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Operator space tensor products and the second dual of a Banach algebra.

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Operator Space Tensor Products and the Second Dual of a Banach Algebra

by

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Abstract

This thesis explores a possible operator space framework for the study of the second dual of a Banach algebra A. We prove some new characterizations for A to be Arens regular and we try to unify, for the Arens regularity problem, two of current approaches: by considering weakly almost periodic functionals on A and by considering the topological center of A^{**}. Motivated by this study, we define two operator space tensor products, namely, the extended projective tensor product and the normal projective tensor product. We investigate the properties of these two products, and compare them with other operator space tensor products. It is shown that the extended projective tensor product is injective, and the normal projective tensor product can linearize a class of bilinear maps under the condition that the pair of operator spaces has certain type of Kaplansky density property.

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Contents

CHAPTER 1

Introduction

In the operator space theory, three most interesting operator space tensor products are frequently considered: the projective tensor product $\overset{\wedge}{\otimes}$, the injective tensor product $\overset{\vee}{\otimes}$, and the Haagerup tensor product $\overset{h}{\otimes}$ — see [11] for overview. All of these tensor products are norm closures of the algebraic tensor product with the underlying operator space norms. Projective tensor product is closely related to completely bounded bilinear maps: $CB(V \overset{\wedge}{\otimes} W, X) \cong CB(V \times W, X)$, and it has the dual relationship with the injective tensor product via $V \overset{\vee}{\otimes} W \hookrightarrow \left(V^* \overset{\wedge}{\otimes} W^*\right)^*$. The Haagerup tensor product, however, is dual to itself: $V \overset{h}{\otimes} W \hookrightarrow \left(V^* \overset{h}{\otimes} W^*\right)^*$ (or $V^* \overset{h}{\otimes} W^* \hookrightarrow \left(V \overset{h}{\otimes} W\right)^*$), and linearizes the multiplicatively bounded bilinear maps, that is, $CB(V \overset{h}{\otimes} W, X) \cong MB(V \times W, X)$.

When V and W are dual operator spaces, algebraic tensor product $V \otimes W$ will naturally inherits the relatively weak*-topology from $\left(V_* \overset{h}{\otimes} W_*\right)^*$. Taking the weak*-closure gives the weak*-Haagerup tensor product, which turned out to be same as the extended Haagerup tensor product $V \overset{eh}{\otimes} W$ since they have the same predual $V_* \overset{h}{\otimes} W_*$ (cf. [4], [11]). In fact, the extended Haagerup tensor product has general form: for any two operator spaces V and W, $V \overset{e\wedge}{\otimes} W = MB^\sigma(V^* \times W^*, \mathbb{C})$, which is a subspace of $MB(V^* \times W^*, \mathbb{C})$. The extended Haagerup tensor product has many same properties as the Haagerup tensor product has, such as injectivity, self-duality, preserving complete contraction, etc. Effros-Kishimoto in [9] defined the normal Haagerup tensor product $V \overset{\sigma\wedge}{\otimes} W$ of two dual operator spaces V and W as $\left(V_* \overset{eh}{\otimes} W_*\right)^*$. It is finally connected with normal bilinear maps: for any dual operator space X, $CB^\sigma(V \overset{\sigma h}{\otimes} W, X) \cong MB^\sigma(V \times W, X)$. $\overset{\sigma\wedge}{\otimes}$ is automatically projective for weak*-closed subspaces due to the dual relationship with the extended Haagerup tensor product. The details about the tensor products $\overset{e\wedge}{\otimes}$ and $\overset{\sigma\wedge}{\otimes}$ are presented in Chapter 2 and Chapter 3.

In Chapter 4, first we review some well-known results on the second dual of a Banach algebra. Then we explore some new characterizations of Arens regularity. The second dual A^{**} of a Banach algebra A has two natural products extending the multiplication on A, namely the first Arens product and the second Arens product. Generally, these two products may not coincide. When they coincide, A is called Arens regular. There are already some characterizations for A to be Arens regular expressed at A^*-level and A^{**}-level. At these two levels, there are two concepts, i.e., the space $wap(A)$ of weakly almost periodic functionals on A and the topological center $Z(A^{**})$ of A^{**} with respect to each Arens product, to describe the non-Arens regularity of A. It is known that $A \subseteq Z(A^{**}) \subseteq A^{**}$, $wap(A) \subseteq A^*$, and A is Arens regular iff $Z(A^{**}) = A^{**}$ iff $wap(A) = A^*$, and A is strongly Arens irregular iff $Z(A^{**}) = A$. We attempt to unify these approaches. Via certain bilinear map, some interesting subspaces of A^* are introduced such as $\varphi(\widetilde{S}), \varphi(W)$, and $\varphi(\widetilde{Z})$, from which a candidate to $wap(A)$ playing a similar role as A to $Z(A^{**})$ is investigated.

Suppose A is a Banach algebra with an operator space structure. It is shown that the multiplication m on A is in $CB(A \times A, A)$ if and only if the first and the second Arens products are in $CB(A^{**} \times A^{**}, A^{**})$. A more general conclusion is obtained for bilinear maps $m : X \times Y \to Z$. From these observations, we realize that the study of A^{**} may be related to some generalized operator space projective tensor products.

Motivated by the study of the second dual of a Banach algebra, in Chapter 5, we define and study the extended projective tensor product $\overset{e\wedge}{\otimes}$ and the normal projective tensor product $\overset{\sigma\wedge}{\otimes}$. They do not have many nice properties as extended and normal Haagerup tensor products have any more. Even if some properties like injectivity still hold, the way to get them is totally different. This is mainly owing to the lack of self-duality of the projective tensor product. In this chapter, we also prove a few identifications, such as $CB^\sigma(V^* \times W^*, \mathbb{C}) \cong CB^{\sigma-w}(V^*, W)$, and the conditional identification $CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*) \cong CB^\sigma(V_1^* \times V_2^*, W^*)$. As subspaces of $\left(V^* \overset{\wedge}{\otimes} W^*\right)^*$, the extended projective tensor product, the normal spatial tensor product, and the injective tensor product are related to each other. The extended projective tensor product and the extended Haagerup tensor product are also compared.

Owing to the time limit, we leave some interesting questions open at the end of this thesis.

CHAPTER 2

Haagerup Tensor Product

Operator space Haagerup tensor product is one of important objects in the pertinent fields. It is projective, injective, and self-duality. It linearizes the multiplicatively bounded bilinear map. Besides, it has the multilinear decomposition property which plays a key role in the later study of extended Haagerup tensor product. This chapter reviews most of these interesting properties of Haagerup tensor product, some of which are proved in a way different from the original one.

2.1. Multiplicatively bounded bilinear mappings

In this section, we give a quick review of multiplicatively bounded norm, and two decompositions of an element in $M_n(V \otimes W)$, where V and W both are operator spaces (cf. [11]). We give a proof of the second decomposition in Lemma 2.1.4.

Let V, W and X be operator spaces, $v \in M_{m,r}(V)$ and $w \in M_{r,n}(W)$. The **matrix inner product** of v and w is $v \odot w \in M_{m,n}(V \otimes W)$ given by

$$v \odot w := \left[\sum_{k=1}^{r} v_{ik} \otimes w_{kj} \right].$$

If $v = \alpha \otimes v_0$ and $w = \beta \otimes w_0$ with $\alpha \in M_{m,r}, \beta \in M_{r,n}$, then we can get another useful formula for the matrix inner product.

LEMMA 2.1.1. *$(\alpha \otimes v_0) \odot (\beta \otimes w_0) = \alpha\beta \otimes v_0 \otimes w_0$, where $\alpha \in M_{m,r}, \beta \in M_{r,n}, v_0 \in V$, and $w_0 \in W$.*

PROOF. Since $v_{ik} = \alpha_{ik} \otimes v_0$ and $w_{kj} = \beta_{kj} \otimes w_0$ for $1 \leq i, j \leq n$ and $1 \leq k \leq r$, we have

$$\begin{aligned}(\alpha \otimes v_0) \odot (\beta \otimes w_0) &= v \odot w = \left[\sum_{k=1}^{r} v_{ik} \otimes w_{kj} \right] \\ &= \left[\sum_{k=1}^{r} \alpha_{ik} \otimes v_0 \otimes \beta_{kj} \otimes w_0 \right]\end{aligned}$$

$$= \left[(\sum_{k=1}^{r} \alpha_{ik}\beta_{kj}) \otimes v_0 \otimes w_0 \right] = \alpha\beta \otimes v_0 \otimes w_0.$$

□

Here we list some properties of matrix inner product without proof.

(1) For any $\alpha \in M_{m,r}, \beta \in M_{r,n}$, and $w \in M_{r,n}(W)$,

$$\alpha \odot w = \alpha w \quad \textit{and} \quad v \odot \beta = v\beta.$$

(2) For any $\alpha \in M_{m,r}$ and $w = \gamma \otimes w_0 \in M_{r,n} \otimes W$,

$$\alpha \odot w = \alpha\gamma \odot w_0.$$

(3) For any $v \in M_{m,r}(V), w \in M_{r,s}(W)$, and $x \in M_{s,n}(X)$,

$$(v \odot w) \odot x = v \odot (w \odot x).$$

(4) For any $\alpha \in M_{g,m}$, $\beta \in M_{n,h}$, $v \in M_{m,r}(V)$, and $w \in M_{r,n}(W)$,

$$\alpha(v \odot w)\beta = (\alpha v) \odot (w\beta).$$

(5) For any $v' \in M_{m,r}(V), v'' \in M_{n,s}(V), w' \in M_{r,m}(W)$, and $w'' \in M_{s,n}(W)$, let $v = v' \oplus v''$ and $w = w' \oplus w''$. Then

$$v \odot w = (v' \oplus v'') \odot (w' \oplus w'') = (v' \odot w') \oplus (v'' \odot w'').$$

Notice that the fifth property follows from

$$((v' \oplus v'') \odot (w' \oplus w''))_{ij} = \begin{cases} \sum\limits_{1 \le k \le r} v'_{ik} \otimes w'_{kj} & \text{if } 1 \le i, j \le m, \\ \sum\limits_{r+1 \le k \le r+s} v''_{ik} \otimes w''_{kj} & \text{if } m+1 \le i, j \le m+n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varphi : V \times W \to X$ be a bilinear mapping and $\widetilde{\varphi} : V \otimes W \to X$ its linearization. Then we have the (n,l)**-th amplification** of φ, namely $\varphi^{n,l} : M_{n,l}(V) \times M_{l,n}(W) \to M_n(X)$, which is defined by

$$\varphi^{n,l}(v, w) = \widetilde{\varphi}^{(n)}(v \odot w) = \left[\sum_{k=1}^{l} \varphi(v_{ik}, w_{kj}) \right] \in M_n(X),$$

where $\widetilde{\varphi}^{(n)}$ is the n-th amplification of the linear mapping $\widetilde{\varphi}$. When $l = n$, we shortly denote $\varphi^{n,l}$ by φ^n.

The **multiplicatively bounded norm** of φ is defined by

$$\|\varphi\|_{mb} = \sup\{\|\varphi^{n,l}\| : n, l \in \mathbb{N}\} = \sup\{\|\varphi^n\| : n \in \mathbb{N}\}.$$

We say that φ is multiplicatively bounded (resp., multiplicatively contractive) if $\|\varphi\|_{mb} < \infty$ (resp., $\|\varphi\| \leq 1$). Let $MB(V \times W, X)$ denote the linear space of all multiplicatively bounded bilinear mappings $\varphi : V \times W \to X$ with the norm $\|\cdot\|_{mb}$.

Using the linear space identifications $M_n(MB(V \times W, X)) \cong MB(V \times W, M_n(X))$, we may define an operator space matrix norm on $MB(V \times W, X)$.

LEMMA 2.1.2. *Let V and W be operator spaces, $v \in M_p(V)$, and $w \in M_q(W)$. Then $v \otimes w = (v \otimes I_q) \odot (I_p \otimes w)$. That is, we can express the Kronecker product in terms of the matrix inner product.*

PROOF. Suppose that

$$v = \alpha \otimes v_0 \in M_p \otimes V$$

and

$$w = \beta \otimes w_0 \in M_q \otimes W.$$

Since, by the scalar matrix tensor product, we may write $\alpha \otimes \beta$ as a matrix product

$$\alpha \otimes \beta = (\alpha \otimes I_q)(I_p \otimes \beta),$$

we have

$$\begin{aligned} v \otimes w &= (\alpha \otimes v_0) \otimes (\beta \otimes w_0) = (\alpha \otimes \beta) \otimes (v_0 \otimes w_0) \\ &= (\alpha \otimes I_q)(I_p \otimes \beta) \otimes v_0 \otimes w_0 = ((\alpha \otimes I_q) \otimes v_0) \otimes ((I_p \otimes \beta) \otimes w_0) \\ &= (v \otimes I_q) \odot (I_p \otimes w). \end{aligned}$$

In the fourth step, we used Lemma 2.1.1. □

LEMMA 2.1.3. *Given linear spaces V and W and $u \in M_n(V \otimes W)$, there exist $r \in \mathbb{N}$, $v \in M_{n,r}(V)$, and $w \in M_{r,n}(W)$ such that*

$$u = v \odot w = \left[\sum_{k=1}^{r} v_{ik} \otimes w_{kj}\right].$$

PROOF. Let $U_n = \{v \odot w : \ v \in M_{n,r}(V), w \in M_{r,n}(W), r \in \mathbb{N}\}$. We want to show that $U_n \supseteq M_n(V \otimes W) = M_n \otimes V \otimes W$.

In fact, $M_n \otimes V \otimes W = span\{E_{i,j} \otimes v \otimes w : v \in V, w \in W, i, j = 1, \cdots, n\}$, where $E_{i,j}$ is the ij-th unit matrix in M_n. Note that $E_{i,j} = e_i^{[n,1]} e_j^{[1,n]}$. So, by Lemma 2.1.1, $E_{i,j} \otimes v \otimes w = (e_i^{[n,1]} \otimes v) \odot (e_j^{[1,n]} \otimes w) \in U_n$. It remains to show that U_n is a linear space. U_n is clearly closed under the scalar multiplication.

Now given $u_i = v_i \odot w_i \ \ (i = 1, 2)$ with $v_1 \in M_{n,r}(V), w_1 \in M_{r,n}(W), v_2 \in M_{n,s}(V)$, and $w_2 \in M_{s,n}(W)$. Let $v = (v_1 \ \ v_2)$ and $w = (w_1 \ \ w_2)^T$. Then $v \in M_{n,r+s}(V), w \in M_{r+s,n}(W)$, and

$$v \odot w = v_1 \odot w_1 + v_2 \odot w_2 = u_1 + u_2.$$

Then we complete the proof. □

LEMMA 2.1.4. *Given linear spaces V and W and $u \in M_n(V \otimes W)$, there exist $p, q \in \mathbb{N}$, $v \in M_p(V)$, $w \in M_q(W), \alpha \in M_{n,pq}$, and $\beta \in M_{pq,n}$ such that*

$$u = \alpha(v \otimes w)\beta.$$

PROOF. Let $U_n = \{\alpha(v \otimes w)\beta : \alpha \in M_{n,pq}, \beta \in M_{pq,n}, v \in M_p(V), w \in M_q(W)\}$. Now we show $U_n \supseteq M_n(V \otimes W) = span\{E_{i,j} \otimes v \otimes w : v \in V, w \in W, i, j = 1, \cdots, n\}$. Note that $E_{i,j} \otimes v \otimes w = e_i^{[n,1]}(v \otimes w) e_j^{[1,n]} \in U_n$, it remains to show that U_n is a linear space.

Clearly, U_n is closed under the scalar multiplication. Let $u_1 = \alpha_1(v_1 \otimes w_1)\beta_1$ and $u_2 = \alpha_2(v_2 \otimes w_2)\beta_2$. Then we have

$$\begin{aligned} u_1 + u_2 &= \alpha_1(v_1 \otimes w_1)\beta_1 + \alpha_2(v_2 \otimes w_2)\beta_2 \\ &= \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} v_1 \otimes w_1 & 0 & 0 & 0 \\ 0 & v_1 \otimes w_2 & 0 & 0 \\ 0 & 0 & v_2 \otimes w_1 & 0 \\ 0 & 0 & 0 & v_2 \otimes w_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ 0 \\ 0 \\ \beta_2 \end{pmatrix} \\ &= \alpha(v \otimes w)\beta, \end{aligned}$$

where $\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & \alpha_2 \end{pmatrix}, v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$, and $\beta = \begin{pmatrix} \beta_1 \\ 0 \\ 0 \\ \beta_2 \end{pmatrix}$.

That completes the proof. □

2.2. Haagerup tensor product and its properties

Before going to the properties, we recall the operator space Haagerup tensor product norm. The readers can find most results of this section in [11] and [3, Lemma 2.2.6]. We deduce Corollary 2.2.3 and Proposition 2.2.4 from Proposition 2.2.2 and [11, Theorem 7.1.2], respectively. Both Lemma 2.2.7 and Lemma 2.2.8 were used in the proof of [11, Theorem 9.2.5] without proof. From our point of view, they are not trivial. So, we present detailed proofs here.

Given operator spaces V and W and $u \in M_n(V \otimes W)$, the operator space **Haagerup tensor norm** of u is defined by

$$\|u\|_h = \inf\{\|v\|\|w\| : \ u = v \odot w, v \in M_{n,r}(V), w \in M_{r,n}(W), r \in \mathbb{N}\}.$$

THEOREM 2.2.1. *Let V and W be operator spaces. Then $\|\cdot\|_h$ is an operator space matrix norm on $V \otimes W$, and for any $u \in M_n(V \otimes W)$,*

$$\|u\|_\vee \leq \|u\|_h \leq \|u\|_\wedge.$$

PROOF. Suppose $u_1 \in M_m(V \otimes W), u_2 \in M_n(V \otimes W)$, and $\varepsilon > 0$. Then there exist $v_1 \in M_{m,r}(V), w_1 \in M_{r,m}(W), v_2 \in M_{n,l}(V)$, and $w_2 \in M_{l,n}(W)$ such that $u_i = v_i \odot w_i$ with $\|w_i\| = 1$ and $\|v_i\| \leq \|u_i\|_h + \varepsilon \ \ (i = 1, 2)$. So,

$$\begin{aligned} \|u_1 \oplus u_2\| &= \|(v_1 \oplus v_2) \odot (w_1 \oplus w_2)\| \leq \|v_1 \oplus v_2\| \\ &= \max\{\|v_1\|, \ \|v_2\|\} \leq \max\{\|u_1\|_h, \ \|u_2\|_h\} + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have obtained **M1$'$**. For any $u \in M_n(V \otimes W)$ and $\varepsilon > 0$, we may choose $v \in M_{n,r}(V)$ and $w \in M_{r,n}(W)$ with $u = v \odot w$ and $\|v\|\|w\| < \|u\|_h + \varepsilon$. Then for $\alpha, \ \beta \in M_n$, we have

$$\|\alpha u \beta\| = \|(\alpha v) \odot (w \beta)\| \leq \|\alpha v\|\|w\beta\| \leq \|\alpha\|\|v\|\|w\|\|\beta\| \leq \|\alpha\|(\|u\|_h + \varepsilon)\|\beta\|.$$

Again, since ε is arbitrary, we obtained **M2**.

Let us suppose that $f \in M_p(V^*)$ and $g \in M_q(W^*)$ are complete contractions. Then by Lemma 2.1.2 and property (1) of the matrix inner product in Section 1, we have

$$\begin{aligned}(f \otimes g)^{(n)}(v \odot w) &= \left[\sum_{k=1}^{r} f(v_{ik}) \otimes g(w_{kj})\right]_{n,n} \\ &= \left[\sum_{k}^{r} (f(v_{ik}) \otimes I_q) \odot (I_p \otimes g(w_{kj}))\right]_{n,n} \\ &= \left[\sum_{k}^{r} (f(v_{ik}) \otimes I_q)(I_p \otimes g(w_{kj}))\right]_{n,n} \\ &= [f(v_{ik}) \otimes I_q)]_{n,r} [I_p \otimes g(w_{kj})]_{r,n} .\end{aligned}$$

Hence,

$$\begin{aligned}\|(f \otimes g)^{(n)}(v \odot w)\| &\leq \| [f(v_{ik}) \otimes I_q)]_{n,r} \| \| [I_p \otimes g(w_{kj})]_{r,n} \| \\ &= \| [f(v_{ik})]_{n,r} \otimes I_q \| \| I_p \otimes [g(w_{kj})]_{r,n} \| \\ &\leq \| f^{(n,r)}(v) \| \| g^{(r,n)}(w) \| \leq \|v\| \|w\|.\end{aligned}$$

It follows from the definition of the injective tensor matrix norm that

$$\|u\|_{\vee} \leq \|v\| \|w\| \leq \|u\|_h + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, we have $\|u\|_{\vee} \leq \|u\|_h$.

For any matrices $v \in M_m(V)$ and $w \in M_n(W)$, we have from Lemma 2.1.1 that

$$\|v \otimes w\|_h \leq \|v\| \|w\|.$$

That is, the Haargerup tensor norm is a subcross norm. Since the projective tensor norm is the largest subcross norm (cf. [11, Theorem 7.1.1]), $\|u\|_h \leq \|u\|_{\wedge}$. □

We let

$$V \otimes_h W = (V \otimes W, \| \cdot \|_h),$$

and define the **Haagerup tensor product** $V \overset{h}{\otimes} W$ of V and W to be the completion of the operator space $V \otimes_h W$.

