \ldots, k; j = 1, 2, \ldots, n_i$. Let $i_1, i_2, \ldots, i_u$ be the positions of the zeros as they appear in the k-tuple state for a given path. Thus, $r_{i_m}^{(m)} = 0$, $m = 1, 2, \ldots, k$. For $1 \leq m \leq u$, define $A(m) = \{1, 2, \ldots, k\} \setminus \{i_1, i_2, \ldots, i_{m-1}\}$, for some $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, k\}$ and $m \leq k$. Define the function f_m by

$$f_m = f_m(i_1, i_2, \ldots, i_m, r_1^{(m-1)}, r_2^{(m-1)}, \ldots, r_k^{(m-1)}, r_1^{(m)}, r_2^{(m)}, \ldots, r_k^{(m)})$$

$$= \binom{C}{C_1, \ldots, C_{i_m}, \ldots, C_k} \prod_{i=1}^{k} \left(\frac{\lambda_i}{\sum_{a \in A(m)} \lambda_a + s} \right)^{r_i^{(m-1)} - r_i^{(m)}}.$$

Then,

$$L_{(u)}(s) = \sum_{i_1 \in A(1)} \sum_{\underset{\sim}{r}^{(1)}} f_1 \sum_{i_2 \in A(2)} \sum_{\underset{\sim}{r}^{(2)}} f_2 \cdots \sum_{i_u \in A(u)} \sum_{\underset{\sim}{r}^{(u)}} f_u$$
$$= \prod_{m=1}^{u} \left(\sum_{i_m \in A(m)} \sum_{\underset{\sim}{r}^{(m)}} f_m \right), u = 1, 2, \ldots, k,$$

where $\prod_{m=1}^{u} \left(\sum_{i_m \in A(m)} \sum_{\underset{\sim}{r}^{(m)}} f_m \right)$ *is a nested summation, where* $\sum_{\underset{\sim}{r}^{(m)}}$ *represents the multiple summation over all possible values of* $r_1^{(m)}, r_2^{(m)}, \ldots, r_k^{(m)}$ *(i.e. over all possible states* $\underset{\sim}{r}^{(m)}$*) where* $r_{i_1}^{(m)}, r_{i_2}^{(m)}, \ldots, r_{i_m}^{(m)}$ *all equal zero and the values* $r_1^{(m-1)}, r_2^{(m-1)}, \ldots, r_k^{(m-1)}$ *are known. That is,*

$$\sum_{\underset{\sim}{r}^{(m)}} = \sum_{b \in A(m+1)} \sum_{r_b^{(m)}=1}^{r_b^{(m-1)}}$$

where we define

$$\sum_{\underset{\sim}{r}^{(k)}} f_k = f_k = \left(\frac{\lambda_{i_k}}{\lambda_{i_k} + s} \right)^{r_{i_k}^{(k-1)}}.$$

Proof. We begin in the state $(n_1\, n_2 \ldots n_k) = (r_1^{(0)}\, r_2^{(0)} \ldots r_k^{(0)})$. If we decrease the i-th component by s (i.e. complete s stages of the i-th Erlang random variable), we shall say that we have moved s steps in the direction D_i, $i = 1, 2, \ldots, k$. When we say we have reached a state, we shall mean that the state has been reached prior to catastrophe. When we say we have reached a state where the i-th component is zero ($i = 1, 2, \ldots, k$), we shall mean that this is the first state in which that component has been zero unless it was stated to be zero in a previous state.

To calculate $L_{(u)}(s) = P(Y_{(u)} < Z)$ for $u = 1, 2, \ldots, k$, we want to ensure that at least u of the k Erlang variables have occurred prior to Z. That is, u of the k components of the state reach zero prior to catastrophe. Let $I = \{1, 2, \ldots, k\}$. For $m \in I$, let $f_m = P(\text{component } i_m \text{ reaches zero at state } \underset{\sim}{r}^{(m)} | \underset{\sim}{r}^{(m-1)})$.

Now,

$$
\begin{aligned}
P(Y_{(u)} < Z) &= P(\text{at least } u \text{ of } Y_1, Y_2, \ldots, Y_k \text{ are less than } Z) \\
&= P(\text{we reach a state with } u \text{ zeros}) \\
&= \sum_{i_1=1}^{k} \sum_{\underline{r}^{(1)}} P(\text{component } i_1 \text{ reaches zero at state } \underline{r}^{(1)}) \\
&\quad \times P(\text{u-1 more components reach zero} | \underline{r}^{(1)}) \\
&= \sum_{i_1=1}^{k} \sum_{\underline{r}^{(1)}} f_1 P(\text{u-1 more components reach zero} | \underline{r}^{(1)}) \\
&= \sum_{i_1=1}^{k} \sum_{\underline{r}^{(1)}} f_1 \sum_{i_2 : i_2 \neq i_1} \sum_{\underline{r}^{(2)}} P(\text{component } i_2 \text{ reaches zero at state } \underline{r}^{(2)} | \underline{r}^{(1)}) \\
&\quad \times P(\text{u-2 more components reach zero} | \underline{r}^{(2)}) \\
&= \ldots \\
&= \sum_{i_1 \in A(1)} \sum_{\underline{r}^{(1)}} f_1 \sum_{i_2 \in A(2)} \sum_{\underline{r}^{(2)}} f_2 \cdots \sum_{i_u \in A(u)} \sum_{\underline{r}^{(u)}} f_u.
\end{aligned}
$$

Now f_m can be computed as follows for $m = 1, 2, \ldots u$. Note that $r_{i_m}^{(m-1)} \neq 0$. To reach state $(r_1^{(m)} \ldots r_{i_m-1}^{(m)}\ 0\ r_{i_m+1}^{(m)} \ldots r_k^{(m)})$ from state $(r_1^{(m-1)} \ldots r_k^{(m-1)})$ we moved a total of $r_i^{(m-1)} - r_i^{(m)}$ steps in direction D_i for $i = 1, 2, \ldots, k$. The number of ways in which we can do this, saving one step in direction D_{i_m} until the last step, is

$$
\frac{(r_{i_m}^{(m-1)} - 1 + \sum_{a \neq i_m} (r_a^{(m-1)} - r_a^{(m)}))!}{(r_{i_m}^{(m-1)} - 1)! \prod_{a \neq i_m} (r_a^{(m-1)} - r_a^{(m)})!} = \binom{C}{C_1, \ldots, C_{i_m}, \ldots, C_k} \tag{3.21}
$$

with probability along any of these paths of

$$
\prod_{i=1}^{k} \left(\frac{\lambda_i}{\sum_{a \in A(m)} \lambda_a + s} \right)^{r_i^{(m-1)} - r_i^{(m)}}. \tag{3.22}
$$

Thus, by (3.21) and (3.22) we obtain

$$
\begin{aligned}
f_m &= P(\text{component } i_m \text{ reaches zero at state } \underline{r}^{(m)} | \underline{r}^{(m-1)}) \\
&= (\text{the number of ways to reach } \underline{r}^{(m)} \text{ from } \underline{r}^{(m-1)}) \\
&\qquad \times P(\text{we reach } \underline{r}^{(m)} \text{ from } \underline{r}^{(m-1)} \text{ along any path}) \\
&= \binom{C}{C_1, \ldots, C_{i_m}, \ldots, C_k} \prod_{i=1}^{k} \left(\frac{\lambda_i}{\sum_{a \in A(m)} \lambda_a + s} \right)^{r_i^{(m-1)} - r_i^{(m)}} . \blacksquare
\end{aligned}
$$

Example 3.3. Let Y_1, Y_2 and Y_3 represent three Erlang random variables with respective parameters $(2,5), (3,5)$ and $(1,2)$. Calculate $L_{(1)}(1)$.

