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FOURTH MOMENT AND SIMULATED POWERS OF MRPP STATISTICS.

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ABSTRACT

In classical tests of hypotheses, assumptions concerning normality and homogeneity of variances are needed. Often practical data do not meet some of these assumptions and the idea of robustness is advanced. To avoid making assumptions underlying a tests, Mielke, Berry and Johnson introduced the MRPP (Multi-response Permutation Procedures) test. The test statistic δ is simply a weighted average of some distance measure between pairs of observations within a group. It tests H_0: Classification of data into g groups is random against H_1: Classification is done according to some a priori scheme. Special cases of δ are equivalent to some well-known test statistics. When the distance measure is the Euclidean distance between ranks of observations and the weights are proportional to the size of groups, in the 2-sample case, the MRPP statistic, called δ_1, performs better than the Wilcoxon test for some underlying distributions.

The null distribution of δ is often highly negatively skewed, and is, in general, asymptotically non-normal. To account for the skewness, Mielke and others have recommended the use of the Pearson Type III approximation, determined by the first three moments of δ.

We define 23 symmetric functions in order to obtain the fourth moment of δ. In the case of two equal samples, an explicit result for the fourth moment of δ_1 is obtained. We obtain empirical powers of δ_1, considering 10,000 samples for both small and large samples, using a Pearson type approximation based on four moments, as well as the Type III approximation. These powers are compared with those of the Wilcoxon test against various shifts in location for several underlying distributions, viz., uniform, normal, logistic, 10% 3N, 10% 10N, Laplace, U-shaped, Cauchy, and exponential.

We conclude the dissertation by discussing the scope for further work with the use of the fourth moment.

DEDICATED

To my parents

and

my wife

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INTRODUCTION

1.1 The Testing Problem

The classical approach to testing a hypothesis concerning a set of parameters assumes that we know all about the population but for the values of the parameters under the null hypothesis. Often the underlying population is assumed to be normal. In testing equality of several means, homogeneity of variances is assumed. We assume that the k populations have common variance, in order to arrive at the F statistic. Sampling distributions of many test statistics are obtained on the assumption that the parent population is normal. A large proportion of multivariate theory is based on normal theory. Often, in the classical methods, the data are subjected to some appropriate transformations in order to meet the assumptions of normality and homogeneity. But, sometimes, it is not easy to find a suitable transformation. Kendall (1979, p. 496) indicates that it is difficult to decide whether the standard procedures are likely to be approximately valid or misleading. He points out that the difficulty in using a transformation is that we must have knowledge of the underlying distribution before we know which transformation is best applied. As an alternate approach, he discusses permutation tests, which we describe in section 1.3.

Many articles have been written about the shortcomings of classical tests. In a review paper, Huber (1972, p. 1042) points out, "The theory of estimation originated with problems where almost all of the statistical variability is due to measurements of errors." If we consider an estimator $\hat{\theta}$ of θ, we see that $\hat{\theta}$ is expressed in the form $\hat{\theta} = k\theta + \varepsilon$, where $k = 1$ for an unbiased estimator and ε is assumed to be $N(0,\sigma^2)$. Huber quotes Gauss (1821) as saying that he had obtained the normal distribution as a distribution of errors to suit the mean. The mean as a location parameter is much criticized in the literature, since it is highly affected by extreme values and thus any inference based on the mean, concerning heavy tailed-populations, becomes less powerful.

Boneau (1960) remarks that psychological data too frequently have an exasperating tendency to manifest themselves in a form which violates one or more of the assumptions underlying the usual tests of significance. The researcher usually attempts to transform them in such a way that the assumptions are tenable, or he may look elsewhere for a statistical test.

A solution to the problem of classical tests is to devise testing procedures that do not make any assumption about the form of the distribution; that is, to follow a distribution-free approach or use non-parametric methods. Here one develops a test statistic whose distribution does not depend on the distribution from which the sample is drawn. An alternative solution is to develop robust tests that are insensitive to the violation of assumptions underlying them. We discuss these in the following section.

1.2 Non-parametric Methods and Robust Procedures

Non-parametric tests assume that the parent population is continuous and, sometimes, that it is symmetric. Most of the tests are based on the ranks of the observations. These tests are easy to use and the theory behind them is simple to understand. However, obtaining the distribution of a test statistic under the alternative hypothesis is often extremely difficult. The non-parametric test that competes with the classical t-test is the Wilcoxon test. The asymptotic relative efficiency (ARE) of the Wilcoxon test relative to the t-test when data is from a normal population is 0.955. That is, we have to take about 5% more observations when using a Wilcoxon test instead of a t-test. On the other hand, the ARE of the Wilcoxon test relative to the t-test is 1.5 when the underlying distribution is a double exponential and 3 when it is an exponential. It is at least 0.864 for any underlying distribution. Therefore, using the Wilcoxon test, we are protected from a heavy loss when the underlying distribution is non-normal but pay a little premium when the data is normal. For this reason some authors (e.g., Keppel 1973) prefer to use the Wilcoxon test instead of the t test all the time. The Kruskal-Wallis H test performs similarly relative to the F test. In section 1.3, we discuss permutation tests that are distribution free tests but are not limited to the rank of the observations. Below, we discuss robust procedures.

By a robust test we mean a test which is not affected severely upon violation of the assumptions underlying it. According to Box and Anderson (1955), a statistical criterion should be insensitive to extraneous factors, e.g., non-normality of data and the presence of outliers, etc., and should be sensitive to a change in factors to be tested; that is, the test should be powerful as well. Andrews et al. (1972) have carried out a study on robustness of several test statistics, e.g., the trimmed mean, the sine wave, Hampel statistic, etc. They have shown empirically that these statistics generally perform better than the classical statistics.

Unlike the classical tests, robust tests can handle non-normal data. However, many do not like this solution because robust procedures are based on empirical studies of some limited number of cases. One determines the performance, say the power, of a robust test, by evaluating it against some limited number of alternatives, for a limited number of underlying distributions, for a specified level of significance and for limited sizes of samples. There is a critical discussion in Bickel's (1976) paper, "Another look at Robustness". Bickel himself and, earlier, Andrews et al (1972), point out that there is no clear understanding about the definition of robustness among statisticians.

1.3 Permutation Tests

Let there be a random sample X_{ij}, $j=1, \ldots, n_i$ of size n_i from a distribution function $F(x_i)$, for $i=1,\ldots, c(\geq 2)$. Let the c samples be mutually independent with distribution functions $F_1,\ldots, F_c$, which may or may not be continuous. Then the hypothesis of interest is

$$H_o: F_1(x) = \ldots = F_c(x) \text{ for all } x \in R.$$

Let $N=\sum_{i=1}^{c} n_i$. Then the null hypothesis implies that the joint distribution of $E_N=(x_{11},\ldots,x_{cn_c})$ remains invariant under the N! permutations of the coordinates of

E_N among themselves. Equivalently, this implies that the joint distribution of E_N remains invariant under all possible partitionings into c subsets of sizes $n_1, \ldots, n_c$, respectively.

A test that considers the above principle, that is, calculates every possible test statistic and compares the values of the observed test statistics on the basis of this, is called a "permutation test."

Detailed theory of permutation tests are given in Puri and Sen (1970) and Hoeffding (1952). Permutation tests are due to Fisher (1935). In using these tests, we do not require any assumption except that the observations are drawn independently. We now discuss a permutation test based on multi-responses, that is, where x_{ij} are r-response vectors for each $j=1, \ldots, n_i$, $i=1, \ldots, c$. This is called the multi-response permutation procedure (MRPP) test statistic.

1.4 The MRPP Test Statistic

The MRPP test statistic, as introduced by Mielke, Berry and Johnson (1976), is used to test the hypothesis

H_o: Classification of a set of data into g subgroups is random against the alternative.

H_1: Classification is done according to some a priori scheme.

The description of the test statistic is as follows.

Let there be a set of r-vector ($r \geq 1$) observations $\{X_1, \ldots, X_N\}$. Suppose $K \leq N$ observations are partitioned into g subgroups such that the i^{th} subgroup has $n_i \geq 2$ observations, $i=1,\ldots,g$. Let Δ_{IJ} be some distance measure between the observations X_I and X_J. Then the MRPP test statistic δ is defined as

$$\delta = \frac{1}{\sum_{i=1}^{g} \binom{n_i}{2}} \sum_{i=1}^{g} \sum_{I<J\leq N} \Delta_{IJ} S_i(I) S_i(J), \qquad (1.1)$$

where $K = \sum_{i=1}^{g} n_i$ and S is an indicator function with $S_i(I)=1$ if X_I is in the i^{th} subgroup and 0 otherwise.

We note that the test statistic is simply a weighted average of the distances between pairs of observations within a subgroup.

The assumptions underlying this test are:

(i) the data are at an ordinal level or a higher level, and

(ii) when $r \geq 2$, responses are measured on each individual. These response measurements are commensurate with one another, i.e., each of the r response measurements have an appropriate scale of measurement.

The test is carried out using a permutation procedure and, since X_I, $I=1,\ldots,N$, are, in general, r-response measurements, on an individual, it is termed "the multi-response permutation procedure (MRPP) test statistic". Throughout this dissertation, we shall denote it by δ.

When classification is done according to some a priori scheme, there will be some cluster pattern and, hence, δ will take on a smaller value. Therefore, the test rejects H_o when δ is smaller than δ_α - the α^{th} percentile of δ. The probability of every possible value of δ under H_o is the same. There are $M = \frac{N!}{\prod_{i=1}^{g} n_i!(N-K)!}$ possible classifications. This gives an exact probability distribution of δ under H_o.

With larger N, M becomes very large. For instance, when $n_1=n_2=\frac{N}{2}=10$, there are 184,756 different possible classifications. It then becomes difficult to obtain an exact distribution. This difficulty is resolved by considering an approximate distribution of δ using the first few moments obtained under the exact distribution. For this purpose Mielke et al. (1976) have defined 12 symmetric functions which give the first three exact moments of δ. These symmetric functions are listed in Table 2.1.

Mielke (1979b, p. 1542) has revised the MRPP test statistic (1.1). Following the same notation, the revised version of δ is

$$\delta = 2 \sum_{i=1}^{g} [K(n_i-1)]^{-1} \sum_{I<J} \Delta_{IJ} S_i(I) S_i(J) \tag{1.2}$$

Versions (1.1) and (1.2) are special cases of the latest version of the MRPP test statistic (Mielke et al. 1981a, 1981b) defined as follows:

Consider the set of observations $\{X_1,\ldots,X_N\}$, which are classified into g disjoint subgroups, not necessarily exhaustive. The i^{th} subgroup, carries $n_i \geq 2$ observations. The left-over observations $(N-\sum_{i=1}^{g} n_i = N-K = n_{g+1} \geq 0)$ belong to the $(g+1)^{th}$ subgroup. Then the MRPP test statistic δ is

$$\delta = \sum_{i=1}^{g} c_i \xi_i , \tag{1.3}$$

where $c_i > o$ are some weights with $\sum_{i=1}^{g} c_i = 1$ and

$\xi_i = \binom{n_i}{2}^{-1} \sum_{I<J} \Delta_{IJ} \; S_i(I) \; S_i(J)$ is an average distance measure within the i^{th} subgroup.

The distance measure Δ_{IJ} could be, for example,

$$\Delta_{IJ} = ||X_I - X_J||^{\nu}, \; \nu > o \tag{1.4}$$

We note that $c_i = \dfrac{\binom{n_i}{2}}{\sum_{i=1}^{g} \binom{n_i}{2}}$ in (1.3) yields (1.1),

while $c_i = \dfrac{n_i}{\sum_{i=1}^{g} n_i} = \dfrac{n_i}{K}$ in (1.3) gives (1.2).

When the subgroups are all equal in size, both choices of c_i $(=\frac{1}{g})$ above give the same δ. However, when the n_i's are not all equal, the first choice is an inefficient choice for detecting location shifts when K=N (Mielke et al. 1981a). Different choices of the $\{c_i\}$ and Δ, in general, give different MRPP test statistics.

1.5 Applications

The MRPP tests have a wide range of applications, because the assumptions underlying these tests are minimal. The tests require that data be at least at ordinal level and, for more than one response from individuals, each of the response measurements should be commensurate with each other. An MRPP test examines whether a classification scheme indicates some difference among the subgroups or not. Suppose that N individuals in a population are asked about their level of education, length of service after completing the education and current income. These N individuals can be classified according to sex or by colour or race or by any scheme of interest. Say they are classified according to the sex of an individual. Then the MRPP test δ tests whether or not there is a difference in socio-economic status of males and females, assuming that the above three responses are indicators of socio-economic status. This illustrates the usefulness of the MRPP test for data commonly encountered by social scientists. Mielke et al. (1976, pp. 1416-1418) have demonstrated applications with actual social science data. Mielke et al. (1981a, 1982) have also shown applications in atmospheric sciences.

There has been an attempt to equate the ordinal variable with an interval variable in order to utilize interval level statistics, especially in multiple regression and ANOVA, e.g., Labovitz (1967). Alternatively, Hawkes (1971) and Somers (1968) have tried to develop multivariate analysis for ordinal data. The MRPP test gives a better approach to resolving some of the problems associated with these data.

When all observations are classified into g subgroups, the classification problem can be related to the g-sample problem. Under H_o, classification is random and one may look upon the g samples as coming from the same parent population. Mielke, Berry, Brockwell and Williams (1981b) consider the univariate case, i.e., $r=1$ and $N=K$. In case the distance measure is the square of the Euclidean distance between ranks of observations and the weights are proportional to the number of observations in a subgroup, the MRPP-test, called δ_2 here, is equivalent to the Wilcoxon test for $g=2$ and is asymptotically equivalent to the Kruskal Wallis test for $g>2$. When the distance measure is simply the Euclidean distance between the ranks of observations, the MRPP test, called δ_1, performs better than δ_2-

the Wilcoxon test - for some underlying distribution (see results, Chapter IV). Mielke and Berry (1982) have demonstrated that two MRPP tests jointly perform better than the sign test or the Wilcoxon signed-ranks test in the case of paired observations for the following underlying populations - Normal, Uniform, Laplace, Logistic and U-shaped. Mielke and Iyer (1982) have shown that the permutation version of the classical univariate technique and the Friedman test for randomized blocks with Pearson's and Spearman's correlation measures are special cases of MRPP tests. Still and White (1981) have emphasized the use of randomized tests in psychological data. Berry and Mielke (1983) have shown the suitability of MRPP tests in ANOVA.

We find applications of MRPP tests in different disciplines where the classical tests are not robust against violation of assumptions or where the data is on an ordinal scale.

1.6 Asymptotic Behavior of δ

Mielke et al. (1976, p. 1421) give simulated values of skewness for different size configurations and different proportions p of unclassified observations such that $n_{g+1}=Np$, $o \le p < 1$. The results indicate that for p=o, the distribution of δ is highly negatively skewed and skewness increases with increase in N for the same value of g.

When p=o, the MRPP test statistic is a special case of Mantel and Valand's (1970) test statistic T. We note that T and δ are conceptually the same, both being based on distance measures between pairs of observations. However, Mantel and Valand (1970) used a normal approximation to carry out the test, which is shown to be incorrect and thus any inference drawn is invalid (Mielke 1978). Because of the skewness that remains in the asymptotic case also, an approximation based on at least 3 moments is recommended. O'Reilly and Mielke (1980) discuss the situation of asymptotic normality of the distribution of δ and recommend an approximation by the Beta Type I distribution in the absence of normality of δ. Brockwell et al. (1982) consider the univariate case of the MRPP test with $\Delta_{xy}=|x-y|^{\nu}$, $\nu>0$. They establish that for $\nu=2$, the asymptotic distribution of δ is the chi-square distribution, while for $\nu\neq2$, it is dependent on the underlying distribution of observations; i.e., the asymptotic distribution is non-invariant.

Mielke et al. (1976), Mielke (1978, 1979 a, b), O'Reilly and Mielke (1980) have recommended the use of the Beta Type I for the approximate distribution of δ. In using this approximation with three moments, one has to choose arbitrarily a value of one of the parameters of the Beta density which is determined by four moments.

However, Mielke et al. (1981a, b and later) have confined themselves to the use of the Pearson Type III distribution as an approximation to the null distribution, since this also accounts for skewness and it is determined by three moments. In this dissertation, we obtain the fourth moment of δ and choose an appropriate approximation from among the Pearson family of distributions.

In the next section, we discuss the Pearson family of distributions.

1. 7 The Pearson Family of Distributions

The Pearson family of curves is defined by the differential equation,

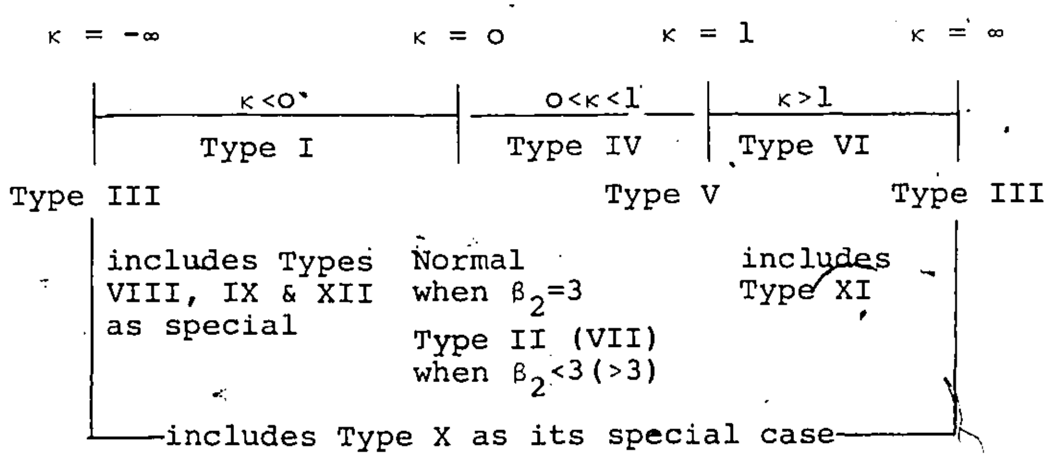
$$\frac{dy}{dx} = \frac{y(x+a)}{b_o+b_1x+b_2x^2}, \tag{1.5}$$

where $y = f(x)$ is a probability function or a frequency function. The above equation is a limiting case of the hypergeometric series. Elderton and Johnson (1969) give a detailed exposition of fitting of Pearson curves. We present here only the points pertaining to our study.

Pearson curves are divided into 12 types - 3 are main types: Type I, Type IV and Type VI, depending on the roots of $b_o+b_1x + b_2x^2 = o$. The other 9 are called "transition types". These are either limiting forms of the main types or are their special cases. Fitting of a particular type to given data is decided by either looking at a (β_1, β_2) plot or considering the Pearson's criterion κ as given below:

$$\kappa = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)} \qquad (1.6)$$

Classification on the basis of κ is shown schematically in the following



The major advantage of the use of the Pearson system of curves is the existence of tables of their percentiles, indexed as functions of β_1 and β_2 (Shapiro and Gross 1981, p. 220).

1.8 Scope of Our Study

In section 1.5, we discussed applications of MRPP tests. However, in section 1.6, we noticed that the asymptotic distribution of δ is, in general, non-normal and non-invariant. Mielke et al. (1976) have used an approximation to the null distribution of δ by a Beta Type I distribution. They used the first three exact moments with an arbitrary value of one of the parameters of the Beta density. The complete determination of a Beta density requires the first four moments. Mielke (1979b) and O'Reilly and Mielke (1980) have recommended its use in order to account for the high value of skewness associated with the null distribution of δ. However, since 1981, Mielke et al. have adopted the use of the Pearson Type III distribution which is determined by using the first three moments of the test statistic. MRPP tests generally cover a wide range of situations and it is not known whether or not the Pearson Type III approximation will be good in every situation. This leads us to calculate the exact fourth moment of δ, with the motivation that the use of the fourth moment will give a better approximation to the null distribution of δ.

The aim of Chapter II is to derive the general expression for the fourth moment of δ, defined by (1.3).

This derivation involves heavy, cumbersome algebra. Therefore, to reduce the labour, we adopt a simple notation in section 2.2 If we look at (1.3), it can be noted that δ^4 will involve sums of the distance functions of the form $\Delta_{I_1J_1}\Delta_{I_2J_2}\Delta_{I_3J_3}\Delta_{I_4J_4}$ associated with the product of indicator functions. The expectations of the product of indicator functions depend on whether $I_k=J_\ell$ or $I_k=I_\ell$ or $J_k=J_\ell (k\neq\ell)$, and whether the observations are in the same subgroup or in different subgroups. This situation is simplified by considering two types of sums:

(i) Sums with no restriction on the indices of the Δ's

(ii) Sums with all the indices of the Δ's being different.

We obtain relationships between these two types of sums in section 2.4. In section 1.4, we indicated that Mielke et al. (1976) required 12 symmetric functions in order to express the first three moments of δ in terms of these functions. We need an additional 23 symmetric functions, which will be defined in section 2.3. These symmetric functions and earlier ones from (Mielke et al. 1976) are expressed in terms of a second type of sum. The relationships between the two types of sums, which we obtain in section 2.4, can also be seen by the careful use of a combinatorial method that we describe in section 2.5. The combinatorial method serves as a check of some of the computations we make in section 2.4. Finally, in the last section of Chapter II, we obtain an expression of the fourth moment in terms of the symmetric functions defined. We partially check the result of the fourth moment through an independent computer program which calculates every possible value of δ for small sets of data.

In Chapter III, we obtain the fourth moment of δ_1, in the case of $n_1=n_2=N/2$. Since the distance measure is the Euclidean distance between the ranks of observations, this involves some further algebra in the simplification of the 23 symmetric functions involved in the expression of the fourth moment. We show these calculations in section 3.2. The symmetric functions in the case of δ_1 are simple polynomials in N. The long expression of the fourth moment of δ, for the case of δ_1, is simplified to a simple polynomial in N. The last section of this chapter indicates that for small samples, i.e., $N \le 34$, Pearson Type I approximation will be suitable, while for large samples, i.e., $N > 34$, the Pearson Type VI seems to be appropriate.

In Chapter IV, we carry out a simulation study, on the basis of 10,000 independent samples, to obtain powers of δ_1 and δ_2, against various shifts in locations and for different underlying populations. Empirical powers of δ_1 are obtained using the Pearson Type suggested in Chapter III, as well as the Pearson Type III distribution.

Finally, in the last chapter we discuss the scope for further work. In this dissertation, powers of δ_1 and δ_2 are calculated for the case of two equal samples. The cases of unequal samples and g (>2) samples, both with equal and unequal sizes, are proposed for future study. We also discuss applications of MRPP tests in one-way analysis of variance where the use of four moments yields better results than when only three moments are used.

CHAPTER II

FOURTH MOMENT OF MRPP STATISTIC

2.1 Nature of Computations

The MRPP statistic δ defined by (1.3) is written as

$$\delta = \sum_{i=1}^{g} K_i \Sigma \Delta_{J_1 J_2} S_i(J_1) S_i(J_2), \qquad (2.1)$$

where the second summation extends over J_1 and J_2, each running from 1 to N and $K_i = c_i/[n_i(n_i-1)]$, $i=1,\ldots,g$, the c_i's being weights. The indices J_1 and J_2 need not be distinct, since $\Delta_{J_1 J_2} = 0$ for $J_1=J_2$.

From (2.1),

$$\delta^4 = \sum_{i_1,i_2,i_3,i_4} \Sigma \prod_{\ell=1}^{4} \{K_{i_\ell} \Delta_{J_{2\ell-1} J_{2\ell}} S_{i_\ell}(J_{2\ell-1}) S_{i_\ell}(J_{2\ell})\}, \qquad (2.2)$$

where the inner summation extends over $J_1,\ldots,J_8$, each running from 1 to N. The indices i_1, i_2, i_3, i_4 run from 1 to g.

The expectation of δ^4 involves the expectations of

$\prod_{\ell=1}^{4}\{S_{i_\ell}(J_{2\ell-1}) S_{i_\ell}(J_{2\ell})\}$ for various choices of i_ℓ,

$\ell=1,\ldots,4$. To simplify the sum of these expectations, the sums over unrestricted values of indices are expressed in terms of the sums over distinct values. It can be seen from (2.1) or (2.2) that for $i_1=i_2=i_3=i_4$, one of the terms in the expansion of δ^4 is

$$\sum_i K_i^4 \left(\sum_{J_1 J_2} \Delta_{J_1 J_2}\right)^4 \prod_{\ell=1}^{8} S_i(J_\ell).$$

Before the expectation of the above term can be taken, the factor $\left(\sum_{J_1 J_2} \Delta_{J_1 J_2}\right)^4$ requires to be expanded. This expansion involves 23 different types of distance function, defined in section 2.3. Most of the terms in the expansion carry coefficients which are not easily obtained by combinatorial methods. Actual enumeration of possible ways of arriving at a specific term is needed. Another approach to expanding $\left(\sum_{J_1 J_2} \Delta_{J_1 J_2}\right)^4$ is to first write it as $\sum \Delta_{J_1 J_2} \Delta_{J_3 J_4} \Delta_{J_5 J_6} \Delta_{J_7 J_8}$ with sum over $J_1,\ldots,J_8$, running from 1 to N. Then the sum is split in a systematic manner, into two sums - one with two indices taking on the same values, and the other with two different values of indices. Since, in the beginning, it is difficult to know with certainty how many different terms are in the expansion, the latter cumbersome approach is used. The combinatorial method is employed to check some of the results of the massive calculations.

To reduce the use of multi-level symbols and for ease of computation, the following notation is adapted.

2.2 Notation

(i) A simple summation sign "Σ" means that the sum is taken over every index that is involved in the summand as a subscript of Δ. Each index runs from 1 to N.

(ii) A summation sign with "$\neq$" beneath the summation "$\sum_{\neq}$" implies that the sum is taken over indices taking on distinct values from 1 to N.

(iii) An index J_i is indicated by i, e.g., $\Sigma\Delta_{J_1J_2}$ is indicated by $\Sigma\Delta_{12}$. This convention is used only with summation signs.

(iv) Let $J_1,J_2,...,J_K$ be the indices of a sum. Suppose $\ell<K$ of the indices take on values distinct from all the others. Then without any loss it can be assumed that an index $J_i(i=1,2,...,\ell)$ takes on a value 1 to N distinct from every index $J_j(j\neq i)$.

(a) A sum with indices $J_1,J_2,...,J_K$ taking on values 1 to N, with the restriction that indices $J_i,i=1,2,...,\ell$, take on distinct value from every other index $J_j(j\neq i)$, is denoted by $\sum_{\ell\neq}..$.

(b) A sum with indices $J_1,\ldots,J_K$ taking on values 1 to N with the restriction that an index $J_i, i=1,\ldots,\ell-1$, takes a distinct value from every index $J_j (j \neq i)$ and the index J_ℓ takes on a distinct value from Index J_1 to J_S, is denoted by ${}_{\ell}\underset{\neq .S}{\Sigma}$. Note that when S=K then ${}_{\ell}\underset{\neq .S}{\Sigma}$ is ${}_{\ell}\underset{\neq ..}{\Sigma}$.

2.3 Symmetric Functions Arising in Computing the Fourth Moment

As pointed out in section 2.1, there are 23 different types of distance function. These are symmetric functions needed to extend Mielke's (1976, p. 1412) model, which are necessary to obtain the fourth moment of an MRPP statistic. For the sake of completeness, the 12 symmetric functions defined by Mielke are first presented both in Mielke's notation and then the notation of section 2.2. This is then followed by a table of additional symmetric functions required in computing the fourth moment of an MRPP test statistic.

Table 2.1

The Twelve Symmetric Functions as Defined by Mielke

Notation of Sec. 2.2	Mielke's Notation
$D(1) = \frac{1}{N^{(2)}} \sum_{\neq} \Delta_{12}$	$D(1) = \frac{1}{N^{(2)}} \sum_{J_1 J_2} \Delta_{J_1 J_2}$
$D(2) = \frac{1}{N^{(2)}} \sum_{\neq} \Delta^2_{12}$	$D(2) = \frac{1}{N^{(2)}} \sum_{J_1 J_2} \Delta^2_{J_1 J_2}$
$D(3) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12} \Delta_{13}$	$D(2') = \frac{1}{N^{(3)}} \sum_{J_1 J_2 J_3} \Delta_{J_1 J_2} \Delta_{J_1 J_3}$
$D(4) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12} \Delta_{34}$	$D(2'') = \frac{1}{N^{(4)}} \sum_{J_1 J_2 J_3 J_4} \Delta_{J_1 J_2} \Delta_{J_3 J_4}$
$D(5) = \frac{1}{N^{(2)}} \sum_{\neq} \Delta^3_{12}$	$D(3) = \frac{1}{N^{(2)}} \sum_{J_1 J_2} \Delta^3_{J_1 J_2}$
$D(6) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta^2_{12} \Delta_{13}$	$D(3') = \frac{1}{N^{(3)}} \sum_{J_1 J_2 J_3} \Delta^2_{J_1 J_2} \Delta_{J_1 J_3}$
$D(7) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{23}$	$D(3^*) = \frac{1}{N^{(3)}} \sum_{J_1 J_2 J_3} \Delta_{J_1 J_2} \Delta_{J_1 J_3} \Delta_{J_2 J_3}$
$D(8) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta^2_{12} \Delta_{34}$	$D(3'') = \frac{1}{N^{(4)}} \sum_{J_1 J_2 J_3 J_4} \Delta^2_{J_1 J_2} \Delta_{J_3 J_4}$
$D(9) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{24}$	$D(3^{**}) = \frac{1}{N^{(4)}} \sum_{J_1 J_2 J_3 J_4} \Delta_{J_1 J_2} \Delta_{J_1 J_3} \Delta_{J_2 J_4}$
$D(10) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{14}$	$D(3^{***}) = \frac{1}{N^{(4)}} \sum_{J_1 J_2 J_3 J_4} \Delta_{J_1 J_2} \Delta_{J_1 J_3} \Delta_{J_1 J_4}$
$D(11) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{45}$	$D(3''') = \frac{1}{N^{(5)}} \sum_{J_1 J_2 J_3 J_4 J_5} \Delta_{J_1 J_2} \Delta_{J_1 J_3} \Delta_{J_4 J_5}$
$D(12) = \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12} \Delta_{34} \Delta_{56}$	$(D3'''') = \frac{1}{N^{(6)}} \sum_{J_1 J_2 J_3 J_4 J_5 J_6} \Delta_{J_1 J_2} \Delta_{J_3 J_4} \Delta_{J_5 J_6}$,

where J_i $(i=1,\ldots 6)$ takes on distinct values from 1 to N

Table 2.2

List of 23 Symmetric Functions Required in Expressing the Fourth Moment of MRPP Statistic

$D(13) = \frac{1}{N^{(2)}} \sum_{\neq} \Delta_{12}^{4}$

$D(14) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{3}\Delta_{13}$

$D(15) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}^{2}$

$D(16) = \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}\Delta_{23}$

$D(17) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{3}\Delta_{34}$

$(D18) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{34}^{2}$

$D(19) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}\Delta_{14}$

$D(20) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}\Delta_{24}$

$D(21) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}\Delta_{34}$

$D(22) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}$

$D(23) = \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$

$D(24) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{34}\Delta_{35}$

$D(25) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{13}\Delta_{45}$

$D(26) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$

$D(27) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{34}\Delta_{45}$

$D(28) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$

$D(29) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$

$D(30) = \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12}^{2}\Delta_{34}\Delta_{56}$

$D(31) = \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}$

$D(32) = \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46}$

$D(33) = \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56}$

$D(34) = \frac{1}{N^{(7)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$

and

$D(35) = \frac{1}{N^{(8)}} \sum_{\neq} \Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78}$

As discussed in section 2.1, to evaluate the expectation of (2.2) the sums over unrestricted indices are to be expressed in terms of sums with distinct values of indices. This leads to the following section.

2.4 Sums Over All Indices Running From 1 to N in Terms of Sums Over Indices Taking on Distinct Values

In this section, the relationship between Σ and $\sum_{\neq}$ for various summands is given. The calculations involved are of varied difficulty. Some are straightforward, while some require a systematic approach.

This section is divided into two sub-sections. In the first sub-section general results are obtained and the relationships of interest are then deduced from them. The latter section involves some tedious calculations. Therefore results of interest are obtained directly.

