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Viscous compressible aligned MFD flows.

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VISCOUS COMPRESSIBLE ALIGNED MFD FLOWS

BY

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To my beloved parents

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ABSTRACT

This thesis is devoted to the mathematical study of steady plane flow of a viscous compressible, electrically and thermally conducting fluid, in the presence of a magnetic field. We assume that the flow is aligned, that is, the magnetic vector is everywhere parallel to the velocity vector. We consider both finitely and infinitely electrically conducting fluids.

We transform the flow equations governing the flow above to a streamlined coordinate system and using an inverse method, we seek exact solutions of these equations by assuming a priori certain functional forms of speed, when the flow geometry is specified to be parallel, radial or circular. Several examples are worked out in each case to illustrate the technique. In the case of parallel flows, some boundary value problems have been treated to indicate the applicability of solutions to physically possible situations.

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CHAPTER 1

INTRODUCTION

1.1. Historical Review

The physical properties of gases, the compressible fluids, serve as a foundation for the fluid dynamics of viscous compressible fluids and also show the limitations of the analysis of the fluid dynamics of continuous media. The fundamental equations for the fluid dynamics of viscous compressible fluids are the Navier-Stokes equations. There is no general method for solving these fundamental equations because these equations are non-linear. Only for a small number of special cases can exact solutions of these Navier-Stokes equations be found. In these cases, some assumptions about the state of the fluid have to be made and at the same time simple configurations of the flow pattern have to be considered. Nevertheless, it is very instructive to derive and discuss these fundamental equations because they give many insights, yield several particular solutions and can be examined for modelling laws.

Little has been known about the mathematical properties of the fundamental equations of a viscous compressible fluids because of the high non-linearity due to the coefficients of viscosity and heat conduction as functions of velocity in general. It is common practice to assume that their properties are the same as that for viscous incompressible fluids with constant viscosity, namely, the Navier-Stokes equations in a restricted sense.

As stated, the Navier-Stokes equations may be simplified by considering the limiting cases of very large viscosity, namely, the theory of very small motion or by considering the limiting cases of vanishingly small viscosity, the boundary layer theory.

One of the most important contributions to the fluid dynamics of a viscous fluid is the concept of boundary layer introduced by *Prandtl* in 1904. This concept is the consequence of the fact that for very large Reynolds numbers, the viscous effects are confined to a very thin layer near the boundary. The Navier-Stokes equations may be simplified by the approximations of a boundary layer so that viscous effects of many practically important problems may be evaluated.

Due to the fact that very little is known about the mathematical properties of the fundamental equations for viscous compressible fluids, Lagerstrom(1949) and his associates showed that many essential features of these equations may be brought out by considering the linearized equations of viscous compressible fluids with heat conduction neglected. This approach may be regarded as an initial step toward the understanding of the complex equations for a viscous compressible fluid.

In the study of ordinary viscous, compressible fluid flow, we wish to find the velocity distribution as well as the state of the fluid over the whole space for all time. There are six unknowns, namely, the three components of velocity (u, v, w), the temperature T, the pressure p and the density ρ of the fluid, which are functions of spatial coördinates and time. In order to determine these unknowns we have to find six relations connecting the unknowns, which are as follows:

(a) Equation of state which connects the temperature, the pressure and the density of the fluid.

(b) Equation of continuity which gives the relation of conservation of mass of the fluid.

(c) Equations of motion which are three in general and which give the relations of conservation of momentum of the fluid.

(d) Equation of energy which gives the relation of conservation of energy of the fluid.

In addition, for compressible flows, we note the following complications:

(1) Variable density, which depends upon temperature and pressure.

(2) Variable transport properties, all of which, depend upon temperature and pressure.

(3) Energy and momentum equations, coupled and nonlinear.

These are far from trivial matters and hence there are very few exact solutions of any sort for compressible viscous flow. All the exact solutions known are for degenerate cases where one velocity component varies with one coordinate. An extensive survey of possible viscous compressible solutions was made by Illingworth (1950) who found the following amenable cases: Coutte flow between plates, flow past a plate with suction, flow around a cylinder with suction and flow between rotating cylinders.

Illingworth also found that only solutions similar to Coutte flows of an incompressible fluid are obtained in simple closed form and that no simple solutions corresponding to Poiseuille flow or other exact solutions of a incompressible fluid could be found. We briefly state some of the findings of Illingworth, namely, simple flows of a viscous compressible fluid for which exact solutions have been obtained:

(a) Plane Couette flow

In this problem, we consider the flow of a compressible fluid between two infinite parallel plates, the lower one fixed and the upper plate moving at some velocity. Thus the flow is generated entirely by the moving upper plate. All the variables are a function of y only.

(b) Flow past a porous flat plate.

In this problem, we consider the flow past an infinite porous plate situated in a steady stream parallel to its plane when there is a steady suction applied uniformly over the plate.

(c) Simple shearing motion between co-rotating cylinders

In this problem, we consider the two-dimensional steady laminar flow between two concentric cylinders which are rotating. By keeping one cylinder at rest, we have Couette flow.

(d) Circulating flow around a circular cylinder

In this problem, we consider a steady circulating flow of a compressible fluid around a porous circular cylinder with fixed axis and with suction applied at the surface.

We turn our attention now to *Magnetohydrodynamics*. When a conductor carrying an electric current moves in the presence of a magnetic field, it experiences a force tending to move it at right-angles to the electric field. In the case when the conductor is either a liquid or a gas, electromagnetic forces will be generated which may be of the same order of magnitude as the hydrodynamical and inertial forces. Thus the fundamental equations of motion will have to take these electromagnetic forces into account as well as the other forces. The science which treats these phenomena is called magnetohydrodynamics or magnetogasdynamics.

In magnetogasdynamics, we assume that there are no free charges and that frequencies are not high enough to make displacement current important. In addition, we assume that the velocity of the flow **q** is much smaller than the velocity of light, c, so that the relativistic effects may be neglected and the energy in the electric field is negligible in comparison with that in the magnetic field. Hence, we consider essentially the interaction between the magnetic field **H** and the gas dynamical variables: velocity **q**, temperature T, density ρ and pressure p. We assume that the fluid is a perfect gas for which the equation of state is

$$p = \rho RT$$

where R is a gas constant. We consider flows in which the fluid is viscous (coefficient of viscosity = ν), heat conducting (coefficient of heat conductivity = k) and electrically conducting (electrical conductivity = σ).

The one-dimensional flow of magnetogasdynamics has been investigated by many authors. Most of the problems investigated are either one-dimensional steady flow which describes the shock wave structure or one-dimensional wave motion of small amplitude in which the fundamental equations may be linearized. Little has been done about the one-dimensional wave motion of finite amplitude. Pai(1957) investigated the one-dimensional unsteady flow of a viscous, heat conducting, electrically conducting and compressible gas under planar magnetic field perpendicular to the velocity vector.

More recently, Martin (1971) developed an elegant method for the study of steady, plane, incompressible, viscous, electrically non-conducting fluid flows. He departed from the usual practise in the treatment of such flows. In his method, he reduced the order of the governing equations from second order to first order by introducing two functions, the vorticity function ω and the energy function h. He then introduced curvilinear coordinates Φ, Ψ in the plane of flow, in which the coordinates lines Ψ = constant are the streamlines of the flow and the coordinate lines Φ = constant are left arbitrary.

This method was first applied to MHD by Nath and Chandna (1973) in establishing certain geometrical properties and obtaining solutions for viscous, incompressible, infinitely conducting MHD flows when the velocity and magnetic field are everywhere orthogonal. Following this work, extensive use of

Martin's method has been made in MHD. Chandna, Barron and Naeem (1984) extended Martin's method to steady, plane, viscous, compressible thermally conducting fluids.

1.2 Outline of Current Work

The present work is devoted entirely to the mathematical study of steady, plane flow of a viscous compressible, electrically and thermally conducting fluid in the presence of a magnetic field. We assume that the flow is aligned, that is, the magnetic field vector is everywhere parallel to the velocity vector. We consider both finitely and infinitely electrically conducting fluids.

A brief outline of the present work is as follows:

In Chapter II we present the flow equations for magnetofluiddynamics. We then discuss some required results from differential geometry. Then we formulate the flow equations in terms of the (Φ, Ψ) coordinate system. Lastly, we formulate the flow equations again in terms of the (ξ, η) natural coordinate system.

In Chapter III, assuming a priori certain functional forms of speed, we solve the new system of flow equations in the (ξ, η) coordinate system when the flow geometry is specified to be parallel. Some boundary value problems have been solved to indicate the applicability of solutions to physically possible situations.

In Chapter IV, we solve the new flow equations in the (ξ, η)

coordinate system when the flow geometry is specified to be circular or vortex.

Chapter V is devoted to solving the new flow equations when the flow geometry is specified to be radial.

CHAPTER 2

THEORY OF FLOW EQUATIONS

2.1 Flow Equations

The steady plane flow of a viscous, compressible, electrically and thermally conducting fluid, in the presence of a magnetic field, when the influence of radiation heat flux is negligible is governed by the following system of equations:

$$\operatorname{div}(\rho V) = 0 \tag{2.1}$$

$$\rho(V \cdot \operatorname{grad})V + \operatorname{grad} p = \tfrac{1}{3}\nu \operatorname{grad}(\operatorname{div} V) + \nu\nabla^2 V + \mu(\operatorname{curl} H)\times H \tag{2.2}$$

$$Q + \operatorname{div}(k \operatorname{grad} T) = \rho(V \cdot \operatorname{grad})e + p \operatorname{div} V + \tfrac{1}{\sigma}|\operatorname{curl} H|^2 \tag{2.3}$$

$$\operatorname{curl}(V\times H) + \frac{1}{\mu\sigma}\nabla^2 H = 0 \tag{2.4}$$

$$p = f(\rho, T) \tag{2.5}$$

This is a system of seven equations wherein u,v are the components of the velocity vector field V, p is the fluid pressure function, ρ the density function, T the temperature function, H the magnetic field vector, ν_0 the constant coefficient of viscosity, μ the constant magnetic permeability and σ the electrical conductivity of the fluid. The specific internal energy $e = e(T)$ is a known function of T for any ideal gas.

The mechanical dissipation function Q is given by

$$Q = \nu[2(u_x)^2 + 2(v_y)^2 + (u_y + v_x)^2 - \frac{2}{3}(u_x + v_y)^2].$$

The magnetic field vector **H** satisfies the additional equation

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0$$

expressing the absence of magnetic poles in flow where

$$\mathbf{H} = (H_1, H_2)$$

2.2 Martin's Approach

Following M.H. Martin[1971], we introduce the vorticity function ω, the current density function j, and a new pressure function P defined by

$$\omega = v_x - u_y$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

$$P = p - \frac{4}{3}\nu(u_x + v_y)$$

into the previous system, so that equations (2.1) to (2.5) are replaced by the following system:

$$(\rho u)_x + (\rho v)_y = 0 \tag{2.6}$$

$$\frac{1}{2}\rho(q^2)_x - \rho v\omega + \nu\omega_y + \mu H_2 j = -P_x \tag{2.7}$$

$$\frac{1}{2}\rho(q^2)_y + \rho u\omega - \nu\omega_x - \mu H_1 j = -P_y \tag{2.8}$$

$$\nu[\omega^2 + 4(u_y v_x - u_x v_y)] = \rho(ue_x + ve_y) - p[u(\ln\rho)_x + v(\ln\rho)_y] - k(T_{xx} + T_{yy}) + \frac{j^2}{\sigma} \tag{2.9}$$

$$(uH_2 - vH_1)_y - \frac{1}{\mu\sigma} j_y = 0 \tag{2.10}$$

$$-(uH_2 - vH_1)_x + \frac{1}{\mu\sigma} j_x = 0 \tag{2.11}$$

$$\omega = v_x - u_y \tag{2.12}$$

$$j = (H_2)_x - (H_1)_y \tag{2.13}$$

$$P = f(\rho, T) - \frac{4}{3}\nu[u(\ln\rho)_x + v(\ln\rho)_y] \tag{2.14}$$

$$(H_1)_x + (H_2)_y = 0 \tag{2.15}$$

wherein $q^2 = u^2 + v^2$. This is a system of ten equations in nine unknowns. The unknowns in this system are $u(x,y)$, $v(x,y)$, $\omega(x,y)$, $j(x,y)$, $P(x,y)$, $\rho(x,y)$, $T(x,y)$, $H_1(x,y)$ and $H_2(x,y)$. The number of equations has increased from eight in the initial system to ten in the new system but the order of the linear momentum equations of this new system is reduced to one from two in the starting system.