PROPOSITION 2.2.2. *Let V, W and X be operator spaces. Then we have a complete isometry*

$$MB(V \times W, X) \cong CB(V \overset{h}{\otimes} W, X).$$

PROOF. Let $\varphi \in MB(V \times W, X)$ and $\widetilde{\varphi}$ its unique linearization. Then by the definition of the n-th amplification of φ and $\widetilde{\varphi}$, we have $\varphi^n(v, w) = \widetilde{\varphi}^{(n)}(v \odot w)$. We want to show $\|\varphi^n\| = \|\widetilde{\varphi}^{(n)}\|$ for each $n \in \mathbb{N}$, and hence $\|\varphi\|_{mb} = \|\widetilde{\varphi}\|_{cb}$. In fact,

$$\begin{aligned} \|\varphi^n\| &= \sup\{\|\varphi^n(v, w)\| : \|v\| \leq 1, \|w\| \leq 1, v \in M_n(V), w \in M_n(W)\} \\ &= \sup\{\|\widetilde{\varphi}^{(n)}(v \odot w)\| : \|v\| \leq 1, \|w\| \leq 1, v \in M_n(V), w \in M_n(W)\} \\ &\leq \sup\{\|\widetilde{\varphi}^{(n)}(u)\| : \|u\|_h \leq 1, u \in M_n(V \otimes W)\} = \|\widetilde{\varphi}^{(n)}\|. \end{aligned}$$

Conversely, for every $u \in M_n(V \otimes W)$ with $\|u\| \leq 1$ and $\varepsilon > 0$, we can find $v \in M_{n,r}(V), w \in M_{r,n}(W)$ such that $u = v \odot w$ and $\|v\|\|w\| \leq 1 + \varepsilon$. Then

$$\begin{aligned} \|\widetilde{\varphi}^{(n)}(u)\| &= \|\widetilde{\varphi}^{(n)}(v \odot w)\| = \|\varphi^n(v, w)\| \\ &\leq \|\varphi^n\|\|v\|\|w\| \leq \|\varphi^n\|(1 + \varepsilon). \end{aligned}$$

Since ε is arbitrary, $\|\widetilde{\varphi}^{(n)}\| = \sup_{\|u\| \leq 1} \|\widetilde{\varphi}^{(n)}(u)\| \leq \|\varphi^n\|$. Therefore, $\|\varphi^n\| = \|\widetilde{\varphi}^{(n)}\|$. ☐

COROLLARY 2.2.3. *Let V and W be operator spaces and $\varphi : V \times W \to M_n$ a bilinear map. Then $\|\varphi\|_{mb} = \|\varphi^n\|$, where $\varphi^n : M_n(V) \times M_n(W) \to M_n(M_n)$ is the n-th amplification of φ.*

PROOF. Let $\widetilde{\varphi} : V \overset{h}{\otimes} W \to M_n$ be the unique linearization of φ. Then $\|\varphi\|_{mb} = \|\widetilde{\varphi}\|_{cb} = \|\widetilde{\varphi}^{(n)}\|$, where $\widetilde{\varphi}^{(n)}$ is the n-th amplification of the linear map $\widetilde{\varphi}$. From the proof of the identification $MB(V \times W, X) \cong CB(V \overset{h}{\otimes} W, X)$, we have $\|\widetilde{\varphi}^{(n)}\| = \|\varphi^n\|$. Therefore, $\|\varphi\|_{mb} = \|\varphi^n\|$. ☐

When $n = 1$, we have the following property of completely bounded bilinear maps.

PROPOSITION 2.2.4. *Let V and W be operator spaces and $\varphi : V \times W \to \mathbb{C}$ a bilinear map. Then $\|\varphi\|_{cb} = \|\varphi\|$.*

PROOF. Recall that $\|\varphi\|_{cb} = \sup_{n \in \mathbb{N}}\{\|\varphi_n\|\}$, where $\varphi_n : M_n(V) \times M_n(W) \to M_{n^2}$ is the n-th joint amplification of φ. By the operator space identification $CB(V \times W, \mathbb{C}) \cong CB(V \overset{\wedge}{\otimes} W, \mathbb{C})$, $\|\varphi\|_{cb} = \|\widetilde{\varphi}\|_{cb} = \|\widetilde{\varphi}\| = \|\varphi\|$, where $\widetilde{\varphi}$ is the unique linearization of φ such that $\widetilde{\varphi}(v \otimes w) = \varphi(v, w)$. ☐

PROPOSITION 2.2.5. *Let V and W be operator spaces. For any u in $V \otimes_h W$ with $\|u\|_h \leq 1$, there exists a representation*

$$u = v \odot w = \sum_{k=1}^{r} v_j \otimes w_j$$

with $\|v\| \leq 1, \|w\| \leq 1$ such that $v_1, \cdots, v_r$ are linearly independent in V, and $w_1, \cdots, w_r$ are linearly independent in W.

LEMMA 2.2.6. *Let V and W be operator spaces and $u \in V \otimes W$. Let $u = \sum_{i=1}^{n} v_i \otimes w_i = \sum_{k=1}^{m} v'_k \otimes w'_k$ be two representations of u such that each of the sets $\{v_1, \cdots, v_n\}, \{w_1, \cdots, w_n\}, \{v'_1, \cdots, v'_m\}$, and $\{w'_1, \cdots, w'_m\}$ is linearly independent. Then $span\{v_1, \cdots, v_n\} = span\{v'_1, \cdots, v'_m\}$ and $span\{w_1, \cdots, w_n\} = span\{w'_1, \cdots, w'_m\}$.*

PROOF. By Hahn-Banach Theorem, we can choose $f_s \in V^*$ such that $f_s(v'_k) = \delta_{sk}$ $(s, k = 1, \cdots, m)$. Now the map $f_s \otimes id : V \otimes W \to W$ is given by $\sum_{j=1}^{r} x_j \otimes y_j \mapsto \sum_{j=1}^{r} f_s(x_j) y_j$. Then $(f_s \otimes id)(u) = \sum_{k=1}^{m} f_s(v'_k) w'_k = w'_k$. On the other hand, $(f_s \otimes id)(u) = \sum_{i=1}^{n} f_s(v_i) w_i$. Hence, $w'_k \in span\{w_1, \cdots, w_m\}$. The remaining cases follow similarly. □

LEMMA 2.2.7. *Let V' and W' be operator spaces, $V \subseteq V'$ and $W \subseteq W'$ subspaces of V and W, respectively. Then the inclusion map $V \overset{h}{\otimes} W \to V' \overset{h}{\otimes} W'$ is an isometry.*

PROOF. Let $u \in V \otimes W$. Then its Haagerup tensor product norm in $V \otimes W$ is same as its Haagerup norm in $V' \otimes W'$. In fact,

$$\|u\|_{V \overset{h}{\otimes} W} = \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{1,r}(V), w \in M_{r,1}(W), r \in \mathbb{N}\}$$

and

$$\|u\|_{V' \overset{h}{\otimes} W'} = \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{1,r}(V'), w \in M_{r,1}(W'), r \in \mathbb{N}\}.$$

So, it is clear that $\|u\|_{V \overset{h}{\otimes} W} \geq \|u\|_{V' \overset{h}{\otimes} W'}$. By Lemma 2.2.6, we have $\|u\|_{V' \overset{h}{\otimes} W'} \geq \|u\|_{V \overset{h}{\otimes} W}$ as well. □

LEMMA 2.2.8. *Let V, V', W, and W' be operator spaces. If $\varphi : V \to V'$ and $\psi : W \to W'$ are complete isometries, then $\varphi \otimes \psi$ is an isometry.*

PROOF. The inclusion mapping $\varphi(V)\overset{h}{\otimes}\psi(W)\to V'\overset{h}{\otimes}W'$ is isometric by Lemma 2.2.7. It suffices to show the map

$$V\otimes_h W\to\varphi(V)\otimes_h\psi(W),\ \sum_{i=1}^{n}v_i\otimes w_i\mapsto\sum_{i=1}^{n}\varphi(v_i)\otimes\psi(w_i)$$

is an isometry.

Suppose $u'\in\varphi(V)\otimes_h\psi(W)$. Then u' has a representation $\sum_{i=1}^{n}\varphi(v_i')\otimes\psi(w_i')$, where $v_i'\in V$ and $w_i'\in W$. Let $u=\sum_{i=1}^{n}v_i'\otimes w_i'$. Then $u\in V\otimes_h W$ and $\varphi\otimes\psi(u)=u'$, i.e., $\varphi\otimes\psi: V\otimes_h W\to\varphi(V)\otimes_h\psi(W)$ is onto.

Let $u_1=\sum_{i=1}^{n}v_i^1\otimes w_i^1$ and $u_2=\sum_{k=1}^{m}v_k^2\otimes w_k^2$ be two elements in $V\otimes_h W$ with $u_1\neq u_2$. Then we can write u_1-u_2 as $\sum_{j=1}^{l}\widetilde{v_j}\otimes\widetilde{w_j}\neq 0$ with $\widetilde{w_j}$ linearly independent, where $1\leq l\leq m+n$. So, $\widetilde{v_{j_0}}\neq 0$ for some j_0. Then

$$(\varphi\otimes\psi)(u_1-u_2)=(\varphi\otimes\psi)(\sum_{j=1}^{l}\widetilde{v_j}\otimes\widetilde{w_j})=\sum_{j=1}^{l}\varphi(\widetilde{v_j})\otimes\psi(\widetilde{w_j}).$$

Since $\widetilde{w_j}$ are linearly independent and ψ is isometric, $\psi(\widetilde{w_j})$ are linearly independent. Again since $\varphi(\widetilde{v_{j_0}})\neq 0,(\varphi\otimes\psi)(u_1-u_2)\neq 0$. So, $\varphi\otimes\psi$ is one-one.

Now we show that $\varphi\otimes\psi: V\otimes_h W\to\varphi(V)\otimes_h\psi(W)$ is isometric. First we show that $\|(\varphi\otimes\psi)(u)\|_h\geq\|u\|_h$. For each $u\in V\otimes_h W$, $(\varphi\otimes\psi)(u)\in\varphi(V)\otimes_h\psi(W)$, so, by Lemma 2.2.7

$$\|(\varphi\otimes\psi)(u)\|_h=\inf\{\|[\varphi(v_i)]_{1,r}\|\|[\psi(w_i)]_{r,1}\|\},$$

where the infimum is taken over all decompositions $(\varphi\otimes\psi)(u)=[\varphi(v_i)]\odot[\psi(w_i)]$ with $[\varphi(v_i)]\in M_{1,r}(\varphi(V)),[\psi(w_i)]\in M_{r,1}(\psi(W))$, and $r\in\mathbb{N}$. But this infimum is just

$$\inf\{\|\varphi^{(1,r)}(v)\|\|\psi^{(r,1)}(w)\|:(\varphi\otimes\psi)(u)=\varphi^{(1,r)}(v)\odot\psi^{(r,1)}(w)\},$$

where $v=[v_i]\in M_{1,r}(V),w=[w_i]\in M_{r,1}(W)$, and $r\in\mathbb{N}$.

Thus for any $\varepsilon>0$, there exist $v\in M_{1,r}(V)$ and $w\in M_{r,1}(W)$ such that $(\varphi\otimes\psi)(u)=\varphi^{(1,r)}(v)\odot\psi^{(r,1)}(w)$ and $\|(\varphi\otimes\psi)(u)\|_h\geq\|\varphi^{(1,r)}(v)\|\|\psi^{(r,1)}(w)\|+\varepsilon$. Since $\varphi^{(1,r)}$ and $\psi^{(r,1)}$ are isometries, $\|(\varphi\otimes\psi)(u)\|_h\geq\|v\|\|w\|+\varepsilon\geq\|u\|_h$, where we use the fact that $u=v\odot w$ since $(\varphi\otimes\psi)(u)=\varphi^{(1,r)}(v)\odot\psi^{(r,1)}(w)=(\varphi\otimes\psi)(v\odot w)$ and $\varphi\otimes\psi$ is one-one. Therefore, $\|(\varphi\otimes\psi)(u)\|_h\geq\|u\|_h$.

On the other hand, for $u = v \odot w \in V \otimes_h W$ with $v \in M_{1,r}(V)$ and $w \in M_{n,1}(W)$, we have

$$\begin{aligned}
\|(\varphi \otimes \psi)(u)\|_h &= \|\sum_{k=1}^{r} \varphi(v_k) \otimes \psi(w_k)\|_h \\
&= \|\varphi^{(1,r)}(v) \odot \psi^{(r,1)}(w)\|_h \\
&\leq \|\varphi^{(1,r)}(v)\|\|\psi^{(r,1)}(w)\| \\
&= \|v\|\|w\|.
\end{aligned}$$

Taking the infimum over all such representations of u gives $\|(\varphi\otimes\psi)(u)\|_h \leq \|u\|_h$. □

PROPOSITION 2.2.9. *Let V, V', W and W' be operator spaces. For all complete contractions $\varphi : V \to V'$ and $\psi : W \to W'$, the corresponding mapping*

$$\varphi \otimes \psi : V \overset{h}{\otimes} W \to V' \overset{h}{\otimes} W'$$

is a complete contraction.

If φ and ψ are complete isometries (resp., completely quotient mappings), then the same is true for $\varphi \otimes \psi$.

PROOF. We have the commutative diagram

$$\begin{array}{ccc}
M_n(V \otimes_h W) & \xrightarrow{(\varphi\otimes\psi)^{(n)}} & M_n(V' \otimes_h W') \\
\downarrow & & \downarrow \\
M_{n,1}(V) \otimes_h M_{1,n}(W) & \xrightarrow{\varphi^{(n,1)}\otimes\psi^{(1,n)}} & M_{n,1}(V') \otimes_h M_{1,n}(W')
\end{array}.$$

By [11, Theorem 9.2.4], the two vertical mappings are isometries. Now we note that if φ and ψ are completely contractive, isometric, or complete quotient mappings, then that is also the case for the mappings $\varphi^{(n,1)}$ and $\psi^{(1,n)}$. Thus, it suffices to show that the mapping $\varphi \otimes \psi$ is a contraction, isometry, or quotient mapping.

Suppose that $\|\varphi\|_{cb} \leq 1$ and $\|\psi\|_{cb} \leq 1$. Then the proof of $\|\varphi\otimes\psi\| \leq 1$ is contained in the last part in the proof of Lemma 2.2.8.

The case of isometry is just Lemma 2.2.8.

Finally, given $u' \in V' \otimes_h W'$ with $\|u'\|_h < 1$, there exist $v' \in M_{1,r}(V'), w' \in M_{r,1}(W')$ such that $u' = v' \odot w'$ and $\|v'\|, \|w'\| < 1$. If φ, ψ are complete quotient mappings, then $\varphi^{(1,r)}$ and $\psi^{(r,1)}$ are quotient mappings. So, there exist $v \in M_{1,r}(V)$

with $\|v\| < 1$ and $w \in M_{r,1}(W)$ with $\|w\| < 1$ such that

$$v' = \varphi^{(1,r)}(v) \quad and \quad w' = \psi^{(r,1)}(w).$$

It follows that $u = v \odot w \in V \otimes_h W$ satisfying $\|u\|_h < 1$ and $(\varphi \otimes \psi)(u) = u'$. So, $\varphi \otimes \psi$ is a quotient mapping. □

The above assertion that the Haagerup tensor product preserves both quotient maps and complete isometries shows that it is both projective and injective. The proposition below shows that Haagerup tensor product also possesses associativity.

PROPOSITION 2.2.10. *Let V, W and X be operator spaces. Then we have the following complete isometry*

$$(V \overset{h}{\otimes} W) \overset{h}{\otimes} X \cong V \overset{h}{\otimes} (W \overset{h}{\otimes} X).$$

2.3. Row and Column Hilbert Operator Spaces

Let H be a Hilbert space. In this section, we consider two natural operator space structures on a H.

First, we use the column identification

$$C : H \cong B(\mathbb{C}, H),$$

where $C(\xi)(a) = a\xi \quad (\xi \in H, a \in \mathbb{C})$, to determine an operator space structure on H. To be more specific, for $\xi \in M_n(H)$, we have the amplification

$$C^{(n)}(\xi) : \mathbb{C}^n \to H^n,$$

and we define the column matrix norm of ξ by

$$\|\xi\|_c = \|C^{(n)}(\xi)\|.$$

Let H_c denote H with this operator structure, and we refer to it as the **column Hilbert operator space** or simply the column Hilbert space determined by H. That is

$$H_c \cong B(\mathbb{C}, H).$$

For each $\xi \in M_{m,n}(H_c)$,

$$\begin{aligned} \|\xi\|_c &= \|C^{(m,n)}(\xi)\| = \|C^{(m,n)}(\xi)^* C^{(m,n)}(\xi)\|^{1/2} = \|[C(\xi_{ji})^*]_{n,m}[C(\xi_{ij})]_{m,n}\|^{1/2} \\ &= \|[\sum_{k=1}^{m} C(\xi_{ki})^* C(\xi_{kj})]\|^{1/2} = \|[\sum_{k=1}^{m} \langle \xi_{kj} | \xi_{ki} \rangle]\|^{1/2}. \end{aligned}$$

From the definition, we have the natural complete isometry

$$M_{m,n}(H_c) \cong B(\mathbb{C}^n, H^m)$$

for all $m, n \in \mathbb{N}$, since for all $k \in \mathbb{N}$,

$$M_k(M_{m,n}(H_c) = M_{km,kn}(H_c) \cong B(\mathbb{C}^{kn}, H^{km}) = M_k(B(\mathbb{C}^n, H^m)).$$

This shows that $M_{m,n}(H_c)$ is also an operator space.

In particular, $(H_c)^m = M_{m,1}(H_c) \cong B(\mathbb{C}, H^m) \cong (H^m)_c$, which means that the sum of column Hilbert space is also a column Hilbert space.

Recall that if H is a Hilbert space, then we may define the complex conjugate space $\overline{H}$ by the identity map

$$J : H \to \overline{H},\ x \mapsto \overline{x}$$

with the usual addition and conjugate multiplication, that is

$$\overline{x} + \overline{y} = \overline{x+y} \quad \text{and} \quad a \cdot \overline{x} = \overline{\bar{a}x}.$$

Then $\overline{H}$ is a Hilbert space with the inner product given by

$$\langle \overline{x} | \overline{y} \rangle = \langle y | x \rangle.$$

Now we use the Banach space identification $\theta : \overline{H} \to H^*$, where $\theta(\overline{\xi})(\eta) = \langle \eta \mid \xi \rangle$. The natural isometry

$$R : H \to H^{**} = B(H^*, \mathbb{C}) = B(\overline{H}, \mathbb{C})$$

given by

$$R(\eta)(\overline{\xi}) = \theta(\overline{\xi})(\eta) = \langle \eta \mid \xi \rangle$$

determines an operator space matrix norm on H. We denote H with this operator structure by H_r, and refer to it as **row Hilbert operator space**. That is

$$H_r \cong B(\overline{H}, \mathbb{C}) \cong B(H, \mathbb{C}).$$

For $\xi \in M_{m,n}(H_r)$, then

$$\|\xi\|_r = \|R_{m,n}(\xi)\| = \|[\sum_{k=1}^{n} \langle \xi_{ik} | \xi_{jk} \rangle]\|^{1/2}.$$

Similarly, we have $(H_r)^n = M_{1,n}(H_r) \cong (H^n)_r$.

THEOREM 2.3.1. *For any Hilbert spaces H and K, there are natural completely isometric identifications*

$$B(H, K) \cong CB(H_c, K_c)$$

and

$$B(K^*, H^*) \cong CB(H_r, K_r).$$

The operator duals of column and row Hilbert spaces are related with their Banach duals in the following way.

$$(H_c)^* = CB(H_c, \mathbb{C}) = B(H, \mathbb{C}) = B(H^{**}, \mathbb{C}) = (H^*)_r.$$

Let $K = H^*$ in the above identities. Then

$$(K_r)^* = (H_c)^{**} = H_c = (K^*)_c, \quad i.e., \quad (H_r)^* = (H^*)_c.$$

2.4. Multilinear decomposions

In this section, we summarize a few nice properties of Haagerup tensor product without proof. A different description of the Haagerup tensor product norm is also presented.

PROPOSITION 2.4.1. *Let V and W be operator spaces. Then a linear functional*

$$F : V \overset{h}{\otimes} W \to \mathbb{C}$$

is bounded if and only if there exist a Hilbert space H and completely bounded linear mappings

$$\varphi : V \to (H_c)^* \quad \text{and} \quad \psi : W \to H_c$$

such that

$$F(v \otimes w) = \varphi(v)\psi(w).$$

In this case, we can choose φ and ψ such that

$$\|F\| = \|\varphi\|_{cb}\|\psi\|_{cb}.$$

More general, we have the following decomposition theorem for multilinear mappings.

THEOREM 2.4.2. *Let $V_1, \cdots, V_n$ be operator spaces and H_0, H_n Hilbert spaces. Then a linear mapping*

$$\varphi : V_1 \overset{h}{\otimes} \cdots \overset{h}{\otimes} V_n \to B(H_n, H_0)$$

is completely bounded if and only if there exist Hilbert spaces $H_1, \cdots, H_{n-1}$ and completely bounded mappings $\psi_k : V_k \to B(H_k, H_{k-1})$ $(k = 1, \cdots, n)$ such that

$$\varphi(v_1 \otimes \cdots \otimes v_2) = \psi_1(v_1) \cdots \psi_n(v_n).$$

In this case we can choose ψ_k $(k = 1, \cdots, n)$ such that

$$\|\varphi\|_{cb} = \|\psi_1\|_{cb} \cdots \|\psi_n\|_{cb}.$$

THEOREM 2.4.3. *Let V and W be operator spaces. Then the natural embedding*

$$V^* \overset{h}{\otimes} W^* \hookrightarrow (V \overset{h}{\otimes} W)^*$$

is completely isometric.

This property of Haagerup tensor product is called **self-duality**. When one of the two underlying operator spaces is finite-dimensional, the above embedding actually becomes surjective. This fact was observed in [11]. Here we give a complete proof.

COROLLARY 2.4.4. *Let V and W be operator spaces. If either V or W is finite-dimensional, then we have the complete isometry*

$$V^* \overset{h}{\otimes} W^* \cong (V \overset{h}{\otimes} W)^*.$$

PROOF. Assume that either V or W is finite-dimensional. Then $V \overset{h}{\otimes} W = V \otimes_h W$. It is easy to see that every functional in $(V \otimes_h W)^*$ has the form $\sum_{i=1}^{n} f_i \otimes g_i$ for some $f_i \in V^*$ and $g_i \in W^*$ $(1 \leq i \leq n)$, where $n = min\{dim(V), dim(W)\}$. So, the natural embedding in Theorem 2.4.3 is surjective. Therefore, $V^* \otimes_h W^* = (V \otimes_h W)^*$. □

The relationship between the Haagerup tensor product and the injective tensor product is also indicated by the form of the Haagerup tensor product norm given as follows.

PROPOSITION 2.4.5. *Let V and W be operator spaces. For each $u \in M_n(V \otimes W)$, there exist contractive elements $f \in M_{n,r}(V^*)$ and $g \in M_{r,n}(W^*)$ such that*

$$\|u\|_h = \|(f \odot g)^{(n)}(u)\|.$$

Thus

$$\|u\|_h = \sup\{\|(f \odot g)^{(n)}(u)\| : f \in M_{n,r}(V^*)_{\|\cdot\| \leq 1}, g \in M_{r,n}(W^*)_{\|\cdot\| \leq 1}, r \in \mathbb{N}\}.$$

PROOF. First, we need to explain $f \odot g$. It is an element of $M_n(V^* \overset{h}{\otimes} W^*) \subseteq M_n((V \overset{h}{\otimes} W)^*) \cong CB(V \overset{h}{\otimes} W, M_n)$. So, $(f \odot g)^{(n)} : M_n(V \overset{h}{\otimes} W) \to M_{n^2}$. For $u \in M_n(V \otimes_h W)$, there exist finite-dimentional subspaces of V and W, say, V_1 and W_1, respectively, such that $u \in M_n(V_1 \otimes_h W_1)$. Then u has the same norm in $M_n(V_1 \overset{h}{\otimes} W_1)$ as in $M_n(V \overset{h}{\otimes} W)$. By Effros-Ruan [11, Lemma 2.3.4], there exists a complete contraction $\varphi : V_1 \overset{h}{\otimes} W_1 \to M_n$ such that $\|u\|_h = \|\varphi^{(n)}(u)\|$.

