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A class of multi-server queueing systems with unreliable servers: Models and application.

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A class of multi-server queueing systems with unreliable servers: models and application

by

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ABSTRACT

In the literature on queueing systems, most of the papers assume that the servers are available all the time. But in realistic situations, there are many cases in which servers may fail and need to be repaired. For example, a machining center in a job shop may break down; a channel in a CDMA system may become noisy or lose a connection. The performance of a queueing system may be heavily affected by the server breakdown and limited repair capacity. So the research on queueing systems with server failures has to be done to meet the requirements of industries.

Where queueing systems with unreliable servers are concerned, most research that has been done focuses on one-server systems or systems with a Poisson arrival process and exponential service time. However, in some situations we need to consider non-exponential service time or service rate changes with the number of available servers. These are the queueing systems that are discussed in this thesis, none of which has ever been discussed in the literature.

First, the queueing system M/M/n with server failures leading to possible change of service rate is discussed. First, a mathematical model is built and the stability condition is analyzed. Then the matrix geometric method is used to obtain the stationary distribution. Computer programs are developed to implement calculation for the stability condition analysis, stationary distribution and performance measurements. Numerical examples are given to test the validation of the mathematical model and analysis thereafter.

Since the phase type distribution is more general than the exponential distribution and captures most features of a general distribution, the phase type distributed service time is considered in unreliable queueing systems such as M/PH/n and M/PH/n/c. For the M/PH/n queueing system with unreliable servers, the mathematical model, stability condition analysis, stationary distribution calculation, computer programs and examples are all presented. For the M/PH/n/c queueing system with server failures, a finite birth-and-death mathematical model is built and the stationary distribution and performance evaluation measurements are calculated. Computer programs are developed and an example is given to demonstrate the application of this queueing system.

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CHAPTER 1 INTRODUCTION

1.1 BACKGROUND

Queueing systems that are subject to service interruptions are commonly encountered in the real world. For example, a machining center in a job shop may break down due to a variety of reasons: preventive maintenance, power failure, tool replacement, raw material quality, and so on. A teller in a bank may be temporarily absent from his or her location in order to perform some background transactions. A queueing system with server interruptions is called an unreliable queueing system, a queueing system with server vacations, or a queueing system with server failures[5,6,12]. In this thesis, we call all these kinds of service interruptions server failures. We also name the corresponding queueing systems as queueing systems with server failures. Figure 1.1 illustrates the single-line queueing system with server failures[39].

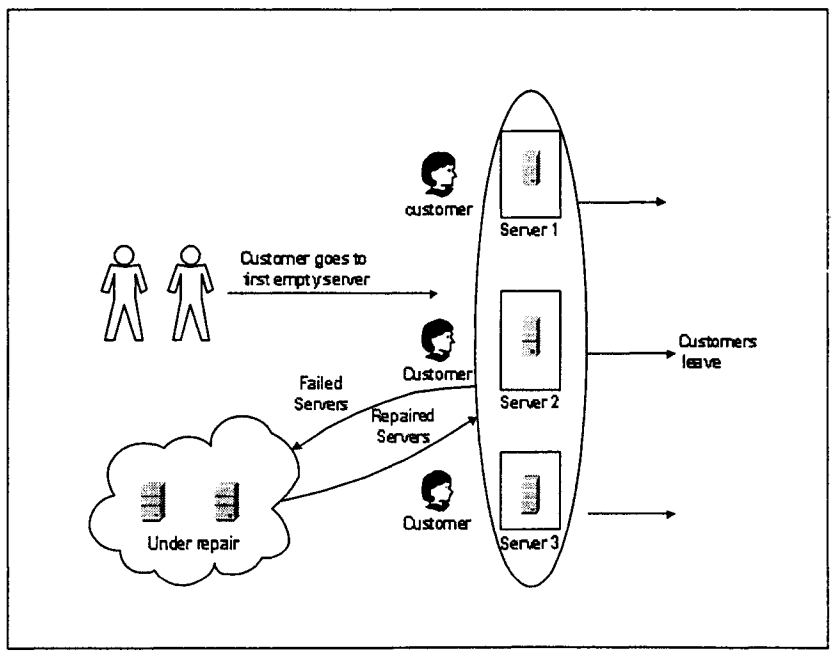

Figure 1.1 Single-line queueing system with server failures

We can describe a queueing system with server failures with the following notation, which uses two more characteristics than the notation Kendall devised in 1951.

$$1/2(7, 8)/3/4/5/6$$

The seventh characteristic specifies the nature of the server failure times. The eighth characteristic specifies the nature of the server repair times.