2.3 Aligned Flow

We now assume our flow to be aligned, that is, the velocity vector field is everywhere parallel to the magnetic vector field. Then there exist some scalar function $f(x,y)$ such that

$$\mathbf{H} = f(x,y)\, \rho\, \mathbf{V}$$

that is

$$H_1 = f\rho u \quad , \quad H_2 = f\rho v \tag{2.16}$$

For aligned flow, we have

$$\mathbf{V} \times \mathbf{H} = 0 \quad \Longrightarrow \quad uH_2 - vH_1 = 0$$

and our two diffusion eqns (2.10) and (2.11) reduce to

$$(1/\mu\sigma)\, j_y = 0$$

$$(1/\mu\sigma)\, j_x = 0$$

Thus for a finitely conducting fluid ($\sigma < \infty$), we have

$$j_x = 0$$

$$j_y = 0 \tag{2.17}$$

Eqn.(2.17) imply that

$$j = \text{constant} = j_o \text{ (say)} \tag{2.18}$$

Using eqns.(2.16) and (2.18) in the previous system, equations (2.6) to (2.15) are replaced by the following system of equations:

Finite Electrical Conductivity

$$(\rho u)_x + (\rho v)_y = 0 \tag{2.19}$$

$$\frac{1}{2}\rho(q^2)_x - \rho v\omega + \nu\omega_y + \mu j_o f\rho v = -P_x \tag{2.20}$$

$$\frac{1}{2}\rho(q^2)_y + \rho u\omega - \nu\omega_x - \mu j_o f\rho u = -P_y \tag{2.21}$$

$$\nu[\ \omega^2 + 4(u_y v_x - u_x v_y)] = \rho(ue_x + ve_y) - p[u(\ln\rho)_x + v(\ln\rho)_y\ ] -k(\ T_{xx} + T_{yy}\ ) + \frac{j_o^2}{\sigma} \tag{2.22}$$

$$f\rho\omega + \rho(\ vf_x - uf_y\ ) + f(\ v\rho_x - u\rho_y\ ) = j_o \tag{2.23}$$

$$uf_x + vf_y = 0 \tag{2.24}$$

$$\omega = v_x - u_y \tag{2.25}$$

$$P = f(\rho,T) - \frac{4}{3}\nu[u(\ln\rho)_x + v(\ln\rho)_y\ ] \tag{2.26}$$

This is a system of eight equations in seven unknowns. The unknowns are $u(x,y)$, $v(x,y)$, $\omega(x,y)$, $f(x,y)$, $P(x,y)$, $\rho(x,y)$ and $T(x,y)$.

For an infinitely conducting fluid, our two diffusion eqns.(2.10) and (2.11) vanish identically. Then using eqn. (2.16), eqns.(2.6) to eqns.(2.15) in the previous system are replaced by the following system of equations:

Infinite Electrical Conductivity

$$(\rho u)_x + (\rho v)_y = 0 \tag{2.27}$$

$$\frac{1}{2}\rho(q^2)_x - \rho v\omega + \nu\omega_y + \mu j f\rho v = -P_x \tag{2.28}$$

$$\frac{1}{2}\rho(q^2)_y + \rho u\omega - \nu\omega_x - \mu j f\rho u = -P_y \tag{2.29}$$

$$\nu[\,\omega^2 + 4(u_y v_x - u_x v_y)] = \rho(ue_x + ve_y) - p[u(\ln\rho)_x + v(\ln\rho)_y\,] -k(\,T_{xx} + T_{yy}\,) \tag{2.30}$$

$$f\rho\omega + \rho(\,vf_x - uf_y\,) + f(\,v\rho_x - u\rho_y\,) = j \tag{2.31}$$

$$uf_x + vf_y = 0 \tag{2.32}$$

$$\omega = v_x - u_y \tag{2.33}$$

$$P = f(\rho,T) - \frac{4}{3}\nu[u(\ln\rho)_x + v(\ln\rho)_y\,] \tag{2.34}$$

This is a system of eight equations in eight unknowns. The unknowns are $u(x,y)$, $v(x,y)$, $\omega(x,y)$, $j(x,y)$, $P(x,y)$, $\rho(x,y)$, $T(x,y)$ and $f(x,y)$.

Equation of continuity implies the existence of a streamfunction $\psi(x,y)$ such that

$$\psi_x = -\rho v \quad , \quad \psi_y = \rho u \tag{2.35}$$

We take $\phi(x,y)$ = constant to be some arbitrary family of curves which generates with the streamlines $\psi(x,y)$ = constant a curvilinear net (ϕ,ψ) so that in the physical plane x,y can be replaced by ϕ,ψ:

Let

$$x = x(\phi,\psi) \quad , \quad y = y(\phi,\psi) \tag{2.36}$$

define the curvilinear net with the squared element of arc length along any curve given by

$$ds^2 = E(\phi,\psi)d\phi^2 + 2F(\phi,\psi)d\phi\, d\psi + G(\phi,\psi)d\psi^2 \tag{2.37}$$

wherein

$$E = x_\phi^2 + y_\phi^2 \quad , \quad F = x_\phi x_\psi + y_\phi y_\psi \quad , \quad G = x_\psi^2 + y_\psi^2 \tag{2.38}$$

Equation (2.36) can be solved to determine ϕ,ψ as functions of x,y so that

$$x_\phi = J\psi_y \quad , \quad x_\psi = -J\phi_y \quad , \quad y_\phi = -J\psi_x \quad , \quad y_\psi = J\phi_x \tag{2.39}$$

where $0 < |J| < \infty$ and by (2.38) we have

$$J = x_\phi\, y_\psi - x_\psi\, y_\phi = \mp\sqrt{EG - F^2} = \mp W \text{ (say)} \tag{2.40}$$

Denoting by α the angle of inclination of the tangent to the coordinate line ψ = constant directed in the sense of increasing ϕ, we have from differential geometry the following:

$$x_\phi = \sqrt{E}\cos\alpha \quad , \quad y_\phi = \sqrt{E}\sin\alpha$$

$$x_\psi = F(\sqrt{E})^{-1}\cos\alpha - J(\sqrt{E})^{-1}\sin\alpha$$

$$y_\psi = F(\sqrt{E})^{-1}\sin\alpha + J(\sqrt{E})^{-1}\cos\alpha \tag{2.41}$$

$$\alpha_\phi = J(E)^{-1} \Gamma^2_{11} \qquad , \qquad \alpha_\psi = J(E)^{-1} \Gamma^2_{12} \tag{2.42}$$

$$k = \frac{1}{W} \left[\frac{\partial}{\partial\psi} \left(\frac{W}{E} \Gamma^2_{11}\right) - \frac{\partial}{\partial\phi} \left(\frac{W}{E} \Gamma^2_{12}\right) \right] = 0 \tag{2.43}$$

where

$$\Gamma^2_{11} = \frac{-FE_\phi + 2EF_\phi - EE_\psi}{2W^2} \tag{2.44}$$

$$\Gamma^2_{12} = \frac{EG_\phi - FE_\psi}{2W^2} \tag{2.45}$$

and k is the Gaussian curvature.

Having recorded the above results, we now take up first equations (2.19) to (2.26) and develop these flow equations in the new variables ϕ and ψ, thus obtaining

THEOREM I

If the streamlines $\psi(x,y)$ = constant of the flow of a viscous compressible fluid in the presence of a magnetic field are taken as one set of coordinate lines in a curvilinear coordinate system ϕ, ψ in the physical plane, then the system of eight partial differential equations (2.19) to (2.26) for u, v, ω, f, ρ, P, T as functions of x and y may be replaced by the system of nine equations for E, F, G, q, ω, ρ, f, P and T as functions of ϕ and ψ as follows:

Finite Electrical Conductivity

$$q = \frac{\gamma E}{\rho W} \tag{2.46}$$

$$\nu\,[F\,\omega_\phi - E\,\omega_\psi] - \tfrac{1}{2}\rho\,J\,q_\phi^2 = J\,P_\phi \tag{2.47}$$

$$\nu\,[G\,\omega_\phi - F\,\omega_\psi] - J\,\omega - \tfrac{1}{2}\rho\,J\,q_\psi^2 + \mu j_o f\,J = J\,P_\psi \,. \tag{2.48}$$

$$\nu\,[\omega^2 W^2 + 4\,W^2E^{-1}\Gamma_{11}^2\,q\,q_\psi + 4\,W^2E^{-1}\Gamma_{12}^2\,q\,q_\phi] = J\,e_\phi$$

$$-J\,P\,\rho^{-1}(\ln\rho)_\phi - k\,[G\,T_{\phi\phi} + E\,T_{\psi\psi} - 2\,F\,T_{\phi\psi} + T_\phi(J(GJ^{-1})_\phi$$

$$-J(FJ^{-1})_\psi) + T_\psi(J(EJ^{-1})_\psi - J(FJ^{-1})_\phi)] + \frac{j_o^2}{\sigma} \tag{2.49}$$

$$J^2 f\rho\omega + Ff_\phi - Ef_\psi + f\rho^{-1}[F\rho_\phi - E\rho_\psi] = J^2 j_o \tag{2.50}$$

$$f_\phi = 0 \tag{2.51}$$

$$\omega = W^{-1}[(F\rho^{-1}W^{-1})_\phi - (E\rho^{-1}W^{-1})_\psi] \tag{2.52}$$

$$(WE^{-1}\Gamma_{11}^2)_\psi + (WE^{-1}\Gamma_{12}^2)_\phi = 0 \tag{2.53}$$

$$P = f(\rho,T) - \tfrac{4}{3}\nu\,(\rho J)^{-1}(\ln\rho)_\phi \tag{2.54}$$

In the case of the infinitely conducting fluid, the system of eight partial differential equations (2.27) to (2.34) for u, v, ω, j, ρ, P, T and f as functions of x and y may be replaced by the system of nine equations for $E, F, G, q, \omega, j, \rho, T, P$ and f as follows:

Infinite Electrical Conductivity

$$q = \frac{\gamma E}{\rho W} \tag{2.55}$$

$$\nu [F \omega_\phi - E \omega_\psi] - \tfrac{1}{2} \rho J q_\phi = J P_\phi \tag{2.56}$$

$$\nu [G \omega_\phi - F \omega_\psi] - J \omega - \tfrac{1}{2} \rho J q_\psi^2 + \mu j f J = J P_\psi \tag{2.57}$$

$$\nu [\omega^2 W^2 + 4 W^2 E^{-1} \Gamma_{11}^2 q q_\psi + 4 W^2 E^{-1} \Gamma_{12}^2 q q_\phi] = J e_\phi$$

$$-J P \rho^{-1} (\ln \rho)_\phi - k [G T_{\phi\phi} + E T_{\psi\psi} - 2 F T_{\phi\psi} + T_\phi (J (G J^{-1})_\phi$$

$$-J (F J^{-1})_\psi) + T_\psi (J (E J^{-1})_\psi - J (F J^{-1})_\phi)] \tag{2.58}$$

$$J^2 f \rho \omega + F f_\phi - E f_\psi + f \rho^{-1} [F \rho_\phi - E \rho_\psi] = J^2 j \tag{2.59}$$

$$f_\phi = 0 \tag{2.60}$$

$$\omega = W^{-1} [(F \rho^{-1} W^{-1})_\phi - (E \rho^{-1} W^{-1})_\psi] \tag{2.61}$$

$$(W E^{-1} \Gamma_{11}^2)_\psi + (W E^{-1} \Gamma_{12}^2)_\phi = 0 \tag{2.62}$$

$$P = f(\rho, T) - \tfrac{4}{3} \nu (\rho J)^{-1} (\ln \rho)_\phi \tag{2.63}$$

The system of equations (2.46) to (2.54) or (2.55) to (2.63) is underdetermined and obviously reflects the arbitrariness inherent in the choice of the coordinates lines ϕ = constant. In the following work, we consider the fluid flowing towards higher parameter values of ϕ so that $J = W > 0$.