Since V_1 and W_1 are finite-dimensional, by Corollary 2.4.4, we have

$$CB((V_1 \overset{h}{\otimes} W_1)^*, M_n) \cong M_n((V_1 \overset{h}{\otimes} W_1)^*) \cong M_n(V_1^* \overset{h}{\otimes} W_1^*),$$

and hence we may regard φ as a contractive element in $M_n(V_1^* \overset{h}{\otimes} W_1^*)$. Then there exist $f \in M_{n,r}(V_1^*)$ and $g \in M_{r,n}(W_1^*)$ such that $\varphi = f \odot g$ with $\|f\| \leq 1$ and $\|g\| \leq 1$. Since $B(\mathbb{C}^n, \mathbb{C}^r)$ and $B(\mathbb{C}^r, \mathbb{C}^n)$ are injective operator spaces (cf. [2, Theorem 1.2.10]), f and g have corresponding extensions, namely $\widetilde{f} \in M_n(V^*)$ and $\widetilde{g} \in M_n(W^*)$, respectively, such that $\|\widetilde{f}\| \leq 1, \|\widetilde{g}\| \leq 1$ and

$$\|(\widetilde{f} \odot \widetilde{g})^{(n)}(u)\| = \|(f \odot g)^{(n)}(u)\| = \|\varphi^{(n)}(u)\| = \|u\|_h,$$

where the first step is true since $u \in M_n(V_1 \otimes W_1)$.

Now suppose $f \in M_{n,r}(V^*)$ and $g \in M_{r,n}(W^*)$ are contractive. Then

$$\begin{aligned} \|(f \odot g)^{(n)}(u)\| &\leq \|f \odot g\|_{cb}\|u\|_h \\ &= \|f \odot g\|_h\|u\|_h \\ &\leq \|f\|\|g\|\|u\|_h \leq \|u\|_h. \end{aligned}$$

Therefore,

$$\|u\|_h = \sup\{\|(f \odot g)^{(n)}(u)\|\},$$

where the supremum is taken over all $f \in M_{n,r}(V^*), g \in M_{r,n}(W^*), \|f\| \leq 1, \|g\| \leq 1$, and $r \in \mathbb{N}$. □

2.5. Some tensor product computations

The following proposition can be found in [11, Proposition 9.3.1]. In [11], the identifications (1) and (2) were proved in different ways. In light of the similarity of these identifications, we give a unified proof of Proposition 2.5.1, which is consistent with the proof of (2) given in [11].

PROPOSITION 2.5.1. *Let V be an operator space and H a Hilbert space. Then we have the following natural complete isometries.*

$$\begin{aligned} H_c \overset{h}{\otimes} V &\cong H_c \overset{\vee}{\otimes} V. && (1) \\ V \overset{h}{\otimes} H_r &\cong V \overset{\vee}{\otimes} H_r. && (2) \\ V \overset{h}{\otimes} H_c &\cong V \overset{\wedge}{\otimes} H_c. && (3) \\ H_r \overset{h}{\otimes} V &\cong H_r \overset{\wedge}{\otimes} V. && (4) \end{aligned}$$

PROOF. For (1), it suffices to show that for all $u \in M_n(H_c \otimes V)$,

$$\|u\|_h \leq \|u\|_\vee.$$

So, it suffices to show that $\|u\|_\vee \leq 1$ implies $\|u\|_h \leq 1$.

Suppose $u \in M_n(H_c \otimes V)$ with $\|u\|_\vee \leq 1$. Now for all $f \in M_{n,s}((H_c)^*) = M_{n,s}((H^*)_r)$ and $g \in M_{s,n}(V^*)$ with $\|f\| \leq 1, \|g\| \leq 1$, let $H_1 = span\{f_{ij}, i = 1, \cdots, n, j = 1, \cdots, r\}$ and $e_1, \cdots, e_p$ its orthonormal basis. Writing $f_{ij} = \sum_{k=1}^{p} c_{ij}^k e_k$, we have by the discussion in [10, Section 3.4] that $\|f\| = \|\,[C^1 \cdots C^p]\,\|$, where $C^k = \left[c_{ij}^k\right] \in M_{n,s}$ $(k = 1, \cdots, p)$.

Following the notation in [3], $e = \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix} \in M_{p,1}((H^*)_r) = B(H, \mathbb{C}^p)$ with $\|e\|_r = 1$. Since $e \otimes g = \begin{pmatrix} e_1 \otimes g \\ \vdots \\ e_p \otimes g \end{pmatrix} \in M_{sp,n}(V^* \otimes (H^*)_r) \subseteq M_{sp,n}((V \otimes H_c)^*)$, it follows that

$$\begin{aligned} f \odot g &= \left[\sum_{l=1}^{r} f_{il} \otimes g_{lj}\right] = \left[\sum_{l=1}^{r}\sum_{k=1}^{p} c_{il}^k e_k \otimes g_{lj}\right] \\ &= \sum_{k=1}^{p} C^k(e_k \otimes g) = C(e \otimes g), \end{aligned}$$

where $C = [C^1 \cdots C^p] \in M_{n,sp}$. So, we have

$$\begin{aligned} (C(e \otimes g))^{(n)}(u) &= [C(e \otimes g)(u_{ij})] \\ &= \begin{pmatrix} C & & & \\ & C & & \\ & & \ddots & \\ & & & C \end{pmatrix} [(e \otimes g)(u_{ij})] \\ &= (I_n \otimes C)((e \otimes g)^{(n)}(u)). \end{aligned}$$

Then,

$$\begin{aligned} \|(f \odot g)^{(n)}(u)\| &= \|(I_n \otimes C)((e \overset{\vee}{\otimes} g)^{(n)}(u))\| \\ &\leq \|(I_n \otimes C)\| \|(e \overset{\vee}{\otimes} g)^{(n)}(u)\| \\ &\leq \|C\| \|(e \overset{\vee}{\otimes} g)^{(n)}\| \|u\|_\vee \\ &= \|f\| \|(e \overset{\vee}{\otimes} g)^{(n)}\| \|u\|_\vee \\ &\leq \|e \overset{\vee}{\otimes} g\|_{cb} \leq 1, \end{aligned}$$

where the fourth step follows from $\|C\| = \|f\|$ (≤ 1) and in the last step we use the fact that both e and g are contractive. So, $\|u\|_h \leq 1$ by Proposition 2.4.5 and the proof of (1) is complete.

(2), (3), and (4) can be similarly proved. □

The following propositions and their proofs can be found in [11].

PROPOSITION 2.5.2. *Let V be an operator space, H and K Hilbert spaces. Then we have a complete isometry*

$$((K_c)^* \overset{h}{\otimes} V \overset{h}{\otimes} H_c)^* \cong CB(V, B(H,K)).$$

PROPOSITION 2.5.3. *Let H and K be operator spaces. Then we have the complete isometries*

$$H_c \overset{h}{\otimes} (K_c)^* \cong \mathcal{K}(K,H)$$

and

$$(K_c)^* \overset{h}{\otimes} H_c \cong \mathcal{T}(K,H).$$

PROPOSITION 2.5.4. *Let H and K be Hilbert spaces. Then we have the complete isometries*

$$H_c \overset{\wedge}{\otimes} K_c \cong H_c \overset{h}{\otimes} K_c \cong H_c \overset{\vee}{\otimes} K_c \cong (H \otimes K)_c$$

and

$$H_r \overset{\wedge}{\otimes} K_r \cong H_r \overset{h}{\otimes} K_r \cong H_r \overset{\vee}{\otimes} K_r \cong (H \otimes K)_r.$$

2.6. Comparison with $\Gamma_c(V,W)$ and $\Gamma_r(V,W)$

Let V and W be operator spaces. we say that a linear map $\varphi : V \to W$ **factors through column Hilbert space** if there is a Hilbert space H and a commutative diagram of completely bounded maps

$$\begin{tikzcd} H_c \arrow[dr, "\psi"] & \\ V \arrow[u, "\phi"] \arrow[r, "\varphi"] & W. \end{tikzcd}$$

We define

$$\gamma_c(\varphi) = \inf\{\|\psi\|_{cb}\|\phi\|_{cb} : \varphi = \psi \circ \phi,\ \phi : V \to H_c,\ \psi : H_c \to W\}.$$

If no such a factorization exists, we set $\gamma_c(\varphi) = \infty$.

If $\varphi_1, \varphi_2 : V \to W$ factor through the column Hilbert space $(H_1)_c$ and $(H_2)_c$, respectively, that is, there exist $\phi_k : V \to (H_k)_c$ and $\psi_k : (H_k)_c \to W$ $(k = 1, 2)$ such that $\varphi_1 = \psi_1 \circ \phi_1$ and $\varphi_2 = \psi_2 \circ \phi_2$, then let $L = H_1 \oplus H_2$, $\phi(v) = (\phi_1(v), \phi_2(v))$, and $\psi(\xi_1, \xi_2) = \psi_1(\xi_1) + \psi_2(\xi_2)$ for all $v \in V$ and $(\xi_1, \xi_2) \in H_1 \oplus H_2$. It is clear that $\varphi_1 + \varphi_2 = \psi \circ \phi$. Now let $\Gamma_c(V,W)$ be the linear space of linear maps $\varphi : V \to W$ with $\gamma_c(\varphi) < \infty$. Effros-Ruan proved that γ_c really determines a norm on $\Gamma_c(V,W)$ (cf. [10, Lemma 5.1]), and so $\Gamma_c(V,W)$ becomes a normed space.

If $\varphi = [\varphi_{ij}] \in M_n(\Gamma_c(V,W))$, then we may define a map $\widetilde{\varphi} : V \to M_n(W)$ by $\widetilde{\varphi}(v) = [\varphi_{ij}(v)]$, each entry of which factors through a column Hilbert space as given in the following commutative diagram

$$\begin{tikzcd} (H_{ij})_c \arrow[dr, "\psi_{ij}"] & \\ V \arrow[u, "\phi_{ij}"] \arrow[r, "\varphi_{ij}"'] & W. \end{tikzcd}$$

We want to find a factorization of $\widetilde{\varphi}$ through some column Hilbert space K in a natural way. That is of the form $\widetilde{\varphi} = \psi \circ \phi$, where $\phi : V \to K_c$ and $\psi : K_c \to M_n(W)$. Since $M_{n^2,1}(W) \cong M_n(W)$ as operator spaces, there exists a linear map from V to $M_{n^2,1}(W)$ corresponding to $\widetilde{\varphi}$, we still denote it by $\widetilde{\varphi}$.

Let $K = \oplus H_{ij}$. Then $K_c = \oplus (H_{ij})_c$. We define $\phi : V \to K_c$ and $\psi : K_c \to M_{n^2,1}(W)$ by

$$\phi(v) = (\phi_{11}(v), \cdots, \phi_{nn}(v))^t$$

and

$$\psi(\xi_{11}, \cdots, \xi_{nn})^t = (\psi_{11}(\xi_{11}), \cdots, \psi_{nn}(\xi_{nn}))^t$$

for $v \in V$, $\xi_{ij} \in H_{ij}$. Then $\psi \circ \phi = \widetilde{\varphi}$. This shows that $\widetilde{\varphi} \in \Gamma_c(V, M_n(W))$.

Conversely, suppose a linear map $\widetilde{\varphi} : V \to M_n(W)$ factors through H_c, i.e., $\widetilde{\varphi} = \psi \circ \phi$, where $\phi : V \to H_c$ and $\psi : H_c \to M_n(W)$ are completely bounded. We define $\varphi = [\varphi_{ij}] \in M_n(\Gamma_c(V,W))$ by

$$\varphi_{ij}(v) = (\widetilde{\varphi}(v))_{ij}$$

and $\psi_{ij} : H_c \to W$ by

$$\psi_{ij}(\xi) = (\psi(\xi))_{ij}$$

for all $v \in V$ and $\xi \in H$. Then

$$\varphi_{ij}(v) = (\psi \circ \phi(v))_{ij} = \psi_{ij}(\phi(v)) = (\psi_{ij} \circ \phi)(v).$$

Clearly, each ψ_{ij} is completely bounded. So, φ_{ij} factors also through H_c and then $\varphi \in M_n(\Gamma_c(V,W))$.

Therefore, we have the linear space identifications $M_n(\Gamma_c(V,W)) \cong \Gamma_c(V, M_n(W))$ ($n \in \mathbb{N}$), and so we can define a natural operator space matrix norm on $\Gamma_c(V,W)$ to make $\Gamma_c(V,W)$ an operator space.

In general, $\|\varphi\|_{cb} \le \gamma_c(\varphi)$, and hence $\Gamma_c(V,W) \subseteq CB(V,W)$. If either V or W is a column Hilbert space, then $\|\varphi\|_{cb} = \gamma_c(\varphi)$ and $\Gamma_c(V,W) = CB(V,W)$.

PROPOSITION 2.6.1. *Let V and W be operator spaces and W_1 a subspace of W. Then the corresponding inclusion*

$$\Gamma_c(V,W_1) \hookrightarrow \Gamma_c(V,W)$$

is completely isometric.

THEOREM 2.6.2. *Let V and W be operator spaces. Then we have a complete isometry*

$$(W \overset{h}{\otimes} V)^* \cong \Gamma_c(V,W^*).$$

COROLLARY 2.6.3. *Let V, W, and X be operator spaces. Then we have a complete isometry*

$$\Gamma_c((W \overset{h}{\otimes} V), X) \cong \Gamma_c(V, \Gamma_c(W,X)).$$

PROOF. First we suppose X is a dual operator space, say, $X = (X_*)^*$. Then from Theorem 2.6.2 we have the natural complete isometries

$$\Gamma_c((W \overset{h}{\otimes} V), X) \cong (X_* \overset{h}{\otimes} W \overset{h}{\otimes} V)^* \cong \Gamma_c(V, (X_* \overset{h}{\otimes} W)^*) \cong \Gamma_c(V, \Gamma_c(W,X)).$$

For a general X, we have the following commutative diagram

$$\begin{array}{ccc} \Gamma_c(W \overset{h}{\otimes} V, X) & \longrightarrow & \Gamma_c(V, \Gamma_c(W,X)) \\ \downarrow & & \downarrow \\ \Gamma_c(W \overset{h}{\otimes} V, X^{**}) & \longrightarrow & \Gamma_c(V, \Gamma_c(W, X^{**})), \end{array}$$

where the bottom map is completely isometric and the vertical maps are completely isometric injections by [10, Proposition 5.2]. We need to show the top map

$$\Phi : \Gamma_c(W \overset{h}{\otimes} V, X) \to \Gamma_c(V, \Gamma_c(W, X)) \text{ determined by } \Phi(\varphi)(v)(w) = \varphi(w \otimes v)$$

is onto.

For any $\widetilde{\varphi} \in \Gamma_c(V, \Gamma_c(W, X))$, by the argument in the first paragraph of this proof, there exists a map $\varphi \in \Gamma_c(W \overset{h}{\otimes} V, X^{**})$ such that $\widetilde{\varphi}(v)(w) = \varphi(w \otimes v)$. Due to the fact that $W \otimes V$ is dense in $W \overset{h}{\otimes} V$, φ is valued in X, and hence it is in $\Gamma_c(W \overset{h}{\otimes} V, X)$. □

Let V and W be operator spaces. we say that a linear map $\varphi : V \to W$ **factors through row Hilbert space** if there is a Hilbert space H and a commutative diagram of completely bounded maps

$$\begin{tikzcd} H_r \arrow[dr, "\psi"] & \\ V \arrow[u, "\phi"] \arrow[r, "\varphi"] & W. \end{tikzcd}$$

Let $\Gamma_r(V, W)$ be the corresponding operator space. Then we have the following result

$$(V \overset{h}{\otimes} W)^* \cong \Gamma_r(V, W^*),$$

and

$$\Gamma_r(V \overset{h}{\otimes} W), X) \cong \Gamma_r(V, \Gamma_r(W, X)).$$

The proofs are similar to the corresponding parts of $\Gamma_c(V, W)$. In particular, we have now the identification

$$\Gamma_r(V, W^*) \cong \Gamma_c(W, V^*).$$

CHAPTER 3

Extended and Normal Haagerup Tensor Products

Based on the Haagerup tensor product, two more operator space tensor products are introduced – the extended and normal Haagerup tensor products. These two tensor products are not usual tensor products any more in the sense that they are not norm closures of the correponding algebraic tensor products. However, due to the self–duality of the Haagerup tensor product, they have some nice properties and both have the Haagerup tensor product as certain weak*-dense subspace. Most of results in the chapter can be found in [12]. We start with the general theory of infinite matrices, which is a bridge between mapping spaces and matrix spaces.

3.1. Infinite matrices

Given an operator space V, and index sets I and J, we let $M_{I,J}(V)$ denote the vector space of matrices $F = [v_{ij}]_{i\in I, j\in J}$, for which finite submatrices are uniformly bounded in norm, i.e., $\sup\limits_{F'} \|F'\| < \infty$, where the supremum is taken over all finite submatrices F' of F.

As usual, we denote $M_{J,J}(V)$ by $M_J(V)$. It can be seen that, as linear spaces, $M_{I,J}$ can be identified with $B(l^2(J), l^2(I))$, and in particular, $M_J = M_J(\mathbb{C})$ can be identified with $B(l^2(J))$ as linear spaces.

In fact, suppose $\{e_j\}$ and $\{f_i\}$ are orthonormal bases of $l^2(J)$ and $l^2(I)$, respectively. We define a linear map $\varphi : M_{I,J} \longrightarrow B(l^2(J), l^2(I))$ by $\langle \varphi(B)e_j | f_i\rangle = b_{ij}$, where $B = [b_{ij}] \in M_{I,J}$. Obviously, φ is one-one.

For each $b \in B(l^2(J), l^2(I))$, let $B = [b_{ij}]_{i\in I, j\in J}$, where $b_{ij} = \langle be_j | f_i\rangle$. We want to show $B \in M_{I,J}$.

Suppose S and T are finite subsets of I and J, respectively. Let $B^{S,T}$ denote the $S \times T$ submatrix of B. Then

$$\|B^{S,T}\| \quad = \quad \sup\{\|B^{S,T}\alpha\| : \alpha \in \mathbb{C}^T \ and \ \|\alpha\| \leq 1\}$$

$$= \sup\{\langle B^{S,T}\alpha | B^{S,T}\alpha\rangle^{1/2} : \alpha \in \mathbb{C}^T \textit{ and } \sum_{j\in T} |\alpha_j|^2 \leq 1\}$$

$$= \sup\{\left(\sum_{i\in S}(|\sum_{j\in T} b_{ij}\alpha_j|^2)\right)^{1/2} : \alpha \in \mathbb{C}^T \textit{ and } \sum_{j\in T}|\alpha_j|^2 \leq 1,\},$$

where we use the fact that $B^{S,T}$ is a finite matrix and $B^{S,T}\alpha \in \mathbb{C}^S$.

On the other hand, we have

$$\begin{aligned}
\|b\| &= \sup\{\|b\xi\| : \xi \in l^2(J) \textit{ and } \|\xi\| \leq 1,\} \\
&= \sup\{\left(\sum_{i\in I}(|\sum_{j\in J}\langle bc_je_j|f_i\rangle|^2)\right)^{1/2} : \xi = \sum_{j\in J} c_je_j \textit{ and } \sum_{j\in J}\|c_j\|^2 \leq 1\} \\
&= \sup\{\left(\sum_{i\in I}(|\sum_{j\in J} c_j\langle be_j|f_i\rangle|^2)\right)^{1/2} : \xi = \sum_{j\in J} c_je_j \textit{ and } \sum_{j\in J}\|c_j\|^2 \leq 1\} \\
&= \sup\{\left(\sum_{i\in I}(|\sum_{j\in J} c_jb_{ij}|^2)\right)^{1/2} : \xi = \sum_{j\in J} c_je_j \textit{ and } \sum_{j\in J}\|c_j\|^2 \leq 1\}.
\end{aligned}$$

Clearly, $\sup_{S\subseteq I, T\subseteq J} \|B^{S,T}\| = \|b\| < \infty$, and hence $B \in M_{I,J}$. Obviously, $b = \varphi(B)$. Therefore, $\varphi : M_{I,J} \to B(l^2(J), l^2(I))$ is onto.

Now we can define the operator space matrix norm on $M_{I,J}$ by using the above linear space identification. So far, $M_{I,J}$ is an operator space with the norm

$$\|F\| = \sup_{F'} \|F'\|,$$

where F' is taken over all finite submatries of F. In particular, if we order the set of finite submatries of F by inclusion, then it is a directed set and

$$\|F\| = \lim_{F'} \|F'\|.$$

Let $a \in M_{I,K}, b \in M_{K,L}$, and $c \in M_{L,J}$. Then abc makes sense by the above identification. For a subset $S \subseteq K$, we let $P_K(S) : l^2(K) \to l^2(S)$ be the orthogonal projection. Similarly, we can define $P_L(T) : l^2(L) \to l^2(T)$ for $T \subseteq L$. Now we restrict to finite subsets $F \subseteq K$ and $G \subseteq L$, and we may regard $\{P_K(F)\}_F$ and $\{P_L(G)\}_G$ as nets of projections, where finite sets are ordered by inclusion relationship. Now both nets converge to the identity operator in the strong operator topology.

REMARK 3.1.1. *As an infinite matrix, $P_K(F)$ has its (i,j)th entry $(P_K(F))_{ij} = \langle P_K(F)f_i | f_j \rangle$, where $\{f_i\}_{i\in K}$ and $\{f_i\}_{i\in F}$ are the orthonormal bases of $l^2(K)$ and $l^2(F)$, respectively. If $i \in F$, then $P_K(F)f_i = f_i$. Otherwise, $P_K(F)f_i = 0$. So,*

$$(P_K(F))_{ij} = \begin{cases} \delta_{ij} & \text{if } i \in F \\ 0 & \text{otherwise} \end{cases}.$$

Now, $aP_K(F)bP_L(G)c$ is a well-defined operator in $B(l^2(J), l^2(I))$. Note that all $a, P_K(F), b, P_L(G), c$ are bounded, so, $aP_K(F)bP_L(G)c \to abc$ in SOT when $P_K(F) \overset{SOT}{\to} id$ and $P_L(G) \overset{SOT}{\to} id$. Thus

$$\sum_{k\in F,\, l\in G} a_{ik}b_{kl}c_{lj} \to (abc)_{ij},$$

i.e., we can express the entry of an infinite matrix product as a limit of finite sums. This fact will be used in the sequel when we consider an infinite matrix product.

If H and K are Hilbert spaces with bases $(e_j)_{j\in J}$ and $(f_i)_{i\in I}$, then $H \cong B(l^2(J))$ and $K \cong B(l^2(I))$. So, we may identify $B(H,K)$ with $M_{I,J}$, i.e., $B(H,K) \cong M_{I,J}$, which is important in the later discussion.