M = Server failure (repair) times are iid and exponentially distributed.

PH = Server failure (repair) times are iid and phase type distributed.

D = Server failure (repair) times are iid and deterministic.

G = Server failure (repair) times are iid and follow some general distribution.

For example, M/M(M,M)/n represents a queueing system with *n* servers, exponential interarrival times, exponentially distributed service times, exponentially distributed server failure times and exponentially distributed server repair times. M/PH(M,M)/n represents a queueing system with *n* servers, exponential interarrival times, phase type distributed service times, exponentially distributed server failure times and exponentially distributed server repair times. M/PH(M,M)/n/c represents a queueing system with *n* servers, exponential interarrival times, phase type distributed service times, exponentially distributed server failure times, exponentially distributed server repair times and a total capacity of *c* customers.

1.2 OBJECTIVES

Although research of queueing systems with server failures has been worked on for about half a century, some areas still need to be discussed to meet the requirements of industries. For instance, the service time of secretaries may be shorter if one of them is absent and

the customers are in a long line waiting to be served. The service time of a flexible manufacturing cell would be longer if one of the workstations is broken. So, the change of the service rate is considered for M/M/n queueing systems with server failures in this thesis. As a more general distribution, phase type distributed service time for multi-server queueing systems with server failures is discussed too. The objectives of the research are:

(1) To discuss the queueing system M/M/n with server failures when the service rate changes with the number of available servers.

(2) To discuss the queueing systems M/PH/n and M/PH/n/c with server failures.

(3) To build a mathematical model, to analyze the stability condition (for infinite capacity queueing systems only), to find a method to obtain the stationary distribution, and to calculate some performance measurements for every queueing system.

(4) To develop computer programs to implement the analysis and calculation in (3).

(5) To validate the stability condition analysis and calculation method by examples for every queueing system.

1.3 THESIS OVERVIEW

Chapter 1 provides the introduction of queueing systems with server failures. Different kinds of queueing systems are introduced. The objectives of this research are also presented.

In chapter 2, a thorough literature review of important research work related to queueing systems, especially to the unreliable server queueing systems, is provided.

Chapter 3 analyzes the M/M/n queueing system with server failures and different service rates compared to the M/M/n queueing system with server failures and the same service rate. Mathematical models are set up, stability conditions are analyzed, a stationary distribution calculation method is discussed, computer programs are developed, and application examples are presented.

Chapter 4 presents an analysis of M/PH/n and M/PH/n/c queueing systems. The assumptions, notation, model, stability condition analysis (for the M/PH/n queueing system), and performance measurement calculations are discussed. Computer programs and examples are also provided to demonstrate the analysis and calculation.

The thesis is concluded with a discussion of conclusions, contributions, and suggestions for further research work in chapter 5.

CHAPTER 2 LITERATURE REVIEW

The problem of queueing systems with server failures is of continuing interest to many researchers. Several models have been built and analyzed. For a survey of this work, emphasis has been given to the models themselves.

2.1 OVERVIEW OF QUEUEING SYSTEMS WITH SERVER FAILURES

The work of White and his team[38]

The problem of queueing systems with server failures was introduced when researchers studied the priority queueing systems.

White et al. (1958) introduced queueing systems with server interruptions. They pointed out the similarity of queueing with breakdown to queueing with preemptive priority and worked out two models of breakdown effects when they discussed the preemptive priority queueing system M/G/1 and found the steady state solutions to it. They assumed that the server breakdown distribution is the distribution of the time to failure, which is exponential (i.e., the breakdown process is a Poisson process), the distribution of repair times is exponential, and the interrupted item reenters the facility after repairs.

The work of Keilson and his team[14,15,16]

Keilson (1962) discussed the M/G (M, G)/1 queue with interruptions by server breakdowns or the arrival of customers with higher priority. The time dependent behavior of the system is discussed in a complete state space and the joint density of all system variables of this space is constructed systematically from the densities associated with a

set of simpler first-passage problems. Equilibrium distributions are available as limiting forms and server busy period distributions are obtained.

The work of Thiruvengadam[35]

Thiruvengadam (1963) considered a single-server M/G (M, G)/1 queueing system with server breakdowns. Three models were discussed. Model 1 permits a queue of breakdowns, Model 2 assumes that no latent breakdown can build up when the system is out of order, but still the breakdowns can happen anytime (i.e., when the server is busy or idle). Model 3 assumes that breakdowns are restricted to the server busy time.