We consider now an orthogonal coordinate system consisting of the streamlines $\eta(x,y)$ = constant and their orthogonal trajectories $\xi(x,y)$ = constant. The squared element of the arc length in this coordinate system is given by

$$ds^2 = g_1^2(\xi,\eta)\, d\xi^2 + g_2^2(\xi,\eta)\, d\eta^2 \tag{2.64}$$

Referring to these coordinates, with $\psi = \psi(\eta)$ and $\phi = \phi(\xi)$, eqn.(2.37) becomes

$$ds^2 = E\phi'(\xi)^2 d\xi^2 + 2F\phi'(\xi)\psi'(\eta)\, d\xi d\eta + G\psi'(\eta)^2 d\eta^2 \tag{2.65}$$

Comparing eqn.(2.64) to eqn.(2.65), we obtain

$$E = g_1^2\phi'(\xi)^{-2} \quad , \quad F = 0 \quad , \quad G = g_2^2\psi'(\eta)^{-2} \tag{2.66}$$

where $g_1(\xi,\eta)$ and $g_2(\xi,\eta)$ are known functions once the flow configuration is prescribed.

Employing the above transformations, with

$$W = J = g_1 g_2(\phi'(\xi)\psi'(\eta))^{-1} > 0$$

in equations (2.46) to (2.54) and equations (2.55) to (2.63), we obtain the following theorem:

THEOREM II

If an orthogonal coordinate net (ξ,η) where $\eta(x,y)$ = constant are the streamlines of the flow of a viscous compressible fluid, in the presence of a magnetic field, is chosen, then the flow equations of theorem I are replaced by:

Finite Electrical Conductivity

$$q = \frac{\psi'(\eta)}{\rho g_2} \tag{2.67}$$

$$\nu g_1 g_2^{-1}\omega_\eta + (1/2)\rho(q^2)_\xi = -P_\xi \tag{2.68}$$

$$\nu g_2 g_1^{-1}\omega_\xi - \omega\,\psi'(\eta) - (1/2)\rho(q^2)_\eta + \mu j_o f\psi'(\eta) = P_\eta \tag{2.69}$$

$$\begin{aligned}&\nu[\omega^2 - 2(g_1 g_2^2)^{-1}(g_1)_\eta(q^2)_\eta + 2(g_1^2 g_2)^{-1}(g_2)_\xi(q^2)_\xi]\\ &= \psi'(\eta)\,(g_1 g_2)^{-1}\,[e_\xi - (P/\rho)(\ln\rho)_\xi] - k\,[g_1^{-2}T_{\xi\xi} + g_2^{-2}T_{\eta\eta} +\\ &+ (g_1 g_2)^{-1}(g_2/g_1)_\xi T_\xi + (g_1 g_2)^{-1}(g_1/g_2)_\eta T_\eta\,] + (j_o^2/\sigma)\end{aligned} \tag{2.70}$$

$$f\rho\omega - \psi'(\eta)g_2^{-2}f_\eta - \psi'(\eta)g_2^{-2}(f/\rho)\rho_\eta = j_o \tag{2.71}$$

$$f_\xi = 0 \tag{2.72}$$

$$\omega = -(g_1 g_2)^{-1}[g_1'\psi'(\eta)/\rho g_2]_\eta \tag{2.73}$$

$$[g_1^{-1}(g_2)_\xi]_\xi + [g_2^{-1}(g_1)_\eta]_\eta = 0 \tag{2.74}$$

$$P = f(\rho, T) + (4/3)\ \nu\ \psi'(\eta)(\rho g_1 g_2)^{-1}(\ln\rho)_\xi \qquad (2.75)$$

Equation (2.74) is the Gauss equation which is identically satisfied when a natural net (ξ,η) is chosen. Therefore, equations (2.67) to (2.75) form a system of eight equations in eight unknowns.

Infinite Electrical Conductivity

$$q = \frac{\psi'(\eta)}{\rho g_2} \qquad (2.76)$$

$$\nu g_1 g_2^{-1}\omega_\eta + (1/2)\rho(q^2)_\xi = -P_\xi \qquad (2.77)$$

$$\nu g_2 g_1^{-1}\omega_\xi - \omega\,\psi'(\eta) - (1/2)\rho(q^2)_\eta + \mu j f \psi'(\eta) = P_\eta \qquad (2.78)$$

$$\nu[\omega^2 - 2(g_1 g_2^2)^{-1}(g_1)_\eta(q^2)_\eta + 2(g_1^2 g_2)^{-1}(g_2)_\xi(q^2)_\xi]$$
$$= \psi'(\eta)\ (g_1 g_2)^{-1}\ [e_\xi - (P/\rho)(\ln\rho)_\xi] - k\ [g_1^{-2}T_{\xi\xi} + g_2^{-2}T_{\eta\eta} +$$
$$+ (g_1 g_2)^{-1}(g_2/g_1)_\xi T_\xi + (g_1 g_2)^{-1}(g_1/g_2)_\eta T_\eta\ ] \qquad (2.79)$$

$$f\rho\omega - \psi'(\eta) g_2^{-2} f_\eta - \psi'(\eta) g_2^{-2}(f/\rho)\rho_\eta = j \qquad (2.80)$$

$$f_\xi = 0 \qquad (2.81)$$

$$\omega = -(g_1 g_2)^{-1}[g_1'\psi'(\eta)/\rho g_2]_\eta \qquad (2.82)$$

$$[g_1^{-1}(g_2)_\xi]_\xi + [g_2^{-1}(g_1)_\eta]_\eta = 0 \tag{2.83}$$

$$P = f(\rho,T) + (4/3)\,\nu\,\psi'(\eta)(\rho g_1 g_2)^{-1}(\ln\rho)_\xi \tag{2.84}$$

Equation (2.83) is the Gauss equation which identically satisfied when a natural (ξ,η) is chosen. Therefore, equations (2.76) to (2.84) form a system of eight equations in nine unknowns.

Using the integrability condition $P_{\xi\eta} = P_{\eta\xi}$, P can be eliminated from equation (2.68) and (2.69) yielding

$$[\nu g_1 g_2^{-1}\omega_\eta + (1/2)\rho(q^2)_\xi]_\eta = [(1/2)\rho(q^2)_\eta + \omega\psi'(\eta) - \nu g_2 g_1^{-1}\omega_\xi - \mu j_0 f\psi'(\eta)]_\xi \tag{2.85}$$

We proceed now to solve the systems of equations (2.67) to (2.75) and equations (2.78) to (2.84) along with equation (2.85). We assume that the specific internal energy e in the energy equation is a known function of T, that is,

$$e = C_v T \implies \text{polytropic gas} \tag{2.86}$$

and the compressible fluid is a ideal gas and has the state equation given by

$$p = \rho RT \tag{2.87}$$

where R is a gas constant.

We also assume for the finitely electrically conducting fluid, σ, the electrical conductivity is non-zero.

We consider solutions of system (2.67) to (2.75) and

(2.85) by assuming a priori various forms for $q = q(\xi, \eta)$. In the next three chapters, we solve the above system of equations when the flow geometry is specified to be parallel, radial or circular.

We follow the same procedure in solving the system (2.76) to (2.84) for the infinitely electrically conducting fluid.

CHAPTER 3

PARALLEL FLOW

We consider flow parallel to the x-axis. We take (ξ,η) to be the cartesian coordinate system (x,y), that is, we choose

$$\xi = x \quad , \quad \eta = y. \tag{3.1}$$

The squared element of the arc length in the cartesian coordinate system is given by

$$ds^2 = dx^2 + dy^2.$$

Comparing this equation with the following equation,

$$ds^2 = g_1^2\, d\xi^2 + g_2^2\, d\eta^2$$

we have

$$g_1 = 1 \quad \text{and} \quad g_2 = 1. \tag{3.2}$$

With this choice of g_1 and g_2, the Gauss equation is identically satisfied.

Employing (3.1) and (3.2) in the system of equations (2.67) to (2.75), we find that the equations governing parallel flow are given as follows:

<u>**Finite Electrical Conductivity**</u>

$$q = \frac{\psi'(y)}{\rho} \tag{3.3}$$

$$\nu\omega_y + \psi'(y)q_x = -P_x \tag{3.4}$$

$$\nu\omega_x + \mu j_o f\psi'(y) = P_y \tag{3.5}$$

$$f\rho\omega - \psi'(y)f_y - \psi'(y)f\rho^{-1}\rho_y = j_o \tag{3.6}$$

$$\nu\omega^2 = \psi'(y)e_x + pq_x - k[T_{xx} + T_{yy}] + j_o^2/\sigma \tag{3.7}$$

$$f_x = 0 \tag{3.8}$$

$$\omega = -q_y \tag{3.9}$$

$$P = f(\rho,T) - \tfrac{4}{3}\nu q_x \tag{3.10}$$

From equations (3.4) and (3.5), $P_{xy} = P_{yx}$ gives

$$\psi'(y)q_x - \nu q_{yy} - \nu q_{xx} = L(x) \tag{3.11}$$

where L(x) is an arbitrary function of x.

The equations governing infinitely conducting parallel flow are similar to the ones for the finitely conducting case with the exception that j, the current density is no longer a constant but a function of x and y and the last term in the energy equation vanishes.

Assuming now, a priori certain functional forms of the speed q, we solve the system of equations (3.3) to (3.10).

Example 1

Let $q = q(y)$, $q'(y) \neq 0$

Then eqn (3.11) gives

$$q(y) = C_1 y^2 + C_2 y + C_3 \tag{3.12}$$

and
$$L(x) = 2C_1 \tag{3.13}$$

where C_1, C_2 and C_3 are arbitrary constants.

Then eqn (3.9) becomes,

$$\omega = -2C_1 y - C_2 \tag{3.14}$$

and eqn. (3.3) implies,

$$\rho = \frac{\psi'(y)}{C_1 y^2 + C_2 y + C_3} \tag{3.15}$$

where $\psi'(y)$ is an arbitrary function of y and C_1, C_2 and C_3 are not zero simultaneously.

Eqns. (3.4) and (3.5) imply

$$P_x = 2\nu C_1 \tag{3.16}$$

$$P_y = \mu j_o \psi'(y) f(y) \tag{3.17}$$

From eqns. (3.16) and (3.17), we get,

$$P(x,y) = 2C_1 \nu x + \mu j_o \int f(y)\psi'(y)dy \tag{3.18}$$

Eqn. (3.6) becomes

$$\psi'(y) f_y + \psi''(y) f = -j_o \tag{3.19}$$

Eqn. (3.8) imply that $f = f(y)$. Then eqn. (3.19) can be rewritten as

$$\psi'(y) f' + \psi''(y) f = -j_o \tag{3.20}$$

Eqn. (3.20) is an linear first order O.D.E. in f. The solution to eqn. (3.20) is given by

$$f(y) = \frac{1}{\psi'(y)} (-j_o y + c) \tag{3.21}$$

where c is an arbitrary constant.