Solution. We have that $k = 3, n_1 = 2, n_2 = 3, n_3 = 1, \lambda_1 = \lambda_2 = 5$, and $\lambda_3 = 2$. Note that since we want to calculate $L_{(1)}(1)$, our (artificial) catastrophe variable is exponentially distrbuted with rate parameter s=1. Then by Property 3.7 we have

$$
\begin{aligned}
L_{(1)}(1) &= \sum_{i_1 \in A(1)} \sum_{\underline{r}^{(1)}} f_1 \\
&= \sum_{i_1 \in A(1)} \sum_{\underline{r}^{(1)}} \binom{C}{C_1, C_2, C_3} \prod_{i=1}^{3} \left(\frac{\lambda_i}{\sum_{a=1}^{3} \lambda_a + s} \right)^{r_i^{(0)} - r_i^{(1)}} \\
&= \sum_{i_1 \in A(1)} \sum_{\underline{r}^{(1)}} \binom{C}{C_1, C_2, C_3} \prod_{i=1}^{3} \left(\frac{\lambda_i}{13} \right)^{n_i - r_i^{(1)}} \\
&= \sum_{i_1=1}^{3} \sum_{b \in A(2)} \sum_{r_b^{(1)}=1}^{n_b} \binom{C}{C_1, C_2, C_3} \prod_{i=1}^{3} \left(\frac{\lambda_i}{13} \right)^{n_i - r_i^{(1)}} . \\
&= \sum_{r_2^{(1)}=1}^{3} \sum_{r_3^{(1)}=1}^{1} \binom{C}{1, C_2, C_3} \left(\frac{5}{13} \right)^{n_1 + n_2 - r_2^{(1)}} \left(\frac{2}{13} \right)^{n_3 - r_3^{(1)}} \\
&\quad + \sum_{r_1^{(1)}=1}^{2} \sum_{r_3^{(1)}=1}^{1} \binom{C}{C_1, 2, C_3} \left(\frac{5}{13} \right)^{n_1 - r_1^{(1)} + n_2} \left(\frac{2}{13} \right)^{n_3 - r_3^{(1)}} \\
&\quad + \sum_{r_1^{(1)}=2}^{2} \sum_{r_2^{(1)}=1}^{3} \binom{C}{C_1, C_2, 0} \left(\frac{5}{13} \right)^{n_1 - r_1^{(1)} + n_2 - r_2^{(1)}} \left(\frac{2}{13} \right)^{n_3}
\end{aligned}
$$

$$
\begin{aligned}
&= \binom{3}{1,2,0}\frac{5^4 2^0}{13^4} + \binom{2}{1,1,0}\frac{5^3 2^0}{13^3} + \binom{1}{1,0,0}\frac{5^2 2^0}{13^2} + \binom{3}{1,2,0}\frac{5^4 2^0}{13^4} \\
&+ \binom{2}{0,2,0}\frac{5^3 2^0}{13^3} + \binom{3}{1,2,0}\frac{5^3 2^1}{13^4} + \binom{2}{1,1,0}\frac{5^2 2^1}{13^3} + \binom{2}{0,2,0}\frac{5^2 2^1}{13^3} \\
&+ \binom{1}{0,1,0}\frac{5^1 2^1}{13^2} + \binom{1}{1,0,0}\frac{5^1 2^1}{13^2} + \binom{0}{0,0,0}\frac{5^0 2^1}{13^1} \\
&= \frac{1875}{28561} + \frac{250}{2197} + \frac{25}{169} + \frac{1875}{28561} + \frac{125}{2197} + \frac{750}{28561} + \frac{100}{2197} + \frac{50}{2197} \\
&\quad + \frac{10}{169} + \frac{10}{169} + \frac{2}{13} \\
&= \frac{23324}{28561} \\
&\approx 0.8166. \blacksquare
\end{aligned}
$$

Corollary 3.2. *In Property 3.7, let $\lambda_i = \lambda$ for $i = 1, 2, \ldots, k$. Define the function g_m by*

$$
\begin{aligned}
g_m =& g_m(i_1, i_2, \ldots, i_m, r_1^{(m-1)}, r_2^{(m-1)}, \ldots, r_k^{(m-1)}, r_1^{(m)}, r_2^{(m)}, \ldots, r_k^{(m)}) \\
=& \binom{C}{C_1, \ldots, C_{i_m}, \ldots, C_k}\left(\frac{\lambda}{(k-m+1)\lambda + s}\right)^{\sum_{i=1}^{k}(r_i^{(m-1)} - r_i^{(m)})}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
L_{(u)}(s) &= \sum_{i_1 \in A(1)} \sum_{\underline{r}^{(1)}} g_1 \sum_{i_2 \in A(2)} \sum_{\underline{r}^{(2)}} g_2 \cdots \sum_{i_u \in A(u)} \sum_{\underline{r}^{(u)}} g_u \\
&= \prod_{m=1}^{u} \left(\sum_{i_m \in A(m)} \sum_{\underline{r}^{(m)}} g_m \right), u = 1, 2, \ldots, k,
\end{aligned}
$$

where $\prod_{m=1}^{u}\left(\sum_{i_m \in A(m)} \sum_{\underline{r}^{(m)}} g_m\right)$ is a nested summation, where $\sum_{\underline{r}^{(m)}}$ is as defined in Property 3.7 and we define

$$
\sum_{\underline{r}^{(k)}} g_k = g_k = \left(\frac{\lambda}{\lambda + s}\right)^{r_{i_k}^{(k-1)}}.
$$

Corollary 3.3. *In Property 3.7, let $n_i = n$ and $\lambda_i = \lambda$ for $i = 1, 2, \ldots, k$. Define the function h_m by*

$$h_m = h_m(r_m^{(m-1)}, r_{m+1}^{(m-1)}, \ldots, r_k^{(m-1)}, r_{m+1}^{(m)}, r_{m+2}^{(m)}, \ldots, r_k^{(m)})$$

$$= \binom{D}{D_m, D_{m+1}, \ldots, D_k} \left(\frac{\lambda}{(k-m+1)\lambda + s}\right)^{\sum_{i=m}^{k}(r_i^{(m-1)} - r_i^{(m)})}.$$