The work of Avi-Itzhak and his team[3,4]

Avi-Itzhak et al. (1963) also discussed M/G (M, G)/1 queueing system with server breakdowns. Five models which deal with different assumptions about breakdown situations are considered.

Until this time, all research was focused on various single-server systems, while very little was known about the many-server systems with service interruptions.

The work of Mitrany and Avi-Itzhak[28]

Mitrany and Avi-Itzhak (1968) studied and analyzed a steady-state M/M/N queueing system where each server is subject to random breakdowns of exponentially distributed duration. The moment generating function of the queue size is obtained in explicit form

for N<=2. For larger values of N, a numerical method is suggested. They assumed that there is the same number of repairmen as the number of servers in the system.

The method used in their research is tedious. It is based on transforms and involves the determination of N-1 distinct roots of a certain polynomial in the interval (0, 1). Besides, generalization for an arbitrary number of servers is not straightforward. This prompted the need for alternative techniques that would simplify the analysis and provide significant computational improvements over the classical methods.

The work of Neuts and his team[29]

Neuts et al. (1979) also discussed the model of M/M (M, M)/N queue with server failures. In this system, it is explicitly pointed out and studied that there are c (c<=N) repair persons available to repair the breakdown servers. The new matrix geometric methodological results are discussed and the utility of interactive computation in answering questions on the behavior, design and control of certain service systems is demonstrated.

The work of Vinod and his team[36]

Vinod (1985) considered the M/M (M, M)/N model with ample repair (K=N). For N=1, he discussed two models which have different assumptions about the server down-periods, either independent of the queue length or only occurring when the server is active. The matrix geometric method is used here to demonstrate the computational tractability of an unreliable queueing system. More computational results are given in his research than in Neuts' research. In addition, numerical examples are given.

The work of Federgruen and his team[7,8,9,10]

Federgruen et al. (1986) discussed an M/G (G, G)/1 queueing system with server interruptions. Bounds and approximations for the mean waiting time, probability of delay and steady-state distribution of the number in system are derived. These results are exact for the case of an M/G (G, M)/1 queueing system. In 1988, Federgruen and Green presented an exact solution method for an M/G (PH, G)/1 queueing system with server breakdowns. In 1990, Federgruen and So studied the corrective maintenance policy that minimizes the long-run average operating costs of the M/G/1 queueing system with server breakdowns.

The work of Sengupta[31]

Sengupta (1990) considered a single server queue that operates in a random environment defined by an alternating renewal process with states 1 and 2. The arrival and service rates of the periods when the server is up can be different from those of the periods when the server is down. Thus, it is a generalization of other single server queueing system with server breakdowns. For an M/G (M, M)/1 queueing system, a closed form of the mean waiting time is obtained. But for an M/G (G, M)/1 system, only a calculable approximation is proposed and compared to the simulation results.

The work of Wartenhorst[37]

Wartenhorst (1995) studied the M / M (M, M) / N queueing system with K (K<N) parallel repairmen. There is a queue at each server. He was interested in the mutual influence of different queueing systems via the limited repair capacity. The marginal

queue length distribution and the conditional distributions of the queue length are given and an approximating model is also developed to save time and memory for calculation by giving some assumptions.

The work of Tang[32,33,34]

Tang (1995) studied an GI/G(M, G)/1 queueing system and some reliability quantities of the service station are obtained. Tang (1997) considered a single server M/G/1 queueing system with server breakdowns subject to different distributions according to the state (working or idle) of the server. Some reliability problems as well as queueing problems are studied and some transform results are obtained. In 1999, Tang considered an M/G/1 queueing system with a repairable station and server vacations. Some reliability problems of the service station were discussed.

The work of Li and his team[22]

Li et al. (1997) studied the PH/PH(M,PH)/1 queueing system. The steady state availability and steady state failure frequency of the server are obtained. Some reliability problems are also discussed. He also studied MAP/PH (M/PH)/2 queueing systems with one repair crew. The stable availability and the stable failed frequency of system, the distribution of time to failure and its mean are given.

The work of Yue and Cao[40]

Yue and Cao (1997) did some work on a priority queueing system $M_1^{x_1}$, $M_2^{x_2}$/G_1, G_2/1 with a repairable service station. In this system, the server has a constant failure rate and

arbitrary repair time distribution. Some reliability measurements are obtained via Laplace transform and some queueing measurements are also obtained.

The work of Hsieh and his team[13]

Hsieh et al. (1995) studied the M/M (M, M)/1 system with server breakdown. A simple partition balance approach (can only solve the queueing system with one server) is presented. The expressions for the availability, steady-state queue length distribution, mean queue length, and server utilization of the system subject to multi-mode, bi-level are derived and numerical examples are given.