Substituting eqn. (3.21) into eqn. (3.18) yields

$$P(x,y) = 2C_1 \nu x + \mu j_o (-j_o y^2/2 + cy + d) \tag{3.22}$$

where d is an arbitrary constant.

Then eqn.(3.10) implies

$$p = P \tag{3.23}$$

Now invoking the state equation for a perfect gas, we have

$$T = \frac{p}{\rho R} \tag{3.24}$$

or

$$T(x,y) = \frac{C_1 y^2 + C_2 y + C_3}{R\psi'(y)}(2C_1 \nu x - \mu j_0^2 y^2/2 + c\mu j_0 y + d\mu j_0) \tag{3.25}$$

The energy equation (3.7) implies

$$\nu(4C_1^2 y^2 + 4C_1 C_2 y + C_2^2) = C_v \psi'(y) T_x - k[T_{xx} + T_{yy}] + j_0^2/\sigma \tag{3.26}$$

Eqn.(3.26) can then be used to determine the arbitrary function $\psi'(y)$.

For the infinitely electrically conducting fluids, the unknowns in the system of equations (3.3) to (3.10) above are found as follows:

Eqn. (3.8) implies

$$f = \phi(y)$$

Then eqn. (3.6) yields

$$j = -[\psi'(y)\phi'(y) + \psi''(y)\phi(y)]$$

The pressure is, then, given by

$$p(x,y) = 2C_1 \nu x + \int \mu j \phi(y)\psi'(y)\, dy$$

or

$$p(x,y) = 2C_1 \nu x - (1/2)\mu[\phi(y)\psi'(y)]^2 + c$$

where c is an arbitrary constant.

The temperature is given by

$$T(x,y) = \frac{C_1 y^2 + C_2 y + C_3}{R\psi'(y)}(2C_1 \nu x - (1/2)\mu[\phi(y)\psi'(y)]^2 + c)$$

The other unknowns q, ω and ρ remain the same as in the finitely conducting case.

Particular Solution [finitely conducting case]

If we take $C_1 = 0$ in eqn. (3.12), then employing eqns. (3.22) and (3.24) in eqn. (3.26), we get

$$(-j_o y^2/2 + cy + d)\left(\frac{q(y)}{\psi'(y)}\right)'' + 2(-j_o y + c)\left(\frac{q(y)}{\psi'(y)}\right)' - j_o\left(\frac{q(y)}{\psi'(y)}\right)$$

$$= \frac{R}{\mu j_o k}\,[\, j_o^2/\sigma - \nu C_2^2 \,] \qquad (3.27)$$

If we let the right-hand side of eqn. (3.27) to be another constant C^* (say), then the solution to eqn. (3.27) is given by

$$\frac{q(y)}{\psi'(y)} = \frac{C^* y^2/2 + Dy + E}{-j_o y^2/2 + cy + d} \qquad (3.28)$$

Then

$$\psi'(y) = \frac{(-j_o y^2/2 + cy + d)(C_2 y + C_3)}{C^* y^2/2 + Dy + E} \qquad (3.29)$$

Thus, with $C_1 = 0$, we have

$$q(y) = C_2 y + C_3$$

$$\omega = -C_2$$

$$\rho(y) = \frac{-j_o y^2/2 + cy + d}{C^* y^2/2 + Dy + E}$$

$$p(y) = -\mu j_o^2 y^2/2 + \mu j_o c y + \mu j_o d$$

$$T(y) = \frac{C_2 y + C_3}{R\psi'(y)}(-\mu j_o^2 y^2/2 + \mu j_o c y + \mu j_o d)$$

where $\psi'(y)$ is as given in eqn.(3.29).

Boundary Value Problem

Consider the flow of a viscous compressible fluid in the presence of a magnetic field, between two plates, separated by a distance h, when

(1) Both plates are fixed

The boundary conditions are

$$q = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = h$$

Using these conditions on eqn.(3.12), we obtain

$$C_3 = 0$$

$$C_2 = -C_1 h$$

Hence

$$q = C_1 y(y - h)$$

which implies that the velocity profile is parabolic in shape. In the case above, q_{max} occurs at $y = h/2$.

(2) Upper plate moving with velocity u^* and lower plate fixed.

The boundary conditions are

$$q(0) = 0 \quad \text{and} \quad q(h) = u^*$$

In this case

$$C_3 = 0$$

$$C_2 = \frac{u^* - C_1 h^2}{h}$$

Hence

$$q(y) = \frac{u^*}{h} y + C_1 y(y - h) \quad , \quad C_1 < 0 .$$

and

$$q_{max} \text{ occurs at } y = \frac{h}{2} - \frac{u^*}{2hC_1}$$

(3) Both plates moving.

The boundary conditions in this case are

$$q(0) = u_1$$
$$q(h) = u_2$$

Using these conditions on eqn.(3.12), we get

$$C_3 = u_1$$
$$C_2 = \frac{u_2 - u_1 - C_1 h^2}{h}$$

and hence

$$q = C_1 y^2 + [(u_2 - u_1 - C_1 h^2)/h] y + u_1$$

Example 2

Let $q = q(x) \quad , \quad q'(x) \neq 0$

Then eqn.(3.11) gives

$$q''(x) - \frac{1}{\nu} \psi'(y) q'(x) = L(x) \qquad (3.30)$$

which implies that

$$\psi'(y) = \text{a constant} = k\nu \text{ (say)} \quad , \quad k \neq 0 \qquad (3.31)$$

and

$$q(x) = L(x) + k \int L(x) dx + A e^{-kx} + B \qquad (3.32)$$

where A and B are arbitrary constants.

Employing eqns.(3.31) and (3.32) in the system of equations (3.3) to (3.10), we obtain

$$\omega = 0$$

$$\rho = \frac{k\nu}{L(x) + k\int L(x)dx + Ae^{-kx} + B}$$

$$f = -j_o y/k\nu + c$$

$$P(x,y) = -k\nu q(x) + \mu\, j_o k\nu(-j_o y^2/2k\nu + cy + d)$$

$$p(x,y) = P + (4/3)\nu q'(x)$$

$$T(x,y) = \frac{q(x)}{Rk\nu}(P + (4/3)\nu q'(x))$$

where c and d are arbitrary constants.

In the infinitely conducting case, the above results are modified to

$$f = \gamma(y)$$

where $\gamma(y)$ is an arbitrary function of y.

$$j = - k^* \gamma\,'(y)$$

$$P(x,y) = - k^* q(x) - (1/2)\mu k^{*2}[\gamma(y)]^2 + m$$

where m is an arbitrary constant.

$$p(x,y) = - k^* q(x) + (4/3)\nu q'(x) - (1/2)\mu k^{*2}[\gamma(y)]^2 + m$$

$$T(x,y) = (1/k^* R)\int\phi(x)dx\ (p(x,y))$$

where $k^* = k\nu$.

Example 3

Let $q(x,y) = F(y) + G(x)$, $F'(y)\neq 0$, $G'(x)\neq 0$

Eqn.(3.9) becomes

$$\omega = -F'(y) \tag{3.33}$$

Then with eqn.(3.33), eqn.(3.11) becomes

$$F''(y) + G''(x) - \frac{1}{\nu}\psi'(y)G'(x) = L(x) \tag{3.34}$$

Differentiating eqn.(3.34) with respect to y, we get

$$F'''(y) - \frac{1}{\nu}\psi''(y)G(x) = 0 \tag{3.35}$$

Eqn.(3.35) leads to the following two cases:

(a) $\psi''(y) = 0$

(b) $G'(x) = \text{constant} = A_o$ (say)

Case (a)

$\psi''(y) = 0$ gives

$$\psi'(y) = \text{constant} = d\nu \tag{3.36}$$

$$F(y) = ay^2 + by + c \tag{3.37}$$

where a,b,c,d are arbitrary constants.

Eqns.(3.36) and (3.37) must satisfy eqn.(3.34) which gives,

$$G''(x) - k^*G'(x) = L(x) \tag{3.38}$$

where $k^* = d/2a$. The solution to eqn.(3.38) is given by

$$G(x) = L(x) + k^*\int L(x)dx + Ce^{-k^*x} + D \tag{3.39}$$

where C and D are arbitrary constants. Hence

$$q(x,y) = ay^2 + by + c + L(x) + k^*\int L(x)dx + Ce^{-k^*x} + D \tag{3.40}$$

Employing eqns.(3.36),(3.37) and (3.39) in the system of equations (3.3) to (3.10), we obtain

$$\omega = -2ay - b$$

$$\rho = \frac{d\nu}{q(x,y)}$$

$$f = -j_o y/d\nu + c$$

$$P = 2\nu ax - d\nu G(x) + \mu\, j_o d\nu(-j_o y^2/2d\nu + cy + d)$$

$$p = P + (4/3)\nu G'(x)$$

$$T = \frac{F(y)+G(x)}{Rd\nu}\left(P + (4/3)\nu G'(x) \right)$$

For the infinitely conducting case, the above results are modified to

$$f = \gamma(y)$$

where $\gamma(y)$ is an arbitrary function of y.

$$j = -D_o\gamma'(y)$$

$$P(x,y) = 2\nu C_o x - D_o G(x) - (1/2)\mu D_o^2[\gamma(y)]^2 + k$$

where k is a arbitrary constant.

$$p(x,y) = 2\nu C_o x - D_o G(x) + (4/3)\nu G'(x) - (1/2)\mu[\gamma(y)\psi'(y)]^2 + c$$

where c is an arbitrary constant.

$$T(x,y) = (1/D_o R)[F(x) + G(x)]p(x,y)$$

Case (b)

$$G'(x) = A_o$$

Then

$$G(x) = A_o x + B_o \tag{3.41}$$

where A_o and B_o are arbitrary constants.

Then eqn.(3.35) becomes

$$F'''(y) - \frac{1}{\nu}\psi''(y)A_o = 0$$

Whose solution is given by

$$F(y) = (A_o/\nu)\int\psi(y)dy + l_o y^2 + l_1 y + l_2 \tag{3.42}$$

Therefore,

$$q(x,y) = (A_o/\nu)\int\psi(y)dy + l_o y^2 + l_1 y + l_2 + A_o x + B_o \tag{3.43}$$

Employing eqns.(3.41) and (3.42) into the system of equations

(3.3) to (3.10), we obtain

$$\omega = -[(A_o/\nu)\psi(y) + l_o y + l_1]$$

$$\rho = \psi'(y)/q(x,y)$$

$$f(y) = (1/\psi'(y))(-j_o y + c)$$

$$P(x,y) = \nu F''(y)x - \psi'(y)G(x) - \mu j_o^2 y^2/2 + \mu j_o cy + d$$

$$p(x,y) = \nu l_o x + (4/3)\nu A_o - \mu j_o^2 y^2/2 + \mu j_o cy + d$$

$$T(x,y) = \frac{F(y + G(x)}{R\psi'(y)} p(x,y)$$

For the infinitely conducting case, the above results are modified to

$$f = \gamma(y)$$

$$j = -\psi'(y)\gamma'(y) - \psi''(y)\gamma(y)$$

$$P(x,y) = \nu F''(y)x - \psi'(y)G(y) - (1/2)\mu[\gamma(y)\psi'(y)]^2 + c$$

where c is an arbitrary constant.

$$p(x,y) = \nu l_o x + (4/3)\nu A_o - (1/2)\mu[\gamma(y)\psi'(y)]^2 + c$$

$$T(x,y) = [\{F(x) + G(x)\}/R\psi'(y)]\, p(x,y)$$

Example 4

Let $q(x,y) = F(y)G(x)$, $F'(y) \neq 0$, $G'(x) \neq 0$.