Given operator spaces V and W, if V and W are dual operator spaces, then we let $CB^\sigma(V,W)$ be the space of weak*-weak* continuous maps in $CB(V,W)$.

We already knew that $CB(V, M_n) \cong M_n(V^*)$ as operator spaces. In fact, as shown in the following, it has a more general version

$$CB(V, M_{I,J}) \cong M_{I,J}(V^*).$$

To see this, we define a linear map $\varphi : M_{I,J}(V^*) \to CB(V, M_{I,J})$ by $\varphi(F)(v) = [F_{ij}(v)]$ for each $F = [F_{ij}] \in M_{I,J}(V^*)$. Then that φ is one-one and onto can be similarly proved as we did for $M_{I,J} \cong B(l^2(J), l^2(I))$. It remains to show that $\varphi(F)(v) \in M_{I,J}$ and $\varphi(F) \in CB(V, M_{I,J})$. Actually, for $v \in V$,

$$\begin{aligned} \|\varphi(F)(v)\| &= \sup_{F'} \|\varphi(F')(v)\| \le \sup_{F'} \|\varphi(F')\|_{cb}\|v\| \\ &= \sup_{F'} \|F'\|\|v\| = \|F\|\|v\| < \infty, \end{aligned}$$

where F' is a finite submatrix of F, $F' \in M_n(V^*)$ for some $n \in \mathbb{N}$, and hence $\varphi(F') \in CB(V, M_n) \cong M_n(V^*)$. In particular, $\|\varphi(F)\|_{cb} = \|F\| \le \infty$, i.e., $\varphi(F) \in CB(V, M_{I,J})$.

Using this linear space (in fact, Banach space) identification, we can define the operator space matrix norm on $M_{I,J}(V^*)$ and $M_{I,J}(V^*)$ becomes an operator space.

Since $M_{I,J} \cong B(l^2(J), l^2(I))$ and $B(l^2(J), l^2(I))$ is an dual operator space (cf. [2, Theorem 1.4.5]), $CB^\sigma(V^*, M_{I,J})$ makes sense.

PROPOSITION 3.1.2. *Let V be an operator space. Then we have a natural linear space identification $CB^\sigma(V^*, M_{I,J}) \cong M_{I,J}(V)$ induced by $CB(V^*, M_{I,J}) \cong M_{I,J}(V^{**})$.*

PROOF. Let $\varphi \in CB^\sigma(V^*, M_{I,J})$. Then there exists a matrix $F = [F_{ij}] \in M_{I,J}(V^{**})$ such that $\varphi(f) = [F_{ij}(f)]$ for all $f \in V^*$. By the hypothesis that φ is weak*-weak* continuous, F_{ij} is continuous in the weak*-topology on V^* for all i, j. It follows that $F_{ij} \in V$, and thus $F = [F_{ij}] \in M_{I,J}(V)$.

Conversely, let $F = [v_{ij}] \in M_{I,J}(V)$ and $\varphi(f) = [f(v_{ij})]$ $(f \in V^*)$. Then $\varphi \in CB(V^*, M_{I,J})$. Now we want to show that $\varphi : V^* \to M_{I,J}$ is weak*-weak* continuous, which is equivalent to showing that $P_{i,j} \circ \varphi : V^* \to \mathbb{C}$ is weak*-continuous (i.e., $P_{i,j} \circ \varphi \in V$) for all $(i,j) \in I \times J$, where $P_{i,j} : M_{I,J} \to \mathbb{C}$ is the canonical (i,j)th projection. Note that $(P_{i,j} \circ \varphi)(f) = f(v_{ij}) = \langle v_{ij}, f\rangle$. So, $P_{i,j} \circ \varphi = v_{ij} \in V$ for all $(i,j) \in I \times J$. $\square$

3.2. Extended Haagerup tensor product

Recall we used $MB(V_1 \times V_2, W)$ to denote the linear space of all multiplicatively bounded bilinear maps $\varphi : V_1 \times V_2 \to W$ with the norm $\|\cdot\|_{mb}$ and we have the operator space identification $MB(V_1 \times V_2, W) \cong CB(V_1 \overset{h}{\otimes} V_2, W)$. If V_1, V_2, and W are dual operator spaces, then we say $\varphi \in MB(V_1 \times V_2, W)$ is normal if it is weak*-weak* continuous in each variable. Let $MB^\sigma(V_1 \times V_2, W)$ be the operator subspace of $MB(V_1 \times V_2, W)$ consisting of normal maps in $MB(V_1 \times V_2, W)$.

The **extended Haagerup tensor product** $V_1 \overset{eh}{\otimes} V_2$ of V_1 and V_2 is defined as the space of all normal multiplicatively bounded bilinear functionals $u : V_1^* \times V_2^* \to \mathbb{C}$ and we use $(V_1^* \overset{h}{\otimes} V_2^*)^*_\sigma$ to denote the subspace of $CB(V_1^* \overset{h}{\otimes} V_2^*, \mathbb{C})$ $(= (V_1^* \overset{h}{\otimes} V_2^*)^*)$ corresponding to $MB^\sigma(V_1^* \times V_2^*, \mathbb{C})$, i.e.,

$$V_1 \overset{eh}{\otimes} V_2 = (V_1^* \overset{h}{\otimes} V_2^*)^*_\sigma = MB^\sigma(V_1^* \times V_2^*, \mathbb{C}).$$

We use $\|\cdot\|_{eh}$ to denote the operator space matrix norm on $V_1 \overset{eh}{\otimes} V_2$ induced by the identification $M_n(MB^\sigma(V_1^* \times V_2^*, \mathbb{C})) = MB^\sigma(V_1^* \times V_2^*, M_n)$.

Similar to the decomposition theorem as stated in Theorem 2.4.2, we have the following version of decomposition theorem for multilinear maps.

THEOREM 3.2.1. *Let $V_1, \cdots, V_p$ be operator spaces. Then a multilinear map*

$$\varphi : V_1 \times \cdots \times V_p \to B(H_p, H_0),$$

is multiplicatively contractive if and only if there exist Hilbert spaces $H_1, \cdots, H_{p-1}$ and complete contractions $\varphi_k : V_k \to B(H_k, H_{k-1})$ such that

$$\varphi(v_1, \cdots, v_p) = \varphi_1(v_1) \cdots \varphi_p(v_p)$$

and

$$\|\varphi\|_{mb} = \|\varphi_1\|_{cb} \cdots \|\varphi_p\|_{cb}.$$

If each V_k is a dual space and φ is normal, then we may assume that each φ_k is weak-continuous.*

For Banach spaces X and Y, any bounded linear map $T : X \to Y^*$ has a unique weak*-weak* continuous extension $\widetilde{T} : X^{**} \to Y^*$ with $\|\widetilde{T}\| = \|T\|$. In fact, $\widetilde{T}$ is given by $\widetilde{T} = \pi^* \circ T^{**}$, where $\pi : Y \to Y^{**}$ is the canonical embedding. According to Blecher-Le Merdy [2, 1.4.8], this statement has its operator space version. That is, each completely bounded linear map $T : X \to Y^*$ has a unique weak*-weak* continuous extension $\widetilde{T} : X^{**} \to Y^*$ such that $\|\widetilde{T}\|_{cb} = \|T\|_{cb}$. In the following, we show that there is a corresponding extension theorem for bilinear maps.

PROPOSITION 3.2.2. *Let $V_1, V_2,$ and W be operator spaces, and $\varphi : V_1 \times V_2 \to W^*$ a multiplicatively bounded bilinear map. Then φ admits a (necessarily unique) normal extension $\widetilde{\varphi} : V_1^{**} \times V_2^{**} \to W^*$. This extension is multiplicatively bounded and $\|\widetilde{\varphi}\|_{mb} = \|\varphi\|_{mb}$.*

PROOF. We may assume that W^* is a weak*-closed subspace of some $B(H)$ (cf. [2, Lemma 1.4.7]). By Theorem 3.2.1, there exist a Hilbert space L and two completely bounded maps $\psi_1 : V_1 \to B(L, H)$ and $\psi_2 : V_2 \to B(H, L)$ such that $\varphi(v_1, v_2) = \psi_1(v_1)\psi_2(v_2)$ for all $v_1 \in V_1$, $v_2 \in V_2$, and $\|\varphi\|_{mb} = \|\psi_1\|_{cb}\|\psi\|_{cb}$. By the argument preceding this proposition, ψ_1 and ψ_2 admit weak*-weak* continuous extensions $\widetilde{\psi_1}: V_1^{**} \to B(L,H)$ and $\widetilde{\psi_2}: V_2^{**} \to B(H,L)$ with $\|\widetilde{\psi_i}\|_{cb} = \|\psi_i\|_{cb}$ $(i = 1,2)$. We define $\widetilde{\varphi}: V_1^{**} \times V_2^{**} \to B(H)$ by $\widetilde{\varphi}(\widetilde{v_1}, \widetilde{v_2}) = \widetilde{\psi_1}(\widetilde{v_1})\widetilde{\psi_2}(\widetilde{v_2})$ ($\widetilde{v_1} \in V_1^{**}$ and $\widetilde{v_2} \in V_2^{**}$). Since $\|\widetilde{\varphi}\|_{mb} \leq \|\widetilde{\psi_1}\|_{cb}\|\widetilde{\psi_2}\|_{cb} = \|\psi_1\|\|\psi_2\| = \|\varphi\|_{mb}$, we have $\|\widetilde{\varphi}\|_{mb} = \|\varphi\|_{mb}$. Finally, $\widetilde{\varphi}$ is valued in W^*, since φ is valued in W^*, $V_1 \times V_2$ is weak*-dense in $V_1^{**} \times V_2^{**}$, and $\widetilde{\varphi}$ is normal. $\square$

Now Let V_1, V_2, and W be operator spaces. From Proposition 3.2.2, it follows immediately that

$$MB(V_1 \times V_2, W^*) = MB^\sigma(V_1^{**} \times V_2^{**}, W^*).$$

In particular,

$$(V_1 \overset{h}{\otimes} V_2)^* = (V_1^{**} \overset{h}{\otimes} V_2^{**})^*_\sigma.$$

By the definition of the extended Haagerup tensor product, we have the operator space identification

$$(V_1 \overset{h}{\otimes} V_2)^* \cong V_1^* \overset{eh}{\otimes} V_2^*.$$

PROPOSITION 3.2.3. *Let V_1 and V_2 be operator spaces. Then the inclusion map $V_1 \overset{h}{\otimes} V_2 \to V_1 \overset{eh}{\otimes} V_2$ is a completely isometric injection.*

PROOF. We have the following commutative diagram

$$\begin{array}{ccc} V_1 \overset{h}{\otimes} V_2 & \longrightarrow & V_1^{**} \overset{h}{\otimes} V_2^{**} \\ \downarrow & & \downarrow \\ V_1 \overset{eh}{\otimes} V_2 & \longrightarrow & (V_1^* \overset{h}{\otimes} V_2^*)^*, \end{array}$$

in which the top and right mappings are complete isometries by the injectivity and self-duality of the Haargerup tensor product. The bottom mapping is the completely isometric embedding owing to the definition of the extended Haargerup tensor product. So, the left map is a completely isometric injection. $\square$

LEMMA 3.2.4. *Let V_1 and V_2 be operator spaces. Then each $u \in M_n(V_1 \overset{eh}{\otimes} V_2)$ has a representation of the form $u = v_1 \odot v_2$, where $v_1 \in M_{n,J}(V_1)$ and $v_2 \in M_{J,n}(V_2)$. In particular, if $\|u\|_{eh} \leq 1$, then we can choose v_1 and v_2 such that $u = v_1 \odot v_2$ and $\|u\|_{eh} = \|v_1\|\|v_2\|$, $\|v_1\| \leq 1$ and $\|v_2\| \leq 1$.*

PROOF. Apparently, it suffices to show the last part of the statements. Suppose $\|u\|_{eh} \leq 1$. By Thereom 3.2.1 and the identifications

$$\begin{aligned} M_n(V_1 \overset{eh}{\otimes} V_2) &\cong M_n((V_1^* \overset{h}{\otimes} V_2^*)_\sigma^*) \subseteq M_n(CB(V_1^* \overset{h}{\otimes} V_2^*, \mathbb{C})) \\ &\cong CB(V_1^* \overset{h}{\otimes} V_2^*, M_n) \cong MB(V_1^* \times V_2^*, M_n), \end{aligned}$$

there exist H_1 and contractions $v_1 : V_1^* \to B(H_1, \mathbb{C}^n) = M_{n,J}$ and $v_2 : V_2^* \to B(\mathbb{C}^n, H_1) = M_{J,n}$, i.e., $v_1 \in CB^\sigma(V_1^*, M_{n,J}) = M_{n,J}(V_1)$ and $v_2 \in CB^\sigma(V_2^*, M_{J,n}) = M_{J,n}(V_2)$, such that $u(f_1, f_2) = v_1(f_1)v_2(f_2)$ and $\|u\|_{eh} = \|u\|_{cb} = \|v_1\|_{cb}\|v_2\|_{cb} = \|v_1\|\|v_2\|$. In this case, we use the notation $u = v_1 \odot v_2$. $\square$

If $v_1 \in M_{n,J}(V_1)$ and $v_2 \in M_{J,n}(V_2)$, then $v_1 \odot v_2$ can be written into $\left[\sum_{k \in J} v_{ik}^1 \otimes v_{kj}^2\right]$.

According to Proposition 3.2.3, $V_1 \overset{h}{\otimes} V_2$ can be treated as a subspace of $V_1 \overset{eh}{\otimes} V_2$. The following lemma shows that in this case, the index set J in Lemma 3.2.4 can be chosen to be the set $\mathbb{N}$ of natural numbers.

LEMMA 3.2.5. *Let V and W be operator spaces. Then each $u \in V \overset{h}{\otimes} W$ with $\|u\|_h < 1$ has a representation $u = v \odot w$ with $\|v\| < 1$, $\|w\| < 1$ and $\|u\|_h = \|v\|\|w\|$, where $v \in M_{1,\mathbb{N}}(V)$, $v_2 \in M_{\mathbb{N},1}(W)$.*

PROOF. Suppose $\|u\|_h < 1$. Since $V \otimes W$ is norm dense in $V \overset{h}{\otimes} W$, there exists $u_1 = \sum_{k=1}^{n_1} v_k^1 \otimes w_k^1 \in V \otimes W$ such that $\|u - u_1\| < \frac{1-\|u\|_h}{2}$ with $\|u_1\|_h \leq \|u\|_h$. Let $v_1 = (v_1^1, \cdots, v_{n_1}^1)$ and $w_1 = (w_1^1, \cdots, w_{n_1}^1)^t$. Then $u_1 = v_1 \odot w_1$ and v_1, w_1 can be chosen such that $\|v_1\| \leq \|u\|_h^{1/2}$ and $\|w_1\| \leq \|u\|_h^{1/2}$. For $u - u_1 \in V \overset{h}{\otimes} W$, there exists $u_2 = \sum_{k=n_1+1}^{n_2} v_k^2 \otimes w_k^2 \in V \otimes W$ such that $\|u - u_1 - u_2\|_h < \frac{1-\|u\|_h}{2^2}$ with $\|u_2\|_h < \frac{1-\|u\|_h}{2}$. Let $v_2 = (v_{n_1+1}^2, \cdots, v_{n_2}^2)$ and $w_2 = (w_{n_1+1}^2, \cdots, w_{n_2}^2)^t$. Then $u_2 = v_2 \odot w_2$ and v_2, w_2 are chosen such that $\|v_2\| < \left(\frac{1-\|u\|_h}{2}\right)^{1/2}$ and $\|w_2\| < \left(\frac{1-\|u\|_h}{2}\right)^{1/2}$. Continuing this process, for each $m \in \mathbb{N}$, we can find $u_m = \sum_{k=n_{m-1}+1}^{n_m} v_k^m \otimes w_k^m \in V \otimes W$ such that $\|u - u_1 - \cdots - u_m\| < \frac{1-\|u\|_h}{2^m}$ with $\|u_m\|_h \leq \frac{1-\|u\|_h}{2^{m-1}}$. Let $v_m = (v_{n_{m-1}+1}^m, \cdots, v_{n_m}^m)$ and $w_m = (w_{n_{m-1}+1}^m, \cdots, w_{n_m}^m)^t$. Then $u_m = v_m \odot w_m$, $\|v_m\| < \left(\frac{1-\|u\|_h}{2^{m-1}}\right)^{1/2}$ and $\|w_m\| < \left(\frac{1-\|u\|_h}{2^{m-1}}\right)^{1/2}$. Let $v = (v_1, v_2, v_3, \cdots)$ and $w = (w_1, w_2, w_3, \cdots)^t$. Then $u = v \odot w$ is uniformly convergent in norm. Now $\|v\| = \sqrt{\sum_{m=1}^{\infty} \|v_m\|^2} < \sqrt{\|u\|_h + \frac{1-\|u\|_h}{2} + \frac{1-\|u\|_h}{2^2} + \cdots} = 1$, i.e., $\|v\| < 1$. Similarly, $\|w\| < 1$. $\square$

Let $u \in M_n(V_1 \overset{eh}{\otimes} V_2) \subseteq MB(V_1^* \times V_2^*, M_n)$ and $u = v_1 \odot v_2$, where $v_1 \in M_{n,J}(V_1)$ and $v_2 \in M_{J,n}(V_2)$. For all $f_1 \in V_1^*$ and $f_2 \in V_2^*$,

$$\begin{aligned} \langle f_1 \otimes f_2, u \rangle &= \langle f_1, v_1 \rangle \langle f_2, v_2 \rangle \\ &= \lim_F \left[\sum_{k \in F} f_1(v_{ik}^1) f_2(v_{kj}^2) \right], \end{aligned}$$

where the limit is taken over finite subsets F of J. We let v_i^F be the submatrix of v_i $(i = 1, 2)$ corresponding to F. Then $v_1^F \odot v_2^F \in MB(V_1^* \times V_2^*, M_n)$ and

$$\langle f_1 \otimes f_2, u \rangle = \lim_F \langle f_1, v_1^F \rangle \odot \langle f_2, v_2^F \rangle = \lim_F \langle f_1 \otimes f_2, v_1^F \odot v_2^F \rangle.$$

Thus, $\|u\|_{mb} \leq \underline{\lim}_F \|v_1^F \odot v_2^F\|_{mb}$.

On the other hand, $v_1^F \odot v_2^F \in M_n(V_1^* \otimes_h V_2^*)$, So,

$$\|v_1^F \odot v_2^F\|_{mb} = \|v_1^F \odot v_2^F\|_h \leq \|v_1^F\| \|v_2^F\| \leq \|v_1\| \|v_2\|.$$

Then $\|u\|_{eh} = \|u\|_{mb} \leq \underline{\lim}_F \|v_1^F \odot v_2^F\|_{mb} \leq \|v_1\| \|v_2\|$, and hence the lemma below is immediate by Lemma 3.2.4.

LEMMA 3.2.6. *Let V_1 and V_2 be operator spaces. Then for each $u \in M_n(V_1 \overset{eh}{\otimes} V_2)$, we have*

$$\|u\|_{eh} = \inf\{\|v_1\| \|v_2\|\},$$

where the infimum is taken over all the decompositions of the form $u = v_1 \odot v_2, v_1 \in M_{n,J}(V_1), v_2 \in M_{J,n}(V_2)$, and J is any index set.

REMARK 3.2.7. *By the argument immediately previous to Lemma 3.2.6, we get for each $u \in V_1 \overset{eh}{\otimes} V_2$,*

$$u^F = v_1^F \odot v_2^F \to u$$

in the weak-topology determined by $V_1^* \otimes V_2^*$. But $v_1^F \odot v_2^F \in V_1 \otimes V_2$. So, the algebraic tensor product $V_1 \otimes V_2$ is weak*-dense in $V_1 \overset{eh}{\otimes} V_2$.*

Let $\varphi_1 : V_1 \to W_1$ and $\varphi_2 : V_2 \to W_2$ be completely bounded maps. Then the completely bounded map

$$(\varphi_1^* \overset{h}{\otimes} \varphi_2^*)^* : \left(V_1^* \overset{h}{\otimes} V_2^* \right)^* \to \left(W_1^* \overset{h}{\otimes} W_2^* \right)^*$$

sends $V_1 \overset{eh}{\otimes} V_2$ to $W_1 \overset{eh}{\otimes} W_2$ since φ_1^* and φ_2^* are weak*-weak* continuous. Note that $(\varphi_1^* \overset{h}{\otimes} \varphi_2^*)^*$ is the unique weak*-weak* continuous extension of the algebraic tensor product $\varphi_1 \otimes \varphi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$, where the algebraic tensor product $X \otimes Y$ is embedded into $X \overset{eh}{\otimes} Y$ via $x \otimes y \mapsto \widetilde{x \otimes y}$ by Proposition 3.2.3. We let $\varphi_1 \overset{eh}{\otimes} \varphi_2$ denote the restriction of $(\varphi_1^* \overset{h}{\otimes} \varphi_2^*)^*$ to $V_1 \overset{eh}{\otimes} V_2$. We show that the injectivity of $\overset{eh}{\otimes}$ in the following.

THEOREM 3.2.8. *Let V_1, V_2, W_1, and W_2 be operator spaces. If $\varphi_k : V_k \to W_k$ $(k = 1, 2)$ are complete isometries (resp. contractions), then*

$$\varphi_1 \overset{eh}{\otimes} \varphi_2 : V_1 \overset{eh}{\otimes} V_2 \to W_1 \overset{eh}{\otimes} W_2$$

is a complete isometry (resp. contraction).

PROOF. Suppose φ_k $(k = 1, 2)$ are complete isometries. We have the following commutative diagram

$$\begin{array}{ccc} V_1 \overset{eh}{\otimes} V_2 & \longrightarrow & (V_1^* \overset{h}{\otimes} V_2^*)^* \\ \downarrow & & \downarrow \\ W_1 \overset{eh}{\otimes} W_2 & \longrightarrow & (W_1^* \overset{h}{\otimes} W_2^*)^*, \end{array}$$

in which the top and bottom mappings are completely isometric inclusions by the definition of the extended Haagerup tensor product. By Effros-Ruan [11, Corollary 4.1.9], φ_k^* $(k = 1, 2)$ are complete quotient maps. Then $\varphi_1^* \overset{h}{\otimes} \varphi_2^* : W_1^* \overset{h}{\otimes} W_2^* \to V_1^* \overset{h}{\otimes} V_2^*$ is also a complete quotient map, since the Haagerup tensor product is projective, and hence $(\varphi_1^* \overset{h}{\otimes} \varphi_2^*)^*$ is a complete isometry. That means the right column mapping is complete isometry, so is the left column mapping.