The work of Zhang and his team[41,42,43,44]

Zhang et al. (1997) treated two-threshold policies for an M/G/1 queue with two types of generally distributed random vacations: type 1 (long) and type 2 (short) vacations, studied an M/G/1 queue with an exceptional first vacation in 1998, and considered a single server queueing system with Poisson arrivals and multiple vacation types, in which the server can choose one of several types of vacations to take when he finishes serving all customers in the system, in 2001. For every model, they tried to determine the optimal service policy to minimize the long-term average cost of this vacation system.

The work of Madan and his team[1,23-27]

Madan et al. (2001) studied a single server vacation queue with Poisson arrivals, deterministic service of constant duration b(> 0) and general vacations and designated this model as M/D/G/1 and obtained the time-dependent as well as steady state

probability generating functions for the number in the system in 2001. They obtained the steady state probability generating function of the queue length for various states of the server in an $M/G_1,G_2/D/1$ queue. They also studied a single server vacation queue with Poisson arrivals, deterministic service of constant duration b (> 0) and deterministic vacations of constant duration d (> 0) M/D/D/1, analyzed the steady state behavior of an M/D/1 queue with Bernoulli schedules and Coxian-2 server vacations, and investigated the steady state behavior of an M/G/1 queue with modified Bernoulli schedule server vacations in 2002.

2.2 SUMMARY OF QUEUEING SYSTEMS WITH SERVER FAILURES

The Summary of the literature review of the queueing systems with server failures is illustrated in table 2.1 on page 12.

2.3 DISCUSSION

This chapter began with the literature overview of queueing systems with server failures. A detailed literature review was presented first. Then a table of the summary of the literature overview was provided at the end of this chapter. Many queueing systems have been covered in the literature. However, for some realistic problems, queueing systems with different service rates or non-exponential service times may need to be considered. These are queueing systems that will be discussed in the next two chapters.

Table 2.1 Summary of queueing systems with server failures

Researchers	Queueing systems	Objectives
White et al. (1958)	M/G/1	Pointed out the similarity of queueing with breakdowns to queueing with preemptive priority
Keilson et al. (1962)	M/G (M, G)/1	Discussed the method to obtain the joint density, equilibrium distributions, and server busy period distributions.
Thiruvengadam (1963)	M/G (M, G)/1	Discussed three queueing systems models with different assumptions about server failure styles.
Avi-Itzhak et al. (1963)	M/G (M, G)/1	Considered five models which deal with different assumptions about the server breakdown situations.
Mitrany and Avi-Itzhak (1968)	M/M/N	Obtained the moment generating function of the queue size in explicit form for N<=2; Suggested a numerical method for larger values of N.
Neuts et al. (1979)	M/M (M, M)/N	Pointed out the number of repair persons c (c<=N) in the queueing systems; Discussed the new matrix geometric methodological results; Demonstrated the utility of interactive computation in analysis of the behavior, design and control of certain service systems.
Vinod (1985)	M/M (M, M)/N	Discussed two models with different assumptions about the server down-periods for N=1; Gave mo re computational results and numerical examples.
Federgruen et al. (1986-1990)	M/G (G, G)/1 M/G (PH, G)/1	Derived the mean waiting time, probability of delay and steady-state distribution; Studied the corrective maintenance policy to minimize the long-run average operating costs.
Sengupta (1990)	M/G (M, M)/1 M/G (G, M)/1	Obtained the closed form of mean waiting time for M/G (M, M)/1; Proposed a calculable approximation for M/G (G, M)/1.
Wartenhorst (1995)	M/M (M, M)/N	Gave the marginal queue length distribution and the conditional distributions; Developed an approximating model to save time and memory for calculation by giving some assumptions.
Tang (1995,1997,1999)	M/G/1	Studied some reliability problems as well as queueing problems; Obtained some transform results.
Li et al. (1997)	PH/PH(M,PH)/1	Obtained the steady availability and steady failure frequency of the server.
	MAP/PH (M/PH)/2	Gave the stable availability and the stable failed frequency of the system.
Yue and Cao (1997)	$M_1^{x_1}, M_2^{x_2}/G_1, G_2/1$	Obtained some reliability measurements in Laplace transform and some queueing measurements.
Hsieh et al. (1995)	M/M (M, M)/1	Derived the expressions for the availability, steady-state queue length distribution, mean queue length, and server utilization of the system.
Zhang (1997-2001)	M/G/1	Considered different server vacations type, determined the optimal service policy to minimize the long-term average cost of this vacation system.
Madan (2001-2002	M/D/G/1, $M/G_1,G_2/D/1$, M/D/D/1, M/G/1	Obtained the time-dependent as well as the steady state behavior of the server for every queueing system.

