1975

STEADY MAGNETO AND ELECTROMAGNETO FLUID DYNAMIC FLOWS.

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STEADY MAGNETO AND ELECTROMAGNETO

FLUID DYNAMIC FLOWS

by

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A Dissertation
Submitted to the Faculty of Graduate Studies through the
Department of Mathematics in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy at the
University of Windsor

Windsor, Ontario

1975

Respectfully Dedicated to my beloved parents

ABSTRACT

Steady constantly inclined flows in Magneto Fluid Dynamics are studied. Also, some work is done for Electromagneto Fluid Dynamic Flows.

We proceed to separately outline the flow types considered.

(1) Incompressible, viscous, perfectly electrically conducting and constantly inclined plane flows

Physical conditions are obtained for flows with zero current density as well as for irrotational flows. Complete solutions and geometries are determined for flows with an isometric streamline pattern. Flows for which the magnetic lines and their orthogonal trajectories form an isometric net are completely solved. It is shown that the only permissible straight line geometries are parallel straight lines and concurrent straight lines. Finally, a uniqueness theorem is proved.

(2) Compressible, nonviscous, perfectly electrically conducting and constantly inclined plane flows

A new system of flow equations is obtained by eliminating the magnetic field. Geometric implications, subject to derived physical conditions, are obtained for flows with velocity magnitude constant on each individual streamline and also for sonic flows. A theorem relating density,

pressure and the local speed of sound is obtained. It is shown that for sonic or subsonic flows, a flow geometry of concentric circles or parallel straight lines always implies constant speed on each streamline. The special case of polytropic fluids is considered. Partial solutions are obtained for vortex flow and parallel straight line flow.

(3) Compressible, nonviscous and finitely conducting plane flows

For flows with nonzero charge density, it is shown that the current density is proportional to the vector product of the velocity vector with the magnetic vector. Integrability conditions are derived for flows with nonzero current density. Geometries are determined for irrotational flows, straight line flows and flows whose streamlines are the involutes to a curve. Partial solutions are obtained for incompressible flows with an isometric streamline pattern. Isometric aligned and isometric orthogonal flows are completely solved. Some of the results for nonviscous flows are extended to viscous flows.

ACKNOWLEDGEMENT

The author wishes to express his most sincere thanks and appreciation to Dr. O. P. Chandna. His capable guidance, consideration and friendship will always remain a treasured memory.

He thanks his wife, parents, mother-in-law and son for their understanding and encouragement.

The author expresses his most humble thanks to Dr. R. S. Mishra for his advice and concern.

He thanks Dr. A. C. Smith, Dr. P. N. Kaloni, Dr. G. P. Mathur, Dr. G. S. Guram, and M. R. Garg for their help.

He expresses his gratitude to Mrs. Doreen Truant for her careful typing of the manuscript.

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CHAPTER I

INTRODUCTION

A Historical Sketch

Electromagneto Fluid Dynamics is the study of flows of highly ionized fluids in the presence of an external electromagnetic field. The interaction of the electromagnetic field and the ionized fluid gives rise to mechanical forces which alter the fluid flow. Many new phenomena occur due to the interaction of the fluid dynamic and the electromagnetic forces.

For many cosmic problems, the energy in the electric field is much smaller than the energy in the magnetic field. These problems belong to the realm of the well known Magneto Fluid Dynamics, which is a subfield of the more general Electromagneto Fluid Dynamics. Astrophysicists and Geophysicists have long studied Magneto Fluid Dynamic flows in connection with problems such as sunspot theory and the origin of the earth's magnetism. In recent years Magneto Fluid Dynamics has been applied in engineering to the construction of flow meters and electromagnetic pumps. At present, it is being applied in the design of direct energy converters and fusion type thermonuclear reactors.

The mathematical study of Electromagneto Fluid Dynamics is concerned primarily with the partial differential equations which arise from the well known physical conservation laws. Most of the research to date has been restricted to Magneto Fluid Dynamics in that the assumption of infinite electrical conductivity is made. S. Lunquist (1952) investigated unsteady flows with infinite electrical conductivity. W. R. Sears (1959) and E. L. Resler (1959) did some work in linearized Magneto Fluid Dynamics. S. Chankrasekhar (1961) worked on related stability problems.

Due to the complexity of the subject and because it is a relatively new branch of dynamics, much of the research has consisted in isolating special flows which are accessible via the existing methods of Fluid Dynamics. H. Grad (1960) established the reducibility of a number of Magneto Fluid Dynamic flows to problems in Fluid Dynamics. Aligned flows, Orthogonal flows and Transverse flows are examples of such special flows.

In aligned flow, the magnetic vector and the velocity vector are assumed to be everywhere parallel to one another. Steady aligned plane flows were one of the first flows studied. Many of the results for rotational gases are applicable to these flows. M. Vinokur (1961) and P. Smith (1963) produced many results for these flows. Chandna and Nath (1972) extended Prim's substitution principle to aligned flows with an arbitrary equation of state.

For orthogonal flow, the magnetic vector and the velocity vector are everywhere at right angles to one another.

Orthogonal flows of nonviscous fluids with infinite electrical conductivity have received much study. Ladikov (1962) obtained two Bernoulli equations for such flows. Kingston and Talbot (1969) completely classified the possible flow configurations for the incompressible case. Chandna and Nath (1973) obtained a number of geometric results for the case of compressible fluids. Orthogonal flows of viscous fluids with infinite electrical conductivity were studied by G. Power and his group (1965, 1967, 1969). They were able to relate these flows to flows in ordinary gas dynamics. Nath and Chandna (1973) used M. H. Martin's (1971) approach to study such flows. M. R. Garg and O. P. Chandna have also studied orthogonal flows.

Plane flows are said to be transverse if the magnetic field is normal to the plane of flow. H. Grad (1960) derived two integrals for transverse flows. R. M. Gunderson (1966) studied simple waves for transverse flows. O. P. Chandna (1972) obtained a compatibility equation for such flows. H. Toews and O. P. Chandna (1974) derived a method for solving the transverse flow problem.

A variety of other problems in Magneto Fluid Dynamics have been studied by K. B. Ranger (1969), J. A. Shercliff (1953) and R. H. Wasserman (1967), among others.

One of the few works in Electromagneto Fluid Dynamics to appear in the literature is by Kingston and Power (1968). They considered aligned flows and showed that the charge density is zero or the magnetic field is irrotational. In both cases, they were able to obtain partial solutions.

B Outline of Current Work

This work deals primarily with constantly inclined flows in Magneto Fluid Dynamics. Constantly inclined flows are defined as flows for which the angle between the velocity vector and the magnetic vector is everywhere a nonzero constant. Thus, orthogonal flows may be viewed as a special case of constantly inclined flows. Until 1973, there appears to be no mention of flows corresponding to constantly inclined flows in the literature. J. S. Waterhouse and J. G. Kingston in 1973 published a paper on constantly inclined flows of incompressible nonviscous fluids. In 1974, we published a paper on compressible nonviscous constantly inclined flows. Since 1974, we have submitted three papers for publication and M. R. Garg and O. P. Chandna have submitted one paper. These papers deal with constantly inclined flows of viscous incompressible fluids and will soon appear in print.

In addition to constantly inclined Magneto Fluid Dynamic flow, the present work also gives some results for Electromagneto Fluid Dynamic Flows. In this, the work of Kingston and Power (1968), is extended to nonaligned flows.

We now proceed to give a detailed outline of the thesis.

In section 1 of chapter II, we give the flow equations for Electromagneto Fluid Dynamics. Also, in this section we list the Magneto Fluid Dynamic approximations and give the resulting system of equations. In section 2 of this chapter, we discuss some required results from differential geometry.

Chapter III deals with constantly inclined viscous incompressible plane flows. In section 1, we indicate how either the magnetic field or the velocity field may be eliminated from the flow equations. Sections 2 and 3 consider flows with zero current density and flows with zero vorticity respectively. For both of these flows, we derive general solutions and establish equivalent physical conditions. In section 4 it is shown that the only possible straight streamline geometries are concurrent straight lines or parallel straight lines. Flows with an isometric streamline pattern are considered in section 5. Using a complex variable technique from Berker (1963), general solutions are obtained for the isometric flow problem. From the general solutions, the possible geometries are found. In section 6 a similar study is done for flows in which the magnetic lines and their orthogonal trajectories form an isometric net. Again complete solutions and geometries are obtained. Finally, in section 7 it is shown that a flow is uniquely defined by the fluid properties and the streamline pattern.

In chapter IV, we consider constantly inclined non-viscous compressible flows. In section 1 the magnetic field is eliminated from the flow equations and the resulting equations are transformed to natural or streamline coordinates. In section 2, we study flows with constant speed on each individual streamline. Physical conditions are derived, which relate these flows to flows whose streamlines are concentric circles or parallel straight lines. In particular, it is shown that:

(i) for polytropic gases, the geometry of these flows is always concentric circles or parallel straight lines.

(ii) a sonic or subsonic flow, with a streamline pattern of concentric circles or parallel straight lines, always has constant speed on each individual streamline.

In section 3, we establish physical conditions which relate sonic flows to flow geometries of concentric circles or parallel straight lines. Also, we derive relationships between the fluid pressure, the fluid density, the local speed of sound, the velocity and the vorticity. In section 4 of this chapter, we obtain partial solutions for vortex flows and parallel straight line flows.

Chapter V deals with finitely conducting flows. In section 1, we eliminate the electric field and the current density from the flow equations. Also, in this section, we show that for flows with nonzero charge density the current density is proportional to the vector product of the velocity field with the magnetic field. Several integrability conditions for flows with nonvanishing charge density are derived in section 2. In section 3 the possible geometries, for flows with straight streamlines and flows whose streamlines are the involutes to a curve, are determined. In section 4, we derive the possible geometries for incompressible irrotational flows. Flows with an isometric streamline pattern are considered in section 5. These isometric flows are partially solved in the general care and completely solved for the particular cases of aligned and orthogonal flows. Finally, in Section 6 we briefly consider the case of finitely conducting viscous flows.

CHAPTER II

PRELIMINARIES

Section 1. General Equations of Electromagnetogasdynamics and Magnetogasdynamic Approximations.

Equations of Electromagnetogasdynamics

The fundamental equations governing the motion of an electrically conducting gas as given by Pai (1962) are:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \vec{v} = 0 \tag{21.01}$$

(Conservation of mass)

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \operatorname{grad}) \vec{v} + \operatorname{grad} p =$$

$$\operatorname{div} \tau + \mu \vec{J} \times \vec{H} + q \vec{E} \tag{21.02}$$

(Conservation of linear momentum)

$$\rho \frac{\partial e}{\partial t} + \rho (\vec{v} \cdot \operatorname{grad}) e = -p \operatorname{div} \vec{v} + \Phi$$

$$+ \operatorname{div.} (\kappa \operatorname{grad} T) + I^2/\sigma \tag{21.03}$$

(Conservation of energy)

$$\operatorname{curl} \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \tag{21.04}$$

$$\operatorname{curl} \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{21.05}$$

(Maxwell's equations)

$$\vec{I} = \sigma (\vec{E} + \mu \vec{v} \times \vec{H}) \tag{21.06}$$

(Generalized Ohm's law)

$$\frac{\partial q}{\partial t} + \text{div} \vec{J} = 0 \tag{21.07}$$

(Conservation of Electrical Charge)

$$\rho = \rho (p, s) \tag{21.08}$$

(Equation of state)

where τ denotes the stress tensor, $\vec{v}$ the velocity, $\vec{H}$ the magnetic intensity, $\vec{E}$ the electric intensity, $\vec{J}$ the total electrical current density, $\vec{I}$ the electrical conduction current, ρ the gas density, p the pressure function, s the specific entropy, q the charge density, e the specific internal energy, Φ the viscous dissipation, T the absolute temperature, κ the coefficient of heat conductivity, σ the electrical conductivity, μ the constant magnetic permeability and ε the constant inductive capacity.

If $\vec{v} = (v_1, v_2, v_3)$ and ν is the kinematic coefficient of viscosity, then the components τ_{ij} of the stress tensor τ and the viscous dissipation Φ are given by:

$$\begin{aligned} \tau_{ij} &= \rho\nu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3} \rho\nu \frac{\partial v_K}{\partial x_K} \delta_{ij} \\ \Phi &= \tau_{ij} \frac{\partial v_i}{\partial x_j} \end{aligned} \tag{21.09}$$

where δ_{ij} is the Kronecker delta.

Taking the divergence of both sides of (21.05) gives $\frac{\partial}{\partial t}(\mathrm{div}\,\vec{H}) = 0$. This implies that div $\vec{H}$ is constant relative to time and consequently the magnetic field is usually assumed to be solenoidal.

Magnetogasdynamic Approximations

The theory of magnetogasdynamics makes the following three assumptions:

(1) The time scale of the phenomena is of the same order of magnitude as L/U, where L is the characteristic length and U is the characteristic velocity.

(2) The applied electric field is of the same order of magnitude as the induced magnetic field.

(3) The speed of the flow is much smaller than the speed of light.

Subject to these assumptions, equations (21.01) to (21.08) are replaced by

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \vec{v}) = 0 \tag{21.01}$$

$$\rho = \rho\,(p, s) \tag{21.08}$$

$$\frac{\partial \vec{H}}{\partial t} = \mathrm{curl}\,(\vec{v} \times \vec{H}) - \mathrm{curl}\,(\mathrm{curl}\,\vec{H}/\mu\sigma) \tag{21.10}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\,\vec{v} + \frac{1}{\rho}\,\mathrm{grad}\,p = \frac{\mu}{\rho}\,(\mathrm{curl}\,\vec{H}) \times \vec{H} + \frac{1}{\rho}\,(\mathrm{div}\,\tau) \tag{21.11}$$

$$\rho \left[\frac{\partial h_o}{\partial t} + (\vec{v} \cdot \text{grad})\, h_o\right] = \frac{\partial p}{\partial t} + \text{div}\,(\vec{v} \cdot \tau) + \text{div}\,(\kappa\,\text{grad}\,T) + (\text{curl}\,\vec{H}) . \left[(\text{curl}\,\vec{H})/\sigma - \mu\,\vec{v} \times \vec{H}\right] \tag{21.12}$$

where $h_o = e + p/\rho + \frac{1}{2}|\vec{v}|^2$ is the stagnation enthalpy of the gas.

If the fluid of the flow is incompressible the linear momentum equation (21.11) is replaced by the following Navier Stokes equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \text{grad})\,\vec{v} + \frac{1}{\rho}\,\text{grad}\,p = \frac{\mu}{\rho}\,(\text{curl}\,\vec{H}) \times \vec{H} + \nu\,\nabla^2\,\vec{v} \tag{21.13}$$

Section 2. Results from Differential Geometry

Orthogonal Nets in the x - y Plane

Let

$$x = x(\alpha,\beta), \; y = y(\alpha,\beta) \tag{22.01}$$

define a system of orthogonal curvilinear coordinates in the x - y plane. If $h_1(\alpha,\beta)\, d\alpha^2$ and $h_2(\alpha,\beta)\, d\beta^2$ are the squared components of the vector of arc length, we have

$$ds^2 = h_1(\alpha,\beta)\, d\alpha^2 + h_2(\alpha,\beta)\, d\beta^2 \tag{22.02}$$

where ds is the arc length in the x - y plane and $h_1(\alpha,\beta)$, $h_2(\alpha,\beta)$ are given by

$$\left.\begin{aligned} h_1(\alpha,\beta) &= \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2, \\ h_2(\alpha,\beta) &= \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 \end{aligned}\right\} \tag{22.03}$$

The necessary and sufficient condition for $h_1(\alpha,\beta)$ and $h_2(\alpha,\beta)$ to be the metric coefficients of an orthogonal curvilinear net is that they satisfy the Gauss equation

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{\sqrt{h_1}}\frac{\partial \sqrt{h_2}}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{\sqrt{h_2}}\frac{\partial \sqrt{h_1}}{\partial \beta}\right) = 0 \tag{22.04}$$

The invertibility of the transformation (22.01) implies

$$\frac{\partial x}{\partial \alpha} = J\frac{\partial \beta}{\partial y}, \; \frac{\partial x}{\partial \beta} = -J\frac{\partial \alpha}{\partial y}, \; \frac{\partial y}{\partial \alpha} = -J\frac{\partial \beta}{\partial x}, \; \frac{\partial y}{\partial \beta} = J\frac{\partial \alpha}{\partial x} \tag{22.05}$$

where $J = \left|\frac{\partial(x,y)}{\partial(\alpha,\beta)}\right|$ is the Jacobian of the transformation.

The Jacobian J is related to the metric coefficients by

$$J^2 = h_1 h_2 \tag{22.06}$$

Letting $z = x + iy$ be the complex variable, Θ the angle of inclination of the β = constant curves to the positive x-axis and $e^{i\Theta}$ the unit tangent vector to the β = constant curves, we have

$$\frac{\partial z}{\partial \alpha} = \sqrt{h_1}\, e^{i\Theta}, \quad \frac{\partial z}{\partial \beta} = i\sqrt{h_2} e^{i\Theta} \tag{22.07}$$

Imposition of the integrability condition $\frac{\partial z}{\partial\alpha\partial\beta} = \frac{\partial z}{\partial\beta\partial\alpha}$ in (22.07) yields

$$\frac{\partial \Theta}{\partial \alpha} = -\frac{1}{2J}\frac{\partial h_1}{\partial \beta}, \quad \frac{\partial \Theta}{\partial \beta} = \frac{1}{2J}\frac{\partial h_2}{\partial \alpha} \tag{22.08}$$

From (22.07) and (22.08), we obtain the following form for the transformation from the $\alpha - \beta$ plane to the $x - y$ plane

$$\left.\begin{aligned} \Theta &= \int \frac{1}{2J}\left(\frac{\partial h_2}{\partial \alpha}\, d\beta - \frac{\partial h_1}{\partial \beta}\, d\alpha\right) \\ z &= \int e^{i\Theta}\left(\sqrt{h_1}\, d\alpha + i\sqrt{h_2}\, d\beta\right) \end{aligned}\right\} \tag{22.09}$$

<u>Isometric Nets</u>

An orthogonal curvilinear net is said to be isometric if the metric coefficients $h_1(\alpha,\beta)$, $h_2(\alpha,\beta)$ are everywhere equal. Thus, we have

$$h_1(\alpha,\beta) = h_2(\alpha,\beta) = h(\alpha,\beta) \tag{22.10}$$

where $h(\alpha,\beta)$ is the common value.

Using (22.10) in (22.03), (22.04), and (22.07), we obtain

$$h\ (\alpha,\beta) = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 = \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 \qquad (22.11)$$

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{h}\ \frac{\partial h}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{h}\ \frac{\partial h}{\partial \beta}\right) = 0 \qquad (22.12)$$

$$\left.\begin{aligned} \frac{\partial x}{\partial \alpha} &= \sqrt{h}\ \cos\Theta, & \frac{\partial y}{\partial \beta} &= \sqrt{h}\ \cos\theta \\ \frac{\partial y}{\partial \alpha} &= \sqrt{h}\ \sin\Theta, & \frac{\partial x}{\partial \beta} &= -\sqrt{h}\ \sin\theta \end{aligned}\right\} \qquad (22.13)$$

Equation (22.13) yields the Cauchy Riemann conditions

$$\frac{\partial x}{\partial \alpha} = \frac{\partial y}{\partial \beta},\ \frac{\partial y}{\partial \alpha} = -\frac{\partial x}{\partial \beta} \qquad (22.14)$$

The Cauchy Riemann conditions expressed by (22.14), and the fact that $\frac{\partial \alpha}{\partial x}$, $\frac{\partial \alpha}{\partial y}$, $\frac{\partial \beta}{\partial x}$ and $\frac{\partial \beta}{\partial y}$ are all continuous functions, implies that f(z) defined as

$$f(z) = \alpha + i\beta \qquad (22.15)$$

is an analytic function of z.

By (22.05), (22.11) and (22.14), it follows that

$$|f'(z)|^2 = \left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \beta}{\partial x}\right)^2 = \frac{1}{h} \qquad (22.16)$$

Therefore, the inverse of (22.15) is also analytic and consequently z is an analytic function of $\alpha + i\beta$.