2.4.1 Relationships Between Σ and $\sum_{\neq}$ When the Summand is a Product of At Most Three Terms

Let $\alpha,\ \beta,\ \gamma \in S=\{1,2,\ldots\}$ - the set of natural numbers. Then

G1: $\forall\ \alpha \in S \qquad \Sigma\Delta_{12}^{\alpha} = \sum_{\neq}\Delta_{12}^{\alpha}.$

G2: $\forall\ \alpha, \beta \in S \qquad \Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta} = \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}.$

G3: $\forall\ \alpha, \beta \in S \qquad \Sigma\Delta_{12}^{\alpha}\Delta_{34}^{\beta} = \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{34}^{\beta} + 2\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}$

$$= \sum_{\neq}\Delta_{12}^{\beta}\Delta_{34} + 4\sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta} + 2\sum_{\neq}\Delta_{12}^{\alpha+\beta}.$$

G4: $\forall\ \alpha,\ \beta,\ \gamma \in S,\ \Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{23}^{\gamma} = \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{23}^{\gamma}.$

G5: $\forall\ \alpha,\ \beta,\ \gamma \in S,$

$$\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma} = {}_{2}\sum_{\neq 3}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma} + \Sigma\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma}$$

$$= {}_{2}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\gamma}\Delta_{13}^{\beta}$$

$$+ \sum_{\neq}\Delta_{12}^{\beta+\gamma}\Delta_{13}^{\alpha} + \sum_{\neq}\Delta_{12}^{\alpha+\beta+\gamma}$$

G6: $\forall\ \alpha,\ \beta,\ \gamma \in S,$

$$\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma} = {}_{1}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma} + \Sigma\Delta_{12}^{\alpha+\gamma}\Delta_{13}^{\beta}$$

$$= {}_{2}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma} + \Sigma\Delta_{12}^{\alpha+\gamma}\Delta_{13}^{\beta}$$

$$= \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{23}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma}$$

$$+ \sum_{\neq}\Delta_{12}^{\alpha+\gamma}\Delta_{13}^{\beta} + \sum_{\neq}\Delta_{12}^{\alpha+\beta+\gamma}.$$

G7: $\forall\ \alpha,\ \beta,\ \gamma \in S,$

$$\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{45}^{\gamma} = {}_{1}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{45}^{\gamma} + 2\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma}$$

$$= {}_{2}\sum_{\neq 3}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{45}^{\gamma} + {}_{1}\sum_{\neq ..}\Delta_{12}^{\alpha+\beta}\Delta_{34}^{\gamma} + 2\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma}$$

$$= {}_{2}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{45}^{\gamma} + 2\cdot{}_{2}\sum_{\neq ..}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{34}^{\gamma}$$

$$+ 2\sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma} + 2\Sigma\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma}.$$

$$= \sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{45}^{\gamma} + 2\sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{34}^{\gamma} + 2\sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{24}^{\gamma}$$

$$+ 2\sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{14}^{\gamma} + 2\sum_{\neq}\Delta_{12}^{\alpha}\Delta_{13}^{\beta}\Delta_{23}^{\gamma} + \sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{34}^{\gamma}$$

$$+ 4\sum_{\neq}\Delta_{12}^{\alpha+\beta}\Delta_{13}^{\gamma} + 2\sum_{\neq}\Delta_{12}^{\alpha+\gamma}\Delta_{13}^{\beta} + 2\sum_{\neq}\Delta_{12}^{\beta+\gamma}\Delta_{13}^{\alpha}$$

$$+ 2\sum_{\neq}\Delta_{12}^{\alpha+\beta+\gamma} \quad \text{(Using G5).}$$

G8: $\forall\ p \in S$,

$$\Sigma\Delta_{12}^{p}\Delta_{34}\Delta_{56} = {}_{1}\sum_{\neq .4}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 2\Sigma\Delta_{12}^{p}\Delta_{13}\Delta_{45}$$

$$= {}_{1}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 4\,{}_{1}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{13}\Delta_{45} + 4\Sigma\Delta_{12}^{p}\Delta_{13}\Delta_{14}$$

$$= {}_{2}\sum_{\neq .4}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 2\,{}_{1}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{23}\Delta_{45} + 4\,{}_{2}\sum_{\neq 3}\Delta_{12}^{p}\Delta_{13}\Delta_{45}$$

$$+ 4\,{}_{1}\sum_{\neq ..}\Delta_{12}^{p+1}\Delta_{34} + 4\Sigma\Delta_{12}^{p}\Delta_{13}\Delta_{14}$$

$$= {}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 4\,{}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{23}\Delta_{45} + 4\,{}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{23}\Delta_{24}$$

$$+ 4\,{}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{13}\Delta_{45} + 8\,{}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{13}\Delta_{24} + 4\sum_{\neq}\Delta_{12}^{p+1}\Delta_{34}$$

$$+ 8\sum_{\neq}\Delta_{12}^{p+1}\Delta_{13} + 4\Sigma\Delta_{12}^{p}\Delta_{13}\Delta_{14}$$

$$= {}_{3}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 2\,{}_{2}\sum_{\neq ..}\Delta_{12}^{p}\Delta_{34}\Delta_{35} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{45}$$

$$+ 16\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{34} + 4\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^{p}\Delta_{13}^{2}$$

$$+ 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{24} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{23} + 4\sum_{\neq}\Delta_{12}^{p+1}\Delta_{34}$$

$$+ 8\sum_{\neq}\Delta_{12}^{p+1}\Delta_{13} + 4\Sigma\Delta_{12}^{p}\Delta_{13}\Delta_{14}$$

$$= \sum_{\neq}\Delta_{12}^{p}\Delta_{34}\Delta_{56} + 4\sum_{\neq}\Delta_{12}^{p}\Delta_{34}\Delta_{35} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{45}$$

$$+ 16\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{34} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{24} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{14}$$

$$+ 2\sum_{\neq}\Delta_{12}^{p}\Delta_{34}^{2} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}\Delta_{23} + 4\sum_{\neq}\Delta_{12}^{p+1}\Delta_{34}$$

$$+ 16\sum_{\neq}\Delta_{12}^{p+1}\Delta_{13} + 8\sum_{\neq}\Delta_{12}^{p}\Delta_{13}^{2} + 4\sum\Delta_{12}^{p+2} \quad \text{(Using G5).}$$

The relationships given by G1 through G4 are trivial. There is however, a need to deduce the following specific results from G5 through G8.

$$\sum\Delta_{12}\Delta_{13}\Delta_{14} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14} + 3\sum_{\neq}\Delta_{12}^{2}\Delta_{13} + \sum\Delta_{12}^{3}. \qquad (2.3)$$

$$\sum\Delta_{12}^{2}\Delta_{13}\Delta_{14} = \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 2\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + \sum_{\neq}\Delta_{12}^{4}. \qquad (2.4)$$

$$\sum\Delta_{12}\Delta_{13}\Delta_{24} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24} + \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13} + \sum_{\neq}\Delta_{12}^{3}. \qquad (2.5)$$

$$\sum\Delta_{12}^{2}\Delta_{13}\Delta_{24} = \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + \sum_{\neq}\Delta_{12}^{4}. \qquad (2.6)$$

$$\sum\Delta_{12}^{2}\Delta_{13}\Delta_{34} = \sum\Delta_{12}\Delta_{13}^{2}\Delta_{24}$$

$$= \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + \sum_{\neq}\Delta_{12}^{3}\Delta_{13} + \sum_{\neq}\Delta_{12}^{4}. \qquad (2.7)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{45} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^2\Delta_{34} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13} + 2\sum_{\neq}\Delta_{12}^3. \quad (2.8)$$

$$\Sigma\Delta_{12}^2\Delta_{13}\Delta_{45} = \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{45} + 2(\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23}) + \sum_{\neq}\Delta_{12}^3\Delta_{34} + 6\sum_{\neq}\Delta_{12}^3\Delta_{13} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 2\sum_{\neq}\Delta_{12}^4. \quad (2.9)$$

$$\Sigma\Delta_{12}^2\Delta_{34}\Delta_{35} = \Sigma\Delta_{12}\Delta_{13}\Delta_{45}^2$$

$$= \sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{35} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^2\Delta_{34}^2 + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 4\sum_{\neq}\Delta_{12}^3\Delta_{13} + 2\sum_{\neq}\Delta_{12}^4. \quad (2.10)$$

$$\Sigma\Delta_{12}\Delta_{34}\Delta_{56} = \sum_{\neq}\Delta_{12}\Delta_{34}\Delta_{56} + 12\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45} + 24\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24} + 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14} + 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23} + 6\sum_{\neq}\Delta_{12}^2\Delta_{34} + 24\sum_{\neq}\Delta_{12}^2\Delta_{13} + 4\sum_{\neq}\Delta_{12}^3. \quad (2.11)$$

$$\Sigma\Delta_{12}^2\Delta_{34}\Delta_{56} = \sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{56} + 4\sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{35} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{45} + 16\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^2\Delta_{34}^2 + 4\sum_{\neq}\Delta_{12}^3\Delta_{34}$$

$$+8\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 16\sum_{\neq}\Delta_{12}^3\Delta_{13} + 4\sum_{\neq}\Delta_{12}^4 \ . \qquad (2.12)$$

2.4.2 Relationships Between Σ and $\sum_{\neq}$ When the Summand is a Product of Four Terms

In this subsection, the relationships between Σ and $\sum_{\neq}$ involve some tedious computations. The computational work is reduced considerably by employing the following rules of switching the indices.

Rules of Switching Indices

Let a sum be extended over a set of indices $S = \{i_1, i_2, \ldots, i_k\}$.

Rule 1: Let $S_1 \subseteq S$, so that the indices of S_1 take on values independent of each other. Then any two indices of S_1 can be switched without altering the conditions over which the sum is extended. E.g.:

$$1\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} = 1\sum_{\neq}..\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{17},$$

while

$$1\sum_{\neq}..\Delta_{12}\Delta_{23}\Delta_{45}\Delta_{67} \neq 1\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}.$$

Here $S_1 = \{2,3,4,5,6,7\}$ or $\{1\}$ or any of the subsets of $\{2,3,4,5,6,7\}$. In the first case index 3 is switched with index 7 and index 4 is switched with index 6 while in the second case index 1 is switched with index 2, which together do not belong to S_1.

Rule 2: Let $S_2 \subseteq S$, such that for any index $k \in S_2$ and any $\ell \in S_2$ $\ell \neq k$, k and ℓ take on distinct values. Then any two indices of S_2 can be switched.

e.g.:

$$2\sum_{\neq}\cdot\cdot\ \Delta_{12}\Delta_{23}\Delta_{45}\Delta_{67} = 2\sum_{\neq}\cdot\cdot\ \Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}.$$

Using the above rules, when required, the following results are obtained:

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + \sum_{\neq}\Delta_{12}^4. \qquad (2.13)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} = 2\sum_{\neq}.4\ 4\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + {}^{2}2\sum_{\neq}3\Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ \Sigma\Delta_{12}^3\Delta_{13}$$

$$= 2\sum_{\neq}\cdot\cdot\ \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + {}^{3}2\sum_{\neq}\cdot\cdot\ \Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ 3\sum_{\neq}\Delta_{12}^3\Delta_{13} + \sum_{\neq}\Delta_{12}^4$$

$$= 3\sum_{\neq}\cdot\cdot\ \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + 5\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^3\Delta_{13}$$

$$+ 3\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + \sum_{\neq}\Delta_{12}^4$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + 6\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^3\Delta_{13}$$

$$+ 3\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + \sum_{\neq}\Delta_{12}^4. \qquad (2.14)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} + 2\Sigma\Delta_{12}^2\Delta_{13}\Delta_{23}. \qquad (2.15)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} = {}_{1}\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 2\Sigma\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$= {}_{2}\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 2\,{}_{1}\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{24}$$

$$+ 2\Sigma\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 6\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}. \qquad (2.16)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} = {}_{2}\sum_{\neq}{}_{3}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + {}_{1}\sum_{\neq}..\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+\Sigma\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$= {}_{2}\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\,{}_{2}\sum_{\neq}..\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+ \sum_{\neq}\Delta_{12}^{3}\Delta_{13} + \Sigma\Delta_{12}^{2}\Delta_{13}\Delta_{14}.$$

$$= {}_{3}\sum_{\neq}{}_{4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + {}_{2}\sum_{\neq}..\Delta_{12}\Delta_{13}^{2}\Delta_{24}$$

$$+ 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^{3}\Delta_{13}$$

$$+ \Sigma\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$= {}_{3}\sum_{\neq}..\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + 3\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+ \sum_{\neq}\Delta_{12}^{3}\Delta_{13} + \Sigma\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ 3\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + 3\sum_{\neq}\Delta_{12}^3\Delta_{13} + \sum_{\neq}\Delta_{12}^2\Delta_{13}^2$$

$$+ \Sigma\Delta_{12}^4 \quad \text{(Using (2.4))}. \qquad (2.17)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} = \sum_{1\neq\cdots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + \sum_{1\neq\cdots}\Delta_{12}\Delta_{13}^2\Delta_{24}$$

$$+ \Sigma\Delta_{12}^2\Delta_{13}\Delta_{34}$$

$$= \sum_{2\neq 3}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + \sum_{1\neq\cdots}\Delta_{12}^2\Delta_{23}\Delta_{24}$$

$$+ \sum_{2\neq\cdots}\Delta_{12}\Delta_{13}^2\Delta_{24} + \sum_{1\neq\cdots}\Delta_{12}^3\Delta_{23} + \Sigma_{12}^2\Delta_{13}\Delta_{34}$$

$$= \sum_{2\neq\cdots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + \sum_{2\neq\cdots}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ \sum_{\neq}\Delta_{12}\Delta_{13}^2\Delta_{14} + \sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34}$$

$$+ \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^3\Delta_{13} + \Sigma\Delta_{12}^2\Delta_{13}\Delta_{34}$$

(Using Rule 2).

$$= \sum_{3\neq\cdots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + \sum_{\neq}\Delta_{12}^2\Delta_{13}^2$$

$$+ \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + \sum_{\neq}\Delta_{12}^3\Delta_{13} + \Sigma\Delta_{12}^2\Delta_{13}\Delta_{34}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}$$

$$+ 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 3\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}^2$$

$$+ 2\sum_{\neq}\Delta_{12}^3\Delta_{13} + \sum_{\neq}\Delta_{12}^4. \quad \text{(Using 2.7)} \qquad (2.18)$$

$$\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} = \sum_{1\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 2\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \sum_{2\neq .4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 2\sum_{2\neq 3}\Delta_{12}^2\Delta_{13}\Delta_{45}$$

$$+ \sum_{1\neq ..}\Delta_{12}^3\Delta_{34} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ 2\sum_{2\neq ..}\Delta_{12}^2\Delta_{13}\Delta_{45} + 4\sum_{2\neq ..}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+ \sum_{\neq}\Delta_{12}^3\Delta_{34} + 2\sum_{\neq}\Delta_{12}^3\Delta_{13} + 2\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}^2\Delta_{45}$$

$$+ 2\sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}^2\Delta_{24}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{45} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+ 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^3\Delta_{34} + 2\sum_{\neq}\Delta_{12}^3\Delta_{13}$$

$$+ 2\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 6\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + 6\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 3\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}$$

$$+ 12\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^{3}\Delta_{34}$$

$$+ 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 10\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 2\sum_{\neq}\Delta_{12}^{4}.$$

(Using 2.14)). (2.19)

$$\sum\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} = {}_1\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + 2\,{}_1\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ \sum\Delta_{12}^{2}\Delta_{13}\Delta_{45}$$

$$= {}_2\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + 2\,{}_2\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25}$$

$$+ {}_2\sum_{\neq\cdot\cdot}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 2\,{}_2\sum_{\neq\cdot\cdot}\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$+ 2\,{}_2\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 4\,{}_2\sum_{\neq\cdot\cdot}\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+ 2\sum_{\neq}\Delta_{12}\Delta_{13} + \sum\Delta_{12}^{2}\Delta_{13}\Delta_{45}$$

$$= {}_3\sum_{\neq 4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + {}_2\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$$

$$+ 2\,{}_3\sum_{\neq 4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25} + 2\,{}_2\sum_{\neq\cdot\cdot}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{24}$$

$$+ \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$+ 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 2\,{}_3\sum_{\neq 4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ 2\,{}_2\sum_{\neq}\ldots \Delta_{12}\Delta_{13}^2\Delta_{24} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23}$$

$$+ 2\sum_{\neq}\Delta_{12}^3\Delta_{13} + \Sigma\Delta_{12}^2\Delta_{13}\Delta_{45}$$

Using the result of (2.9), on simplification

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35}$$

$$+ \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34} + 10\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{45} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 6\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+ 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 10\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^3\Delta_{34}$$

$$+ 4\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 8\sum_{\neq}\Delta_{12}^3\Delta_{13} + 2\sum_{\neq}\Delta_{12}^4. \qquad (2.20)$$

$$\Sigma\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} = {}_1\sum_{\neq}\ldots\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + 2\,{}_1\sum_{\neq}\ldots\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ \Sigma\Delta_{12}^2\Delta_{13}\Delta_{14} + \Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= {}_2\sum_{\neq}{}_3\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + {}_1\sum_{\neq}\ldots\Delta_{12}^2\Delta_{34}\Delta_{35}$$

$$+ 2\,{}_2\sum_{\neq}{}_3\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\,{}_1\sum_{\neq}\ldots\Delta_{12}^2\Delta_{13}\Delta_{24} + A,$$

where $A = \Sigma\Delta_{12}^2\Delta_{13}\Delta_{14} + \Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$.

$$= {}_2\sum_{\neq}{}_{.4}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + {}_2\sum_{\neq}{}_3\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25}$$

$$+ \sum_{2\neq 3}\Delta_{12}^2\Delta_{34}\Delta_{35} + \sum_{2\neq ..}\Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ 2\sum_{2\neq .4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 4\sum_{2\neq ..}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+ 2\sum_{\neq}\Delta_{12}^3\Delta_{13} + A$$

$$= \sum_{2\neq .5}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + \sum_{2\neq .4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45}$$

$$+ \sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25} + \sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{24}$$

$$+ 2\sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}^2\Delta_{24}$$

$$+ \sum_{2\neq .4}\Delta_{12}^2\Delta_{34}\Delta_{35} + \sum_{2\neq 3}\Delta_{12}^2\Delta_{23}\Delta_{34}$$

$$+ \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + \sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24}$$

$$+ 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^3\Delta_{13} + A$$

$$= \sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45}$$

$$+ \sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}^2 + \sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25}$$

$$+ 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{23} + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23}$$

$$+ 2\sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{23}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + \sum_{2\neq .5}\Delta_{12}^2\Delta_{34}\Delta_{35}$$

$$+ 2\sum_{\neq}\Delta_{12}^2\Delta_{23}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + \sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14}$$

$$+ 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + A$$

$$= \sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + 2\sum_{2\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45}$$

$$+ 2\sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + \sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{25}$$

$$+ 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} + 5\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + 8\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}$$

$$+ \sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35} + \sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14}$$

$$+ 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 2\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + A$$

$$= \sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + \sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{35}$$

$$+ 2\sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45} + 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 3\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 6\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + \sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$$

$$+ 8\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+ \sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 2\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + A \qquad (2.21a)$$

$$\sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} = \sum_{3\neq 5}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + \sum_{3\neq 4}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{45}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{45} + \sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}.$$

$$\sum_{3\neq ..}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{35} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + \sum\Delta_{12}^{2}\Delta_{13}\Delta_{34}.$$

$$3\sum_{\neq 4}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45} = \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45} + \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}.$$

Using these results, (2.4) and (2.14) in (2.21a), we get

$$\begin{aligned}\Sigma\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} = {} & \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} \\ & + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{45} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} \\ & + 2\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34} + 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} \\ & + 2\sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{35} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} \\ & + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} \\ & + \sum_{\neq}\Delta_{12}^2\Delta_{34}^2 + 6\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 8\sum_{\neq}\Delta_{12}^3\Delta_{13} + 2\sum_{\neq}\Delta_{12}^4 .\end{aligned} \tag{2.21}$$

$$\begin{aligned}\Sigma\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} = {} & {}_1\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} + 4\,{}_1\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} \\ & + 4\Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} \\ = {} & {}_2\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} + 4\,{}_2\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} \\ & + 4\,{}_2\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + {}_2\sum_{\neq\ldots}\Delta_{12}^2\Delta_{34}\Delta_{56} \\ & + 12\,{}_2\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 12\,{}_2\sum_{\neq\ldots}\Delta_{12}^2\Delta_{13}\Delta_{45} \\ & + 4\,{}_2\sum_{\neq\ldots}\Delta_{12}^2\Delta_{13}\Delta_{14} + 16\,{}_2\sum_{\neq\ldots}\Delta_{12}^2\Delta_{13}\Delta_{24} \\ & + 4\sum_{\neq}\Delta_{12}^3\Delta_{34} + 8\sum_{\neq}\Delta_{12}^3\Delta_{13} + 4\Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}\end{aligned}$$

$$= \sum_{3\neq\cdot\cdot} \Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} + 8\sum_{3\neq\cdot\cdot} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56}$$

$$+ 4\sum_{3\neq\cdot\cdot} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + \sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56}$$

$$+ 24\sum_{3\neq\cdot\cdot} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 8\sum_{3\neq\cdot\cdot} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35}$$

$$+ 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 4\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$$

$$+ 16\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 20\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 36\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2} + 24\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24}$$

$$+ 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 4\sum_{\neq}\Delta_{12}^{3}\Delta_{34} + 28\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}$$

$$+ 4\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 8\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 4\Sigma\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46}$$

$$+ 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}$$

$$+ \sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56} + 32\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$+ 24\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35} + 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$$

$$+ 4\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + 6\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$$

$$+ 16\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 40\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 8\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34} + 40\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}$$

$$
\begin{aligned}
&+ 28\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 24\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 2\sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2} \\
&+ 4\sum_{\neq}\Delta_{12}^{3}\Delta_{34} + 28\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + 16\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} \\
&+ 24\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 4\sum_{\neq}\Delta_{12}^{4}. \qquad (2.22)
\end{aligned}
$$

Now, we calculate

$$
\begin{aligned}
\sum\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} &= {}_{1}\sum_{\neq\ldots}\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} + 6\,{}_{1}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} \\
&+ 12\,{}_{1}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 8\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} \\
&= {}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} + 12\,{}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} \\
&+ 24\,{}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56} + 24\,{}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} \\
&+ 6\,{}_{2}\sum_{\neq\ldots}\Delta_{12}^{2}\Delta_{34}\Delta_{56} + 8\,{}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} \\
&+ 48\,{}_{2}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 48\,{}_{2}\sum_{\neq\ldots}\Delta_{12}^{2}\Delta_{13}\Delta_{45} \\
&+ 24\,{}_{2}\sum_{\neq\ldots}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 48\,{}_{2}\sum_{\neq\ldots}\Delta_{12}^{2}\Delta_{13}\Delta_{24} \\
&+ 12\sum_{\neq}\Delta_{12}^{3}\Delta_{34} + 24\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 8\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} \\
&= {}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} + 4\,{}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{34}\Delta_{35}\Delta_{67} \\
&+ 12\,{}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67} + 4\,{}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{34}\Delta_{35}\Delta_{36} \\
&+ 72\,{}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56} + 24\,{}_{3}\sum_{\neq\ldots}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}
\end{aligned}
$$

$$+ 6\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56} + 48\,{}_{3}\sum_{\neq ..}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{35}$$

$$+ 96\,{}_{3}\sum_{\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{35} + 48\,{}_{3}\sum_{\neq ..}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35}$$

$$+ 8\,{}_{3}\sum_{\neq ..}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15} + 72\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45}$$

$$+ 24\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45} + 24\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$$

$$+ 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 96\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$+ 12\sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2} + 40\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 144\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}$$

$$+ 12\sum_{\neq}\Delta_{12}^{3}\Delta_{34} + 32\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}$$

$$+ 24\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2} + 8\sum\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

Upon further simplification and using the result of (2.14), we get

$$\sum\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} = \sum_{\neq}\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78} + 24\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$$

$$+ 12\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56} + 96\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56}$$

$$+ 48\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46} + 32\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}$$

$$+ 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45} + 32\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$$

$$+ 48\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35} + 192\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35}$$

$$+ 192\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25} + 16\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$+ 16\sum_{\neq}\Delta_{12}^{3}\Delta_{34} + 192\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}$$

$$+ 192\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14} + 48\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}$$

$$+ 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14} + 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24} + 12\sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2}$$

$$+ 64\sum_{\neq}\Delta_{12}^{3}\Delta_{13} + 96\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23} + 48\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2}$$

$$+ 8\sum_{\neq}\Delta_{12}^{4}. \qquad (2.23)$$

2.5 Combinatorial Methods

This section presents an occupancy model which is applied to obtain the relationships between Σ and $\sum_{\neq}$. This gives an alternative approach to the method of section 2.4 and hence can be used to check the results of lengthy calculations made in that section.

An Occupancy Model

Let there be M pairs of identical items, labelled A_{ij}, $i = 1,\ldots, M$, $j=1,2$. Let A_{i1} be linked with A_{i2} through a string S_i for $i = 1,\ldots, M$. Suppose all strings S_i $(i=1,\ldots, M)$ are identical and stretchable to any length. Suppose also that there are 2M identical urns, each big enough to carry any number of items.

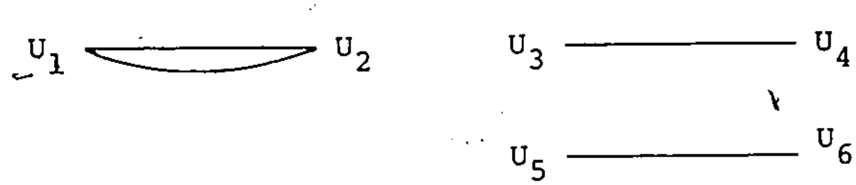
All the items occupy K ($\leq$2M) urns with a condition that each member of a pair of items A_{ij}, $j = 1,2$, must occupy a different urn.

Given any specific occupancy pattern of K urns (K>2), J urns (J = 1,..., K-2) are to be vacated. The items of an urn to be vacated move together into one of the K-J occupied urns without violating the condition of the model. How many different occupancy patterns of K-J urns are there for a given K (K>2) and for J = 1,..., K-2? If items are distinguished by their labels, what is the number of ways of arriving at a specific occupancy pattern of K-J urns?

We consider the case for M = 4, which is pertinent here in obtaining the relationships between Σ and $\sum_{\neq}$, for some cases in the following.

As an illustration, we consider the expansion of $\Sigma\Delta_{12}^{2}\Delta_{34}\Delta_{56}$ in terms of sums over distinct values of indices.

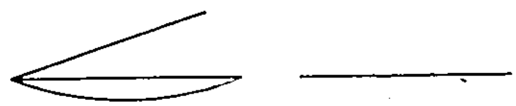
The indices of the above term correspond to K = 6 occupied urns - two of the urns carrying each of the two pairs of items, while the remaining 4 urns carry the other 4 items. This occupancy pattern is represented by the following graph:



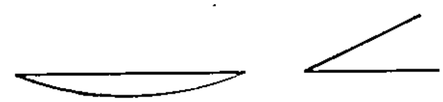
The labelling is immaterial, since the urns are all identical (urns are labelled for reference purposes only).

Given the above occupancy pattern of 6 urns, we find various possible occupancy patterns of 6-J urns, for J = 1, 2, 3, 4, by moving all the items of an urn into one of the 6-J urns, still meeting the conditions of the model.

When J = 1, there can be only two different occupancy patterns of 5 urns, which correspond to the following graphs:



or

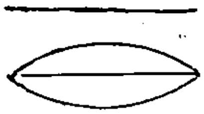


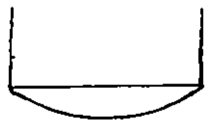
The first graph can occur when any of the urns U_3, U_4, U_5 or U_6 could have been vacated and its contents transferred to U_1 or U_2; equivalently the contents of U_1 or U_2 might have transferred to one of the urns U_3, U_4, U_5, U_6. If items are distinguished, the first graph can happen in 4 x 2 = 8 ways.

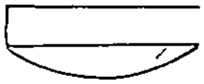
Therefore, it is apparent that the expansion of $\Sigma\Delta^2_{12}\Delta_{34}\Delta_{56}$ will have $\sum_{\neq}\Delta^2_{12}\Delta_{13}\Delta_{45}$ as one of its terms with a coefficient of 8.

The second graph is a result of combining the contents of U_3 or U_4 with U_5 or U_6. This can happen in 2 x 2 = 4 ways, indicating that another term in the expansion of $\Sigma\Delta^2_{12}\Delta_{34}\Delta_{56}$ is $\sum_{\neq}\Delta^2_{12}\Delta_{34}\Delta_{35}$ with a coefficient of 4.

When J = 2, the different possible occupancy patterns of 4 urns can have the following graphs:

2a) 

2b) 

2c) 

2d) 

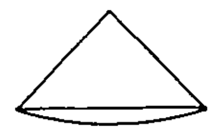
2e) 

The terms corresponding to the above graphs and their coefficients are given below:

Graph	Corresponding Term	Coefficient of the Term	One of the Possible Ways of Arriving at the Coefficient
2a)	$\sum_{\neq} \Delta_{12}^{3} \Delta_{34}$	2 x 2 = 4	The contents of U_3, U_4 are combined with the contents of U_1, U_2 or U_2, U_1.
2b)	$\sum_{\neq} \Delta_{12}^{2} \Delta_{13} \Delta_{24}$	4 x 2 = 8	The contents of U_3 or U_4 with the contents of U_5 or U_6 are combined with the contents of U_1 and U_2.
2c)	$\sum_{\neq} \Delta_{12}^{2} \Delta_{13} \Delta_{34}$	8 x 2 = 16	First the contents of either U_3, U_4, U_5 or U_6 are combined with the contents of U_1 or U_2. Then the contents of a member of the pair (U_3, U_4 or U_5, U_6) not combining with U_1 or U_2, combines with the counterpart of the urn combining U_1 or U_2.
2d)	$\sum_{\neq} \Delta_{12}^{2} \Delta_{34}^{2}$	2	The contents of U_5 and U_6 are combined with the contents of U_3 and U_4 or U_4 and U_3 respectively.

Graph	Corresponding Term	Coefficient of the Term	One of the Possible Ways of Arriving at the Coefficient
2e)	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14}$	$4 \times 2 = 8$	The contents of each member of the pairs $U_3\ U_4$, $U_5\ U_6$ combine with the contents of U_1 or U_2.

When J = 3, the different possible occupancy patterns of 3 urns correspond to the following graphs:

3a)  3b)  3c) 

With considerations similar to the cases of J = 1 and 2, the above graphs correspond to the terms of $\sum_{\neq}\Delta_{12}^{3}\Delta_{13}$, $\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2}$ and $\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}$ respectively. The coefficients of these terms are 16, 8 and 4 respectively.

Finally when J = 4, the only possible graph is , which corresponds to the term $\sum_{\neq}\Delta_{12}^{4}$ having coefficient 4.

In the expansion of $\sum\Delta_{12}^{2}\Delta_{34}\Delta_{56}$, one considers various cases of equality of indices. The above sum is partitioned into sums over distinct values of indices. There are 6 indices and we considered above J indices (J=1, 2, 3, 4) to be equal to the other 6-J indices. The case for J = 0 is a trivial case for which the corresponding term is

$\sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{56}$ with a coefficient of 1. Now, considering the terms arising in each of the cases J=0, 1, 2, 3, 4 we obtain

$$\begin{aligned}\Sigma\Delta_{12}^2\Delta_{34}\Delta_{56} = {} & \sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{56} + 8\sum_{=}\Delta_{12}^2\Delta_{13}\Delta_{45} + 4\sum_{\neq}\Delta_{12}^2\Delta_{34}\Delta_{35} \\ & + 4\sum_{\neq}\Delta_{12}^3\Delta_{34} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{24} + 16\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{34} \\ & + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{14} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}\Delta_{23} + 2\sum_{\neq}\Delta_{12}^2\Delta_{34}^2 \\ & + 16\sum_{\neq}\Delta_{12}^3\Delta_{13} + 8\sum_{\neq}\Delta_{12}^2\Delta_{13}^2 + 4\sum_{\neq}\Delta_{12}^4.\end{aligned}$$

This confirms the relationship given by (2.12).

The combinatorial method can be used in this manner to obtain any relationship of the last section. One needs to enumerate carefully the different graphs, then determining the coefficient of a term corresponding to any graph is straightforward.

The expressions for $\Sigma\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$ and $\Sigma\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78}$ by the method of the last section, involve very cumbersome calculations. With the use of the combinatorial method, one needs only to identify different possible terms of the expansion and compute the coefficients using the occupancy model.

The expansion of these terms are indicated by the following tables:

Table 2.3

Use of the Occupancy Model to Obtain Different Graphs and Coefficients Corresponding to the Terms in the Expansion of $\Sigma\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$

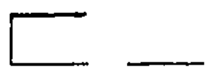
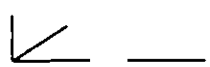
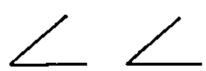
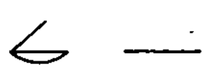
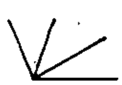
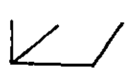
No. of Urns to be Vacated J	Different Graphs Corresponding to the Value of J	No. of Ways of Arriving at the Graph	The Term Corresponding to the Graph
0		1	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$
1		1	$\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56}$
		8	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56}$
		4	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}$
		4	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46}$
		4	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$
		16	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45}$
		6	$\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$
		24	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{45}$
		4	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$
		32	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$

Table 2.3 (cont'd.)

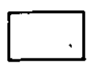
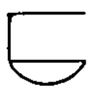
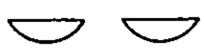
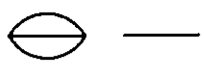
No. of Urns to be Vacated J	Different Graphs Corresponding to the value of J	No. of Ways of Arriving at the Graph	The Term Corresponding to the Graph
3		40	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$
		8	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}$
		28	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{14}$
		40	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{34}$
		24	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{24}$
		2	$\sum_{\neq}\Delta_{12}^{2}\Delta_{34}^{2}$
		4	$\sum_{\neq}\Delta_{12}^{3}\Delta_{34}$
4		28	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{23}$
		16	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}^{2}$
		24	$\sum_{\neq}\Delta_{12}^{3}\Delta_{13}$
5		4	$\sum_{\neq}\Delta_{12}^{4}$

Table 2.4

Use of the Occupancy Model to Obtain Different Graphs and Coefficients Corresponding to the Terms in the Expansion of $\Sigma\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78}$

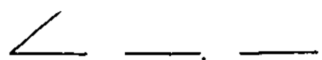
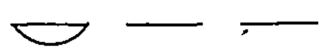
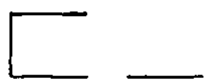
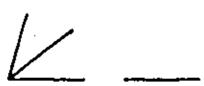
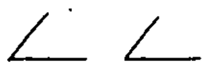
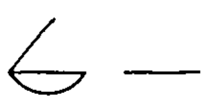
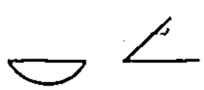
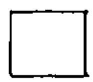
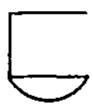
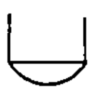
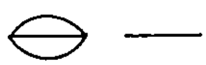
No. of Urns to be Vacated J	Different Graphs Corresponding to the Value of J	No. of Ways of Arriving at the Graph	The Term Corresponding to the Graph
0		1	$\sum_{\neq}\Delta_{12}\Delta_{34}\Delta_{56}\Delta_{78}$
1		24	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{67}$
		12	$\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{56}$
		96	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{24}\Delta_{56}$
		32	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{56}$
		48	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{45}\Delta_{46}$
2		32	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}\Delta_{45}$
		96	$\sum_{\neq}\Delta_{12}^{2}\Delta_{13}\Delta_{45}$
		48	$\sum_{\neq}\Delta_{12}^{2}\Delta_{34}\Delta_{35}$
		192	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{34}\Delta_{45}$
		16	$\sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$

Table 2.4 (cont'd.)

No. of Urns to be Vacated J	Different Graphs Corresponding to the Value of J	No. of Ways Arriving at the Graph	The Term Corresponding to the Graph
		192	$\sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$
3		192	$\sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$
		48	$\sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}$
		96	$\sum_{\neq} \Delta_{12}^2\Delta_{13}\Delta_{14}$
		192	$\sum_{\neq} \Delta_{12}^2\Delta_{13}\Delta_{34}$
		96	$\sum_{\neq} \Delta_{12}^2\Delta_{13}\Delta_{24}$
		12	$\sum_{\neq} \Delta_{12}^2\Delta_{34}^2$
		16	$\sum_{\neq} \Delta_{12}^3\Delta_{34}$
4		96	$\sum_{\neq} \Delta_{12}^2\Delta_{13}\Delta_{23}$
		48	$\sum_{\neq} \Delta_{12}^2\Delta_{13}^2$
		64	$\sum_{\neq} \Delta_{12}^3\Delta_{13}$
5		8	$\sum_{\neq} \Delta_{12}^4$

2.6 The Fourth Moment Under the Null Hypothesis

In this section we obtain an expression for the fourth moment of δ, defined by (2.1). The result is expressed in terms of the 23 symmetric functions defined in section 2.3. The computation of $E(\delta^4)$ requires the relationships given in section 2.4.

The test statistic is

$$\delta = \sum_{i=1}^{g} K_i \Sigma \Delta_{J_1 J_2} S_i(J_1) S_i(J_2).$$

The fourth power of δ can be expressed as a sum of products of r-part partitions (r= 1, 2, 3, 4) and their coefficients.

Dwyer and Tracy (1964) define a combinatorial coefficient $C(P) = (p_1^{\pi_1} \ldots p_s^{\pi_s})$ of P as the number of ways that the distinct units of P may be collected into distinct parcels described by the specified partition of p. For the r-part partition $p_1^{\pi_1} \ldots p_s^{\pi_s}$, the partition coefficient

$$C(P) = (p_1^{\pi_1} \ . \ . \ . \ p_s^{\pi_s}) = \frac{P!}{(p_1!)^{\pi_1} \ . \ . \ . \ (p_s!)^{\pi_s} \pi_1! \ . \ . \ . \ \pi_s!}$$

where $\sum_{i=1}^{s} p_i \pi_i = p$ (weight) and $\sum_{i=1}^{s} \pi_i = r$ (order).