Then,

$$L_{(u)}(s) = \frac{k!}{(k-u)!} \sum_{\mathfrak{r}^{(1)}} h_1 \sum_{\mathfrak{r}^{(2)}} h_2 \cdots \sum_{\mathfrak{r}^{(u)}} h_u$$

$$= \frac{k!}{(k-u)!} \sum_{b_1=2}^{k} \sum_{r_{b_1}{}^{(1)}=1}^{r_{b_1}{}^{(0)}} h_1 \sum_{b_2=3}^{k} \sum_{r_{b_2}{}^{(2)}=1}^{r_{b_2}{}^{(1)}} h_2 \cdots \sum_{b_u=u+1}^{k} \sum_{r_{b_u}{}^{(u)}=1}^{r_{b_u}{}^{(u-1)}} h_u$$

$$= \prod_{m=1}^{u} \left[(k-m+1) \sum_{b_m=m+1}^{k} \sum_{r_{b_m}{}^{(m)}=1}^{r_{b_m}{}^{(m-1)}} h_m \right], u = 1, 2, \ldots, k,$$

where $\prod_{m=1}^{u}\left[(k-m+1)\sum_{b_m=m+1}^{k}\sum_{r_{b_m}{}^{(m)}=1}^{r_{b_m}{}^{(m-1)}} h_m\right]$ is a nested summation, where $\sum_{\mathfrak{r}^{(m)}}$ is as defined in Property 3.7 and we define

$$\sum_{\mathfrak{r}^{(k)}} h_k = h_k = \left(\frac{\lambda}{\lambda + s}\right)^{r_k^{(k-1)}}.$$

Proof. We will let the first zero to appear in the first state component, the second zero in the second state component, and so on, finally putting the u-th zero in the u-th state component. Using the style of argument from the proof of Property 3.7 with i_m fixed equal to m for $m = 1, 2, \ldots, u$, the result follows.■

We shall now verify that formula (3.22) given in Property 3.7 gives the same results as those obtained in the two variable example solved earlier. Setting $k = 2$,

$n_1 = n_2 = n = 2$, $\lambda_1 = \lambda_2 = \lambda$ and $u = 1$, we use Corollary 3.3 to obtain

$$\begin{aligned}
L_{(1)}(s) &= 2\sum_{\Gamma^{(1)}} h_1 \\
&= 2\sum_{b_1=2}^{2}\sum_{r_{b_1}^{(1)}=1}^{r_{b_1}^{(0)}} \binom{D}{D_1, D_2}\left(\frac{\lambda}{k\lambda+s}\right)^{\sum_{i=1}^{2}(r_i^{(0)}-r_i^{(1)})} \\
&= 2\sum_{r_2^{(1)}=1}^{n} \binom{2n-1-r_2^{(1)}}{n-r_2^{(1)}}\left(\frac{\lambda}{2\lambda+s}\right)^{\sum_{i=1}^{2}(n-r_i^{(1)})} \\
&= 2\sum_{r_2^{(1)}=1}^{2} \binom{3-r_2^{(1)}}{2-r_2^{(1)}}\left(\frac{\lambda}{2\lambda+s}\right)^{4-r_2^{(1)}} \\
&= 2\left[\binom{2}{1}\left(\frac{\lambda}{2\lambda+s}\right)^3 + \binom{1}{0}\left(\frac{\lambda}{2\lambda+s}\right)^2\right] \\
&= \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3}
\end{aligned}$$

which agrees with (2.6), (3.10) and the result given by Corollary 3.1.

Also, setting $u = 2$ we obtain

$$\begin{aligned}
L_{(2)}(s) &= 2\sum_{\Gamma^{(1)}} h_1(1)\sum_{\Gamma^{(2)}} h_2 \\
&= 2\sum_{\Gamma^{(1)}} h_1 h_2 \\
&= 2\sum_{b_1=2}^{2}\sum_{r_{b_1}^{(1)}=1}^{r_{b_1}^{(0)}} \binom{D}{D_1, D_2}\left(\frac{\lambda}{k\lambda+s}\right)^{\sum_{i=1}^{2}(r_i^{(0)}-r_i^{(1)})} h_2 \\
&= 2\sum_{r_2^{(1)}=1}^{2} \binom{2n-1-r_2^{(1)}}{n-r_2^{(1)}}\left(\frac{\lambda}{2\lambda+s}\right)^{\sum_{i=1}^{2}(n-r_i^{(1)})}\left(\frac{\lambda}{\lambda+s}\right)^{r_2^{(1)}} \\
&= 2\sum_{r_2^{(1)}=1}^{2} \binom{3-r_2^{(1)}}{2-r_2^{(1)}}\left(\frac{\lambda}{2\lambda+s}\right)^{4-r_2^{(1)}}\left(\frac{\lambda}{\lambda+s}\right)^{r_2^{(1)}} \\
&= 2\left[\binom{2}{1}\frac{\lambda^4}{(2\lambda+s)^3(\lambda+s)} + \binom{1}{0}\frac{\lambda^4}{(2\lambda+s)^2(\lambda+s)^2}\right]
\end{aligned}$$

$$= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2} + \frac{4\lambda^4}{(2\lambda+s)^3(\lambda+s)}$$

which agrees with (2.5), (3.7) and the result given by Corollary 3.1.

In summary, by using a probabilistic interpretation, we have found a method to calculate the Laplace transform of an Erlang order statistic under the most general conditions. Rather than deriving the p.d.f. of each order statistic and applying the definition of a Laplace transform, we can represent the calculation as a multiple, nested summation. While the notation for this summation is somewhat cumbersome, calculating such a summation may often be preferable to the long and tedious process of applying the definition of a Laplace transform directly.

4. A Markov Chain Approach

We can also compute the Laplace transforms of order statistics of Erlang random variables by viewing movements between states as a Markov chain. We shall first define some notation.

Notation. Let $Y_1, Y_2, \ldots, Y_k$ represent k arbitrarily distributed Erlang random variables with respective parameters $(n_1, \lambda_1), (n_2, \lambda_2), \ldots, (n_k, \lambda_k)$. Let Z be an (artificial) catastrophe variable, where $Z \sim exp(s)$. Let the states of the system be vectors of the form $(r_1 \ldots r_k), 0 \le r_1 \le n_1, \ldots, 0 \le r_k \le n_k$ together with a state C (representing a catastrophe). There are $1 + \prod_{i=1}^{k}(n_i + 1)$ possible states. We could order these states and label them as $1, 2 \ldots, 1 + \prod_{i=1}^{k}(n_i + 1)$. Depending on the order statistic of interest and on symmetry, some states can be merged into a single state. Define a sequence of random variables $\{S_n\}$, where S_n represents the state of the system on step n. Let $Y_i = \sum_{j=1}^{n_i} X_{ij}$, where $\{X_{ij}\}$ are independent exponential (λ_i). If the system is in state s_n on step n, then the system will be in state s_{n+1} on step $(n+1)$, where (a) s_{n+1} differs from s_n by one unit in only one of the k components, or (b) s_{n+1} represents the catastrophe, or (c) $s_{n+1} = s_n$ if the system is in an absorbing state.