Letting ψ be the argument of f'(z) and using (22.16), we obtain

$$\ln\ [f'(z)] = -\frac{1}{2}\ \ln h + i\,\psi \qquad (22.17)$$

Differentiation of (22.17) with respect to z gives

$$\frac{f''(z)}{f'(z)} = \frac{d}{df}\left[-\frac{1}{2} \ln h + i\,\psi\right] f'(z)$$

or

$$\frac{f''(z)}{[f'(z)]^2} = \frac{d}{df}\left[-\frac{1}{2} \ln h + i\,\psi\right] \qquad (22.18)$$

Since $\frac{f''(z)}{[f'(z)]^2}$ is an analytic function of $f = \alpha + i\beta$, it follows by applying the Cauchy Riemann conditions to the right side of (22.18) that

$$\frac{f''(z)}{f'(z)} = -\frac{1}{2h}\frac{\partial h}{\partial \alpha} + i\,\frac{1}{2h}\frac{\partial h}{\partial \beta} \qquad (22.19)$$

Summarizing these results, we have the following Lemma:

<u>Lemma 2.1</u>

If $f(z) = \alpha(x,y) + i\beta(x,y)$ is a complex function so that the curves α = const. and β = const. generate an isometric net, then $f(z) = \alpha(x,y) + i\,\beta(x,y)$, $z(\xi) = x(\alpha,\beta) + i\,y\,(\alpha,\beta)$ are analytic functions of $z = x + i\,y$, $\xi = \alpha + i\,\beta$ respectively and, furthermore,

(i) $$\frac{f''(z)}{[f'(z)]^2} = -\frac{1}{2h}\frac{\partial h}{\partial \alpha} + i\,\frac{1}{2h}\frac{\partial h}{\partial \beta} \qquad (22.20)$$

(ii) $$\frac{\partial W_1}{\partial \alpha} = \frac{\partial W_2}{\partial \beta}, \quad \frac{\partial W_2}{\partial \alpha} = -\frac{\partial W_1}{\partial \beta}$$

where

$$W_1 = -\frac{1}{2h}\frac{\partial h}{\partial \alpha}, \quad W_2 = \frac{1}{2h}\frac{\partial h}{\partial \beta} \qquad (22.21)$$

and $h\,(\alpha,\beta)$ is defined by (22.11).

We next consider an arbitrary harmonic function Ψ and show that the orthogonal net, generated by the Ψ = constant curves and their orthogonal trajectories, is isometric.

Since Ψ is harmonic, we have

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0$$

or

$$\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial\Psi}{\partial y}\right) \qquad (22.22)$$

Equation (22.22) is the integrability condition for the harmonic conjugate ψ of Ψ, given by

$$\frac{\partial\psi}{\partial y} = \frac{\partial\Psi}{\partial x}, \quad \frac{\partial\psi}{\partial x} = -\frac{\partial\Psi}{\partial y} \qquad (22.23)$$

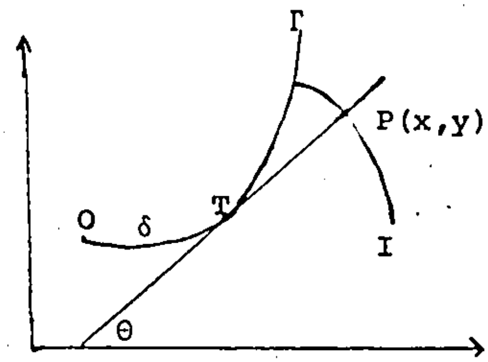
From (22.23) and (22.05), we obtain

$$\frac{\partial\psi}{\partial x}\frac{\partial\Psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\Psi}{\partial y} = 0 \qquad (22.24)$$

$$\left(\frac{\partial x}{\partial\beta}\right)^2 = \left(\frac{\partial y}{\partial\alpha}\right)^2, \quad \left(\frac{\partial y}{\partial\beta}\right)^2 = \left(\frac{\partial x}{\partial\alpha}\right)^2 \qquad (22.25)$$

Equation (22.24) implies that the $\psi - \Psi$ net is orthogonal, and (22.25) that it is isometric.

Orthogonal curvilinear net generated by tangents and involutes to a curve Γ



Letting Γ be a curve in the x-y plane, Θ the angle the tangents to Γ make with the positive x axis, O a fixed point on Γ, we have

$$\left.\begin{aligned} x &= \bar{x}(\delta) + (\xi-\delta)\cos\Theta \\ y &= \bar{y}(\delta) + (\xi-\delta)\sin\Theta \end{aligned}\right\} \qquad (22.26)$$

where $(\bar{x}(\bar{s}), \bar{y}(\bar{s}))$ are the generators of Γ in terms of the arc length $\bar{s}$ from O, δ is the arc length from O to the contact point of the tangent through (x,y) and $\xi = \delta + TP$ is the string length used to generate the involute I through (x,y).

Applying Frenet's formulas,

$$\frac{d\vec{r}}{d\bar{s}} = \vec{t}\,, \quad \frac{d\vec{t}}{d\bar{s}} = \frac{\vec{n}}{R} \qquad (22.27)$$

(where $\vec{t}$ is the unit tangent, $\vec{n}$ the unit principal normal and R the radius of curvature), to (22.26), we obtain

$$\frac{d\bar{x}}{d\delta} = \cos\Theta, \quad \frac{d\bar{y}}{d\delta} = \sin\Theta, \quad \frac{d\Theta}{d\delta} = \frac{1}{R} \qquad (22.28)$$

It follows from (22.26) and (22.28) that the squared element $d\bar{s}^2$, of arc length in the x-y plane, is given by

$$d\bar{s}^2 = d\xi^2 + \left(\frac{\xi-\delta}{R}\right)^2 d\delta^2$$

or

$$d\bar{s}^2 = d\xi^2 + (\xi-\delta)^2\, d\Theta^2 \qquad (22.29)$$

Since ξ is constant on involutes and δ is constant on tangents, the metric coefficients, for the orthogonal net generated by the tangents and their involutes, are given by

$$\left.\begin{aligned} h_1 &= 1 \\ h_2 &= \left(\frac{\xi-\delta}{R}\right)^2 \end{aligned}\right\} \qquad (22.30)$$

CHAPTER III

CONSTANTLY INCLINED INCOMPRESSIBLE FLOWS

Section 1. Flow Equations

The steady motion of an incompressible, viscous and electrically conducting fluid is governed by the equations:

$$\operatorname{div} \vec{v} = 0 \tag{31.01}$$

$$(\vec{v} \cdot \operatorname{grad}) \vec{v} + \frac{1}{\rho} \operatorname{grad} p = \nu \nabla^2 \vec{v} + \frac{\mu}{\rho} (\operatorname{curl} \vec{H}) \times \vec{H} \tag{31.02}$$

$$\vec{J} = \operatorname{curl} \vec{H} = \sigma (\vec{E} + \mu \vec{v} \times \vec{H}) \tag{31.03}$$

$$\operatorname{curl} \vec{E} = \vec{0} \tag{31.04}$$

$$\operatorname{div} \vec{H} = 0 \tag{31.05}$$

For fluids with infinite electrical conductivity, on eliminating $\vec{E}$ between (31.03) and (31.04), we obtain

$$\operatorname{curl} (\vec{v} \times \vec{H}) = \vec{0} \tag{31.06}$$

In this chapter, we study infinitely electrically conducting, nonaligned and steady plane flows for which the magnetic field vector lies in the plane of flow, and the angle between the velocity vector and the magnetic vector is constant throughout the flow.

Assuming $\phi \neq 0$ to be the constant angle between $\vec{v} = (V_1, V_2)$, $\vec{H} = (H_1, H_2)$ in the (x, y)-plane and employing (31.06), we get

$$V_1 H_2 - V_2 H_1 = VH \sin\phi = A \tag{31.07}$$

where V and H are the magnitudes of the velocity and magnetic intensity vectors, and A is an arbitrary non-zero constant due to the exclusion of aligned flows.

Equation (31.07) implies the existence of a constant B such that

$$V_1 H_1 + V_2 H_2 = VH \cos\phi = B \tag{31.08}$$

where

$$B = A \cot\phi \text{ and } V^2 H^2 = A^2 + B^2 \tag{31.09}$$

The constant B is zero if and only if $\vec{v}$ and $\vec{H}$ are everywhere mutually orthogonal.

Solving (31.07) and (31.08) for $\vec{v}$ and $\vec{H}$, we have

$$\vec{v} = \frac{A}{H^2} \vec{H} \times \vec{k} + \frac{B}{H^2} \vec{H} \tag{31.10}$$

$$\vec{H} = \frac{A}{V^2} \vec{k} \times \vec{v} + \frac{B}{V^2} \vec{v} \tag{31.11}$$

where $\vec{k}$ is the unit vector normal to the plane of flow.

Equations (31.10) and (31.11) may be used to eliminate either $\vec{v}$ or $\vec{H}$ from the basic flow equations.

The flows under consideration are solutions to equations (31.01), (31.02), (31.05), (31.10) and (31.11). Having solved for $\vec{v}$, $\vec{H}$ and p, $\vec{E}$ and $\vec{J}$ are given by

$$\vec{E} = -\mu\vec{v} \times \vec{H}, \quad \vec{J} = \text{curl}\ \vec{H} \tag{31.12}$$

Section 2. Flows with Zero Current Density

In the first part of this section, we derive the necessary and sufficient physical conditions for flows with zero current density. In the second part, we obtain the general solutions for such flows.

Part I: Physical Conditions

Employing (31.10) in (31.01) and using (31.05), we have

$$\text{div}\ (\vec{H} \times \vec{k}) = \frac{2}{A}\ \text{grad}\ \ln H \cdot (A\ \vec{H} \times \vec{k} + B\ \vec{H}) \qquad (32.01)$$

or

$$\vec{J} \cdot \vec{k} = \frac{2}{A}\ \text{grad}\ \ln H \cdot (A\ \vec{H} \times \vec{k} + B\ \vec{H}) \qquad (32.02)$$

Using (31.01), (31.09) and (31.10) in (32.02), we obtain

$$J = \frac{(B^2 + A^2)}{A}\ \text{grad}\ (\frac{1}{V^2}) \cdot \vec{v} \qquad (32.03)$$

where $|J|$ is the magnitude of the current density at any point in the flow region.

From equations (31.09) and (32.03), we can state the following result:

Theorem 3.1

For constantly inclined, steady, incompressible plane flow of a perfectly conducting fluid, the current density is everywhere zero in the flow region if and only if the velocity magnitude and, therefore, the magnetic intensity magnitude are constant on each individual streamline.

By (31.12), we have the following corollary to this theorem.

Corollary 3.1

Flows of Theorem 3.1 are force-free, have constant electric intensity normal to the plane of flow and the total energy per unit volume remains constant on each individual streamline.

Part II: Solutions

Since we want to obtain general solutions for incompressible flows with zero current density, equations (31.01) and (32.03) give

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0 \qquad (32.04)$$

$$V_1^2 \frac{\partial V_1}{\partial x} + V_1 V_2 \left(\frac{\partial V_2}{\partial x} + \frac{\partial V_1}{\partial y}\right) + V_2^2 \frac{\partial V_2}{\partial y} = 0 \qquad (32.05)$$

Eliminating $\vec{H}$ between (31.05) and (31.11), we find

$$A \left[2 V_1 V_2 \frac{\partial V_1}{\partial x} + (V_2^2 - V_1^2) \left(\frac{\partial V_1}{\partial y} + \frac{\partial V_2}{\partial x}\right) - 2 V_1 V_2 \frac{\partial V_2}{\partial y} \right] - B \left[2 V_1^2 \frac{\partial V_1}{\partial x} + 2 V_1 V_2 \frac{\partial V_2}{\partial x} + 2 V_1 V_2 \frac{\partial V_1}{\partial y} + 2 V_2 \frac{\partial V_2}{\partial y} - (V_1^2 + V_2^2) \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}\right) \right]$$

$$= 0 \qquad (32.06)$$

Using (32.04) and (32.05) in (32.06), we obtain

$$2 V_1 V_2 \frac{\partial V_1}{\partial x} + (V_2^2 - V_1^2) \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial x}\right) - 2 V_1 V_2 \frac{\partial V_2}{\partial y} = 0 \tag{32.07}$$

Substitution of $\frac{\partial V_2}{\partial y} = -\frac{\partial V_1}{\partial x}$ from (32.04) into (32.05) and (32.07) yields

$$(V_1^2 - V_2^2) \frac{\partial V_1}{\partial x} + V_1 V_2 \left(\frac{\partial V_2}{\partial x} + \frac{\partial V_1}{\partial y}\right) = 0 \tag{32.08}$$

$$4 V_1 V_2 \frac{\partial V_1}{\partial x} + (V_2^2 - V_1^2) \left(\frac{\partial V_2}{\partial x} + \frac{\partial V_1}{\partial y}\right) = 0 \tag{32.09}$$

Equations (32.04), (32.08) and (32.09) imply that V_1, V_2 satisfy

$$\left.\begin{aligned} V_1 &= V_1 (y) \\ V_2 &= V_2 (x) \\ V_1' (y) + V_2' (x) &= 0 \end{aligned}\right\} \tag{32.10}$$

Integrating (32.10) for $\vec{v}$, we obtain

$$V_1 = Cy + D_1, \quad V_2 = -Cx + D_2 \tag{32.11}$$

where C, D_1 and D_2 are arbitrary constants.

Taking the curl of (31.02) and using $\vec{J} = \vec{0}$, we obtain the following integrability condition for the pressure function

$$\text{curl} [\vec{v} \times (\text{curl } \vec{v})] + \nu \text{ curl } [\nabla^2 \vec{v}] = 0 \tag{32.12}$$

The solutions given in (32.11) satisfy (32.12). Therefore, (32.11) gives the general solution for $\vec{v}$ of our flow problem.

Employing (32.11) in (31.02) and using $\vec{J} = 0$, we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) &= C^2 x - C D_2 \\ \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) &= C^2 y + C D_1 \end{aligned}\right\} \qquad (32.13)$$

Integrating (32.13), we find that the pressure function is given by

$$p(x, y) = \frac{C^2 \rho}{2}(x^2 + y^2) + C\rho\,(D_1 y - D_2 x) + D_3 \qquad (32.14)$$

where D_3 is an arbitrary constant.

Using (32.11) in (31.11), we obtain

$$\left.\begin{aligned} H_1 &= \frac{C\,(Ax + By) + B\,D_1 - A\,D_2}{C^2(x^2 + y^2) + 2C\,(D_1 y - D_2 x) + (D_1^2 + D_2^2)} \\ H_2 &= \frac{C\,(Ay - Bx) + A\,D_1 + B\,D_2}{C^2(x^2 + y^2) + 2C\,(D_1 y - D_2 x) + (D_1^2 + D_2^2)} \end{aligned}\right] \qquad (32.15)$$

Section 3. Irrotational Flows

In this section, we obtain the physical implications of flows whose vorticity vector field vanishes in the flow region and obtain the general solutions for these flows.

For the flows of this chapter $\vec{H}$ is given by (31.11). Substituting (31.11) in (31.05) and using (31.01), we get

$$\operatorname{div}(\vec{v} \times \vec{k}) = -\frac{2}{A} \operatorname{grad} \ln V \cdot (A \vec{k} \times \vec{v} + B \vec{v}) \qquad (33.01)$$

By vector calculus,

$$\operatorname{div}(\vec{v} \times \vec{k}) = \vec{k} \times \vec{\omega} \qquad (33.02)$$

where $\vec{\omega}$ is the vorticity vector.

Using (31.05), (31.09), (31.11) and (33.02) in (33.01), we find

$$\omega = -\frac{(B^2 + A^2)}{A} \operatorname{grad} \left(\frac{1}{H^2}\right) \cdot \vec{H} \qquad (33.03)$$

where $|\omega| = |\vec{\omega}|$.

From relations (31.09) and (33.03), we obtain the following theorem

Theorem 3.2

For constantly inclined, steady, incompressible plane flow of a perfectly conducting fluid, the flow is irrotational if and only if the velocity magnitude and the magnetic intensity magnitude are constant on each individual magnetic line.

In order to find the general solution for these irrotational flows, we express (31.05) and (33.03) in the following equivalent forms

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \tag{33.04}$$

$$H_1^{\,2} \frac{\partial H_1}{\partial x} + H_1 H_2 \left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) + H_2^{\,2} \frac{\partial H_2}{\partial y} = 0 \tag{33.05}$$

Employing (31.10) in (31.01) and using (33.04), (33.05), we get

$$2 H_1 H_2 \frac{\partial H_1}{\partial x} + (H_2^{\,2} - H_1^{\,2}) \left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) - 2 H_1 H_2 \frac{\partial H_2}{\partial y} = 0 \tag{33.06}$$

Equations (33.04), (33.05) and (33.06) in $\vec{H}$ are identically similar to (32.04), (32.05) and (32.07) in $\vec{v}$. Solving (33.04) to (33.06) in a manner similar to the method of section 2, we obtain

$$H_1 = Ky + N_1, \quad H_2 = -Kx + N_2 \tag{33.07}$$

where K, N_1 and N_2 are arbitrary constants.

Taking the curl of (31.02) and using $\vec{\omega} = 0$, we find the following integrability condition for the pressure function

$$\text{curl } [(\text{curl } \vec{H}) \times \vec{H}] = \vec{0} \tag{33.08}$$

Since the solution given by (33.07) satisfies (33.08), it constitutes the general solution for $\vec{H}$ to our irrotational flow.

Employing (33.07), (31.09) and $\vec{\omega} = \vec{0}$ in (31.02), we obtain

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) &= -\frac{\partial}{\partial x}\left[\frac{(A^2 + B^2)}{2H^2}\right] - \frac{2\mu}{\rho}(Kx - N_2) \\ \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) &= -\frac{\partial}{\partial y}\left[\frac{(A^2 + B^2)}{2H^2}\right] - \frac{2\mu}{\rho}(Ky + N_2) \end{aligned}\right\} \quad (33.09)$$

Integration of (33.09) yields

$$p(x, y) = N_3 - K^2\rho\,(x^2 + y^2) + 2\,K\,\rho\,(N_2x - N_1y) - \frac{\rho\,(A^2 + B^2)}{2\,K^2(x^2 + y^2) + 4\,K\,(N_1y - N_2x + 2\,N_1^{\,2} + 2\,N_2^{\,2})} \quad (33.10)$$

where N_3 is an arbitrary constant.

Using (33.07) in (31.10), we obtain

$$\left.\begin{aligned} V_1\,(x,y) &= \frac{K\,(By - Ax) + B\,N_1 + A\,N_2}{K^2(x^2 + y^2) + 2\,K\,(N_1y - N_2x) + (N_1^{\,2} + N_2^{\,2})} \\ V_2\,(x,y) &= \frac{B\,N_2 - A\,N_1 - K\,(Ay + Bx)}{K^2(x^2 + y^2) + 2\,K\,(N_1y - N_2x) + (N_1^{\,2} + N_2^{\,2})} \end{aligned}\right] \quad (33.11)$$

Section 4. Flows with Non-Parallel Straight Streamlines

In this section, we study the question whether the only possible non-parallel flows with straight streamlines are source flows. To answer this question, we assume that the flows are not parallel but envelope to a curve Γ. We take the streamlines and their orthogonal trajectories, the involutes of Γ, as the system of orthogonal curvilinear coordinates. Taking ξ, δ and Θ as in section (2.2), we have Θ constant on the streamlines and ξ constant on the involutes of Γ.

Letting (ξ, Θ) be the natural net for our flow, we have that the squared element of arc length is given by

$$ds^2 = d\xi^2 + [\xi - \delta(\Theta)]^2 \, d\Theta^2 \qquad (22.29)$$

and, therefore the metric coefficients are

$$h_1 = 1, \; h_2 = \xi - \delta \; (\Theta) \qquad (22.30)$$

If $\vec{e}_1$, $\vec{e}_2$ are the unit tangents to the tangent lines and the involutes of Γ respectively, it follows by (31.11) that

$$\left.\begin{aligned} \vec{v} &= V \, \vec{e}_1 \\ \vec{H} &= \frac{1}{V} (B \, \vec{e}_1 + A \, \vec{e}_2) \end{aligned}\right\} \qquad (34.01)$$

Using (34.01) in the continuity equation and the solenoidal condition on $\vec{H}$, we find

$$\frac{\partial}{\partial \xi} [(\xi - \delta) \; V] = 0 \qquad (34.02)$$

$$B \frac{\partial}{\partial \xi} \left[\frac{(\xi - \delta)}{V}\right] + A \frac{\partial}{\partial \Theta} \left[\frac{1}{V}\right] = 0 \qquad (34.03)$$

Equation (34.02) implies that

$$V = \frac{f(\Theta)}{(\xi - \delta)} \qquad (34.04)$$

where f (Θ) is an arbitrary integrable function of Θ.