The multinomial theorem is then expressed as

$$[1]^p = \Sigma (p_1 \ . \ . \ . \ p_r) \ [p_1 \ . \ . \ . \ p_r],$$

where the summation applies to every r-part partition of p and r=1,..., p.

Using the multinomial theorem, δ^4 can be written as

$$\delta^4 = \Sigma(p_1 \cdot \cdot \cdot p_r)\ [p_1 \cdot \cdot \cdot p_r],$$

where $K_i \Sigma \Delta_{J_1 J_2} S_i(J_1) S_i(J_2)$ takes the place of x_i and the summation applies to every r-part partition of 4, r = 1, 2, 3, 4.

In the notation of Dwyer and Tracy (1964),

$$\delta^4 = [4] + 4[31] + 3[22] + 6[211] + [1111], \quad (2.24)$$

where x_i is $K_i \Sigma \Delta_{J_1 J_2} S_i(J_1) S_i(J_2)$.

The expectation of each of the terms on the right side of (2.24) is calculated as follows:

$$E([4]) = E(\sum_i K_i^4 (\Sigma \Delta J_1 J_2 S_i(J_1) S_i(J_2))^4)$$

$$= E(\sum_i K_i^4 \Sigma \Delta_{J_1 J_2} \Delta_{J_3 J_4} \Delta_{J_5 J_6} \Delta_{J_7 J_8} \prod_{\ell=1}^{4} \{S_i(J_{2\ell-1}) S_i(J_{2\ell})\})$$

$$= \sum_i K_i^4\ \Sigma\{\Delta_{J_1 J_2} \Delta_{J_3 J_4} \Delta_{J_5 J_6} \Delta_{J_7 J_8} E(\prod_{\ell=1}^{4} S_i(J_{2\ell-1}) S_i(J_{2\ell}))\}$$

Expressing the sum over J_1 to J_8 of the sum over distinct values of indices and then taking expectations of the product of indicator functions,

$$E([4]) = \sum_i K_i^4 \Big(\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_3J_4}\Delta_{J_5J_6}\Delta_{J_7J_8}\, n_i^{(8)}/N^{(8)}$$

$$+ 24\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_4J_5}\Delta_{J_6J_7}\, n_i^{(7)}/N^{(7)}$$

$$+ 12\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_3J_4}\Delta_{J_5J_6}\, n_i^{(6)}/N^{(6)}$$

$$+ 32\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_1J_4}\Delta_{J_5J_6}\, n_i^{(6)}/N^{(6)}$$

$$+ 96\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_4}\Delta_{J_5J_6}\, n_i^{(6)}/N^{(6)}$$

$$+ 48\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_4J_5}\Delta_{J_4J_6}\, n_i^{(6)}/N^{(6)}$$

$$+ 96\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_4J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 32\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_3}\Delta_{J_4J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 48\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_3J_4}\Delta_{J_3J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 192\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_4}\Delta_{J_3J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 192\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_1J_4}\Delta_{J_2J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 16\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_1J_4}\Delta_{J_1J_5}\, n_i^{(5)}/N^{(5)}$$

$$+ 16\sum_{\neq} \Delta^3_{J_1J_2}\Delta_{J_3J_4}\, n_i^{(4)}/N^{(4)}$$

$$+ 192\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_3J_4}\, n_i^{(4)}/N^{(4)}$$

$$+ 192\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_3}\Delta_{J_1J_4}\, n_i^{(4)}/N^{(4)}$$

$$+ 48\sum_{\neq} \Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_4}\Delta_{J_3J_4}\, n_i^{(4)}/N^{(4)}$$

$$+ 96\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_1J_4}\, n_i^{(4)}/N^{(4)}$$

$$+ 96\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_4}\, n_i^{(4)}/N^{(4)}$$

+ [illegible] $\Delta^2_{J_1J_2}\Delta^2_{J_3J_4}\, n_i^{(4)}/N$ [illegible]

+ [illegible] $\Delta^3_{J_1J_2}\Delta_{J_1J_3}\, n_i^{(3)}$ [illegible]

$$+ 96\sum_{\neq} \Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_3}\, n_i^{[illegible]}/N^{(3)}$$

$$+ 48\sum_{\neq} \Delta^2_{J_1J_2}\Delta^2_{J_1J_3}\, n_i^{(3)}/N^{(3)} + 8\sum_{\neq} \Delta^4_{J_1J_2}\, n_i^{(2)}/N^{(2)}$$

Using symmetric functions defined in section 2.3 and rearranging the terms, we get

$$E([4]) = \sum_i K_i^4 \{8n_i^{(2)} D(13) + 16n_i^{(3)} [4D(14) + 3D(15)$$
$$+ 6D(16)] + 4n_i^{(4)}[4D(17) + 3D(18)]$$
$$+ 48n_i^{(4)} [2D(19) + 2D(20) + 4D(21) + D(22)$$
$$+ 4D(23)] + 16n_i^{(5)} [3D(24) + 6D(25) + D(26)$$
$$+ 12D(27)+12D(28) + 2D(29)] + 4n_i^{(6)} [3D(30)$$
$$+ 8D(31)] + 48n_i^{(6)} [D(32) + 2D(33)]$$
$$+ 24n_i^{(7)} D(34) + n_i^{(8)} D(35)\}. \qquad (2.25)$$

The expectation of the 1/4 second term of R.H.S. of (2.24) is

$$E([31]) = \sum_{i \neq j} K_i^3 K_j \; E(\Sigma\Delta_{J_1 J_2} S_i(J_1) S_i(J_2))^3$$
$$(\Sigma\Delta_{J_7 J_8} S_j(J_7) S_j(J_8))$$

The first factor in the braces can be written as

$$(\Sigma\Delta_{J_1 J_2} S_i(J_1) S_i(J_2))^3$$
$$= \Sigma\Delta_{J_1 J_2} \Delta_{J_3 J_4} \Delta_{J_5 J_6} \prod_{\ell=1}^{6} S_i(J_\ell)$$

$$= \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_3J_4} \Delta_{J_5J_6} \prod_{\ell=1}^{6} S_i(J_\ell)$$

$$+ 12 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_4J_5} \prod_{\ell=1}^{5} S_i(J_\ell)$$

$$+ 24 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_2J_4} \prod_{\ell=1}^{4} S_i(J_\ell)$$

$$+ 8 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_1J_4} \prod_{\ell=1}^{4} S_i(J_\ell)$$

$$+ 6 \sum_{\neq} \Delta^2_{J_1J_2} \Delta_{J_3J_4} \prod_{\ell=1}^{4} S_i(J_\ell)$$

$$+ 8 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_2J_3} S_i(J_1) S_i(J_2) S_i(J_3)$$

$$+ 24 \sum_{\neq} \Delta^2_{J_1J_2} \Delta_{J_1J_3} S_i(J_1) S_i(J_2) S_i(J_3)$$

$$+ 4 \sum_{\neq} \Delta^2_{J_1J_2} S_i(J_1) S_i(J_2).$$

Multiplying each of the terms of the right side above by $\Sigma \Delta_{J_7J_8} S_j(J_7) S_j(J_8)$ and taking expectations on the product of indicator functions, we get

$$E([31]) = \sum_{i \neq j} K_i^3 K_j \; \{ \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_3J_4} \Delta_{J_5J_6} \Delta_{J_7J_8} n_i^{(6)} n_j^{(2)} / N^{(8)}$$

$$+ 12 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_4J_5} \Delta_{J_6J_7} n_i^{(5)} n_j^{(2)} / N^{(7)}$$

$$+ 24 \sum_{\neq} \Delta_{J_1J_2} \Delta_{J_1J_3} \Delta_{J_2J_4} \Delta_{J_5J_6} n_i^{(4)} n_j^{(2)} / N^{(6)}$$

$$+ 8\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_1J_4}\Delta_{J_5J_6} n_i^{(4)} n_j^{(2)}/N^{(6)}$$

$$+ 6\sum_{\neq}\Delta^2_{J_1J_2}\Delta_{J_3J_4}\Delta_{J_5J_6} n_i^{(4)} n_j^{(2)}/N^{(6)}$$

$$+ 8\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_2J_3}\Delta_{J_4J_5} n_i^{(3)} n_j^{(2)}/N^{(5)}$$

$$+ 24 \sum_{\neq}\Delta^2_{J_1J_2}\Delta_{J_1J_3}\Delta_{J_4J_5} n_i^{(3)} n_j^{(2)}/N^{(5)}$$

$$+ 4\sum_{\neq}\Delta^2_{J_1J_2}\Delta_{J_3J_4} n_i^{(2)} n_j^{(2)}/N^{(4)}$$

In terms of symmetric functions, the above becomes

$$E([31]) = \sum_{i\neq j} K_i^3 K_j n_j^{(2)}\{4n_i^{(2)}D(17) + 24n_i^{(3)}D(25)$$

$$+ 8n_i^{(3)}D(29) + 6n_i^{(4)}D(30) + 8n_i^{(4)}[D(31)$$

$$+ 3D(33)] + 12n_i^{(5)}D(34) + n_i^{(6)}D(35)\}. \qquad (2.26)$$

The expectation of 1/3 the third term on right side of (2.24) is

$$E([22]) = \sum_{i\neq j} K_i^2 K_j^2 \, E\,\{(\Sigma\Delta_{J_1J_2}S_i(J_1)S_i(J_2))^2$$

$$(\Sigma\Delta_{J_5J_6}S_j(J_5)S_j(J_6))^2\}$$

The term in the braces is

$$(\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_3J_4}\prod_{\ell=1}^{4}S_i(J_\ell) + 4\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_1J_3}\prod_{\ell=1}^{3}S_i(J_\ell)$$

$$+ 2\sum_{\neq}\Delta^2_{J_1J_2}\prod_{\ell=1}^{2}S_i(J_\ell)) \cdot (\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_3J_4}\prod_{\ell=5}^{8}S_j(J_\ell)$$

$$+ 4\sum_{\neq}\Delta_{J_5J_6}\Delta_{J_5J_7}\prod_{\ell=5}^{7}S_j(J_\ell) + 2\sum_{\neq}\Delta^2_{J_1J_2}\prod_{\ell=5}^{6}S_j(J_\ell)).$$

Taking the expectation of each of the terms in the above product, E([22]), in terms of symmetric functions, becomes

$$E([22]) = \sum_{i\neq j}K_i^2K_j^2\ \{4n_i^{(2)}n_j^{(2)}D(18) + 8(n_i^{(3)}n_j^{(2)}$$

$$+ n_i^{(2)}n_j^{(3)})D(24) + 2(n_i^{(4)}n_j^{(2)} + n_i^{(2)}n_j^{(4)})D(30)$$

$$+ 16n_i^{(3)}n_j^{(3)}D(32) + 4(n_i^{(4)}n_j^{(3)} + n_i^{(3)}n_j^{(4)})$$

$$\cdot D(34) + n_i^{(4)}n_j^{(4)}D(35)\}. \qquad (2.27)$$

The expectation of 1/6 the fourth term of the R.H.S. of (2.24) is

$$E([211]) = \sum_{i\neq j\neq k}K_i^2K_jK_k\ E\{(\Sigma\Delta_{J_1J_2}S_i(J_1)S_i(J_2))^2 \cdot$$

$$(\Sigma\Delta_{J_5J_6}S_j(J_4)S_j(J_6)(\Sigma\Delta_{J_7J_8}S_k(J_7)S_k(J_8))\}.$$

The factor $(\Sigma\Delta_{J_1J_2}S_i(J_1)S_i(J_2))^2$ in the braces is

$$\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_3J_4}\prod_{\ell=1}^{4}S_i(J_\ell) + 4\sum_{\neq}\Delta_{J_1J_2}\Delta_{J_1J_3}\prod_{\ell=1}^{3}S_i(J_\ell)$$

$$+ 2\sum_{\neq}\Delta^2_{J_1J_2}\prod_{\ell=1}^{2} S_i(J_\ell).$$

Using this, we get

$$E([211]) = \sum_{i\neq j\neq k} K_i^2K_jK_kn_j^{(2)}n_k^{(2)}\{2n_i^{(2)}D(30)$$

$$+ 4n_i^{(3)}D(34) + n_i^{(4)}D(35)\}. \qquad (2.28)$$

The expectation of the last term of the R.H.S. of (2.24) is

$$E([1111]) = \sum_{i\neq j\neq k\neq \ell} K_iK_jK_kK_\ell \Delta_{J_1J_2}\Delta_{J_3J_4}\Delta_{J_5J_6}\Delta_{J_7J_8}$$

$$\cdot\ S_i(J_1)S_i(J_2)S_j(J_3)S_j(J_4)S_k(J_5)S_k(J_6)$$

$$\cdot\ S_\ell(J_7)S_\ell(J_8).$$

$$= \sum_{i\neq j\neq k\neq \ell} K_iK_jK_kK_\ell n_i^{(2)}n_j^{(2)}n_k^{(2)}n_\ell^{(2)}D(35). \qquad (2.29)$$

Combining (2.25) through (2.29) and placing $K_i = c_i/n_i^{(2)}$, we get

$$E(\delta^4) = \sum_i [(\frac{C_i}{n_i^{(2)}})^4 \{8n_i^{(2)} D(13) + 64n_i^{(3)} D(14) + 48n_i^{(3)} D(15)$$

$$+ 96n_i^{(3)} D(16) + 16n_i^{(4)} D(17) + 12n_i^{(4)} D(18) + 96n_i^{(4)}$$

$$\cdot [D(19) + D(20) + 2D(21)] + 48n_i^{(4)} [D(22) + 4D(23)]$$

$$+ 48n_i^{(5)} [D(24) + 2D(25)] + 16n_i^{(5)} D(26)$$

$$+ 192n_i^{(5)} [D(27) + D(28)] + 32n_i^{(5)} D(29) + 12n_i^{(6)} D(30)$$

$$+ 32n_i^{(6)} D(31) + 48n_i^{(6)} [D(32) + 2D(33)] + 24n_i^{(7)} D(34)$$

$$+ n_i^{(8)} D(35)\}]$$

$$+ 4 \sum_{i \neq j} [\frac{C_i^3}{(n_i^{(2)})^3} \frac{C_j}{n_j^{(2)}} \{4n_i^{(2)} n_j^{(2)} D(17)$$

$$+ 24n_i^{(3)} n_j^{(2)} D(25) + 8n_i^{(3)} n_j^{(2)} D(29)$$

$$+ 6n_i^{(4)} n_j^{(2)} D(30) + 8n_i^{(4)} n_j^{(2)} D(31) + 24n_i^{(4)} n_j^{(2)} D(33)$$

$$+ 12n_i^{(5)} n_j^{(2)} D(34) + n_i^{(6)} n_j^{(2)} D(35)\}]$$

$$+ 3 \sum_{i \neq j} [(\frac{C_i C_j}{n_i^{(2)} n_j^{(2)}})^2 \{4n_i^{(2)} n_j^{(2)} D(18) + 8(n_i^{(3)} n_j^{(2)}$$

$$+ n_i^{(2)} n_j^{(3)}) D(24) + 2(n_i^{(4)} n_j^{(2)} + n_i^{(2)} n_j^{(4)}) D(30)$$

$$+ 16n_i^{(3)} n_j^{(3)} D(32)$$

$$+ 4(n_i^{(4)}n_j^{(3)} + n_i^{(3)}n_j^{(4)}D(34) + n_i^{(4)}n_j^{(4)}D(35)\}]$$

$$+ 6 \sum_{i \neq j \neq k} \frac{C_i^2 . C_j . C_k}{(n_i^{(2)})^2 n_j^{(2)} n_k^{(2)}} \{2n_i^{(2)}n_j^{(2)}n_k^{(2)}D(30)$$

$$+ 4n_i^{(3)}n_j^{(2)}n_k^{(2)}D(34) + n_i^{(4)}n_j^{(2)}n_k^{(2)}D(35)\}$$

$$+ \sum_{i \neq j \neq k \neq \ell} C_i C_j C_k C_\ell D(35). \qquad (2.30)$$

The above result has been checked for several cases in the following manner.

A computer program, named APS, in Appendix A.1 generates all possible samples of sizes n_1, n_2 with $2 \leq n_1 \leq 10$ and $n_1 + n_2 = N$. The fourth moment is obtained considering every possible value of the test statistic and is compared with the one obtained using (2.30). Checks are made for various 2-sample configurations and for the following distance functions:

a) $\Delta_{xy} = |x-y|$; b) $\Delta_{xy} = |R(x) - R(y)|$ and
c) $\Delta_{xy} = |R(x) - R(y)|^2$, where R(x) is the rank of x in the combined sample.

The test statistic for the case of (b) above is discussed in detail in the next chapter.

CHAPTER III

SOME RESULTS CONCERNING A SPECIAL MRPP STATISTIC

3.1 The Special MRPP Statistic

An interesting case of the MRPP test statistic is obtained from (1.3) by letting g=2, $C_1=\frac{n_1}{N}$, $C_2=1-C_1$ and $\Delta_{IJ}=|R(x_I)-R(x_J)|$, where R(y) is the rank of y in the combined sample. This test statistic is designated by δ_1. When the underlying sampling distribution is Laplace, logistic or a U-shaped distribution, then the empirical power of δ_1 is higher than that of the Wilcoxon test (Chapter IV).

The symmetric functions for the Δ as given above are simply functions of N - the total number of observations. The long expression for the fourth moment, as given by (2.30), is simplified to a simple expression involving n_i (i=1,...,g) and N. We obtain a simplified form of the fourth moment of δ_1 taking $n_1=n_2=N/2$. In this case the fourth moment of δ_1 is a simple polynomial in N.

3.2 Symmetric Functions When $\Delta_{IJ} = |R(x_I)-R(x_J)|$

When the distance function is the Euclidean distance between ranks of observations, symmetric functions are simple polynomials in N - the number of observations. Mielke (1981b, p. 723) has listed the results for the first twelve symmetric functions. We calculate all the symmetric functions, defined in section 2.3, for the above case and for completeness sake show the calculations carried on in the first twelve symmetric functions as well.

From section 2.3,

$$D(1) \equiv \frac{1}{N^{(2)}} \sum_{\neq} \Delta_{12} = \frac{1}{N^{(2)}} \sum_{I,J} |I-J|$$

$$= \frac{2}{N^{(2)}} \sum_{I<J} (J-I)$$

$$= \frac{N+1}{3}. \qquad (3.1)$$

$$D(2) \equiv \frac{1}{N^{(2)}} \sum_{\neq} \Delta_{12}^{2} = \frac{1}{N^{(2)}} \sum_{I,J} (I-J)^{2}$$

$$= \frac{2}{N^{(2)}} [N \sum_{I} I^{2} - (\sum_{I} I)^{2}]$$

$$= \frac{N(N+1)}{6}. \qquad (3.2)$$

$$D(3) \equiv \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12} \Delta_{13} = \frac{1}{N^{(3)}} \sum_{\neq} |I-J| |I-K|$$

$$= \frac{1}{N^{(3)}} [\sum_{I} (\sum_{J} |I-J|)^{2} - \sum_{I,J} (I-J)^{2}]$$

$$= \frac{N+1}{60} (7N+1). \qquad (3.3)$$

$$D(4) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}\Delta_{34}$$

$$= \frac{1}{N^{(4)}} [(\sum_{\neq}\Delta_{12})^2 - 4\sum_{\neq}\Delta_{12}\Delta_{13} - 2\sum_{\neq}\Delta_{12}^2]$$

$$= \frac{1}{N^{(4)}} [(\Sigma|I-J|)^2 - 4\Sigma|I-K| - 2\sum_{I,J}(I-J)^2].$$

Using previous results, on simplification we get

$$D(4) = \frac{(N+1)}{45}(5N+4). \quad (3.4)$$

$$D(5) \equiv \frac{1}{N^{(2)}} \sum_{\neq}\Delta_{12}^3$$

$$= \frac{1}{N^{(2)}} \sum_{I,J}|I-J|^3 = \frac{(N+1)(3N^2-2)}{30}. \quad (3.5)$$

$$D(6) \equiv \frac{1}{N^{(3)}} \sum_{\neq}\Delta_{12}^2\Delta_{13}$$

$$= \frac{1}{N^{(3)}} \sum_{\neq}(I-J)^2|I-K|$$

$$= \frac{1}{N^{(3)}} [\sum_{I,J,K}(I-J)^2|I-K| - \sum_{I,J}|I-J|^3]$$

$$= \frac{(N+1)}{180}(11N^2+4N-6). \quad (3.6)$$

$$D(7) \equiv \frac{1}{N^{(3)}} \sum_{\neq}\Delta_{12}\Delta_{13}\Delta_{23}$$

$$= \frac{1}{N^{(3)}} \sum_{I,J,K}|I-J||I-K||J-K|$$

$$= \frac{6}{N^{(3)}} \sum_{I>J>K} (I-J)(I-K)(J-K) \quad \text{(using symmetry)}$$

$$= \frac{N(N+1)(N+2)}{30}. \qquad (3.7)$$

$$D(8) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{34}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} (I-J)^2 |K-L|$$

$$= \frac{1}{N^{(4)}} \left[\sum_{I,J} (I-J)^2 \sum_{K,L} |K-L| - 4N^{(3)} D(6) - 2N^{(2)} D(5) \right]$$

$$= \frac{N+1}{90} (5N^2+3N-2). \qquad (3.8)$$

$$D(9) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{24}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} |I-J|\,|I-K|\,|J-L|$$

$$= \frac{1}{N^{(4)}} \left[\sum |I-J|\,|I-K|\,|J-L| - N^{(3)} D(7) - 2N^{(3)} D(6) \right.$$

$$\left. - N^{(2)} D(5) \right].$$

Now,

$$\sum_{I,J,K,L} |I-J|\,|I-K|\,|J-L|$$

$$= 2 \sum_{I>J} (I-J) \left(I^2-(N+1)I+\frac{N(N+1)}{2}\right) \left(J^2-(N+1)J+\frac{N(N+1)}{2}\right)$$

$$= \frac{1}{6} \sum_{I} [I^6 - 3(N+1)I^5 + \frac{1}{2}(11N^2+15N+2)I^4 - \frac{(N+1)}{2}$$

$$\cdot (8N^2+14N-6)I^3 + \frac{(N+1)}{2}(3N^3+9N^2+N-4)I^2 - \frac{N}{2}$$

$$\cdot (N+1)^2(3N-2)I]$$

$$= \frac{N(N^2-1)}{420}(17N^4-39N^2+24).$$

Using the above result with previous results, we get

$$D(9) = \frac{N+1}{1260}(51N^2+59N-2). \tag{3.9}$$

In a similar way, we find

$$D(10) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} |I-J||I-K||I-L|$$

$$= \frac{1}{N^{(4)}} [\sum_{I}(\sum_{J}|I-J|)^3 - 3N^{(3)}D(6) - N^{(2)}D(5)]$$

$$= \frac{N+1}{420}(18N^2+13N-4). \tag{3.10}$$

$$D(11) \equiv \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{45}$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} |I-J||I-K||L-M|$$

$$= \frac{1}{N^{(5)}} \left[\sum_{I,J,K} |I-J||I-K| \sum_{I,J} |I-J| - 4N^{(4)}D(9) \right.$$

$$-2N^{(4)}D(10) - N^{(4)}D(8) - 2N^{(3)}D(7)$$

$$\left. - 8N^{(3)}D(6) - 2N^{(2)}D(5) \right]$$

$$= \frac{N+1}{1260}(49N^2+59N+6). \qquad (3.11)$$

$$D(12) \equiv \frac{1}{N^{(6)}} \sum_{\neq} \Delta_{12}\Delta_{34}\Delta_{56}$$

$$= \frac{1}{N^{(6)}} \sum_{\neq} |I-J||K-L||M-P|$$

$$= \frac{1}{N^{(6)}} \left[\left(\sum_{I,J} |I-J| \right)^3 - 12N^{(5)}D(11) - N^{(4)} \right.$$

$$\cdot \{24D(9) + 8D(10) + 6D(8)\} - N^{(3)}\{8D(7)$$

$$\left. + 24D(6)\} - 4N^{(2)}D(5) \right]$$

$$= \frac{N+1}{945}(35N^2+49N+12). \qquad (3.12)$$

We now proceed to obtain the symmetric functions D(13) through D(35).

$$D(13) \equiv \frac{1}{N^{(2)}} \sum_{\neq} \Delta_{12}^4$$

$$= \frac{1}{N^{(2)}} \sum_{I,J} (I-J)^4 = \frac{N(N+1)(2N^2-3)}{30}. \qquad (3.13)$$

$$D(14) \equiv \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{3} \Delta_{13}$$

$$= \frac{1}{N^{(3)}} \sum_{\neq} |I-J|^{3} |I-K|$$

$$= \frac{1}{N^{(3)}} \left[\sum_{I,J,K} |I-J|^{3} |I-K| - \Sigma (I-J)^{4} \right]$$

$$= \frac{N+1}{420} (16N^{3}+4N^{2}-25N-8). \qquad (3.14)$$

$$D(15) \equiv \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{13}^{2}$$

$$= \frac{1}{N^{(3)}} \sum_{\neq} (I-J)^{2} (I-K)^{2}$$

$$= \frac{1}{N^{(3)}} \left[\sum_{I,J,K} (I-J)^{2} (I-K)^{2} - \sum_{I,J} (I-J)^{4} \right]$$

$$= \frac{N(N+1)(2N^{2}-3)}{60}. \qquad (3.15)$$

$$D(16) \equiv \frac{1}{N^{(3)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{13} \Delta_{23}$$

$$= \frac{1}{N^{(3)}} \sum_{I,J,K} (I-J)^{2} |I-K| |J-K|.$$

The above is simplified by observing symmetry among different inequalities in I,J,K. We obtain

$$D(16) = \frac{4}{N^{(3)}} \sum_{K<J<I} (I-J)^{2} (I-K)(J-K) + \frac{2}{N^{(3)}}$$

$$\cdot \sum_{I<K<J} (I-J)^{2} (K-I)(J-K)$$

$$= \frac{1}{N^{(3)}} \left[\frac{4N^{(3)}}{2520}(N+1)(N+2)(5N^2-3) + \frac{2N^{(3)}}{2520}\right.$$

$$\left. \cdot (N+1)(N+2)(5N^2-3)\right]$$

$$= \frac{N+1}{315} (N+2)(5N^2-3). \qquad (3.16)$$

$$D(17) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{3} \Delta_{34}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} |I-J|^3 |K-L|$$

$$= \frac{1}{N^{(4)}} \left[\sum_{I,J} |I-J|^3 \sum_{I,J} |I-J| - 4N^{(3)} D(14)\right.$$

$$\left. -2N^{(2)} D(13)\right]$$

$$= \frac{N+1}{630} (21N^3+9N^2-32N-16). \qquad (3.17)$$

$$D(18) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{34}^{2}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} (I-J)^2 (K-L)^2$$

$$= \frac{1}{N^{(4)}} \left[\left(\sum_{I,J} (I-J)^2\right)^2 - 4N^{(3)} D(15) - 2N^{(2)} D(13)\right]$$

$$= \frac{N(N+1)}{180} (5N^2+N-6). \qquad (3.18)$$

$$D(19) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{13} \Delta_{14}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} (I-J)^2 |I-K| |I-L|$$

$$= \frac{1}{N^{(4)}} [\sum_{I,J} (I-J)^2 (\sum_{K} (I-K))^2 - N^{(3)} D(15)$$

$$- 2N^{(3)} D(14) - N^{(2)} D(13)]$$

$$= \frac{N+1}{2520} (59N^3+19N^2-88N-32). \qquad (3.19)$$

$$D(20) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^2 \Delta_{13} \Delta_{24}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} (I-J)^2 |I-K| |J-L|$$

$$= \frac{1}{N^{(4)}} [\Sigma (I-J)^2 |I-K| |J-L| - N^{(3)} \{D(16)$$

$$+ 2D(14)\} - N^{(2)} D(13)]$$

Here,

$$\Sigma (I-J)^2 |I-K| |J-L|$$

$$= \sum_{I,J} (I-J)^2 (I^2-(N+1)I + \frac{N(N+1)}{2}) (J^2-(N+1)J$$

$$+ \frac{N(N+1)}{2}) = \frac{N^2 (N^2-1)^2}{90} (2N^2-3)$$

Using this, we get

$$D(20) = \frac{N+1}{630} (14N^3+12N^2-17N-12). \qquad (3.20)$$

$$D(21) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}^{2} \Delta_{13} \Delta_{34}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} (I-J)^2 |I-K| |K-L|$$

$$= \frac{1}{N^{(4)}} [\Sigma (I-J)^2 |I-K| |K-L| - N^{(3)} \{D(16)$$

$$+ D(15)\} + D(14)\} - N^{(2)} D(13)]$$

The first term in the bracket, $\sum_{I,J,K,L} (I-J)^2 |I-K| |K-L|$

$$= \sum_{I,J,K} (I-J)^2 |I-K| (K^2 - (N+1)K + \frac{N(N+1)}{2})$$

$$= \frac{N}{12} \sum_{I,J} |I-J| (6I^2 - 6(N+1)I + (N+1)(2N+1))(2J^2 - 2(N+1)J + N(N+1))$$

$$= \frac{N^2(N^2-1)}{2520} (53N^4 - 129N^2 + 88) \text{ (on simplification)}$$

Using the above result, we get

$$D(21) = \frac{N+1}{2520} (53N^3 + 45N^2 - 54N - 32). \qquad (3.21)$$

Now, we come to some results which involve some cumbersome algebra.

$$D(22) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12} \Delta_{13} \Delta_{24} \Delta_{34}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} |I-J| |I-K| |J-L| |K-L|$$

We first examine carefully the sum of the above summand over each of the 24 inequalities in I, J, K, L. Noting the symmetry in different inequalities, the above sum can be written as

$$\sum_{\neq} |I-J|\,|I-K|\,|J-L|\,|K-L|$$

$$= 8\{\sum_{I>J>K>L} |I-J|\,|I-K|\,|J-L|\,|K-L|$$

$$+ \sum_{K>J>L>I} |I-J|\,|I-K|\,|J-L|\,|K-L|$$

$$+ \sum_{L>K>I>J} |I-J|\,|I-K|\,|J-L|\,|K-L|\}.$$

We evaluate each of the terms in the bracket above, as follows: To obtain the first term, consider

$$\sum_{I\geq J\geq K\geq L} |I-J|\,|I-K|\,|J-L|\,|K-L|$$

$$= \sum_{I\geq J\geq K} (I-J)(I-K)\{JK^2 - (J+K)\frac{K(K+1)}{2}$$

$$+ \frac{K(K+1)(2K+1)}{6}\}$$

$$= \frac{1}{6}\sum_{I\geq J} (I-J) \sum_{K=1}^{J} \{K^4 - (3J+I)K^3 + (3IJ+3J-1)K^2$$

$$- (3J-1)IK\}$$

$$= \frac{1}{120} \sum_{I \geq J} (J-I)\{11J^2-15IJ^4 + 5(2I-3)J^3$$

$$+ 15IJ^2-2(5I-2)J\}$$

$$= \frac{1}{5040} \sum_{I} I(10I^6-21I^5-35I^4+105I^3-35I^2-84I+60)$$

$$= \frac{N^{(3)}(N+1)}{20160}(5N^4+18N^3-N^2-22N+24)$$

$$\sum_{I>J>K} (I-J)^2(J-K)^2$$

$$= \sum_{I \geq J} \frac{J}{6}\{2J^4-(4I+3)J^3 + (2I^2+6I+1)J^2$$

$$- (3I+2)IJ + I^2\}$$

$$= \sum_{I} \frac{I}{360}(2I^5-6I^4+5I^3-7I+6)$$

$$= \frac{N^{(3)}}{2520} (N+1)(2N^3+4N^2+3N+6)$$

$$\therefore \sum_{I>J>K>L} |I-J||I-K||J-L||K-L|$$

$$= \sum_{I \geq J \geq K \geq L} |I-J||I-K||J-L||K-L| - \sum_{I>J>K} (I-J)^2(J-K)^2$$

$$= \frac{N^{(3)}(N+1)}{20160}\{5N^4 + 18N^3 - N^2 - 22N + 24 - 8(2N^3$$

$$+ 4N^2 + 3N + 6)\}$$

$$= \frac{N^{(4)}(N+1)}{20160}(53N^3+17N^2+18N+8). \qquad (3.22a)$$

Using a similar approach,

$$\sum_{K>J>L>I} |I-J||I-K||J-L||K-L|$$

$$= \sum_{K\geq J\geq L\geq I} |I-J||I-K||J-L||K-L|$$

$$- \sum_{I\geq J\geq K} \{(I-J)^2(I-K)^2 + (I-K)^2(J-K)^2\}$$

$$+ \sum_{J\geq I} (I-J)^4$$

$$= \frac{N^{(2)}(N+1)}{20160}(33N^5+160N^4+131N^3-232N^2-212N+48)$$

$$- \frac{N^{(2)}(N+1)}{5040}\{2(20N^4+42N^3-29N^2-63N+6)$$

$$- 40N^4 + 84N^3 - 58N^2 - 126N + 12\}$$

$$+ \frac{N^2(N^2-1)}{60}(2N^2-3)$$

$$= \frac{N^{(4)}(N+1)}{20160}(33N^4+5N^2-42N-8). \qquad (3.22b)$$

and

$$\sum_{L>K>I>J} |I-J||I-K||J-L||K-L|$$

$$= \sum_{L\geq K\geq I\geq J} |I-J||I-K||J-L||K-L|$$

$$= \frac{N^{(4)}(N+1)}{6720}(N^3+5N^2+6N). \qquad (3.22c)$$

Adding the results of (3.22a) through (3.22b), we get

$$\Sigma|I-J|\,|I-K|\,|J-L|\,|K-L|$$

$$= \frac{N^{(4)}N}{20160}(N+1)(41N^2+37N-6)$$

$$\therefore\ D(22) = \frac{N(N+1)}{2520}(41N^2+37N-6). \qquad (3.22)$$

$$D(23) \equiv \frac{1}{N^{(4)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{23}\Delta_{14}$$

$$= \frac{1}{N^{(4)}} \sum_{\neq} \Delta|I-J|\,|I-K|\,|J-K|\,|I-L|$$

$$= \frac{1}{N^{(4)}}[\Sigma|I-J|\,|I-K|\,|I-L| - 2N^{(3)}D(16)]$$

We can write

$$\Sigma|I-J|\,|I-K|\,|J-K|\,|I-L|$$

$$= \Sigma\left(I^2-(N+1)I+\frac{N(N+1)}{2}\right)|I-J|\,|I-K|\,|J-K|$$

The right side above is evaluated considering every possible inequality in I, J, K. Using the results, we get

$$D(23) = \frac{1}{N^{(4)}}[N^{(3)}\frac{N(N+1)(N+2)(15N^2-23)}{1260}$$

$$- 2N^{(3)}\frac{(N+1)(N+2)(5N^2-3)}{315}]$$

$$= \frac{(N+1)(N+2)}{1260}(15N^2+5N-8). \qquad (3.23)$$

$$D(24) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}^2 \Delta_{34}\Delta_{35}$$

$$= \frac{1}{N^{(5)}}[\Sigma\Delta_{12}^2 \Sigma\Delta_{12}\Delta_{13} - 2N^{(3)} D(13) - 4N^{(3)} D(14)$$

$$- 4N^{(3)} D(15) - 2N^{(3)} D(16) - N^{(4)} D(18)$$

$$- 2N^{(4)} D(19) - 4N^{(4)} D(21)] \text{ (using (2.10))}.$$

But,

$$\Sigma\Delta_{12}^2 \; \Sigma\Delta_{12}\Delta_{13} = \sum_{I,J}(I-J)^2 \sum_{I,J,K}|I-J|\,|I-K|$$

$$= N^{(2)}\frac{N(N+1)}{6} \cdot N^{(2)}\frac{(N+1)(7N^2-8)}{60}.$$

Using the above result and the results of D(13) through D(19) and D(21), upon simplification we get

$$D(24) = \frac{(N+1)}{2520}(49N^3+41N^2-42N-24). \qquad (3.24)$$

$$D(25) \equiv \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}^2 \Delta_{13}\Delta_{45}$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} (I-J)^2 |I-K|\,|L-M|$$

$$= \frac{1}{N^{(5)}}[\Sigma(I-J)^2|I-K|\Sigma|I-J| - 2N^{(2)} D(13)$$

$$- 2N^{(3)} \{3D(14) + D(15) + D(16)\} - N^{(4)} \{D(17)$$

$$+ 2D(19) + 2D(20) + 2D(21)\}] \text{ (from (2.9))}.$$

On simplification, we get

$$D(25) = \frac{(N+1)}{7560}(154N^3 + 126N^2 - 172N - 120). \qquad (3.25)$$

$$D(26) \equiv \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} |I-J|\,|I-K|\,|I-L|\,|I-M|$$

$$= \frac{1}{N^{(5)}}[\sum_I(\sum_J|I-J|)^4 - N^{(2)} D(13) - N^{(3)} \{4D(14)$$

$$+ 3D(15)\} - 6N^{(4)} D(19)] \text{ (from (2.14))}.$$

The first term in the bracket is

$$\sum_I(\sum_J|I-J|)^4$$

$$= \sum_I(I^2-(N+1)I+N\frac{(N+1)}{2})^4$$

$$= \sum_I[I^8 - 4(N+1)I^7 + 2(N+1)(4N+3)I^6 - 2(N+1)(5N^2+7N$$

$$+2)I^5 + \frac{(N+1)^2}{2}(17N^2+16N+2)I^4 - N(N+1)^3(5N+2)I^3$$

$$+ \frac{N^2(N+1)^3}{2}(4N+3)I^2 - \frac{N^3(N+1)^4}{2} + \frac{N^4(N+1)^4}{16}]$$

$$= \frac{N^{(2)}(N+1)}{5040}(83N^6-343N^4+560N^2-384)$$

Using the above and earlier results, we get

$$D(26) = \frac{(N+1)}{5040}(83N^3+39N^2-110N-48). \tag{3.26}$$

$$D(27) \equiv \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{35}$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} |I-J||I-K||J-L||K-M|$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} |I-J||I-K||K-L||L-M|$$

$$= \frac{1}{N^{(5)}} [\Sigma|I-J||I-K||K-L||L-M| - N^{(2)} D(13)$$

$$- N^{(3)} \{D(14) + 2D(15) + 3D(16)\}$$

$$- N^{(4)} \{D(19) + 2D(21) + D(22) + 2D(23)\}]$$

(from (2.18))