If φ_k $(k = 1, 2)$ are complete contractions, then so is $(\varphi_1^* \overset{h}{\otimes} \varphi_2^*)^*$ by Effros-Ruan [11, Proposition 3.2.2] and the property of Haagerup tensor product. Therefore, the left column mapping is also a complete contraction. □

LEMMA 3.2.9. *let V_1, V_2, W_1, and W_2 be operator spaces. If $\varphi_k : V_k \to W_k$ are completely bounded $(k = 1, 2)$, then for any index set J, $v_1 \in M_{1,J}(V_1)$, and $v_2 \in M_{J,1}(V_2)$, we have*

$$(\varphi_1 \overset{eh}{\otimes} \varphi_2)(v_1 \odot v_2) = \varphi_1^{(1,J)}(v_1) \odot \varphi_2^{(J,1)}(v_2).$$

PROOF. By Remark 3.2.7, we have $v_1 \odot v_2 = \lim_{F \subseteq J} v_1^F \odot v_2^F$, where $v_1^F \in M_{n,F}(V_1)$, $v_2^F \in M_{F,n}(V_2)$, and $F \subseteq J$ is finite. Then

$$\begin{aligned}(\varphi_1 \overset{eh}{\otimes} \varphi_2)(v_1 \odot v_2) &= \lim_F (\varphi_1 \overset{eh}{\otimes} \varphi_2)(v_1^F \odot v_2^F) \\ &= \lim_F \varphi_1^{(1,F)}(v_1^F) \odot \varphi_2^{(F,1)}(v_2^F) = \varphi_1^{(1,J)}(v_1) \odot \varphi_2^{(J,1)}(v_2).\end{aligned}$$

□

THEOREM 3.2.10. *Let V_1 and V_2 be operator spaces. Then we have a completely isometric inclusion*

$$V_1^* \overset{eh}{\otimes} V_2^* \hookrightarrow (V_1 \overset{eh}{\otimes} V_2)^*.$$

PROOF. Suppose $\varphi \in (V_1 \overset{h}{\otimes} V_2)^*$ $(= V_1^* \overset{eh}{\otimes} V_2^*)$. Then by Theorem 2.4.2, there exist a Hilbert space H and completely bounded maps $\psi_1 : V_2 \to B(H, \mathbb{C}) = H_r$ and $\psi_2 : V_1 \to B(\mathbb{C}, H) = H_c$ such that $\varphi(v_1 \otimes v_2) = \psi_1(v_1)\psi_2(v_2)$ and $\|\varphi\|_{cb} = \|\psi_1\|_{cb}\|\psi_2\|_{cb}$. Then composing the map

$$\psi_1 \overset{eh}{\otimes} \psi_2 : V_1 \overset{eh}{\otimes} V_2 \to B(H, \mathbb{C}) \overset{eh}{\otimes} B(\mathbb{C}, H)$$

and the multiplication map

$$m : B(\mathbb{C}, H) \overset{eh}{\otimes} B(H, \mathbb{C}) \to \mathbb{C}$$

given by $a \otimes b \mapsto ab$ gives a completely bounded map

$$\widetilde{\varphi} = m \circ (\psi_1 \overset{eh}{\otimes} \psi_2) : V_1 \overset{eh}{\otimes} V_2 \to \mathbb{C},$$

since m is contractive. So, we get an extension $\widetilde{\varphi}$ of φ and $\|\widetilde{\varphi}\|_{cb} \geq \|\varphi\|_{cb}$. On the other hand,

$$\begin{aligned}\|\widetilde{\varphi}\|_{cb} &\leq \|m\|_{cb}\|\psi_1 \overset{eh}{\otimes} \psi_2\| \leq \|m\|_{cb}\|\psi_1\|_{cb}\|\psi_2\|_{cb} \\ &\leq \|\psi_1\|_{cb}\|\psi_2\|_{cb} = \|\varphi\|_{cb},\end{aligned}$$

where the second step follows from Theorem 3.2.8. Therefore, $\|\widetilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$, i.e., $(V_1 \overset{h}{\otimes} V_2)^* \to (V_1 \overset{eh}{\otimes} V_2)^*$, $\varphi \mapsto \widetilde{\varphi}$, is isometric.

Let $n \in \mathbb{N}$. Then $M_n((V_1 \overset{h}{\otimes} V_2)^*) = CB(V_1 \overset{h}{\otimes} V_2, M_n)$ and $M_n = B(\mathbb{C}^n)$. By the decomposition theorem for operators in $CB(V_1 \overset{h}{\otimes} V_2, M_n)$ and the same argument as above, we see that the map $[\varphi_{ij}] \mapsto [\widetilde{\varphi}_{ij}]$ is isometric. Therefore, the embedding $(V_1 \overset{h}{\otimes} V_2)^* \to (V_1 \overset{eh}{\otimes} V_2)^*$, $\varphi \mapsto \widetilde{\varphi}$, is completely isometric. $\square$

3.3. Normal Haagerup tensor product

Given dual operator spaces V_1^* and V_2^*, the **normal Haagerup tensor product** of V_1^* and V_2^* is defined as the operator space dual of $(V_1^* \overset{h}{\otimes} V_2^*)^*_\sigma$ and denoted by $V_1^* \overset{\sigma h}{\otimes} V_2^*$. That is to say

$$V_1^* \overset{\sigma h}{\otimes} V_2^* = ((V_1^* \overset{h}{\otimes} V_2^*)^*_\sigma)^*.$$

According to the definition of extended Haagerup tensor product, we have the complete isometry

$$V_1^* \overset{\sigma h}{\otimes} V_2^* = (V_1 \overset{eh}{\otimes} V_2)^*.$$

From this identification and self-duality of the extended Haagerup tensor product, we conclude immediately that

$$V_1^* \overset{eh}{\otimes} V_2^* \hookrightarrow V_1^* \overset{\sigma h}{\otimes} V_2^*$$

is a complete isometry, and hence

$$V_1^* \overset{h}{\otimes} V_2^* \hookrightarrow V_1^* \overset{\sigma h}{\otimes} V_2^*$$

is also a complete isometric embedding. In fact, we can say more about this embedding.

PROPOSITION 3.3.1. *Let V_1 and V_2 be operator spaces. Then $V_1^* \overset{h}{\otimes} V_2^*$ is dense in $V_1^* \overset{\sigma h}{\otimes} V_2^*$ in the weak*-topology determined by $V_1 \overset{eh}{\otimes} V_2$ and hence $V_1^* \otimes V_2^*$ is weak*-dense in $V_1^* \overset{\sigma h}{\otimes} V_2^*$.*

PROOF. Let us consider the inclusion map $i : V_1 \overset{eh}{\otimes} V_2 \to (V_1^* \overset{h}{\otimes} V_2^*)^*$. Then $ker(i) = \{0\}$ and hence $i^*((V_1^* \overset{h}{\otimes} V_2^*)^{**})$ is weak*-dense in $(V_1 \overset{eh}{\otimes} V_2)^* = V_1^* \overset{\sigma h}{\otimes} V_2^*$. But $V_1^* \overset{h}{\otimes} V_2^*$ is weak*-dense in $(V_1^* \overset{h}{\otimes} V_2^*)^{**}$. Therefore, $V_1^* \overset{h}{\otimes} V_2^*$ is weak*-dense in $V_1^* \overset{\sigma h}{\otimes} V_2^*$. $\square$

By the definition of the extended Haagerup tensor product and the normal Haagerup tensor product, we have $V_1^* \overset{\sigma h}{\otimes} V_2^* = (V_1 \overset{eh}{\otimes} V_2)^*$ and hence

$$CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, \mathbb{C}) \cong V_1 \overset{eh}{\otimes} V_2 \cong MB^\sigma(V_1^* \times V_2^*, \mathbb{C}).$$

We show in the following that the above $\mathbb{C}$ can be replaced by any dual operator space.

THEOREM 3.3.2. *Let V_1, V_2, and W be operator spaces. Then we have a complete isometry*

$$CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, W^*) \cong MB^\sigma(V_1^* \times V_2^*, W^*).$$

PROOF. Let $\varphi \in MB^\sigma(V_1^* \times V_2^*, W^*)$ and $w \in W \subseteq W^{**}$. Then $w \circ \varphi : V_1^* \times V_2^* \to \mathbb{C}$ is normal and multiplicatively bounded since w is weak*-continuous and φ is multiplicatively bounded, and hence it is an element of $V_1 \overset{eh}{\otimes} V_2$. Then we may define a complete bounded map

$$\varphi_* : W \to V_1 \overset{eh}{\otimes} V_2, \quad w \mapsto w \circ \varphi,$$

since $\|\varphi_*\|_{cb} \leq \|\varphi\|_{mb}$.

Let $\varphi_{\sigma h} = (\varphi_*)^*$. Then $\varphi_{\sigma h} : V_1^* \overset{\sigma h}{\otimes} V_2^* \to W^*$ is weak*-weak* continuous and completely bounded with $\|\varphi_{\sigma h}\|_{cb} = \|\varphi_*\|_{cb} \leq \|\varphi\|_{mb}$.

For all $f_1 \in V_1^*, f_2^* \in V_2^*$ and $w \in W$,

$$\begin{aligned} \varphi_{\sigma h}(f_1 \otimes f_2)(w) &= (f_1 \otimes f_2)(\varphi_*(w)) = (f_1 \otimes f_2)(w \circ \varphi) \\ &= (w \circ \varphi)(f_1, f_2) = \varphi(f_1, f_2)(w). \end{aligned}$$

So, $\varphi_{\sigma h}$ is the unique weak*-extension of φ on $V_1^* \overset{\sigma h}{\otimes} V_2^*$ satisfying $\varphi_{\sigma h}(f_1 \otimes f_2) = \varphi(f_1, f_2)$. In particular, we have $\|\varphi\|_{mb} \leq \|\varphi_{\sigma h}\|_{cb}$. So, $\|\varphi_{\sigma h}\|_{cb} = \|\varphi\|_{mb}$. Therefore, $MB^\sigma(V_1^* \times V_2^*, W^*) \to CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, W^*)$, $\varphi \mapsto \varphi_{\sigma h}$, is an isometry.

For the surjectivity of this map, we can restrict $\varphi_{\sigma h}$ to $V_1^* \overset{h}{\otimes} V_2^*$ and denote the restricted map by $\varphi_{\sigma h}|_{\overset{h}{\otimes}}$. Then $\varphi_{\sigma h}|_{\overset{h}{\otimes}} \in CB^\sigma(V_1^* \overset{h}{\otimes} V_2^*, W^*) \cong MB^\sigma(V_1^* \times V_2^*, W^*)$ by the property of Haagerup tensor product.

For each $n \in \mathbb{N}$, we have the commutative diagram

$$\begin{array}{ccc} M_n(CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, W^*)) & \longrightarrow & M_n(MB^\sigma(V_1^* \times V_2^*, W^*)) \\ \downarrow & & \downarrow \\ CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, T_n(W)^*) & \longrightarrow & MB^\sigma(V_1^* \times V_2^*, T_n(W)^*), \end{array}$$

where the bottom and the two vertical maps are isometric and hence so is the top map. Therefore, $CB^\sigma(V_1^* \overset{\sigma h}{\otimes} V_2^*, W^*) \cong MB^\sigma(V_1^* \times V_2^*, W^*)$ as operator spaces. □

If V_1^* and V_2^* are replaced by V_1^{**} and V_2^{**}, respectively, then $V_1^{**} \overset{\sigma h}{\otimes} V_2^{**} = \left(V_1 \overset{h}{\otimes} V_2\right)^{**}$. So, $CB^\sigma(V_1^{**} \overset{\sigma h}{\otimes} V_2^{**}, W^*) = CB^\sigma\left(\left(V_1 \overset{h}{\otimes} V_2\right)^{**}, W^*\right) = CB(V_1 \overset{h}{\otimes} V_2, W^*)$ and $MB^\sigma(V_1^{**} \times V_2^{**}, W^*) = MB(V_1 \times V_2, W^*)$. Then the above identification is exactly Proposition 2.2.2.

CHAPTER 4

The Second Dual of a Banach Algebra

In this chapter, we start with a few facts and characterizations concerning Arens regularity and topological centers of Banach algebras. In Section 4.4, we prove some new results on Arens products. The second dual of a completely contractive Banach algebra is briefly discussed in Section 4.5.

4.1. Preliminaries

Let X, Y, and Z be normed spaces over $\mathbb{C}$ and m a bounded bilinear map from $X \times Y$ into Z. We define two adjoint maps of m, namely $m^* : Z^* \times X \to Y^*$ and $m_* : Y \times Z^* \to X^*$ as follows.

For $f \in Z^*, x \in X$, and $y \in Y$, let

$$m^*(f,x)(y) = f(m(x,y)).$$

$$m_*(y,f)(x) = f(m(y,x)).$$

In particular, if $X = Y = Z$ and $m : X \times X \to X$, then we have

$$m^* : X^* \times X \to X^*,$$

$$m^{**} : X^{**} \times X^* \to X^*,$$

$$m^{***} : X^{**} \times X^{**} \to X^{**}.$$

It can be seen that in general m^{***} is a natural extension of m. Obviously, we have the counterparts of m_* and another natural extension of m, namely m_{***}.

DEFINITION 4.1.1. *Let A be a Banach algebra and let $m : A \times A \to A$ be the multiplication on A. Then the first Arens product on A^{**} is m^{***}, denoted by $*_1$. The second Arens product on A^{**} is m_{***}, denoted by $*_2$.*

More precisely, if $a \in A, f \in A^*, F, G \in A^{**}$, then $f *_1 a$, $G *_1 f \in A^*$ and $F *_1 G \in A^{**}$ are defined as follows.

$$\begin{aligned} f *_1 a(b) &= f(ab) \quad (b \in A). \\ G *_1 f(a) &= G(f *_1 a). \\ F *_1 G(f) &= F(G *_1 f). \end{aligned}$$

Similarly, $a *_2 f$, $f *_2 F \in A^*$ and $F *_2 G \in A^{**}$ are defined as follows.

$$\begin{aligned} a *_2 f(b) &= f(ba) \quad (b \in A). \\ f *_2 F(a) &= F(a *_2 f). \\ F *_2 G(f) &= G(f *_2 F). \end{aligned}$$

4.2. Characterizations of Arens regularity

The main references for this section are Arens [1] and Duncan-Hosseiniun [7]. By the definition of the Arens products, the following lemma is immediate.

LEMMA 4.2.1. *The first (resp. second) Arens product is weak*-weak* continuous on the left (resp. right). That is,*

(a) *if $F_\alpha \to F$ in the weak*- topology, then $F_\alpha *_1 G \to F *_1 G$ in the weak*-topology;*

(b) *if $G_\beta \to G$ in the weak*- topology, then $F *_2 G_\beta \to F *_2 G$ in the weak*-topology.*

LEMMA 4.2.2. *The two Arens products agree if one of the factors is in A. That is, if $G \in A$ and $F \in A^{**}$, then $F *_1 G = F *_2 G$ and $G *_1 F = G *_2 F$.*

PROOF. Let $\pi : A \to A^{**}$ be the canonical embedding of A into A^{**}. Then we can get the following equalities.

$$f *_1 a = f *_2 \pi(a). \tag{1}$$

$$(f *_2 F) *_1 a = (f *_2 F) *_2 \pi(a). \tag{2}$$

$$F *_1 \pi(a) = F *_2 \pi(a). \tag{3}$$

And

$$a *_2 f = \pi(a) *_1 f. \tag{4}$$

$$a *_2 (F *_1 f) = \pi(a) *_1 (F *_1 f). \tag{5}$$

$$\pi(a) *_1 F = \pi(a) *_2 F. \tag{6}$$

Obviously, (1) and (4) hold. (2) and (5) follows from (1) and (4), respectively.

For (3), let $f \in A^*$. Then $[F *_1 (\pi(a))](f) = F(\pi(a) *_1 f) = F(a *_2 f)$ by (4). But $[F *_2 \pi(a)](f) = \pi(a)(f *_2 F) = (f *_2 F)(a) = F(a *_2 f)$. Therefore, (3) is true. Similarly, (6) is true. □

DEFINITION 4.2.3. *Let A be a Banach algebra. A is called Arens regular if the two Arens products agree on A^{**}.*

Let A be a commutative Banach algebra. Then for $F \in A^{**}, f \in A^*$ and $a \in A, f *_1 a = a *_2 f$ and thus $F *_1 f = f *_2 F$. So, $F *_1 G = G *_2 F$ for all $F, G \in A^{**}$. Immediately, we have the following

PROPOSITION 4.2.4. *Let A be a commutative Banach algebra. Then A^{**} is commutative under either Arens product if and only if A is Arens regular.*

THEOREM 4.2.5. *Let A be a Banach algebra. Then the following statements are equivalent.*

(1) *A is Arens regular.*
(2) *For each $F \in A^{**}$, the mapping $G \mapsto F *_1 G$ is weak*-continuous.*
(3) *For each $F \in A^{**}$, the mapping $G \mapsto G *_2 F$ is weak*-continuous.*
(4) *For each $f \in A^*$, the mapping $b \mapsto f *_1 b$ is weakly compact.*
(5) *For each $f \in A^*$, the mapping $b \mapsto b *_2 f$ is weakly compact.*
(6) *Given bounded sequences $\{a_n\}$, $\{b_m\}$ in A and $f \in A^*$, the iterated limits $\lim_n \lim_m f(a_n b_m)$ and $\lim_m \lim_n f(a_n b_m)$ are equal when they both exist.*

PROOF. (1) $\Rightarrow$ (2). Let A be Arens regular and $\{G_\beta\}$ be a net in A^{**} which is weak*-convergent to G. Then for $f \in A^*$,

$$F *_1 G(f) \quad = \quad F *_2 G(f) = G(f *_2 F) = \lim_\beta G_\beta(f *_2 F)$$

$$= \lim_\beta F *_2 G_\beta(f) = \lim_\beta F *_1 G_\beta(f).$$

(2) $\Rightarrow$ (1). Suppose (2) holds and let $F, G \in A^{**}$. Since A is weak*-dense in A^{**}, there exists a net $\{G_\beta\}_\beta$ in A weak*-convergent to G in the weak*-topology. So, by Lemma 4.2.2,

$$F *_1 G = \lim_\beta F *_1 G_\beta = \lim_\beta F *_2 G_\beta = F *_2 G.$$

It follows that A is Arens regular.

(1) $\Leftrightarrow$ (3). Similar to (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4). Let A be Arens regular and $f \in A^*$. Let $T_f : A \longrightarrow A^*$ be defined by $a \mapsto f *_1 a$ and $\pi : A^* \longrightarrow A^{***}$ the canonical embedding of A^* into A^{***}. Then

$$\begin{aligned} T_f^{**}(F)(G) &= F(T_f^*(G)) = F(G *_1 f) = F *_1 G(f) \\ &= F *_2 G(f) = G(f *_2 F) = \pi(f *_2 F)(G). \end{aligned}$$

So, $T_f^{**}(A^{**}) \subseteq \pi(A^*)$. By Dunford-Schwartz [8, Theorem VI. 4.2], T_f is weakly compact.

(4) $\Rightarrow$ (2). Suppose $T_f : A \longrightarrow A^*$, $b \mapsto f *_1 b$ is weakly compact for all $f \in A^*$. Then $T_f^* : A^{**} \longrightarrow A^*$, $F \mapsto F *_1 f$. Let $\pi : A^* \longrightarrow A^{***}$ be the canonical embedding. By [8, Theorem VI. 4.2], for each $F \in A^{**}$, there exists an $f \in A^*$ such that

$$F(T_f^*(G)) = (T_f^{**}(F))(G) = (\pi(f))(G) = G(f).$$

Now let $\{G_\beta\}$ be weak*- convergent to G. Then $T_f^*(G_\beta)$ is weakly convergent to $T_f^*(G)$. So, for any $F \in A^{**}, F(G_\beta *_1 (f)) = F(T_f^*(G_\beta))$ weak*-converges to $F(G *_1 f) = F *_1 G(f)$ in the weak*-topology. i.e., the mapping $G \mapsto F *_1 G$ is weak*-weak* continuous.

(4) $\Leftrightarrow$ (6) and (5) $\Leftrightarrow$ (6). It follows from the Grothendieck's criterion for weakly compactness (cf. [13, Theorem 3.1]). $\square$

COROLLARY 4.2.6. *Let A be an Arens regular Banach algebra, B a closed subalgebra of A and J a closed bi-ideal of A. Then B and A/J are Arens regular.*

Before proving this corollary, we give two useful lemmas.

LEMMA 4.2.7. *Let A_1 and A_2 be Banach algebras. Let T be a continuous homomorphism of A_1 into A_2. Then T^{**} is a homomorphism of A_1^{**} with the first (resp. second) Arens product into A_2^{**} with the first (resp. second) Arens product.*

The proof of this lemma can be found in [5].

LEMMA 4.2.8. *Let A and B be Banach algebras. If A is Arens regular and $h : A \to B$ is a continuous homomorphism of A onto B, then B is Arens regular.*

PROOF. First, we show h^{**} is onto. Since h is continuous linear mapping from A onto B, $B = h(A) = ker(h^*)^{\perp}$ (cf. [8, Lemma VI. 2.8]) and then $ker(h^*) = \{0\}$, i.e., h^* is one-one continuous linear mapping. By Dunford-Schwartz [8, Theorem VI. 6.2], we also know that if the range of h^* is closed, then the range of h^{**} is $ker(h^*)^{\perp} = B^{**}$. But the range of h^* is closed if and only if the range of h is closed. Therefore, h^{**} is onto.

Now we show that B is Arens regular. By Lemma 4.2.7,

$$h^{**}(F) *_1 h^{**}(G) = h^{**}(F *_1 G) = h^{**}(F *_2 G) = h^{**}(F) *_2 h^{**}(G)$$

for all F, G in B^{**}. But as we proved above, the range of h^{**} is exactly B^{**}, Thus B is Arens regular. □

Proof of Corollary 4.2.6. Let T be the inclusion map of B into A. Then it is a continuous homomorphism of B into A. By Lemma 4.2.7, for all $F, G \in B^{**}$,

$$T^{**}(F *_1 G) = T^{**}(F) *_1 T^{**}(G) = T^{**}(F) *_2 T^{**}(G) = T^{**}(F *_2 G).$$

Since T^{**} is one-one, $F *_1 G = F *_2 G$ for all $F, G \in B^{**}$. So, B is Arens regular.

Since the canonical mapping $q : A \to A/J$ is a continuous homomorphism of A onto A/J and A is Arens regular, A/J is Arens regular by Lemma 4.2.8. □

In the sequel, we always use $(A^{**}, *_1)$ (resp. $(A^{**}, *_2)$) to denote the second dual of A with the first (resp. second) Arens product.

PROPOSITION 4.2.9. *Let A be a Banach algebra. Then A has a bounded right (resp. left) approximate identity if and only if $(A^{**}, *_1)$ (resp. $(A^{**}, *_2)$) has a right (resp. left) identity.*

PROOF. Suppose A has a bounded right approximate identity $\{e_\lambda\}$ and $\|e_\lambda\| \leq M$ for all λ. Since the bounded ball in A^{**} is weak*-compact, there exists a subnet $\{e_{\lambda_\beta}\}$ of $\{e_\lambda\}$ weak*-convergenet to a point E in A^{**}. We show that E is the right identity of $(A^{**}, *_1)$, i.e., for all $f \in A^*$, $F *_1 E(f) = F(f)$. But $F *_1 E(f) = F(E *_1 f)$, it suffice to show $E *_1 f = f$.