Using this in equation (3.11), we get

$$\psi'(y)F(y) - \nu F''(y)\frac{G(x)}{G'(x)} - \nu F(y)\frac{G''(x)}{G'(x)} = \frac{L(x)}{G'(x)} \tag{3.44}$$

Differentiating eqn.(3.44) w.r.t. x, we get

$$\nu F''(y)\left[\frac{G(x)}{G'(x)}\right]' - \nu F(y)\left[\frac{G''(x)}{G'(x)}\right]' = \left[\frac{L(x)}{G'(x)}\right]' \tag{3.45}$$

Differentiating eqn. (3.45) w.r.t. y, we get

$$-\nu F'''(y)\left[\frac{G(x)}{G'(x)}\right]' - \nu F'(y)\left[\frac{G''(x)}{G'(x)}\right]' = 0 \tag{3.46}$$

Eqn.(3.46) implies

$$\frac{F'''(y)}{F'(y)} = - \left[\frac{G''(x)}{G'(x)} \right]' / \left[\frac{G(x)}{G'(x)} \right]' = \text{constant} = A \text{ (say)}$$

Thus we have

$$F''(y) = AF(y) + B \tag{3.47}$$

$$G''(x) - CG'(x) + AG(x) = 0 \tag{3.48}$$

where A, B, C are arbitrary constants.

The solution to eqn. (3.47) is given by

$$F(y) = C_1 \exp(\sqrt{A}y) + C_2 \exp(-\sqrt{A}y) - B/A \tag{3.49}$$

where C_1 and C_2 are arbitrary constants.

The solution of eqn. (3.48) is given by

$$G(x) = \begin{cases} D_1 \exp(m_1 x) + D_2 \exp(m_2 x) & , \text{ if } C^2 > 4A \\ D_3 \exp(m_1 x) + D_4 x \exp(m_1 x) & , \text{ if } C^2 = 4A \\ E_1 \cos(\sqrt{4A - C^2}) + E_2 \sin(\sqrt{4A - C^2}) & , \text{ if } C^2 < 4A \end{cases} \tag{3.50}$$

where

$$m_1, m_2 = [C \pm (c^2 - 4A)^{1/2}]/2$$

and D_1, D_2, D_3, D_4, E_1, E_2 are arbitrary constants.

If in the equation following eqn.(3.46), we have

$$\left[\frac{G(x)}{G'(x)} \right]' = 0$$

then

$$G(x) = A_1 \exp(1/D)x \tag{3.50'}$$

where A_1 and D are arbitrary constants.

From eqn. (3.45), we have

$$\frac{G(x)}{G'(x)} = -\frac{1}{\nu B}\frac{L(x)}{G'(x)} + C \qquad (3.51)$$

Employing eqn. (3.51) in (3.44), we get

$$\psi'(y) = \frac{\nu BC}{F(y)} + \nu C \qquad (3.52)$$

Then

$$\omega = - G(x)\ F'(y)$$

$$\rho = \frac{\nu BC}{F^2(y)\ G(x)} + \frac{\nu C}{F(y)G(x)}$$

$$f(y) = [1/\psi'(y)](-j_o y + c)$$

$$P(x,y) = -\nu G'(x)F(y) \quad \mu j_o^2 y^2/2 + \mu j_o cy + \nu B\int G(x)dx - \nu BCG(x)$$

$$p(x,y) = P(x,y) + (4/3)\nu G'(x)F(y)$$

$$T(x,y) = \frac{F(y)G(x)}{R\psi'(y)}\, p(x,y)$$

where G(x) is given by eqn.(3.50) or (3.50').

In the infinitely conducting case, the above results are modified to

$$f = \phi(y)$$

where $\phi(y)$ is an arbitrary function of y.

$$j = -\psi'(y)\phi'(y) - \psi''(y)\phi(y)$$

$$P(x;y) = -\nu G'(x)F(y) + \nu B\int G(x)\ dx \quad \nu BCG(x) - (1/2)\mu[\phi(y)\psi'(y)]^2 + c$$

where c is an arbitrary constant.

$$p(x,y) = (1/3)\nu G'(x)F(y) + \nu B\int G(x)\ dx - \nu BcG(x) - (1/2)\mu[\phi(y)\psi'(y)]^2 + c$$

$$T(x,y) = (1/R)\{[\nu Bc/F^2(y)G(x)] + [\nu c/F(y)G(x)\}\ p(x,y)$$

Example 5

$$\text{Let} \quad q(x,y) = xG(y) + H(y) \quad , \quad G'(y) \neq 0 \; , H'(y) \neq 0.$$

Using this in eqn. (3.11), we get

$$\psi'(y)G(y) - \nu H''(y) = \nu x G''(y) + L(x) \tag{3.53}$$

For the right-hand side of eqn.(3.53) to be a function of x only, that is, for eqn.(3.53) to hold for all x and y, we must have

$$G''(y) = \text{constant} = 2a \text{ (say)} \tag{3.54}$$

Then $G(y) = ay^2 + by + c$

and $L(x) = 2ax + d$ (3.55)

Differentiating eqn. (3.53) with respect to y, we obtain

$$H''(y) = (1/\nu)[\psi'(y)G(y)] \tag{3.56}$$

Integrating eqn. (3.56) twice with respect to y, we get

$$H(y) = (1/\nu)\int\int\psi'(y)G(y)\,dy + Ay + B \tag{3.57}$$

where A and B are constants of integration and $\psi'(y)$ is an arbitrary function of y.

Then

$$\omega = - xG'(y) - H'(y)$$

$$\rho = \psi'(y)/[xG(y) + H(y)]$$

$$f = [1/\psi'(y)]\{- j_o y + c\}$$

$$P(x,y) = - \nu G(y) - \mu j_o^2 y^2/2 + \mu j_o cy + d + \nu a x^2$$

$$p(x,y) = P(x,y) + (4/3)\nu G(y)$$

$$T(x,y) = (1/R)\{[xG(y) + H(y)]/\psi'(y)\}\, p(x,y)$$

For the infinitely conducting fluids, the above results are modified to

$$f = \phi(y)$$

where $\phi(y)$ is an arbitrary function of y.

$$j = -\psi'(y)\phi'(y) - \psi''(y)\phi(y)$$

$$P(x,y) = -\nu G(x) + \nu a x^2 - (1/2)\mu[\phi(y)\psi'(y)]^2 + c$$

where c is an arbitrary constant.

$$p(x,y) = (1/3)\nu G(y) + \nu a x^2 - (1/2)\mu[\phi(y)\psi'(y)]^2 + c$$

$$T(x,y) = (1/R)[\{xG(y) + H(y)\}/\psi'(y)]\ p(x,y)$$

Example 6

Let $q(x,y) = yG(x) + H(x)$, $G'(x) \neq 0, H'(x) \neq 0$

Employing this in eqn.(3.11), we get

$$y\psi'(y)G'(x) + \psi'(y)H'(x) - \nu yG''(x) - \nu H''(x) = L(x) \quad (3.58)$$

Differentiating eqn. (3.58) with respect to y, we get

$$\psi'(y) + y\psi''(y) = [1/G'(x)]\{\nu G''(x) - \psi''(y)H'(x)\} \quad (3.59)$$

For eqn. (3.59) to hold for all x and y, we must have

$$\psi''(y) = \text{constant} = k \text{ (say)}$$

Then

$$\psi'(y) = ky + l \quad (3.60)$$

Then eqn. (3.59) becomes

$$\nu G''(x) - (2ky + l)G'(x) - kH'(x) = 0 \quad (3.61)$$

Eqn. (3.61) implies that

$$k = 0$$

and

$$\nu G''(x) - lG'(x) = 0 \quad (3.62)$$

The solution to eqn. (3.62) is given by

$$G(x) = C_1 + C_2 \exp((l/\nu)x) \quad (3.63)$$

Since k = 0 eqn. (3.60) reduces to

$$\psi'(y) = 1 \tag{3.64}$$

Then eqn.(3.58) becomes

$$\nu H''(x) - 1H'(x) = -L(x) \tag{3.65}$$

The solution to eqn.(3.65) is given by

$$H(x) = A + B\exp[(1/\nu)x] + (\nu/1)\int L(x)dx - (\nu/1)\exp[(1/\nu)x]* \int\exp[-(1/\nu)x]L(x)\,dx \tag{3.66}$$

where A and B are arbitrary constants of integration.

Differentiating eqn.(3.59) with respect to x, we obtain

$$[\ H'(x)/G'(x)\ ]'/\psi''(y) = \nu[\ G''(x)/G'(x)\ ] \tag{3.67}$$

Eqn.(3.67) implies either $\psi''(y)$ is a constant, which has been considered above or that

$$[\ H'(x)/G'(x)\ ]' = 0 \tag{3.68}$$

Eqn.(3.68) in turn implies

$$H(x) = A_0G(x) + B_0 \tag{3.69}$$

where A_0 and B_0 are arbitrary constants.

Substituting (3.69) into (3.59), we get

$$\nu G''(x) - [\psi'(y) + (y + A_0)\psi''(y)]G'(x) = 0 \tag{3.70}$$

Since $G'(x) \neq 0$, eqn.(3.70) implies

$$\psi'(y) + (y + A_0)\psi''(y) = \text{constant} = C_0 \text{ (say)} \tag{3.71}$$

Then eqn.(3.70) becomes

$$\nu G''(x) - C_0G'(x) = 0 \tag{3.72}$$

Solving eqn.(3.71) yields $\psi'(y)$. The solution to eqn.(3.72) is given by

$$G(x) = C_3 + C_4\exp[(C_0/\nu)x] \tag{3.73}$$

where C_3 and C_4 are arbitrary constants. Then eqn.(3.69) gives

Then eqn.(3.69) gives

$$H(x) = D + E\exp[(C_o/\nu)x] + B_o \tag{3.74}$$

where D and E are arbitrary constants.

Then

$$\omega = -G(x)$$

$$\rho = 1/[yG(x) + H(x)]$$

$$f = -(j_o/l)y + c$$

$$P(x,y) = -lyG(x) - lH(x) - \mu j_o^2(y^2/2) + \mu j_o lcy + d$$

$$p(x,y) = P(x,y) + (4/3)\nu[yG'(x) + H'(x)]$$

$$T(x,y) = (1/R)\{[yG(x) + H(x)]/l\}p(x,y)$$

where G(x) is given by either eqns.(3.63) or (3.73) and H(x) is given by either eqns.(3.66) or (3.74).

For the infinitely conducting fluids, the above results are modified to

$$f = \phi(y)$$

where $\phi(y)$ is an arbitrary function of y

$$j = -l\phi'(y)$$

$$P(x,y) = -l[yG(x) + F(x)] - (1/2)\mu l^2[\phi(y)]^2 + m$$

where m is an arbitrary constant.

$$p(x,y) = -l[yG(x) + H(x)] + (4/3)\nu[yG'(x) + H'(x)] - (1/2)\mu l^2[\phi(y)]^2 + m$$

$$T(x,y) = (1/R)\{[yG(x) + H(x)]/l\}\, p(x,y)$$

CHAPTER 4

VORTEX FLOW

In this type of flow, we take (ζ,η) to be (θ,r), that is, we choose

$$\zeta = \theta \qquad , \qquad \eta = r \tag{4.1}$$

where r,θ are polar coordinates and therefore we have

$$ds^2 = dr^2 + r^2 d\theta^2 .$$

Comparing this equation with the following equation,

$$ds^2 = g_1^2\, d\zeta^2 + g_2^2\, d\eta^2$$

we have,

$$g_1 = r \qquad \text{and} \qquad g_2 = 1. \tag{4.2}$$

With this choice of g_1 and g_2, the Gauss equation is identically satisfied.