Substituting (34.04) into (34.03), we obtain

$$A \frac{\delta' (\Theta)}{f (\Theta)} + (\delta - \xi) \left[\frac{2B}{f(\Theta)} - \frac{Af'(\Theta)}{\{f (\Theta)\}^2}\right] = 0 \qquad (34.05)$$

Since ξ and Θ are independent, (34.05) implies

$$A \frac{\delta'(\Theta)}{f(\Theta)} = \frac{2B}{f(\Theta)} - \frac{A f'(\Theta)}{\{f(\Theta)\}^2} = 0 \qquad (34.06)$$

In view of the fact that $A \neq 0$, (34.06) implies that $\delta'(\Theta) = 0$. Therefore, by (22.24) the radius of curvature of Γ is everywhere zero and this proves that:

<u>Theorem 3.3</u>

For a constantly inclined, steady, incompressible flow of a perfectly conducting fluid, the only possible straight streamline patterns are parallel straight lines or concurrent straight lines.

Section 5. Flows with Isometric Geometry

In the first part of this section, we derive general solutions for flows with an isometric streamline pattern. In the second part, we classify these solutions and identify the corresponding geometries.

Part I: Solutions

Let

$$x = x(\alpha,\beta)$$
$$y = y(\alpha,\beta) \qquad (35.01)$$

define a system of isometric curvilinear coordinates in the plane of flow such that the $\beta(x,y)$ = const. curves represent the streamlines and the $\alpha(x,y)$ - const. curves represent the orthogonal trajectories to the streamlines. Letting $\vec{e}_1$ be the unit tangent vector to β = const. in the direction of increasing α, $\vec{e}_2$ the unit tangent vector to α = const., $h(\alpha,\beta)\ d\alpha^2$ and $h(\alpha,\beta)\ d\beta^2$ the squared components of the vector element of arc length, we have

$$\vec{v} = V(\alpha,\beta)\ \vec{e}_1 \qquad (35.02)$$

$$d\overline{s}^2 = h(\alpha,\beta)\ [d\alpha^2 + d\beta^2] \qquad (35.03)$$

where h satisfies

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{h}\ \frac{\partial h}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{h}\ \frac{\partial h}{\partial\beta}\right) = 0 \qquad (22.12)$$

Substitution of (35.02) into (31.01) and (31.11) yields

$$\frac{\partial}{\partial\alpha}\ (\sqrt{h}\ V) = 0 \qquad (35.04)$$

$$\vec{H} = \frac{B}{V} \vec{e}_1 + \frac{A}{V} \vec{e}_2 \tag{35.05}$$

Eliminating $\vec{H}$ between (31.05) and (35.05), we obtain

$$B \frac{\partial}{\partial \alpha} \left(\frac{\sqrt{h}}{V}\right) + A \frac{\partial}{\partial \beta} \left(\frac{\sqrt{h}}{V}\right) = 0 \tag{35.06}$$

Since $\vec{J}$ = curl $\vec{H}$, equation (32.03) implies that

$$\text{curl } \vec{H} = [\frac{(B^2 + A^2)}{A} \text{ grad } (\frac{1}{V^2}) \cdot \vec{V}] \vec{k} \tag{35.07}$$

Using (31.01), (31.11) and (35.07) in (31.02), the linear momentum equation takes the form

$$(\vec{v} \cdot \text{grad}) \vec{v} + \text{grad } (\frac{p}{\rho}) = \nu \nabla^2 \vec{v} + \frac{\mu}{\rho} \{\frac{B^2 + A^2}{2A} \text{ grad } (\frac{1}{V^4}) \cdot \vec{v}\} \{B \vec{k} \times \vec{v} - A \vec{v}\} \tag{35.08}$$

Employing (35.02) in (35.08), we find

$$V \frac{\partial V}{\partial \alpha} + \frac{1}{\rho} \frac{\partial p}{\partial \alpha} = \nu \frac{\partial}{\partial \beta} \{\frac{1}{h} \frac{\partial}{\partial \beta} (\sqrt{h}\, V)\} + \frac{2\mu (A^2 + B^2}{\rho V^3} \frac{\partial V}{\partial \alpha} \tag{35.09}$$

$$\frac{V}{\sqrt{h}} \frac{\partial}{\partial \beta} (\sqrt{h}\, V) - V \frac{\partial V}{\partial \beta} - \frac{1}{\rho} \frac{\partial p}{\partial \beta} = \nu \frac{\partial}{\partial \alpha} \{\frac{1}{h} \frac{\partial}{\partial \beta} (\sqrt{h}\, V)\} + \frac{2 \mu B (A^2 + B^2)}{A \rho V^3} \frac{\partial V}{\partial \alpha} \tag{35.10}$$

Equations (35.04) and (35.06) are two equations in V and can be employed to solve for the velocity field. However, the solution thus obtained must satisfy the integrability condition for the pressure function which is derived by taking the curl of (35.08). Having obtained $\vec{v}$, we solve for the pressure function and the magnetic field by employing (35.09), (35.10) and (31.11).

It follows from (35.04) that

$$V(\alpha,\beta) = \frac{\psi(\beta)}{\sqrt{h}} \tag{35.11}$$

where $\psi(\beta)$ is an arbitrary differentiable function of β.

Substituting (35.11) into (35.06) yields

$$B \frac{\partial}{\partial\alpha}\left(\frac{h}{\psi(\beta)}\right) + A \frac{\partial}{\partial\beta}\left(\frac{h}{\psi(\beta)}\right) = 0 \tag{35.12}$$

or

$$A\, W_2 - B\, W_1 = \frac{A\, \psi'(\beta)}{2\psi(\beta)} \tag{35.13}$$

where $W_1 = -\frac{1}{2h}\frac{\partial h}{\partial\alpha}$, $W_2 = \frac{1}{2h}\frac{\partial h}{\partial\beta}$.

By Lemma (2.1), $W_1 + i\, W_2$ is an analytic function of $\alpha + i\,\beta$, and therefore

$$\frac{\partial W_1}{\partial\alpha} = \frac{\partial W_2}{\partial\beta}, \quad \frac{\partial W_2}{\partial\alpha} = -\frac{\partial W_1}{\partial\beta} \tag{35.14}$$

Differentiating (35.13) with respect to α, we obtain

$$A \frac{\partial W_2}{\partial\alpha} - B \frac{\partial W_1}{\partial\alpha} = 0 \tag{35.15}$$

Combining (35.14) and (35.15), we find the following uncoupled first order partial differential equations

$$A \frac{\partial W_1}{\partial\beta} + B \frac{\partial W_1}{\partial\alpha} = 0 \tag{35.16}$$

$$B \frac{\partial W_2}{\partial\beta} - A \frac{\partial W_2}{\partial\alpha} = 0 \tag{35.17}$$

Therefore, the characteristic curves for W_1, W_2 are given by

$$\left.\begin{aligned} \frac{d\beta}{A} &= \frac{d\alpha}{B} \\ \frac{d\beta}{A} &= -\frac{d\alpha}{A} \end{aligned}\right\} \qquad (35.18)$$

respectively.

Integrating (35.18), the general solutions for W_1 and W_2 are given by

$$\left.\begin{aligned} W_1\ (\alpha,\beta) &= g_1(A\alpha - B\beta) \\ W_2\ (\alpha\ \beta) &= g_2(A\beta + B\alpha) \end{aligned}\right\} \qquad (35.19)$$

where g_1 and g_2 are arbitrary differentiable functions of their respective arguments.

Defining $\xi = A\alpha - B\beta$ and $\eta = A\beta + B\alpha$, it follows that the transformation Jacobian from the (α,β) - plane to the (ξ,η) - plane is given by

$$J = \left|\frac{\partial(\xi,\eta)}{\partial(\alpha,\beta)}\right| = A^2 + B^2 \qquad (35.20)$$

Since $A \neq 0$ for our flows, it follows from (35.20) that ξ and η may be taken as independent variables.

Using (35.19) in (35.14), we get

$$W_1'\ (\xi) = W_2'\ (\eta) \qquad (35.21)$$

Therefore, we have

$$W_1 = C(A\alpha - B\beta) + C_1,\ W_2 = C(A\beta + B\alpha) + C_2 \qquad (35.22)$$

where C, C_1 and C_2 are arbitrary constants.

Substitution of (35.22) into (35.13) and use of the definitions for W_1, W_2 yields

$$\left.\begin{aligned} \frac{\partial}{\partial\alpha}(\ln h) &= 2\,C\,(B\beta - A\alpha) - 2\,C_1 \\ \frac{\partial}{\partial\beta}(\ln h) &= 2\,C\,(A\beta + B\alpha) + 2\,C_2 \\ \frac{d}{d\beta}(\ln \psi) &= \frac{2C}{A}(A^2 + B^2)\beta - \frac{2\,B\,C_1}{A} + 2\,C_2 \end{aligned}\right\} \quad (35.23)$$

Integration of (35.23) gives

$$h(\alpha,\beta) = \exp\{AC\,(\beta^2 - \alpha^2) + 2\,C\,B\alpha\beta + 2\,C_2\,\beta - 2\,C_1\alpha + C_3\} \quad (35.24)$$

$$\psi(\beta) = \exp\{\frac{C}{A}(A^2 + B^2)\,\beta^2 + (2\,C_2 - \frac{2\,B\,C_1}{A})\,\beta + C_4\} \quad (35.25)$$

where C_3 and C_4 are arbitrary constants.

The solutions given by (35.24) and (35.25) satisfy (34.04) and (35.06). However, these solutions must also satisfy the integrability equation for p (α,β) obtained from (35.09), (35.10) and given by

$$\frac{\partial}{\partial\alpha}\{\frac{V}{\sqrt{h}}\,\frac{\partial}{\partial\beta}(\sqrt{h}\,V)\} - \nu\,(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2})\,\{\frac{1}{h}\,\frac{\partial}{\partial\beta}(\sqrt{h}\,V)\}$$

$$- \frac{2\mu}{\rho}(A^2 + B^2)\,\{\frac{\partial}{\partial\beta}(\frac{1}{V^3}\,\frac{\partial V}{\partial\alpha}) + \frac{B}{A}\,\frac{\partial}{\partial\alpha}(\frac{1}{V^3}\,\frac{\partial V}{\partial\alpha})\} = 0 \quad (35.26)$$

Eliminating V (α,β) between (35.11) and (35.26), we find that $\psi(\beta)$ and h (α,β) must satisfy

$$\psi\,\psi'\,\{\frac{1}{h^2}\frac{\partial h}{\partial \alpha}\} + \nu\,[\frac{\psi'''}{h} - 2\,\psi''\,\{\frac{1}{h^2}\frac{\partial h}{\partial \beta}\} +$$

$$\psi'\,\{\frac{\partial^2}{\partial \alpha^2}(\frac{1}{h}) + \frac{\partial^2}{\partial \beta^2}(\frac{1}{h})\}] - \frac{\mu(A^2 + B^2)\,h}{2\,A\,\rho\,\psi^2}$$

$$\{\frac{2A}{h}\frac{\partial^2 h}{\partial\alpha\partial\beta} + \frac{2B}{h}\frac{\partial^2 h}{\partial\alpha^2} - \frac{4A}{h}\frac{\partial h}{\partial \alpha}\frac{\psi'}{\psi}\} = 0 \qquad (35.27)$$

We note that

$$\frac{\partial^2}{\partial \alpha^2}(\frac{1}{h}) = \frac{\partial}{\partial \alpha}(-\frac{1}{h^2}\frac{\partial h}{\partial \alpha}) = -\frac{1}{h}\frac{\partial}{\partial \alpha}(\frac{1}{h}\frac{\partial h}{\partial \alpha}) + \frac{1}{h^3}(\frac{\partial h}{\partial \alpha})^2$$

and (35.28)

$$\frac{\partial^2}{\partial \beta^2}(\frac{1}{h}) = \frac{\partial}{\partial \beta}(-\frac{1}{h^2}\frac{\partial h}{\partial \beta}) = -\frac{1}{h}\frac{\partial}{\partial \beta}(\frac{1}{h}\frac{\partial h}{\partial \beta}) + \frac{1}{h^3}(\frac{\partial h}{\partial \beta})^2$$

Adding the equations in (35.28) and using (22.12), we obtain

$$\frac{\partial^2}{\partial \alpha^2}(\frac{1}{h}) + \frac{\partial^2}{\partial \beta^2}(\frac{1}{h}) = \frac{1}{h^3}[(\frac{\partial h}{\partial \alpha})^2 + (\frac{\partial h}{\partial \beta})^2] \qquad (35.29)$$

Substituting (35.29) in (35.27) and multiplying the resulting equation by h, we get

$$\psi\psi'\{\frac{1}{h}\frac{\partial h}{\partial \alpha}\} + \nu\,[\psi''' - 2\,\psi''\,\{\frac{1}{h}\frac{\partial h}{\partial \beta}\} +$$

$$\psi'\,\{(\frac{1}{h}\frac{\partial h}{\partial \alpha})^2 + (\frac{1}{h}\frac{\partial h}{\partial \beta})^2\}] - \mu\,\frac{(A^2 + B^2)}{2\,A\,\rho} \qquad (35.30)$$

$$\frac{h^2}{\psi^2}\{\frac{2A}{h}\frac{\partial^2 h}{\partial\alpha\partial\beta} + \frac{2B}{h}\frac{\partial^2 h}{\partial\alpha^2} - \frac{4A}{h}\frac{\partial h}{\partial \alpha}\frac{\psi'}{\psi}\} = 0$$

Employing (35.24) in (35.30), we find

$$\psi\psi'\,(2C\,B\beta - 2\,A\,C\,\alpha - 2\,C_1) + \nu\,\psi''' -$$

$$2\nu\,\psi''\,(2A\,C\beta + 2B\,C\alpha + 2\,C_2) + \nu\,\psi'\,\{(2CB\beta - 2AC\alpha -$$

$$2\,C_1)^2 + (2AC\beta + 2BC\alpha + 2\,C_2)^2\} - \frac{4\,\mu\,(A^2 + B^2)}{A\,\rho\,\psi^2}$$

$$\{h^2 \, (B C \beta - A C \alpha - C_1) \, (A^2 C \beta + B^2 C \beta + A C_2 - B C_1)$$

$$- \frac{A\psi'}{\psi} h^2 \, (B C \beta - A C \alpha - C_1) \} = 0 \qquad (35.31)$$

Differentiating (35.31) thrice with respect to α, we obtain

$$\frac{\partial^3}{\partial \alpha^3} [h^2 \, (B C \beta - A C \alpha - C_1) \, (A^2 C \beta + B^2 C \beta + A C_2 -$$

$$B C_1)] - \frac{A\psi'}{\psi} \frac{\partial^3}{\partial \alpha^3} [h^2 \, (B C \beta - A C \alpha - C_1)] = 0 \quad (35.32)$$

Using (35.24), (35.25) in (35.32) and simplifying, we have

$$\{C(A^2 + B^2) \, \beta + AC_2 - B C_1\} \, \{16(B C \beta - A C \alpha - C_1)^4$$

$$- 24 \, AC \, (B C \beta - A C \alpha - C_1)^2 + 3 \, A^2 C^2\} = 0 \; (35.33)$$

Equation (35.33) is of fifth degree in α, β and is satisfied throughout the flow region. This requires that all its coefficients must be zero. In particular, equating the coefficient of $\alpha^4\beta$ to zero, we have

$$A^4 \, C^5 \, (A^2 + B^2) = 0 \qquad (35.34)$$

Since $A \neq 0$ and (35.34) must hold true, we find that $C = 0$. Setting $C = 0$ in (35.33), it follows that for our flows the constants C_1 and C_2 must satisfy

$$C_1 \, (AC_2 - BC_1) = 0 \qquad (35.35)$$

If $C = 0$ and (35.35) is true, (35.24) and (35.25) satisfy equation (35.31). Therefore, we have the following theorem:

Theorem 3.4

If the natural net is isometric in a steady incompressible viscous perfectly conducting constantly inclined non-aligned plane flow, then the metric of this net, the flow speed, the magnetic intensity, the current density and the electric intensity are given by

$$
\left.\begin{aligned}
h\ (\alpha,\beta) &= \exp\ \{2\ C_2\beta - 2\ C_1\alpha + C_3\},\\
V\ (\alpha,\beta) &= \exp\ (C_4 - \tfrac{1}{2}\ C_3)\ \exp\ \{(C_2 - \frac{2\ B\ C_1}{A})\ \beta + C_1\alpha\},\\
\vec{H}\ (\alpha,\beta) &= [\exp\ (\tfrac{1}{2}\ C_3 - C_4)\ \exp\ \{(\frac{2\ B\ C_1}{A} - C_2)\beta\\
&\quad - C_1\ \alpha\}]\ (B\ \vec{e}_1 + A\ \vec{e}_2),\\
\vec{J}\ (\alpha,\beta) &= -\frac{2\ C_1}{A}\ (A^2 + B^2)\ \exp\ (-C_4)\ \exp\ \{\frac{2}{A}\\
&\quad (B\ C_1 - A\ C_2)\beta\}\ \vec{k}\\
&\text{and}\\
\vec{E} &= -\mu\ A\ \vec{k}
\end{aligned}\right\} \qquad (35.36)
$$

where the constants C_1 and C_2 must satisfy (35.35).

The pressure function has no general form and will be derived separately in the next part of this section.

Part II Classification and Geometries

For flows with an isometric streamline pattern, $C = 0$ and C_1, C_2 must satisfy (35.35). Therefore, we have one of the following three possibilities:

(i) $C = 0$, $C_1 \neq 0$ and $C_2 = \frac{B}{A} C_1$

(ii) $C = 0$, $C_1 = 0$ and $C_2 \neq 0$

(iii) $C = C_1 = C_2 = 0$

In order to determine the geometries for these three flow types, we employ the following formula from Lemma (2.1)

$$\frac{f''(z)}{[f'(z)]^2} = W_1 + i\, W_2 \tag{22.20}$$

where $f(z) = \alpha + i\, \beta$.

Using $C = 0$ and (35.22) in (22.20), we obtain

$$\frac{f''(z)}{[f'(z)]^2} = C_1 + i\, C_2 \tag{35.37}$$

Integrating of (35.37) with respect to z yields

$$-\frac{1}{f'(z)} = (C_1 + i\, C_2)\ (z - D) \tag{35.38}$$

where $D = D_1 + i\, D_2$ is an arbitrary constant.

For possibilities (i) and (ii), $C_1 + i\, C_2 \neq 0$, and (35.38) may be inverted to give

$$f'(z) = -\frac{1}{(C_1 + i\, C_2)\ (z - D)} \tag{35.39}$$

Integrating (35.39), we get

$$f(z) = -\frac{1}{(C_1 + i\, C_2)}\ \ln\ (z - D) + E \tag{35.40}$$

where $E = E_1 + i\, E_2$ is an arbitrary constant.

Letting $z - D = r \exp(i\,\Theta)$, where (r, Θ) are polar coordinates, and using $f(z) = \alpha + i\,\beta$, separation of real and imaginary parts of (35.40) yields

$$\left.\begin{aligned}\alpha(r, \Theta) &= -\frac{1}{(C_1^{\,2} + C_2^{\,2})}\{C_1\,(\ln r + E_1) + C_2\,(\Theta + E_2)\}\\ \beta(r, \Theta) &= \frac{1}{(C_1^{\,2} + C_2^{\,2})}\{C_2\,(\ln r + E_1) - C_1\,(\Theta + E_2)\}\end{aligned}\right] \quad (35.41)$$

For possibility (iii), $C_1 = C_2 = 0$ and therefore by (35.37)

$$f''(z) = 0 \qquad (35.42)$$

[note - $f'(z) \neq 0$ since $|f'(z)| = 1/h$ by section (2.2)]

Integration of (35.42) and separation into real and imaginary parts yields

$$\left.\begin{aligned}\alpha(x, y) &= L_1\,x - L_2\,y + M_1\\ \beta(x, y) &= L_2\,x + L_1\,y + M_2\end{aligned}\right\} \qquad (35.43)$$

where L_1, L_2, M_1 and M_2 are real arbitrary constants.

We now take each of the three types individually and indicate their solutions and geometries.