We have $P(S_{n+1} = s_{n+1} | S_n = s_n, \ldots, S_1 = s_1) = P(S_{n+1} = s_{n+1} | S_n = s_n)$ so $\{S_n\}$ is a Markov chain.

Suppose state s_n has form $(r_1 \ldots r_k)$. The transition probabilities are

$$P(S_{n+1} = s_{n+1} | S_n = s_n)$$
$$= \begin{cases} \dfrac{\lambda_i}{\sum_j \lambda_j + s} & \text{if } s_n = (r_1 \ldots r_i - 1 \ldots r_k) \text{ and } s_n \text{ is not absorbing} \\ 1 & \text{if } s_n \text{ is absorbing} \\ \dfrac{s}{\sum_j \lambda_j + s} & \text{if } s_{n+1} \text{ represents the catastrophe and } s_n \text{ is not absorbing} \end{cases}$$

We are now ready to model the process of our "exponential relay race" as a transition matrix. In order to understand how we will accomplish this task, consider the two variable example considered in the previous two sections. Suppose we wish to calculate $L_{(1)}(s)$ or equivalently, by Property 3.1, $P(Y_{(1)} < Z)$. That is, we wish to calculate the probability of eventually entering a state with at least one zero component in one or more of the first $k = 2$ positions. Let all such states belong to the set A_1 (i.e. $A_1 = \{(0\ 0), (1\ 0), (0\ 1), (2\ 0), (0\ 2)\}$). Note that $P(Y_{(1)} \geq Z)$ corresponds to states in the class C. Thus the transition matrix corresponding to computing $L_{(1)}(s)$ is

$$P_1 = \begin{array}{c} (2\ 2) \\ (2\ 1) \\ (1\ 2) \\ (1\ 1) \\ A_1 \\ C \end{array} \begin{bmatrix} 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & 0 & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & \frac{2\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A_1 is considered to be an absorbing state for two reasons. The first reason is that once the process enters a state with at least one zero component, it can never enter a state with no zero components. Secondly, the process is considered to be finished once we know that the first order statistic has occurred prior to catastrophe. C is considered to be an absorbing state. Thus, the problem of computing $L_{(1)}$ becomes that of finding the limiting probability of entering into a state in A_1.

In order to simplify calculations, we can reduce the dimension of P_1. All states, except for A_1 and C, are transient states since once we leave one of these states we never return to it. We now define some notation and present a method by which to reduce the dimension of our transition matrix.

Notation. Let $Y_1, Y_2, \ldots, Y_k$ represent k Erlang random variables with respective parameters $(n_1, \lambda), (n_2, \lambda), \ldots, (n_k, \lambda)$. Let $r = (r_1\ r_2 \ldots r_k)$ represent a typical state, $0 \le r_i \le n_i, i = 1, 2, \ldots, k$. Let $r' = (r_1\ r_2 \ldots r_k)'$ represent the set of all states which are allowable permutations of the state $r = (r_1\ r_2 \ldots r_k)$. By a permutation of a state we mean a state formed by rearrangement of the components of the original state. By an allowable permutation of a state we mean a permuted state with $0 \le r_i \le n_i, i = 1, 2, \ldots, k$. For example, if we start in the state (3 4 5), a possible state would be (1 1 4). Permutations of (1 1 4) are (1 1 4), (1 4 1) and (4 1 1). Allowable permutations of (1 1 4) are (1 1 4) and (1 4 1). Hence, $(1\ 1\ 4)' = \{(1\ 1\ 4), (1\ 4\ 1)\}$. Note that the initial state has only one allowable permutation.

To reduce the dimension of the transition matrix, let $Y_1, Y_2, \ldots, Y_k$ represent k Erlang random variables with respective parameters $(n_1, \lambda), (n_2, \lambda), \ldots, (n_k, \lambda)$. To compute $L_{(u)}(s), u = 1, 2, \ldots, k$, we let A_u be the set of all states with at least u zero components. Let P_u be the transition matrix that models the process, letting the states be A_u, C, and any states which do not fall into these two sets. A_u and C are to be considered absorbing states. For each of the remaining, transient states $r = (r_1\ r_2 \ldots r_k), 0 \le r_i \le n_i, i = 1, 2, \ldots, k$, replace it and its allowable permutations by the single state $r' = (r_1\ r_2 \ldots r_k)'$. Each of these states reduces the original number of states by up to $k! - 1$ states.

Consider any two "permutation states", r'_1 and r'_2. The one step transition probability of reaching r'_2 from r'_1 can be shown to simply be the probability of reaching r'_2 in one step from any element in r'_1. The one step transition probability of reaching A_u or C from r'_1 can also be shown to be the probability of reaching A_u or C from any element in r'_1.

Reducing the dimension of the transition matrix is only feasible when all variables share a common rate parameter. Otherwise, the probability of reaching r_2' from r_1' is not equal to the probability of reaching r_2' from any element in r_1'. Instead, the probability of reaching r_2' from r_1' is a weighted average of the probabilities of reaching r_2' from each state in r_1'. Thus, when not all variables share the same rate parameter, reducing the dimension of the transition matrix is possible but can often require more steps than it saves.

Unless otherwise stated, the transition matrix $P_u, u = 1, 2, \ldots, k$, shall now refer to the transition matrix that has been reduced in dimension by the above described method. Using this method, we replace the original transition matrix P_1 above with

$$P_1 = \begin{array}{c} (2\ 2)' \\ (2\ 1)' \\ (1\ 1)' \\ A_1 \\ C \end{array} \begin{bmatrix} 0 & \frac{2\lambda}{2\lambda+s} & 0 & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & \frac{2\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $(2\ 2)' = \{(2\ 2)\}, (2\ 1)' = \{(2\ 1), (1\ 2)\}$ and $(1\ 1)' = \{(1\ 1)\}$.

Method 1: Fundamental Matrix Method. (ref: Winston, pp 984-987)

A transition matrix P with s states, of which m are absorbing states, can be written as

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

where Q is $(s-m) \times (s-m)$, R is $(s-m) \times m$, 0 is $m \times (s-m)$ and I is the $m \times m$ identity matrix.

Property 4.1. *Given that the system begins in state i, the probability of absorption into absorbing state j is the (i, j) component of $(I-Q)^{-1}R$, for $i = 1, 2, \ldots s, j = 1, 2, \ldots m$.*

Definition. $(I-Q)^{-1}$ is called the fundamental matrix.