Employing (4.1) and (4.2) in the system of equations (2.67) to (2.75), we find that the equations governing finitely conducting vortex flow are given as follows:

Finite Electrical Conductivity

$$q = \frac{\psi'(r)}{\rho} \tag{4.3}$$

$$\nu r\omega_r + \psi'(r) q_\theta = -P_\theta . \tag{4.4}$$

$$(\nu/r)\omega_{\Theta} - \psi'(r)\omega - \psi'(r)q_r + \mu j_o\psi'(r)f = P_r \qquad (4.5)$$

$$\nu[\omega^2 - (4q/r)q_r] = (\psi'(r)/r)e_{\Theta} + (p/r)q_{\Theta}$$
$$-k[(1/r^2)T_{\Theta\Theta} + T_{rr} + (1/r)T_r] + j_o^2/\sigma \qquad (4.6)$$

$$f\rho\omega - \psi'(r)f_r - \psi'(r)f\rho^{-1}\rho_r = j_o \qquad (4.7)$$

$$f_{\Theta} = 0 \qquad (4.8)$$

$$\omega = -(1/r)[rq]_r \qquad (4.9)$$

$$P = f(\rho,T) - (4/3)\nu(1/r)q_{\Theta} \qquad (4.10)$$

From equations (4.4) and (4.5), $P_{\Theta r} = P_{r\Theta}$ gives

$$[\psi'(r)q_{\Theta} + \nu r\omega_r]_r = [\psi'(r)q_r + \psi'(r)\omega - (\nu/r)\omega_{\Theta} - \mu j_o\psi'(r)f]_{\Theta} \qquad (4.11)$$

The equations governing infinitely conducting vortex flow are similar to the ones for the finitely conducting case with the exception that j, the current density is no longer a constant but a function of Θ and r and the last term in the energy equation vanishes.

Assuming now, a priori certain forms of $q(\Theta,r)$, we solve the system of equations (4.3) to (4.10).

Example 1

Assume $q = q(\Theta)$ such that $q'(\Theta) \neq 0$.

Eqn. (4.9) then becomes

$$\omega = -(1/r)q(\Theta) \qquad (4.12)$$

Substituting (4.8) and (4.12) into (4.11) and simplifying, we get

$$q''(\theta) - (1/\nu)[r^2\psi''(r) + r\psi'(r)]q'(\theta) + q(\theta) = 0 \qquad (4.13)$$

Since $q = q(\theta)$, we must have in eqn.(4.13) that

$$r^2\psi''(r) + r\psi'(r) = \text{constant} = a \text{ (say)} \qquad (4.14)$$

and thus eqn.(4.13) becomes

$$q''(\theta) - (a/\nu)q'(\theta) + q(\theta) = 0 \qquad (4.15)$$

The solutions to eqns.(4.14) and (4.15) respectively, are

$$\psi(r) = A_1(\ln r)^2 + A_2 \ln r + A_3 \qquad (4.16)$$

where A_1, A_2 and A_3 are arbitrary constants and .

$$q(\theta) = \begin{cases} A\exp(m_1\theta) + B\exp(m_2\theta) & ; \text{ if } a^2 > 4\nu^2 \\ (A + B\theta)\exp[(a/2\nu)\theta] & ; \text{ if } a^2 = 4\nu^2 \\ \exp[(a/2\nu)\theta][C \cos(m\theta/2\nu) + D \sin(m\theta/2\nu)] & ; \text{ if } a^2 < 4\nu^2 \end{cases} \qquad (4.17)$$

where

$$m_1, m_2 = (1/2\nu)[a \mp (\alpha^2 - 4\nu^2)^{1/2}] \text{ and } m^2 = 4\nu^2 - a^2.$$

Thus the vorticity is given by eqn.(4.12) and the density ρ is

$$\rho = \frac{2A_1 \ln r + A_2/r}{q(\theta)} \qquad (4.18)$$

where $q(\theta)$ is as given in (4.17).

Eqn.(4.7) then implies

$$f'(r) + [a/(r\,\psi'(r))] = -j_o/\psi'(r) \qquad (4.19)$$

The solution to eqn.(4.19) is given by

$$f(r) = [-j_o r]/[2\psi'(r)] + c/[r\psi'(r)] \qquad (4.20)$$

The two generalized momentum eqns.(4.4) and (4.5) imply

$$P_r = (-\nu/r^2)q' + (\psi'(r)/r)q + \mu j_o \psi'(r) f(r)$$

$$P_\theta = (-\nu/r)q - \psi'(r)q'$$

from which, we obtain

$$P(\theta,r) = (\nu/r)q' - \psi'(r)q + (2A_1/r)q + \mu j_o \int \psi'(r)f(r)dr + d$$

where d is an arbitrary constant.

The pressure is then given by

$$p(\theta,r) = P(\theta,r) + (4\nu/3r)q' \quad \text{or}$$

$$p(\theta,r) = (7\nu/3r)q' - \psi'(r)q + (2A_1/r)q + \mu j_o \int \psi'(r)f(r)dr + d. \tag{4.21}$$

Then the temperature is given by

$$T(\theta,r) = \frac{q(\theta)}{R\psi'(r)} \{p(\theta,r)\} \tag{4.22}$$

In the case of infinite conductivity, eqn. (4.8) implies

$$f = \phi(r)$$

where ϕ is an arbitrary function.

Then eqn. (4.7) yields

$$j = -\psi'(r)\phi'(r) - (a/r^2)\phi(r)$$

The pressure is given by

$$p(\theta,r) = (7\nu/3r)q' - \psi'(r)q + (2A_1/r)q$$
$$- \mu\int[\psi'(r)^2\phi(r)\phi'(r) - (a/r^2)\psi'(r)\phi(r)^2]dr + d$$

Finally, the temperature is given by

$$T(\theta,r) = \frac{q(\theta)}{R\psi'(r)}\{p(\theta,r)\}$$

Example 2

We assume that $q = q(r)$, $q'(r) \neq 0$.

Eqn. (4.9) then becomes

$$\omega = -(1/r)q(r) - q'(r). \tag{4.23}$$

Substituting eqn.(4.23) into eqn.(4.11), we get

$$r^2q'' + rq' - q = ar \tag{4.24}$$

where a is an arbitrary constant.

The general solution to eqn.(4.24) is given by

$$q(r) = C_1 r + C_2/r + (ar/2)\ln r - ar/4. \tag{4.25}$$

Then from eqn.(4.3), we have

$$\rho = \psi'(r)/q(r)$$

where $\psi'(r)$ is an arbitrary function of r.

Eqn.(4.23) then yields

$$\omega = -(2C_1 + a\ln r)$$

Eqn.(4.7) gives

$$f(r) = [-j_o r/2\psi'(r)] + [c/r\psi'(r)]$$

From eqns.(4.4) and (4.5), the generalized pressure is

$$P(\theta,r) = \nu\theta rq'' + \nu\theta q' - (\nu\theta/r)q + \int[\psi'(r)/r]q\,dr + \mu j_o\int\psi'(r)f(r)dr$$

Then from eqn.(4.10), the pressure is

$$p(\theta,r) = P(\theta,r)$$

The temperature is then given by

$$T(\theta,r) = \frac{q(r)}{R\psi'(r)}\{p(\theta,r)\}$$

For the case of infinite conductivity, the above results are modified to

$$f = \phi(r)$$

where ϕ is an arbitrary function.

$$j = -\psi'(r)\phi'(r) - \left[\frac{\psi'(r)}{r} + \psi''(r)\right]\phi(r)$$

$$p(\theta,r) = \nu\theta r q'' + \nu\theta q' - (\nu\theta/r)q + \int[\psi'(r)/r]q\,dr - \mu\int\psi'(r)^2\phi'(r)^2 dr$$
$$- \mu\int[\psi'(r)^2/r]\phi(r)^2 dr - \mu\int\psi'(r)\psi''(r)\phi(r)^2 dr$$

$$T(\theta,r) = \frac{q(r)}{R\psi'(r)}\{p(\theta,r)\}.$$

<u>Example 3</u>

Let $q(\theta,r) = F(r) + G(\theta)$, $F'(r)\neq 0$, $G'(\theta)\neq 0$

The vorticity in this case takes the form

$$\omega = -\frac{1}{r}[F(r) + G(\theta) + rF'(r)] \tag{4.26}$$

Substituting q and ω in eqn.(4.11), we obtain

$$r^2\psi''(r)G'(\theta) - \nu F(r) + \nu rF'(r) - 2\nu r^2F''(r) - \nu r^3F'''(r)$$
$$- \nu G(\theta) = -r\psi'(r)G'(\theta) + \nu G''(\theta) \tag{4.27}$$

Differentiating eqn.(4.27) with respect to θ, we get

$$r^2\psi''(r)G''(\theta) - \nu G'(\theta) = -r\psi'(r)G''(\theta) + \nu G'''(\theta) \tag{4.28}$$

or

$$G'''(\theta) - (1/\nu)[r\psi'(r) + r^2\psi''(r)]G''(\theta) + G'(\theta) = 0 \tag{4.29}$$

Since $G = G(\theta)$, eqn.(4.29) implies

$$r^2\psi''(r) + r\psi'(r) = \text{constant} = b \text{ (say)} \tag{4.30}$$

and therefore eqn.(4.29) becomes

$$G'''(\theta) - (b/\nu)G''(\theta) + G'(\theta) = 0. \tag{4.31}$$

If $G''(\theta) = 0$ in eqn.(4.29) then $G(\theta)$ is given by

$$G(\theta) = C_1\theta + C_2 \tag{4.32}$$

If $G''(\theta) \neq 0$ in eqn.(4.29) then the solutions of eqns.(4.30) and (4.31) respectively, are

$$\psi(r) = A(\ln r)^2 + B\ln r + C \tag{4.33}$$

and

$$G(\theta) = \begin{cases} D + E\exp(m_2\theta) + F\exp(m_3\theta) & ; \text{ if } b^2 > 4\nu^2 \\ D^* + [E^* + F^*\theta]\exp[(a/2\nu)\theta] & ; \text{ if } b^2 > 4\nu^2 \\ D_1 + \exp[(a/2\nu)\theta][E_1\cos(m\theta/2\nu) + F_1\sin(m\theta/2\nu)] & ; \text{ if } b^2 < 4\nu^2 \end{cases} \quad (4.34)$$

where

$$m_1, m_2 = [b \mp (b^2 - 4\nu^2)^{1/2}]/2\nu \quad \text{and} \quad m^2 = 4\nu^2 - b^2$$

Using eqn.(4.31) in eqn.(4.27), we obtain

$$r^3F'''(r) + 2r^2F''(r) - rF'(r) + F(r) = \text{constant} = A_o$$

whose solution is

$$F(r) = A_1 r + A_2 r\ln r + A_3/r + A_o. \quad (4.35)$$

Hence, we have

$$\rho = \frac{A(\ln r)^2 + B\ln r + C}{F(r) + G(\theta)} \quad (4.36)$$

$$f = [-j_o r]/2\psi'(r) + [c/r\psi'(r)] \quad (4.37)$$

$$P(\theta,r) = \nu\theta rF''(r) + \nu\theta F'(r) - (\nu\theta/r)F(r) - (\nu/r)\int G(\theta)d\theta$$
$$- \psi'(r)G(\theta) + \int\{\psi'(r)/r[F(r) + F'(r) - rF'(r)]\}dr$$
$$+ \mu j_o\int\psi'(r)f(r)dr \quad (4.38)$$

$$p(\theta,r) = P(\theta,r) + (4\nu/3r)G'(\theta) \quad (4.39)$$

$$T(\theta,r) = [\psi'(r)/\{R(F(r) + G(\theta))\}]p(\theta,r) \quad (4.40)$$

where $G(\theta)$ is as given in either eqn.(4.32) or (4.34).