CHAPTER 3 M/M/n QUEUEING SYSTEM WITH SERVER FAILURES

The simplest and most extensively studied queueing models are those having a Poisson arrival process and exponentially distributed service times. In these cases the queue size forms a birth and death process, and the corresponding stationary distribution is readily found. In this chapter, the reliability of the servers is considered. The M/M(M, M)/1, M/M(M, M)/2 and M/M(M, M)/n queueing systems (We use M/M/1, M/M/2 and M/M/n respectively later for abbreviation) are discussed.

Notation in this chapter:

- λ Intensity or rate of the Poisson arrival process
- μ_i Parameter of the exponential distribution of the service time
- θ Parameter of the exponential distribution of the server failure time
- γ Parameter of the exponential distribution of the server repair time
- L Mean queue length
- W Mean of the exponential waiting time distribution
- n Number of the servers
- k Number of the repair persons $(1<=k<=n)$
- π Limiting or equilibrium distribution of queue length

Assumptions:

- The servers cannot help each other, even if some of them are idle.

- A server can fail when it is idle (since failure here refers to non-working condition of any reason) unless other assumptions are mentioned explicitly.
- Infinite capacity of the system.
- *k(1<=k<=n)* repair persons

3.1 MODELING OF M/M(M, M)/n QUEUEING SYSTEM

3.1.1 M/M/n Queueing System with the Same Service Rates

The M/M/1, M/M/2 and M/M/n queueing systems are discussed respectively. When we consider the server failures, the infinitesimal matrices will have the following structures. In the matrices, an element is omitted if it is 0.

3.1.1.1 M/M/1 queueing system with server failures

For the M/M/1 queueing system with server failures, the state space of the system is *E={(i, j); i>=0, j=0,1}*. the infinitesimal matrix is the following matrix (the server can fail when the system is empty):

$$\begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \cdots \\ & A_2 & A_1 & A_0 & \\ & & & \cdots & \end{bmatrix},$$

where

$$B_0 = \begin{bmatrix} -(\gamma+\lambda) & \gamma \\ \theta & -(\theta+\lambda) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\gamma+\lambda) & \gamma \\ \theta & -(\mu+\theta+\lambda) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & \\ & \mu \end{bmatrix}.$$

All the matrices are 2×2 matrices.

If we consider the case where the server cannot fail when the system is empty, the state space of the system is *E={(i, j); i>=0, j=0,1}*. The infinitesimal matrix becomes the following:

$$\begin{bmatrix} B & C & & & \\ E & A_1 & A_0 & & \cdots \\ & A_2 & A_1 & A_0 & \\ & & & \cdots & \end{bmatrix},$$

where

$$B = -\lambda,$$

$$C = [0 \quad \lambda],$$

$$E = \begin{bmatrix} 0 \\ \mu \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\gamma+\lambda) & \gamma \\ \theta & -(\mu+\theta+\lambda) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & \\ & \mu \end{bmatrix}.$$

3.1.1.2 M/M/2 system with server failures

If there is only one repair person, the state space of the M/M/2 queueing system is $E=\{(i, j);\ i>=0,\ j=0,1,2\}$. The infinitesimal matrix for the system with server failures is as follows:

$$\begin{bmatrix} B_{00} & A_0 & & \\ B_{10} & B_{11} & A_0 & \cdots \\ & A_2 & A_1 & A_0 \\ & & \cdots & \end{bmatrix},$$

where

$$B_{00} = \begin{bmatrix} -(\gamma+\lambda) & \gamma & \\ \theta & -(\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(2\theta+\lambda) \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} & & \\ & \mu & \\ & & \mu \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -(\gamma+\lambda) & \gamma & \\ \theta & -(\mu+\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(\mu+2\theta+\lambda) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\gamma+\lambda) & \gamma & \\ \theta & -(\mu+\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(2\mu+2\theta+\lambda) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & \mu & \\ & & 2\mu \end{bmatrix}.$$

All the matrices are 3×3 matrices.