In fact, for all $a \in A$, $E *_1 f(a) = E(f *_1 a) = \lim_{\lambda_\beta} e_{\lambda_\beta}(f *_1 a) = \lim_{\lambda_\beta}(f *_1 a)(e_{\lambda_\beta}) = \lim_{\lambda_\beta} f(ae_{\lambda_\beta}) = f(\lim_{\lambda_\beta} ae_{\lambda_\beta}) = f(a)$.

Conversely, suppose $(A^{**}, *_1)$ has a right identity E. Then since the unit ball of A is weak*-dense in the unit ball of A^{**}, there is a net $\{e_\lambda\}$ in A with $\|e_\lambda\| \leq \|E\|$ such that $\lim_\lambda e_\lambda = E$ in the weak*-topology of A^{**}.

Let $f \in A^*$. Since $F(f) = F *_1 E(f) = F(E *_1 f)$ holds for all $F \in A^{**}, f = E *_1 f$. For all $x \in A$ and $f \in A^*$, we have

$$\begin{aligned} f(x) &= E *_1 f(x) = E(f *_1 x) = \lim_\lambda e_\lambda(f *_1 x) \\ &= \lim_\lambda f(xe_\lambda) = f(\lim_\lambda xe_\lambda). \end{aligned}$$

Hence $x = \lim_\lambda xe_\lambda$ in the weak topology of A. Then there exists a net $\{a_\gamma\}$ such that each a_γ is a convex combination of e_λ and $\|xa_\gamma - x\| \to 0$ for all $x \in A$. Therefore, A has a right bounded approximate identity. □

4.3. Topological centers

Although Arens regular Banach algebras are nice, unfortunately, many important Banach algebras are not Arens regular. For example, the group algebra $L_1(G)$ is never Arens regular unless G is finite. A natural question is how to describe the non-Arens regularity of a Banach algebra. As we will discuss, the topological center with respect to each Arens product is one of such measurements. The main reference for this section is [15].

DEFINITION 4.3.1. *Let A be a Banach algebra. The topological centers of A^{**} are defined as follows.*

$Z_1(A^{**}) = \{F \in A^{**} : G \mapsto F *_1 G$ *is weak*-weak* continuous on* $A^{**}\}$.

$Z_2(A^{**}) = \{G \in A^{**} : F \mapsto F *_2 G$ *is weak*-weak* continuous on* $A^{**}\}$.

Clearly, $A \subseteq Z_1 \bigcap Z_2 \subseteq A^{**}$. Furthermore, $Z_1 = A^{**}$ or $Z_2 = A^{**}$ if and only if A is Arens regular.

Denote the algebraic center of A^{**} with respect to the first (resp. second) Arens product by $C_1(A^{**})$ (resp. $C_2(A^{**})$). Then

$$C_1(A^{**}) = \{F \in A^{**} : F *_1 G = G *_1 F \textit{ for all } G \in A^{**}\}$$

and

$$C_2(A^{**}) = \{F \in A^{**} : F *_2 G = G *_2 F \textit{ for all } G \in A^{**}\}.$$

PROPOSITION 4.3.2. *$C_1(A^{**}) \subseteq Z_1(A^{**})$ and $C_2(A^{**}) \subseteq Z_2(A^{**})$.*

PROOF. For any $F \in C_1(A^{**})$, the mapping $G \mapsto F *_1 G$ is just the mapping $G \mapsto G *_1 F$. But the latter is automatically weak*-weak* continuous on A^{**}, so $F \in Z_1(A^{**})$.

Similarly, we can get the second inclusion. □

COROLLARY 4.3.3. *If A is a commutative Banach algebra, then $C_1 = C_2 = Z_1 = Z_2$.*

PROOF. We show $Z_1 = C_1$. By Proposition 4.3.2, it suffice to show $Z_1 \subseteq C_1$. Let $Z_1' = \{F \in A^{**} : F *_1 G = F *_2 G \text{ for all } G \in A^{**}\}$.

Claim. $Z_1 = Z_1'$. Clearly, $Z_1' \subseteq Z_1$. Conversely, for each $G \in A^{**}$, there exists a net $\{G_\alpha\}$ in A such that $G_\alpha \to G$ in the weak*-topology. For any $F \in Z_1$, $F *_1 G_\alpha \to F *_1 G$ in the weak*-topology. On the other hand, $F *_1 G_\alpha = F *_2 G_\alpha \to F *_2 G$ in the weak*-topology. So, for all $F \in Z_1$ and $G \in A^{**}$, $F *_1 G = F *_2 G$.

By the argument immediate preceding to Proposition 4.2.4, $Z_1' \subseteq C_1$, therefore, $Z_1 \subseteq C_1$. The proof that $Z_2 = C_2$ is similar. □

PROPOSITION 4.3.4. *Z_1 and Z_2 are subalgebras of A^{**}.*

PROOF. It follows from the associativity of the Arens products. □

Let A be a Banach algebra. We define

$$A^*A = \{f *_1 a : f \in A^*, a \in A\}$$

and

$$AA^* = \{a *_2 f : f \in A^*, a \in A\}.$$

If $A^* = A^*A$ (resp. $A^* = AA^*$), we say A^* factors on the left (resp. right).

DEFINITION 4.3.5. *An element E of A^{**} is said to be a mixed identity if for all $F \in A^{**}$, $F *_1 E = E *_2 F = F$.*

By Proposition 4.2.9, we have the following result immediately.

PROPOSITION 4.3.6. *Let A be a Banach algebra. Then A^{**} has a mixed identity E if and only if A has a BAI $\{e_\alpha\}$ such that $e_\alpha \to E$ in the weak* topology.*

PROOF. It follows from Proposition 4.2.9 and its proof. □

For convenience, we use $\langle f, a\rangle$ or $\langle a, f\rangle$ to denote the duality between A^* and A.

PROPOSITION 4.3.7. *Let A be a Banach algebra. If $(A^{**}, *_1)$ (resp. $(A^{**}, *_2)$) has an identity E, then E is a mixed identity of A^{**}.*

PROOF. Let E be the identity of $(A^{**}, *_1)$. We need to show that for all $F \in A^{**}$, $E *_2 F = F$. For $F \in A^{**}$, there exists a bounded net $\{F_\alpha\}$ in A such that $F_\alpha \to F$ in the weak*- topology. Then $E *_2 F_\alpha \to E *_2 F$ in the weak*-topology. But $E *_2 F_\alpha = E *_1 F_\alpha = F_\alpha \to F$ in the weak*-topology. So, $E *_2 F = F$ and E is a mixed identity of A^*.

The $(A^{**}, *_2)$ case can be similarly proved. □

PROPOSITION 4.3.8. *Let A be a Banach algebra with a BAI. Then the following statements are true.*

(a) A^* factors on the left if and only if $(A^{**}, *_1)$ is unital.

(b) A^* factors on the right if and only if $(A^{**}, *_2)$ is unital.

(c) If A^* factors on both sides, then the identities of $(A^{**}, *_1)$ and $(A^{**}, *_2)$ are the same.

PROOF. (a). Suppose A^* factors on the left. Since A has a BAI, A^{**} has a mixed identity E. Then for all $F \in A^{**}$, $F *_1 E = F$. We want to show $E *_1 F = F$. i.e., for each $f \in A^*, \langle f, E *_1 F\rangle = \langle f, F\rangle$.

Since each $f \in A^*$ has the form $g *_1 a$ for some $g \in A^*$ and $a \in A$, we have

$$\begin{aligned}\langle E *_1 F, f\rangle &= \langle E *_1 F, g *_1 a\rangle = \langle a *_2 (E *_1 F), g\rangle = \langle a *_1 E *_1 F, g\rangle \\ &= \langle a *_1 F, g\rangle = \langle a *_2 F, g\rangle = \langle F, g *_1 a\rangle = \langle F, f\rangle.\end{aligned}$$

Therefore, $E *_1 F = f$ for all $F \in A^{**}$.

Conversely, suppose E is the identity of $(A^{**}, *_1)$. We show $A^* \subseteq A^*A$.

Since $E \in A^{**}$, there exists a net $\{e_\alpha\}$ in A such that $e_\alpha \to E$ in the weak*-topology. Then for each $F \in A^{**}, e_\alpha *_1 F \to E *_1 F = F$ in the weak*-topology of A^{**}. So, for each $f \in A^*$,

$$\langle F, f *_1 e_\alpha \rangle = \langle F, f *_2 e_\alpha \rangle = \langle e_\alpha *_2 F, f \rangle = \langle e_\alpha *_1 F, f \rangle \to \langle F, f \rangle,$$

i.e., $f *_1 e_\alpha \to f$ in the weak topology of A^*. By the Cohen's factorization Theorem (cf. [14, Theorem 32.22]), A^*A is norm (and hence weakly) closed in A^*. So, f is in A^*A.

(b). The proof is similar to the proof of (a).

(c). Let E_1, E_2 be the identities of $(A^{**}, *_1)$ and $(A^{**}, *_2)$, respectively. Then we show E_1 is the identity of $(A^{**}, *_2)$. By Proposition 4.3.7, it suffices to show that for each $F \in A^{**}$, $F *_2 E_1 = F$. This is true since $F *_2 E_1 = F *_2 (E_1 *_2 E_2) = F *_2 E_2 = F$. For the second and the fourth steps we use the assumption that E_2 is a unit of $(A^{**}, *_2)$, and for the third step we use the fact that E_1 is a mixed unit of A^{**}. □

DEFINITION 4.3.9. *Let X be a normed space. A sequence $\{x_n\}$ in X is said to be weakly Cauchy if $\{f(x_n)\}$ is Cauchy in $\mathbb{C}$ for all $f \in X^*$.*

X said to be weakly sequentially complete if every weakly Cauchy sequence in X is weakly convergent.

PROPOSITION 4.3.10. *Let A be weakly sequentially complete Banach algebra with a sequential BAI. Then the following statements are equivalent.*

(a) A^* factors on the left.

(b) A^* factors on the right.

(c) A is unital.

PROOF. $(a) \Rightarrow (c)$. Assume (a) holds. Let $\{e_n\}$ be a sequential BAI. Then any $f \in A^*$ is of the form $f = g *_1 a$ for some $g \in A^*$ and $a \in A$. Since $ae_n \to a$ in the norm topology of A, $\langle f, e_n \rangle = \langle g, ae_n \rangle \to \langle g, a \rangle$ in $\mathbb{C}$. This show that the sequence $\{e_n\}$ is weakly Cauchy. Since A is weakly sequentially complete, $\{e_n\}$ is weakly convergent to some element e of A. It is immediate to see that e is the identity of A.

$(c) \Rightarrow (b)$. It is trivial.

$(b) \Rightarrow (c)$. The proof is similar to the proof of $(a) \Rightarrow (c)$.

$(c) \Rightarrow (a)$. It follows from Proposition 4.3.8, since the identity of A is also the identity of $(A^*, *_1)$. □

COROLLARY 4.3.11. *A weakly sequentially complete Banach algebra A with a sequential BAI can not be Arens regular unless it is unital.*

4.4. Some new characterizations of Arens regularity

In this section, we prove some new characterizations for a Banach algebra A to be Arens regular, which were obtained when we attempted to unify some of existing approches to the study of Arens regularity.

Given a Banach algebra A, let

$$Z = \{G \in A^{**} : \text{for each } f \in A^*, G *_1 F_\alpha(f) \to G *_1 F(f) \text{ whenever } F_\alpha \xrightarrow{w*} F\}$$

and

$$S = \{f \in A^* : \text{for each } G \in A^{**}, G *_1 F_\alpha(f) \to G *_1 F(f) \text{ whenever } F_\alpha \xrightarrow{w*} F\}.$$

Then $Z = Z_1(A^{**})$ and it is easy to see that $S = wap(A)$, where

$$wap(A) = \{f \in A^* : a \mapsto f *_1 a, A \mapsto A^*, \text{ is weakly compact}\}.$$

By Dunford-Schwartz [8, Theorem VI.4.7], for each $f \in A^*$, T_f is weakly compact if and only if $T_f^* : A^{**} \to A^*$ is weak*-weakly continuous, where $T_f(a) = f *_1 a$. So, $f \in wap(A)$ if and only if $f \in S$.

Let $W = \{(G, f) \in A^{**} \times A^* : G *_1 F_\alpha(f) \to G *_1 F(f) \text{ whenever } F_\alpha \xrightarrow{w*} F\}$, $\widetilde{Z} = Z \times A^*$, and $\widetilde{S} = A^{**} \times S$. Then $\widetilde{Z} \subseteq W$, $\widetilde{S} \subseteq W$, $P_1(\widetilde{Z}) = Z$, and $P_2(\widetilde{S}) = S$, where P_1, P_2 are the natural projections. Clearly,

$$A \text{ is Arens regular} \iff W = A^{**} \times A^*.$$

For $(G, f) \in A^{**} \times A^*$, we define a map $\varphi_{G,f} : A^{**} \to \mathbb{C}$ by

$$\varphi_{G,f}(F) = \langle G *_1 F, f \rangle.$$

Obviously, $\varphi_{G,f}$ is linear and bounded. So, $\varphi_{G,f} \in A^{***}$. In particular, $(G, f) \in W$ if and only if $\varphi_{G,f}$ is weak*-continuous, i.e., $\varphi_{G,f} \in A^*$.

Then we get a bilinear map $\varphi : A^{**} \times A^* \to A^{***}$ given by $\varphi(G, f) = \varphi_{G,f}$. We note that $\varphi(a, f) = f *_1 a$ and φ is weak*-weak* continuous with respect to the first variable. Also, we have $W = \varphi^{-1}(A^*)$. Therefore,

$$A \text{ is Arens regular} \iff \varphi^{-1}(A^*) = A^{**} \times A^*.$$

LEMMA 4.4.1. *Let A be a Banach algebra and let φ be defined as above. Then $A^*A \subseteq \varphi(\widetilde{Z}) \subseteq \varphi(W) \subseteq A^*$.*

PROOF. The last two inclusions are clear by the arguments above. Recall that $A^*A = \{f *_1 a : f \in A^*, a \in A\}$. So, to get the first inclusion, let $f \in A^*$ and $a \in A$. Then $a \in Z$ and thus $(a, f) \in \widetilde{Z}$. Therefore, $f *_1 a = \varphi(a, f) \in \varphi(\widetilde{Z})$. □

We consider now the relation between A^*A and $\varphi(\widetilde{S})$.

PROPOSITION 4.4.2. *Let A be a Banach algebra. Then*

(1) $\varphi(\widetilde{S}) \subseteq S$.

(2) $\varphi(\widetilde{S}) = S \subseteq A^*A$ if A has a BAI.

(3) $A^*A \subseteq S$ if A is a right ideal in A^{**} (i.e., $AA^{**} \subseteq A$). In particular, if A has a BAI and A is a right ideal in A^{**}, then $\varphi(\widetilde{S}) = S = A^*A$.

PROOF. (1) Let $(G, f) \in \widetilde{S}$. For each $E \in A^{**}$, if $F_\alpha \xrightarrow{w*} F$, then $\langle E *_1 F_\alpha, \varphi(G, f)\rangle = \varphi_{G,f}(E *_1 F_\alpha) = \langle G *_1 (E *_1 F_\alpha), f\rangle = \langle (G *_1 E) *_1 F_\alpha), f\rangle \to \langle (G *_1 E) *_1 F), f\rangle$, since $f \in S$. Therefore, $\varphi(G, f) \in S$.

(2) Suppose A has a BAI (e_α). We first prove that $S \subseteq A^*A$. Let $f \in S$. Then $f *_1 e_\alpha$ is relatively weakly compact in A^*. Without loss of generality, we may assume that $f *_1 e_\alpha \to g$ weakly in A^*. Since (e_α) is a **right** BAI, by the Cohen's Factorization Theorem (cf. [14, Theorem 32.22]), A^*A is a norm (and hence weakly) closed linear subspace of A^*. In particular, we have $g \in A^*A$.

On the other hand, since (e_α) is a **left** BAI of A, for all $a \in A$,

$$\langle f *_1 e_\alpha, a\rangle = \langle f, e_\alpha a\rangle \to \langle f, a\rangle,$$

i.e., $f *_1 e_\alpha \to f$ in the weak*-topology of A^*. It follows that $f = g \in A^*A$. Therefore, $S \subseteq A^*A$.

To get the equality $\varphi(\widetilde{S}) = S$, we only need to prove that $S \subseteq \varphi(\widetilde{S})$. Let $f \in S$ and E be a weak*-cluster point of (e_α) in A^*. From the above arguments, we may assume that $e_\alpha \xrightarrow{w^*} E$ in A^{**} and $f *_1 e_\alpha \xrightarrow{w} f$ in A^*. Then, for all $G \in A^{**}$, we have

$$\begin{aligned}\langle \varphi_{E,f}, G\rangle &= \langle E *_1 G, f\rangle = \lim_\alpha \langle e_\alpha, G *_1 f\rangle \\ &= \lim_\alpha \langle G, f *_1 e_\alpha\rangle = \langle G, f\rangle.\end{aligned}$$

Therefore, $f = \varphi_{E,f} = \varphi(E,f) \in \varphi(\widetilde{S})$.

(3) Let $f \in A^*$ and $a \in A$. For any $G \in A^{**}$, if $F_\alpha \xrightarrow{w^*} F$ in A^{**}, we have $\langle G *_1 F_\alpha, f *_1 a\rangle = \langle a *_1 G *_1 F_\alpha, f\rangle \to \langle a *_1 G *_1 F_\alpha, f\rangle$ since $a *_1 G \in A$. □

REMARK 4.4.3. *A proof to the inclusion $(S =)\ wap(A) \subseteq A^*A$ can be found in the proof of [16, Theorem 3.1], which contains an oversight on (e_α): (e_α) was only assumed to be a left BAI of A there.*

COROLLARY 4.4.4. *Let A be a Banach algebra with a BAI. If A is Arens regular, then A^* factors on both sides.*

PROOF. Under the Hypothesis that A is Arens regular, $S = A^*$ is obviously. From Proposition 4.4.2(2), $S \subseteq A^*A \subseteq A^*$. So, A^* factors on the left. The right case can be proved similarly, since one can also prove that $S \subseteq AA^*$ when A has a BAI. □

PROPOSITION 4.4.5. *Let A be a Banach algebra. Then $A^*A \subseteq \varphi(\widetilde{S})$ if A is Arens regular. In particular, if A has a BAI, then A is Arens regular if and only if $A^*A = \varphi(\widetilde{S})$ and $(A^{**}, *_1)$ is unital.*

PROOF. Assume A is Arens regular. Then $S = wap(A) = A^*$. In this case, for all $f \in A^*$ and $a \in A$, $(a,f) \in \widetilde{S}$ and $f *_1 a = \varphi(a,f) \in \varphi(\widetilde{S})$. Therefore, $A^*A \subseteq \varphi(\widetilde{S})$.

Now suppose that A has a BAI. Then $(A^{**}, *_1)$ has a right identity E. Assume A is Arens regular. Then, by Proposition 4.4.2(2), $\varphi(\widetilde{S}) = S \subseteq A^*A \subseteq \varphi(\widetilde{S})$, i.e., $A^*A = \varphi(\widetilde{S})$. In this case, E is also a left identity of $(A^{**}, *_1)$. So, $(A^{**}, *_1)$ is unital.

Conversely, assume $A^*A = \varphi(\widetilde{S})$ and $(A^{**}, *_1)$ is unital. Then $A^* = A^*A$ (see Lau-Ülger [15, Proposition 2.2(a)]). Therefore, by Proposition 4.2.2(1), $A^* = A^*A = \varphi(\widetilde{S}) \subseteq S$, i.e., A is Arens regular. □

4.5. The second dual of a completely contractive Banach algebra

We start this section with the following two definitions, which are adopted from [11]. In Proposition 4.5.3, we consider the completely bounded norm of the adjoints of a bilinear map. We present a characterization of a completely contractive Banach algebra in Proposition 4.5.5.

DEFINITION 4.5.1. *Let A be an associative algebra over $\mathbb{C}$. We call A a completely contractive Banach algebra if A is a complete operator space and the multiplication is a completely contractive bilinear mapping, i.e., for all $m, n \in \mathbb{N}$ and for all $a = [a_{ij}] \in M_m(A)$ and $b = [b_{kl}] \in M_n(A)$,*

$$\|[a_{ij}b_{kl}]\| \leq \|a\|\|b\|.$$

DEFINITION 4.5.2. *Let A be a completely contractive Banach algebra and V an A-bimodule. Then V is called an operator A-bimodule if V is a complete operator space and the left and right A-module operations*

$$\rho_l : A \times V \to V, \quad (a, v) \mapsto av$$

and

$$\rho_r : V \times A \to V, \quad (v, a) \mapsto va$$

are completely bounded.

PROPOSITION 4.5.3. *Let X, Y and Z be operator spaces and $m : X \times Y \to Z$ a bilinear map. Then $\|m^*\|_{cb} \leq \|m\|_{cb}$ and $\|m_*\|_{cb} \leq \|m\|_{cb}$.*

PROOF. We only prove the inequality $\|m^*\|_{cb} \leq \|m\|_{cb}$. Recall that $m^* : Z^* \times X \to Y^*$ is defined by $\langle m^*(f, x), y\rangle = \langle f, m(x, y)\rangle$.

Let $n \in \mathbb{N}, f = [f_{ij}] \in M_n(Z^*)$ and $x = [x_{kl}] \in M_n(X)$. Then $(m^*)_n : M_n(Z^*) \times M_n(X) \to M_{n^2}(Y^*)$ sends (f, x) to $[m^*(f_{ij}, x_{kl})] \in M_{n^2}(Y^*) \cong CB(Y, M_{n^2})$. So,

$$\|(m^*)_n(f, x)\| = \|[m^*(f_{ij}, x_{kl})]\|_{cb} = \|[m^*(f_{ij}, x_{kl})]^{(n^2)}\|,$$

where $[m^*(f_{ij}, x_{kl})]^{(n^2)}$ is the n^2-th amplification of $[m^*(f_{ij}, x_{kl})]$ which is treated as a map from Y to M_{n^2}.