For the case of infinite conductivity, the above results are modified to:

$$f = \phi(r)$$

where ϕ is an arbitrary function.

$$j = -\psi'(r)\phi'(r) - [\psi'(r)/r + \psi''(r)]\phi(r).$$

The pressure is given by

$$p(\theta,r) = \nu\theta rF''(r) + \nu\theta F'(r) - (\nu\theta/r)F(r) - (\nu/r)\int G(\theta)d\theta$$
$$- \psi'(r)G(\theta) + \int\{\psi'(r)/r[F(r) + F'(r) - rF'(r)]\}dr$$
$$+ \mu\int\psi'(r)^2\phi'(r)^2 dr - \mu\int\psi'(r)^2/r\ \phi(r)^2$$
$$- \mu\int\psi'(r)\psi''(r)\phi(r)^2 dr + (4\nu/3r)G'(\theta)$$

and finally

$$T(\theta,r) = [\psi'(r)/Rq(\theta,r)]p(\theta,r).$$

CHAPTER 5

RADIAL FLOW

In this type of flow, we take (ξ,η) to be (r,θ), that is, we choose

$$\xi = r \qquad , \qquad \eta = \theta \tag{5.1}$$

where r,θ are the polar coordinates and therefore we have

$$ds^2 = dr^2 + r^2 d\theta^2$$

Comparing this equation with the following equation,

$$ds^2 = g_1^2\, d\xi^2 + g_2^2\, d\eta^2$$

we obtain

$$g_1 = 1 \qquad \text{and} \qquad g_2 = r \tag{5.2}$$

With this choice of g_1 and g_2, the Gauss equation is identically satisfied.

Employing (5.1) and (5.2) in the system of equations (2.67) to (2.75), we find that the equations governing finitely conducting radial flow are as follows:

Finite Electrical Conductivity

$$q = \frac{\psi'(\theta)}{\rho r} \tag{5.3}$$

$$[\psi'(\theta)/r]q_r + (\nu/r)\omega_\theta = -P_r \tag{5.4}$$

$$\nu r\omega_r - \psi'(\theta)\omega - [\psi'(\theta)/r]q_\theta + \psi'(\theta)\mu j_o f = P_\theta \tag{5.5}$$

$$f\rho\omega - [\psi'(\theta)/r^2]f_\theta - (fq/r^2)[\psi'(\theta)/q]_\theta = j_o \tag{5.6}$$

$$\nu[\omega^2 + (4q/r)q_r] = [\psi'(\theta)/r]e_r + (p/r)[rq]_r - k[T_{rr} + (1/r^2)T_{\theta\theta} + (1/r)T_r] + (j_0^2/\sigma) \quad (5.7)$$

$$f_r = 0 \quad (5.8)$$

$$\omega = -(1/r)q_\theta \quad (5.9)$$

$$P = f(\rho,T) - (4/3)\nu[rq]_r \quad (5.10)$$

From equations (5.4) and (5.5), $P_{r\theta} = P_{\theta r}$ gives

$$[(\psi'(\theta)/r) + (\nu/r)\omega_\theta]_\theta = [(\psi'(\theta)/r)q_\theta + \psi'(\theta)\omega - \nu r\omega_r + \psi'(\theta)\mu j_0 \hat{f}]_r \quad (5.11)$$

After simplification, equation (5.11) can be written as

$$\nu[r^2q_{rr} - rq_r + q_{\theta\theta} + q] - r\psi'(\theta)q_r = r^2L(r) \quad (5.12)$$

OR

$$\nu[r^2q_{rr} + rq_r + q_{\theta\theta}] - 2\nu rq_r + \nu q - \psi'(\theta)rq_r = r^2L(r)$$

The equations governing infinitely conducting radial flow are similar to the ones for the finitely conducting case with the exception that j, the current density is no longer a constant but a function of r and θ and the last term in the energy equation vanishes.

Assuming, now, a priori certain forms of $q(r,\theta)$, we solve the system of equations (5.3) to (5.10).

Example 1

Assume q is such that in eqn. (5.12),

$$r^2q_{rr} + rq_r + q_{\theta\theta} = 0 \quad (5.13)$$

Then by the method of separation of variables, we seek a solution of the form

$$q(r,\theta) = R(r)\Theta(\theta) \tag{5.14}$$

Substitution of (5.14) in (5.13) yields

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda \text{ (say)}$$

where λ is a constant.

Hence, we have

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \tag{5.15}$$

and

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \tag{5.16}$$

To ensure that the function $\Theta(\theta)$ is single-valued, the case $\lambda < 0$ does not yield an acceptable solution. Thus $\lambda \geq 0$.

We first consider the case $\lambda = 0$.

Then eqns.(5.15) and (5.16) become

$$r^2R''(r) + rR'(r) = 0 \tag{5.17}$$

$$\Theta''(\theta) = 0 \tag{5.18}$$

The solutions of eqns.(5.17) and (5.18) respectively, are

$$R(r) = A + B\ \ln r \tag{5.19}$$

$$\Theta(\theta) = C\theta + D \tag{5.20}$$

where A,B,C and D are arbitrary constants.

Thus the complete solution is given by

$$q(r,\theta) = (A + B\ \ln r)(C\theta + D) \tag{5.21}$$

Now, substituting (5.21) into (5.12), we get

$$2\nu B(C\theta + D) + \nu(A + B\ \ln r)(C\theta + D) - \psi'(\theta)B(C\theta + D) = r^2L(r)$$

Differentiating the above eqn. with respect to r, we get

$$\nu B(C\theta + D) = 2r^2L(r) + r^3L'(r) \tag{5.22}$$

For eqn. (5.22) to hold, we must have $C = 0$.

Then, we have

$$[2\nu - \psi'(\theta)]BD + \nu AD + \nu BD\ln r = r^2L(r)$$

where we define

$$2\nu BD - \psi'(\theta)BD + \nu AD = k \tag{5.23}$$

where k is some constant. Then eqn. (5.23) implies

$$\psi'(\theta) = \text{constant} = k^* \text{ (say)}$$

Then since $C = 0$, eqn. (5.21) becomes

$$q(r,\theta) = A_1 + B_1\ln r \tag{5.24}$$

Then eqn. (5.9) becomes

$$\omega = 0 \tag{5.25}$$

and eqn. (5.3) yields

$$\rho = k^*/[r(A_1 + B_1\ln r)] \tag{5.26}$$

Eqns. (4.4) and (5.5) take the form

$$P_r = (k^*/r)q'(r) \tag{5.27}$$

$$P_\theta = - k^*\mu j_o f \tag{5.28}$$

To obtain P_θ in eqn. (5.28), we must, first, determine f.

Eqns. (5.6) and (5.8) imply

$$(k^*/r^2)f'(\theta) = - j_o \tag{5.29}$$

Since $f = f(\theta)$, we must have $j_o = 0$. Thus

$$f(\theta) = \text{constant} = c \tag{5.30}$$

From eqns. (5.27), (5.28) and (5.30), we obtain

$$P(r,\theta) = - k^*\mu j_o c\theta + k^*A_1\ln r - (k^*B_1/r) + d \tag{5.31}$$

Then from eqn. (5.10), the pressure is given by

$$p(r,\theta) = P(r,\theta) + (4/3)(\nu/r)[A_1 + (1 + \ln r)B_1] \tag{5.32}$$

Hence, the temperature is given by

$$T(r,\theta) = p(r,\theta)/\rho R \quad (5.33)$$

where p and ρ are given by (5.32) and (5.26) respectively.

Finally, the energy eqn. implies

$$4\nu(B_1/r^2)q = (k^*/r)C_v T_r + (p/r)[rq]_r - k\,[T_{rr} + (1/r^2)T_{\theta\theta} + (1/r)T_r] + (j_0^2/\sigma) \quad (5.34)$$

In the case of infinite conductivity, the above results can be similarly obtained.

Example 2

In example 1, we considered the case where $\lambda = 0$. Now, we consider the case where $\lambda>0$. Then eqns. (5.15) and (5.16) are

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \quad (5.35)$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad (5.36)$$

Eqn. (5.35) is the Euler eqn. and, therefore, the general solution is

$$R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}}$$

The general solution of eqn. (5.36), using the separation of variable method is given by

$$\Theta(\theta) = C\cos\sqrt{\lambda}\,\theta + D\sin\sqrt{\lambda}\,\theta$$

For the solution to be periodic, we must have

$$\sqrt{\lambda} = m \quad \text{for} \quad m = 1,2,3,\ldots$$

Thus ,the complete solution is

$$q(r,\theta) = (Ar^m + Br^{-m})(C\cos m\theta + D\sin m\theta) \quad (5.37)$$

Substituting eqn. (5.37) into (5.12), we obtain

$$-2\nu m(Ar^m - Br^{-m}) + \nu(Ar^m + Br^{-m}) - \psi'(\theta)m(Ar^m - Br^{-m})$$

$$= r^2L(r)/M(\theta) \qquad (5.38)$$

where

$$M(\theta) = C \cos m\theta + D \sin m\theta$$

Differentiating eqn. (5.38) with respect to θ, we get

$$-m(Ar^m - Br^{-m})\psi''(\theta) = r^2L(r)\{M'(\theta)/M^2(\theta)\} \qquad (5.39)$$

This eqn. can be rewritten as

$$\psi''(\theta) = r^2L(r)/[-m(Ar^m - Br^{-m})]\ \{M'(\theta)/M^2(\theta)\} \qquad (5.40)$$

where

$$r^2L(r) = mC_1(Ar^m - Br^{-m})$$

where C_1 is a constant. Then eqn. (5.40) becomes

$$\psi''(\theta) = -C_1[M'(\theta)/M^2(\theta)] \qquad (5.41)$$

Integrating eqn. (5.41) with respect to θ yields

$$\psi'(\theta) = -[C_1/M(\theta)] + C_2 \qquad (5.42)$$

Then using (5.42) in (5.38), we obtain

$$-2\nu m(Ar^m - Br^{-m}) + \nu(Ar^m + Br^{-m}) - C_2m(Ar^m - Br^{-m}) = 0 \qquad (5.43)$$

which holds for all r if and only if

$$A(\nu - 2\nu m - C_2m) = 0 \qquad (5.44)$$

and

$$B(\nu + 2\nu m + C_2m) = 0 \qquad (5.45)$$

Eqns. (5.44) and (5.45) lead to the following two cases:

Case 1 $\quad A = 0 \quad , \quad B \neq 0$

In this case,

$$q(r,\theta) = Br^{-m}M(\theta)$$

$$\psi'(\theta) = -[C_1/M(\theta)] + (1/m)[\nu - 2\nu m]$$

$$\rho = \frac{-C_1m + (\nu - 2\nu m)M(\theta)}{mBr^{-m+1}M^2(\theta)}$$

$$\omega = -Br^{-m-1}M'(\theta)$$

Eqn.(5.6) implies

$$\psi'(\theta)f'(\theta) + \psi''(\theta)f(\theta) = -j_o r^2 \qquad (5.46)$$

which imply that j_o must be zero and hence

$$f(\theta) = k/\psi'(\theta)$$

where k is some constant.

Then

$$P(r,\theta) = -\nu(m+1)Br^{-(m+1)}M(\theta) + [m/(m+1)]C_1Br^{-(m+1)} - \mu j_o k\theta + p_o$$

where p_o is an arbitrary constant and $m \neq -1$.

Thus,

$$p(r,\theta) = P(r,\theta) + (4/3)(1-m)\nu Br^{-(m+1)}M(\theta)$$

$$T(r,\theta) = [p(r,\theta)]/\rho R$$

where p and ρ are as given above.