Type (i): Taking $C_1 \neq 0$ and $C_2 = \frac{B}{A}\,C_1$ in (35.41), we obtain

$$\beta(r, \Theta) = \frac{A}{C_1(A^2 + B^2)}\{B(\ln r + E_1) - A(\Theta + E_2)\} \qquad (35.44)$$

By (35.44), the streamlines β = constant are given by

$$B\,\ln r - A\,\Theta = \text{constant} \qquad (35.45)$$

Therefore, the streamlines are concurrent straight lines for orthogonal flows and logarithmic spirals for non-orthogonal flows. (We recall that $B = 0$ for orthogonal flows.)

The flow speed, the magnetic intensity, the current density, the electrical intensity and the metric of the natural isometric net for this flow are given by (35.36) with $C_2 = \frac{B}{A} C_1$.

To obtain the pressure function, we substitute V, h from (35.36), with $C_2 = \frac{B}{A} C_1$, into (35.09) and (35.10). This yields

$$\frac{1}{\rho}\frac{\partial p}{\partial \alpha} = \frac{2\,\mu\,(A^2 + B^2)}{\rho} C_1 \exp\,(C_3 - 2\,C_4)\exp\left\{\frac{2\,C_1}{A}(B\beta - A\alpha)\right\} - C_1 \exp\,(2\,C_4 - C_3)\exp\left\{\frac{2\,C_1}{A}(A\alpha - B\beta)\right\} \quad (35.46)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial \beta} = -\frac{2\,\mu\,B\,C_1\,(A^2 + B^2)}{A\rho}\exp\,(C_3 - 2\,C_4)\exp\left\{\frac{2\,C_1}{A}(B\beta - A\alpha)\right\} + \frac{B\,C_1}{A}\exp\,(2\,C_4 - C_3)\exp\left\{\frac{2\,C_1}{A}(A\alpha - B\beta)\right\} \quad (35.47)$$

Integrations of (35.46) and (35.47) yields

$$p = -\frac{\rho}{2}\exp\left\{\frac{2\,C_1}{A}(A\alpha - B\beta) + 2\,C_4 - C_3\right\} - \mu\,(A^2 + B^2)\exp\left\{\frac{2\,C_1}{A}(B\beta - A\alpha) + C_3 - 2\,C_4\right\} + C_5 \quad (35.48)$$

where C_5 is an arbitrary constant.

Type (ii): Letting $C_1 = 0$ and $C_2 \neq 0$ in (35.41), we find

$$\beta(r, \theta) = \frac{1}{C_2} (\ln r + E_1) \qquad (35.49)$$

By (35.49), the β = constant curves have the form

$$\ln r = \text{constant} \qquad (35.50)$$

Therefore, the streamlines, for this type, are a family of concentric circles. The flow speed, the magnetic intensity, the current density, the electrical intensity and the metric of the natural isometric net for this flow are given by (35.36) with $C_1 = 0$.

To obtain the pressure function, we substitute V, h from (35.36), with $C_1 = 0$, into (35.09), (35.10). This gives

$$\frac{1}{\rho} \frac{\partial p}{\partial \alpha} = 0 \qquad (35.51)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial \beta} = C_2 \exp (2 C_2 \beta + 2 C_4 - C_3) \qquad (35.52)$$

Integration of (35.51) and (35.52) yields

$$p = \frac{\rho}{2} \exp (2 C_2 \beta + 2 C_4 - C_3) + C_6 \qquad (35.53)$$

where C_6 is an arbitrary constant.

Type (iii): For this type of flow, equations (35.43) implies the following form for the streamlines

$$L_2 x + L_1 y = \text{constant} \qquad (35.54)$$

Thus, the streamlines are parallel straight lines for the case where $C_1 = C_2 = 0$. The solutions for this parallel flow are obtained from (35.36) and (35.53) with C_1, C_2 both equal to zero.

Summarizing the results of this section into a theorem yields:

Theorem 3.5

If the natural net is isometric in a steady, incompressible, viscous, perfectly conducting and constantly inclined plane flow, then the flow is characterized by one of the following three alternatives:

	Geometry	Solutions
(i)	Logarithmic Spirals	Equations (35.36) and (35.48) with $C_1 \neq 0$, $C_2 = \frac{B}{A} C_1$.
(ii)	Concentric Circles	Equations (35.36) and (35.53) with $C_1 = 0$, $C_2 \neq 0$.
(iii)	Parallel Straight Lines	Equations (35.36) and (35.53) with $C_1 = C_2 = 0$.

In the case of orthogonal flows, the Logarithmic Spirals of alternative (i) degenerate into a family of concurrent straight lines.

Section 6. Flows for Which the Magnetic Lines and Their Orthogonal Trajectories Form an Isometric Net.

The solenoidal condition on $\vec{H}$, given by (31.05), implies the existence of a magnetic streamfunction such that

$$\frac{\partial\Psi}{\partial y} = H_1 \ , \ \frac{\partial\Psi}{\partial x} = -H_2 \tag{36.01}$$

where $\vec{H} = (H_1, H_2)$

If the current density vanishes for a flow, we have

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = 0 \tag{36.02}$$

Equations (36.01) and (36.02) imply that the magnetic streamfunction Ψ is harmonic. Therefore, by section (2.2) the magnetic lines, given by the Ψ = constant curves, and their orthogonal trajectories form an isometric net.

In this section, we completely solve our constantly inclined flow problems for the case when the magnetic lines and their orthogonal trajectories form an isometric net. In particular, it is shown that all such flows have constant current density.

Paralleling the development of section (3.5), we let

$$\left.\begin{aligned} x &= x\,(\alpha,\beta) \\ y &= y\,(\alpha,\beta) \end{aligned}\right\} \tag{36.03}$$

define an isometric net such that the $\beta(x,y)$ = constant curves are the magnetic lines and the $\alpha(x,y)$ = constant curves are the orthogonal trajectories to the magnetic lines.

Letting $\vec{e}_1$ be the unit tangent vector to the magnetic lines in the direction of increasing α and $h(\alpha,\beta)$ the metric coefficient of the isometric net, we have

$$\vec{H} = H(\alpha,\beta)\ \vec{e}_1 \tag{36.04}$$

$$ds^2 = h(\alpha,\beta)\ [d\alpha^2 + d\beta^2] \tag{36.05}$$

where ds is the element of arc length in the physical plane.

Substituting (36.04) in (31.05) and (31.10), we obtain

$$\frac{\partial}{\partial\alpha}(\sqrt{h}\ H) = 0 \tag{36.06}$$

$$\vec{v} = \frac{B}{H}\vec{e}_1 - \frac{A}{H}\vec{e}_2 \tag{36.07}$$

Use of (31.10) and (31.09) in (31.02) yields the following form for the linear momentum equation

$$\text{grad}\ [\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}] = (\frac{A}{H^2}\vec{H}\times\vec{k} + \frac{B}{H^2}\vec{H}) \times [\text{curl}\ (\frac{A}{H^2}\vec{H}\times\vec{k} + \frac{B}{H^2}\vec{H})] + \nu\nabla^2 (\frac{A}{H^2}\vec{H}\times\vec{k} + \frac{B}{H^2}\vec{H}) + \frac{u}{\rho}(\text{curl}\ \vec{H})\times\vec{H} \tag{36.08}$$

Employing (36.04) in (36.08), we get

$$\frac{\partial}{\partial\alpha}[\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}] = \frac{A}{\sqrt{h}\ H}[\frac{\partial}{\partial\alpha}(\frac{\sqrt{h}\ A}{H}) - \frac{\partial}{\partial\beta}(\frac{\sqrt{h}\ B}{H})] - \nu\frac{\partial}{\partial\beta}\{\frac{1}{h}[\frac{\partial}{\partial\alpha}(\frac{\sqrt{h}\ A}{H}) - \frac{\partial}{\partial\beta}(\frac{\sqrt{h}\ B}{H})]\} \tag{36.09}$$

$$\frac{\partial}{\partial\beta}\left[\frac{p}{\rho}+\frac{(A^2+B^2)}{2H^2}\right] = -\frac{B}{\sqrt{h}\,H}\left[\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{h}\,A}{H}\right)-\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}\,B}{H}\right)\right] +$$

$$\nu\frac{\partial}{\partial\alpha}\left\{\frac{1}{h}\left[\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{h}\,A}{H}\right)-\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}\,B}{H}\right)\right]\right\} - \frac{\mu H}{\rho\sqrt{h}}\frac{\partial}{\partial\beta}(\sqrt{h}\,H) \qquad (36.10)$$

Elimination of $\vec{v}$ between (31.01) and (36.07) yields

$$B\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{h}}{H}\right) - A\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}}{H}\right) = 0 \qquad (36.11)$$

Equations (36.06), (36.11) can be solved for H and h. The resulting solutions must also satisfy the integrability condition for p, which is derived from (36.09), (36.10) and has the form

$$\left(A\frac{\partial}{\partial\beta}+B\frac{\partial}{\partial\alpha}\right)\left\{\frac{1}{\sqrt{h}\,H}\left[\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{h}\,A}{H}\right)-\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}\,B}{H}\right)\right]\right\} -$$

$$\nu\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\left\{\frac{1}{h}\left[\frac{\partial}{\partial\alpha}\left(\frac{\sqrt{h}\,A}{H}\right)-\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}\,B}{H}\right)\right]\right\} +$$

$$\frac{\mu}{\rho}\frac{\partial}{\partial\alpha}\left\{\frac{H}{\sqrt{h}}\frac{\partial}{\partial\beta}\sqrt{h}\,H\right\} = 0 \qquad (36.12)$$

Equation (36.06) implies that

$$H(\alpha,\beta) = \frac{\psi(\beta)}{\sqrt{h}} \qquad (36.13)$$

where $\psi(\beta)$ is an arbitrary differentiable function of β.

Substitution of (36.13) into (36.11) yields

$$A\,W_2 + B\,W_1 = \frac{A\,\psi'(\beta)}{2\,\psi(\beta)} \qquad (36.14)$$

where $W_1 = -\frac{1}{2h}\frac{\partial h}{\partial\alpha}$, $W_2 = \frac{1}{2h}\frac{\partial h}{\partial\beta}$

We note that (36.13) and (36.14) have the same form, with A replaced by -A, as equations (35.11), (35.13). Therefore, the solutions of (36.13), (36.14) are obtained by replacing A by -A in (35.24) and (35.25). This yields

$$h\ (\alpha,\beta) = \exp\ [A\ K\ (\alpha^2 - \beta^2) + 2\ K\ B\ \alpha\ \beta + 2\ K_2\ \beta - 2\ K_1\ \alpha + K_3] \qquad (36.15)$$

$$\psi(\beta) = \exp\ \{-\frac{K}{A}\ (A^2 + B^2)\ \beta^2 + (2\ K_2 + \frac{2BK_1}{A})\ \beta + K_4\} \qquad (36.16)$$

where K, K_1, K_2, K_3 and K_4 are arbitrary constants.

H (α,β) and h (α,β) as defined by (36.13), (36.15) and (36.16) satisfy equations (36.06) and (36.11). It remains to establish the conditions for which these solutions satisfy the integrability condition, (36.12), for p (α,β).

Employing (36.13), (36.15) and (36.16) in (36.12), we find

$$P_1\ (\alpha,\beta)\ \exp\ [P_2\ (\alpha,\beta)] - \nu\ P_3\ (\alpha,\beta)\ \exp\ [P_4(\alpha,\beta)] + \frac{\mu}{\rho}\ P_5\ (\alpha,\beta)\ \exp\ [-P_2(\alpha,\beta)] = 0 \qquad (36.17)$$

where

$$P_1\ (\alpha,\beta) = [-2\ K\ (A^2 + B^2)\ \beta + 2\ A\ K_2 + 2\ B\ K_1]\ [-2\ K\ (A^2 + B^2)\ \alpha - 2\ K\ (AB + \frac{B^3}{A})\ \beta + 2\ K_1\ A + \frac{2\ B^2}{A}\ K_1]\ ,$$

$$P_2\ (\alpha,\beta) = \frac{K}{A}\,(A^2 + 2B^2)\ \beta^2 + A\,K\alpha^2 + 2\,K\,B\,\alpha\,\beta$$

$$- (2K_2 + \frac{4B}{A})\ \beta - 2K_1\ \alpha + K_3 - 2K_4\ ,$$

$$P_3\ (\alpha,\beta) = [-2\ K\ (A^2 + B^2)\ \alpha - 2\ K\ (AB + \frac{B^3}{A})\ \beta$$

$$+ 2\ K_1\ A + \frac{2B^2}{A}\ K_1]\ \{\frac{2K}{A}\ (A^2 + B^2) +$$

$$[\frac{2K}{A}\ (A^2 + B^2)\ \beta - (2\ K_2 + \frac{2B}{A}\ K_1)]^2\} -$$

$$4\ K\ (AB + \frac{B^3}{A})\ [\frac{2K}{A}\ (A^2 + B^2)\ \beta -$$

$$(2\ K_2 + \frac{2B}{A}\ K_1)]\ ,$$

$$P_4\ (\alpha,\beta) = \frac{K}{A}\ (A^2 + B^2)\ \beta^2 - (2\ K_1 + \frac{2B}{A}\ K_1)\beta - K_4\ ,$$

and

$$P_5\ (\alpha,\beta) = [-\frac{2K}{A}\ (A^2 + B^2)\ \beta + (2\ K_2 + \frac{2B}{A}\ K_1)]$$

$$[-2\ A\ K\ \alpha - 2\ K\ B\ \beta + 2\ K_1]$$

In order to manipulate equation (36.17), we require several simple results which are summarized in the following lemma.

Lemma 3.1

If $M(\alpha,\beta)$, $N(\alpha,\beta)$ are two arbitrary polynomials of degree m and n respectively, then for all positive integers i and j

$$\frac{\partial^{(i+j)}}{\partial\alpha^i\ \partial\beta^j}\,M\ (\alpha,\beta)\ \exp\ [N(\alpha,\beta)] = Q\ (\alpha,\beta)\ \exp\ [N(\alpha,\beta)]$$

where $Q(\alpha,\beta)$ is a polynomial of degree equal or less than $[m + (i + j)(n - 1)]$. Moreover, the high order terms of $Q(\alpha,\beta)$ are contained in

$$M(\alpha,\beta)\left[\frac{\partial N}{\partial \alpha}\right]^{i}\left[\frac{\partial N}{\partial \beta}\right]^{j}.$$

Returning to equation (36.17), we eliminate the first term by first dividing out the exponential factor and then taking the second order derivative with respect to α. This operation yields

$$- \nu \frac{\partial^2}{\partial \alpha^2}\{P_3(\alpha,\beta)\exp[P_4(\alpha,\beta) - P_2(\alpha,\beta)]$$
$$+ \frac{\mu}{\rho}\frac{\partial^2}{\partial \alpha^2}\{P_5(\alpha,\beta)\exp[-2P_2(\alpha,\beta)] = 0$$

Using Lemma 3.1, this equation can be put in the following form

$$P_6(\alpha,\beta)\exp[P_4(\alpha,\beta) - P_2(\alpha,\beta)]$$
$$+ \frac{\mu}{\rho}\frac{\partial^2}{\partial \alpha^2}\{P_5(\alpha,\beta)\exp[-2P_2(\alpha,\beta)] = 0 \qquad (36.18)$$

where $P_6(\alpha,\beta)$ is a polynomial of a most degree five in α,β.

Performing a similiar operation to (36.18), we find

$$\frac{\partial^6}{\partial \alpha^6}\{\exp[P_2(\alpha,\beta) - P_4(\alpha,\beta)]\frac{\partial^2}{\partial \alpha^2}[$$
$$P_5(\alpha,\beta)\exp[-2P_2(\alpha,\beta)]]\} = 0 \qquad (36.19)$$

Employing Lemma 3.1, this equation becomes

$$P_7(\alpha,\beta)\exp[-P_2(\alpha,\beta) - P_4(\alpha,\beta)] = 0 \qquad (36.20)$$

where P_7 (α,β) is a polynomial of degree ten.

Since the exponential function is always nonzero and (36.20) holds everywhere in the flow region, the polynomial P_7 (α,β) and consequently all its coefficients are identically zero. The second part of Lemma 3.1 implies that the tenth order terms of P_7 (α,β) are contained in

$$4\, P_5\, (\alpha,\beta)\, \{\frac{\partial P_2}{\partial \alpha}\, (\alpha,\beta)\}^2\, \{\frac{\partial}{\partial \alpha}\, [-P_2\, (\alpha,\beta)\, -\, P_4\, (\alpha,\beta)]\}^6$$

or

$$4\, [-\frac{2k}{A}\, (A^2 + B^2)\, \beta + (2\, K_2 + \frac{2B}{A}\, K_1)]\, [-2\, A\, K\, \alpha\, -$$

$$2\, K\, B\, \beta + 2\, K_1]\, [2\, A\, K\, \alpha + 2\, K\, B\, \beta - 2\, K_1]^8 \qquad (36.21)$$

Selecting the coefficient of $\alpha^9\beta$ and equating it to zero, we obtain

$$4096\, (A^2 + B^2) K^{10} A^8 \;=\; 0 \qquad (36.22)$$

Therefore, $K = 0$ by virtue of the fact that $A \neq 0$ for the flows under investigation.

Setting $K = 0$ in (36.21), it follows that for our flows the arbitrary constants K_1 and K_2 must satisfy

$$K_1\, (AK_2 + BK_1) \;=\; 0 \qquad (36.23)$$

If $K = 0$ and K_1, K_2 satisfy (36.23), then P_1 (α,β) $= P_3$ $(\alpha,\beta) = P_5$ $(\alpha,\beta) = 0$ and consequently (36.17) is identically satisfied.

Summing up the above results, we have

Theorem 3.6

If the natural net for the magnetic lines is isometric in a steady, incompressible, viscous, perfectly electrically conducting and constantly inclined non-aligned plane flow, then the metric of the net, the magnetic intensity, the velocity, the current density and electrical intensity are given by

$$\left.\begin{aligned}
h(\alpha,\beta) &= \exp[2K_2\beta - 2K_1\alpha + K_3] \\
H(\alpha,\beta) &= \exp[K_4 - \frac{K_3}{2}]\exp[(K_2 + \frac{2B}{A}K_1)\beta + \\
&\quad + K_1\alpha] \\
\vec{v}(\alpha,\beta) &= \{\exp[\frac{K_3}{2} - K_4]\exp[(-\frac{2B}{A}K_1 - K_2)\beta \\
&\quad - K_1\alpha]\}(B\vec{e}_1 - A\vec{e}_2) \\
\vec{J} &= -(2K_2 + \frac{2B}{A}K_1)\exp[\frac{2K_1}{A}(B\beta + A\alpha) + \\
&\quad K_4 - K_3]\vec{k} \\
\vec{E} &= -\mu A\vec{k}
\end{aligned}\right] \qquad (36.24)$$

where K_1, K_2, K_3, K_4 are arbitrary constants with K_1 and K_2 satisfying (36.23).

For flows with isometric magnetic lines, $K = 0$ and equation (36.23) imply the following three types:

(i) $K = 0,\ K_1 \neq 0$ and $K_2 = -\frac{B}{A}K_1$

(ii) $K = 0,\ K_1 = 0$ and $K_2 \neq 0$

(iii) $K = K_1 = K_2 = 0$

Since equations (36.15) and (36.23) have the same form as (35.24) and (35.35), the geometric implications follow as in section (3.5). Therefore, it remains to calculate the pressure function for the three types.

We now treat each of the three types separately.

Type (i)

For this type the magnetic lines are concurrent straight lines for orthogonal flows and logarithmic spirals for nonorthogonal flows. All the flow variables, with the exception of pressure, are obtained by letting $K_2 = -\frac{B}{A} K_1$ in (36.24).

To obtain the pressure function, we substitute H and h from (36.24), with $K_2 = -\frac{B}{A} K_1$, into (36.09) and (36.10). This gives

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha}\left[\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}\right] &= -2 K_1 (A^2 + B^2) \exp \\ &\left[-\frac{2K_1}{A}(B\beta + A\alpha) + K_3 - 2 K_4\right] \\ \frac{\partial}{\partial \beta}\left[\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}\right] &= -\frac{2BK_1}{A}(A^2 + B^2) \exp \\ &\left[-\frac{2K_1}{A}(B\beta + A\alpha) + K_3 - 2 K_4\right] \end{aligned}\right\} \quad (36.25)$$

Integration of (36.25) yields

$$p = \rho \frac{(A^2 + B^2)}{2} \exp\left[-\frac{2K_1}{A}(B\beta + A\alpha) + K_3 - 2K_4\right] + K_5 \tag{36.26}$$

where K_5 is an arbitrary constant.