In order to facilitate this computation, we need to divide the matrix P_1 into separate blocks or matrices. Define the four matrices $Q_1, R_1, 0_1$ and I_1 as follows. Q_1 is the 3×3 matrix obtained by deleting the row and columns of P_1 which correspond to the absorbing states A_1 and C. R_1 is the 3×2 matrix obtained by deleting the rows which do correspond and the columns which do not correspond to the absorbing states. That is,

$$Q_1 = \begin{bmatrix} 0 & \frac{2\lambda}{2\lambda+s} & 0 \\ 0 & 0 & \frac{\lambda}{2\lambda+s} \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0 & \frac{s}{2\lambda+s} \\ \frac{\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ \frac{2\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \end{bmatrix}.$$

Also, 0_1 is the 2×3 matrix of zeros obtained by deleting the rows which do not correspond and the columns which do correspond to the absorbing states. I_1 is the 2×2 identity matrix obtained by deleting the rows and columns not corresponding to the absorbing states. Therefore, we can rewrite P_1 as

$$P_1 = \begin{bmatrix} Q_1 & R_1 \\ 0_{2\times3} & I_{2\times2} \end{bmatrix}.$$

Now,

$$I_{3\times3} - Q_1 = \begin{bmatrix} 1 & -\frac{2\lambda}{2\lambda+s} & 0 \\ 0 & 1 & -\frac{\lambda}{2\lambda+s} \\ 0 & 0 & 1 \end{bmatrix}$$

and so, using Maple V, we calculate $(I_{3\times3} - Q_1)^{-1}$ to be

$$(I_{3\times3} - Q_1)^{-1} = \begin{bmatrix} 1 & \frac{2\lambda}{2\lambda+s} & \frac{2\lambda^2}{(2\lambda+s)^2} \\ 0 & 1 & \frac{\lambda}{2\lambda+s} \\ 0 & 0 & 1 \end{bmatrix}.$$

Again with the assistance of Maple V, we obtain

$$(I_{3\times3} - Q_1)^{-1}R_1 = \begin{matrix} (2\ 2)' \\ (2\ 1)' \\ (1\ 1)' \end{matrix} \begin{bmatrix} \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3} & \frac{s}{2\lambda+s} + \frac{2\lambda s}{(2\lambda+s)^2} + \frac{2\lambda^2 s}{(2\lambda+s)^3} \\ \frac{\lambda}{2\lambda+s} + \frac{2\lambda^2}{(2\lambda+s)^2} & \frac{s}{2\lambda+s} + \frac{2\lambda s}{(2\lambda+s)^2} \\ \frac{2\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \end{bmatrix}. \quad (4.1)$$

The Maple V commands needed to derive (4.1) are

```
> with(linalg):

> I3:=diag(1$3):

> R1:=matrix(3,2,[0,s/(2*lambda+s),lambda/(2*lambda+s),s/(2*lambda+s),
        2*lambda/(2*lambda+s),s/(2*lambda+s)]):

> Q1:=matrix(3,3,[0,2*lambda/(2*lambda+s),0,0,0,lambda/(2*lambda+s),0,0,0]):

> evalm(inverse(I3-Q1)&*R1);.
```

We wish to know the probability that, starting in state (2 2), we are absorbed into state A_1 rather than state C. According to Property 4.1, this is given by the $(1,1)$ entry of $(I_{3\times3} - Q_1)^{-1}R_1$. That is,

$$P(\text{reach } A_1|\text{at } (2\ 2)) = \frac{2\lambda^2}{(2\lambda+s)^2} + \frac{4\lambda^3}{(2\lambda+s)^3}$$

which is the same result we computed for $L_{(1)}(s)$ in the previous chapters for the two variable example. We note that, because we reduced the dimension of the transition matrix, we were only required to invert a 3×3 matrix rather than a 4×4 matrix.

The Fundamental Matrix Method (FMM) also gives us some additional information. That is, each element in column 1 of $(I-Q)^{-1}R$ represents the Laplace transform of a different random variable. For our example, the $(2,1)$ component of $(I_{3\times3} - Q_1)^{-1}R_1$ represents the probability that starting in state (2 1) (or (1 2)) we are absorbed into A_1. This is clearly the Laplace transform of the first order statistic of two Erlang random variables with respective parameters $(2, \lambda)$ and $(1, \lambda)$. Thus, in deriving one Laplace transform, we simultaneously find derivations for several Laplace transforms.

In general, consider two Erlang variables with respective parameters (n_1, λ_1) and (n_2, λ_2). If we create a transition matrix P_1, which has not been reduced in dimension, for the purpose of deriving $L_{(1)}(s)$ given we start in state $(n_1\ n_2)$, the first column of $(I-Q)^{-1}R$ will contain $L_{(1)}(s)$ for any two Erlang variables with respective parameters (r_1, λ_1) and (r_2, λ_2) for $1 \leq r_1 \leq n_1$ and $1 \leq r_2 \leq n_2$. When we reduce the dimension of the transition matrix in cases where there is a common rate parameter, the element in the first column of $(I-Q)^{-1}R$ corresponding to a particular permutation state, r_1', represents $L_{(1)}(s)$ starting at any state in r_1'. However, if we reduce the transition matrix in cases where there is not a common rate parameter, the element in the first column of $(I-Q)^{-1}R$ corresponding to a particular permutation state, r_1', represents a weighted average of $L_{(1)}(s)$'s starting over all states in r_1'.

We now use FMM to calculate $L_{(2)}(s) = P(Y_{(2)} < Z)$ in the two variable example. That is, we wish to calculate the probability of eventually entering a state with at least two zero components in one or more of the first $k = 2$ positions. Let all such states belong to the set A_2. Note that $A_2 = \{(0\ 0)\}$. C is as defined earlier. The reduced transition matrix corresponding to computing $L_{(2)}(s)$ is

$$P_2 = \begin{array}{c} (2\ 2)' \\ (2\ 1)' \\ (1\ 1)' \\ (2\ 0)' \\ (1\ 0)' \\ A_2 \\ C \end{array} \begin{bmatrix} 0 & \frac{2\lambda}{2\lambda+s} & 0 & 0 & 0 & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & 0 & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & \frac{2\lambda}{2\lambda+s} & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & \frac{\lambda}{\lambda+s} & 0 & \frac{s}{\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{\lambda+s} & \frac{s}{\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since we are only interested in the probability of being absorbed into A_2 from a given state, we only require the first column of $(I_{5\times5} - Q_2)^{-1}R_2$ which is

$$\text{COL1}[(I_{5\times5} - Q_2)^{-1}R_2] = \begin{array}{c} (2\ 2)' \\ (2\ 1)' \\ (1\ 1)' \\ (2\ 0)' \\ (1\ 0)' \end{array} \begin{bmatrix} \frac{2\lambda^4(4\lambda+3s)}{(2\lambda+s)^3(\lambda+s)^2} \\ \frac{2\lambda^3(4\lambda+3s)}{(2\lambda+s)^2(\lambda+s)^2} \\ \frac{2\lambda^2}{(2\lambda+s)(\lambda+s)} \\ \frac{\lambda^2}{(\lambda+s)^2} \\ \frac{\lambda}{\lambda+s} \end{bmatrix} \tag{4.2}$$

and so following the method used for computing $L_{(1)}(s)$ we find that

$$P(\text{reach } A_2|\text{at } (2\ 2)) = \frac{2\lambda^4(4\lambda+3s)}{(2\lambda+s)^3(\lambda+s)^2}$$

which, with a little algebraic manipulation, one can show matches the results for $L_{(2)}(s)$ from the previous chapters. We note that, because we reduced the dimension of the transition matrix, we were only required to invert a 5×5 matrix rather than a 9×9 matrix.