To evaluate the first term in the bracket, we first find

$$\sum_{\neq} |I-K||K-L|(I^2-(N+1)I+\frac{N(N+1)}{2})(L^2-(N+1)L+\frac{N(N+1)}{2})$$

$$= 2 \sum_{K>L>I} \{(K-L)(K-I)(I^2-(N+1)I+\frac{N(N+1)}{2})$$

$$\cdot(L^2-(N+1)L+\frac{N(N+1)}{2}) + (L-I)(K-L)(I^2-(N+1)I$$

$$+ \frac{N(N+1)}{2})(K^2-(N+1)K+\frac{N(N+1)}{2}) + (L-I)(K-I)$$

$$(L^2-(N+1)L+\frac{N(N+1)}{2})(K^2-(N+1)K+\frac{N(N+1)}{2})\}]$$

$$= 2\sum_{K>L>I}[\{L^3-(K+(N+1))L^2+(K^2+N(N+1))L-K^3$$

$$+ (N+1)K^2 - N(N+1)K\}I^3 - \{(K+N+1)L^3$$

$$- (K^2+2(N+1)K + (N+1)^2)L^2-(K^3-4(N+1)K^2$$

$$+ (N+1)^2K - \frac{3N(N+1)^2}{2})L - (N+1)K^3 + (N+1)K^2$$

$$- \frac{N(N+1)^2}{2}K - \frac{N^2(N+1)^2}{4}\}I^2 - \{(K-2N-2)KL^3$$

$$+ (K^3-2(N+1)K^2 + (N+1)(4N+3)K - \frac{N(N+1)^2}{2})L^2$$

$$- (N+1)^2(K^2+NK+\frac{N^2}{4})L + N(N+1)K(K^2-\frac{(N+1)}{2}K$$

$$+ \frac{3}{4}N(N+1))\}I + (K-N-1)K^2L^3 - (N+1)(K^3-(N+1)K^2$$

$$- \frac{N(N+1)}{2}K + \frac{N^2(N+1)}{4})L^2 + N(N+1)(K^3-\frac{3}{2}(N+1)K^2$$

$$+ \frac{N}{4}(N+1)K)L + \frac{N^2(N+1)^2}{4}K^2]$$

$$= \frac{1}{6} \sum_{K \geq L} [3L^7 - (7K+7N+13)L^6 + (K^2+(20N+32)K$$

$$+ (7N^2+23N+19))L^5 + \{7K^3-(13N+19)K^2$$

$$- (23N^2+61N+43)K - (N+1)(3N^2+15N+11)\}L^4$$

$$- \{2(4N+7)K^3 - (18N^2+50N+37)K^2 - 2(N+1)$$

$$\cdot (7N^2+19N+8)K + \frac{(N+1)}{2}(N^3-11N^2-22N-4)\}L^3$$

$$+ \{(6N^2+12N+5)K^3 - (N+1)(15N^2+33N+17)K^2$$

$$- \frac{(N+1)}{2}(3N^3+33N^2+32N-4)K - 3N(N+1)^2\}L^2$$

$$- \frac{(N+1)}{2}\{4(3N-1)K^3 - 2(3N^3+18N^2+15N-2)K^2$$

$$- N(N+1)(3N+2)K - N^2(N+1)\}L - 3N^2(N+1)^2K^2]$$

$$= \frac{1}{5040} \sum_{K} [791K^8 - 4(476N+1223)K^7 + 2(1288N^2+5096N+5663)K^6$$

$$- 4(441N^3+2856N^2+5012N+3122)K^5 + (735N^4+6510N^3$$

$$+ 17255N^2+18340N+7399)K^4 - 2(1155N^4+3780N^3$$

$$+ 5355N^2+4228N+1414)K^3 + (1785N^4+3570N^3$$

$$+ 2849N^2+1708N+644)K^2 - 6(35N^4+126N^3$$

$$+ 91N^2-28N-8)K]$$

$$= \frac{N^{(2)}(N+1)(N-2)}{22680}(319N^5+107N^4-883N^3-281N^2+756N+288)$$

(3.27a).

Now,

$$\sum_{I,K}(I-K)^2\left(I^2-(N+1)I + \frac{N(N+1)}{2}\right)^2$$

$$= N\sum_I\left(I^2-(N+1)I + \frac{N(N+1)}{2}\right)^2\left(I^2-(N+1)I + \frac{(N+1)(2N+1)}{6}\right)$$

$$= N\sum_I[I^6-3(N+1)I^5 + \frac{(N+1)}{6}(26N+19)I^4 - \frac{(N+1)^2}{3}$$

$$(11N+4)I^3 + \frac{(N+1)^2}{12}(23N^2+20N+2)I^2 - \frac{N(N+1)^3}{12}$$

$$(7N+2)I + \frac{N^2(N+1)^3(2N+1)}{24}]$$

$$= \frac{N^2(N^2-1)}{2520}(59N^4-165N^2+136) \qquad (3.27b)$$

Adding (3.27a) and (3.27b), we get $\Sigma|I-J||I-K||K-L||L-M|$. Using this and earlier results, we obtain

$$D(27) = \frac{(N+1)}{22680}(319N^2+477N^2-22N-156) \tag{3.27}$$

$$D(28) = \frac{1}{N^{(5)}} \sum_{\neq} \Delta_{12}\Delta_{13}\Delta_{14}\Delta_{25}$$

$$= \frac{1}{N^{(5)}} \sum_{\neq} |I-J|\,|I-K|\,|I-L|\,|J-M|$$

$$= \frac{1}{N^{(5)}}[\Sigma|I-J|\,|I-K|\,|I-L|\,|J-M| - N^{(2)}\ D(13)$$

$$- N^{(3)}\ \{3D(14) + D(15) + 3D(16)\} - N^{(4)}\ \{D(19)$$

$$+ 2D(20) + D(21) + 2D(23)\}] \text{ (using (2.17))}$$

The first term in the bracket $\Sigma|I-J|\,|I-K|\,|I-L|\,|J-M|$ is

$$\Sigma|I-J|\,(J^2-(N+1)J + \frac{N(N+1)}{2})\,(I^2-(N+1)I + \frac{N(N+1)}{2})^2$$

$$= \sum_{I>J}(I-J)\,(I^2-(N+1)I + \frac{N(N+1)}{2})\,(J^2-(N+1)J$$

$$+ \frac{N(N+1)}{2}\{I^2+J^2 - (N+1)(I+J) + N(N+1)\}$$

$$= \frac{N(N^2-1)}{15120}(223N^6-815N^4+1096N^2-576)$$

Using this and earlier results, we obtain

$$D(28) = \frac{(N+1)}{15120}(223N^3+303N^2-118N-168) \tag{3.28}$$

The remaining symmetric functions D(29) through (D35) are obtained easily using relationships given in section (2.4.2). Since calculations are straightforward, we give below the simplified results.

$$D(29) \equiv \frac{1}{N^{(5)}} \sum_{\neq} |I-J|\,|I-K|\,|J-K|\,|L-M|$$

$$= \frac{(N+1)(N+2)}{630}(7N^2+4N-3) \qquad (3.29)$$

$$D(30) \equiv \frac{1}{N^{(6)}} \sum_{\neq} (I-J)^2\ |K-L|\,|M-P|$$

$$= \frac{(N+1)}{1890}(35N^3+35N^2-28N-24) \qquad (3.30)$$

$$D(31) \equiv \frac{1}{N^{(6)}} \sum_{\neq} |I-J|\,|I-K|\,|I-L|\,|M-P|$$

$$= \frac{(N+1)}{630}(9N^3+11N^2-5N-6) \qquad (3.31)$$

$$D(32) \equiv \frac{1}{N^{(6)}} \sum_{\neq} |I-J|\,|I-K|\,|L-M|\,|L-P|$$

$$= \frac{(N+1)}{226800}(3087N^3+4523N^2-218N-1392) \qquad (3.32)$$

$$D(33) \equiv \frac{1}{N^{(6)}} \sum_{\neq} |I-J|\,|I-K|\,|J-L|\,|M-P|$$

$$= \frac{(N+1)}{11340}(153N^3+247N^2+2N-84) \qquad (3.33)$$

$$D(34) \equiv \frac{1}{N^{(7)}} \sum_{\neq} |I-J|\,|I-K|\,|L-M|\,|P-Q|$$

$$= \frac{(N+1)}{18900}(245N^3+401N^2+40N-104) \qquad (3.34)$$

$$D(35) \equiv \frac{1}{N^{(8)}} \sum_{\neq} |I-J|\,|K-L|\,|M-P|\,|Q-R|$$

$$= \frac{(N+1)}{14175}(175N^3+315N^2+86N-48) \qquad (3.35)$$

The values of the above symmetric functions are required in obtaining an expression of the fourth moment of δ when the distance function is the Euclidean distance between the ranks of observations.

3.3 Fourth Moment of δ_1

In the general expression of the fourth moment of the MRPP statistic, given by (2.30), if we put g=2, $C_i = \frac{n}{N}$, i=1, 2, with N=2n, we get

$$(2n^{(2)})^3 E(\delta_1^4) = 8D(13) + 16(n-2)\{4D(14) + 3D(15)$$

$$+ 6D(16)\} + 4\{(n-2)(n-3) + n(n-1)\}.\{4D(17)$$

$$+ 3D(18)\} + 48(n-2)(n-3)\{2D(19) + 2D(20)$$

$$+ 4D(21) + D(22) + 4D(23)\} + 48(n-2)$$

$$+ \{(n-3)(n-4) + n(n-1)\}\{D(24) + 2D(25)\}$$

$$D(34) \equiv \frac{1}{N^{(7)}} \sum_{\neq} |I-J|\,|I-K|\,|L-M|\,|P-Q|$$

$$= \frac{(N+1)}{18900}(245N^3+401N^2+40N-104) \qquad (3.34)$$

$$D(35) \equiv \frac{1}{N^{(8)}} \sum_{\neq} |I-J|\,|K-L|\,|M-P|\,|Q-R|$$

$$= \frac{(N+1)}{14175}(175N^3+315N^2+86N-48) \qquad (3.35)$$

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$$(2n^{(2)})^3 E(\delta_1^4) = 8D(13) + 16(n-2)\{4D(14) + 3D(15) + 6D(16)\} + 4\{(n-2)(n-3) + n(n-1)\}.\{4D(17) + 3D(18)\} + 48(n-2)(n-3)\{2D(19) + 2D(20) + 4D(21) + D(22) + 4D(23)\} + 48(n-2) + \{(n-3)(n-4) + n(n-1)\}\{D(24) + 2D(25)\}$$

$$+ 16(n-2)(n-3)(n-4)\{D(26) + 12D(27) + 12D(28)\}$$

$$+ 32(n-2)\{(n-3)(n-4) + n(n-1)\}D(29) + 12(n-2)$$

$$(n-3)\{(n-4)(n-5) + 3n(n-1)\}D(30) + 32(n-2)$$

$$(n-3)\{(n-4)(n-5) + n(n-1)\}\{D(31) + 3D(33)\}$$

$$+ 48(n-2)\{(n-3)(n-4)(n-5) + n(n-1)(n-2)\}D(32)$$

$$+ 24(n-2)(n-3)\{(n-4)(n-5)(n-6) + n(n-1)(n-2)$$

$$+ 2n(n-1)(n-4)\}D(34) + (n-2)(n-3)\{(n-4)(n-5)$$

$$(n-6)(n-7) + 4n(n-1)(n-4)(n-5) + 3n(n-1)$$

$$(n-2)(n-3)\}D(35).$$

Using the results of last section for D(13)-D(35), upon simplification we get

$$E(\delta_1^4) = \frac{(N+1)}{14175N^3(N-2)^3}(175N^8-525N^7+315N^6-1035N^5$$

$$+4818N^4-6180N^3+1640N^2+8640N-4608) \qquad (3.36)$$

We note the following from Mielke et al. (1981b, p. 722).

$$\mu(\delta_1) = \frac{N+1}{3}, \quad \sigma^2_{\delta_1} = \frac{4(N+1)}{45(N-2)}$$

and $$\gamma_{\delta_1} = -\frac{\sqrt{20}(4-11N^{-1}-6N^{-2})}{7(1-N^{-1}-2N^{-2})^{0.5}}$$

Using these, with the result (3.36), we obtain

$$\mu_4(\delta_1) = \frac{16(N+1)}{4725N^2(N-2)^3}(31N^4-175N^3+180N^2+260N-96) \tag{3.37}$$

$$\beta_1(\delta_1) = \frac{20}{49}\frac{(4N^2-11N-6)^2}{N^2(N-2)(N+1)} \tag{3.38}$$

$$\beta_2(\delta_1) = \frac{3}{7}\frac{(31N^4-175N^3+180N^2+260N-96)}{N^2(N-2)(N+1)} \tag{3.39}$$

$$2\beta_2-3\beta_1-6 = \frac{24}{49}\frac{(2N^4-74N^3+157N^2+125N-258)}{N^2(N-2)(N+1)} \tag{3.40}$$

In the following we present a table of β_1, β_2, $2\beta_2-3\beta_1-6$ and the Pearson criterion κ, given by (1.6), for the test statistic δ_1, for various values of N.

Table 3.1
Values of β_1, β_2, $2\beta_2-3\beta_1-6$ and κ for Selected N

N	β_1	β_2	$2\beta_2-3\beta_1-6$	κ
4	0.5000	1.5000	-4.5000	-0.1250
6	2.0991	4.3878	-3.5218	-0.7227
8	3.0995	6.3095	-2.6794	-1.5724
10	3.7410	7.5732	-2.0765	-2.6403
20	5.0963	10.3093	-0.6704	-12.9733
30	5.5664	11.2764	-0.1462	-68.2866
32	5.6257	11.3991	-0.0788	-128.7936
34	5.6782	11.5077	-0.0190	-541.0672
36	5.7249	11.6045	0.0344	303.7323
38	5.7668	11.6913	0.0824	128.2103
40	5.8045	11.7696	0.1257	84.8653
50	5.9483	12.0685	0.2920	38.0022
60	6.0446	12.2690	0.4041	28.1763
70	6.1136	12.4127	0.4848	23.9192
80	6.1654	12.5209	0.5457	21.5428
90	6.2058	12.6052	0.5932	20.0267
100	6.2381	12.6728	0.6313	18.9754
1000	6.5012	13.2240	0.9443	13.5664
∞	6.5306	13.2857	0.9796	13.1750

The above table indicates that for $N \le 20$ and $N \ge 80$, the value of $|2\beta_2-3\beta_1-6|$ is more than 0.5 while the condition for Pearson Type III distribution is "$2\beta_2-3\beta_1-6 = 0$". On the basis of the values of Pearson criterion and referring to (β_1, β_2) plot, we note that Pearson Type VI is recommended for $N>34$ and Pearson Type I for $N \le 34$.

We obtain empirical powers of δ_1 using above approximations as well as Pearson Type III approximation and compare these with the power of δ_2, in the following chapter.

CHAPTER IV

POWER OF SOME MRPP TESTS

4.1 Test Statistics

For the MRPP test statistic δ, defined by (1.3), we consider the univariate case with g=2, $n_1=n_2=N/2$, and the distance function given by the following:

$$\text{(i)} \quad \Delta_{xy} = |R(x) - R(y)|$$

$$\text{(ii)} \quad \Delta_{xy} = \{R(x) - R(Y)\}^2,$$

where R(x) is the rank of x in the combined sample. The above cases of distance functions give δ_1 and δ_2 respectively. As mentioned in section 1.5, δ_2 is an equivalent to the Wilcoxon test with which δ_1 competes.

The asymptotic distribution of δ_1 is not known. Therefore, we obtain powers of δ_1 using the Pearson Type III approximation as well as the approximation suggested by Table 3.1, that is, the Pearson Type VI for N>34 and Type I for N≤34. We compare these powers with the power of δ_2 - the Wilcoxon test, using the Pearson Type III distribution.

For the case of large samples, we consider two samples of size 40 each, while for the smaller sample case, we take samples of size 10 each. We conduct a simulation study on the basis of 10,000 independent samples. We describe the details in section 4.3. In the following section, we indicate transformations that enable us to use IMSL routines in order to obtain p values under different Pearson type approximations.

4.2 Transformations

This section is divided into three subsections according to the Pearson Type approximation used in calculating empirical powers of the test.

4.2.1 Pearson Type I Approximation

The Pearson Type I distribution with origin at the mean is given by

$$f(x) = K\left(1+\frac{x}{A_1}\right)^{m_1}\left(1-\frac{x}{A_2}\right)^{m_2}, \qquad -A_1 < x < A_2,$$

where

$$\frac{m_1+1}{A_1} = \frac{m_2+1}{A_2}$$

and

$$K = \frac{1}{A_1+A_2}\,\frac{(m_1+1)^{m_1}(m_2+1)^{m_2}}{(m_1+m_2+2)^{m_1+m_2}}\,\frac{\Gamma(m_1+m_2+2)}{\Gamma(m_1+1)\ \Gamma(m_2+1)}.$$

The transformation $u = \frac{A_1}{A_1+A_2}\left(1+\frac{x}{A_1}\right)$ gives

$$f(u) = \frac{1}{\beta(m_1+1, m_2+1)} u^{m_1}(1-u)^{m_2}, \quad 0<u<1.$$

Therefore, under Type I approximation,

$$\text{Prob}\,\{\delta_1 < \delta_1'\} = \text{Prob}\,\{u < u_0\},$$

where

$$u_0 = \frac{A_1}{A_1+A_2}\left(1+\frac{x_0}{A_1}\right) \quad \text{and } x_0 = \delta_1' - \mu_{\delta_1}.$$

$$\text{Prob}\,\{u<u_0\} = \int_0^{u_0} \frac{1}{\beta(m_1+1, m_2+1)} u^{m_1}(1-u)^{m_2} du, \quad 0 \le u_0 \le 1$$

The above probability is calculated using IMSL routine MDBETA. The constants, A_1, A_2, m_1, m_2 are obtained in a manner described in Elderton and Johnson (1969, pp. 51-52).

4.2.2 Pearson Type VI Approximation

The Pearson Type VI distribution, with origin at the mean and for $\mu_3<0$, is

$$f(x) = K\left(1+\frac{x}{A_1}\right)^{-q_1}\left(1+\frac{x}{A_2}\right)^{q_2}, \qquad x<-A_2,$$

where

$$K = \frac{1}{A_2-A_1} \frac{(q_2+1)^{q_2}(q_1-q_2-2)^{q_1-q_2}\,\Gamma(q_1)}{(q_1-1)^{q_1}\,\Gamma(q_1-q_2-1)\,\Gamma(q_2+1)}, \quad A_2 > A_1$$

and

$$\frac{q_1-1}{A_1} = \frac{q_2+1}{A_2}.$$

In this case, if we let $u = \frac{A_2+x}{A_1+x}$, we get

$$f(u) = \frac{\Gamma(q_1)}{\Gamma(q_2+1)\ \Gamma(q_1-q_2-1)} u^{q_2}(1-u)^{q_1-q_2-2}, \quad 0<u<1$$

The probability Prob $\{\delta_1<\delta_1'\}$ under this approximation is Prob $\{u>u_0\}$, where $u_0 = \frac{A_2+x_0}{A_1+x_0}$ and $x_0 = \delta_1' - \mu_{\delta_1}$.

Thus evaluating the constants, we can calculate the above probability, again using IMSL routine MDBETA.

4.2.3 Pearson Type III Approximation

The Pearson Type III distribution with origin at the mean and for $\mu_3<0$, is

$$f(x) = -\gamma\frac{(p+1)^p}{e^{p+1}\ \Gamma(p+1)}\left(1+\frac{x}{A}\right)^p e^{-\gamma x}, \qquad x<-A,$$

where $\gamma = \frac{2\mu_2}{\mu_3}<0$, $p = \gamma a = \frac{4}{\beta_1} -1$, $A = \frac{p+1}{\gamma}$ and

$$a = \frac{2\mu_2^2}{\mu_3} - \frac{\mu_3}{2\mu_2}.$$

The transformation $u = (p+1)+\gamma x$ gives

$$f(u) = \frac{1}{\Gamma(p+1)} u^p e^{-u}, \qquad u>0.$$

$\therefore$ Prob $\{\delta_1<\delta_1'\}$ = Prob $\{u>\frac{4}{\gamma^2} + \gamma x_0\}$, where $x_0 = \delta_1'-\mu_{\delta_1}$.

The RHS above is calculated using IMSL routing MDGAM, after calculation of constants.

4.3 Methodology

We consider the following symmetric underlying distributions for our simulation study:

1. Uniform
2. Normal
3. Logistic
4. $0.1N(0,9) + 0.9N(0,1)$
5. $0.1N(0,100) + 0.9N(0,1)$
6. Laplace (Double Exponential)
7. U-Shaped
8. Cauchy

In addition to the above, in order to observe the power performance of the tests in the case of asymmetric situations, we consider an exponential distribution as the underlying distribution. The U-shaped distribution that we consider is $f(x) = \frac{9\sqrt{15}}{50}x^2, -\sqrt{\frac{5}{3}}<x<\sqrt{\frac{5}{3}}$. The other distributions are standard distributions.

We generate 10 sets of 1000 independent pairs of samples from each of the above populations. Samples from the uniform, normal, Cauchy and exponential distributions are obtained by invoking IMSL routines GGUBS, GGNML, GGCAY and GGEXN, respectively. Samples from logistic, Laplace and U-shaped distributions are obtained by transforming the uniform random numbers. For a mixture of normal distributions, we obtain a random number U from uniform (0,1) for every random number drawn from N(0,1). When U is less than 0.1, we multiply the corresponding random number drawn from a normal population by σ; otherwise, we do not multiply. This gives a random observation from $0.1N(0,\sigma^2) + 0.9N(0,1)$.

To calculate the power of the tests against a shift of $K\sigma$, we shift first N_1 observations by $K\sigma$ and count the number of rejections. For Cauchy as underlying distribution, we take σ to be the solution of $F(\sigma) = 0.8413$; i.e., 1.83672. This approach is adopted by Randles and Hogg (1973).

We calculate standard deviations of the estimates of powers based on 1000 samples and obtain estimates of powers based on 10,000 samples. Results are indicated in Tables 4.1 to 4.9 for the large samples ($N_1=N_2=40$) and in Tables 4.10 to 4.18 for small samples ($N_1=N_2=10$). Following each table we provide a power plot which is obtained using the spline method of interpolation. The plot gives an overall view of the information contained in the table.

In the following section, we present our results and draw conclusions.

4.4 Results for Large Samples

We first present empirical powers of the tests for large samples.

In the case of large samples, powers of δ_1 are computed using the Pearson Type VI, as well as the Pearson Type III

Table 4.1

Empirical Powers, When Underlying Distribution is Uniform, $N_1=N_2=40$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0011 (.0010)	.0012 (.0010)	.0012 (.0010)	.0015 (.0018)	.0017 (.0017)	.0020 (.0016)
0.01	.0117 (.0036)	.0115 (.0036)	.0116 (.0032)	.0154 (.0041)	.0150 (.0042)	.0172 (.0037)
0.05	.0512 (.0076)	.0505 (.0078)	.0515 (.0072)	.0663 (.0075)	.0651 (.0075)	.0722 (.0062)
0.10	.1000 (.0041)	.0998 (.0041)	.0998 (.0066)	.1198 (.0082)	.1194 (.0080)	.1325 (.0074)

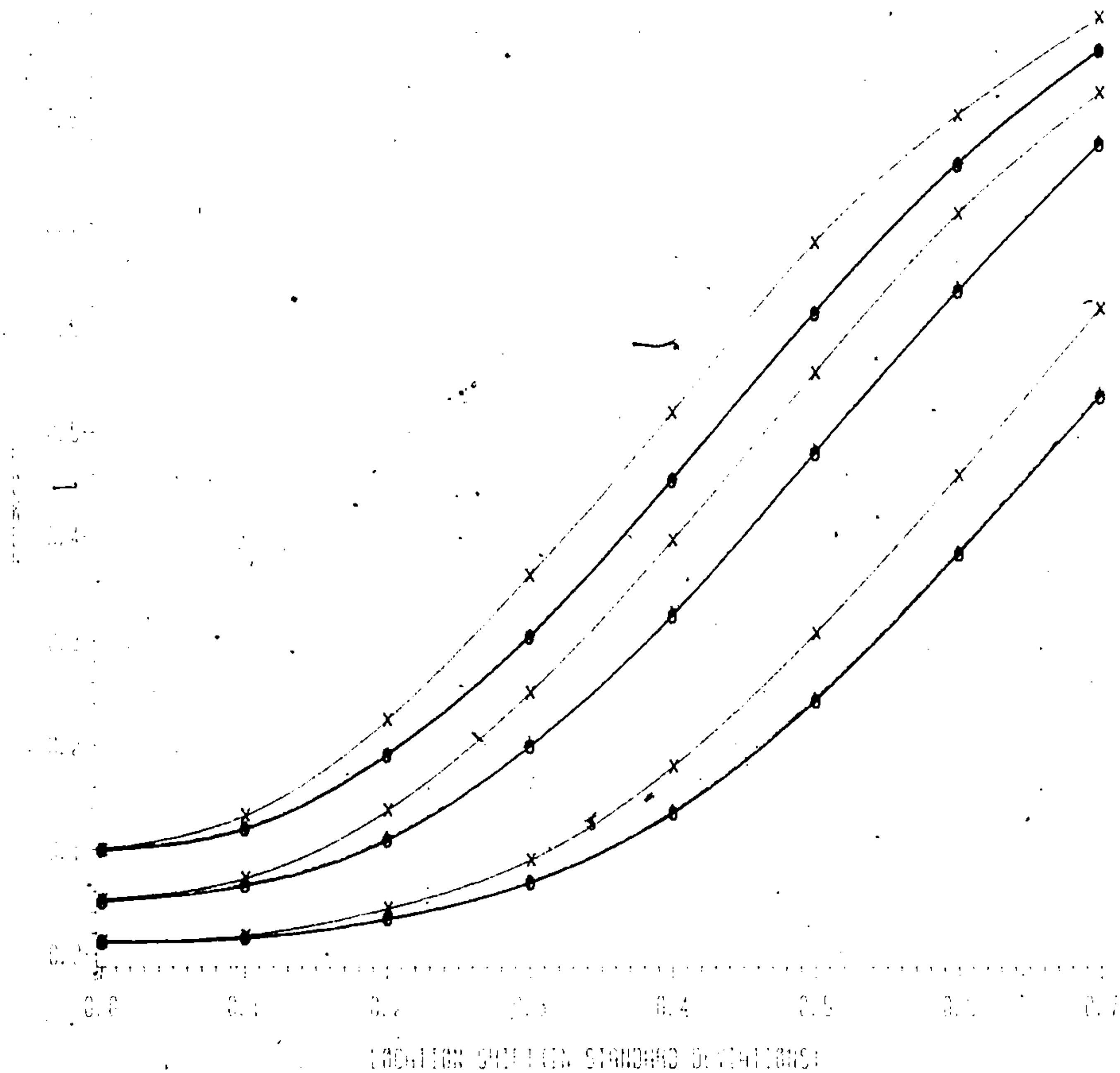
Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0053 (.0022)	.0053 (.0022)	.0071 (.0030)	.0133 (.0041)	.0142 (.0044)	.0191 (.0039)
0.01	.0342 (.0039)	.0336 (.0041)	.0435 (.0056)	.0689 (.0065)	.0682 (.0060)	.0898 (.0104)
0.05	.1103 (.0112)	.1089 (.0112)	.1377 (.0082)	.2014 (.0100)	.1986 (.0098)	.2500 (.0134)
0.10	.1908 (.0113)	.1902 (.0112)	.2244 (.0128)	.3040 (.0194)	.3033 (.0190)	.3621 (.0164)

Table 4.1 (cont'd.)

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0331 (.0027)	.0345 (.0030)	.0471 (.0045)	.0714 (.0071)	.0729 (.0075)	.0976 (.0090)
0.01	.1365 (.0097)	.1348 (.0097)	.1791 (.0082)	.2442 (.0118)	.2418 (.0116)	.3064 (.0179)
0.05	.3260 (.0171)	.3232 (.0162)	.3957 (.0201)	.4816 (.0194)	.4784 (.0202)	.5557 (.0233)
0.10	.4536 (.0215)	.4526 (.0212)	.5179 (.0248)	.6129 (.0198)	.6121 (.0198)	.6802 (.0166)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1410 (.0087)	.1450 (.0090)	.1914 (.0100)	.2469 (.0142)	.2534 (.0141)	.3177 (.0170)
0.01	.3848 (.0187)	.3824 (.0178)	.4578 (.0197)	.5367 (.0182)	.5326 (.0186)	.6170 (.0210)
0.05	.6373 (.0176)	.6341 (.0185)	.7082 (.0167)	.7767 (.0148)	.7742 (.0142)	.8238 (.0141)
0.10	.7557 (.0172)	.7551 (.0171)	.8028 (.0124)	.8644 (.0102)	.8630 (.0105)	.8957 (.0099)

POWER OF MRPP TESTS
UNDERLYING DISTRIBUTION: UNIFORM
$N_1 = N_2 = 40$



LEGEND: $\delta 16$ ---- $\delta 13$ ---- $\delta 23$ ----

FIGURE: 4.1

Table 4.2

Empirical Powers, When Underlying Distribution is Normal, $N_1=N_2=40$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0012 (.0011)	.0013 (.0011)	.0012 (.0010)	.0024 (.0013)	.0026 (.0013)	.0027 (.0013)
0.01	.0119 (.0028)	.0118 (.0030)	.0120 (.0025)	.0181 (.0041)	.0176 (.0038)	.0180 (.0054)
0.05	.0488 (.0065)	.0482 (.0063)	.0515 (.0089)	.0686 (.0061)	.0682 (.0061)	.0719 (.0066)
0.10	.0963 (.0101)	.0960 (.0101)	.1008 (.0103)	.1213 (.0096)	.1211 (.0096)	.1265 (.0089)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0069 (.0026)	.0075 (.0027)	.0077 (.0024)	.0209 (.0053)	.0219 (.0057)	.0212 (.0047)
0.01	.0417 (.0060)	.0414 (.0058)	.0437 (.0051)	.0887 (.0071)	.0876 (.0068)	.0936 (.0066)
0.05	.1265 (.0092)	.1248 (.0086)	.1347 (.0094)	.2329 (.0128)	.2307 (.0126)	.2506 (.0107)
0.10	.2049 (.0128)	.2040 (.0128)	.2220 (.0131)	.3428 (.0177)	.3416 (.0169)	.3639 (.0162)

Table 4.2 (cont'd).