Type (ii)

For type (ii) flows the magnetic lines are concentric circles. The pressure function, for this case, is obtained by substituting H and h from (36.24), with $K_1 = 0$, into (36.09) and (36.10). This yields

$$\frac{\partial}{\partial\alpha}\left[\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}\right] = 0$$

$$\frac{\partial}{\partial\beta}\left[\frac{p}{\rho} + \frac{(A^2 + B^2)}{2H^2}\right] = -\frac{2\mu K_2}{\rho} \exp\left[2K_2\beta + 2K_4 - K_3\right] \tag{36.27}$$

Integrating (36.27), we obtain

$$p = -\mu \exp\left[2K_2\beta + 2K_4 - K_3\right] - \rho \frac{(A^2 + B^2)}{2} \exp\left[-2K_2\beta - 2K_4 + K_3\right] + K_6 \tag{36.28}$$

where K_6 is an arbitrary constant.

The other flows variables are obtained by letting $K_1 = 0$ in (36.24).

Type (iii)

For this type of flow, the magnetic lines are a family of parallel straight lines. The solution to this flow are given by (36.24) and (36.28) with $K_1 = K_2 = 0$.

Section 7. Uniqueness Theorem

In this section, we show that the flows of this chapter are uniquely determined by the fluid's properties, the streamline geometry and the value of the pressure at one point of the flow region.

Letting $\vec{v}$, $\vec{v}^*$ be the velocity fields for two distinct flows, there exists by the continuity equation, (31.01), two streamfunctions Ψ and Ψ^* such that

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial y} &= V_1 \quad , \quad \frac{\partial \Psi}{\partial x} = -V_2 \\ \frac{\partial \Psi^*}{\partial y} &= V_1^* \quad , \quad \frac{\partial \Psi^*}{\partial x} = -V_2^* \end{aligned}\right\} \qquad (37.01)$$

where $\vec{v} = (V_1, V_2)$ and $\vec{v}^* = (V_1^*, V_2^*)$.

Employing (37.01) in (31.11), we find

$$\left.\begin{aligned} \vec{H} &= \frac{1}{(\Psi_x^2 + \Psi_y^2)} [B\,\Psi_y + A\Psi_x, -B\,\Psi_x + A\,\Psi_y] \\ \vec{H}^* &= \frac{1}{(\Psi_x^{*2} + \Psi_y^{*2})} [B\,\Psi_y^* + A\,\Psi_x^*, -B\,\Psi_x^* + A\,\Psi_y^*] \end{aligned}\right\} \qquad (37.02)$$

where $\vec{H}$, $\vec{H}^*$ are the respective magnetic intensities for the two flows.

Substitution of (37.02) into (31.05) yields

$$[-4A\,\Psi_y\,\Psi_x + 2B(\Psi_x^2 - \Psi_y^2)]\,\Psi_{yx} + [A\Psi_y^2 - A\Psi_x^2 - 2B\,\Psi_x\,\Psi_y]\,\Psi_{xx} + [A\Psi_x^2 - A\Psi_y^2 + 2B\,\Psi_x\,\Psi_y]\,\Psi_{yy} = 0 \quad (37.03)$$

and

$$[-4A\,\Psi^*_y\,\Psi^*_x + 2B\,(\Psi^{*2}_x - \Psi^{*2}_y)]\;\Psi^*_{yx} +$$
$$[A\,\Psi^{*2}_y - A\,\Psi^{*2}_x - 2B\,\Psi^*_x\,\Psi^*_y]\;\Psi^*_{xx} +$$
$$[A\,\Psi^{*2}_x - A\,\Psi^{*2}_y + 2B\,\Psi^*_x\,\Psi^*_y]\;\Psi^*_{yy} = 0 \qquad (37.04)$$

Assuming the two flows have the same stream line pattern, we have

$$\Psi^* = f(\Psi) \qquad (37.05)$$

where $f\,(\Psi)$ is a suitably differentiable function.

Eliminating Ψ^* between (37.04) and (37.05), we obtain

$$f'(\Psi)^3\;\{[-4A\,\Psi_y\,\Psi_x + 2B\,(\Psi_x^{\,2} - \Psi_y^{\,2})]\;\Psi_{yx} +$$
$$[A\,\Psi_y^{\,2} - A\,\Psi_x^{\,2} - 2B\,\Psi_x\,\Psi_y]\;\Psi_{xx} + [A\,\Psi_x^{\,2} -$$
$$A\,\Psi_y^{\,2} + 2B\,\Psi_x\,\Psi_y]\;\Psi_{yy}\} + f'(\Psi)^2\,f''(\Psi)$$
$$\{[-4A\,\Psi_y\,\Psi_x + 2B\,(\Psi_x^{\,2} - \Psi_y^{\,2})]\;\Psi_x\,\Psi_y +$$
$$[A\,\Psi_y^{\,2} - A\,\Psi_x^{\,2} - 2B\,\Psi_x\,\Psi_y]\;\Psi_x^{\,2} + [A\,\Psi_x^{\,2} -$$
$$A\,\Psi_y^{\,2} + 2B\,\Psi_x\,\Psi_y]\;\Psi_y^{\,2}\} = 0 \qquad (37.06)$$

Equations (37.03) and (37.06) imply

$$f'(\Psi)^2\;f''(\Psi)\;\{[-4A\,\Psi_y\,\Psi_x + 2B\,(\Psi_x^{\,2} - \Psi_y^{\,2})]$$
$$\Psi_x\,\Psi_y + [A\,\Psi_y^{\,2} - A\,\Psi_x^{\,2} - 2B\,\Psi_x\,\Psi_y]\;\Psi_x^{\,2} +$$
$$[A\,\Psi_x^{\,2} - A\,\Psi_y^{\,2} + 2B\,\Psi_x\,\Psi_y]\;\Psi_y^{\,2}\} = 0 \qquad (37.07)$$

or

$$-A\ f'(\Psi)^2\ f''(\Psi)\ (\Psi_x^{\ 2} + \Psi_y^{\ 2}) = 0$$

Since $A \neq 0$ for our flows and $f'(\Psi)$, $(\Psi_x^{\ 2} + \Psi_y^{\ 2})$ are nonzero for nontrivial flows, equation (37.07) implies

$$f''(\Psi) = 0 \tag{37.08}$$

Equation (37.05) and (37.08) yield

$$\Psi^* = R\,\Psi + T \tag{37.09}$$

where R and T are arbitrary constants.

Using (37.09) in (37.01) and (37.02), we obtain

$$\vec{v}^* = R\,\vec{v}\ , \quad \vec{H}^* = \frac{1}{R}\,\vec{H} \tag{37.10}$$

Therefore, for flows with the same streamline pattern, we have the theorem:

<u>Theorem 3.6</u>

If two steady, incompressible, viscous, perfectly conducting and constantly inclined plane flows have the same streamlines, then their respective velocity fields $\vec{v}$, $\vec{v}^*$ and magnetic fields $\vec{H}$, $\vec{H}^*$ are related by

$$\vec{v}^* = R\,\vec{v}\ , \quad \vec{H}^* = \frac{1}{R}\,\vec{H}$$

where R is an arbitrary constant.

If both flows take place in identical fluids, we consider the integrability condition for p obtained from (31.02), and having the form

$$\text{curl } [\vec{v} \times (\text{curl } \vec{v})] + \nu \text{ curl } [\nabla^2 \vec{v}] +$$

$$\frac{\mu}{\rho} \text{ curl } [(\text{curl } \vec{H}) \times \vec{H}] = 0 \qquad (37.11)$$

Taking (37.11) as the integrability condition for the flow with velocity field $\vec{v}$ and magnetic field $\vec{H}$, and by using (37.10) in (37.11), we obtain the following equation for the integrability condition of the other flow

$$R^2 \text{ curl } [\vec{v} \times (\text{curl } \vec{v})] + \nu^* R [\nabla^2 \vec{v}] +$$

$$\frac{\mu^*}{\rho^*} \text{ curl } [(\text{curl } \vec{H}) \times \vec{H}] = 0 \qquad (37.12)$$

where ν^*, μ^* and ρ^* are the kinematic viscosity, the magnetic permeability and the fluid density for the flow whose velocity and magnetic field are $\vec{v}^*$ and $\vec{H}^*$.

If the fluids of the two flows are the same then $\nu = \nu^*$, $\mu = \mu^*$, $\rho = \rho^*$ and equation (37.11) and (37.12) imply that $R = 1$.

For a given velocity field and magnetic field, equation (31.02) defines the pressure function to within an additive constant. Therefore, we have the following Corollary:

Corollary 3.2

A steady, incompressible, viscous, perfectly conducting and constantly inclined nonaligned plane flow is uniquely defined by its streamline pattern, fluid properties and the value of the pressure function at one point of the flow region.

CHAPTER IV

CONSTANTLY INCLINED COMPRESSIBLE FLOWS

Section 1. Flow Equations

The flow of an adiabatic, steady, compressible, nonviscous, thermally nonconducting fluid, with infinite electrical conductivity in the presence of a magnetic field, is governed by:

$$\text{div } (\rho\vec{v}) = 0 \tag{41.01}$$

$$\rho\ (\vec{v}\cdot\text{grad})\ \vec{v} + \text{grad } p = \mu\ (\text{curl } \vec{H}) \times \vec{H} \tag{41.02}$$

$$\vec{v}\cdot\text{grad } s = 0 \tag{41.03}$$

$$\text{curl } (\vec{v} \times \vec{H}) = \vec{0} \tag{41.04}$$

$$\text{div } (\vec{H}) = 0 \tag{41.05}$$

$$p = p\ (\rho, s) \tag{41.06}$$

We restrict our attention to constantly inclined non-aligned flows. Therefore, by the logic used in section (3.1), equation (41.04) can be replaced by

$$\vec{v} = \frac{A}{H^2}\ \vec{H} \times \vec{k} + \frac{B}{H^2}\ \vec{H} \tag{31.10}$$

$$\vec{H} = \frac{A}{v^2}\ \vec{k} \times \vec{v} + \frac{B}{v^2}\ \vec{v} \tag{31.11}$$

where H, V are the respective magnitudes of the magnetic intensity and velocity vectors, $A = V\,H \sin(\widehat{\vec{v}, \vec{H}})$ and $B = V\,H \cos(\widehat{\vec{v}, \vec{H}})$.

Since we exclude aligned flows, $A \neq 0$. The constant B will be zero if and only if the flow is orthogonal. By the definition of A and B, we have

$$V^2\,H^2 = A^2 + B^2 \tag{41.07}$$

Letting (α,β) be the natural or streamline coordinates, $\sqrt{h_1(\alpha,\beta)}\,d\alpha$ and $\sqrt{h_2(\alpha,\beta)}\,d\beta$ the components of a vector element of arc length and $\vec{e}_1$, $\vec{e}_2$ the unit tangent vectors to the $\beta(x,y)$ = const. and $\alpha(x,y)$ = const. curves respectively, we have

$$\vec{v} = V\,\vec{e}_1 \tag{41.08}$$

$$d\bar{s}^2 = h_1(\alpha,\beta)\,d\alpha^2 + h_2(\alpha,\beta)\,d\beta^2 \tag{41.09}$$

where $d\bar{s}^2$ is the squared element of arc length in the $x - y$ plane.

Substitution of (41.08) into (31.11) yields

$$\vec{H} = \frac{B}{V}\,\vec{e}_1 + \frac{A}{V}\,\vec{e}_2 \tag{41.10}$$

Employing (41.08), (41.09) and (41.10) in equations (41.01), (42.02), (41.03) and (41.05), we obtain

$$\frac{\partial}{\partial\alpha}\left(\rho\,V\,\sqrt{h_2}\right) = 0 \tag{41.11}$$

$$\rho V \frac{\partial V}{\partial \alpha} + \frac{\partial p}{\partial \alpha} + \frac{A\mu}{\sqrt{h_2}\, V} \left[A \frac{\partial}{\partial \alpha} \left(\frac{\sqrt{h_2}}{V}\right) - B \frac{\partial}{\partial \beta} \left(\frac{\sqrt{h_1}}{V}\right)\right] = 0 \quad (41.12)$$

$$\rho V^2 \frac{\partial \sqrt{h_1}}{\partial \beta} - \frac{\partial p}{\partial \beta} + \frac{B\mu}{\sqrt{h_1} V} \left[A \frac{\partial}{\partial \alpha}\left(\frac{\sqrt{h_2}}{V}\right) - B \frac{\partial}{\partial \beta} \left(\frac{\sqrt{h_1}}{V}\right)\right] = 0 \quad (41.13)$$

$$\frac{\partial s}{\partial \alpha} = 0 \quad (41.14)$$

$$\frac{\partial}{\partial \alpha} \left(\frac{\sqrt{h_2}\, B}{V}\right) + \frac{\partial}{\partial \beta} \left(\frac{\sqrt{h_1}\, A}{V}\right) = 0 \quad (41.15)$$

Also, the metric coefficients $h_1\ (\alpha,\beta)$, $h_2\ (\alpha,\beta)$ satisfy the Gauss equation

$$\frac{\partial}{\partial \alpha} \left(\frac{1}{\sqrt{h_1}} \frac{\partial \sqrt{h_2}}{\partial \alpha}\right) + \frac{\partial}{\partial \beta} \left(\frac{1}{\sqrt{h_2}} \frac{\partial \sqrt{h_1}}{\partial \beta}\right) = 0 \quad (22.04)$$

Section 2. Flows with Velocity Magnitude Constant along Each Streamline.

In this section, we assume

$$V(\alpha,\beta) = V(\beta) \tag{42.01}$$

Using (41.15) and (42.01) in (41.12), we find

$$\frac{\partial p}{\partial \alpha} + \mu \frac{(A^2 + B^2)}{\sqrt{h_2}\ V^2} \frac{\partial}{\partial \alpha} \sqrt{h_2} = 0 \tag{42.02}$$

The adiabatic condition, (41.14), and the state equation, (41.06), imply

$$\frac{\partial p}{\partial \alpha} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial \alpha} = C^2 \frac{\partial \rho}{\partial \alpha} \tag{42.03}$$

where C is the local speed of sound.

Employing (42.01) in (41.11), we obtain

$$\frac{\partial \rho}{\partial \alpha} = -\frac{\rho}{\sqrt{h_2}} \frac{\partial}{\partial \alpha} \sqrt{h_2} \tag{42.04}$$

Equations (42.02), (42.03) and (42.04) yield

$$\left[\mu \frac{(A^2 + B^2)}{V^2} - \rho C^2\right] \frac{\partial \sqrt{h_2}}{\partial \alpha} = 0 \tag{42.05}$$

Therefore, if

$$\mu (A^2 + B^2) \neq \rho C^2 V^2 \tag{42.06}$$

then

$$\frac{\partial \sqrt{h_2}}{\partial \alpha} = 0 \tag{42.07}$$

Since $(A^2 + B^2) = V^2 H^2$, the inequality (42.06) is equivalent to the physical condition $\mu H^2 \neq \rho C^2$. Assuming this condition to be true, we use (42.01), (42.07) in (41.15) and obtain

$$\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h_1}}{V}\right) = 0 \tag{42.08}$$

Integrating (42.07) and (42.08), we find

$$\sqrt{h_2} = f_1(\beta), \quad \sqrt{h_1} = f_2(\alpha)\ V(\beta) \tag{42.09}$$

where $f_1(\beta)$ and $f_2(\alpha)$ are arbitrary differentiable functions of their respective arguments. Substitution of (42.09) into the Gauss equation, 22.04, yields

$$\frac{\partial}{\partial\beta}\left[\frac{1}{f_1(\beta)}\frac{\partial V(\beta)}{\partial\beta}\right] = 0 \tag{42.10}$$

Equation (42.10) implies that $V(\beta) = C\int f_1(\beta)\, d\beta$, where K is an arbitrary constant. Substitution of this result into (42.09) gives

$$\sqrt{h_2} = f_1(\beta), \quad \sqrt{h_1} = K f_2(\alpha)\int f_1(\beta)\, d\beta \tag{42.11}$$

From section (2.2), we have

$$\frac{\partial\theta}{\partial\alpha} = -\frac{1}{\sqrt{h_2}}\frac{\partial\sqrt{h_1}}{\partial\alpha}, \quad \frac{\partial\theta}{\partial\beta} = \frac{1}{\sqrt{h_1}}\frac{\partial\sqrt{h_2}}{\partial\alpha} \tag{42.12}$$

$$z = \int e^{i\theta}(\sqrt{h_1}\, d\alpha + i\sqrt{h_2}\, d\beta) \tag{42.13}$$

where $\theta(\alpha,\beta)$ is the angle of inclination the streamlines make with respect to the positive x - axis and $z = x + iy$

is the complex variable.

Substituting (42.11) into (42.12) and integrating the resulting equations, we obtain

$$\Theta = \Theta(\alpha) \; , \; \Theta'(\alpha) = -K f_2(\alpha) \qquad (42.14)$$

If K = 0, equation (42.14) implies Θ = constant, and, consequently, the streamlines are parallel straight lines. For $K \neq 0$, we substitute (42.11), (42.14) into (42.13) and obtain

$$z = \int e^{i\Theta(\alpha)} \{[K f_2(\alpha) \int f_1(\beta)\, d\beta]\, d\alpha + i f_1(\beta)\, d\beta\} \qquad (42.15)$$

Integrating (42.15) and separating the result into real and imaginary parts, we get

$$\begin{aligned} x &= C_1 - \sin[\Theta(\alpha)] \int f_1(\beta)\, d\beta \\ y &= C_2 + \cos[\Theta(\alpha)] \int f_1(\beta)\, d\beta \end{aligned} \qquad (42.16)$$

Where C_1, C_2 are arbitrary real constants. The streamlines generated by (42.16) are concentric circles about $(-C_1, -C_2)$.

We have proved the Theorem:

Theorem 4.1

If for a constantly inclined, nonviscous and compressible plane flow, the velocity magnitude V is constant on each individual streamline and $\mu H^2 \neq \rho C^2$, then the streamlines are either concentric circles or parallel straight lines.

By (41.07), V is constant on each streamline if and only if H is constant on each streamline. Therefore, Theorem (4.1) holds true if V is replaced by H.

If ρ is constant on each streamline, then by using $\rho = \rho(\beta)$ and (42.01) in (41.11), we obtain

$$\sqrt{h_2} = g_1(\beta) \tag{42.17}$$

where $g_1(\beta)$ is an arbitrary function of its argument. Employing (42.01) and (42.17) in (41.15), we obtain

$$\sqrt{h_1} = g_2(\alpha)\, V(\beta) \tag{42.18}$$

where $g_2(\alpha)$ is an arbitrary function of α.

The form of the metric coefficients defined by (42.17) and (42.18) have the same form as in (42.09). Therefore, as in the case of (42.09), these metric coefficients imply a geometry consisting of concentric circles or parallel straight lines. This yields the corollary:

<u>Corollary 4.1</u>

If for a constantly inclined, nonviscous and compressible plane flow, the velocity magnitude V and the density ρ are constant on each individual streamline, then the streamlines are either concentric circles or parallel straight lines.

We next take another look at the physical condition "$\mu H^2 \neq \rho C^2$" of Theorem (4.1). Using $C^2 = \frac{\partial p}{\partial \rho}$, it follows that this condition is equivalent to

$$\frac{\partial p}{\partial \rho} \neq \frac{\mu H^2(\beta)}{\rho} \tag{42.19}$$

, or to

$$\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial \alpha} \neq \frac{\mu H^2(\beta)}{\rho}\frac{\partial \rho}{\partial \alpha} \tag{42.20}$$

Integrating both sides of (42.20) with respect to α and using (41.14), we obtain

$$p \neq \mu H^2(\beta) \ln \rho + K(\beta) \tag{42.21}$$

where $K(\beta)$ is an arbitrary function of β.

Therefore, we have the following Corollary to Theorem (4.1).

Corollary 4.2

If for a constantly inclined, nonviscous and compressible plane flow, the velocity is constant on each individual streamline and the pressure $p \neq \mu H^2(\beta) \ln \rho + K(\beta)$ for an arbitrary function $K(\beta)$, then the flow geometry is either concentric circles or parallel straight lines.

In particular, for a polytropic gas

$$p = A(s) \rho^{\gamma} \tag{42.22}$$

where $A(s)$ is a function of the specific entropy s and γ is the gas constant.