We note that $f \in M_n(Z^*) \cong CB(Z, M_n)$ and hence $\|f\|_{cb} = \|f^{(n)}\| = \|f^{(n^3)}\|$. Also, we have $m_{n,n^2} : M_n(X) \times M_{n^2}(Y) \to M_{n^3}(Z)$. Now, for $y = [y_{st}] \in M_{n^2}(Y)$, we have

$$\begin{aligned}
\|[m^*(f_{ij},x_{kl})]^{(n^2)}(y)\|_{M_{n^4}} &= \|\,[\langle m^*(f_{ij},x_{kl}),y_{st}\rangle]\,\|_{M_{n^4}} \\
&= \|\,[\langle f_{ij}, m(x_{kl},y_{st})\rangle]\,\|_{M_{n^4}} \\
&= \|f^{(n^3)}\left([m(x_{kl},y_{st})]\right)\|_{M_{n^4}} \\
&\le \|f^{(n^3)}\|\|\,[m(x_{kl},y_{st})]\,\|_{M_{n^3}(Z)} \\
&= \|f\|\|m_{n,n^2}(x,y)\| \\
&\le \|f\|_{cb}\|m_{n,n^2}\|\|x\|\|y\| \\
&\le \|m\|_{cb}\|x\|\|f\|\|y\|,
\end{aligned}$$

i.e.,

$$\|[m^*(f_{ij},x_{kl})]^{(n^2)}\| \le \|m\|_{cb}\|x\|\|f\|.$$

Therefore,

$$\|(m^*)_n(f,x)\| \le \|m\|_{cb}\|x\|\|f\|$$

for all $n \in \mathbb{N}, f \in M_n(Z^*)$ and $x \in M_n(X)$. It follows that $\|m^*\|_{cb} = \sup_{n\in\mathbb{N}} \|(m^*)_n\| \le \|m\|_{cb}$. □

COROLLARY 4.5.4. *Let A be a completely contractive Banach algebra and V an operator A-bimodule. Then V^* is an operator A-bimodule under the natural A-module operations.*

PROOF. In Proposition 4.5.3, we let $X = A$, $Y = Z = V$, and $m = \rho_l : A \times V \to V$ the left A-module action. Then it is seen immediately that $m^* : V^* \times A \to V^*$ is the right A-module action which is completely bounded. Similarly, the left A-module action on V^* is also completely bounded. Therefore, V^* is an operator A-bimodule. □

We note that if A is a completely contractive Banach algebra, i.e., A is a complete operator space and the multiplication

$$m : A \times A \to A, \quad (a,b) \mapsto ab$$

is completely contractive, then A itself is an operator A-bimodule. By Proposition 4.5.3, $m^*, m_*, m^{**}, m_{**}, m^{***}$ and m_{***} are all completely bounded with cb-norm bounded by $\|m\|_{cb}$.

Note that m^{***} and m_{***} are the first and the second Arens products, respectively, and $m^{***}|_{A\times A} = m_{***}|_{A\times A} = m$. So, combining with $\|m^{***}\|_{cb} \leq \|m\|_{cb}$ and $\|m_{***}\|_{cb} \leq \|m\|_{cb}$, we have $\|m^{***}\|_{cb} = \|m_{***}\|_{cb} = \|m\|_{cb}$. Therefore, we have the following proposition.

PROPOSITION 4.5.5. *Let A be a Banach algebra together with an operator space structure. Then A is a completely contractive Banach algebra if and only if A^{**} is a completely contractive Banach algebra under either of Arens products.*

CHAPTER 5

Extended and normal projective tensor products

Inspired by the study of the second dual of a Banach algebra, in this chapter, we define and study the extended and normal projective tensor products, which are based on the projective tensor product and parallel to the extended and normal Haagerup tensor products.

5.1. Extended and normal projective tensor products

Given operator spaces V and W, let $V \overset{e\wedge}{\otimes} W$ denote the subspace of $(V^* \overset{\wedge}{\otimes} W^*)^* = CB(V^* \overset{\wedge}{\otimes} W^*, \mathbb{C}) \cong CB(V^* \times W^*, \mathbb{C})$ corresponding to $CB^\sigma(V^* \times W^*, \mathbb{C})$, which is called the **extended projective tensor product** of V and W. And we let the **normal projective tensor product** $V^* \overset{\sigma\wedge}{\otimes} W^*$ of dual operator spaces V^* and W^* be $\left(V \overset{e\wedge}{\otimes} W\right)^*$. That is

$$V \overset{e\wedge}{\otimes} W \cong CB^\sigma(V^* \times W^*, \mathbb{C})$$

and

$$V^* \overset{\sigma\wedge}{\otimes} W^* = \left(V \overset{e\wedge}{\otimes} W\right)^*.$$

PROPOSITION 5.1.1. *Let V and W be operator spaces. Under the operator space identifications $CB(V^* \overset{\wedge}{\otimes} W^*, \mathbb{C}) \cong CB(V^* \times W^*, \mathbb{C}) \cong CB(V^*, W^{**})$, we have*

$$V \overset{e\wedge}{\otimes} W \cong CB^\sigma(V^* \times W^*, \mathbb{C}) \cong CB^{\sigma-w}(V^*, W),$$

where $CB^{\sigma-w}(V^*, W)$ denotes the space of weak-weakly continuous completely bounded linear maps from V^* to W.*

PROOF. Let $\Phi : CB(V^* \times W^*, \mathbb{C}) \to CB(V^*, W^{**})$ be the natural complete isometry. Then Φ is given by $\langle \Phi(T)(f), g\rangle = T(f, g)$ ($T \in CB(V^* \times W^*, \mathbb{C}), f \in V^*$ and $g \in W^*$).

Let $T \in CB^{\sigma}(V^* \times W^*, \mathbb{C})$. We claim that $\Phi(T)(f) \in W$ for all $f \in V^*$. Indeed, whenever $g_\alpha \xrightarrow{w*} g$ in W^*, we have $\langle \Phi(T)(f), g_\alpha \rangle = T(f, g_\alpha) \to T(f,g) = \langle \Phi(T)(f), g \rangle$, i.e., $\Phi(T)(f) : W^* \to \mathbb{C}$ is normal. Therefore, $\Phi(T)(f) \in W$.

Next, we show that $\Phi(T) : V^* \to W$ is weak*-weakly continuous. Let $f_\alpha \xrightarrow{w*} f$ in V^*. For any $g \in W^*$, $\langle \Phi(T)(f_\alpha), g \rangle = T(f_\alpha, g) \to T(f,g)$. So, $\Phi(T) \in CB^{\sigma-w}(V^*, W)$. Therefore, $\Phi\left(CB^{\sigma}(V^* \times W^*), \mathbb{C})\right) \subseteq CB^{\sigma-w}(V^*, W)$.

Finally, we have $\Phi\left(CB^{\sigma}(V^* \times W^*), \mathbb{C})\right) = CB^{\sigma-w}(V^*, W)$. In fact, for any $\widetilde{T} \in CB^{\sigma-w}(V^*, W)$, if we define $T(f,g) = \langle \widetilde{T}(f), g \rangle$ ($f \in V^*$ and $g \in W^*$), then $\Phi(T) = \widetilde{T}$. Clearly, $T \in CB^{\sigma}(V^* \times W^*, \mathbb{C})$. □

PROPOSITION 5.1.2. *Let V, W, and X be operator spaces. If W is reflexive, then we have a natural completely isometric identification*

$$CB^{\sigma}(V^* \times W^*, X^*) \cong CB^{\sigma}(V^*, CB^{\sigma}(W^*, X^*)).$$

PROOF. Since W is reflexive, $CB^{\sigma}(W^*, X^*) = CB^{\sigma}(W^{***}, X^*) \cong CB(W^*, X^*) \cong \left(W^* \hat{\otimes} X\right)^*$. So, the space on the right hand side makes sense.

Let $\Phi : CB(V^* \times W^*, X^*) \to CB(V^*, CB(W^*, X^*))$ be the natural complete isometry given by $\langle \Phi(T)(f)(g), x \rangle = \langle T(f,g), x \rangle$ ($T \in CB(V^* \times W^*, X^*), f \in V^*$ and $g \in W^*, x \in X$).

Let $T \in CB^{\sigma}(V^* \times W^*, X^*)$. Apparently, for $f \in V^*$, $\Phi(T)(f) \in CB^{\sigma}(W^*, X^*)$. The surjectivity of $\Phi|_{CB^{\sigma}(V^* \times W^*, X^*)} : CB^{\sigma}(V^* \times W^*, X^*) \to CB^{\sigma}(V^*, CB^{\sigma}(W^*, X^*))$ can be proved in a similar way as used in Proposition 5.1.1. We only need to show $\Phi(T) : V^* \to CB(W^*, X^*) = \left(W^* \hat{\otimes} X\right)^*$ is weak*-weak* continuous on a bounded ball.

Let $f_\alpha \xrightarrow{w*} f$ in $\mathrm{Ball}(V^*)$. For any elementary tensor $g \otimes x$ in $W^* \otimes X$,

$$\langle \Phi(T)(f_\alpha), g \otimes x \rangle = \langle T(f_\alpha, g), x \rangle \to \langle T(f,g), x \rangle = \langle \Phi(T)(f), g \otimes x \rangle.$$

Therefore, we have $\Phi(T)(f_\alpha) \xrightarrow{w^*} \Phi(T)(f)$ in $\left(W^* \hat{\otimes} X\right)^*$. So, $\Phi(T) \in CB^{\sigma}(V^*, (W^* \hat{\otimes} X)^*) = CB^{\sigma}(V^*, CB^{\sigma}(W^*, X^*))$. □

So, under the assumption that W is reflexive, we have

$$\begin{aligned} CB^{\sigma}(V^{**} \times W^{**}, X^*) &\cong CB^{\sigma}(V^{**}, CB(W^{**}, X^*)) \cong CB(V, CB(W^{**}, X^*)) \\ &\cong CB(V \times W^{**}, X^*) = CB(V \times W, X^*). \end{aligned}$$

This is parallel to Proposition 3.2.2. From here, we may define some kind of tensor product to linearize the normal completely bounded bilinear maps.

When the above X is $\mathbb{C}$, the first space in the identification sequence is exactly $V^* \overset{e\wedge}{\otimes} W^*$ we defined before and the last space is $\left(V \overset{\wedge}{\otimes} W\right)^*$. So, if W is reflexive, then the extended projective tensor product is the dual of projective tensor product.

LEMMA 5.1.3. *The algebraic tensor product $V^* \otimes W^*$ is weak*-dense in $V^* \overset{\sigma\wedge}{\otimes} W^*$.*

PROOF. First we observe that for $v \in V$ and $w \in W$, $\widetilde{v \otimes w} \in CB^\sigma(V^* \times W^*, \mathbb{C})$, where $\widetilde{v \otimes w}(f,g) = f(v)g(w)$ ($f \in V^*$ and $g \in W^*$). Obviously, $\widetilde{v \otimes w} : V^* \times W^* \to \mathbb{C}$ is separately weak*-continuous.

For $f = [f_{ij}] \in M_n(V^*)$ and $g = [g_{kl}] \in M_n(W^*)$, we have

$$\begin{aligned} \|(\widetilde{v \otimes w})_n(f,g)\|_{M_{n^2}} &= \|\,[f_{ij}(v)g_{kl}(w)]\,\|_{M_{n^2}} = \|[f_{ij}(v)] \otimes [g_{kl}(w)]\|_{M_{n^2}} \\ &= \|[f_{ij}(v)]\|_{M_n}\|[g_{kl}(w)]\|_{M_n} = \|f(v)\|\,\|g(w)\| \\ &\leq \|f\|\|g\|\|v\|\|w\|. \end{aligned}$$

So, for all $n \in \mathbb{N}$, $\|(\widetilde{v \otimes w})_n\| \leq \|v\|\|w\|$, and hence $\|\widetilde{v \otimes w}\|_{cb} \leq \|v\|\|w\|$. That is to say, $\widetilde{v \otimes w}$ is completely bounded.

Next, we define a natural linear injection $V^* \otimes W^* \hookrightarrow (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$. For an elementary tensor $f \otimes g \in V^* \otimes W^*$, let

$$\Phi(f \otimes g)(T) = T(f,g) \; (T \in CB^\sigma(V^* \times W^*, \mathbb{C})).$$

Then $|\Phi(f \otimes g)(T)| = |T(f,g)| \leq \|T\|\|f\|\|g\| \leq \|f\|\|g\|\|T\|_{cb}$ for all $T \in CB^\sigma(V^* \times W^*, \mathbb{C})$, i.e., $\Phi(f \otimes g) \in (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$. We then extend Φ to a linear map $V^* \otimes W^* \to (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$, which is still denoted by Φ.

Claim. $\Phi : V^* \otimes W^* \hookrightarrow (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$ is injective. Suppose $f_i \in V^*$ and $g_i \in W^*$ such that $\Phi\left(\sum_{i=1}^n f_i \otimes g_i\right) = 0$. We may assume that $f_1, \cdots, f_n$ are linearly independent. Let $w \in W$ be fixed. Then for all $v \in V$, $\Phi\left(\sum_{i=1}^n f_i \otimes g_i\right)(\widetilde{v \otimes w}) = \sum_{i=1}^n f_i(v)g_i(w) = 0$, i.e., $\sum_{i=1}^n g_i(w)f_i = 0$. Since $f_1, \cdots, f_n$ are linearly independent, $g_i(w) = 0$ ($i = 1, \cdots, n$). Since $w \in W$ is arbitrary, we have $g_i = 0$ ($i = 1, \cdots, n$). Therefore, $\sum_{i=1}^n f_i \otimes g_i = 0$.

We want to point out here that, by the same arguments, one can see that the map $v \otimes w \to CB^\sigma(V^* \times W^*, \mathbb{C})$, $v \otimes w \mapsto \widetilde{v \otimes w}$, is also injective.

For the weak*-density, we note that $\overline{V^* \otimes W^*}^{w^*} = \left[{}^\perp(V^* \otimes W^*)\right]^\perp$, where ${}^\perp(V^* \otimes W^*) = \{T \in CB^\sigma(V^* \times W^*, \mathbb{C}) : \Phi(f,g)(T) = 0$ for all $f \in V^*$ and $g \in W^*\}$. So, $V^* \otimes W^*$ is weak*-dense in $V^* \overset{\sigma\wedge}{\otimes} W^*$ if and only if ${}^\perp(V^* \otimes W^*) = \{0\}$. But that is true by the definition of Φ. □

For operator spaces V and W, let $V^{**} \overline{\otimes} W^{**}$ be the abstract normal spatial tensor product of V^{**} and W^{**}, i.e., the weak* closure of the algebraic tensor product $V^{**} \otimes W^{**}$ in $\left(V^* \overset{\wedge}{\otimes} W^*\right)^*$. We call (V, W) a **bi-normal pair** if $V^{**} \overline{\otimes} W^{**} = \left(V^* \overset{\wedge}{\otimes} W^*\right)^*$.

For example, if V^{**} and W^{**} are both von Neumann algebras, then (V, W) is a bi-normal pair (cf. [11, Theorem 7.2.4]). In particular, for all C^*-algebras A and B, (A, B) is a bi-normal pair.

It is not clear for us whether there are bi-normal pairs (V, W) such that V^{**} and W^{**} are not von Neumann algebras.

PROPOSITION 5.1.4. *Let V and W be a bi-normal pair of operator spaces. Then the algebraic tensor product $V \otimes W$is weak*-dense in $CB(V^* \times W^*, \mathbb{C})$, and hence $CB^\sigma(V^* \times W^*, \mathbb{C})$ is weak*-dense in $CB(V^* \times W^*, \mathbb{C})$. Therefore, for all $n \in \mathbb{N}$, $M_n(CB^\sigma(V^* \times W^*, \mathbb{C}))$ is weak*-dense in $M_n(CB(V^* \times W^*, \mathbb{C}))$.*

PROOF. As pointed out in the proof of Lemma 5.1.3, we know that the linear map $v \otimes w \mapsto \widetilde{v \otimes w}, V \otimes W \to CB^\sigma(V^* \times W^*, \mathbb{C}) \subseteq CB(V^* \times W^*, \mathbb{C}) \cong (V^* \overset{\wedge}{\otimes} W^*)^*$ is injective, and hence it suffices to show the weak*-density.

Note that

$$V \otimes W \subseteq V^{**} \otimes W^{**} \subseteq V^{**} \overline{\otimes} W^{**} \subseteq \left(V^* \overset{\wedge}{\otimes} W^*\right)^*.$$

Since $V_{\|\cdot\| \le 1}$ (resp. $W_{\|\cdot\| \le 1}$) is weak*-dense in $V^{**}_{\|\cdot\| \le 1}$ (resp. $W^{**}_{\|\cdot\| \le 1}$), $V \otimes W$ is weak*-dense in $V^{**} \overline{\otimes} W^{**}$. By the assumption, $V^{**} \overline{\otimes} W^{**} = \left(V^* \overset{\wedge}{\otimes} W^*\right)^*$. It follows that $V \otimes W$ is weak*-dense in $\left(V^* \overset{\wedge}{\otimes} W^*\right)^*$. Consequently, $(V \otimes W \subseteq)$ $CB^\sigma(V^* \times W^*, \mathbb{C})$ is weak*-dense in $CB(V^* \times W^*, \mathbb{C})$.

To consider the weak*-density at level n, let $f \in [f_{ij}] \in M_n(CB(V^* \times W^*, \mathbb{C}))$. For each (i,j), there exists a net $(f_{\alpha_{ij}})_{\alpha_{ij} \in A_{ij}}$ in $CB^\sigma(V^* \times W^*, \mathbb{C})$ such that $f_{\alpha_{ij}} \xrightarrow{w*} f_{ij}$ in $\left(V^* \hat{\otimes} W^*\right)^*$. Let $A = \prod_{1 \le i,j \le n} A_{ij}$ and order A by $\beta \succ \alpha$ if and only if $\beta_{ij} \succ \alpha_{ij}$ for all $1 \le i,j \le n$. For each $\alpha = (\alpha_{ij}) \in A$, let $f_\alpha = [f_{\alpha_{ij}}]$. Then $f_\alpha \in M_n(CB^\sigma(V^* \times W^*, \mathbb{C}))$. We claim that $f_\alpha \xrightarrow{w*} f$ in $M_n\left(\left(V^* \hat{\otimes} W^*\right)^*\right) \cong \left(T_n\left(V^* \hat{\otimes} W^*\right)\right)^*$.

Note that the duality between $M_n\left(\left(V^* \hat{\otimes} W^*\right)^*\right)$ and $T_n\left(V^* \hat{\otimes} W^*\right)$ is given by

$$\langle f, x \rangle = \sum_{1 \le i,j \le n} f_{ij}(x_{ij})$$

for $f = [f_{ij}] \in M_n\left(\left(V^* \hat{\otimes} W^*\right)^*\right)$ and $x = [x_{ij}] \in T_n\left(V^* \hat{\otimes} W^*\right)$. Now for all $x = [x_{ij}] \in T_n\left(V^* \hat{\otimes} W^*\right)$, we have

$$\langle f_\alpha - f, x \rangle = \sum_{1 \le i,j \le n} \langle f_{\alpha_{ij}} - f_{ij}, x_{ij} \rangle \to 0.$$

Therefore, $f_\alpha \xrightarrow{w*} f$ in $M_n\left(\left(V^* \hat{\otimes} W^*\right)^*\right)$. ☐

LEMMA 5.1.5. *Let V_1, V_2 and W be operator spaces, $\varphi : V_1 \times V_2 \to W$ a bilinear map and $\psi : W \to X$ a linear map. Then*

$$\|\psi \circ \varphi\|_{cb} \le \|\psi\|_{cb} \|\varphi\|_{cb}$$

and

$$\|\psi \circ \varphi\|_{mb} \le \|\psi\|_{cb} \|\varphi\|_{mb}.$$

PROOF. For $v_1 = \left[v_{ij}^1\right] \in M_n(V_1)$ and $v_2 = [v_{kl}^2] \in M_n(V_2)$, we have

$$\begin{aligned}
\|(\psi \circ \varphi)_n(v_1, v_2)\|_{M_{n^2}(X)} &= \| \left[\psi \circ \varphi(v_{ij}^1, v_{kl}^2)\right] \|_{M_{n^2}(X)} \\
&= \| \left[\psi(\varphi(v_{ij}^1, v_{kl}^2))\right] \|_{M_{n^2}(X)} \\
&= \|\psi^{(n^2)}\left(\left[\varphi(v_{ij}^1, v_{kl}^2)\right]\right) \|_{M_{n^2}(X)} \\
&\le \|\psi^{(n^2)}\| \| \left[\varphi(v_{ij}^1, v_{kl}^2)\right] \|_{M_{n^2}(W)} \\
&= \|\psi\|_{cb} \|\varphi_n(v_1, v_2)\|_{M_{n^2}(W)} \\
&\le \|\psi\|_{cb} \|\varphi_n\| \|v_1\| \|v_2\| \\
&\le \|\psi\|_{cb} \|\varphi\|_{cb} \|v_1\| \|v_2\|.
\end{aligned}$$

Therefore, $\|\psi \circ \varphi\|_{cb} = \sup_{n \in \mathbb{N}} \|(\psi \circ \varphi)_n\| \leq \|\psi\|_{cb}\|\varphi\|_{cb}$.

For the second inequality, we get now

$$\begin{aligned}
\|(\psi \circ \varphi)^n(v_1, v_2)\|_{M_n(X)} &= \| \left[\sum_{k=1}^{n} \psi \circ \varphi(v^1_{ik}, v^2_{kj}) \right] \|_{M_n(X)} \\
&= \| \left[\psi(\sum_{k=1}^{n} \varphi(v^1_{ik}, v^2_{kj})) \right] \|_{M_n(X)} \\
&= \|\psi^{(n)} \left(\left[\sum_{k=1}^{n} \varphi(v^1_{ik}, v^2_{kj}) \right] \right) \|_{M_n(X)} \\
&\leq \|\psi^{(n)}\| \| \left[\sum_{k=1}^{n} \varphi(v^1_{ik}, v^2_{kj}) \right] \|_{M_n(W)} \\
&= \|\psi^{(n)}\| \|\varphi^n(v_1, v_2)\|_{M_n(W)} \\
&\leq \|\psi\|_{cb}\|\varphi^n\|\|v_1\|\|v_2\| \\
&\leq \|\psi\|_{cb}\|\varphi\|_{mb}\|v_1\|\|v_2\|.
\end{aligned}$$

Therefore, $\|\psi \circ \varphi\|_{mb} = \sup_{n \in \mathbb{N}} \|(\psi \circ \varphi)^n\| \leq \|\psi\|_{cb}\|\varphi\|_{mb}$. □

DEFINITION 5.1.6. *Let V and W be operator spaces. We say that (V, W) satisfies* ***condition (*)*** *if the unit ball of $CB^\sigma(V^* \times W^*, M_n)$ is weak*-dense in the unit ball of $CB(V^* \times W^*, M_n)$ $\left(= \left(T_n(V^* \hat{\otimes} W^*)\right)^*\right)$ for all $n \in \mathbb{N}$.*

It is not clear for us whether a bi-normal pair (V, W) automatically satisfies condition (*) (cf. Proposition 5.1.4). However, it is the case at least for the following bi-normal pairs.