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0489 (.0052)	.0511 (.0047)	.0523 (.0043)	.1050 (.0091)	.1078 (.0086)	.1126 (.0069)
0.01	.1778 (.0108)	.1753 (.0110)	.1895 (.0101)	.3094 (.0111)	.3072 (.0110)	.3302 (.0115)
0.05	.3826 (.0177)	.3800 (.0172)	.4044 (.0148)	.5448 (.0166)	.5415 (.0171)	.5747 (.0195)
0.10	.5003 (.0163)	.4999 (.0162)	.5268 (.0150)	.6666 (.0192)	.6660 (.0192)	.6948 (.0181)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2053 (.0103)	.2102 (.0101)	.2190 (.0086)	.3432 (.0112)	.3500 (.0100)	.3646 (.0081)
0.01	.4701 (.0173)	.4676 (.0177)	.4919 (.0146)	.6324 (.0201)	.6294 (.0203)	.6579 (.0158)
0.05	.7035 (.0181)	.7010 (.0177)	.7345 (.0174)	.8318 (.0169)	.8305 (.0171)	.8551 (.0160)
0.10	.8037 (.0176)	.8034 (.0176)	.8291 (.0177)	.8981 (.0086)	.8977 (.0088)	.9148 (.0085)

POWER OF LRPP TESTS

UNDERLYING DISTRIBUTION: NORMAL

$N_1 = N_2 = 40$

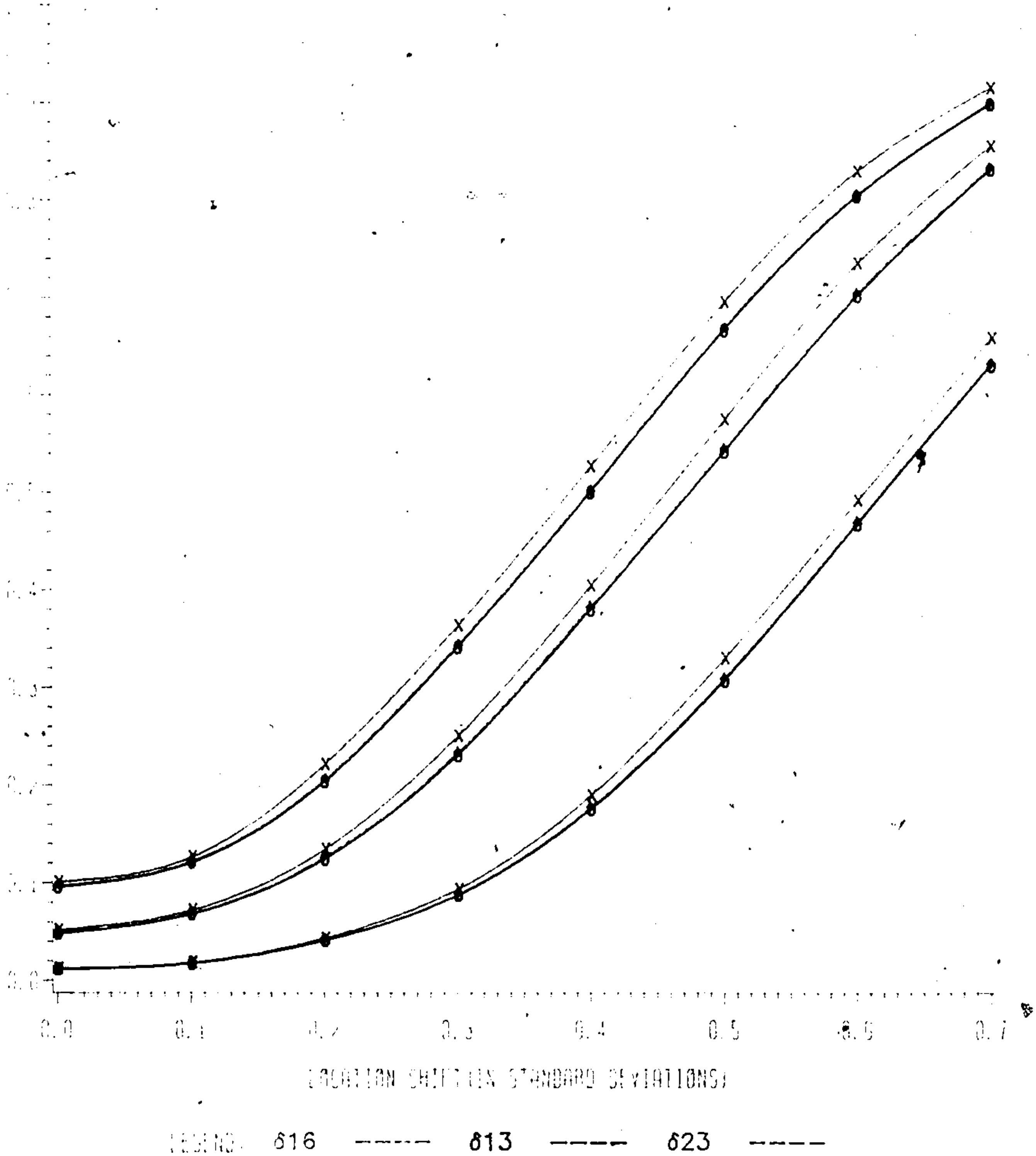


FIGURE: 4.2

Table 4.3
Empirical Powers, When Underlying Distribution is Logistic, $N_1 = N_2 = 40$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0009 (.0011)	.0009 (.0011)	.0010 (.0011)	.0027 (.0016)	.0029 (.0017)	.0029 (.0015)
0.01	.0106 (.0037)	.0106 (.0037)	.0118 (.0041)	.0192 (.0036)	.0190 (.0037)	.0196 (.0051)
0.05	.0527 (.0079)	.0517 (.0077)	.0552 (.0095)	.0781 (.0059)	.0772 (.0057)	.0790 (.0075)
0.10	.1067 (.0115)	.1064 (.0115)	.1106 (.0121)	.1370 (.0120)	.1368 (.0125)	.1417 (.0095)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0101 (.0032)	.0107 (.0034)	.0095 (.0031)	.0299 (.0047)	.0306 (.0046)	.0274 (.0046)
0.01	.0523 (.0064)	.0521 (.0064)	.0525 (.0067)	.1201 (.0077)	.1184 (.0075)	.1188 (.0063)
0.05	.1531 (.0106)	.1514 (.0103)	.1587 (.0115)	.2783 (.0161)	.2759 (.0153)	.2888 (.0132)
0.10	.2384 (.0112)	.2376 (.0108)	.2470 (.0117)	.3909 (.0123)	.3901 (.0120)	.4051 (.0150)

Table 4.3 (cont'd.)

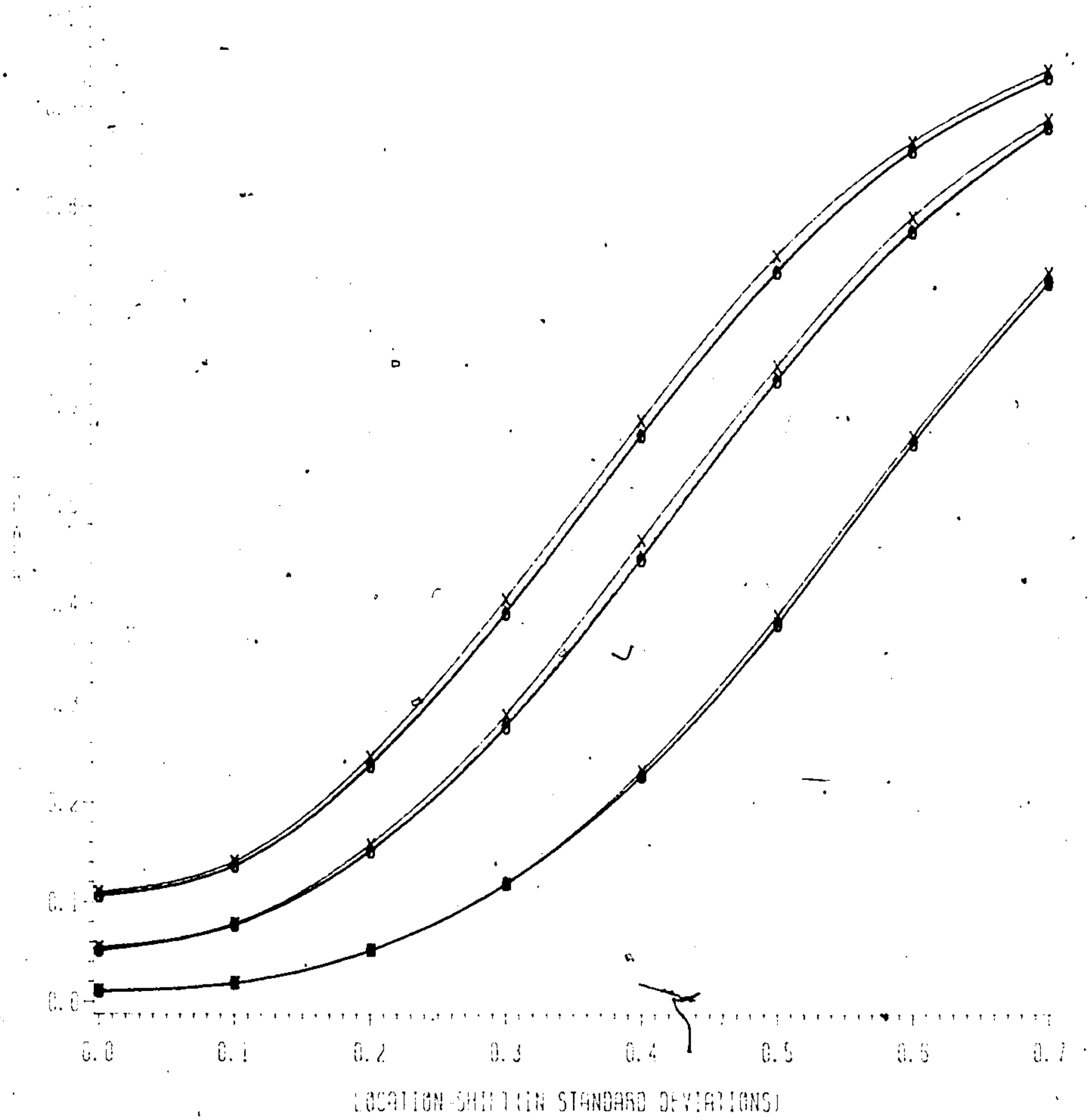
Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0722 (.0045)	.0743 (.0045)	.0714 (.0054)	.1536 (.0077)	.1568 (.0079)	.1532 (.0086)
0.01	.2286 (.0130)	.2268 (.0128)	.2325 (.0153)	.3814 (.0115)	.3792 (.0114)	.3885 (.0151)
0.05	.4479 (.0166)	.4445 (.0166)	.4641 (.0142)	.6263 (.0172)	.6237 (.0167)	.6389 (.0153)
0.10	.5689 (.0179)	.5684 (.0177)	.5840 (.0205)	.7336 (.0119)	.7329 (.0119)	.7501 (.0084)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2811 (.0134)	.2868 (.0130)	.2808 (.0125)	.4398 (.0172)	.4472 (.0125)	.4423 (.0140)
0.01	.5641 (.0184)	.5608 (.0177)	.5687 (.0202)	.7241 (.0120)	.7220 (.0125)	.7328 (.0140)
0.05	.7759 (.0089)	.7739 (.0094)	.7894 (.0088)	.8795 (.0081)	.8783 (.0076)	.8874 (.0083)
0.10	.8554 (.0074)	.8551 (.0076)	.8651 (.0080)	.9287 (.0089)	.9285 (.0089)	.9369 (.0073)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: LOGISTIC

$N_1 = N_2 = 40$



LEGEND: δ16 ------ δ13 ---- δ23 ----

FIGURE: 4.3

Table 4.4

Empirical Powers, When Underlying Distribution is $.1N(0,9) + .9N(0,1), N_1=N_2=40$

Shift: $K\sigma$	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0009 (.0011)	.0010 (.0012)	.0009 (.0009)	.0038 (.0018)	.0039 (.0019)	.0032 (.0014)
0.01	.0106 (.0045)	.0104 (.0043)	.0105 (.0041)	.0226 (.0060)	.0224 (.0059)	.0213 (.0057)
0.05	.0515 (.0075)	.0507 (.0076)	.0521 (.0077)	.0838 (.0081)	.0827 (.0080)	.0861 (.0098)
0.10	.0998 (.0079)	.0993 (.0081)	.1009 (.0111)	.1466 (.0107)	.1459 (.0102)	.1518 (.0131)

Shift: $K\sigma$	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0126 (.0030)	.0128 (.0029)	.0129 (.0034)	.0394 (.0074)	.0414 (.0064)	.0396 (.0054)
0.01	.0641 (.0072)	.0631 (.0073)	.0641 (.0074)	.1527 (.0099)	.1518 (.0102)	.1527 (.0131)
0.05	.1772 (.0106)	.1759 (.0104)	.1840 (.0099)	.3423 (.0116)	.3403 (.0112)	.3512 (.0130)
0.10	.2745 (.0120)	.2737 (.0116)	.2819 (.0134)	.4575 (.0124)	.4564 (.0128)	.4717 (.0119)

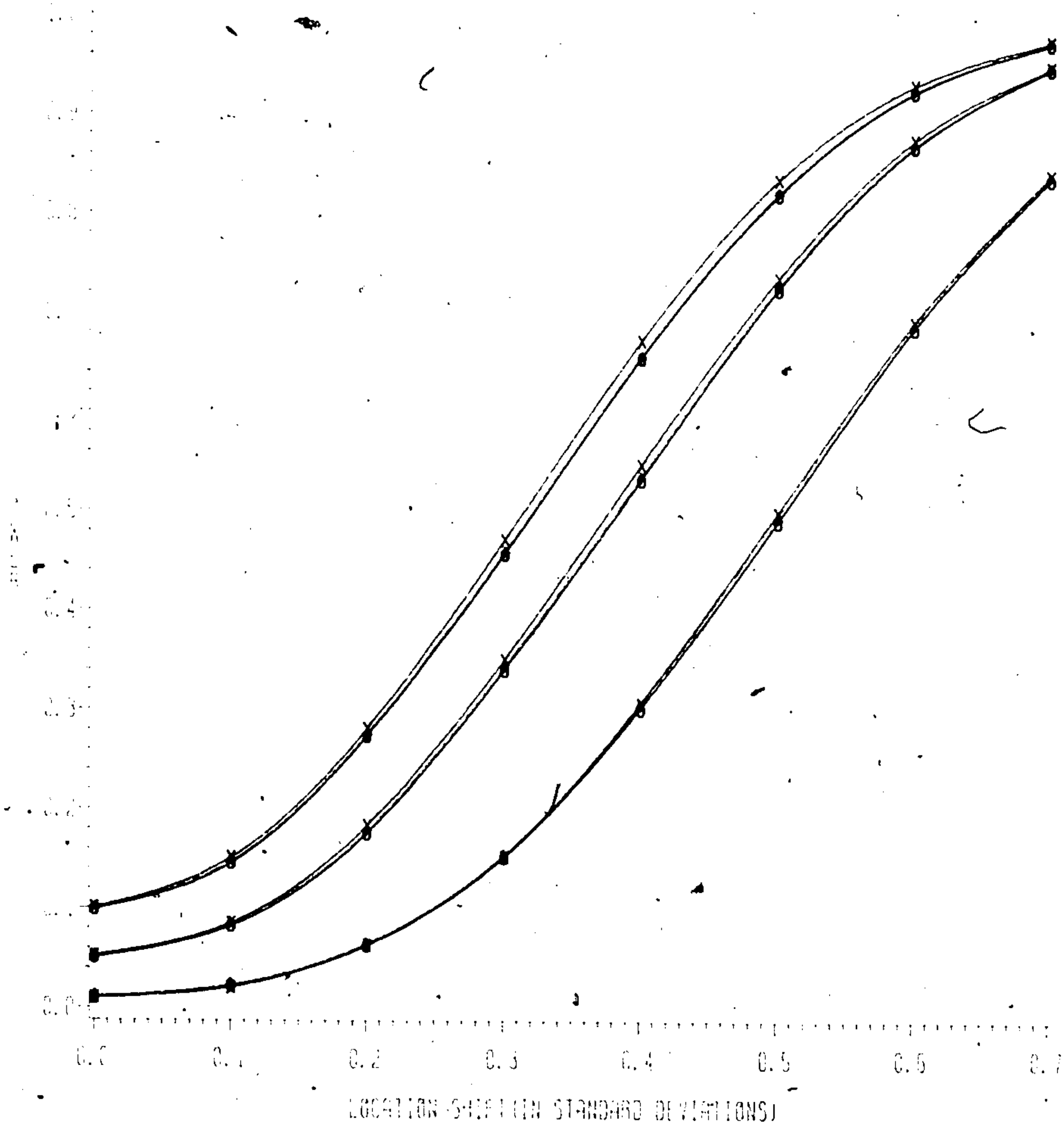
Table 4.4 (cont'd.)

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1067 (.0087)	.1086 (.0085)	.1051 (.0103)	.2255 (.0120)	.2318 (.0106)	.2256 (.0128)
0.01	.3042 (.0095)	.3016 (.0099)	.3072 (.0132)	.4928 (.0159)	.4898 (.0154)	.4995) (.0139)
0.05	.5334 (.0130)	.5314 (.0132)	.5474 (.0147)	.7255 (.0112)	.7235 (.0112)	.7354 (.0126)
0.10	.6543 (.0127)	.6538 (.0128)	.6721 (.0138)	.8192 (.0150)	.8187 (.0149)	.8345 (.0141)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.3951 (.0119)	.4019 (.0130)	.3972 (.0112)	.5961 (.0140)	.6022 (.0121)	.5916 (.0119)
0.01	.6864 (.0121)	.6843 (.0124)	.6916 (.0130)	.8364 (.0148)	.8349 (.0148)	.8398 (.0133)
0.05	.8675 (.0138)	.8672 (.0135)	.8746 (.0123)	.9461 (.0099)	.9458 (.0101)	.9483 (.0082)
0.10	.9223 (.0104)	.9221 (.0102)	.9298 (.0099)	.9711 (.0046)	.9711 (.0046)	.9746 (.0055)

POWER OF LRPP TESTS
UNDERLYING DISTRIBUTION: 10% 3N

$N_1 = N_2 = 40$



LEGEND: δ16 ----- δ13 ----- δ23 -----

FIGURE: 4.4

Table 4.5

Empirical Powers, When Underlying Distribution is .1N(0,100) + .9N(0,1), N_1=N_2=40

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0014 (.0014)	.0016 (.0016)	.0014 (.0012)	.0170 (.0042)	.0174 (.0041)	.0158 (.0037)
0.01	.0103 (.0034)	.0103 (.0034)	.0104 (.0029)	.0819 (.0075)	.0809 (.0074)	.0798 (.0100)
0.05	.0461 (.0068)	.0455 (.0067)	.0477 (.0070)	.2210 (.0140)	.2184 (.0147)	.2216 (.0177)
0.10	.0957 (.0096)	.0954 (.0096)	.0969 (.0081)	.3257 (.0161)	.3248 (.0158)	.3312 (.0177)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1739 (.0126)	.1792 (.0144)	.1608 (.0093)	.5861 (.0200)	.5935 (.0194)	.5485 (.0197)
0.01	.4272 (.0169)	.4243 (.0176)	.4087 (.0198)	.8359 (.0115)	.8339 (.0122)	.8125 (.0135)
0.05	.6652 (.0178)	.6621 (.0178)	.6558 (.0164)	.9451 (.0059)	.9447 (.0057)	.9359 (.0047)
0.10	.7701 (.0160)	.7692 (.0158)	.7694 (.0142)	.9711 (.0056)	.9711 (.0056)	.9682 (.0049)

Table 4.5 (cont'd.)

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.9098 (.0074)	.9124 (.0067)	.8752 (.0104)	.9907 (.0033)	.9911 (.0035)	.9807 (.0053)
0.01	.9818 (.0049)	.9814 (.0047)	.9736 (.0055)	.9992 (.0013)	.9992 (.0013)	.9978 (.0016)
0.05	.9964 (.0025)	.9963 (.0025)	.9948 (.0031)	.9998 (.0010)	.9998 (.0010)	.9997 (.0011)
0.10	.9990 (.0015)	.9989 (.0014)	.9977 (.0017)	1.000 (0)	1.000 (0)	1.000 (0)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.9995 (.0011)	.9995 (.0011)	.9977 (.0016)	1.000 (0)	1.000 (0)	0.999 (.0003)
0.01	1.000 (0)	1.000 (0)	.9999 (.0003)	1.000 (0)	1.000 (0)	1.000 (0)
0.05	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)
0.10	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)	1.000 (0)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: 10% 10N

$N_1 = N_2 = 40$

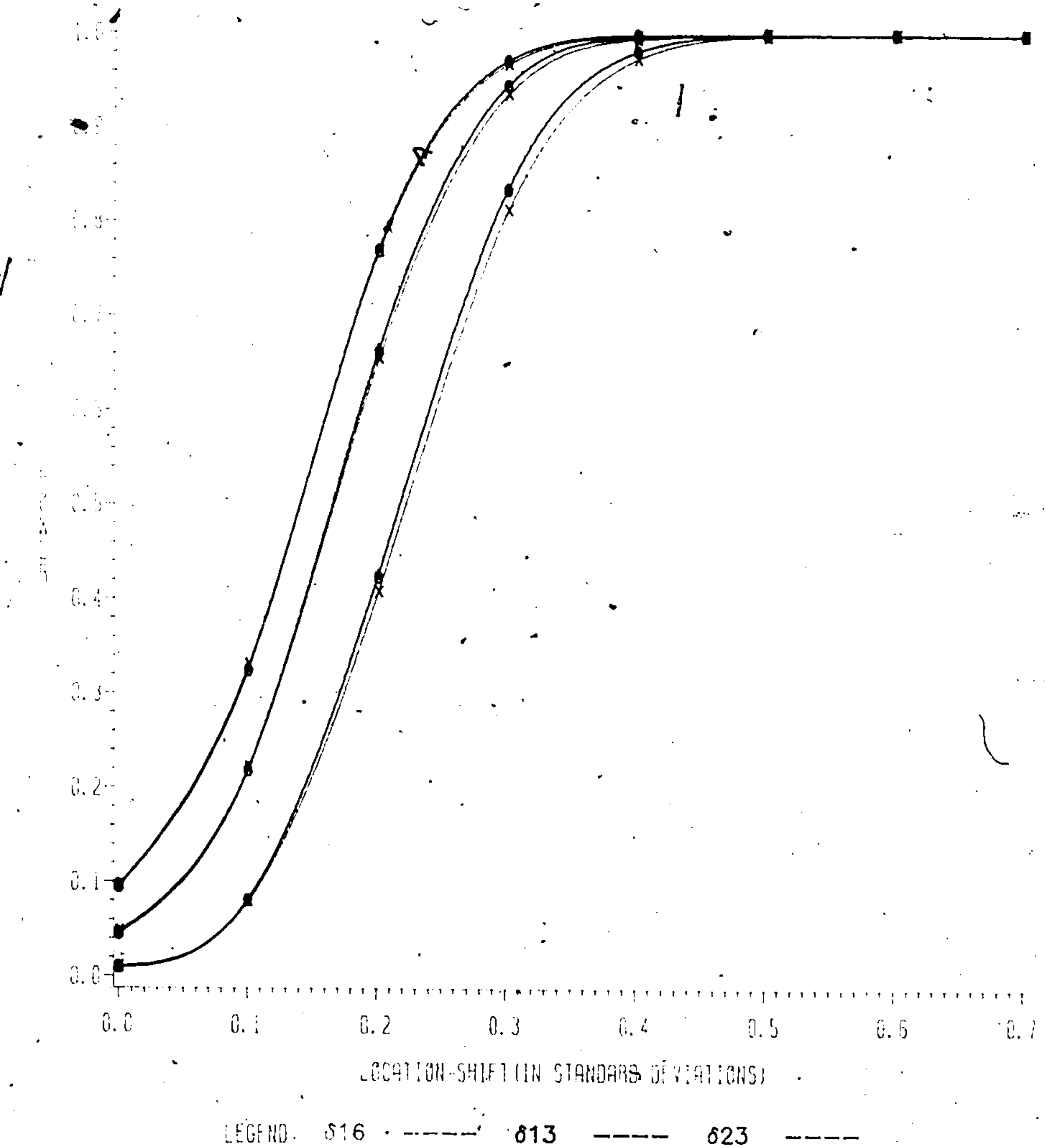


FIGURE: 4.5

Table 4.6
Empirical Powers, When Underlying Distribution is Laplace (Double Exponential); $N_1=N_2=40$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0009 (.0009)	.0010 (.0011)	.0007 (.0008)	.0024 (.0011)	.0027 (.0013)	.0019 (.0010)
0.01	.0108 (.0030)	.0106 (.0030)	.0112 (.0042)	.0216 (.0056)	.0209 (.0050)	.0205 (.0041)
0.05	.0477 (.0055)	.0471 (.0052)	.0502 (.0047)	.0856 (.0087)	.0851 (.0090)	.0871 (.0094)
0.10	.0967 (.0098)	.0960 (.0096)	.0986 (.0079)	.1480 (.0082)	.1477 (.0081)	.1508 (.0110)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0133 (.0037)	.0141 (.0045)	.0122 (.0032)	.0503 (.0076)	.0521 (.0079)	.0422 (.0064)
0.01	.0728 (.0095)	.0722 (.0091)	.0672 (.0089)	.1794 (.0109)	.1774 (.0106)	.1632 (.0120)
0.05	.1982 (.0119)	.1955 (.0121)	.1938 (.0104)	.3782 (.0202)	.3759 (.0199)	.3628 (.0199)
0.10	.2987 (.0166)	.2982 (.0168)	.2947 (.0138)	.4975 (.0219)	.4969 (.0223)	.4851 (.0241)

Table 4.6 (cont'd).

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1281 (.0121)	.1319 (.0121)	.1124 (.0125)	.2652 (.0158)	.2706 (.0155)	.2389 (.0155)
0.01	.3497 (.0209)	.3481 (.0208)	.3195 (.0195)	.5482 (.0222)	.5461 (.0218)	.5119 (.0203)
0.05	.5882 (.0203)	.5854 (.0206)	.5657 (.0198)	.7721 (.0120)	.7707 (.0118)	.7442 (.0114)
0.10	.7058 (.0149)	.7053 (.0150)	.6836 (.0150)	.8530 (.0104)	.8525 (.0105)	.8400 (.0113)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.4506 (.0245)	.4568 (.0230)	.4050 (.0222)	.6359 (.0152)	.6420 (.0151)	.5861 (.0170)
0.01	.7322 (.0120)	.7305 (.0123)	.6915 (.0138)	.8660 (.0088)	.8646 (.0095)	.8375 (.0109)
0.05	.8931 (.0059)	.8917 (.0065)	.8762 (.0086)	.9590 (.0046)	.9584 (.0048)	.9481 (.0050)
0.10	.9415 (.0061)	.9414 (.0062)	.9302 (.0062)	.9785 (.0043)	.9784 (.0043)	.9734 (.0043)

POWER OF LRPP TESTS
UNDERLYING DISTRIBUTION: LAPLACE
$N_1 = N_2 = 40$

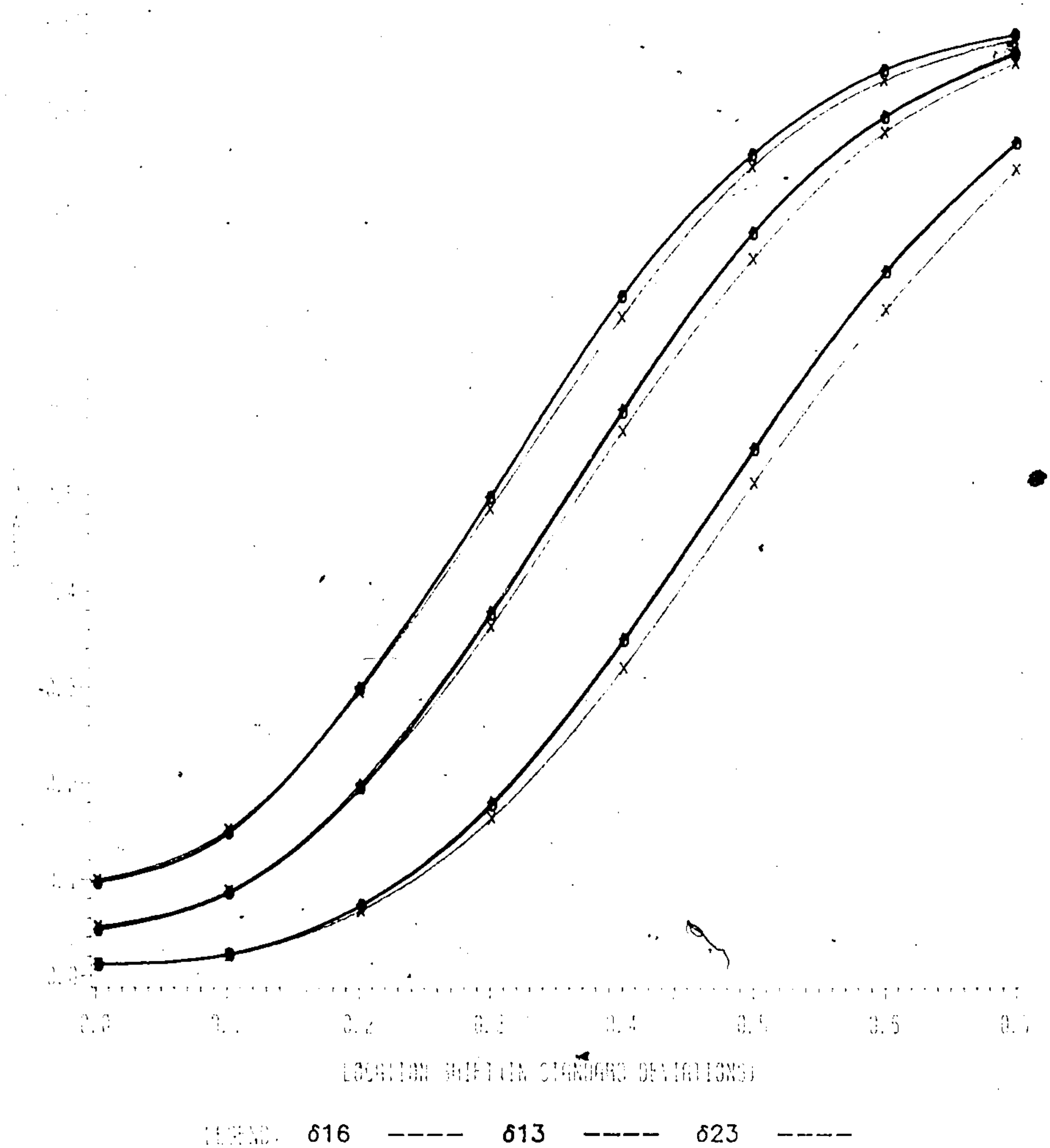


FIGURE: 4.6

Table 4.7

Empirical Powers, When Underlying Distribution is U-shaped, $N_1 = N_2 = 40$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0004 (.0005)	.0005 (.0007)	.0002 (.0004)	.0048 (.0020)	.0050 (.0020)	.0075 (.0024)
0.01	.0079 (.0032)	.0078 (.0033)	.0072 (.0030)	.0322 (.0056)	.0317 (.0054)	.0492 (.0044)
0.05	.0462 (.0078)	.0458 (.0076)	.0469 (.0085)	.1247 (.0100)	.1228 (.0101)	.1566 (.0115)
0.10	.0942 (.0102)	.0941 (.0102)	.0953 (.0086)	.2121 (.0141)	.2109 (.0143)	.2475 (.0121)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0362 (.0051)	.0376 (.0056)	.0531 (.0064)	.1411 (.0123)	.1447 (.0127)	.1743 (.0121)
0.01	.1595 (.0093)	.1581 (.0085)	.2004 (.0119)	.4275 (.0099)	.4243 (.0097)	.4359 (.0086)
0.05	.4099 (.0125)	.4063 (.0121)	.4299 (.0111)	.7404 (.0113)	.7370 (.0118)	.6843 (.0153)
0.10	.5726 (.0088)	.5711 (.0080)	.5592 (.0100)	.8807 (.0099)	.8796 (.0096)	.7840 (.0125)

Table 4.7 (cont'd.)

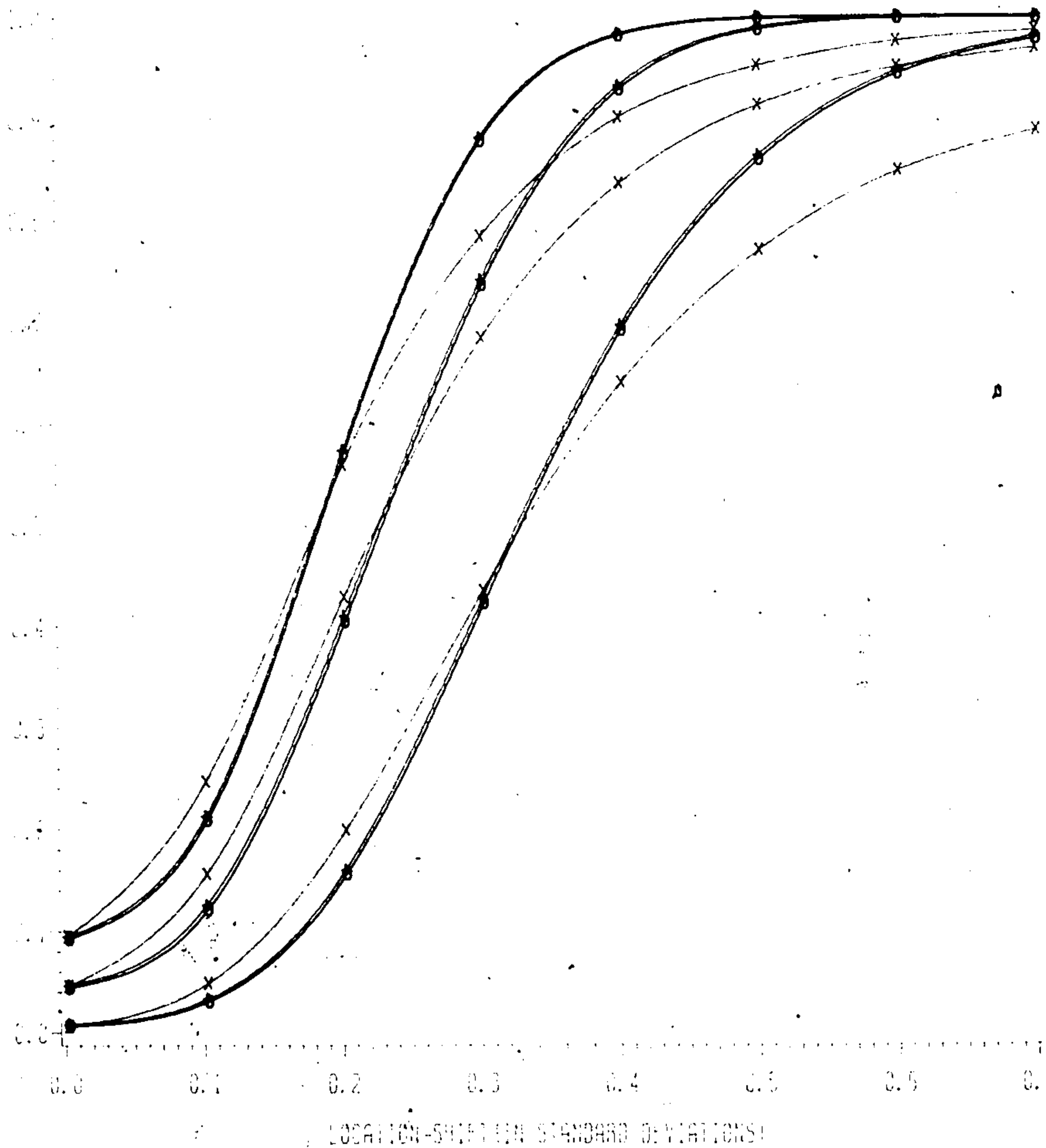
Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.3285 (.0168)	.3361 (.0156)	.3419 (.0179)	.5307 (.0100)	.5393 (.0102)	.4986 (.0108)
0.01	.6959 (.0123)	.6926 (.0126)	.6400 (.0104)	.8632 (.0106)	.8608 (.0105)	.7703 (.0132)
0.05	.9319 (.0084)	.9301 (.0082)	.8366 (.0090)	.9897 (.0027)	.9892 (.0027)	.9141 (.0065)
0.10	.9833 (.0032)	.9830 (.0029)	.9016 (.0061)	.9997 (.0005)	.9995 (.0007)	.9525 (.0053)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.6830 (.0124)	.6912 (.0113)	.6156 (.0094)	.7817 (.0121)	.7887 (.0124)	.6877 (.0108)
0.01	.9465 (.0052)	.9450 (.0051)	.8487 (.0097)	.9810 (.0049)	.9795 (.0047)	.8885 (.0082)
0.05	.9999 (.0003)	.9999 (.0003)	.9514 (.0039)	1.000 (0)	1.000 (0)	.9697 (.0037)
0.10	1.000 (0)	1.000 (0)	.9767 (.0051)	1.000 (0)	1.000 (0)	.9867 (.0025)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: U-SHAPED

$N_1 = N_2 = 40$



LEGEND: δ16 ---- δ13 ---- δ23 ----

FIGURE: 4.7

Table 4.8

Empirical Powers, When Underlying Distribution is Cauchy, $N_1=N_2=40$

Shift: $K\sigma$	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0009 (.0012)	.0009 (.0012)	.0007 (.0009)	.0027 (.0020)	.0028 (.0021)	.0024 (.0018)
0.01	.0091 (.0037)	.0090 (.0036)	.0096 (.0032)	.0179 (.0057)	.0178 (.0056)	.0170 (.0058)
0.05	.0460 (.0093)	.0454 (.0097)	.0478 (.0105)	.0730 (.0076)	.0716 (.0074)	.0702 (.0075)
0.10	.0928 (.0116)	.0923 (.0114)	.0955 (.0146)	.1376 (.0121)	.1367 (.0121)	.1294 (.0129)

Shift: $K\sigma$	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0087 (.0035)	.0092 (.0040)	.0075 (.0040)	.0294 (.0070)	.0305 (.0071)	.0210 (.0066)
0.01	.0524 (.0087)	.0516 (.0085)	.0432 (.0091)	.1292 (.0138)	.1275 (.0134)	.1004 (.0107)
0.05	.1634 (.0111)	.1623 (.0112)	.1442 (.0125)	.3082 (.0165)	.3064 (.0170)	.2688 (.0142)
0.10	.2533 (.0136)	.2527 (.0129)	.2363 (.0148)	.4211 (.0140)	.4204 (.0142)	.3849 (.0120)

Table 4.8 (cont'd.)