Since (42.22) is not of the form $p = \mu H^2(\beta) \ln \rho + K(\beta)$, the following Corollary follows from Corollary (4.2).

Corollary 4.3

If the velocity is constant on each individual streamline for a constantly inclined plane flow of a polytropic gas, then the flow geometry is either concentric circles or parallel straight lines.

Finally, we derive physical conditions for which the assumption $h_2 = h_2(\beta)$ implies $V = V(\beta)$.

Using $h_2 = h_2(\beta)$ in (41.11), we find

$$\frac{\partial \rho}{\partial \alpha} = -\frac{\rho}{V}\frac{\partial V}{\partial \alpha} \qquad (42.23)$$

Employing (41.14), (41.06) and (42.23), we obtain

$$\frac{\partial p}{\partial \alpha} = -\frac{C^2 \rho}{V}\frac{\partial V}{\partial \alpha} \qquad (42.24)$$

Using $h_2 = h_2(\beta)$, (41.15) and (42.24) in (41.12), we get

$$\left[\rho V - \frac{C^2 \rho}{V} - \frac{\mu}{V^3}(A^2 + B^2)\right]\frac{\partial V}{\partial \alpha} = 0 \qquad (42.25)$$

Therefore, if $\rho (V^2 - C^2) \neq \mu H^2$, the flow velocity V is constant on each individual streamline. Since $h_2 = h_2(\beta)$ if and only if the streamline pattern is concentric circles or parallel straight lines, we have the theorem:

Theorem 4.2

If for a constantly inclined, nonviscous and compressible plane flow, the flow geometry is concentric circles or parallel straight lines and $\rho (V^2 - C^2) \neq \mu H^2$, then the velocity magnitude V is constant on each individual streamline.

The condition $\rho\ (V^2 - C^2) \neq \mu\ H^2$ is always true for subsonic or sonic flows. Therefore, we state the following corollary.

Corollary 4.4

If for a constantly inclined, nonviscous, sonic or subsonic plane flow, the flow geometry is parallel straight lines or concentric circles, then the velocity magnitude V is constant on each individual streamline.

Section 3. Sonic Flows

A flow is said to be sonic if

$$V\ (\alpha,\beta) = C\ (\alpha,\beta) \tag{43.01}$$

at all points of the flow region.

Since $C = C\ (\rho,\ s)$, it follows from (41.06) and (41.14) that

$$\frac{\partial p}{\partial \alpha} = C^2 \frac{\partial \rho}{\partial \alpha} \tag{43.02}$$

and

$$\frac{\partial C}{\partial \alpha} = \frac{\partial C}{\partial \rho} \frac{\partial \rho}{\partial \alpha} \tag{43.03}$$

where C^2 and $\frac{\partial C}{\partial \rho}$ are positive definite.

Equations (43.02) and (43.03) imply that if one of C, p, ρ is constant on each streamline, then so also are the other two.

Eliminating $\sqrt{h_1}$ between (41.12) and (41.15) we obtain

$$V \frac{\partial V}{\partial \alpha} + \frac{1}{\rho} \frac{\partial p}{\partial \alpha} + \frac{\mu (A^2 + B^2)}{\rho \sqrt{h_2}\ V} \frac{\partial}{\partial \alpha} \left(\frac{\sqrt{h_2}}{V}\right) = 0 \tag{43.04}$$

In order to eliminate $\sqrt{h_2}$ between (41.11) and (43.04), we note that

$$\frac{\partial}{\partial \alpha} \left(\frac{\sqrt{h_2}}{V}\right) = \frac{1}{V} \frac{\partial \sqrt{h_2}}{\partial \alpha} - \frac{\sqrt{h_2}}{V^2} \frac{\partial V}{\partial \alpha} \tag{43.05}$$

Expansion of (41.11) yields

$$\sqrt{h_2}\ \rho \frac{\partial V}{\partial \alpha} + \sqrt{h_2}\ V \frac{\partial \rho}{\partial \alpha} + \rho\ V \frac{\partial \sqrt{h_2}}{\partial \alpha} = 0 \tag{43.06}$$

If p and ρ are constant on each streamline, then by using (43.04), (43.05) and (43.06), we find

$$[V - 2\mu (A^2 + B^2)/\rho V^3] \frac{\partial V}{\partial \alpha} = 0 \qquad (43.07)$$

Equation (43.07) implies $\frac{\partial V}{\partial \alpha} = 0$ or $V = [\frac{2\mu(A^2 + B^2)}{\rho}]^{1/4}$.

In either case, V is constant on each streamline, since ρ is assumed constant on each streamline.

It now follows from Theorem (4.1), that if one of C, p, ρ is constant on each streamline and $\mu H^2 \neq C^2 \rho$, then the flow geometry is parallel straight lines or concentric circles.

For a flow pattern consisting of parallel straight lines or concentric circles, the natural choice of coordinates gives

$$\sqrt{h_2} = 1 \text{ and } \sqrt{h_1} = 1 \text{ or } \beta \qquad (43.08)$$

In natural coordinates, the nonzero component of the vorticity vector is given by

$$\omega = -\frac{1}{\sqrt{h_1 h_2}} \frac{\partial}{\partial \beta} (\sqrt{h_1}\, V) \qquad (43.09)$$

Differentiation of (43.09) with respect to α yields

$$\frac{\partial \omega}{\partial \alpha} = \frac{1}{h_2} \frac{\partial \sqrt{h_2}}{\partial \alpha} [V \frac{\partial}{\partial \beta} (\ln \sqrt{h_1}) + \frac{\partial V}{\partial \beta}] -$$

$$\frac{1}{\sqrt{h_2}} [\frac{\partial V}{\partial \alpha} \frac{\partial}{\partial \beta}(\ln \sqrt{h_1}) + V \frac{\partial^2}{\partial \alpha \partial \beta} (\ln \sqrt{h_1}) + \frac{\partial^2 V}{\partial \alpha \partial \beta}] \qquad (43.10)$$

We conclude from (43.08) and (43.10), that if $V = V(\beta)$ and the flow is in parallel straight lines or concentric circles, then the vorticity field is constant on each individual streamline.

Summarizing, we have the theorem

<u>Theorem 4.3</u>

If one of the flow variables C, ρ or p is constant on each streamline, for a constantly inclined nonviscous compressible plane flow, then the other two flow variables and the speed are also constant along each streamline. Moreover, if $\mu H^2 \neq C^2 \rho$ in the flow region, then the vorticity is also constant along each streamline and the flow geometry is either concentric circles or parallel straight lines.

We now assume our flow is sonic as defined by (43.01). Using (43.01) in (43.06), we obtain

$$\frac{\partial \sqrt{h_2}}{\partial \alpha} = - \frac{\sqrt{h_2}}{\rho C} \left(\rho \frac{\partial C}{\partial \rho} + C \frac{\partial p}{\partial \alpha} \right) \tag{43.11}$$

Employing (43.01), (43.02), (43.03) and (43.11) in (43.04), we find

$$\left[\rho C^4 \left(\frac{\partial C}{\partial \rho} + \frac{C}{\rho} \right) - (A^2 + B^2) \mu \left(2 \frac{\partial C}{\partial \rho} + \frac{C}{\rho} \right) \right] \frac{\partial \rho}{\partial \alpha} = 0 \tag{43.12}$$

Equations (41.07), (43.12) and Theorem (4.3) yield the following theorem:

Theorem 4.4

If for a constantly inclined, nonviscous and sonic plane flow ,

$$H^2 \mu \neq \rho C^2 \left(\frac{\partial C}{\partial \rho} + \frac{C}{\rho}\right) / \left(2 \frac{\partial C}{\partial \rho} + \frac{C}{\rho}\right) ,$$

then ρ, p, V and C are constant on each streamline. Moreover, if $\mu H^2 \neq \rho C^2$, the flow is either in concentric circles or in parallel straight lines.

Section 4. Solutions

Straight Parallel Flow

For this problem, we choose the natural coordinate system to be the rectangular coordinates in which y = constant are the streamlines, and x = constant are the orthogonal trajectories to the streamlines. For this net $h_1 = h_2 = 1$. We solve this flow for regions in which $\rho\ (V^2 - C^2) \neq \mu\ H^2$. If this condition holds, then by Theorem (4.2)

$$V = V(y) \tag{44.01}$$

Using (44.01) and $h_1 = h_2 = 1$ in (41.15) we have

$$\frac{\partial V}{\partial y} = 0 \tag{44.02}$$

Equation (44.01) and (44.02) imply

$$V = K \tag{44.03}$$

where K is an arbitrary constant.

By substituting (44.03) and $h_1 = h_2 = 1$ in (41.12) and (41.13), we obtain

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \tag{44.04}$$

Therefore, the pressure is constant.

Since the pressure is constant, it follows by (41.06) and (41.14) that

$$\rho = \rho(y) \tag{44.05}$$

Employing (44.03) in (41.10), we find

$$\vec{H} = \frac{B}{K}\,\vec{i} + \frac{A}{K}\,\vec{j}$$

where $\vec{i}$, $\vec{j}$ are unit base vectors for the (x, y) plane.

Vortex Flow

To study vortex flow, we use polar coordinates as the natural coordinate system. Therefore, $\sqrt{h_1} = r$, $\sqrt{h_2} = 1$. Assuming $\rho\,(V^2 - C^2) \neq \mu\,H^2$ and using Theorem (4.2), we have

$$V = V(r) \qquad (44.06)$$

By using (44.06) in (41.15), we obtain

$$-\left(\frac{rA}{V^2}\right)\frac{dV}{dr} + \frac{A}{V} = 0 \qquad (44.07)$$

Integration of (44.07) yields

$$V = Kr \qquad (44.08)$$

where K is an arbitrary constant.

By using $\sqrt{h_2} = 1$ and (44.08) in (41.11) and (43.04), we have

$$\rho = \rho(r)\ ,\ p = p(r) \qquad (44.09)$$

Finally, the magnetic field is given by

$$\vec{H} = \frac{1}{Kr}\,[\,B\,\vec{e}_\theta + A\,\vec{e}_r] \qquad (44.10)$$

where $\vec{e}_\theta$, $\vec{e}_r$ are the unit base vectors of the polar plane.

CHAPTER V

FLOWS OF FINITELY CONDUCTING FLUIDS

Section 1. Flow Equations

The equations governing steady, finitely electrically conducting and inviscid gas flow, in the presence of an electric field and a magnetic field, are:

$$\operatorname{div} \vec{H} = 0 \tag{51.01}$$

$$\operatorname{div} \vec{E} = q/\varepsilon \tag{51.02}$$

$$\operatorname{curl} \vec{E} = \vec{0} \tag{51.03}$$

$$\operatorname{curl} \vec{H} = \vec{I} + q \vec{v} \tag{51.04}$$

$$\vec{I} = \sigma (\vec{E} + \mu \vec{v} \times \vec{H}) \tag{51.05}$$

$$\rho(\vec{v} \cdot \operatorname{grad}) \vec{v} = -\operatorname{grad} p + \mu (\vec{I} + q \vec{v}) \times \vec{H} + q \vec{E} \tag{51.06}$$

$$\operatorname{div} (\rho\vec{v}) = 0 \tag{51.07}$$

In this chapter, we consider plane flows in which the magnetic field lies in the flow plane.

From equations (51.04) and (51.05), we obtain

$$\left.\begin{aligned} \vec{I} &= \operatorname{curl} \vec{H} - q \vec{v} \\ \vec{E} &= \frac{1}{\sigma} (\operatorname{curl} \vec{H} - q \vec{v}) - \mu (\vec{v} \times \vec{H}) \end{aligned}\right\} \tag{51.08}$$

Employing equations (51.08) to eliminate $\vec{E}$ and $\vec{I}$ from (51.02), (51.03) and (51.06), we find

$$\operatorname{div}(q\,\vec{v}) = -\frac{\sigma q}{\varepsilon}, \tag{51.09}$$

$$\operatorname{curl}(q\,\vec{v}) - \operatorname{curl}[\operatorname{curl}\vec{H} - \mu\,\sigma\,\vec{v}\times\vec{H}] = \vec{0}, \tag{51.10}$$

and

$$\rho(\vec{v}\cdot\operatorname{grad})\,\vec{v} = -\operatorname{grad} p + \mu\,(\operatorname{curl}\vec{H})\times\vec{H} + \frac{q}{\sigma}\,(\operatorname{curl}\vec{H} - q\,\vec{v}) - \mu\,q\,(\vec{v}\times\vec{H}) \tag{51.11}$$

respectively.

Decomposition of vector equations (51.10), (51.11), into their vector components in the flow plane and their vector components perpendicular to the flow plane, yields

$$\operatorname{curl}(q\,\vec{v}) = \vec{0} \tag{51.12}$$

$$\operatorname{curl}[\operatorname{curl}\vec{H} - \mu\,\sigma(\vec{v}\times\vec{H})] = 0 \tag{51.13}$$

$$\rho(\vec{v}\cdot\operatorname{grad})\,\vec{v} = -\operatorname{grad} p + \mu\,(\operatorname{curl}\vec{H})\times\vec{H} - \frac{q^2\vec{v}}{\sigma} \tag{51.14}$$

$$q\,[\operatorname{curl}\vec{H} - \mu\,\sigma\,(\vec{v}\times\vec{H})] = 0 \tag{51.15}$$

where equations (51.12), (51.13) are equivalent to (51.10) and (51.14), (51.15) are equivalent to (51.11).

Equation (51.15) implies the flow classification of the following theorem.

Theorem 5.1

For a steady, finitely conducting and inviscid plane flow, at least one of the following alternatives holds

(i) $q = 0$ (51.16)

(ii) $\text{curl}\,\vec{H} = \mu\,\sigma\,(\vec{v} \times \vec{H})$ (51.17)

This classification will be used, throughout the chapter, to study different types of flows. In particular, since curl $\vec{H}$ is the total current density, we immediately have the following result.

Corollary 5.1

If the current density is everywhere zero, for a steady, finitely conducting and inviscid flow with nonzero charge density, the flow is necessarily aligned.

If the charge density is nonzero, the flow is described by equations (51.01), (51.07), (51.09), (51.12), (51.14) and (51.17). For flows with zero charge density, the governing equations are (51.01), (51.07), (51.13), (51.14) and (51.16). Having solved these basic systems for $\vec{v}$, $\vec{H}$, ρ, p and q, $\vec{I}$ and $\vec{E}$ follow from equations (51.08).

Let (α,β) be the natural system of coordinates such that the $\beta(x,y)$ = constant curves represent the streamlines. Letting $h_1(\alpha,\beta)$, $h_2(\alpha,\beta)$ be the metric coefficient of this net and $\vec{e}_1$, $\vec{e}_2$ the unit tangent vectors to $\beta(x,y)$ = constant and $\alpha(x,y)$ = constant respectively, we have the following systems of equations for the two flow classifications.

Flows with Nonzero Charge Density

$$\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,H_1) + \frac{\partial}{\partial\beta}(\sqrt{h_1}\,H_2) = 0 \qquad (51.18)$$

$$\frac{\partial}{\partial\alpha}(\rho\sqrt{h_2}\,V) = 0 \qquad (51.19)$$

$$\frac{1}{\sqrt{h_1 h_2}}\frac{\partial}{\partial\alpha}(q\sqrt{h_2}\,V) = -\frac{\sigma q}{\varepsilon} \qquad (51.20)$$

$$\frac{\partial}{\partial\beta}(q\sqrt{h_1}\,V) = 0 \qquad (51.21)$$

$$\frac{\rho V}{\sqrt{h_1}}\frac{\partial V}{\partial\alpha} + \frac{1}{\sqrt{h_1}}\frac{\partial p}{\partial\alpha} = -\frac{H_2}{\sqrt{h_1 h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_2) - \right.$$

$$\left.\frac{\partial}{\partial\beta}(\sqrt{h_1}\,H_1)\right] - \frac{q^2 V}{\sigma} \qquad (51.22)$$

$$-\rho V^2\frac{\partial\sqrt{h_1}}{\partial\beta} + \frac{\partial p}{\partial\beta} = \frac{H_1}{\sqrt{h_1}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}H_1)\right] \qquad (51.23)$$

$$\frac{1}{\sqrt{h_1 h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_2) - \frac{\partial}{\partial\beta}\sqrt{h_1}H_1)\right] = \mu\,\sigma\,V\,H_2 \qquad (51.24)$$

where $\vec{v} = V\,\vec{e}_1$ and $\vec{H} = H_1\,\vec{e}_1 + H_2\,\vec{e}_2$

Flows with Zero Charge Density

$$\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_1) + \frac{\partial}{\partial\beta}(\sqrt{h_1}H_2) = 0 \qquad (51.18)$$

$$\frac{\partial}{\partial\alpha}(\rho\sqrt{h_2}\,V) = 0 \qquad (51.19)$$

$$\frac{1}{\sqrt{h_1 h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}\,H_1)\right] - \mu\,\sigma\,V\,H_2 = A \qquad (51.24)$$

$$\rho V\frac{\partial V}{\partial\alpha} + \frac{\partial p}{\partial\alpha} = -\frac{H_2}{\sqrt{h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}\,H_1)\right] \qquad (51.25)$$

$$-\rho\, V^2 \frac{\partial \sqrt{h_1}}{\partial \beta} + \frac{\partial p}{\partial \beta} = \frac{H_1}{\sqrt{h_1}} \left[\frac{\partial}{\partial \alpha} (\sqrt{h_2}\, H_2) - \frac{\partial}{\partial \beta} (\sqrt{h_1}\, H_1)\right] \quad (51.23)$$

where A is an arbitrary constant.

Using (51.08), $\vec{I}$ and $\vec{E}$ are given by

$$\vec{I} = -q\, V\, \vec{e}_1 + \frac{1}{\sqrt{h_1 h_2}} \left[\frac{\partial}{\partial \alpha} (\sqrt{h_2}\, H_2) - \frac{\partial}{\partial \beta} (\sqrt{h_1} H_1)\right] \vec{k}$$

$$\vec{E} = -\frac{q}{\sigma} V\, \vec{e}_1 + \left\{\frac{1}{\sigma\sqrt{h_1 h_2}} \left[\frac{\partial}{\partial \alpha} (\sqrt{h_2}\, H_2) - \frac{\partial}{\partial \beta} (\sqrt{h_1}\, H_1)\right] - \mu\, V\, H_2\right\} \vec{k} \quad (51.26)$$

where $\vec{k}$ is the unit normal vector to the flow plane.

Section 2. Integrability Conditions

In this section, we derive several integrability conditions which are used to determine geometries and solutions in subsequent sections.

For flows with nonvanishing charge density, equations (51.09), (51.12) yield

$$\text{grad } (\ln q) \cdot \vec{v} + \text{div } \vec{v} = -\frac{\sigma}{\varepsilon} \tag{52.01}$$

and

$$\text{grad } (\ln q) \times \vec{v} + \text{curl } \vec{v} = \vec{0} \tag{52.02}$$

Taking the cross product of (52.02) with $\vec{v}$, we have

$$\vec{v} \times [\text{grad } (\ln q) \times \vec{v}] + \vec{v} \times \text{curl } \vec{v} = \vec{0}$$

or

$$(\vec{v} \cdot \vec{v}) \text{ grad } (\ln q) - [\vec{v} \cdot \text{grad } (\ln q)] \vec{v} + \vec{v} \times \text{curl } \vec{v} = \vec{0} \tag{52.03}$$

Using (52.01) in (52.03), we obtain

$$(\vec{v} \cdot \vec{v}) \text{ grad } (\ln q) = -\vec{v} \times \text{curl } \vec{v} - (\text{div } \vec{v} + \frac{\sigma}{\varepsilon}) \vec{v} \tag{52.04}$$

Equation (52.04) implies

$$\text{grad } (\ln q) = \frac{\vec{w} \times \vec{v} - (\text{div } \vec{v} + \sigma/\varepsilon) \vec{v}}{\vec{v} \cdot \vec{v}} \tag{52.05}$$

where $\vec{w}$ is the vorticity vector.

Taking the curl of (52.05), it follows that

$$\text{curl}\left[\frac{\vec{w} \times \vec{v} - (\text{div}\,\vec{v} + \sigma/\varepsilon)\,\vec{v}}{\vec{v} \cdot \vec{v}}\right] = 0 \qquad (52.06)$$

Equation (52.06) is an equation in $\vec{v}$ and represents an integrability condition for the charge density.