The Maple V commands needed to derive $(I_{5\times 5} - Q_2)^{-1}R_2$ and, hence, (4.2) are

```
> with(linalg):
> I5:=diag(1$5):
> R2:=matrix(5,2,[0,s/(2*lambda+s),0,s/(2*lambda+s),0,s/(2*lambda+s),
          0,s/(2*lambda+s),lambda/(2*lambda+s),s/(2*lambda+s)]):
> Q2:=matrix(5,5,[0,2*lambda/(2*lambda+s),0,0,0,0,0,lambda/(2*lambda+s),
          lambda/(2*lambda+s),0,0,0,0,0,2*lambda/(2*lambda+s),0,0,0,
          0,lambda/(lambda+s),0,0,0,0,0]):
> evalm(inverse(I5-Q2)&*R2);.
```

In this case, the first column of $(I_{5\times 5} - Q_2)^{-1}R_2$ will contain $L_{(2)}(s)$ for any two Erlang variables with respective parameters (r_1, λ) and (r_2, λ) for $1 \le r_1 \le 2$ and $1 \le r_2 \le 2$.

The General Case for the Fundamental Matrix Method.

Let $Y_1, Y_2, \ldots, Y_k$ represent k arbitrarily distributed Erlang random variables with respective parameters $(n_1, \lambda_1), (n_2, \lambda_2), \ldots, (n_k, \lambda_k)$. To compute $L_{(u)}(s)$, $u = 1, 2, \ldots, k$, we let A_u be the set of all states with at least u zero components. Let P_u be the transition matrix that models the process and has been decreased in dimension if convenient, letting the states be A_u, C, and any states which do not fall into these two sets. A_u and C are to be considered absorbing states.

Let Q_u be the matrix obtained by deleting the row and columns of P_u which correspond to the absorbing states A_u and C. Let R_u be the matrix obtained by deleting the rows which do correspond and the columns which do not correspond to the absorbing states. Let 0_u be the matrix of zeros obtained by deleting the rows which do not correspond and the columns which do correspond to the absorbing states. Let I be the 2×2 identity matrix obtained by deleting the rows and columns not corresponding to the absorbing states.

Example 4.1. Let Y_1, Y_2 and Y_3 represent three Erlang random variables with respective parameters $(2,5),(3,5)$ and $(1,2)$. Calculate $L_{(1)}(1)$. This is identical to Example 3.3.

Solution. We begin at the state (2 3 1) Note that since we want to calculate $L_{(1)}(1)$, our (artificial) catastrophe variable is exponentially distributed with rate parameter 1. Let A_1 be the set of all states with at least one zero. The transition matrix that models this process is

$$P_1 = \begin{array}{c} (2\ 3) \\ (2\ 2) \\ (2\ 1) \\ (1\ 3) \\ (1\ 2) \\ (1\ 1) \\ A_1 \\ C \end{array} \begin{bmatrix} 0 & \frac{5}{13} & 0 & \frac{5}{13} & 0 & 0 & \frac{2}{13} & \frac{1}{13} \\ 0 & 0 & \frac{5}{13} & 0 & \frac{5}{13} & 0 & \frac{2}{13} & \frac{1}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & \frac{7}{13} & \frac{1}{13} \\ 0 & 0 & 0 & 0 & \frac{5}{13} & 0 & \frac{7}{13} & \frac{1}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & \frac{7}{13} & \frac{1}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{12}{13} & \frac{1}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now wish to divide P_1 into the four matrices Q_1, R_1, $0_{2\times 6}$ and $I_{2\times 2}$. We have that

$$Q_1 = \begin{bmatrix} 0 & \frac{5}{13} & 0 & \frac{5}{13} & 0 & 0 \\ 0 & 0 & \frac{5}{13} & 0 & \frac{5}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} \\ 0 & 0 & 0 & 0 & \frac{5}{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 = \begin{bmatrix} \frac{2}{13} & \frac{1}{13} \\ \frac{2}{13} & \frac{1}{13} \\ \frac{7}{13} & \frac{1}{13} \\ \frac{7}{13} & \frac{1}{13} \\ \frac{7}{13} & \frac{1}{13} \\ \frac{12}{13} & \frac{1}{13} \end{bmatrix}$$

and so

$$P_1 = \begin{bmatrix} Q_1 & R_1 \\ 0_{2\times 6} & I_{2\times 2} \end{bmatrix}.$$

Now,

$$(I_{6\times 6} - Q_1) = \begin{bmatrix} 1 & -\frac{5}{13} & 0 & -\frac{5}{13} & 0 & 0 \\ 0 & 1 & -\frac{5}{13} & 0 & -\frac{5}{13} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{13} \\ 0 & 0 & 0 & 1 & -\frac{5}{13} & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{13} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and so, using Maple V, we obtain

$$(I_{6\times 6} - Q_1)^{-1} = \begin{bmatrix} 1 & \frac{5}{13} & \frac{25}{169} & \frac{5}{13} & \frac{50}{169} & \frac{375}{2197} \\ 0 & 1 & \frac{5}{13} & 0 & \frac{5}{13} & \frac{50}{169} \\ 0 & 0 & 1 & 0 & 0 & \frac{5}{13} \\ 0 & 0 & 0 & 1 & \frac{5}{13} & \frac{25}{169} \\ 0 & 0 & 0 & 0 & 1 & \frac{5}{13} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Again, we use Maple V to obtain

$$(I_{6\times 6} - Q_1)^{-1} R_1 = \begin{bmatrix} \frac{23324}{28561} & \frac{5237}{28561} \\ \frac{1848}{2197} & \frac{349}{2197} \\ \frac{151}{169} & \frac{18}{169} \\ \frac{1938}{2197} & \frac{259}{2197} \\ \frac{151}{169} & \frac{18}{169} \\ \frac{12}{13} & \frac{1}{13} \end{bmatrix}.$$

and so $L_{(1)}(1)$ is the $(1,1)$ component of $(I_{6\times 6} - Q_1)^{-1} R_1$. That is,

$$L_{(1)}(1) = \frac{23324}{28561} \approxeq 0.8166$$

which matches the result from Example 3.3.

The Maple V commands needed to derive $(I_{6\times6} - Q_1)^{-1}R_1$ are

```
> with(linalg):
> I6:=diag(1$6):
> R1:=matrix(6,2,[2/13,1/13,2/13,1/13,7/13,1/13,7/13,1/13,7/13,1/13,12/13,1/13]):
> Q1:=matrix(6,6,[0,5/13,0,5/13,0,0,0,0,5/13,0,5/13,0,0,0,0,0,0,5/13,0,0,0,0,5/13,0,
                 0,0,0,0,0,5/13,0,0,0,0,0,0]):
> evalm(inverse(I6-Q1)&*R1);.■
```

Method 2: Recursive Method.