According to Kaplansky Density Theorem, if $\mathcal{A}$ is a weak*-dense $*$-subalgebra of a von Neumann algebra $\mathcal{M}$, then the unit ball of $\mathcal{A}$ is weak*-dense in the unit ball of $\mathcal{M}$. Suppose V and W are $*$-algebras such that V^{**} and W^{**} are von Neumann algebras (e.g., it is the case when V and W are both C^*algebras). Then $CB(V^* \times W^*, \mathbb{C}) = \left(V^* \hat{\otimes} W^*\right)^* = V^{**}\overline{\otimes}W^{**}$ is also a von Neumann algebra, and hence $M_n\left(CB(V^* \times W^*, \mathbb{C})\right)$ is a von Neumann algebra for each $n \in \mathbb{N}$. The algebraic tensor product $V \otimes W$ with the multiplication

$$(\sum_i a_i \otimes b_i)(\sum_j c_j \otimes d_j) = \sum_{i,j} a_i c_j \otimes b_i d_j,$$

and the involution

$$(\sum_k a_k \otimes b_k)^* = \sum_k a_k^* \otimes b_k^*$$

is a weak*-dense $*$-subalgebra of $V^{**}\overline{\otimes}\, W^{**}$. The same is true for $M_n(V \otimes W)$ in $M_n(V^{**}\overline{\otimes}\, W^{**})$.

So, applying Kaplansky Density Theorem shows that *condition (*)* is satisfied in this case.

PROPOSITION 5.1.7. *Let V and W be a pair of operator spaces satisfying condition (*). Then the linear injection*

$$\Phi : V^* \otimes_\wedge W^* \to (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$$

considered in the proof of Lemma 5.1.3 is a completely isometric embedding. Therefore, we have a completely isometric embedding

$$V^* \overset{\wedge}{\otimes} W^* \hookrightarrow V^* \overset{\sigma\wedge}{\otimes} W^*.$$

PROOF. We already have the completely isometric embedding

$$\Psi : V^* \overset{\wedge}{\otimes} W^* \hookrightarrow \left(V^* \overset{\wedge}{\otimes} W^*\right)^{**} \cong (CB(V^* \times W^*, \mathbb{C}))^*$$

given by $\Psi\left(\sum_{i=1}^n f_i \otimes g_i\right)(T) = \sum_{i=1}^n T(f_i, g_i)$ for all $f_i \in V^*, g_i \in W^*$, and $T \in CB(V^* \times W^*, \mathbb{C})$.

Let $i : CB^\sigma(V^* \times W^*, \mathbb{C}) \to CB(V^* \times W^*, \mathbb{C})$ be the inclusion map and let $p = i^*$. Then $p : (CB(V^* \times W^*, \mathbb{C}))^* \to (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$ is completely bounded with $\|p\|_{cb} = \|i\|_{cb} = 1$. We observe that $\Phi = p \circ \Psi$ and hence Φ is a complete contraction. To finish the proof, we only need to show that for all $n \in \mathbb{N}$ and $u \in M_n(V^* \otimes W^*)$, we have $\|\Phi^{(n)}(u)\| \geq \|u\|_\wedge$.

Let $n \in \mathbb{N}$ and $u = [u_{ij}] \in M_n(V^* \otimes_\wedge W^*) \subseteq M_n\left(\left(V^* \overset{\wedge}{\otimes} W^*\right)^{**}\right)$, which is identified with $CB(CB(V^* \times W^*, \mathbb{C}), M_n)$. Then

$$\|u\|_\wedge = \|u\|_{cb} = \|u^{(n)}\|,$$

where u is considered as a linear map from $CB(V^* \times W^*, \mathbb{C})$ to M_n and hence $u^{(n)}$: $CB(V^* \times W^*, M_n) \to M_{n^2}$. Now

$$\Phi : V^* \otimes_\wedge W^* \to (CB^\sigma(V^* \times W^*, \mathbb{C}))^*$$

and

$$\Phi^{(n)} : M_n\left(V^* \otimes_\wedge W^*\right) \to M_n\left((CB^\sigma(V^* \times W^*, \mathbb{C}))^*\right) \cong CB(CB^\sigma(V^* \times W^*, \mathbb{C}), M_n).$$

Thus,

$$\Phi^{(n)}(u) : CB^\sigma(V^* \times W^*, \mathbb{C}) \to M_n$$

and

$$\|\Phi^{(n)}(u)\| = \|\Phi^{(n)}(u)\|_{cb} = \|(\Phi^{(n)}(u))^{(n)}\|,$$

where $(\Phi^{(n)}(u))^{(n)} : CB^\sigma(V^* \times W^*, M_n) \to M_{n^2}$. Therefore, to get the inequality $\|\Phi^{(n)}(u)\| \geq \|u\|_\wedge$, we only have to prove that $\|(\Phi^{(n)}(u))^{(n)}\| \geq \|u^{(n)}\|$. We observe that for $T \in CB^\sigma(V^* \times W^*, \mathbb{C})$, $(\Phi^{(n)}(u))^{(n)}(T) = u^{(n)}(T)$, or $(\Phi^{(n)}(u))^{(n)}$ is really the restriction of $u^{(n)}$ to $CB^\sigma(V^* \times W^*, M_n)$. We also note that $CB(V^* \times W^*, M_n) \cong M_n\left(\left(V^* \overset{\wedge}{\otimes} W^*\right)^*\right) \cong \left(T_n\left(V^* \overset{\wedge}{\otimes} W^*\right)\right)^*$ and $M_{n^2} \cong (T_{n^2})^*$. It can be seen that now

$$u^{(n)} : \left(T_n\left(V^* \overset{\wedge}{\otimes} W^*\right)\right)^* \to (T_{n^2})^*$$

is weak*-weak* continuous. Therefore, it suffices to show that the unit ball of $CB^\sigma(V^* \times W^*, M_n)$ is weak*-dense in the unit ball of $CB(V^* \times W^*, M_n)$. By the assumption of condition *(*)*, the statement follows. $\square$

By the definitions of $\overset{e\wedge}{\otimes}$ and $\overset{\sigma\wedge}{\otimes}$, the following operator space identification are immediate:

$$CB^\sigma(V^* \overset{\sigma\wedge}{\otimes} W^*, \mathbb{C}) = \left((V \overset{e\wedge}{\otimes} W)^{**}\right)_\sigma \cong V \overset{e\wedge}{\otimes} W \cong CB^\sigma(V^* \times W^*, \mathbb{C}).$$

We show below that the above $\mathbb{C}$ can be replaced by W^* if (V, W) satisfies condition (*).

PROPOSITION 5.1.8. *Let V_1, V_2 and W be operator spaces such that (V_1, V_2) satisfies condition (*). Then we have completely isometric identification*

$$CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*) \cong CB^\sigma(V_1^* \times V_2^*, W^*).$$

PROOF. Let $\varphi \in CB^\sigma(V_1^* \times V_2^*, W^*)$ and $w \in W \subseteq W^{**}$. Then $w \circ \varphi : V_1^* \times V_2^* \to \mathbb{C}$ is separately weak*-continuous and jointly completely bounded with $\|w \circ \varphi\|_{cb} \leq \|w\|_{cb}\|\varphi\|_{cb} = \|w\|\|\varphi\|_{cb}$ (by Lemma 5.1.5). So, $w \circ \varphi$ is in $V_1 \overset{e\wedge}{\otimes} V_2$.

Now we define a linear map

$$\varphi_* : W \to V_1 \overset{e\wedge}{\otimes} V_2, \quad w \mapsto w \circ \varphi.$$

Then $\|\varphi_*\|_{cb} \leq \|\varphi\|_{cb}$. To see this, let $w = [w_{ij}] \in M_n(W)$. Then $[w_{ij} \circ \varphi] \in M_n(V_1 \overset{e\wedge}{\otimes} V_2) \subseteq M_n(CB(V_1^* \times V_2^*, \mathbb{C})) \cong CB(V_1^* \times V_2^*, M_n)$. Note that $w \in M_n(W) \subseteq M_n(W^{**}) \cong CB(W^*, M_n)$. So,

$$(\varphi_*)^{(n)}(w) = [w_{ij} \circ \varphi]_{M_n(V_1 \overset{e\wedge}{\otimes} V_2)} = w \circ \varphi,$$

where w is treated as a map from W^* to M_n. Therefore, by Lemma 5.1.5,

$$\|(\varphi_*)^{(n)}(w)\| = \|[w_{ij} \circ \varphi]\|_{M_n(V_1 \overset{e\wedge}{\otimes} V_2)} = \|w \circ \varphi\|_{cb} \leq \|w\|_{cb}\|\varphi\|_{cb}.$$

It follows that $\|\varphi_*\|_{cb} = \sup_{n \in \mathbb{N}} \|(\varphi_*)^{(n)}\| \leq \|\varphi\|_{cb}$. Hence

$$\|(\varphi_*)^*\|_{cb} = \|\varphi_*\|_{cb} \leq \|\varphi\|_{cb}.$$

Let $\overline{\varphi} = (\varphi_*)^*$. Then $\overline{\varphi} : (V_1 \overset{e\wedge}{\otimes} V_2)^* = V_1^* \overset{\sigma\wedge}{\otimes} V_2^* \to W^*$ is weak*-weak* continuous and completely bounded, i.e., $\overline{\varphi} \in CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*)$, and $\|\overline{\varphi}\|_{cb} = \|\varphi_*\|_{cb} \leq \|\varphi\|_{cb}$.

For all $f_1 \in V_1^*, f_2 \in V_2^*$ and $w \in W$,

$$\begin{aligned} \overline{\varphi}(f_1 \otimes f_2)(w) &= (f_1 \otimes f_2)(\varphi_*(w)) = (f_1 \otimes f_2)(w \circ \varphi) \\ &= (w \circ \varphi)(f_1, f_2) = \varphi(f_1, f_2)(w). \end{aligned}$$

So, $\overline{\varphi}$ is an extension of the linearization $\tilde{\varphi}$ of φ. Note that $\|\tilde{\varphi}\|_{CB(V_1^* \hat{\otimes} V_2^*, W^*)} = \|\varphi\|_{cb}$ and

$$V_1^* \overset{\wedge}{\otimes} V_2^* \hookrightarrow (CB^\sigma(V_1^* \times V_2^*, \mathbb{C}))^* \cong V_1^* \overset{\sigma\wedge}{\otimes} V_2^*$$

is a completely isometric embedding (Proposition 5.1.7). Thus, we have

$$\|\overline{\varphi}\|_{cb} \geq \|\tilde{\varphi}\|_{CB(V_1^* \hat{\otimes} V_2^*, W^*)} = \|\varphi\|_{cb}.$$

Therefore, $\|\overline{\varphi}\|_{cb} = \|\varphi\|_{cb}$. So far, we have the completely isometric embedding

$$CB^\sigma(V_1^* \times V_2^*, W^*) \hookrightarrow CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*), \quad \varphi \mapsto \overline{\varphi}.$$

In fact, it is onto.

To show this, let $S \in CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*)$ and $S' : V_1^* \times V_2^* \to W^*$ be the completely bounded bilinear map corresponding to $S|_{V_1^* \hat{\otimes} V_2^*} : V_1^* \hat{\otimes} V_2^* \to W^*$ (cf. Proposition 5.1.7). Then $S' : V_1^* \times V_2^* \to W^*$ is separately weak*-weak* continuous.

Indeed, let $f_\alpha \xrightarrow{w^*} f$ in V_1^* and $g \in V_2^*$. Then $f_\alpha \otimes g \xrightarrow{w^*} f \otimes g$ in $(CB^\sigma(V_1^* \times V_2^*, \mathbb{C}))^*$ under the embedding $V_1^* \otimes V_2^* \hookrightarrow (CB^\sigma(V_1^* \times V_2^*, \mathbb{C}))^*$. Thus,

$$S(f_\alpha \otimes g) \xrightarrow{w^*} S(f \otimes g), \quad \text{i.e., } S'(f_\alpha, g) \xrightarrow{w^*} S(f, g).$$

Similarly, if $f \in V_1^*$ and $g_\alpha \xrightarrow{w^*} g$ in V_2^*, then $S'(f, g_\alpha) \xrightarrow{w^*} S(f, g)$. Therefore, $S' \in CB^\sigma(V_1^* \times V_2^*, \mathbb{C})$. By the definition of $\overline{S'}$, we have $\overline{S'}(f \otimes g) = S(f \otimes g)$ for all $f \in V_1^*$ and $g \in V_2^*$. Due to the weak*-weak* continuity of S and $\overline{S'}$ and the weak*-density of $V_1^* \otimes V_2^*$ in $V_1^* \overset{\sigma\wedge}{\otimes} V_2^*$ (cf. Lemma 5.1.3), we have $S = \overline{S'}$.

Therefore, we have the completely isometric identification

$$CB^\sigma(V_1^* \times V_2^*, W^*) \cong CB^\sigma(V_1^* \overset{\sigma\wedge}{\otimes} V_2^*, W^*)$$

$\square$

Let $\varphi_1 : V_1 \to W_1$ and $\varphi_2 : V_2 \to W_2$ be completely bounded maps. Then the completely bounded map

$$(\varphi_1^* \hat{\otimes} \varphi_2^*)^* : \left(V_1^* \hat{\otimes} V_2^*\right)^* \to \left(W_1^* \hat{\otimes} W_2^*\right)^*$$

sends $V_1 \overset{e\wedge}{\otimes} V_2$ to $W_1 \overset{e\wedge}{\otimes} W_2$ since φ_1^* and φ_2^* are weak*-weak* continuous. Note that $(\varphi_1^* \hat{\otimes} \varphi_2^*)^*$ is the unique weak*-weak* continuous extension of the algebraic tensor product $\varphi_1 \otimes \varphi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$, where the algebraic tensor product $X \otimes Y$ is embedded into $X \overset{e\wedge}{\otimes} Y$ via $x \otimes y \mapsto \widetilde{x \otimes y}$ (see proof of Proposition 5.1.4). We let $\varphi_1 \overset{e\wedge}{\otimes} \varphi_2$ denote the restriction of $(\varphi_1^* \hat{\otimes} \varphi_2^*)^*$ to $V_1 \overset{e\wedge}{\otimes} V_2$. We show that the injectivity of $\overset{e\wedge}{\otimes}$ in the following

Proposition 5.1.9. *Let V_1, V_2, W_1 and W_2 be operator spaces. If $\varphi_i : V_i \to W_i$ is completely isometric (resp. contractive) $(i = 1, 2)$, then so is $\varphi_1 \overset{e\wedge}{\otimes} \varphi_2 : V_1 \overset{e\wedge}{\otimes} V_2 \to W_1 \overset{e\wedge}{\otimes} W_2$.*

Proof. Since φ_1 and φ_2 are completely isometric, φ_1^* and φ_2^* are complete quotient maps, and then $\varphi_1^* \hat{\otimes} \varphi_2^* : W_1^* \hat{\otimes} W_2^* \to V_1^* \hat{\otimes} V_2^*$ is a complete quotient map.

Therefore,

$$(\varphi_1^* \overset{\wedge}{\otimes} \varphi_2^*)^* : \left(V_1^* \overset{\wedge}{\otimes} V_2^*\right)^* \cong CB(V_1^* \times V_2^*, \mathbb{C}) \to \left(W_1^* \overset{\wedge}{\otimes} W_2^*\right)^* \cong CB(W_1 \times W_2, \mathbb{C})$$

is a complete isometry. It follows that as a restriction of $(\varphi_1^* \overset{\wedge}{\otimes} \varphi_2^*)^*$,

$$\varphi_1 \overset{e\wedge}{\otimes} \varphi_2 : V_1 \overset{e\wedge}{\otimes} V_2 \to W_1 \overset{e\wedge}{\otimes} W_2$$

is a complete isometry.

The case for complete contractions follows from the corresponding property of the projective tensor product and the fact that the completely bounded norm of a linear map is the same as the completely bounded norm of its adjoint. □

5.2. Comparison with other operator space tensor products

In this section, we compare the extended projective tensor product with some other existing tensor products.

5.2.1. $\overset{e\wedge}{\otimes}$ and the injective tensor product. Let V and W be operator spaces. Then $V \overset{\vee}{\otimes} W$ is the norm closure of the algebraic tensor product $V \otimes W$ in $\left(V^* \overset{\wedge}{\otimes} W^*\right)^*$.

By definition, $V \overset{e\wedge}{\otimes} W = CB^\sigma(V^* \times W^*, \mathbb{C})$ is a closed subspace of $\left(V^* \overset{\wedge}{\otimes} W^*\right)^*$. Also, $V \otimes W \subseteq CB^\sigma(V^* \times W^*, \mathbb{C})$ (see the proof of Lemma 5.1.2). Therefore, we have the completely isometric embedding

$$V \overset{\vee}{\otimes} W \subseteq V \overset{e\wedge}{\otimes} W.$$

5.2.2. $\overset{e\wedge}{\otimes}$ and the normal spatial tensor product. Recall that for dual operator spaces V^* and W^*,

$$V^* \overline{\otimes} W^* = \overline{V^* \otimes W^*}^{w^*} \subseteq \left(V \overset{\wedge}{\otimes} W\right)^* = CB(V, W^*) \cong CB^\sigma(V^{**}, W^*),$$

and

$$V^* \overset{e\wedge}{\otimes} W^* = CB^\sigma(V^{**} \times W^{**}, \mathbb{C}) = CB^{\sigma - w}(V^{**}, W^*) \subseteq CB^\sigma(V^{**}, W^*).$$

So, both operator space tensor products are subspaces of $CB^\sigma(V^{**}, W^*)$.

If W is reflexive, then $CB^{\sigma-w}(V^{**}, W^*) = CB^{\sigma}(V^{**}, W^*)$ for all operator spaces V. In this case, we have $V^* \overline{\otimes} W^* \subseteq V^* \overset{e\wedge}{\otimes} W^*$.

In fact, the converse is also true.

PROPOSITION 5.2.1. *Let W be an operator spaces. Then W is reflexive if and only if $CB^{\sigma-w}(V^{**}, W^*) = CB^{\sigma}(V^{**}, W^*)$ for all operator spaces V.*

PROOF. We only need to show the sufficiency. Let $V = W^*$. Considering the canonical embedding $i : W \to W^{**}$, we have $i^* \in CB^{\sigma}(W^{***}, W^*) \cong CB^{\sigma-w}(W^{***}, W^*)$. So, i is weakly compact (cf. [8. Theorem VI.4.7]), and thus $\overline{i(W_{\|\cdot\|\leq 1})} = W_{\|\cdot\|\leq 1}$ is weakly compact. By [6, Theorem V.4.2], W is reflexive. $\square$

(3) Considering the symmetry of the projective tensor product with respect to the two underlying operator spaces, we have $CB^{\sigma-w}(V^{**}, W^*) \cong CB^{\sigma-w}(W^{**}, V^*)$ and $CB^{\sigma}(V^{**}, W^*) \cong CB^{\sigma}(W^{**}, V^*)$. Therefore,

$$V^* \overset{e\wedge}{\otimes} W^* = V^* \overline{\otimes} W^*$$

if either V or W is reflexive such that (V, W) is a bi-normal pair. In particular, if V or W is of finite dimension, then the injective tensor product $V^* \overset{\vee}{\otimes} W^*$ and the above two tensor products are all the same.

5.2.3. $\overset{e\wedge}{\otimes}$ and the extended Haargerup tensor product. Recall for each $\varphi \in B(H)_*$, the right slice map $R_\varphi : B(H) \otimes B(K) \to B(K)$ is defined by

$$R_\varphi\left(\sum_i a_i \otimes b_i\right) = \sum_i \varphi(a_i) b_i \text{ for } a_i \in B(H) \text{ and } b_i \in B(K).$$

Similarly, for $\psi \in B(K)_*$, the left slice map $L_\psi : B(H) \otimes B(K) \to B(H)$ is defined by

$$L_\psi\left(\sum_i a_i \otimes b_i\right) = \sum_i a_i \psi(b_i).$$

The right (resp. left) slice map has the unique extension to $\left(B(H)_* \overset{h}{\otimes} B(K)\right)^*$ (or $\left(B(H)_* \overset{\wedge}{\otimes} B(K)\right)^*$) which is still denoted by R_φ (resp. L_ψ).

Let V and W be operator spaces with $V^* \subseteq B(H)$ and $W^* \subseteq B(K)$. The Fubini product $\mathcal{F}(V^*, W^*)$ is defined as the set

$$\{u \in B(H) \overset{eh}{\otimes} B(K) : R_\varphi(u) \in V^* \text{ and } L_\psi(u) \in W^* \text{ for all } \varphi \in B(H)_*, \psi \in B(K)_*\}$$

The normal Fubini tensor product $V^*\overline{\otimes}_{\mathcal{F}}W^*$ $\left(= \left(V \overset{\wedge}{\otimes} W\right)^*\right)$ (cf. [11, Theorem 7.2.3]) is

$$\{u \in B(H \otimes K) : R_\varphi(u) \in V^* \text{ and } L_\psi(u) \in W^* \text{ for all } \varphi \in B(H)_*, \psi \in B(K)_*\}.$$

Since $B(H) \overset{eh}{\otimes} B(K)\left(= \left(B(H)_* \overset{h}{\otimes} B(K)_*\right)^*\right)$ can be treated as a linear subspace of $\left(B(H)_* \overset{\wedge}{\otimes} B(K)_*\right)^*$ $(= B(H \otimes K))$, $\mathcal{F}(V^*, W^*) \subseteq V^*\overline{\otimes}_{\mathcal{F}}W^*$. By [4, Theorem 3.1 (ii)], $\mathcal{F}(V^*, W^*) = V^* \overset{eh}{\otimes} W^*$ (cf. [4, Theorem 3.1(ii)]), i.e., $V^* \overset{eh}{\otimes} W^*$ and $V^* \overset{e\wedge}{\otimes} W^*$ are both subspaces of $V^*\overline{\otimes}_{\mathcal{F}}W^*$ and they may be equal under some conditions.

PROPOSITION 5.2.2. *Let V be an operator space and H a Hilbert space. Then we have the completely isometric identification*

$$V \overset{e\wedge}{\otimes} H_r \cong V \overset{eh}{\otimes} H_r.$$

PROOF. Since $V^*\overset{\wedge}{\otimes}(H_r)^* \cong V^*\overset{h}{\otimes}(H_r)^*$, $CB(V^* \times (H_r)^*, \mathbb{C}) \cong MB(V^* \times (H_r)^*, \mathbb{C})$. Then their normal parts should also be identified, i.e., $V \overset{e\wedge}{\otimes} H_r \cong V \overset{eh}{\otimes} H_r$ by the definitions of the extended projective tensor product and the extended Haagerup tensor product. □

5.3. Some open questions

We conclude the thesis with the following open questions.

1. What is the characterization for a pair (V, W) of operator spaces to satisfy condition *(*)*?

2. Do we have a canonical subspace X of A^* such that $X \subsetneq wap(A) \subsetneq A^*$ and X plays a role similar to what A does in the sequence $A \subsetneq Z_1(A^{**}) \subsetneq A^{**}$ when the strong Arens irregularity of A is concerned?

3. How can we establish the relationship between the Arens regularity of a Banach algebra A and some operator space structures, say on $A^* \times A^{**}$, involving certain tensor products?

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