If the flow is irrotational and incompressible, then (52.06) reduces to

$$\text{grad}\left(\frac{1}{v^2}\right) \times \vec{v} = \vec{0} \qquad (52.07)$$

Equation (52.07) yields the corollary:

Corollary 5.2

For a steady, finitely conducting, incompressible and irrotational plane flow with nonzero charge density, the velocity magnitude is constant on each orthogonal trajectory to the streamlines.

In the case of aligned flows, we have

$$\vec{H} = \lambda\,\vec{v} \qquad (52.08)$$

where λ is an arbitrary function such that λ/ρ is constant on each individual streamline.

Substitution of (52.08) into (51.01), (51.17) yields

$$\text{grad}\,(\ln\lambda) \cdot \vec{v} + \text{div}\,\vec{v} = 0 \qquad (52.09)$$

and

$$\text{grad}\,(\ln\lambda) \times \vec{v} + \text{curl}\,\vec{v} = \vec{0} \qquad (52.10)$$

respectively.

Equations (52.09), (52.10) have the same form as

(52.01) and (52.02). Therefore, employing the method used to derive (52.06), we obtain the following integrability condition for λ.

$$\operatorname{curl} \left[\frac{\vec{w} \times \vec{v} - (\operatorname{div} \vec{v})\ \vec{v}}{\vec{v} \cdot \vec{v}} \right] = \vec{0} \tag{52.11}$$

If a flow is both aligned and has nonzero charge density, equation (52.06), (52.11) imply

$$\operatorname{curl} \left[\frac{\vec{v}}{\vec{v} \cdot \vec{v}} \right] = 0 \tag{52.12}$$

In case an aligned flow with nonzero charge density is also irrotational, equation (52.12) reduces to

$$\operatorname{grad} \left(\frac{1}{V^2} \right) \times \vec{v} = \vec{0} \tag{52.13}$$

Therefore, we have the corollary:

<u>Corollary 5.3</u>

For a steady, finitely conducting, aligned and irrotational plane flow with nonzero charge density, the velocity magnitude is constant on each orthogonal trajectory to the streamlines.

Summarizing, we have the following lemma.

<u>Lemma 5.1</u>

For a steady, finitely conducting and inviscid plane flow with nonzero charge density, we have the integrability condition

$$\text{curl}\left[\frac{\vec{w}\times\vec{v}-(\text{div}\,\vec{v}+\sigma/\varepsilon)\,\vec{v}}{\vec{v}\cdot\vec{v}}\right] = \vec{0}$$

If the flow is also aligned, then we also have

$$\text{curl}\left[\frac{\vec{v}}{\vec{v}\cdot\vec{v}}\right] = \vec{0}$$

In terms of the natural streamline coordinates, (α,β), equations (52.06), and (52.12) take the following forms

$$\frac{\partial}{\partial\beta}\left[\frac{\partial}{\partial\beta}\ln(\sqrt{h_2}\,V)\right] - \frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\beta}\ln(\sqrt{h_1}\,V)\right] + \frac{\sigma}{\varepsilon}\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h_1}}{V}\right) = 0, \tag{52.14}$$

and

$$\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h_1}}{V}\right) = 0 \tag{52.15}$$

respectively.

Section 3. Flows with Straight Streamlines and Flows whose Streamlines are the Involutes of a Curve.

In this section, we consider incompressible flows with nonzero charge density. Therefore, by equations (51.19) and (52.14), the velocity field must satisfy the following two equations

$$\frac{\partial}{\partial \alpha} (\sqrt{h_2}\, V) = 0 \tag{53.01}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} \ln (\sqrt{h_1} \cdot V) - \frac{\sigma}{\varepsilon} \frac{\partial}{\partial \beta} \left(\frac{\sqrt{h_1}}{V}\right) = 0 \tag{53.02}$$

where (α,β) is a natural net such that the β = constant curves represent the streamlines.

To study the flow geometrics, of this section, we let Γ be a curve in the flow plane. Relative to Γ, we have the coordinate system (ξ,Θ) from section (2.2), where the Θ = constant curves represent the tangents of Γ and the ξ = constant curves represent the involutes of Γ. The first fundamental form for this net is

$$d\bar{s}^2 = d\xi^2 + [\xi - \delta(\Theta)]^2\, d\Theta^2 \tag{22.29}$$

where $\delta(\Theta)$ is the radius of curvature along Γ.

We now separately consider the two geometry types.

Flows with Straight Streamlines

If the streamlines are nonparallel straight lines, they must form the envelope of a curve Γ. Taking (ξ,Θ) as the natural coordinates and using (22.29) and (53.01), we obtain

$$\frac{\partial}{\partial \xi}\,[(\xi - \delta)\,V] = 0 \tag{53.03}$$

Equation (53.03) implies

$$V = \frac{f(\Theta)}{(\xi - \delta)} \tag{53.04}$$

where $f(\Theta)$ is an arbitrary differentiable function of Θ.

Substituting (53.04) into (53.02), we get

$$\frac{\partial^2}{\partial \xi \partial \Theta} \ln \left[\frac{f(\Theta)}{(\xi - \delta)}\right] - \frac{\delta}{\varepsilon}\frac{\partial}{\partial \Theta}\left[\frac{(\xi - \delta)}{f(\Theta)}\right] = 0$$

or

$$\frac{\sigma}{\varepsilon}\frac{f'(\Theta)}{[f(\Theta)]^2}(\xi - \delta)^3 + \frac{\sigma}{\varepsilon}\frac{\delta'(\Theta)}{f(\Theta)}(\xi - \delta)^2 - \delta'(\Theta) = 0 \tag{53.05}$$

Since ξ and Θ are independent, equation (35.05) implies

$$\delta'(\Theta) = 0$$

Therefore, the only possible nonparallel straight line flow is flow in concurrent straight lines. This yields the following theorem.

Theorem 5.2

For a steady, incompressible, finitely conducting and nonviscous plane flow with nonzero charge density, the only possible straight streamline patterns are parallel straight lines or concurrent straight lines.

Flows whose Streamlines are the Involutes of a curve Γ.

The coordinates (Θ,ξ), in which the ξ = constant curves are the involutes of Γ and the Θ = constant curves are the tangents of Γ, are a natural system of coordinates for this flow.

By (53.01) and (22.29), it follows that

$$\frac{\partial}{\partial\Theta} \quad (V) \quad = \quad 0 \qquad (53.06)$$

Integration of (53.06) yields

$$V = g\ (\xi) \qquad (53.07)$$

where $g\ (\xi)$ is an arbitrary differentiable function of ξ.

Substitution of (53.06) into (53.02) yields

$$\frac{\partial^2}{\partial\Theta\partial\xi} \ln\ [(\xi - \delta)\ g\ (\xi)] \quad - \frac{\sigma}{\varepsilon} \frac{\partial}{\partial\xi} [\frac{(\xi - \delta)}{g\ (\xi)}] = 0$$

or

$$[g(\xi)]^2\ \delta'(\Theta) - \frac{\sigma}{\varepsilon}\, g(\xi)(\xi - \delta)^2 + \frac{\sigma}{\varepsilon}\, g'(\xi)(\xi - \delta)^3 = 0 \qquad (53.08)$$

Equation (53.08) implies that $\delta'\ (\Theta) = 0$ and therefore, we have

Theorem 5.3

For a steady, incompressible, finitely conducting and nonviscous plane flow with nonzero charge density, the only flow whose streamlines are the involutes of a curve is flow in concentric circles.

Section 4. Irrotational Incompressible Flows with Nonvanishing Charge Density.

In order to derive the possible geometries for the flows of this section, we require the formulas which are summarized in the following lemma.

Lemma 5.2

If we make the natural correspondence between plane vectors and complex numbers, [i.e. the vector $\vec{w} = (w_1, w_2)$ corresponds to the complex number $w = (w_1 + i\, w_2)$], then for every scalar function $f\ (x, y)$ and every plane vector $\vec{w}$, we have the following formulas:

$$\text{grad } f \times \vec{w} = [(w \frac{\partial f}{\partial z} - \bar{w} \frac{\partial f}{\partial \bar{z}})/i]\ \vec{k}, \tag{54.01}$$

$$\text{grad } f \cdot \vec{w} = w \frac{\partial f}{\partial z} + \bar{w} \frac{\partial f}{\partial \bar{z}}, \tag{54.02}$$

$$\frac{\partial w}{\partial z} = \frac{1}{2} [\text{div } \vec{w} + i\ (\text{curl } \vec{w} \cdot \vec{k})] \tag{54.03}$$

and

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{4} [\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - 2\ i\ \frac{\partial^2 f}{\partial x \partial y}] \tag{54.04}$$

where $z = x + i\ y$, $\frac{\partial}{\partial z} = \frac{1}{2} [\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}]$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} [\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}]$ and $\vec{k}$ is the unit normal to the $x - y$ plane.

Using the assumed incompressibility and irrotationality in equations (51.09) and (51.12), we obtain

$$\text{grad } \Phi \cdot \vec{v} = -\sigma/\varepsilon = B \qquad (54.05)$$

$$\text{grad } \Phi \times \vec{v} = \vec{0} \qquad (54.06)$$

where $\Phi = \ln q$.

Application of (54.02) to (54.05) and (54.01) to (54.06) yields

$$v \frac{\partial \Phi}{\partial z} + \bar{v} \frac{\partial \Phi}{\partial \bar{z}} = B \qquad (54.07)$$

$$v \frac{\partial \Phi}{\partial z} - \bar{v} \frac{\partial \Phi}{\partial \bar{z}} = 0 \qquad (54.08)$$

Equations (54.07) and (54.08) are equivalent to the single equation

$$v \frac{\partial \Phi}{\partial z} = \frac{B}{2} \qquad (54.09)$$

Since the flow is incompressible and irrotational, it follows from (54.03) that

$$\frac{\partial v}{\partial z} = 0 \qquad (54.10)$$

Operating on (54.09) with $\frac{\partial}{\partial z}$ and using (54.10), we get

$$\frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (54.11)$$

Formula (54.04) and equation (54.11) imply

$$\left.\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2} &= \frac{\partial^2 \Phi}{\partial y^2} \\ \frac{\partial^2 \Phi}{\partial x \partial y} &= 0 \end{aligned}\right\} \qquad (54.12)$$

Integrating (54.12) for Φ, we obtain

$$\Phi = C(x^2 + y^2) + C_1 x + C_2 y + C_3 \qquad (54.13)$$

where C, C_1, C_2 and C_3 are arbitrary constants. By (54.05), Φ cannot be constant throughout the flow and, therefore, at least one of C, C_1, C_2 must be nonzero.

Substitution of (54.13) into (54.09) yields

$$v = \frac{B\,(2Cx + C_1) + i\,B\,(2\,Cy + C_2)}{(2\,Cx + C_1)^2 + (2\,Cy + C_2)^2} \qquad (54.14)$$

From (54.14), we see that for $C = 0$ the flow is in parallel straight lines in the direction (C_1, C_2), while, for $C \neq 0$ the flow is in concurrent straight lines through the point $\left(-\frac{C_1}{2C}, \frac{C_2}{2C}\right)$.

Employing $\Phi = \ln q$ and (54.13), we find

$$q = \exp\left[C(x^2 + y^2) + C_1 x + C_2 y + C_3\right] \qquad (54.15)$$

Summarizing, we have the following theorem

Theorem 5.4

For incompressible, irrotational plane flows with non-vanishing charge density, the flow geometry is either parallel straight lines or concurrent straight lines. Moreover, for these flows, the velocity field and the charge density are given by

$$v = \frac{B\,(2Cx + C_1)\,\vec{i} + B\,(2\,Cy + C_2)\,\vec{j}}{(2Cx + C_1)^2 + (2Cy + C_2)^2}$$

$$q = \exp\,[C(x^2 + y^2) + C_1 x + C_2 y + C_3]$$

where C, C_1, C_2 and C_3 are arbitrary constants and $\vec{i}$, $\vec{j}$ are the usual unit base vectors of the (x, y) plane.

Section 5. Incompressible Flows with Isometric Geometry and Nonvanishing Charge Density.

Let (α,β) be a natural isometric net such that the β = constant curves represent the streamlines. Letting h be the metric coefficient of this net, $\vec{e}_1$ and $\vec{e}_2$ the unit tangent vectors to the β = constant, α = constant curves respectively and ds the element of arc length in the (x, y) plane, we have

$$ds^2 = h\,[d\alpha^2 + d\beta^2] \tag{55.01}$$

$$\vec{v} = V\,\vec{e}_1 \tag{55.02}$$

$$\vec{H} = H_1\,\vec{e}_1 + H_2\,\vec{e}_2 \tag{55.03}$$

Using the assumed incompressibility and (55.01) in (51.19) and (52.14), we obtain

$$\frac{\partial}{\partial\alpha}\,(\sqrt{h}\,V) = 0 \tag{55.04}$$

$$\frac{\partial^2}{\partial\alpha\partial\beta}\ln\,(\sqrt{h}\,V) + \frac{\sigma}{\varepsilon}\frac{\partial}{\partial\beta}\left(\frac{\sqrt{h}}{V}\right) = 0 \tag{55.05}$$

Equation (55.04) implies that

$$V = \frac{\Psi(\beta)}{\sqrt{h}}, \tag{55.06}$$

where $\Psi(\beta)$ is an arbitrary differentiable function of β.

Substituting (55.06) into (55.05), we get

$$\frac{1}{h}\,\frac{\partial h}{\partial\beta} = \frac{\Psi'(\beta)}{\Psi(\beta)} \tag{55.07}$$

Application of Lemma (2.1) to our isometric net yields

$$\frac{\partial W_1}{\partial \alpha} = \frac{\partial W_2}{\partial \beta}, \quad \frac{\partial W_2}{\partial \alpha} = -\frac{\partial W_1}{\partial \beta}$$

where

$$W_1 = -\frac{1}{2h}\frac{\partial h}{\partial \alpha}, \quad W_2 = \frac{1}{2h}\frac{\partial h}{\partial \beta} \qquad (55.08)$$

Employing (55.07) and (55.08), we obtain

$$\frac{\partial W_2}{\partial \alpha} = -\frac{\partial W_1}{\partial \beta} = 0 \qquad (55.09)$$

Equation (55.09) implies that

$$W_1 = f_1(\alpha), \quad W_2 = f_2(\beta) \qquad (55.10)$$

where $f_1(\alpha)$, $f_2(\beta)$ are arbitrary differentiable functions of their respective arguments. Substitution of (55.10) into the first equation of (55.08) gives

$$f_1'(\alpha) = f_2'(\beta) = K \qquad (55.11)$$

where K is an arbitrary constant. Therefore, the general solutions for W_1, W_2 are

$$W_2 = K\beta + K_1, \quad W_1 = K\alpha + K_2 \qquad (55.12)$$

where K_1, K_2 are arbitrary constants.

Using (55.07), (55.08) and (55.12), we find the following system of equations for $\Psi(\beta)$ and $h(\alpha,\beta)$.

$$\left.\begin{aligned} \frac{\Psi'(\beta)}{\Psi(\beta)} &= 2\,K\,\beta + 2\,K_1 \\ \frac{1}{2h}\,\frac{\partial h}{\partial \alpha} &= -\,K\,\alpha - K_2 \\ \frac{1}{2h}\,\frac{\partial h}{\partial \beta} &= K\,\beta + K_1 \end{aligned}\right\} \tag{55.13}$$

Integrating (55.13), we obtain

$$\Psi(\beta) = \exp\,[\,K\,\beta^2 + 2\,K_1\,\beta + K_3] \tag{55.14}$$

$$h = \exp\,[K\,(\beta^2 - \alpha^2) + 2\,(K_1\,\beta - K_2\alpha) + K_4] \tag{55.15}$$

where K_3 and K_4 are arbitrary constants.

To solve for the charge density, we use (55.01), (55.06), (55.14), (55.15) in equations (51.20) and (51.21). This yields

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha}\,(\ln\,q) &= -\,\frac{\sigma}{\varepsilon}\,\exp\,[-K\alpha^2 - 2\,K_2\alpha + K_4 - K_3] \\ \frac{\partial}{\partial \beta}\,(\ln\,q) &= -\,2\,K\,\beta - 2\,K_1 \end{aligned}\right] \tag{55.16}$$

Integration of (55.16) gives

$$q = \exp\,[\,-\,\frac{\sigma}{\varepsilon}\int \exp\,[-K\alpha^2 - 2\,K_2\alpha + K_4 - K_3]\,d\alpha - K\,\beta^2 - 2\,K_1\,\beta + K_4] \tag{55.17}$$

Although we are not able to solve for the magnetic field, we derive an equation which can be solved in certain cases. Equations (51.18), (55.01) imply the existence of a magnetic streamfunction M such that

$$\frac{\partial M}{\partial \alpha} = -\sqrt{h}\, H_2 \quad , \quad \frac{\partial M}{\partial \beta} = \sqrt{h}\, H_1 \tag{55.18}$$

Employing (55.01), (55.06), (55.14), (55.15) and (55.18) in (51.24), we get

$$\frac{\partial^2 M}{\partial \alpha^2} + \frac{\partial^2 M}{\partial \beta^2} = \mu\, \sigma \exp\, [K\beta^2 + 2\, K_1\beta + K_3]\, \frac{\partial M}{\partial \alpha} \tag{55.19}$$

Summarizing, we have:

Theorem 5.5

For incompressible, finitely conducting plane flow with nonvanishing charge density and isometric flow pattern, the flow speed, the charge density, and the metric of the natural net have the forms

$$\left.\begin{aligned} V &= \exp\, [\tfrac{K}{2}\, (\beta^2 + \alpha^2) + (K_1\beta + K_2\alpha) + K_3 - \tfrac{K_4}{2}] \\ q &= \exp\, [-\tfrac{\sigma}{\varepsilon} \int \exp\, [-K\alpha^2 - 2K_2\alpha + K_4 - K_3]\, d\alpha \\ &\qquad - K\beta^2 - 2\, K_1\beta + K_4] \\ h &= \exp\, [K(\beta^2 - \alpha^2) + 2\, (K_1\beta - K_2\, \alpha) + K_4] \end{aligned}\right] \tag{55.20}$$

Moreover, the magnetic streamfunction M satisfies (55.19).

In case the flow is aligned or orthogonal, we are able to extend the results of Theorem (5.5). We consider these flow types separately.

Aligned Flows

For aligned flows $\vec{H} = H_1 \vec{e}_1$. Therefore, by (55.18), the magnetic streamfunction satisfies

$$\frac{\partial M}{\partial \alpha} = 0 \quad , \quad \frac{\partial M}{\partial \beta} = \sqrt{h}\, H_1 \tag{55.21}$$

Equations (55.21) imply $M = M(\beta)$. Using this result in (55.19), we obtain

$$\frac{\partial M}{\partial \beta} = K_5 \tag{55.22}$$

where K_5 is an arbitrary function.

From (55.18), (55.22) it follows that

$$\vec{H} = K_5 \exp\left[\frac{K}{2}(\alpha^2 - \beta^2) + (K_2\alpha - K_1\beta) - \frac{K_4}{2}\right] \vec{e}_1 \tag{55.23}$$

In order for (55.20) and (55.22) to be solutions, we must be able to integrate for the pressure function. By Corollary (5.1) and equation (51.14), the integrability condition for p is

$$\text{curl}\,[\vec{w} \times \vec{v}] + \text{curl}\,\left[\frac{q^2 \vec{v}}{\sigma}\right] = 0$$

or

$$\frac{\Psi(\beta)\ \Psi'(\beta)}{h^2}\ \frac{\partial h}{\partial \alpha} - \frac{1}{\rho\sigma}\,[q^2\ \Psi(\beta)] = 0 \tag{55.24}$$

Using (55.14) and (55.20) in (55.24), we get

$$-4\ (K\beta + K_1)\ (K\alpha + K_2) + \frac{2}{\rho\sigma}\ (K\beta + K_1)\ \exp\{$$

$$-\frac{2\sigma}{\varepsilon} \int \exp\,[-K\alpha^2 - 2\,K_2\alpha + K_4 - K_3]\, d\alpha - 2\,K\,\beta^2$$

$$- K\alpha^2 - 4 K_1 \beta - 2 K_2 \alpha - K_3 + 3 K_4\} = 0 \quad (55.25)$$

Differentiating (55.25) twice with respect to β and dividing out the exponential term, we obtain

$$32 K^3 \beta^3 + 96 K^2 K_1 \beta^2 + (96 K K_1^2 - 24 K^2) \beta$$

$$+ (32 K_1^3 - 24 K K_1) = 0 \quad (55.26)$$

Since (55.26) holds everywhere in the flow region, all the coefficients of the cubic polynomial must be zero. Therefore, we have

$$K = K_1 = 0 \quad (55.27)$$

To obtain the pressure function, we use (55.20) and (55.23), with $K = K_1 = 0$, in equations (51.22) and (51.23). This gives

$$\frac{\partial p}{\partial \beta} = 0$$

$$\frac{\partial p}{\partial \alpha} = -\rho K_2 \exp [K_2\alpha + K_3 - \frac{K_4}{2}]$$

$$- \frac{1}{\sigma} \exp [K_3 + 2 K_4] \exp [- \frac{2 \sigma}{\varepsilon} \quad \exp (}$$

$$- 2 K_2\alpha + K_4 - K_3) d\alpha]$$

Integration of these equations gives

$$p = -\rho \exp [K_2\alpha + K_3 - \frac{K_4}{2}] - \frac{1}{\sigma} \exp [$$

$$K_3 + 2 K_4] \int \exp [- \frac{2 \sigma}{\varepsilon} \{\exp (-2 K_2\alpha +$$

$$K_4 - K_3) d\alpha]\} d\alpha \quad (55.28)$$

In order to determine the possible streamline patterns, we employ the following formula from Lemma (2.1).

$$\frac{f''(z)}{[f'(z)]^2} = W_1 + i\, W_2 \tag{22.20}$$

where $f(z) = \alpha + i\, \beta$.