Property 4.2. *Let $Y_1, Y_2, \ldots, Y_k$ represent k Erlang random variables with respective parameters $(n_1, \lambda_1), (n_2, \lambda_2), \ldots, (n_k, \lambda_k)$. Let*

$$L_{(u)}(s) = L_{(u),(n_1\ n_2 \ldots n_k)}(s) = p^u_{n_1\ n_2 \ldots n_k}, u = 1, 2, \ldots, k,$$

represent the Laplace transform of the u-th order statistic starting from the state $(n_1\ n_2 \ldots n_k)$. If $n_i \geq 1, i = 1, 2, \ldots, k$, then

$$\begin{aligned} L_{(u)}(s) &= \frac{1}{\sum_{i=1}^k \lambda_i + s} \sum_{i=1}^k \lambda_i L_{(u),(n_1 \ldots n_i - 1 \ldots n_k)}(s) \\ &= \frac{1}{\sum_{i=1}^k \lambda_i + s} \sum_{i=1}^k \lambda_i p^u_{n_1 \ldots n_i - 1 \ldots n_k} \end{aligned}$$

where $L_{(u),*}(s) = 1$ whenever we start in a state that already has u zero components.

Proof. Let Z represent the catastrophe random variable, $Z \sim exp(s)$.

$$
\begin{aligned}
L_{(u)}(s) &= P(Y_{(u)} < Z) \\
&= P(\text{reach a state with } u \text{ zeros}|(n_1\ n_2 \dots n_k)) \\
&= P(Y_1 \text{ completes 1 stage and we reach a state with } u \text{ zeros}) \\
&\quad + P(Y_2 \text{ completes 1 stage and we reach a state with } u \text{ zeros}) \\
&\quad \vdots \\
&\quad + P(Y_k \text{ completes 1 stage and we reach a state with } u \text{ zeros}) \\
&= \frac{\lambda_1}{\sum_{i=1}^k \lambda_i + s} P(\text{reach a state with } u \text{ zeros}|(n_1 - 1\ n_2 \dots n_k)) \\
&\quad + \frac{\lambda_2}{\sum_{i=1}^k \lambda_i + s} P(\text{reach a state with } u \text{ zeros}|(n_1\ n_2 - 1 \dots n_k)) \\
&\quad \vdots \\
&\quad + \frac{\lambda_k}{\sum_{i=1}^k \lambda_i + s} P(\text{reach a state with } u \text{ zeros}|(n_1\ n_2 \dots n_k - 1)) \\
&= \frac{1}{\sum_{i=1}^k \lambda_i + s} \sum_{i=1}^k \lambda_i L_{(u),(n_1 \dots n_i - 1 \dots n_k)}(s) \\
&= \frac{1}{\sum_{i=1}^k \lambda_i + s} \sum_{i=1}^k \lambda_i p^u_{n_1 \dots n_i - 1 \dots n_k}.
\end{aligned}
$$

Note that $L_{(u)}(s) = P(Y_{(u)} < Z)$ and so, whenever we start in a state that already has u zero components, $P(Y_{(u)} < Z) = 1$ and hence $L_{(u)}(s) = 1$.■

If we reach the state $(r_1 \dots r_{i-1}\ 0\ r_{i+1} \dots r_k)$, i.e. a state with a zero component in the i-th position for some $i = 1, 2, \dots, k$, then it is clear that

$$
L_{(u),(r_1 \dots r_{i-1}\ 0\ r_{i+1} \dots r_k)}(s) = L_{(u-1),(r_1 \dots r_{i-1}\ r_{i+1} \dots r_k)}(s), u \geq 2. \tag{4.3}
$$

By this reduction, we never encounter a "state" with more than one zero.

Let us confirm that Property 4.2 results in the same answers already obtained for the two variable case. We have that

$$
\begin{aligned}
L_{(1)}(s) &= L_{(1),(2\ 2)}(s) \\
&= \frac{\lambda}{2\lambda+s}\left[L_{(1),(2\ 1)}(s) + L_{(1),(2\ 1)}(s)\right] \\
&= \left(\frac{\lambda}{2\lambda+s}\right)^2 \left[L_{(1),(0\ 2)}(s) + 2L_{(1),(1\ 1)}(s) + L_{(1),(2\ 0)}(s)\right] \\
&= \left(\frac{\lambda}{2\lambda+s}\right)^2 \left[2 + 2L_{(1),(1\ 1)}(s)\right] \\
&= 2\left(\frac{\lambda}{2\lambda+s}\right)^2 + 2\left(\frac{\lambda}{2\lambda+s}\right)^3 \left[L_{(1),(0\ 1)}(s) + L_{(1),(1\ 0)}(s)\right] \\
&= 2\left(\frac{\lambda}{2\lambda+s}\right)^2 + 4\left(\frac{\lambda}{2\lambda+s}\right)^3
\end{aligned}
$$

and

$$
\begin{aligned}
L_{(2)}(s) &= L_{(2),(2\ 2)}(s) \\
&= \frac{\lambda}{2\lambda+s}\left[L_{(2),(2\ 1)}(s) + L_{(2),(2\ 1)}(s)\right] \\
&= \left(\frac{\lambda}{2\lambda+s}\right)^2 \left[L_{(2),(0\ 2)}(s) + 2L_{(2),(1\ 1)}(s) + L_{(2),(2\ 0)}(s)\right] \\
&= \left(\frac{\lambda}{2\lambda+s}\right)^2 \left[2L_{(1),(2)}(s) + 2L_{(2),(1\ 1)}(s)\right] \text{ (by (4.3))} \\
&= \left(\frac{\lambda}{2\lambda+s}\right)^2 \left[2\left(\frac{\lambda}{\lambda+s}\right)^2 + 2L_{(2),(1\ 1)}(s)\right] \\
&= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2} + 2\left(\frac{\lambda}{2\lambda+s}\right)^2 \left[L_{(2),(0\ 1)}(s) + L_{(2),(1\ 0)}(s)\right] \\
&= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2} + 2\left(\frac{\lambda}{2\lambda+s}\right)^2 \left[2L_{(1),(1)}(s)\right] \\
\text{(by (4.3))} &= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2} + 2\left(\frac{\lambda}{2\lambda+s}\right)^3 2\left(\frac{\lambda}{\lambda+s}\right) \\
&= \frac{2\lambda^4}{(2\lambda+s)^2(\lambda+s)^2} + \frac{4\lambda^4}{(2\lambda+s)^3(\lambda+s)}.
\end{aligned}
$$

We see that both $L_{(1)}(s)$ and $L_{(2)}(s)$ agree with the previously obtained results.

Note that unless otherwise specified, each step in the above two derivations invoked the use of Property 4.2.

Example 4.2. Let Y_1, Y_2 and Y_3 represent three Erlang random variables with common parameters $(2, \lambda)$. Calculate $L_{(2)}(s)$.