If there are two repair persons, the state space of the M/M/2 queueing system is $E=\{(i, j);\ i>=0, j=0,1,2\}$. The infinitesimal matrix for the system with server failures is as follows:

$$\begin{bmatrix} B_{00} & A_0 & & \\ B_{10} & B_{11} & A_0 & \cdots \\ & A_2 & A_1 & A_0 \\ & & \cdots & \end{bmatrix},$$

where

$$B_{00} = \begin{bmatrix} -(\gamma+2\lambda) & 2\gamma & \\ \theta & -(\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(2\theta+\lambda) \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} & \mu & \\ & & \mu \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -(2\gamma+\lambda) & 2\gamma & \\ \theta & -(\mu+\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(\mu+2\theta+\lambda) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(2\gamma+\lambda) & 2\gamma & \\ \theta & -(\mu+\theta+\gamma+\lambda) & \gamma \\ & 2\theta & -(2\mu+2\theta+\lambda) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & & \\ & \mu & \\ & & 2\mu \end{bmatrix}.$$

All the matrices are 3×3 matrices.

3.1.1.3 M/M/n system with server failures[29]

If there is only one repair person in the system, the state space of the system is $E=\{(i, j);\ i>=0,\ j=0,1,2,\ \ldots,\ n\}$. The infinitesimal matrix for the M/M/n queueing system with server failures is as follows, for each block matrix B_{ij}, $i = 0,1,2,\cdots$, $j = 0,1,2,\cdots$.

$$\begin{bmatrix} B_{00} & A_0 & & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & & \\ & & \cdots & & & & & & \\ & & & B_{n-1,n-2} & B_{n-1,n-1} & A_0 & & & \\ & & & & B_{n,n-1} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \cdots & \end{bmatrix},$$

where the block matrices are as follows

$$B_{i,i-1} = \begin{bmatrix} & & & & & & \\ \mu & & & & & & \\ & 2\mu & & & & & \\ & & \cdots & & & & \\ & & & (i-1)\mu & & & \\ & & & & i\mu & & \\ & & & & & \cdots & \\ & & & & & & i\mu \end{bmatrix},$$

$$B_{i,i} = \begin{bmatrix} -(\gamma+\lambda) & \gamma & & & & & \\ \theta & -(\mu+\theta+\gamma+\lambda) & \gamma & & & & \\ & & \cdots & & & & \\ & & i\theta & -(i\mu+i\theta+\gamma+\lambda) & & & \\ & & & \cdots & & & \\ & & & & -(i\mu+(n-1)\theta+\gamma+\lambda) & \gamma & \\ & & & & n\theta & -(i\mu+n\theta+\lambda) & \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \cdots & \\ & & & & \lambda \end{bmatrix},$$

$$A_1 = B_{nn},$$

$$A_2 = B_{n,n-1}.$$

All the block matrices are $(n+1)\times(n+1)$ matrices, corresponding to the number of working servers 0, 1, 2, …, n.

If there are *k (1<k<=n)* repair persons in the system, the infinitesimal matrix for the M/M/n queueing system with server failures is as follows, for each block matrix B_{ij}, $i = 0,1,2,\cdots$, $j = 0,1,2,\cdots$.

$$
\begin{bmatrix}
B_{00} & A_0 & & & & & & & \\
B_{10} & B_{11} & A_0 & & & & & & \\
 & & \cdots & & & & & & \\
 & & & B_{n-1,n-2} & B_{n-1,n-1} & A_0 & & & \\
 & & & & B_{n,n-1} & A_1 & A_0 & & \\
 & & & & & A_2 & A_1 & A_0 & \\
 & & & & & & A_2 & A_1 & A_0 \\
 & & & & & & & \cdots &
\end{bmatrix},
$$

where the block matrices are as follows

$$
B_{i,i-1} = \begin{bmatrix}
\mu & & & & & & \\
 & 2\mu & & & & & \\
 & & \cdots & & & & \\
 & & & (i-1)\mu & & & \\
 & & & & i\mu & & \\
 & & & & & \cdots & \\
 & & & & & & i\mu
\end{bmatrix},
$$

$$
B_{i,i} = \begin{bmatrix}
-(k\gamma+\lambda) & k\gamma & & & & & \\
\theta & -(\mu+\theta+\min(n-1,k)\gamma+\lambda) & \min(n-1,k)\gamma & & & & \\
 & & \cdots & & & & \\
 & & & i\theta & -(i\mu+i\theta+\min(n-i,k)\gamma+\lambda) & & \\
 & & & & \cdots & & \\
 & & & & & -(i\mu+(n-1)\theta+\gamma+\lambda) & \gamma \\
 & & & & & n\theta & -(i\mu+n\theta+\lambda)
\end{bmatrix},
$$

$$
A_0 = \begin{bmatrix}
\lambda & & & & \\
 & \lambda & & & \\
 & & \lambda & & \\
 & & & \cdots & \\
 & & & & \lambda
\end{bmatrix},
$$

$A_1 = B_{nn}$,

$A_2 = B_{n,n-1}$.