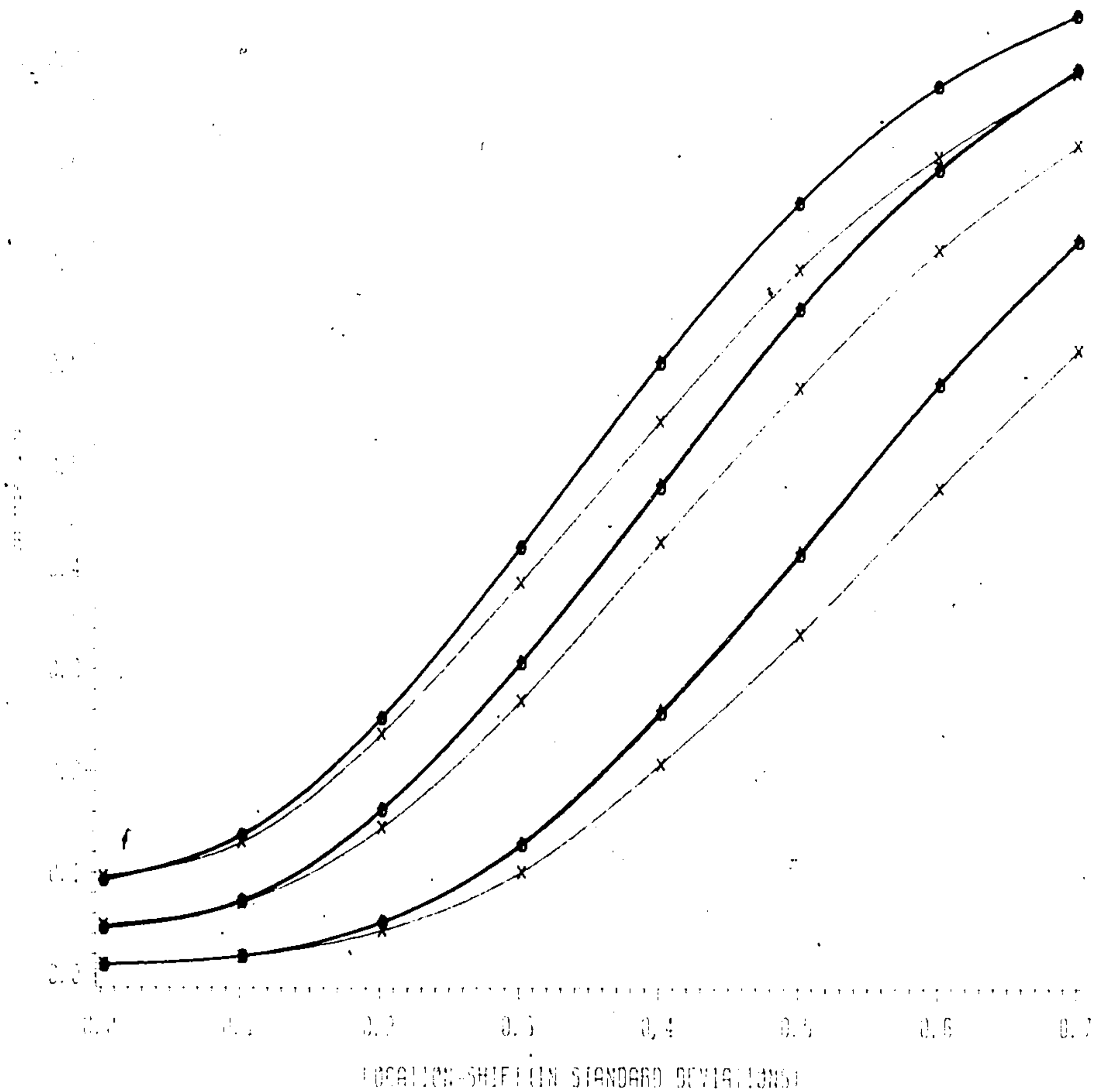
Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0775 (.0085)	.0798 (.0075)	.0555 (.0092)	.1702 (.0118)	.1743 (.0120)	.1163 (.0109)
0.01	.2602 (.0130)	.2575 (.0139)	.2070 (.0143)	.4164 (.0129)	.4137 (.0132)	.3347 (.0169)
0.05	.4829 (.0121)	.4797 (.0128)	.4252 (.0121)	.6563 (.0101)	.6545 (.0102)	.5770 (.0106)
0.10	.6029 (.0101)	.6017 (.0105)	.5438 (.0136)	.7597 (.0097)	.7591 (.0094)	.6933 (.0114)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2989 (.0154)	.3043 (.0153)	.2165 (.0114)	.4447 (.0097)	.4502 (.0102)	.3343 (.0135)
0.01	.5827 (.0074)	.5805 (.0070)	.4783 (.0009)	.7239 (.0114)	.7214 (.0112)	.6142 (.0126)
0.05	.7943 (.0066)	.7919 (.0071)	.7124 (.0115)	.8915 (.0066)	.8908 (.0065)	.8154 (.0104)
0.10	.8747 (.0057)	.8743 (.0056)	.8045 (.0107)	.9441 (.0050)	.9439 (.0052)	.8867 (.0080)

POWER OF LRPP TESTS

UNDERLYING DISTRIBUTION: CAUCHY

$N_1 = N_2 = 40$



LEGEND: δ16 ---- δ13 ---- δ23 ----

FIGURE: 4.8

Table 4.9

Empirical Powers, When Underlying Distribution is Exponential, $N_1=N_2= 40$

Shift: $K\sigma$	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0006 (.0005)	.0006 (.0007)	.0006 (.0007)	.0039 (.0018)	.0040 (.0019)	.0047 (.0013)
0.01	.0088 (.0023)	.0085 (.0022)	.0087 (.0018)	.0255 (.0054)	.0252 (.0055)	.0301 (.0067)
0.05	.0448 (.0063)	.0444 (.0065)	.0481 (.0073)	.1032 (.0138)	.1022 (.0139)	.1106 (.0147)
0.10	.0927 (.0096)	.0924 (.0094)	.0953 (.0112)	.1776 (.0124)	.1767 (.0125)	.1862 (.0143)

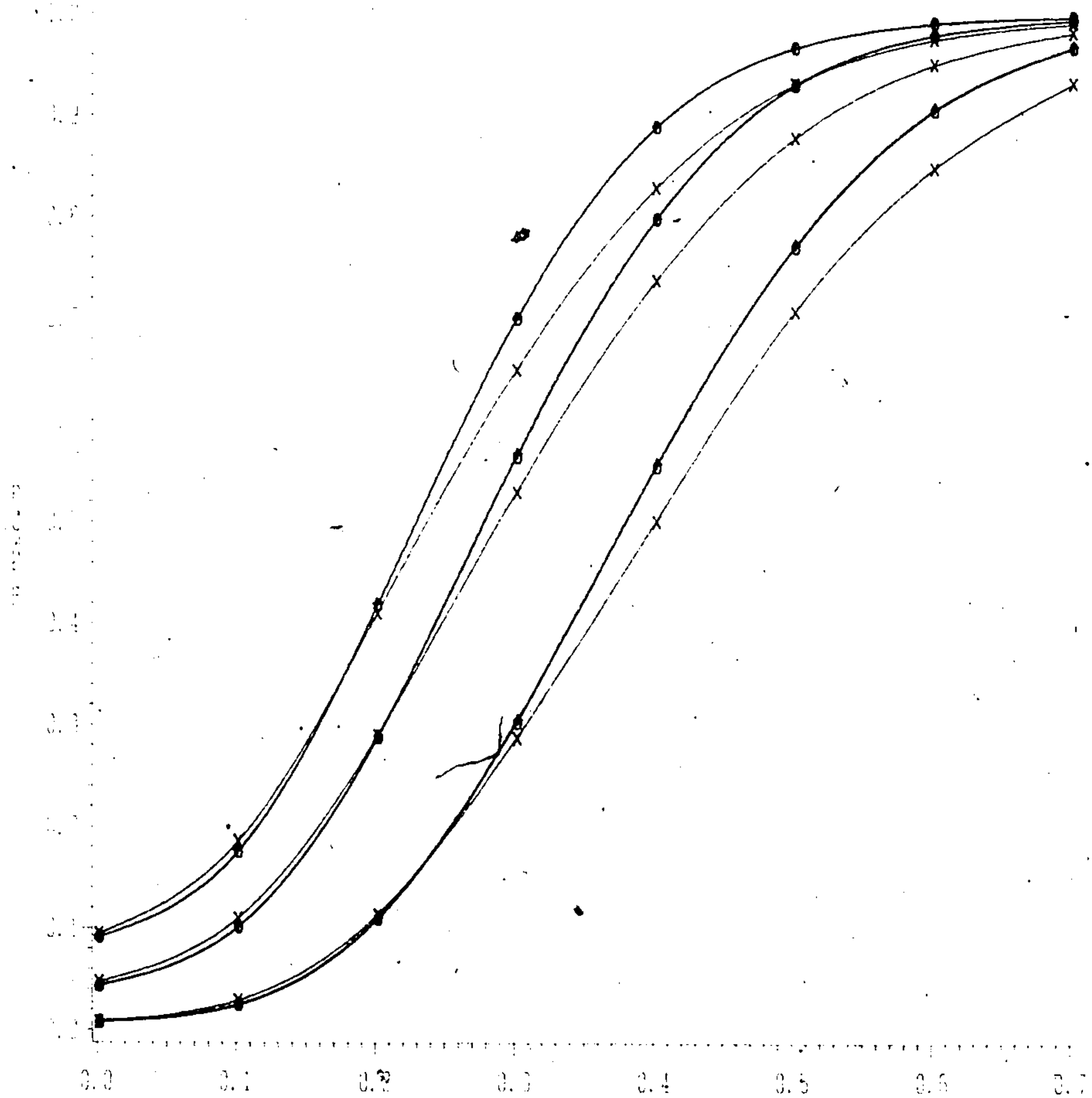
Shift: $K\sigma$	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0221 (.0054)	.0229 (.0050)	.0246 (.0058)	.0933 (.0133)	.0948 (.0133)	.0892 (.0134)
0.01	.1117 (.0145)	.1101 (.0139)	.1140 (.0142)	.3057 (.0183)	.3034 (.0188)	.2877 (.0186)
0.05	.2910 (.0194)	.2890 (.0190)	.2904 (.0159)	.5682 (.0204)	.5648 (.0204)	.5300 (.0257)
0.10	.4207 (.0226)	.4197 (.0233)	.4111 (.0252)	.7013 (.0183)	.7009 (.0183)	.6500 (.0205)

Table 4.9 (cont'd.)

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2563 (.0196)	.2618 (.0199)	.2294 (.0164)	.4741 (.0239)	.4819 (.0239)	.4191 (.0253)
0.01	.5576 (.0226)	.5552 (.0223)	.5016 (.0230)	.7734 (.0218)	.7715 (.0223)	.7079 (.0228)
0.05	.8004 (.0184)	.7988 (.0184)	.7382 (.0203)	.9319 (.0077)	.9308 (.0077)	.8792 (.0133)
0.10	.8904 (.0117)	.8899 (.0115)	.8296 (.0155)	.9684 (.0056)	.9683 (.0056)	.9325 (.0093)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.6933 (.0175)	.6997 (.0186)	.6158 (.0212)	.8431 (.0148)	.8481 (.0149)	.7760 (.0182)
0.01	.9080 (.0108)	.9064 (.0101)	.8498 (.0150)	.9690 (.0072)	.9682 (.0073)	.9332 (.0104)
0.05	.9809 (.0056)	.9807 (.0056)	.9520 (.0086)	.9956 (.0017)	.9956 (.0017)	.9833 (.0046)
0.10	.9930 (.0024)	.9930 (.0024)	.9768 (.0052)	.9990 (.0009)	.9990 (.0009)	.9927 (.0023)

POWER OF LRPP TESTS
UNDERLYING DISTRIBUTION: EXPONENTIAL
$N_1 = N_2 = 40$



LEGEND δ16 ---- δ13 ---- δ23 ----

FIGURE: 4.9

distribution. The tests in these cases are indicated by δ_{16} and δ_{13}, respectively. Powers of δ_2 are obtained using the Pearson Type III distribution and we label the test as δ_{23}.

A table gives empirical powers of the tests against the shifts of one sample by $K\sigma$ for K=0.0(0.1)0.7 and α=0.001, 0.01, 0.05, 0.10 for a specific underlying distribution. The plot gives power curves of δ_1 and δ_2 for a specified underlying distribution. In the case of δ_1, it shows power curves under both approximations of the null distribution of δ_1. Any difference in powers of 0.01 or more is significant, since max $\{2\sqrt{(pq/10000)}\} = 0.01$.

We conclude the following from Tables 4.1 to 4.9 and the corresponding plots.

1. Powers of δ_1 under the Pearson Type VI approximation are almost always more than those obtained under the Pearson Type III approximation for α=0.01, 0.05 and 0.10. For α=0.001, the situation is the other way around, i.e., powers of δ_1 under the Type III approximation are generally higher than those under the Type VI approximation.
2. With the exception of the logistic as an underlying distribution, empirical α's are all within acceptable limits. In most cases these are lower than nominal α's for both δ_1 and δ_2; however, a test using δ_1 is a conservative test relative to one using δ_2.

3. When the underlying distribution is uniform or normal (Tables 4.1, 4.2), empirical powers of δ_2 are significantly higher than those of δ_1 under both approximations.

4. When the logistic is the underlying distribution, Table 4.3 indicates higher powers of δ_2 compared to those of δ_1. But we note also that empirical α's are significantly higher than nominal α's. Therefore, it cannot be concluded that δ_2 performs better than δ_1.

5. When the underlying distribution is 10% 3N; i.e., 0.1N(0,9) + 0.9N(0,1), we note from Table 4.4 that the empirical power of δ_2 is slightly higher than that of δ_1 under both approximations.

6. In the case of 10% 10N (Table 4.5) as underlying distribution, empirical powers of δ_1 under both approximations are higher than those of δ_2 for shifts of $\geq 0.3\sigma$.

7. From Table 4.6, we note that powers of δ_1 under both approximations are significantly higher than those of δ_2 when the underlying distribution is a Laplace distribution.

8. When the underlying distribution is a U-shaped distribution, powers of δ_1 are significantly lower than those of δ_2 for shifts of $\leq 0.2\sigma$, while gain in power

is appreciable for shifts of $\geq 0.3\sigma$.

9. In the case of Cauchy as underlying distribution, powers of δ_1 are almost always significantly higher than those of δ_2.

10 When underlying distribution is exponential, powers of δ_1 are more than those of δ_2 for shifts of $\geq 0.3\sigma$.

4.5 Results for Small Samples

Now, we present the results for small samples. Here we obtain powers of δ_1 and δ_2 against shifts of $K\sigma$ with K=0.0(0.2)1.4 for the underlying distributions already mentioned with the exception of the "contaminated normal" 0.1N(0,100) + 0.9N(0,1) distribution for which we take K=0.0(0.1)0.7. This exception is made because the power increases rapidly for the case of 10% 10N as an underlying distribution.

For δ_1, powers are obtained using both the Pearson Type I and the Pearson Type III distributions. However, we note from Tables 4.10-4.18 and Figures 4.10-4.18 that powers of δ_1 remain the same under both approximations. Following each table, we give a plot to show power curves of δ_1 and δ_2 for α=0.001, 0.01, 0.05, 0.10 for the specific distribution.

Table 4.10

Empirical Powers, When Underlying Distribution is Uniform, $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0013 (.0009)	.0013 (.0009)	.0010 (.0009)	.0018 (.0020)	.0018 (.0020)	.0012 (.0016)
0.01	.0117 (.0034)	.0117 (.0034)	.0127 (.0032)	.0164 (.0039)	.0164 (.0039)	.0184 (.0040)
0.05	.0487 (.0054)	.0487 (.0054)	.0525 (.0049)	.0625 (.0051)	.0625 (.0051)	.0695 (.0042)
0.10	.0959 (.0041)	.0959 (.0041)	.1053 (.0054)	.1215 (.0079)	.1215 (.0079)	.1356 (.0060)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0046 (.0027)	.0046 (.0027)	.0036 (.0023)	.0097 (.0032)	.0097 (.0032)	.0086 (.0033)
0.01	.0292 (.0059)	.0292 (.0059)	.0371 (.0059)	.0610 (.0060)	.0610 (.0060)	.0802 (.0084)
0.05	.1081 (.0084)	.1081 (.0084)	.1297 (.0110)	.1917 (.0128)	.1917 (.0128)	.2323 (.0159)
0.10	.1877 (.0126)	.1877 (.0126)	.2233 (.0113)	.2983 (.0166)	.2983 (.0166)	.3462 (.0197)

Table 4.10 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0210 (.0059)	.0210 (.0059)	.0191 (.0064)	.0448 (.0067)	.0448 (.0067)	.0410 (.0069)
0.01	.1180 (.0074)	.1180 (.0074)	.1505 (.0099)	.2039 (.0126)	.2039 (.0126)	.2538 (.0180)
0.05	.3064 (.0197)	.3064 (.0197)	.3615 (.0192)	.4417 (.0195)	.4417 (.0195)	.5019 (.0173)
0.10	.4325 (.0160)	.4325 (.0160)	.4936 (.0175)	.5761 (.0145)	.5761 (.0145)	.6385 (.0157)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0908 (.0075)	.0908 (.0075)	.0853 (.0070)	.1623 (.0136)	.1623 (.0136)	.1551 (.0126)
0.01	.3246 (.0201)	.3246 (.0201)	.3871 (.0191)	.4617 (.0178)	.4617 (.0178)	.5311 (.0167)
0.05	.5850 (.0133)	.5850 (.0133)	.6515 (.0162)	.7223 (.0094)	.7223 (.0094)	.7773 (.0122)
0.10	.7136 (.0159)	.7136 (.0159)	.7695 (.0131)	.8252 (.0101)	.8252 (.0101)	.8696 (.0111)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: UNIFORM

$N_1 = N_2 = 10$

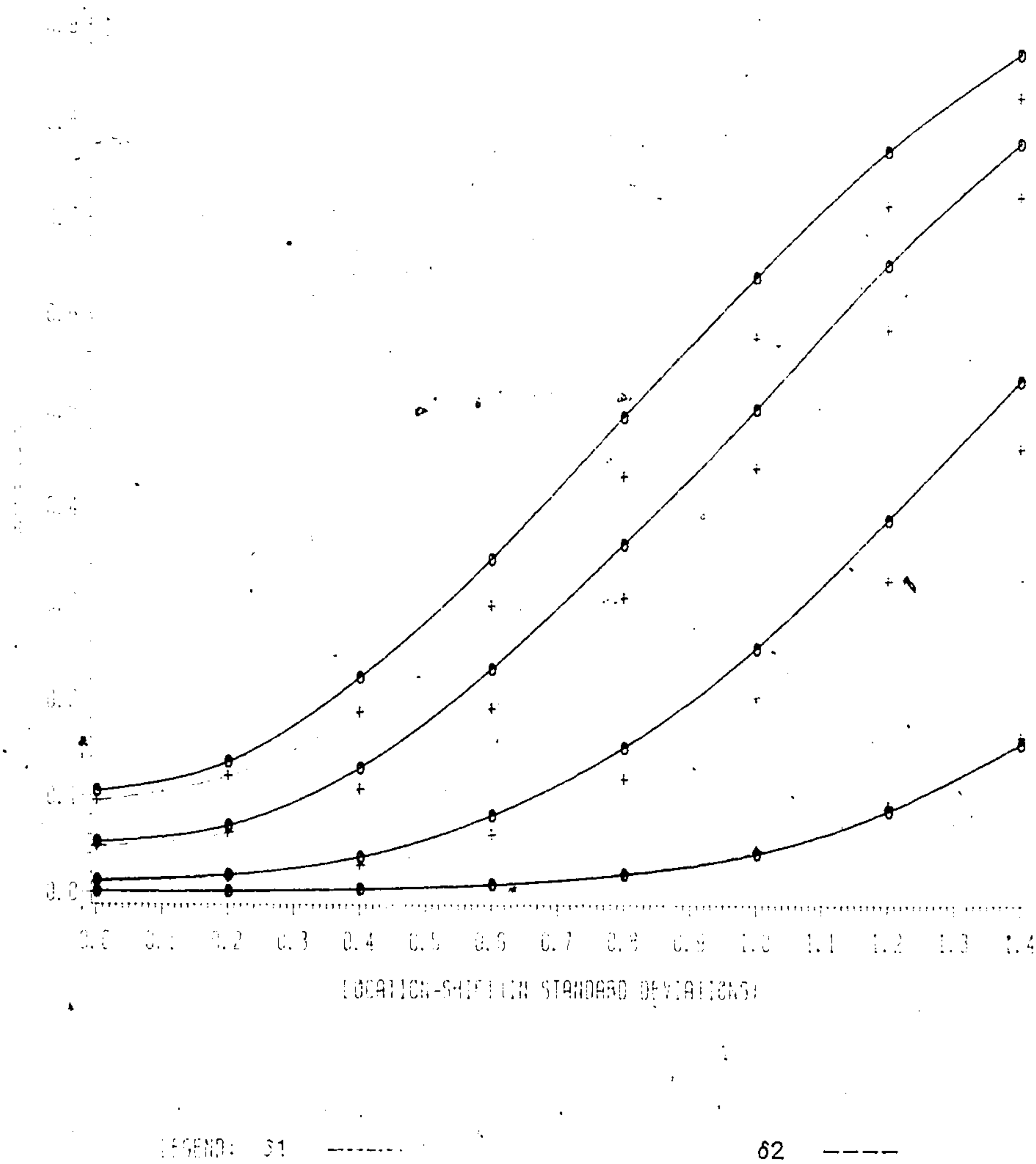


FIGURE: 4.10

Table 4.11

Empirical Powers, When Underlying Distribution is Normal, $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0008 (.0008)	.0008 (.0008)	.0005 (.0007)	.0024 (.0019)	.0024 (.0019)	.0019 (.0014)
0.01	.0102 (.0029)	.0102 (.0029)	.0116 (.0044)	.0162 (.0047)	.0162 (.0047)	.0188 (.0042)
0.05	.0463 (.0044)	.0463 (.0044)	.0505 (.0069)	.0705 (.0080)	.0705 (.0080)	.0755 (.0087)
0.10	.0992 (.0090)	.0992 (.0090)	.1033 (.0080)	.1313 (.0082)	.1313 (.0082)	.1374 (.0097)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0051 (.0029)	.0051 (.0029)	.0037 (.0024)	.0158 (.0035)	.0158 (.0035)	.0125 (.0033)
0.01	.0398 (.0074)	.0398 (.0074)	.0442 (.0085)	.0855 (.0096)	.0855 (.0096)	.0954 (.0093)
0.05	.1317 (.0114)	.1317 (.0114)	.1377 (.0096)	.2298 (.0124)	.2298 (.0124)	.2451 (.0170)
0.10	.2114 (.0109)	.2114 (.0109)	.2248 (.0123)	.3338 (.0156)	.3338 (.0156)	.3606 (.0192)

Table 4.11 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0359 (.0057)	.0359 (.0057)	.0296 (.0052)	.0777 (.0077)	.0777 (.0077)	.0660 (.0057)
0.01	.1593 (.0134)	.1593 (.0134)	.1797 (.0156)	.2656 (.0144)	.2656 (.0144)	.2980 (.0159)
0.05	.3603 (.0184)	.3603 (.0184)	.3868 (.0197)	.5178 (.0131)	.5178 (.0131)	.5493 (.0091)
0.10	.4918 (.0160)	.4918 (.0160)	.5237 (.0135)	.6522 (.0087)	.6522 (.0087)	.6848 (.0086)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1421 (.0142)	.1421 (.0142)	.1240 (.0134)	.2405 (.0123)	.2405 (.0123)	.2149 (.0118)
0.01	.4093 (.0151)	.4093 (.0151)	.4457 (.0145)	.5646 (.0124)	.5646 (.0124)	.6077 (.0090)
0.05	.6758 (.0067)	.6758 (.0067)	.7057 (.0080)	.8031 (.0102)	.8031 (.0102)	.8280 (.0099)
0.10	.7845 (.0095)	.7845 (.0095)	.8138 (.0088)	.8853 (.0111)	.8853 (.0111)	.9090 (.0083)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: NORMAL

$N_1 = N_2 = 10$

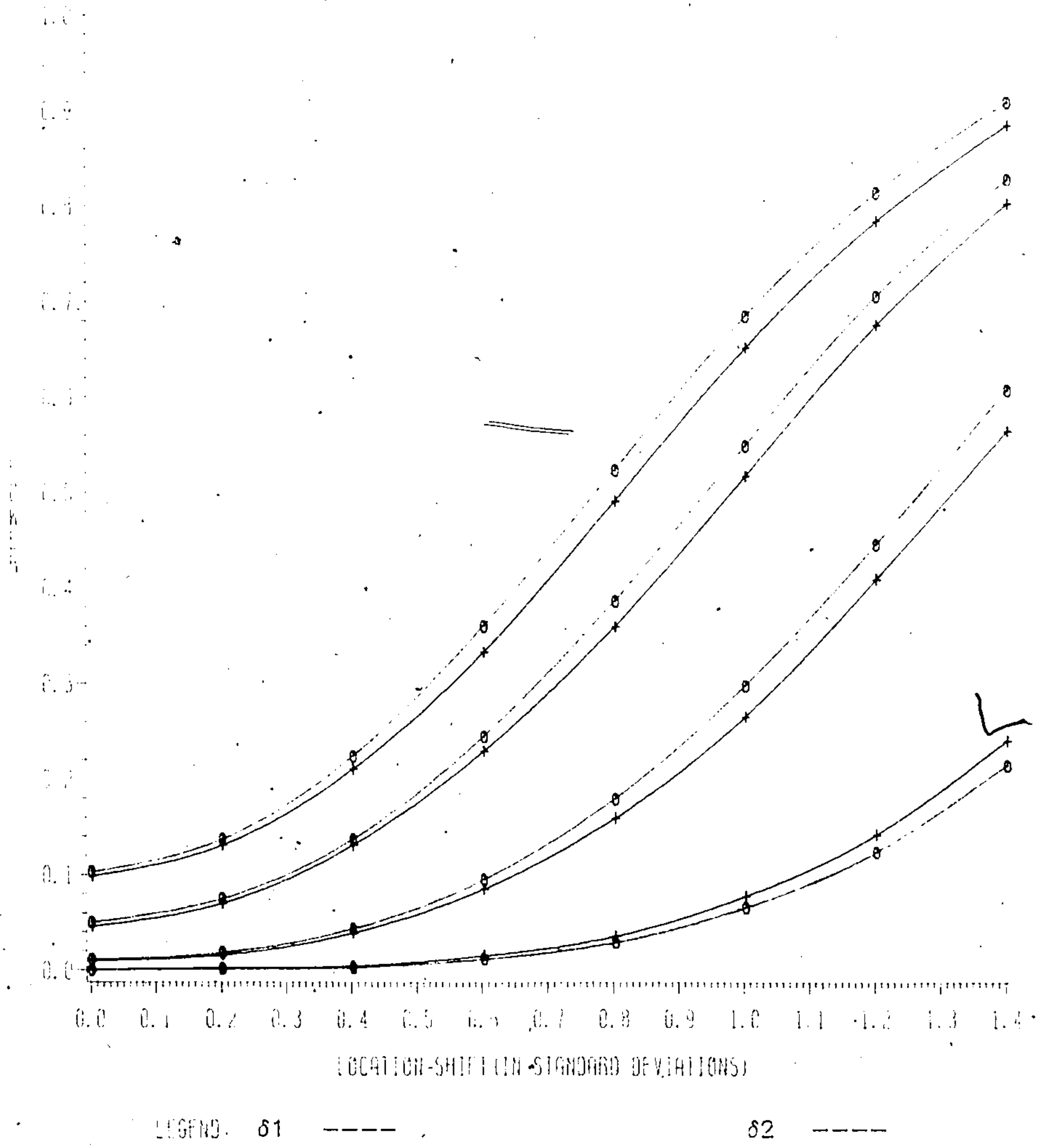


FIGURE: 4.11

Table 4.12

Empirical Powers, When Underlying Distribution is Logistic, $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0014 (.0012)	.0014 (.0012)	.0011 (.0011)	.0023 (.0014)	.0023 (.0014)	.0017 (.0014)
0.01	.0099 (.0033)	.0099 (.0033)	.0103 (.0031)	.0186 (.0063)	.0186 (.0063)	.0191 (.0059)
0.05	.0496 (.0078)	.0496 (.0078)	.0528 (.0060)	.0720 (.0085)	.0720 (.0085)	.0763 (.0068)
0.10	.1005 (.0125)	.1005 (.0125)	.1068 (.0125)	.1293 (.0100)	.1293 (.0100)	.1369 (.0102)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0066 (.0031)	.0066 (.0031)	.0048 (.0030)	.0188 (.0044)	.0188 (.0044)	.0152 (.0038)
0.01	.0425 (.0072)	.0425 (.0072)	.0461 (.0063)	.1023 (.0063)	.1023 (.0063)	.1093 (.0056)
0.05	.1398 (.0096)	.1398 (.0096)	.1469 (.0077)	.2573 (.0113)	.2573 (.0113)	.2667 (.0083)
0.10	.2279 (.0089)	.2279 (.0089)	.2425 (.0103)	.3723 (.0084)	.3723 (.0084)	.3887 (.0083)

Table 4.12 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0456 (.0037)	.0456 (.0037)	.0384 (.0058)	.1029 (.0073)	.1029 (.0073)	.0841 (.0077)
0.01	.1915 (.0130)	.1915 (.0130)	.2038 (.0099)	.3222 (.0138)	.3222 (.0138)	.3378 (.0139)
0.05	.4110 (.0063)	.4110 (.0063)	.4231 (.0093)	.5721 (.0142)	.5721 (.0142)	.5842 (.0104)
0.10	.5412 (.0109)	.5412 (.0109)	.5558 (.0125)	.6920 (.0087)	.6920 (.0087)	.7120 (.0109)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1843 (.0090)	.1843 (.0090)	.1539 (.0082)	.2977 (.0087)	.2977 (.0087)	.2550 (.0090)
0.01	.4670 (.0098)	.4670 (.0098)	.4849 (.0121)	.6143 (.0083)	.6143 (.0083)	.6332 (.0091)
0.05	.7223 (.0086)	.7223 (.0086)	.7358 (.0073)	.8390 (.0093)	.8390 (.0093)	.8482 (.0091)
0.10	.8216 (.0095)	.8216 (.0095)	.8377 (.0099)	.9061 (.0062)	.9061 (.0062)	.9163 (.0059)

POWER OF LRPP TESTS
UNDERLYING DISTRIBUTION: LOGISTIC
$N_1 = N_2 = 10$

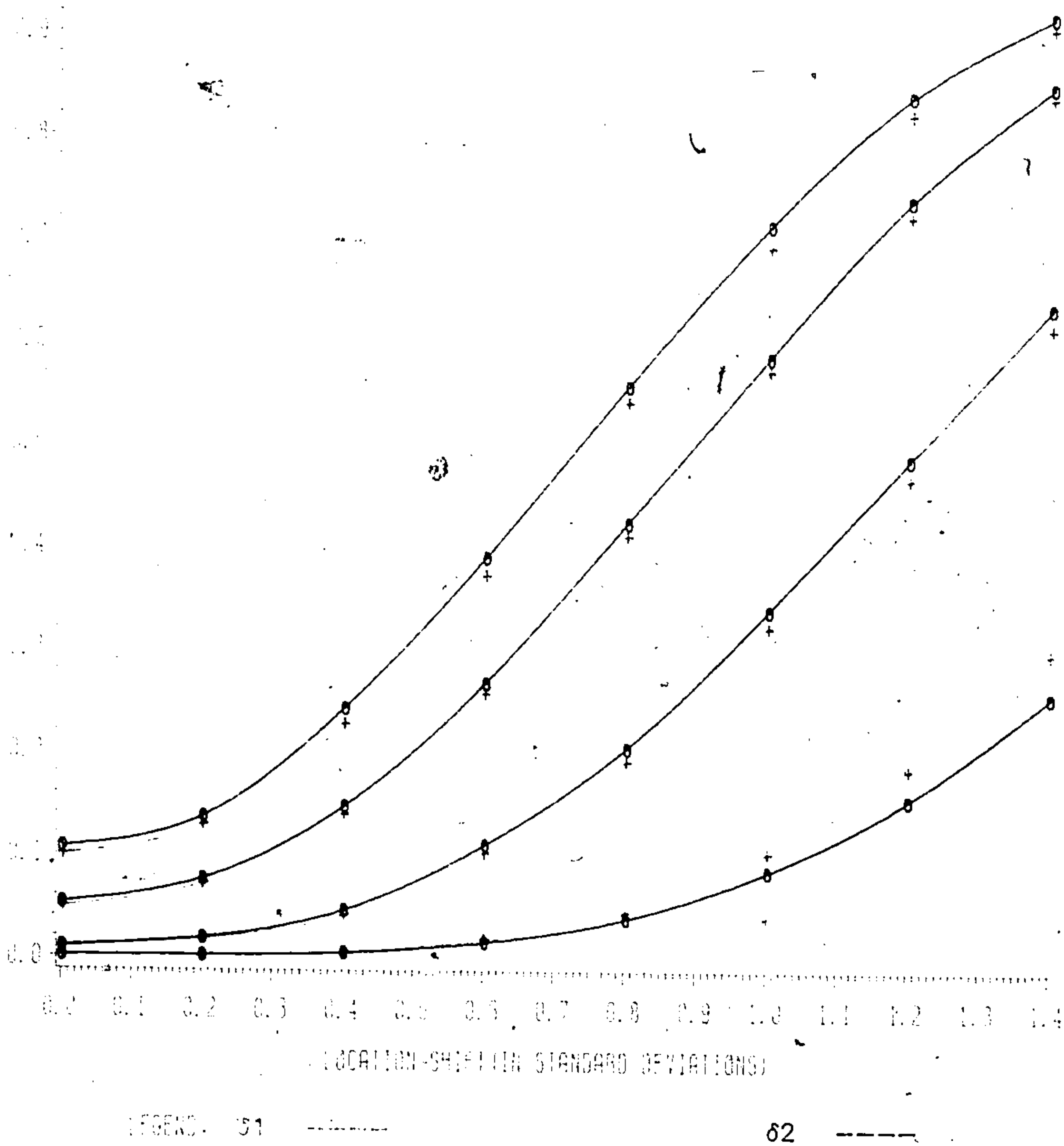


FIGURE: 4.12

Table 4.13

Empirical Powers, When Underlying Distribution is .1N(0,9) + .9N(0,1), $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0017 (.0016)	.0017 (.0016)	.0011 (.0014)	.0036 (.0015)	.0036 (.0015)	.0029 (.0009)
0.01	.0114 (.0044)	.0114 (.0044)	.0120 (.0045)	.0227 (.0057)	.0227 (.0057)	.0253 (.0060)
0.05	.0540 (.0103)	.0540 (.0103)	.0548 (.0095)	.0782 (.0095)	.0782 (.0095)	.0830 (.0110)
0.10	.1014 (.0114)	.1014 (.0114)	.1085 (.0149)	.1371 (.0112)	.1371 (.0112)	.1461 (.0094)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0105 (.0032)	.0105 (.0032)	.0081 (.0030)	.0302 (.0065)	.0302 (.0065)	.0231 (.0053)
0.01	.0559 (.0070)	.0559 (.0070)	.0618 (.0093)	.1237 (.0093)	.1237 (.0093)	.1341 (.0081)
0.05	.1584 (.0120)	.1584 (.0120)	.1650 (.0110)	.3082 (.0220)	.3082 (.0220)	.3209 (.0184)
0.10	.2530 (.0162)	.2530 (.0162)	.2672 (.0212)	.4279 (.0187)	.4279 (.0187)	.4462 (.0206)