Using (55.12), (55.27) and (22.20), we get

$$\frac{f''(z)}{[f'(z)]^2} = i\, K_2 \tag{55.29}$$

It was shown in section (3.5) that (55.29) implies concentric circles or parallel straight lines, in accordance with $K_2 \neq 0$, $K_2 = 0$ respectively.

Therefore, we have the theorem:

Theorem 5.6

For incompressible, finitely conducting aligned plane flow with nonvanishing charge density and isometric flow pattern, the flows are

Geometry	Solutions
Concentric Circles	Equation (55.20), (55.23), (55.28) with $K = K_1 = 0$
Parallel Straight Lines	Equations (55.20), (55.23), (55.28) with $K = K_1 = K_2 = 0$.

Orthogonal Flows

For orthogonal flows $\vec{H} = H_2\, \vec{e}_2$. Therefore, by (55.18), we have

$$\frac{\partial M}{\partial \alpha} = -\sqrt{h}\, H_2 \quad , \quad \frac{\partial M}{\partial \beta} = 0 \tag{55.30}$$

where M is the magnetic streamfunction.

Substitution of (55.30) into (55.19) yields

$$M''(\alpha) = \mu\, \sigma\, \exp\,[K\beta^2 + 2\, K_1\, \beta + K_3]\, M'(\alpha) \tag{55.31}$$

Integrating (55.31), we find

$$M'(\alpha) = \exp\,[\mu\, \sigma\, \exp\,(K\beta^2 + 2\, K_1\, \beta + K_3)\, \alpha + K_6] \tag{55.32}$$

where K_6 is an arbitrary constant. Since $M'(\alpha)$ is a function of α, it follows from (55.32) that

$$K = K_1 = 0 \tag{55.33}$$

Using (55.20), (55.30), (55.32) and (55.33), we have

$$\vec{H} = -\exp\,[\mu\sigma\, \exp\,(K_3)\alpha + K_2\alpha - \frac{K_4}{2} + K_6]\, \vec{e}_2 \tag{55.34}$$

Employing (55.20), (55.33), (55.34), (51.31) and (51.32), we obtain the following equations for the pressure function

$$\left.\begin{aligned}
&\frac{\partial p}{\partial \beta} = 0 \\
&\frac{\partial p}{\partial \alpha} = -\rho\, K_2\, \exp\,[2\, K_2\, \alpha + 2\, K_3 - K_4] \\
&-\mu\, \sigma\, \exp\,(K_3)\, \exp\,[2\, \mu\sigma\, \exp\,(K_3)\alpha + 2\, K_2\alpha - K_4 \\
&+ 2\, K_6] - \frac{\exp\,(K_3)}{\sigma}\, \exp\,\{-\frac{2\,\sigma}{\varepsilon} \int \exp\,[-2\, K_2 + K_4 \\
&- K_3]\, d\alpha + 2\, K_4\}
\end{aligned}\right] \tag{55.35}$$

Integrating (55.35), we get

$$p = -\frac{\rho}{2} \exp [2 K_2 \alpha + 2 K_3 - K_4]$$

$$-\frac{1}{2} \exp [2 \mu \sigma \exp (K_3) \alpha + 2 K_2 \alpha + 2 K_6 - K_4]$$

$$-\frac{\exp (K_3)}{\sigma} \int \{\exp [-\frac{2 \sigma}{\varepsilon} \int \exp (-2 K_2 \alpha + K_4$$

$$- K_3) d \alpha + 2 K_4]\} d\alpha \qquad (55.36)$$

Since the restraint on the constants as given by (55.33), is identical with (55.26), the geometry of orthogonal flows is the same as the geometry for aligned flows. Therefore, we have the theorem:

Theorem 5.7

For incompressible, finitely conducting orthogonal plane flow with nonvanishing charge density and isometric flow pattern, the possible flows are

Geometry	Solutions
Concentric Circles	Equations (55.20), (55.34), (55.36) with $K = K_1 = 0$.
Parallel Straight Lines	Equations (55.20), (55.34), (55.36) with $K = K_1 = K_2 = 0$

Section 6. Viscous Flows

Most of the results established for finitely conducting inviscid flows also apply to finitely conducting incompressible viscous flows.

For viscous incompressible flows, the linear momentum equation, (51.06), and the continuity equation, (51.07), are replaced by

$$(\vec{v} \cdot \text{grad})\ \vec{v} = -\frac{\text{grad}\ p}{\rho} + \frac{\mu}{\rho}(\vec{I} + q\ \vec{v}) \times \vec{H}$$

$$+ \frac{q\ \vec{E}}{\rho} + \nu\ \nabla^2\ \vec{v}\ , \qquad (56.01)$$

$$\text{div}\ \vec{v} = 0 \qquad (56.02)$$

respectively.

Using (51.08) to eliminate $\vec{I}$, $\vec{E}$ from (56.01) and decomposing the resulting equation into its vector component in the flow plane and its vector component perpendicular to the flow plane, we obtain

$$(\vec{v} \cdot \text{grad})\ \vec{v} = -\frac{\text{grad}\ p}{\rho} + \frac{\mu}{\rho}(\text{curl}\ \vec{H}) \times \vec{H}$$

$$-\frac{q^2\ \vec{v}}{\sigma\rho} + \nu\ \nabla^2\ \vec{v} \qquad (56.03)$$

$$q\ [\text{curl}\ \vec{H} - \mu\ \sigma\ (\vec{v} \times \vec{H})] = 0\ . \qquad (56.04)$$

Equation (56.04) gives the following theorem.

Theorem 5.8

For a steady, finitely conducting, incompressible and viscous plane flow, at least one of the following alternatives holds

(i) $q = 0$ (56.05)

(ii) $\text{curl}\ \vec{H} = \mu\ \sigma\ (\vec{v} \times \vec{H})$ (56.06)

We consider flows for which the charge density is non-zero. These flows are governed by equations (51.01), (56.02), (51.09), (51.12), (56.03) and (56.06). Having solved these equations for $\vec{v}$, $\vec{H}$, p and q, $\vec{I}$ and $\vec{E}$ follow from (51.08). In terms of the natural curvilinear coordinates (α,β), the basic flow equations assume the form:

$$\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,H_1) + \frac{\partial}{\partial\beta}(\sqrt{h_1}\,H_2) = 0 \tag{51.18}$$

$$\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,V) = 0 \tag{56.07}$$

$$\frac{1}{\sqrt{h_1h_2}}\frac{\partial}{\partial\alpha}(q\sqrt{h_2}\,V) = \frac{\sigma q}{\varepsilon} \tag{51.20}$$

$$\frac{\partial}{\partial\beta}(q\sqrt{h_1}\,V) = 0 \tag{51.21}$$

$$\frac{V}{\sqrt{h_1}}\frac{\partial V}{\partial\alpha} + \frac{1}{\rho\sqrt{h_1}}\frac{\partial p}{\partial\alpha} = -\frac{H_2}{\rho\sqrt{h_1h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}\,H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}H_1)\right] - \frac{q^2V}{\rho\,\sigma} + \frac{\nu}{\sqrt{h_2}}\frac{\partial}{\partial\beta}\left[\frac{1}{\sqrt{h_1h_2}}\frac{\partial}{\partial\beta}(\sqrt{h_1}V)\right] \tag{56.08}$$

$$-\frac{V^2}{\sqrt{h_2}}\frac{\partial\sqrt{h_1}}{\partial\beta} + \frac{1}{\rho\sqrt{h_2}}\frac{\partial p}{\partial\beta} = \frac{H_1}{\rho\sqrt{h_1h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}H_1)\right] - \frac{\nu}{\sqrt{h_1}}\frac{\partial}{\partial\alpha}\left[\frac{1}{\sqrt{h_1h_2}}\frac{\partial}{\partial\beta}(\sqrt{h_1}V)\right] \tag{56.09}$$

$$\frac{1}{\sqrt{h_1h_2}}\left[\frac{\partial}{\partial\alpha}(\sqrt{h_2}H_2) - \frac{\partial}{\partial\beta}(\sqrt{h_1}H_1)\right] = \mu\,\sigma V\,H_2 \tag{51.24}$$

where $\vec{v} = V\,\vec{e}_1$, $\vec{H} = H_1\,\vec{e}_1 + H_2\vec{e}_2$ and the element of arc length $d\bar{s} = h_1\,d\alpha^2 + h_2\,d\,\beta^2$.

Calculating the integrability condition $\frac{\partial q}{\partial\alpha\partial\beta} = \frac{\partial q}{\partial\beta\partial\alpha}$ from (51.20), (51.21) and using (56.07), we obtain

$$\frac{\partial^2}{\partial\alpha\partial\beta} \ln (\sqrt{h}_1\ V) - \frac{\sigma}{\varepsilon} \frac{\partial}{\partial\beta} (\frac{\sqrt{h}_1}{V}) = 0 \tag{56.10}$$

Since the component of the linear momentum equation in the flow plane is not used in sections (5.3) and (5.4), the results of these sections also apply to viscous incompressible flows. Therefore, we have the theorems:

Theorem 5.9

For a steady, incompressible, finitely conducting and viscous plane flow with nonzero charge density, the only possible straight streamline patterns are parallel straight lines or concurrent straight lines.. Moreover, the only flow whose streamlines are the involutes of a curve is flow in concentric circles.

Theorem 5.10

For incompressible, viscous and irrotational plane flows with nonvanishing charge density, the flow geometry is either parallel straight lines or concurrent straight lines.

For the case of isometric flows with metric coefficient $\sqrt{h}$, equations (56.07), (56.10), take the form

$$\frac{\partial}{\partial\alpha} (\sqrt{h}\ V) = 0 , \tag{56.11}$$

$$\frac{\partial^2}{\partial\alpha\partial\beta} \ln (\sqrt{h}\ V) - \frac{\sigma}{\varepsilon} \frac{\partial}{\partial\beta} (\frac{\sqrt{h}}{V}) = 0 \tag{56.12}$$

respectively.

Since (56.11), (56.12) have the same form as (55.04), (55.05), we obtain exactly as in section (5.5) the following counterpart to theorem (5.5).

Theorem 5.11

For incompressible, finitely conducting and viscous plane flow with nonvanishing charge density and isometric flow pattern, the flow speed, the charge density and the metric of the natural net have the forms

$$V = \exp\left[\frac{K}{2}(\beta^2 + \alpha^2) + (K_1\beta + K_2\alpha) + K_3 - \frac{K_4}{2}\right]$$

$$q = \exp\left[-\frac{\sigma}{\varepsilon}\int \exp\left[-K\alpha^2 - 2K_2\alpha + K_4 - K_3\right] d\alpha \right.$$
$$\left. - K\beta^2 - 2K_1\beta + K_4\right]$$

$$h = \exp\left[K(\beta^2 - \alpha^2) + 2(K_1\beta - K_2\alpha) + K_4\right] \tag{56.13}$$

where K, K_1, K_3 and K_4 are arbitrary constants.

In the case of isometric flows, equation (51.18) implies the existence of a streamfunction M such that

$$\frac{\partial M}{\partial \alpha} = -\sqrt{h}\, H_2\ , \quad \frac{\partial M}{\partial \beta} = \sqrt{h}\, H_1 \tag{56.14}$$

where $\vec{H} = H_1 \vec{e}_1 + H_2 \vec{e}_2$

Employing (56.13) and (56.14) in (51.24), we obtain

$$\frac{\partial^2 M}{\partial \alpha^2} + \frac{\partial^2 M}{\partial \beta^2} = \mu\, \sigma \exp\left[K\beta^2 + 2K_1\beta + K_3\right] \frac{\partial M}{\partial \alpha} \tag{56.15}$$

We next determine the possible flow geometries for constantly inclined isometric flows.

Letting Θ be the constant angle of inclination between $\vec{v}$ and $\vec{H}$, we define A and B as

$$A = \sin \phi \ , \quad B = \cos \Theta \tag{56.16}$$

In terms of A and B, the magnetic vector has the form

$$\vec{H} = H\,B\,\vec{e}_1 + H\,A\,\vec{e}_2 \tag{56.17}$$

In the case of aligned or orthogonal isometric flows, the possible geometries follow as in section (5.5). We extend these results to nonorthogonal constantly inclined flows. Therefore, we assume both A and B to be nonzero constants.

Using (56.14) and (56.17) in (56.15), we get

$$A \frac{\partial}{\partial \alpha} (\ln \sqrt{h}\, H) - B \frac{\partial}{\partial \beta} (\ln \sqrt{h}\, H) = A\,\mu\,\sigma \exp [K\beta^2 + 2\,K_1\beta + K_3] \tag{56.18}$$

Integration of (56.18) yields

$$\ln \sqrt{h}\, H = F_1 (A\,\beta + B\alpha) + A\,\mu\,\sigma \int \exp [K\beta^2 + 2\,K_1\,\beta + K_3]\, d\beta + F_2(\alpha) \tag{56.19}$$

where $F_1(A\,\beta + B\alpha)$, $F_2(\alpha)$ are arbitrary functions of their respective arguments.

Substituting (56.19) into (51.18) and using (56.17), we obtain

$$F_1'(A\beta + B\alpha) + B F_2'(\alpha) + A^2 \mu \sigma \exp [K \beta^2 + 2 K_1\beta + K_3] = 0 \quad (56.20)$$

Differentiating (56.20) with respect to β, we find

$$A F_1''(A\beta + B\alpha) + A^2 \mu \sigma (2K\beta + 2 K_1) \exp [K\beta^2 + 2 K_1 \beta + K_3] = 0 \quad (56.21)$$

Since $A \beta + B\alpha$ and β are independent, equation (56.21) implies

$$K = K_1 = 0 \quad (56.22)$$

Equation (56.22) in conjunction with (22.20) implies that the streamline geometry is concentric circles or parallel straight lines. Therefore, we have the theorem:

Theorem 5.12

For incompressible, finitely conducting, viscous, constantly inclined or aligned plane flows, the only possible isometric flow patterns are concentric circles or parallel straight lines.

REFERENCES

R. Berker (1963) — "Intégration des équations de mouvement d'un fluide visques incompressible." Vol. VIII/2, Strömungsmechanik II, Handbuch der Physik.

O. P. Chandna (1972) — "Steady transverse Magnetogasdynamic flows." Can. J. Phys., 50, pp. 2565-2567.

O. P. Chandna and M. R. Garg (1975) — "The Flow of a Viscous MHD fluid." Quart. Appl. Math, Accepted for publication.

O. P. Chandna and V. I. Nath (1972) — "On the uniqueness of M.H.D. aligned flows with given streamlines." Can. J. Phys., 50, pp. 661-665.

(1972) — "Some properties of aligned Flows with given Streamlines." Jap. J. Appl. Phys., 11, pp. 889-892.

(1973) — "Two dimensional steady Magneto Fluid Dynamic flows with orthogonal magnetic and velocity field distributions." Can. J. Phy., 51, pp. 772-778.

O. P. Chandna and H. Toews (1975) — "Plane Constantly inclined MHD flow with isometric geometry." Quart. Appl. Math., Accepted for publication.

O. P. Chandna, H. Toews and V. I. Nath (1975) — "Plane steady flows with constantly inclined magnetic and velocity fields." Can. J. Phys., Accepted for publication.

S. Chandrasekhar (1961) — "Hydrodynamic and Hydromagnetic Stability." The Claredon Press, Oxford.

M. R. Garg and O. P. Chandna (1974) — "Viscous Orthogonal MHD flows." Siam J. Appl. Math., Accepted for publication.

H. Grad (1960) "Reducible problems in magneto fluid dynamic steady flows." Rev. Mod. Phys., 32, pp. 830-847.

R. M. Gunderson (1966) "Steady two-dimensional magneto-hydrodynamic flow." Z. Angew. Math. Phys., 17, pp. 755-765.

J. G. Kingston and G. Power (1968) "An analysis of two-dimensional aligned field Magnetogasdynamic flows." Z. Angew. Math. Phys., 19, pp. 851-863.

J. G. Kingston and R. Talbot (1969) "The solutions to a class of Magneto-hydrodynamic flows with orthogonal magnetic and velocity field distributions." Z. Angew. Math. Phys., 20, pp. 956-965.

I. P. Ladikov (1962) "Properties of plane and axisymmetrical stationary flows in magnetohydrodynamics." J. Appl. Math. Mech., 26, pp. 1646-1652.

S. Lundquist (1952) "Studies in Magnetohydrodynamics." Arkiv F. Fysik, 5, pp. 297-347.

M. H. Martin (1971) "The flow of a viscous fluid." Arch. Rat. Mech. Anal., 41, pp. 266-246.

Shih-I Pai (1962) "Magnetogasdynamics and Plasma Physics." Prentice-Hall, Inc., Englewood Cliffs, N. J.

G. Power and R. Talbot (1969) "Magnetogasdynamic flows in two dimensions with orthogonal magnetic and velocity field distributions." Z. Angew. Math. Phys., 20, pp. 358-369.

G. Power and D. Walker (1965) "Plane gasdynamic flows with orthogonal magnetic and velocity field distributions." Z. Angew. Math. Phys., 16, pp. 803-817.

(1967) "Reduction of viscous flows having orthogonal magnetic and velocity field distributions." Appl. Sci. Res., 17, pp. 223-232.

K. B. Ranger (1969) "Exact solutions for rotating bodies in a viscous conducting fluid." Phys. Fluids, 12, pp. 776-777.

(1969) "Magnetohydrodynamic Stokes flows for rotating solids of revolution." Proc. Comb. Phil. Soc., 66, pp. 663-674.

E. L. Resler and J. E. McCune (1959)	"Some exact solutions in linearized magnetoaerodynamics for arbitrary magnetic Reynold numbers." Rev. Mod. Phys., 32, pp. 843-854.
W. R. Sears and E. L. Resler (1959)	"Theory of thin airofoils in fluids of high electrical conductivity." J. Fluid Mech., 5, pp. 257-273.
W. R. Sears (1959)	"Magnetohydrodynamic effects in aerodynamic flows." Amer. Rocket Soc. J., 29, pp. 397-406.
J. A. Shercliff (1953)	"Steady motion of conducting fluids in pipes under transverse magnetic fields." Proc. Comb. Phil. Soc., 55, pp. 141-143.
P. Smith (1963)	"Substitution principle for M.H.D. flows." J. Math. Mech., 12, pp. 505-520.
H. Toews and O. P. Chandna (1974)	"Steady transverse plane Magneto-gasdynamic flows." Tensor, 28, pp. 184-188.
(1974)	"Plane magnetofluiddynamic flows with constantly inclined magnetic and velocity fields." Can. J. Phys., 52, pp. 753-758.
(1975)	"Constantly inclined flows with isometric magnetic line pattern." Siam J. Appl. Math., Submitted for publication.
M. Vinokur (1961)	"Kinematic formulation of rotational flow in magnetogasdynamics." Lockheed Aircraft Corp., Tech. Report 6-90-61-10.
J. S. Waterhouse and J. G. Kingston (1973)	"Plane magnetohydrodynamics flows with constantly inclined magnetic and velocity fields." Z. Angew. Math. Phys., 24.

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