All the block matrices are $(n+1)\times(n+1)$ matrices, corresponding to the number of working servers 0, 1, 2, …, n.

3.1.2 M/M/n Queueing System with Different Service Rates

So far we have assumed that service rates are independent of the state of the system. Sometimes as fewer servers become available, the remaining servers start to work faster or slower. Hence service rates may change with the state of the system. We have to consider the change of the service rate of servers. The following part is the discussion about the M/M/n system with different service rates when the number of available servers changes. The service rate is μ_i when the number of available servers is i.

If there is only one repair person in the system, the state space of the system is $E=\{(i, j);\ i>=0,\ j=0,1,2,\ \ldots,\ n\}$. The infinitesimal matrix for the M/M/n queueing system with different service rates and server failures is as follows, for each block matrix B_{ij}, $i=0,1,2,\cdots,\ j=0,1,2,\cdots$.

$$\begin{bmatrix} B_{00} & A_0 & & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & & \\ & & \cdots & & & & & & \\ & & & B_{n-1,n-2} & B_{n-1,n-1} & A_0 & & & \\ & & & & B_{n,n-1} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \cdots & \end{bmatrix},$$

where the block matrices are as follows

$$B_{i,i-1} = \begin{bmatrix} & & & & & & & \\ & \mu_1 & & & & & & \\ & & 2\mu_2 & & & & & \\ & & & \cdots & & & & \\ & & & & (i-1)\mu_{i-1} & & & \\ & & & & & i\mu_i & & \\ & & & & & & \cdots & \\ & & & & & & & i\mu_i \end{bmatrix},$$

$$B_{i,i} = \begin{bmatrix} -(\gamma+\lambda) & \gamma & & & & & \\ \theta & -(\mu_1+\theta+\gamma+\lambda) & \gamma & & & & \\ & & \cdots & & & & \\ & & i\theta & -(i\mu_i+i\theta+\gamma+\lambda) & & & \\ & & & & \cdots & & \\ & & & & & -(i\mu_i+(n-1)\theta+\gamma+\lambda) & \gamma \\ & & & & & s\theta & -(i\mu_i+n\theta+\lambda) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \cdots & \\ & & & & \lambda \end{bmatrix},$$

$A_1 = B_{nn}$,

$A_2 = B_{n,n-1}$.

All the block matrices are $(n+1)\times(n+1)$ matrices, corresponding to the number of working servers 0, 1, 2, $\ldots n$.

If we consider the case that there are k $(1<k<=n)$ repair persons in the system, the state space of the system is $E=\{(i, j);\ i>=0,\ j=0,1,2,\ \ldots,\ n\}$. The infinitesimal matrix for the M/M/n queueing system with different service rates and server failures is as follows, for each block matrix B_{ij}, $i = 0,1,2,\cdots$, $j = 0,1,2,\cdots$.

$$
\begin{bmatrix}
B_{00} & A_0 & & & & & & & \\
B_{10} & B_{11} & A_0 & & & & & & \\
 & & \cdots & & & & & & \\
 & & & B_{n-1,n-2} & B_{n-1,n-1} & A_0 & & & \\
 & & & & B_{n,n-1} & A_1 & A_0 & & \\
 & & & & & A_2 & A_1 & A_0 & \\
 & & & & & & A_2 & A_1 & A_0 \\
 & & & & & & & \cdots &
\end{bmatrix},
$$

where the block matrices are as follows

$$
B_{i,i-1} = \begin{bmatrix}
 & & & & & & \\
 & \mu_1 & & & & & \\
 & & 2\mu_2 & & & & \\
 & & & \cdots & & & \\
 & & & & (i-1)\mu_{i-1} & & \\
 & & & & & i\mu_i & \\
 & & & & & & \cdots \\
 & & & & & & i\mu_n
\end{bmatrix},
$$

$$
B_{i,i} = \begin{bmatrix}
-(k\gamma+\lambda) & k\gamma & & & & & \\
\theta & -(\mu_1+\theta+\min(i-1,k)\gamma+\lambda) & \min(i-1,k)\gamma & & & & \\
 & & \cdots & & & & \\
 & & & i\theta & -(i\mu_i+i\theta+\min(i-i,k)\gamma+\lambda) & & \\
 & & & & \cdots & & \\
 & & & & & -(i\mu_{n-1}+(n-1)\theta+\gamma+\lambda) & \gamma \\
 & & & & & n\theta & -(i\mu_n+n\theta+\lambda)
\end{bmatrix},
$$

$$
A_0 = \begin{bmatrix}
\lambda & & & & \\
 & \lambda & & & \\
 & & \lambda & & \\
 & & & \cdots & \\
 & & & & \lambda
\end{bmatrix},
$$

$A_1 = B_{nn}$,

$A_2 = B_{n,n-1}$.