Table 4.13 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0713 (.0094)	.0713 (.0094)	.0558 (.0091)	.1458 (.0122)	.1458 (.0122)	.1203 (.0087)
0.01	.2484 (.0167)	.2484 (.0167)	.2628 (.0172)	.4121 (.0183)	.4121 (.0183)	.4265 (.0198)
0.05	.4880 (.0160)	.4880 (.0160)	.5035 (.0151)	.6680 (.0143)	.6680 (.0143)	.6774 (.0118)
0.10	.6164 (.0163)	.6164 (.0163)	.6365 (.0143)	.7799 (.0189)	.7799 (.0189)	.7923 (.0112)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2619 (.0147)	.2619 (.0147)	.2193 (.0117)	.4071 (.0164)	.4071 (.0164)	.3457 (.0157)
0.01	.5846 (.0116)	.5846 (.0116)	.5966 (.0114)	.7363 (.0120)	.7363 (.0120)	.7358 (.0130)
0.05	.8114 (.0156)	.8114 (.0156)	.8128 (.0153)	.9066 (.0112)	.9066 (.0112)	.9059 (.0131)
0.10	.8900 (.0110)	.8900 (.0110)	.8979 (.0120)	.9528 (.0081)	.9528 (.0081)	.9543 (.0067)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: 10% 3N

$N_1 = N_2 = 10$

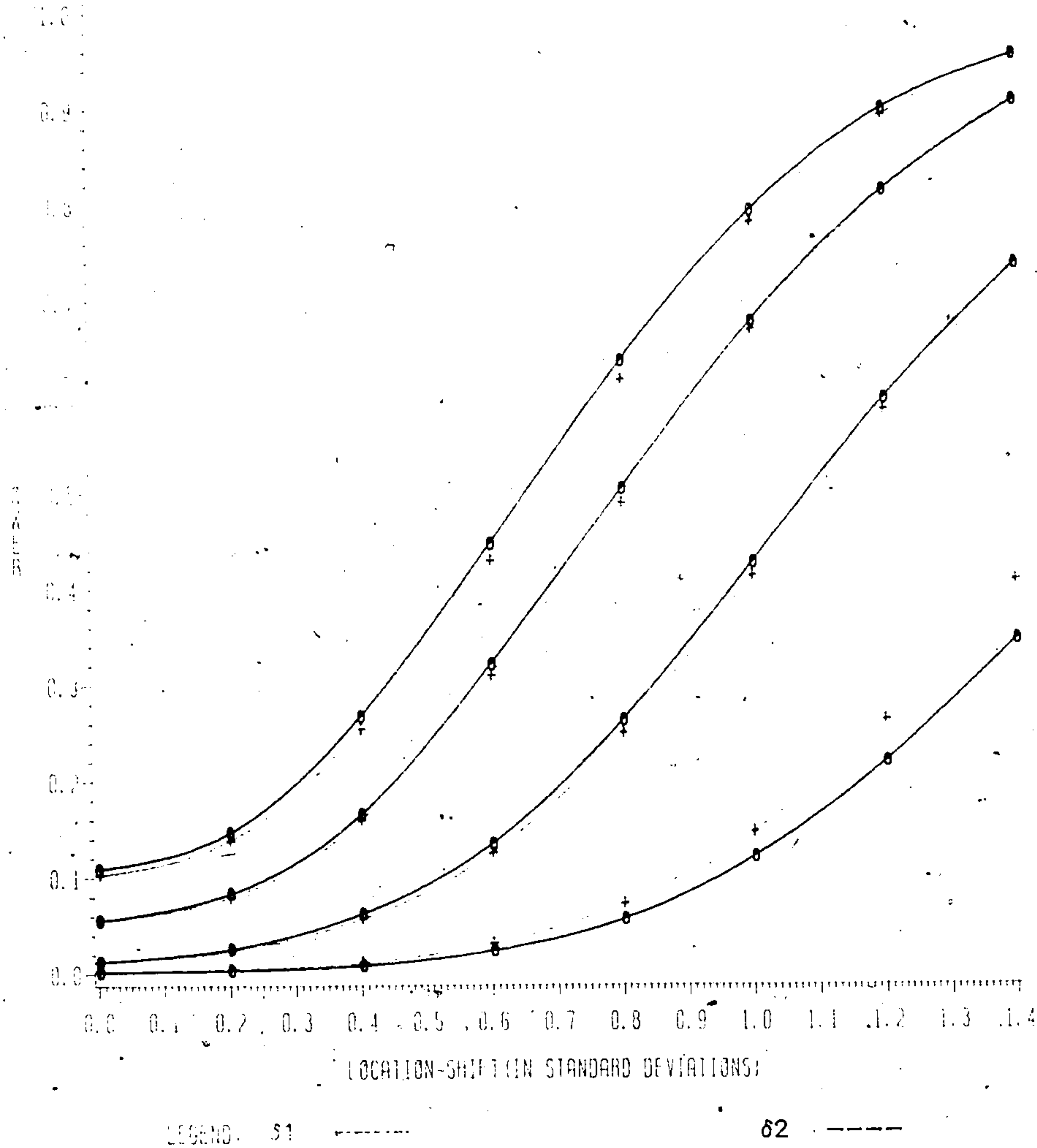


FIGURE: 4.1.3

Table 4.14

Empirical Powers, When Underlying Distribution is .1N(0,100) + .9N(0,1), $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.1		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0010 (.0007)	.0010 (.0007)	.0007 (.0007)	.0043 (.0017)	.0043 (.0017)	.0030 (.0013)
0.01	.0107 (.0020)	.0107 (.0020)	.0121 (.0018)	.0242 (.0039)	.0242 (.0039)	.0253 (.0033)
0.05	.0489 (.0065)	.0489 (.0065)	.0514 (.0042)	.0875 (.0067)	.0875 (.0067)	.0893 (.0068)
0.10	.0976 (.0072)	.0976 (.0072)	.1054 (.0075)	.1533 (.0074)	.1533 (.0074)	.1618 (.0084)

Shift: Kσ	K = 0.2			K = 0.3		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0138 (.0035)	.0138 (.0035)	.0100 (.0020)	.0460 (.0064)	.0460 (.0064)	.0339 (.0060)
0.01	.0739 (.0060)	.0739 (.0060)	.0782 (.0058)	.1861 (.0103)	.1861 (.0103)	.1890 (.0097)
0.05	.2090 (.0118)	.2090 (.0118)	.2125 (.0108)	.3959 (.0125)	.3959 (.0125)	.3929 (.0149)
0.10	.3087 (.0116)	.3087 (.0116)	.3193 (.0145)	.5231 (.0132)	.5231 (.0132)	.5254 (.0110)

Table 4.14 (cont'd.)

Shift: Kσ	K = 0.4			K = 0.5		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1203 (.0108)	.1203 (.0108)	.0903 (.0100)	.2339 (.0123)	.2339 (.0123)	.1783 (.0132)
0.01	.3471 (.0133)	.3471 (.0133)	.3404 (.0182)	.5349 (.0193)	.5349 (.0193)	.5076 (.0188)
0.05	.6021 (.0171)	.6021 (.0171)	.5847 (.0138)	.7728 (.0115)	.7728 (.0115)	.7477 (.0147)
0.10	.7187 (.0136)	.7187 (.0136)	.7119 (.0144)	.8607 (.0114)	.8607 (.0114)	.8409 (.0136)

Shift: Kσ	K = 0.6			K = 0.7		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.3762 (.0128)	.3762 (.0128)	.2802 (.0140)	.5210 (.0124)	.5210 (.0124)	.3711 (.0155)
0.01	.6959 (.0141)	.6959 (.0141)	.6525 (.0127)	.8107 (.0125)	.8107 (.0125)	.7449 (.0162)
0.05	.8860 (.0132)	.8860 (.0132)	.8486 (.0145)	.9419 (.0072)	.9419 (.0072)	.9046 (.0139)
0.10	.9388 (.0089)	.9388 (.0089)	.9140 (.0111)	.9735 (.0049)	.9735 (.0049)	.9501 (.0079)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: 10% 10N

$N_1 = N_2 = 10$

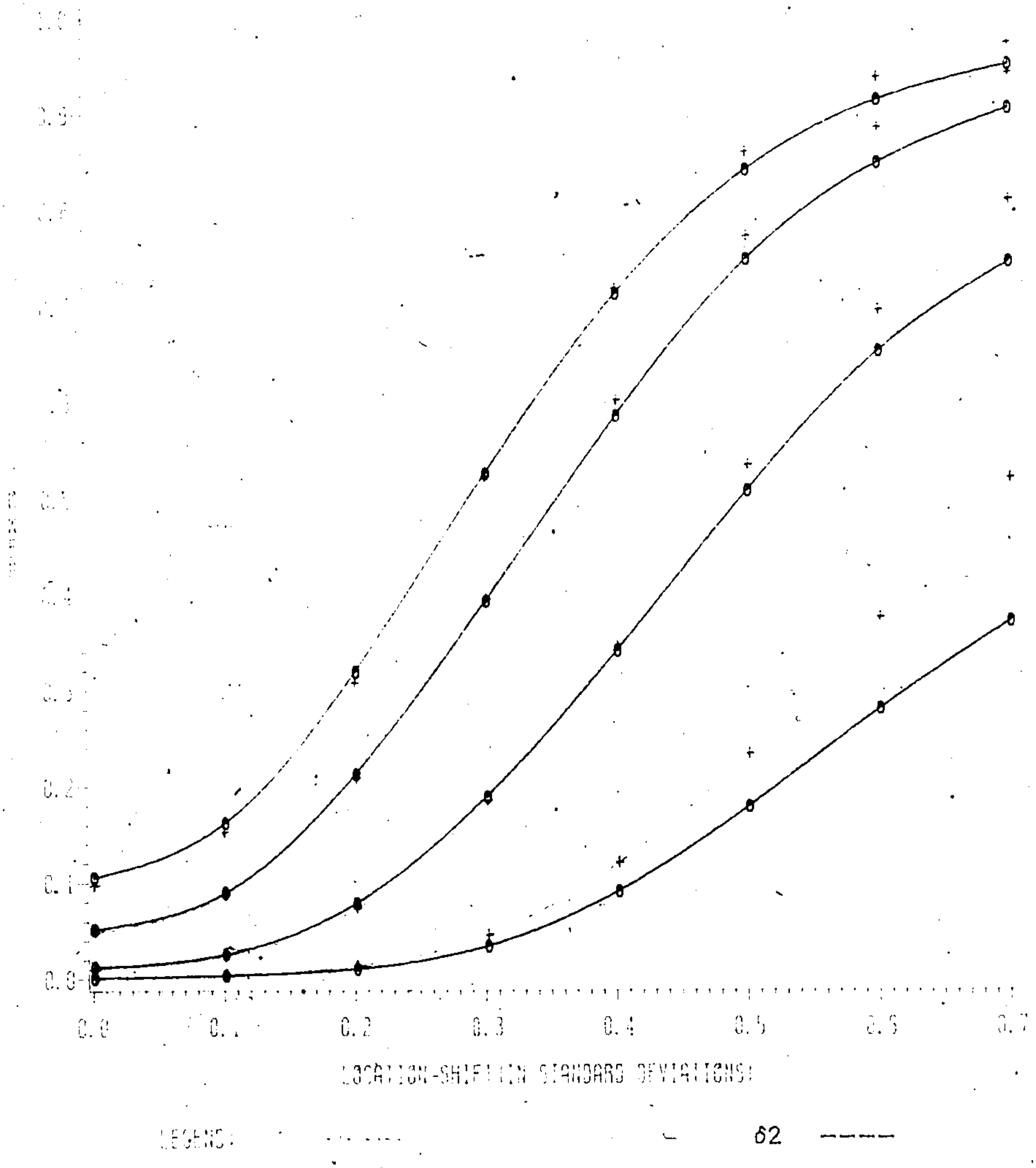


FIGURE: 4.14

Table 4.15

Empirical Powers, When Underlying Distribution is Laplace (Double Exponential), $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0009 (.0006)	.0009 (.0006)	.0005 (.0005)	.0024 (.0016)	.0024 (.0016)	.0015 (.0010)
0.01	.0096 (.0031)	.0096 (.0031)	.0101 (.0021)	.0186 (.0035)	.0186 (.0035)	.0216 (.0029)
0.05	.0481 (.0054)	.0481 (.0054)	.0500 (.0049)	.0805 (.0095)	.0805 (.0095)	.0838 (.0088)
0.10	.0960 (.0099)	.0960 (.0099)	.1018 (.0055)	.1448 (.0109)	.1448 (.0109)	.1508 (.0115)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0103 (.0034)	.0103 (.0034)	.0074 (.0031)	.0330 (.0062)	.0330 (.0062)	.0236 (.0058)
0.01	.0592 (.0055)	.0592 (.0055)	.0622 (.0069)	.1420 (.0071)	.1420 (.0071)	.1461 (.0086)
0.05	.1825 (.0107)	.1825 (.0107)	.1831 (.0093)	.3319 (.0100)	.3319 (.0100)	.3269 (.0104)
0.10	.2789 (.0112)	.2789 (.0112)	.2792 (.0130)	.4542 (.0158)	.4542 (.0158)	.4513 (.0177)

Table 4.15 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0791 (.0078)	.0791 (.0078)	.0588 (.0078)	.1584 (.0080)	.1584 (.0080)	.1185 (.0059)
0.01	.2676 (.0095)	.2676 (.0095)	.2645 (.0087)	.4199 (.0126)	.4199 (.0126)	.4125 (.0141)
0.05	.5086 (.0187)	.5086 (.0187)	.4970 (.0199)	.6702 (.0136)	.6702 (.0136)	.6566 (.0160)
0.10	.6347 (.0145)	.6347 (.0145)	.6259 (.0178)	.7824 (.0102)	.7824 (.0102)	.7715 (.0136)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.2617 (.0089)	.2617 (.0089)	.2067 (.0093)	.3867 (.0122)	.3867 (.0122)	.3127 (.0112)
0.01	.5674 (.0178)	.5674 (.0178)	.5601 (.0154)	.7026 (.0144)	.7026 (.0144)	.6928 (.0159)
0.05	.7988 (.0153)	.7988 (.0153)	.7847 (.0153)	.8902 (.0095)	.8902 (.0095)	.8779 (.0088)
0.10	.8817 (.0104)	.8817 (.0104)	.8736 (.0107)	.9441 (.0064)	.9441 (.0064)	.9364 (.0083)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION LAPLACE

N =N =10

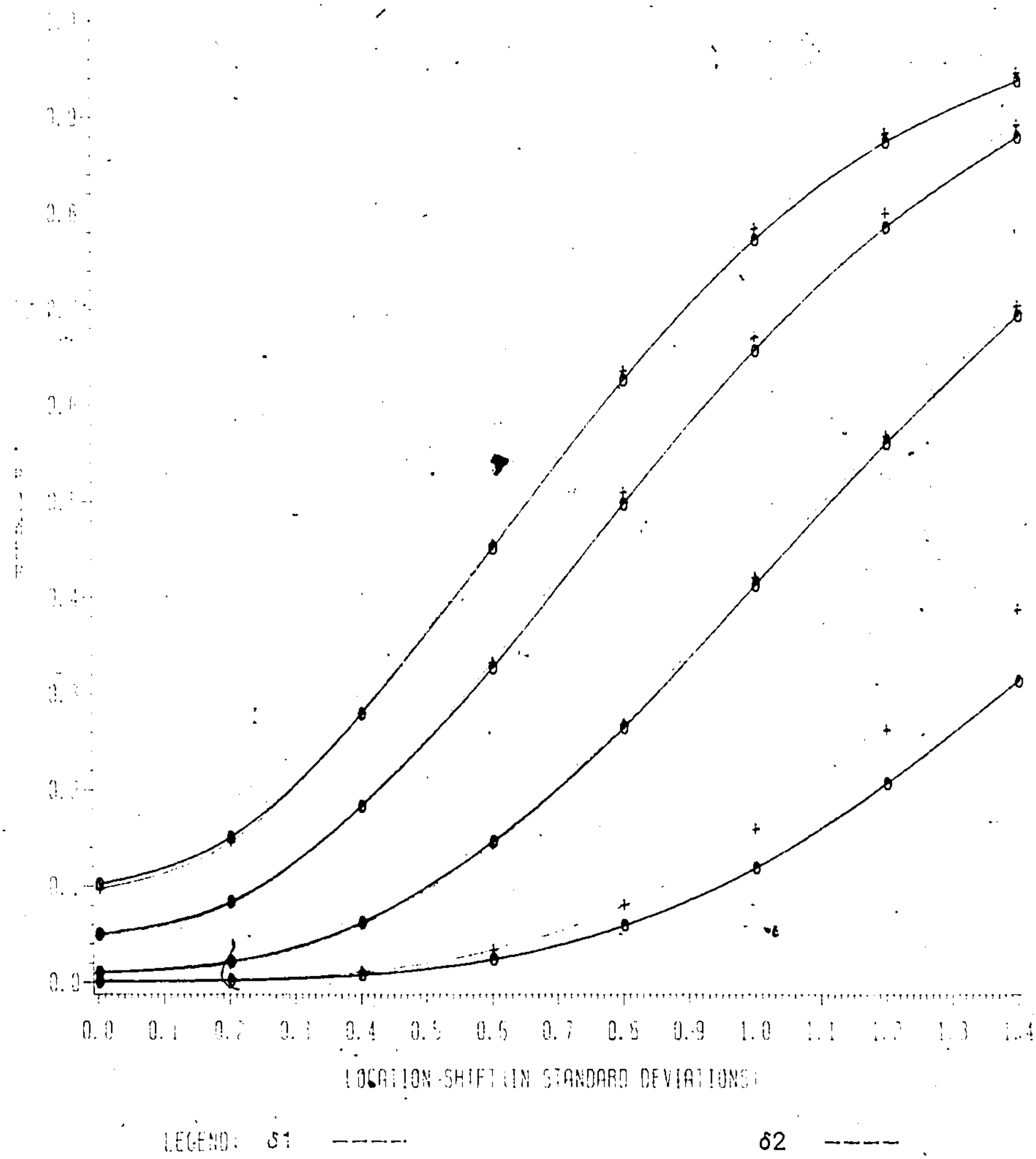


FIGURE: 4.15

Table 4.16

Empirical Powers, When Underlying Distribution is a U-Shaped distribution, $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0012 (.0008)	.0012 (.0008)	.0007 (.0007)	.0035 (.0030)	.0035 (.0030)	.0030 (.0027)
0.01	.0109 (.0030)	.0109 (.0030)	.0121 (.0028)	.0286 (.0024)	.0286 (.0024)	.0375 (.0036)
0.05	.0480 (.0067)	.0480 (.0067)	.0505 (.0056)	.1119 (.0100)	.1119 (.0100)	.1350 (.0082)
0.10	.0995 (.0106)	.0995 (.0106)	.1061 (.0099)	.1989 (.0070)	.1989 (.0070)	.2281 (.0090)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0121 (.0023)	.0121 (.0023)	.0109 (.0028)	.0311 (.0039)	.0311 (.0039)	.0294 (.0042)
0.01	.0904 (.0087)	.0904 (.0087)	.1200 (.0095)	.1595 (.0115)	.1595 (.0115)	.2040 (.0109)
0.05	.2678 (.0122)	.2678 (.0122)	.3050 (.0141)	.4179 (.0157)	.4179 (.0157)	.4352 (.0124)
0.10	.4070 (.0122)	.4070 (.0122)	.4409 (.0138)	.5668 (.0140)	.5668 (.0140)	.5741 (.0084)

Table 4.16 (cont'd.)

Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0436 (.0050)	.0436 (.0050)	.0426 (.0047)	.0506 (.0048)	.0506 (.0048)	.0502 (.0048)
0.01	.2045 (.0103)	.2045 (.0103)	.2482 (.0113)	.2294 (.0111)	.2294 (.0111)	.2676 (.0125)
0.05	.5050 (.0140)	.5050 (.0140)	.5108 (.0123)	.5396 (.0109)	.5396 (.0109)	.5452 (.0101)
0.10	.6500 (.0091)	.6500 (.0091)	.6341 (.0078)	.6754 (.0088)	.6754 (.0088)	.6554 (.0075)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0560 (.0049)	.0560 (.0049)	.0559 (.0047)	.0684 (.0056)	.0684 (.0056)	.0676 (.0056)
0.01	.2449 (.0082)	.2449 (.0082)	.2867 (.0122)	.2788 (.0124)	.2788 (.0124)	.3278 (.0173)
0.05	.5521 (.0117)	.5521 (.0117)	.5660 (.0111)	.5797 (.0116)	.5797 (.0116)	.6056 (.0094)
0.10	.6847 (.0086)	.6847 (.0086)	.6775 (.0075)	.7086 (.0093)	.7086 (.0093)	.7195 (.0091)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: J-SHAPED

$N_1 = N_2 = 10$

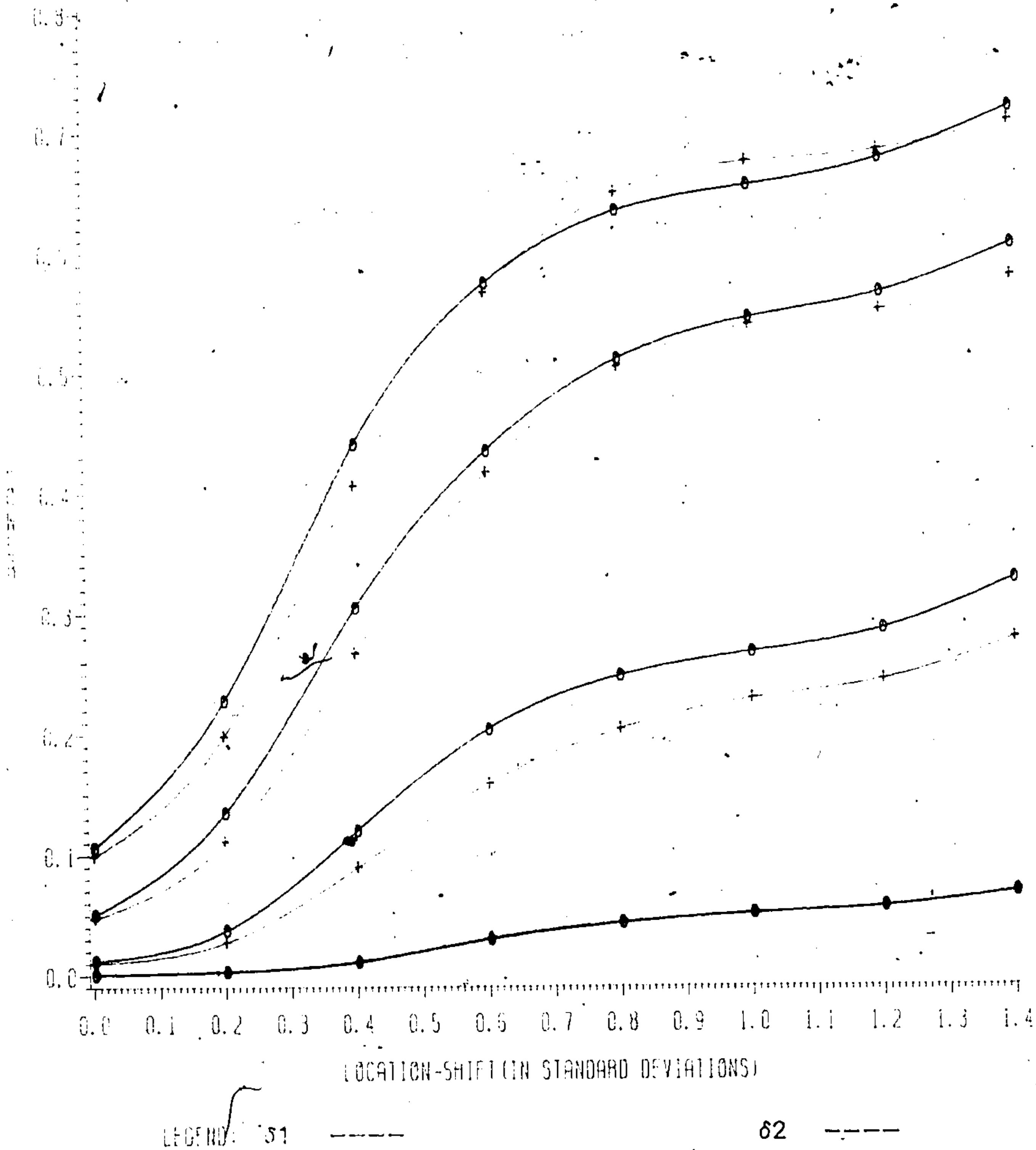


FIGURE: 4.16

Table 4.17

Empirical Powers, When Underlying Distribution is Cauchy, N_1=N_2=10

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0014 (.0014)	.0014 (.0014)	.0011 (.0012)	.0030 (.0017)	.0030 (.0017)	.0025 (.0018)
0.01	.0102 (.0042)	.0102 (.0042)	.0115 (.0048)	.0181 (.0047)	.0181 (.0047)	.0178 (.0041)
0.05	.0485 (.0063)	.0485 (.0063)	.0499 (.0065)	.0743 (.0084)	.0743 (.0084)	.0744 (.0070)
0.10	.0958 (.0086)	.0958 (.0086)	.1058 (.0108)	.1364 (.0099)	.1364 (.0099)	.1381 (.0072)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0086 (.0025)	.0086 (.0025)	.0053 (.0028)	.0219 (.0048)	.0219 (.0048)	.0139 (.0031)
0.01	.0454 (.0041)	.0454 (.0041)	.0458 (.0054)	.1035 (.0055)	.1035 (.0055)	.0930 (.0081)
0.05	.1498 (.0061)	.1498 (.0061)	.1408 (.0067)	.2562 (.0115)	.2562 (.0115)	.2346 (.0094)
0.10	.2367 (.0118)	.2367 (.0118)	.2251 (.0116)	.3766 (.0175)	.3766 (.0175)	.3472 (.0160)

Table 4.17 (cont'd.)

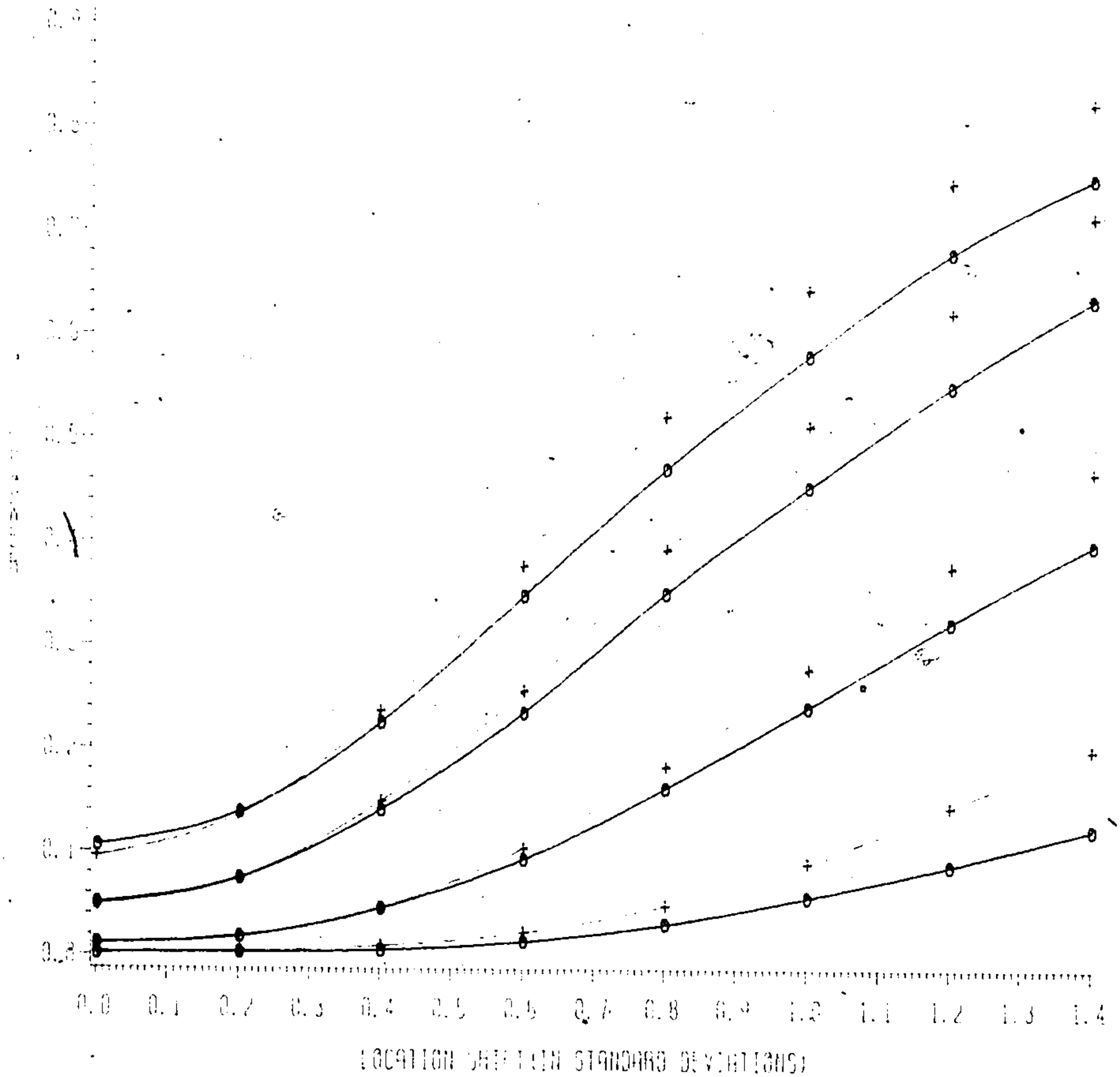
Shift: Kσ	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0476 (.0072)	.0476 (.0072)	.0305 (.0058)	.0889 (.0075)	.0889 (.0075)	.0569 (.0066)
0.01	.1823 (.0091)	.1823 (.0091)	.1617 (.0102)	.2767 (.0134)	.2767 (.0134)	.2403 (.0137)
0.05	.3923 (.0154)	.3923 (.0154)	.3497 (.0174)	.5122 (.0151)	.5122 (.0151)	.4525 (.0171)
0.10	.5211 (.0123)	.5211 (.0123)	.4701 (.0151)	.6435 (.0102)	.6435 (.0102)	.5797 (.0144)

Shift: Kσ	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1430 (.0098)	.1430 (.0098)	.0875 (.0077)	.1985 (.0061)	.1985 (.0061)	.1226 (.0081)
0.01	.3755 (.0128)	.3755 (.0128)	.3220 (.0122)	.4667 (.0178)	.4667 (.0178)	.3971 (.0194)
0.05	.6209 (.0129)	.6209 (.0129)	.5496 (.0119)	.7138 (.0137)	.7138 (.0137)	.6334 (.0121)
0.10	.7471 (.0135)	.7471 (.0135)	.6783 (.0145)	.8242 (.0155)	.8242 (.0155)	.7507 (.0144)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: CAUCHY

$N_1 = N_2 = 10$



LEGEND: δ1 -------- δ2 ----

FIGURE: 4.17

Table 4.18

Empirical Powers, When Underlying Distribution is Exponential, $N_1=N_2=10$

Shift: Kσ	K = 0.0			K = 0.2		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0012 (.0006)	.0012 (.0006)	.0011 (.0007)	.0032 (.0018)	.0032 (.0018)	.0023 (.0015)
0.01	.0113 (.0035)	.0113 (.0035)	.0128 (.0046)	.0281 (.0066)	.0281 (.0066)	.0320 (.0071)
0.05	.0511 (.0077)	.0511 (.0077)	.0538 (.0075)	.0994 (.0060)	.0994 (.0060)	.1089 (.0072)
0.10	.0987 (.0076)	.0987 (.0076)	.1046 (.0089)	.1751 (.0176)	.1751 (.0176)	.1879 (.0162)

Shift: Kσ	K = 0.4			K = 0.6		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.0158 (.0044)	.0158 (.0044)	.0125 (.0033)	.0540 (.0062)	.0540 (.0062)	.0415 (.0056)
0.01	.0877 (.0099)	.0877 (.0099)	.0950 (.0076)	.2074 (.0167)	.2074 (.0167)	.2139 (.0160)
0.05	.2438 (.0163)	.2438 (.0163)	.2539 (.0137)	.4448 (.0184)	.4448 (.0184)	.4375 (.0128)
0.10	.3659 (.0214)	.3659 (.0214)	.3734 (.0168)	.5833 (.0150)	.5833 (.0150)	.5687 (.0143)

Table 4.18 (cont'd.)