Solution. We begin in state (2 2 2). In cases where our variables are identically distributed, it is often easier to read the necessary information from the reduced dimension transition matrix P_2 where

$$P_2 = \begin{array}{c} (2\ 2\ 2)' \\ (2\ 2\ 1)' \\ (2\ 1\ 1)' \\ (1\ 1\ 1)' \\ (2\ 2\ 0)' \\ (2\ 1\ 0)' \\ (1\ 1\ 0)' \\ A_2 \\ C \end{array} \begin{bmatrix} 0 & \frac{3\lambda}{3\lambda+s} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s}{3\lambda+s} \\ 0 & 0 & \frac{2\lambda}{3\lambda+s} & 0 & \frac{\lambda}{3\lambda+s} & 0 & 0 & 0 & \frac{s}{3\lambda+s} \\ 0 & 0 & 0 & \frac{\lambda}{3\lambda+s} & 0 & \frac{2\lambda}{3\lambda+s} & 0 & 0 & \frac{s}{3\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\lambda}{3\lambda+s} & 0 & \frac{s}{3\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & \frac{2\lambda}{2\lambda+s} & 0 & 0 & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{2\lambda+s} & \frac{\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2\lambda}{2\lambda+s} & \frac{s}{2\lambda+s} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, looking at P_2 row by row we see that

$$(1) L_{(2),(2\ 2\ 2)'}(s) = \frac{3\lambda}{3\lambda+s} L_{(2),(2\ 2\ 1)'}(s)$$
$$(2) L_{(2),(2\ 2\ 1)'}(s) = \frac{2\lambda}{3\lambda+s} L_{(2),(2\ 1\ 1)'}(s) + \frac{\lambda}{3\lambda+s} L_{(2),(2\ 2\ 0)'}(s)$$
$$(3) L_{(2),(2\ 1\ 1)'}(s) = \frac{\lambda}{3\lambda+s} L_{(2),(1\ 1\ 1)'}(s) + \frac{2\lambda}{3\lambda+s} L_{(2),(2\ 1\ 0)'}(s)$$
$$(4) L_{(2),(2\ 2\ 0)'}(s) = \frac{2\lambda}{2\lambda+s} L_{(2),(2\ 1\ 0)'}(s)$$
$$(5) L_{(2),(1\ 1\ 1)'}(s) = \frac{3\lambda}{3\lambda+s} L_{(2),(1\ 1\ 0)'}(s)$$
$$(6) L_{(2),(2\ 1\ 0)'}(s) = \frac{\lambda}{2\lambda+s} L_{(2),(1\ 1\ 0)'}(s) + \frac{\lambda}{2\lambda+s}$$

$$(7) L_{(2),(1\ 1\ 0)'}(s) = \frac{2\lambda}{2\lambda + s}$$

Using back substitution of (7) into (5) and (6), we obtain

$$L_{(2),(2\ 1\ 0)'}(s) = \frac{2\lambda^2}{(2\lambda + s)^2} + \frac{\lambda}{2\lambda + s}$$

and

$$L_{(2),(1\ 1\ 1)'}(s) = \frac{6\lambda^2}{(3\lambda + s)(2\lambda + s)}.$$

Hence, substituting these results into (3) and (4), we obtain

$$L_{(2),(2\ 2\ 0)'}(s) = \frac{4\lambda^3}{(2\lambda + s)^3} + \frac{2\lambda^2}{(2\lambda + s)^2}$$

and

$$L_{(2),(2\ 1\ 1)'}(s) = \frac{6\lambda^3}{(3\lambda + s)^2(2\lambda + s)} + \frac{4\lambda^3}{(3\lambda + s)(2\lambda + s)^2} + \frac{2\lambda^2}{(3\lambda + s)(2\lambda + s)}$$

and so, substituting both of these into (2) we obtain

$$L_{(2),(2\ 2\ 1)'}(s) = \frac{12\lambda^4}{(3\lambda + s)^3(2\lambda + s)} + \frac{8\lambda^4}{(3\lambda + s)^2(2\lambda + s)^2} + \frac{4\lambda^3}{(3\lambda + s)^2(2\lambda + s)} + \frac{4\lambda^4}{(3\lambda + s)(2\lambda + s)^3} + \frac{2\lambda^3}{(3\lambda + s)(2\lambda + s)^2}.$$

Therefore, substituting $L_{(2),(2\ 2\ 1)'}(s)$ into (1),

$$L_{(2),(2\ 2\ 2)'} = \frac{36\lambda^5}{(3\lambda + s)^4(2\lambda + s)} + \frac{24\lambda^5}{(3\lambda + s)^3(2\lambda + s)^2} + \frac{12\lambda^5}{(3\lambda + s)^2(2\lambda + s)^3} + \frac{12\lambda^4}{(3\lambda + s)^3(2\lambda + s)} + \frac{6\lambda^4}{(3\lambda + s)^2(2\lambda + s)^2}. \blacksquare$$

In summary, by viewing the states of our "exponential relay race" as a Markov chain, we have discovered two more methods by which to calculate the Laplace transform of an Erlang order statistic under the most general conditions. Rather than directly applying the definition of a Laplace transform, we can represent the calculation as the limiting probability of entering a particular absorbing state in a transition matrix.

5. Conclusions

We have presented several methods by which to calculate the Laplace transform of order statistics of Erlang random variables. We derived these methods using a probabilistic interpretation of the Laplace transform instead of applying the definition of the Laplace transform directly. We invoked combinatorial arguments as well as different methods by which to find the limiting probabilities of a transition matrix with absorbing states. While the number of operations required for each method is unclear, each of the methods discussed within this thesis have their advantages and disadvantages. For example, suppose we have a case where we have many Erlang random variables, each with a different number of stages and a different rate parameter. Then the two methods presented under the Markov analysis would result in an extremely large transition matrix which, because of the lack of identically distributed variables, would be very difficult to decrease in dimension. Also, the direct approach would become difficult because the general formula for the density of an order statistic as given by Hogg and Craig would no longer be applicable; each order statistic density function would need to be calculated individually. Hence, in such a case, the probabilistic and combinatorial approach is preferable since we simply substitute the given parameters of the problem into the general formula of Property 3.7. In cases where all of the Erlang variables are identically distributed, it is certainly feasible to invoke the order statistic density formula as given by Hogg and Craig and then to directly apply the definition of the Laplace transform. However, even in this simple case, the methods of chapters 3 and 4 certainly provide a more systematic and intuitive approach to the problem, and do so without the use of integration.

The techniques discussed in this thesis have many possible applications. Among these, as mentioned in chapter 1, are applications in queueing theory and reliability prediction.

Methods to calculate Laplace transforms of order statistics of Erlang random variables are not the only possible results that can be obtained using the techniques in this thesis. In the past, the probabilistic interpretation of the Laplace transform has been used to find general methods of calculating Laplace transforms of order statistics of exponential random variables (van Danzig; Roy; Kleinrock). One topic that warrants future study is finding general methods of calculating the Laplace transform of any linear combination of order statistics of exponential and, in general, Erlang random variables. For example, one may be interested in calculating the mean or range of such Laplace transforms. In future study, the probabilistic interpretation of the Laplace transform may also be used to find various methods by which to calculate the transform of order statistics of other types of random variables.

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