<u>Case 2</u> $\quad A \neq 0 \quad , \quad B = 0$

In this case, we have

$$q(r,\theta) = Ar^m M(\theta)$$

$$\psi'(\theta) = -[C_1M(\theta)] - (1/m)[\nu + 2\nu m]$$

$$\rho = \frac{-C_1m - (\nu + 2\nu m)M(\theta)}{mAr^{m+1}M^2(\theta)}$$

$$\omega = -Ar^{m-1}M'(\theta)$$

$$f = k/\psi'(\theta)$$

$$P(r,\theta) = \nu(m-1)Ar^{m-1}M(\theta) + [m/m-1]C_1Ar^{m-1} - \mu j_o k\theta + p_o \ , \ m \neq 1$$

Then

$$p(r,\theta) = P(r,\theta) + (4/3)(m+1)\nu Ar^{m-1}M(\theta)$$

$T(r,\theta) = [p(r,\theta)]/\rho R$

In both cases considered above, for the infinitely conducting fluid, the results above can be similarly obtained.

Example 3

We assume $q = q(\theta)$ where $q'(\theta) \neq 0$

Then using q in eqn.(5.12), we get

$$q'''(\theta) + q'(\theta) = 0$$

which gives

$$q(\theta) = A + B\cos\theta + C\sin\theta \tag{5.47}$$

where A,B,C are arbitrary constants.

Then

$$\omega = \frac{1}{r}[B\sin\theta - C\cos\theta]$$

$$\rho = \frac{\psi'(\theta)}{r(A + B\cos\theta + C\sin\theta)}$$

where $\psi'(\theta)$ is an arbitrary function of θ.

$$f = k/\psi'(\theta)$$

where k is an arbitrary constant.

$$P(r,\theta) = (\nu/r)[C\cos\theta - B\sin\theta] - \mu j_o k\theta + p_o$$

Thus

$$p(r,\theta) = P(r,\theta) + (4\nu/3r)[A + B\cos\theta + C\sin\theta]$$

Then

$$T(r,\theta) = [p(r,\theta)]/\rho R$$

where p and ρ are as given above.

Example 4

We now assume $q = q(r)$ such that $q'(r) \neq 0$

Then using $q = q(r)$ in eqn.(5.12), we obtain

$$\nu r^2 q'' + [-\nu r - \psi'(\theta) r] q' + \nu q = r^2 L(r) \quad (5.48)$$

which on simplification gives

$$\psi'(\theta) = a$$

where a is a positive constant. Then eqn.(5.48) becomes

$$\nu r^2 q'' - r[\nu + a] q' + \nu q = r^2 L(r) \quad (5.49)$$

Eqn.(5.49) is a Cauchy-Euler equation. Its solution is given by

$$q(r) = C_1 r^{m_1} + C_2 r^{m_2} + U_1 r^{m_1} + U_2 r^{m_2} \quad (5.50)$$

where C_1 and C_2 are arbitrary constants and

$$U_1 = -[1/\nu(m_2 - m_1)] \int r^{1-m_1} L(r)\, dr$$

$$U_2 = [1/\nu(m_2 - m_1)] \int r^{1-m_2} L(r)\, dr$$

Also

$$m_1, m_2 = \frac{(2\nu + a) \mp [(2\nu + a)^2 - 4\nu^2]^{1/2}}{2\nu}$$

and L(r) is an arbitrary function.

Then

$$\omega = 0$$

$$\rho = \frac{a}{rq}$$

$$f = k$$

where k is some constant.

$$P(r, \theta) = p_0 - a\mu j_0 k\theta + a \int q'(r)/r\, dr$$

Thus

$$p(r,\theta) = P + (4\nu/3)[(1+m_1)(C_1+U_1)r^{m_1-2} + (1+m_2)(C_2+U_2)r^{m_2-2}]$$

Then

$$T(r,\theta) = [p(r,\theta)]/\rho R$$

where p and ρ are as given above.

Example 6

$$\text{Let } q = R(r)\,[Ae^{m\theta} + Be^{-m\theta}] \qquad \text{where } R'(r) \neq 0$$

Substituting q into eqn.(5.12) gives

$$\nu[r^2R'' - rR' + m^2R + R] - r\psi'(\theta)R' = r^2L(r)/M(\theta) \tag{5.51}$$

where

$$M(\theta) = Ae^{m\theta} + Be^{-m\theta}$$

Differentiating eqn.(5.51) with respect to θ and simplifying gives

$$\psi'(\theta) = -a/M(\theta) + c \tag{5.52}$$

and
$$r^2L(r) = arR' \tag{5.53}$$

Substitution of (5.52) and (5.53) into (5.51) yields

$$r^2R'' - r[1 + c/\nu]R' + [1 + m^2]R = 0 \tag{5.54}$$

whose solution is as follows:

Case 1

When $[2 + c/\nu]^2 > 4(1 + m^2)$, then

$$R(r) = C_1r^m + C_2r^n$$

where C_1 and C_2 are arbitrary constants and

$$m,n = \frac{[2+c/\nu] \mp [(2+c/\nu)^2-4(1+m^2)]^{1/2}}{2}$$

are real and distinct. Then

$$q = [C_1r^m + C_2r^n]M(\theta)$$

and
$$L(r) = a[mC_1r^{m-2} + nC_2r^{n-2}]$$

Case 2

When $[2 + c/\nu] = 4(1 + m^2)$, then

$$R(r) = Dr^m + Er^n \ln r$$

where D and E are constants and m, n are real and equal. Then

$$q = R(r)M(\theta)$$

and $$L(r) = a[\, mD + mE \ln r + E\,]r^{m-2}$$

Case 3

When $[2 + c/\nu] < 4(1 + m^2)$, then

$$R(r) = r^{\alpha}\,[D \cos(\beta \ln r) + E \sin(\beta \ln r)]$$

where $m = \alpha + i\beta$ and $n = \alpha - i\beta$ are the complex roots of (5.54).

Then

$$q = R(r)M(\theta)$$

and $$L(r) = (a/r)R'(r)$$

Then

$$\omega = [mR(r)/r]\{\, Be^{-m\theta} - Ae^{m\theta}\}$$

$$\rho = -[a/(rR(r)M^2(\theta)] + [c/rR(r)M(\theta)]$$

$$f = k/\psi'(\theta)$$

The other quantities can be similarly obtained. We make the same remark for the infinitely conducting case.

Example 6

Let $q = F(r) + G(\theta)$

Using q in eqn. (5.12), we obtain

$$\nu[r^2F'' - rF' + F] + \nu[G'' + G] - r\psi'(\theta)F' = r^2L(r) \qquad (5.55)$$

Differentiating (5.55) with respect to θ, we get

$$\nu[G'' + G]' - r\psi''(\theta)F' = 0$$

For $\psi''(\theta) \neq 0$, this gives

$$F(r) = c\ \ln r + k \qquad (5.56)$$

$$G''(\theta) + G(\theta) = (c/\nu)\psi'(\theta) + b \qquad (5.57)$$

where c, k, and b are arbitrary constants. The solution to eqn. (5.57) is given by

$$G(\theta) = A\cos\theta + B\sin\theta + b + (c/\nu)[\sin\theta\int\psi'\cos\theta d\theta - \cos\theta\int\psi'\sin\theta d\theta\ ] \qquad (5.58)$$

Thus,

$$q = c\ \ln r + k + G(\theta)$$

where $G(\theta)$ is given in (5.58).

Substituting (5.56) and (5.57) into (5.55) yields

$$L(r) = \frac{c\nu\ \ln r + \nu k - 2\nu c + \nu b}{r^2}$$

The other quantities now can be easily obtained using the same procedure as in the previous examples.

Particular Solution.

(1) If $\psi''(\theta) = 0$, then we have $\psi'(\theta) =$ constant $= a$ and therefore we have

$$G(\theta) = A\cos\theta + B\sin\theta + b^*$$

and

$$F(r) = C_1 r^{m_1} + C_2 r^{m_2} + U_1 r^{m_1} + U_2 r^{m_2}$$

where U_1, U_2, m_1 and m_2 are as in eqn. (5.50).

(2) If we let

$$G''(\theta) + G(\theta) = 0$$

and

$$r^2F''(r) + rF'(r) + F(r) = 0$$

Then $G(\theta)$ and $F(r)$ are given by

$$G(\theta) = A\cos\theta + B\sin\theta$$

$$F(r) = C\cos(\ln r) + D\sin(\ln r)$$

Example 7

Let $q = F(r)G(\theta)$ such that $F'(r) \neq 0, G'(\theta) \neq 0$.

Using q in eqn.(5.12), we obtain

$$\nu[r^2F''G + rF'G + FG''] - 2\nu rF'G + \nu FG - \psi' rF'G = r^2L(r)$$

which implies

$$\nu r^2\frac{F''}{F} - \nu r\frac{F'}{F} - \psi'(\theta)r\frac{F'}{F} + \nu\frac{G''}{G} + \nu = \frac{r^2L(r)}{FG} \qquad (5.59)$$

Differentiating (5.59) with respect to θ and simplifying, we get

$$-\psi''(\theta) + \nu[F/rF'][G''/G]' = [rL(r)/F'][1/G]' \qquad (5.60)$$

Differentiating (5.60) with respect to r implies

$$\nu[F/rF']'[G''/G]' = [rL(r)/F']'[1/G]' \qquad (5.61)$$

Eqn.(5.61) has solutions if $G''= 0$ or $G'' \neq 0$.

We consider these two cases separately.

Case 1 $\quad G'' = 0$

Then $\quad G(\theta) = A\theta + B \qquad (5.62)$

where A and B are arbitrary constants and eqn.(5.61) gives

$$F(r) = \frac{1}{m}\int rL(r)dr + n \qquad (5.63)$$

where m,n are constants.

Using (5.62) and (5.63) in (5.60), we get

$$\psi'(\theta) = d - \frac{m}{G(\theta)} \qquad (5.64)$$

where d is a constant.

The eqns.(5.62) to (5.64) must satisfy (5.59) to form a solution and therefore we have,

$$ar^2L''(r) + brL'(r) + cL(r) = 0 \quad (5.65)$$

where a, b, c are constants. Eqn.(5.65) yields three different solutions for L(r) depending on the values of a, b and c. After obtaining L(r), substitution into eqn.(5.63) yields F(r).

<u>**Case 2**</u> $G'' \neq 0$

In this case (i) $[G''/G]' = 0$

or (ii) $[G''/G]' \neq 0$

Sub-case (i) leads to the previous case and (ii) implies

$$[rL(r)/F']' = 0 \quad (5.66)$$

which gives

$$F(r) = \frac{1}{b}\int rL(r)dr + B \quad (5.67)$$

where b and B are constants.

Employing (5.66) and (5.67) in (5.60), we get

$$[\nu F/rF'] = b_1 \quad (5.68)$$

and

$$G'' = [\psi'(\theta)/b_1]G + cG + b/b_1 \quad (5.69)$$

From eqn.(5.68), we get

$$L(r) = Ar^{\nu/b_1 - 2}$$

where A is an arbitrary constant. Therefore

$$F(r) = [Ab_1/\nu b]r^{\nu/b_1} + B \quad (5.70)$$

The expression obtained for F and L must satisfy (5.59) to form solutions. Hence after substitution, we find that G(θ) satisfies eqn.(5.69) provided

$$c = \frac{2A\nu}{b_1} - \frac{A\nu^2}{bb_1} - 1$$

After obtaining F(r) and G(θ) in both the cases above, the other quantities can then be easily determined for both the finitely and infinitely conducting cases.

CONCLUSION

In the present thesis, an indirect method has been developed to generate exact solutions of the viscous, compressible, plane, steady-state MHD flow equations when the flow is assumed to be aligned.

Numerous examples have been investigated for parallel, vortex and radial flows.

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