Shift: $K\sigma$	K = 0.8			K = 1.0		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.1243 (.0117)	.1243 (.0117)	.0932 (.0112)	.2282 (.0150)	.2282 (.0150)	.1732 (.0112)
0.01	.3664 (.0141)	.3664 (.0141)	.3617 (.0155)	.5247 (.0147)	.5247 (.0147)	.5109 (.0111)
0.05	.6269 (.0164)	.6269 (.0164)	.6076 (.0167)	.7735 (.0146)	.7735 (.0146)	.7418 (.0135)
0.10	.7563 (.0142)	.7563 (.0142)	.7227 (.0116)	.8682 (.0142)	.8682 (.0142)	.8359 (.0128)

Shift: $K\sigma$	K = 1.2			K = 1.4		
Statistic	δ_{16}	δ_{13}	δ_{23}	δ_{16}	δ_{13}	δ_{23}
α						
0.001	.3460 (.0169)	.3460 (.0169)	.2733 (.0134)	.4677 (.0112)	.4677 (.0112)	.3752 (.0120)
0.01	.6653 (.0144)	.6653 (.0144)	.6420 (.0128)	.7747 (.0119)	.7747 (.0119)	.7464 (.0096)
0.05	.8672 (.0166)	.8672 (.0166)	.8377 (.0115)	.9262 (.0106)	.9262 (.0106)	.9015 (.0126)
0.10	.9356 (.0122)	.9356 (.0122)	.9069 (.0138)	.9701 (.0066)	.9701 (.0066)	.9492 (.0064)

POWER OF MRPP TESTS

UNDERLYING DISTRIBUTION: EXPONENTIAL

$N_1 = N_2 = 10$

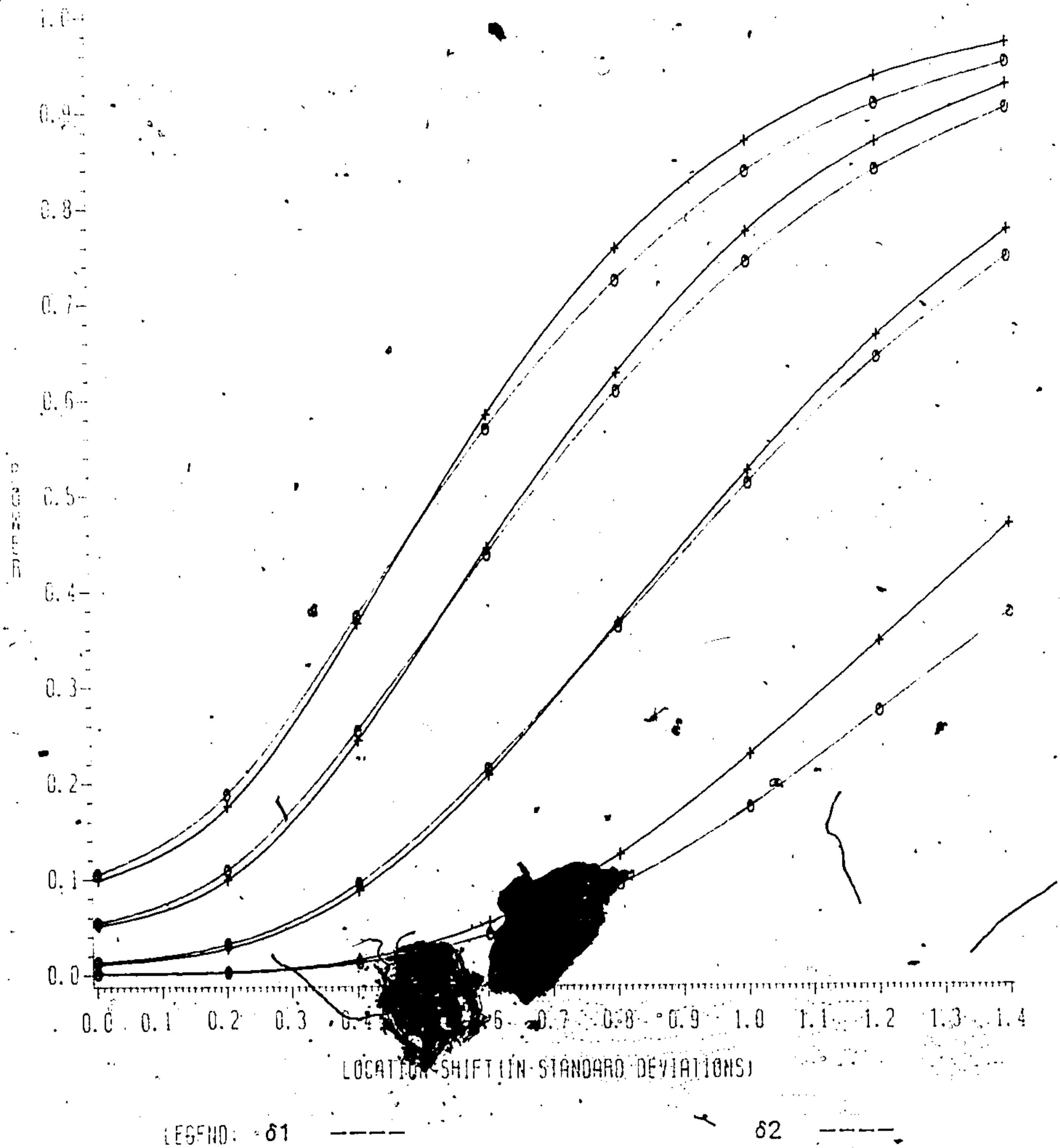


FIGURE: 4.18

The conclusions drawn from Tables 4.10-4.18 and Figures 4.10-4.18 are the following:

1. For α=0.001, the power of δ_1 is higher than that of δ_2 for every shift and each underlying distribution considered.
2. For α=0.01, 0.05, and 0.10, results are less favourable for δ_1 than they are in the case of large samples. However, like the large samples case, δ_1 performs significantly better than δ_2 when the underlying distribution is a Cauchy or an exponential distribution.

CHAPTER V

SCOPE FOR FURTHER STUDY

In Chapter II, we obtained the fourth moment of MRPP statistic δ in terms of an arbitrary choice of weights and sample sizes and the 23 symmetric functions for a distance measure Δ.

In Chapter III, we simplified the 23 symmetric functions in the case where the distance measure is the absolute difference between ranks of the observations. We then obtained a simplified form of the fourth moment of δ_1 for the case of two equal samples.

Following the results of Table 3.1, we carried out an empirical study for the powers of δ_1 and δ_2, for two equal samples. In the case of δ_1, we approximated the null distribution by the Pearson Type III as well as by the Pearson Type I for small samples and by the Pearson Type VI for large samples

Our study has been limited to the case of two equal samples, for large and small sample sizes. Based on the results of Chapter IV, the following recommendations for further study can be made.

Since results with the use of four moments are encouraging in the case of large samples, empirical powers of δ_1 can be studied for the following cases:

i) two unequal samples,

ii) g equal samples ($g>2$), and

iii) g samples not all equal.

In case (i), it will be interesting to investigate whether the performance of δ_1 relative to the Wilcoxon test remains the same as in the case of equal samples. For the cases (ii) and (iii), powers of δ_1 can be compared with those of δ_2 or the Kruskal-Wallis test. The test statistic δ_2 for $g>2$ is only asymptotically equivalent to the Kruskal Wallis test, therefore it may be worth simplifying the expression for its fourth moment for the above cases. This can help reveal the asymptotic equivalence for finite N.

Berry and Mielke (1983) have proposed a moment approximation procedure as an alternative to the F test, using the three exact moments. The example they provide indicates that the p value based on this procedure is closer to the exact p value than are the p values corresponding to the usual F test or a Monte Carlo test. To illustrate an improvement in the approximation on using the four moments, we consider the example of Berry and Mielke and also an additional example from Siegel (1956, p. 187).

Example 1

Data from Berry and Mielke (1983, p. 205)

S_1	S_2	S_3
43.75	46.00	50.50
50.50	61.75	68.50
43.75	46.00	64.00
	52.75	68.50
		50.50
		66.25

The exact p value using Fisher-Pitman randomization test is 0.0411. The usual F statistic for the data is $F_{2,10} = 4.70$ and the corresponding p value is 0.0364.

A program named GENAPP (Appendix A.2) gives the four moments and related constants as follows:

$\mu(\delta) = 183.029,\qquad \mu_2(\delta) = 1045.56,$

$\mu_3(\delta) = -44096.629,\qquad \mu_4(\delta) = 519549.1025,$

$\gamma_1(\delta) = -1.3043,\qquad \beta_2(\delta) = 4.68313,$

$2\beta_2-3\beta_1-6 = -1.7375,$

Pearson criterion $\kappa = -1.06025$.

The criterion suggests a Pearson Type I approximation.

The MRPP test statistic δ is related to the F statistic by the following:

$$\delta = \frac{2Ns^2}{N-g+(g-1)F} ,$$

where

$$s^2 = \frac{\sum_{i=1}^{g} \sum_{j=1}^{ni} (x_{ij} - \bar{x})^2}{N}$$

For the above data, δ is 113.231. The p values, with the aid of the transformations proposed in sections 4.2.1 and 4.2.3, are 0.0426 and 0.0377 under the Type I and the Type III approximations, respectively. We notice that the use of the four moments gives a closer approximation than the one obtained by using the three moments.

Example 2

Data from Siegel (1956, p. 187)

S_1	S_2	S_3
96	82	115
128	124	149
83	132	166
61	135	147
101	109	

Using an algorithm given by Berry (1982), we find the exact p value to be 0.0247. The F statistic for the data is $F_{2,11} = 5.4935$ and the p value under the F test is 0.0222.

For the moment approximation procedure, we have

$$\delta = 1029.5903,$$

$$\mu(\delta) = 1741.3626, \qquad \mu_2(\delta) = 76184.5650,$$

$$\mu_3(\delta) = -26604811.4041,$$

$\mu_4(\delta) = 25764338418.0756$

$\gamma_1(\delta) = -1.2652, \quad \beta_2(\delta) = 4.4390$

$2\beta_2 - 3\beta_1 - 6 = -1.924$, and $\kappa = -0.888$

The Pearson criterion for the above data again suggests a Type I approximation.

The p values under the Type I and the Type III approximations are 0.02438 and 0.02197 respectively.

Here again, the p value 0.02438 based on the four moments is closer to the exact p value 0.0247 than the p value 0.02197 based on the three moments.

Both the examples above indicate that it is worth exploring the use of the fourth moment in the analysis of variance situations. The performance of the moment approximation procedure based on four moments should also be compared with the performance of δ_1 in the g-sample problem.

The Kruskal-Wallis test competes with the classical F test and performs better than it in the absence of normality and homogeneity of variances. Therefore, a test based on δ_1, which competes with δ_2 and the Wilcoxon test, may provide another alternative to the F test. The main advantage in using δ_1 is that we do not have to calculate the moments from the data.

Finally, since the MRPP test covers a wide range of tests as its special cases for various choices of weights and the distance measure, therefore, it is worth exploring the optimal choices of weights and the distance measure for the various situations.

APPENDIX A.1

APS: GENERATING ALL POSSIBLE SAMPLES

```
C          THIS PROGRAM CALCULATES FOUR MOMENTS AND OTHER CONSTANTS BY
C        GENERATING ALL POSSIBLE SAMPLES FOR THE TWO-SAMPLE CASE WHERE
C        N1,N2 ARE MORE THAN 2 AND N2 IS AT MOST 10.THE SIZES ARE TAKEN
C        DEPENDING ON HOW LARGE THE DIMENSION OF D IS DECLARED.
      REAL X1(15),X2(10)
      REAL *8 D(50000),SEED,SD,DEL1,TEMP,S1,DM1,DM2,DM3,DM4,
     C  GC,PK,Y1(15),Y2(10)
      DATA S1,DM2,DM3,DM4,N1,N2/4*0.0,10,5/,SEED/67089.0D0/,L/0/
      PRINT 91,SEED,N1,N2
   91 FORMAT('1','INITIAL SEED VALUE=',F9.1,'(N1,N2)=(',I2,',',I2,'
     C ).'/'  THE TWO SAMPLES FROM U-DISTRIBUTION ARE:')
      CALL GGUBS(SEED,N1,X1)
      CALL GGUBS(SEED,N2,X2)
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      PRINT 93,(X1(I),I=1,N1)
      PRINT 93,(X2(I),I=1,N2)
   93 FORMAT('0',6F15.7/)
      DO 3 K=1,N2
       N2K1=N2+1-K
       N1K1=N1+1-K
       N2K2=N2+2-K
       N1K2=N1+2-K
       N2K3=N2+3-K
       N1K3=N1+3-K
       N2K4=N2+4-K
       N1K4=N1+4-K
       N2K5=N2+5-K
       N1K5=N1+5-K
       N2K6=N2+6-K
       N1K6=N1+6-K
       N2K7=N2+7-K
       N1K7=N1+7-K
       N2K8=N2+8-K
       N1K8=N1+8-K
       N2K9=N2+9-K
       N1K9=N1+9-K
       N2K10=N2+10-K
       N1K10=N1+10-K
        DO 71 I1=1,N2K1
        DO 71 J1=1,N1K1
        Y1(J1)=X2(I1)
```

```
            Y2(I1)=X1(J1)
            IF(K .NE. 1) GO TO 11
            CALL STAT(Y1,Y2,N1,N2,DEL1,L)
            D(L)=DEL1
            GO TO 131
   11       I11=I1+1
            J11=J1+1
            DO 72 I2=I11,N2K2
            DO 72 J2=J11,N1K2
             Y1(J2)=X2(I2)
             Y2(I2)=X1(J2)
             IF (K .NE. 2) GO TO 12
             CALL STAT(Y1,Y2,N1,N2,DEL1,L)
             D(L)=DEL1
             GO TO 132
   12        I21=I2+1
             J21=J2+1
             DO 73 I3=I21,N2K3
             DO 73 J3=J21,N1K3
              Y1(J3)=X2(I3)
              Y2(I3)=X1(J3)
              IF(K .NE. 3) GO TO 13
              CALL STAT(Y1,Y2,N1,N2,DEL1,L)
                D(L)=DEL1
                GO TO 133
   13           I31=I3+1
                J31=J3+1
                DO 74 I4=I31,N2K4
                DO 74 J4=J31,N1K4
                 Y1(J4)=X2(I4)
                 Y2(I4)=X1(J4)
                IF(K .NE. 4) GO TO 14
                CALL STAT(Y1,Y2,N1,N2,DEL1,L)
                D(L)=DEL1
                GO TO 134
   14           I41=I4+1
                J41=J4+1
                DO 75 I5=I41,N2K5
                DO 75 J5=J41,N1K5
                 Y1(J5)=X2(I5)
                 Y2(I5)=X1(J5)
                IF(K .NE. 5) GO TO 15
                CALL STAT(Y1,Y2,N1,N2,DEL1,L)
                D(L)=DEL1
                GO TO 135
   15           I51=I5+1
                J51=J5+1
                DO 76 I6=I51,N2K6
                DO 76 J6=J51,N1K6
```

```
      Y1(J6)=X2(I6)
      Y2(I6)=X1(J6)
      IF (K .NE. 6) GO TO 16
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      GO TO 136
16    I61=I6+1
      J61=J6+1
      DO 77 I7=I61,N2K7
      DO 77 J7=J61,N1K7
      Y1(J7)=X2(I7)
      Y2(I7)=X1(J7)
      IF (K .NE. 7) GO TO 17
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      GO TO 137
17    I71=I7+1
      J71=J7+1
      DO 78 I8=I71,N2K8
      DO 78 J8=J71,N1K8
      Y1(J8)=X2(I8)
      Y2(I8)=X1(J8)
      IF (K .NE. 8) GO TO 18
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      GO TO 138
18    I81=I8+1
      J81=J8+1
      DO 79 I9=I81,N2K9
      DO 79 J9=J81,N1K9
      Y1(J9)=X2(I9)
      Y2(I9)=X1(J9)
      IF (K .NE. 9) GO TO 19
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      GO TO 139
19    I91=I9+1
      J91=J9+1
      DO 80 I10=I91,N2K10
      DO 80 J10=J91,N1K10
      Y1(J10)=X2(I10)
      Y2(I10)=X1(J10)
      IF (K .NE. 10) GO TO 20
      CALL STAT(Y1,Y2,N1,N2,DEL1,L)
      D(L)=DEL1
      GO TO 140
20    I101=I10+1
      J101=J10+1
140 Y1(J10)=X1(J10)
```

```
          Y2(I10)=X2(I10)
   80     CONTINUE
  139     Y1(J9)=X1(J9)
          Y2(I9)=X2(I9)
   79     CONTINUE
  138     Y1(J8)=X1(J8)
          Y2(I8)=X2(I8)
   78     CONTINUE
  137     Y1(J7)=X1(J7)
          Y2(I7)=X2(I7)
   77           CONTINUE
  136         Y1(J6)=X1(J6)
              Y2(I6)=X2(I6)
   76          CONTINUE
  135         Y1(J5)=X1(J5)
              Y2(I5)=X2(I5)
   75         CONTINUE
  134         Y1(J4)=X1(J4)
              Y2(I4)=X2(I4)
   74         CONTINUE
  133         Y1(J3)=X1(J3)
              Y2(I3)=X2(I3)
   73        CONTINUE
  132         Y1(J2)=X1(J2)
              Y2(I2)=X2(I2)
   72       CONTINUE
  131         Y1(J1)=X1(J1)
              Y2(I1)=X2(I1)
   71     CONTINUE
    3    CONTINUE
         DO 111 I=1,L
  111    S1=S1+D(I)
         DM1=S1/(1.0*L)
         DO 102 I=1,L
         TEMP=D(I)-DM1
         DM2=DM2+TEMP*TEMP
         DM3=DM3+TEMP**3
  102    DM4=DM4+TEMP**4
         DM2=DM2/(1.0*L)
         DM3=DM3/(1.0*L)
         DM4=DM4/(1.0*L)
         DB1=(DM3/DM2)**2/DM2
         DB2=(DM4/DM2)/DM2
         FK=DB1*(DB2+3.0)**2/(4.0*(4.0*DB2-3.0*DB1)*(2.0*DB2-3.0*DB1-6.0
))
          GC=2.0*DB2-3.0*DB1-6.0
         PRINT 96,DM1,DM2,DM3,DM4,DB1,DB2,FK,GC
   96    FORMAT('0'//,'         MV1           MV2              MV3              MV4
        CB1            B2             K          2B2-3B1-6   ',/15F12.8)
         STOP
```

```
      END
C       FOLLOWING SUBROUTINE GIVES DELTA FOR TWO SAMPLES.
      SUBROUTINE STAT(Y1,Y2,N1,N2,DEL1,L)
      REAL *8 Y1(N1),Y2(N2),DEL1,SD
      L=L+1
      CALL DIST(Y1,N1,SD)
      DEL1=SD
      CALL DIST(Y2,N2,SD)
      DEL1=(DEL1+SD)/(N1+N2)
      RETURN
      END
C            FOLLOWING SUBROUTINE CALCULATES N TIMES THE
C               AVERAGE EUCLIDEAN DISTANCE WITHIN A GROUP
C               OF SIZE N.
      SUBROUTINE DIST(X,N,SD)
      REAL *8 SD,X(N)
      SD=0.0D0
      DO 31 I=1,N
      IF(I .EQ. N) GO TO 42
      J=I+1
      DO 41 K=J,N
       IF(X(K).GE.X(I)) DE=X(K)-X(I)
       IF(X(K).LT.X(I)) DE=X(I)-X(K)
       SD=SD+DE
   41  CONTINUE
   31 CONTINUE
   42 SD=2.0*SD/(N-1.0)
      RETURN
      END
```

APPENDIX A.2

GENAPP: COMPUTING FOUR MOMENTS

```
C          THE PROGRAM CALCULATES FOUR MOMENTS AND OTHER CONSTANTS
C          OF DELTA IN THE CASE OF FOUR SAMPLES FOR A SPECIFIC CHOICE
C          OF WEIGHTS AND THE DISTANCE MEASURE.IT NEEDS SLIGHT CHANGES
C          FOR ANY NUMBER OF SAMPLES,CHOICE OF WEIGHTS AND THE
C          DISTANCE MEASURE.
C                                DSIJ-DISTANCE BETWEEN X(I) & X(J)
C                                DK-(N,4) MATRIX WITH SUMS OVER I
C                                OF DSIJ*DSIK
C                                DM-(N,N) MATRIX WITH SUMS OVER L
C                                OF DSIJ*DM(J,L)
C                                DP-ARRAY TO GET 35 PARAMETERS
C                                DD1-ARRAY OF SUM OF POWERS OF DK(I,1)
C                                DD2-ARRAY OF SUM OF POWERS OF DK(I,2)
C                                DD3 &DD4 ARE LIKE ABOVE
C                                DP#-INTERMEDIATE VALUE OF A PARAMETER
      IMPLICIT REAL *16  (D)
      REAL *16 NP(8),NK(8),NKP(8,8),C(8)
      DIMENSION DK(80,4),DM(80,80),DSQ(80,80),DP(35),
     CDD1(4),DD2(2),DD3(1),DD4(1)
      REAL *8  X(100),Y(8,12)
      DATA DK,DM,DSQ,DP,DD1,DD2,DD3,DD4,DM2P,DM3P,DM4P/13166*0.0D0/
     C  ,DP7,DP9,DP16,DP20,DP23,DP10,DP14,DP15,DP19,D1211,D1311,D1221
     C ,DP21,DP22,DP27,DP28/16*0.0D0/,IG,N,NU/4,16,2/
      READ (5,11) (NK(I),I=1,IG)
      DO 10 I=1,IG
         C(I)=(NK(I)-1.0)/(1.0*N-IG)
         NKP(I,1)=NK(I)
   10 CONTINUE
      READ (5,11) (X(J),J=1,4)
      READ (5,11) (X(J),J=5,8)
      READ (5,11) (X(J),J=9,12)
      READ (5,11) (X(J),J=13,N)
       WRITE(6,91) (X(J),J=1,N)
      NP(1)=N
      DO 20 I=2,8
         NP(I)=0.0
         DO 25 J=1,IG
            NKP(J,I)=0.0
   25    CONTINUE
   20 CONTINUE
      DO 30 I=2,8
         IF(I .LE. N) NP(I)=NP(I-1)*(N+1.0-I)
```

```
            DO 35 J=1,IG
               IF(I .LE. NK(J)) NKP(J,I)=NKP(J,I-1)*(NK(J)+1.0-I)
   35       CONTINUE
   30 CONTINUE
      DO 40 I=1,N
         DO 50 J=1,N
            IF(X(I) .GE. X(J)) DSIJ=(X(I)-X(J))**NU
            IF(X(I) .LT. X(J)) DSIJ=(X(J)-X(I))**NU
            DK(I,1)=DK(I,1)+DSIJ
            DK(I,2)=DK(I,2)+DSIJ**2.0
            DK(I,3)=DK(I,3)+DSIJ**3.0
            DK(I,4)=DK(I,4)+DSIJ**4.0
            DO 80 K=1,N
            IF(X(I) .GE. X(K)) DSIK=(X(I)-X(K))**NU
            IF(X(I) .LT. X(K)) DSIK=(X(K)-X(I))**NU
            DM(J,K)=DM(J,K)+DSIJ*DSIK
   80       CONTINUE
   50    CONTINUE
   40 CONTINUE
       DO 90 I=1,N
          DMT=0.0
          DO 100 J=1,N
             IF(X(I) .GE. X(J)) DSIJ=(X(I)-X(J))**NU
             IF(X(I) .LT. X(J)) DSIJ=(X(J)-X(I))**NU
             DP7=DP7+DSIJ*DM(I,J)
             DP9=DP9+DSIJ*DK(I,1)*DK(J,1)
             DP16=DP16+DSIJ**2.0*DM(I,J)
             DP23=DP23+DSIJ*DM(I,J)*DK(I,1)
             DP20=DP20+DSIJ**2.0*DK(I,1)*DK(J,1)
             DP28=DP28+DK(I,1)**2.0*DK(J,1)*DSIJ
             DMT=DMT+DM(I,J)
             DO 110 L=1,N
             DSQ(I,L)=DSQ(I,L)+DSIJ*DM(J,L)
  110        CONTINUE
  100     CONTINUE
          DP21=DP21+DK(I,2)*DMT
   90  CONTINUE
       DO 120 I=1,N
          DO 130 J=1,N
             IF(X(I) .GE. X(J)) DSIJ=(X(I)-X(J))**NU
             IF(X(I) .LT. X(J)) DSIJ=(X(J)-X(I))**NU
             DP22=DP22+DSIJ*DSQ(I,J)
             DCH=0.0
             DO 170 K=1,N
             DCH=DCH+DSQ(J,K)
  170        CONTINUE
             DP27=DP27+DCH*DSIJ
  130     CONTINUE
  120  CONTINUE
```

```
      DO 140 I=1,N
        DO 150 K=1,4
          DD1(K)=DD1(K)+DK(I,K)
150     CONTINUE
        DD2(1)=DD2(1)+DK(I,1)**2.0
        DD2(2)=DD2(2)+DK(I,2)**2.0
        DD3(1)=DD3(1)+DK(I,1)**3.0
        DD4(1)=DD4(1)+DK(I,1)**4.0
        D1211=D1211+DK(I,1)*DK(I,2)
        D1311=D1311+DK(I,1)*DK(I,3)
        D1221=D1221+DK(I,1)**2.0*DK(I,2)
        DP10=DP10+DK(I,1)**3.0
        DP19=DP19+DK(I,2)*DK(I,1)**2.0
140   CONTINUE
      DP(1)=DD1(1)/NP(2)
      DP(2)=DD1(2)/NP(2)
      DP3=DD2(1)-DD1(2)
      DP4=DD1(1)**2.0-4.0*DD2(1)+2.0*DD1(2)
      DP(3)=DP3/NP(3)
      DP(4)=DP4/NP(4)
      DP(5)=DD1(3)/NP(2)
      DP6=D1211-DD1(3)
      DP(6)=DP6/NP(3)
      DP8=DD1(1)*DD1(2)-4.0*D1211+2.0*DD1(3)
      DP(8)=DP8/NP(4)
      DP(7)=DP7/NP(3)
      DP9S=DP9
      DP9=DP9-DP7-2.0*DP6-DD1(3)
      DP(9)=DP9/NP(4)
      DP10=DP10-3.0*DP6-DD1(3)
      DP(10)=DP10/NP(4)
      DP11=DP3*DD1(1)-4.0*DP9-2.0*DP10-4.0*DP6-2.0*DP7
      DP(11)=DP11/NP(5)
      DP(12)=(DD1(1)*DP4-8.0*DP11-4.0*DP8-8.0*DP9)/NP(6)
      DP(13)=DD1(4)/NP(2)
      DP14=D1311-DD1(4)
      DP(14)=DP14/NP(3)
      DP15=DD2(2)-DD1(4)
      DP(15)=DP15/NP(3)
      DP17=DD1(3)*DD1(1)-4.0*DP14-2.0*DD1(4)
      DP(17)=DP17/NP(4)
      DP18=DD1(2)**2.0-4.0*DP15-2.0*DD1(4)
      DP(18)=DP18/NP(4)
      DP19=DP19-DP15-2.0*DP14-DD1(4)
      DP(19)=DP19/NP(4)
      DP(16)=DP16/NP(3)
      DP20=DP20-DP16-2.0*DP14-DD1(4)
      DP(20)=DP20/NP(4)
      DP21=DP21-DP16-DP14-DP15-DD1(4)
```

```
      DP(21)=DP21/NP(4)
      DP24=DD1(2)*DD2(1)-2.0*(DP16+DP19+DD1(4))-4.0*
     C (DP21+DP15+DP14)-DP18
      DP(24)=DP24/NP(5)
      DP25=DD1(1)*D1211-2.0*(DP20+DP21+DP19+DP16+DD1(4)+DP15)
     C -6.0*DP14-DP17
      DP(25)=DP25/NP(5)
      DP30=DD1(2)*DD1(1)**2.0-4.0*(DP24+DP17+DD1(4))-8.0*(DP25+
     C DP19+DP20+DP16+DP15)-16.0*(DP21+DP14)-2.0*DP18
      DP(30)=DP30/NP(6)
      DP22=DP22-2.0*DP15-DD1(4)
      DP(22)=DP22/NP(4)
      DP23=DP23-2.0*DP16
      DP(23)=DP23/NP(4)
      DP26=DD4(1)-6.0*DP19-3.0*DP15-4.0*DP14-DD1(4)
      DP(26)=DP26/NP(5)
      DP27=DP27-DP22-2.0*(DP23+DP21+DP15+DP14)-
     C DP19-3.0*DP16-DD1(4)
      DP(27)=DP27/NP(5)
      DP28=DP28-2.0*(DP23+DP20)-3.0*(DP16+DP14)-DP21-
     C DP19-DP15-DD1(4)
      DP(28)=DP28/NP(5)
      DP29=DP7*DD1(1)-6.0*(DP23+DP16)
      DP(29)=DP29/NP(5)
      DP31=DD3(1)*DD1(1)-6.0*(DP28+DP23+DP20+DP21+DP16+DP15)-
     C 2.0*(DP26+DD1(4))-3.0*DP25-12.0*DP19-DP17-10.0*DP14
      DP(31)=DP31/NP(6)
      DP32=DD2(1)**2.0-DP26-2.0*(DP22+DP24+DD1(4))-8.0*(DP23+DP21+
     C DP16+DP19+DP14)-4.0*(DP27+DP28+DP20)-DP18-6.0*DP15
      DP(32)=DP32/NP(6)
      DP33=DP9S*DD1(1)-4.0*(DP27+DP28+DP19+DP15)-DP29-2.0*(DP22+DP25
     C  +DD1(4))-10.0*(DP23+DP16)-8.0*DP21-6.0*DP20-DP17-8.0*DP14
      DP(33)=DP33/NP(6)
      DP34=DD2(1)*DD1(1)**2.0-4.0*(DP32+DP31+DP29+DP26+DP17+DD1(4))-
     C 8.0*(DP33+DP22)-24.0*(DP27+DP20+DP14)-32.0*DP28-40.0*(DP23+
     C DP21)-6.0*DP24-28.0*(DP19+DP16)-DP30-16.0*(DP25+DP15)-2.0*DP18
      IF (N .GE. 7) DP(34)=DP34/NP(7)
      DP35=DD1(1)**4-24.0*DP34-12.0*(DP30+DP18)-96.0*(DP33+DP25+DP19
     C +DP20+DP16)-48.0*(DP32+DP24+DP22+DP15)-32.0*(DP31+DP29)-192.0
     C *(DP27+DP28+DP21+DP23)-16.0*(DP26+DP17)-64.0*DP14-8.0*DD1(4)
      IF (N .GE. 8) DP(35)=DP35/NP(8)
      DM2P=DP(4)
     DO 210 I=1,IG
         DM2P=DM2P+2.0*C(I)**2/NKP(I,2)*(2.0*(NK(I)-2.0)
     C   *(DP(3)-DP(4))+DP(2)-DP(4))
         DM3P=DM3P+(C(I)/NKP(I,2))**3*(NKP(I,6)*DP(12)+12.0*NKP(I,5)
     C   *DP(11)+NKP(I,4)*(24.0*DP(9)+8.0*DP(10)+6.0*DP(8))+8.0*
     C   NKP(I,3)*(DP(7)+3.0*DP(6))+4.0*NKP(I,2)*DP(5))
         DM4P=DM4P+(C(I)/NKP(I,2))**4*(8.0*NKP(I,2)*DP(13)+
```

```
     C    16.0*NKP(I,3)*(4.0*DP(14)+3.0*DP(15)+6.0*DP(16))+4.0*
     C    NKP(I,4)*(4.0*DP(17)+3.0*DP(18)+24.0*(DP(19)+DP(20)+2.0*
     C    DP(21))+12.0*(DP(22)+4.0*DP(23)))+16.0*NKP(I,5)*(3.0*DP(24)
     C    +6.0*DP(25)+DP(26)+12.0*(DP(27)+DP(28))+2.0*DP(29))+4.0*
     C    NKP(I,6)*(3.0*DP(30)+4.0*(2.0*DP(31)+3.0*DP(32)+6.0*DP(33)))
     C    +24.0*NKP(I,7)*DP(34)+NKP(I,8)*DP(35))
          DO 220 J=1,IG
             IF(I .EQ. J) GO TO 220
             DM3P=DM3P+3.0*(C(I)/NKP(I,2))**2*C(J)*(NKP(I,4)*DP(12)+4.
     C      0*NKP(I,3)*DP(11)+2.0*NKP(I,2)*DP(8))
           DM4P=DM4P+4.0*(C(I)/NKP(I,2))**3*C(J)*(4.0*NKP(I,2)*DP(17)+
     C      8.0*NKP(I,3)*(3.0*DP(25)+DP(29))+NKP(I,4)*(6.0*DP(30)+8.0*
     C      DP(31)+24.0*DP(33))+12.0*NKP(I,5)*DP(34)+NKP(I,6)*DP(35))
     C      +3.0*(C(I)*C(J)/(NKP(I,2)*NKP(J,2)))**2*(4.0*NKP(I,2)*
     C      NKP(J,2)*DP(18)+8.0*(NKP(I,3)*NKP(J,2)+NKP(I,2)*NKP(J,3))*
     C      DP(24)+2.0*(NKP(I,4)*NKP(J,2)+NKP(J,4)*NKP(I,2))
     C      *DP(30)+16.0*NKP(I,3)*NKP(J,3)*DP(32)+4.0*(NKP(I,4)*NKP(J,
     C      3)+NKP(I,3)*NKP(J,4))*DP(34)+NKP(I,4)*NKP(J,4)*DP(35))
             IF (IG .EQ. 2) GO TO 220
             DO 230 K=1,IG
                IF(I .EQ. K .OR. J .EQ. K) GO TO 230
                DM3P=DM3P+C(I)*C(J)*C(K)*DP(12)
                DM4P=DM4P+6.0*(C(I)/NKP(I,2))**2*C(J)*C(K)*(2.0*DP(30)
     C         *NKP(I,2)+4.0*NKP(I,3)*DP(34)+NKP(I,4)*DP(35))
                IF (IG .EQ. 3) GO TO 230
                DO 240 L=1,IG
                   IF(I .EQ. L .OR. J .EQ. L .OR. K .EQ. L) GO TO 240
                   DM4P=DM4P+C(I)*C(J)*C(K)*C(L)*DP(35)
  240           CONTINUE
  230        CONTINUE
  220     CONTINUE
  210 CONTINUE
      DV=DM2P-DP(1)**2
      DM3=DM3P-3.0*DP(1)*DV-DP(1)**3
      DSKEW=DM3/(DV**1.5)
   11  FORMAT(10F6.3)
      DM4=DM4P-(4.0*DM3+6.0*DV*DP(1)+DP(1)**3)*DP(1)
      DKUR=DM4/DV/DV
      DB1=DSKEW*DSKEW
       GC1=2.0*DKUR-3.0*DB1-6.0
      PK1=DB1*(DKUR+3.0)**2/(4.0*(4.0*DKUR-3.0*DB1)*(2.0*DKUR-
     C3.0*DB1-6.0))
       WRITE(6,91) N,DSKEW,DKUR,GC1,PK1
      WRITE(6,91) DV,DM3P,DM3,DM4P,DM4
       WRITE(6,91) (DP(I),I=1,35)
   91  FORMAT(' ',3F35.12)
      STOP
      END
//GO.SYSIN DD *
```

239.	4	4	4	4
240.	12	17	13	11
241.	14	13	14	12
242.	10	11	14	13
243.	13	9	8	9
244.	//			

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