All the block matrices are $(n+1)\times(n+1)$ matrices, corresponding to the number of working servers 0, 1, 2, ..., n.

3.2 STABILITY CONDITION

To evaluate the stationary distributions for the M/M/n queueing system with server failures and different service rates, the stability condition has to be established first. If there is only one repair person in the system, the stability condition is as follows. We define $\pi_i = P($ i servers are available). Let $A = A_0 + A_1 + A_2$ and $\pi A = 0$, $\pi e = 1$, where $\pi = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \cdots \quad \pi_n]$.

Then

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} -\gamma & \gamma & & & & & \\ \theta & -(\theta+\gamma) & \gamma & & & & \\ & & \cdots & & & & \\ & & & i\theta & -(i\theta+\gamma) & & \\ & & & & \cdots & & \\ & & & & & -((n-1)\theta+\gamma) & \gamma \\ & & & & & n\theta & -n\theta \end{bmatrix}.$$

The balance equations are

$$\pi_0(-\gamma) + \pi_1(\theta) = 0, \tag{1}$$

$$\pi_0(\gamma) + \pi_1(-\theta-\gamma) + \pi_2(2\theta) = 0, \tag{2}$$

$$\pi_1(\gamma) + \pi_2(-2\theta-\gamma) + \pi_3(3\theta) = 0, \tag{3}$$

$$\ldots,$$

$$\pi_{n-1}(\gamma) + \pi_n(-n\theta) = 0. \tag{4}$$

From (1), we can get:

$$\pi_1 = \pi_0(\gamma)(\theta)^{-1} = \pi_0 \tfrac{\gamma}{\theta} = \tfrac{\gamma}{\theta}\pi_0. \tag{5}$$

Substituting (5) into (2), we can get:

$$\pi_2 = \frac{\gamma^2}{2\theta^2}\pi_0. \tag{6}$$

Substituting (5), (6) into (3),

$$\pi_3 = \frac{\gamma^3}{6\theta^3}\pi_0. \tag{7}$$

From equations (5), (6)and (7), we can know that

$$\pi_i = \frac{1}{i!}(\frac{\gamma}{\theta})^i \pi_0. \tag{8}$$

We know that

$$\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_n = 1. \tag{9}$$

Substituting (5), (6),(7) and (8) into (9), give us

$$\sum_{i=0}^{n}\frac{1}{i!}(\frac{\gamma}{\theta})^i \pi_0 = 1 \quad \text{and} \tag{10}$$

$$\pi_0 = [\sum_{i=0}^{n}\frac{1}{i!}(\frac{\gamma}{\theta})^i]^{-1}.$$

Now $\pi A_0 e = \pi(\lambda I)e = \lambda \pi e = \lambda,$ (11)

$$\pi A_2 e = \pi\begin{pmatrix} o \\ \mu_1 \\ \vdots \\ n\mu_s \end{pmatrix} = \begin{pmatrix}\pi_0 & \pi_1 & \cdots & \pi_n\end{pmatrix}\begin{pmatrix} o \\ \mu_1 \\ \vdots \\ n\mu_n \end{pmatrix}$$

$$= \pi_1\mu_1 + 2\pi_2\mu_2 + \cdots + n\pi_n\mu_n \tag{12}$$

$$= \frac{\sum_{i=0}^{n}\frac{i}{i!}(\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{n}\frac{1}{i!}(\frac{\gamma}{\theta})^i},$$

where e is the column vector with all entries 1.

So, according to the conclusion of page 83 of the reference [30], the stability condition for the M/M/n with server failures and one repair person is

$$\lambda < \frac{\sum_{i=0}^{n} \frac{i}{i!} (\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{n} \frac{1}{i!} (\frac{\gamma}{\theta})^i}. \tag{13}$$

If $\mu_i = \mu$, and $n = 1$, this is one-server queueing system. Then the right hand side of (13) is

$$\frac{\sum_{i=0}^{1} \frac{i}{i!} (\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{1} \frac{1}{i!} (\frac{\gamma}{\theta})^i} = \frac{\frac{\gamma}{\theta}\mu}{1+\frac{\gamma}{\theta}} = \frac{\gamma}{\theta+\gamma}\mu .$$

Then the stability condition is $\lambda < \frac{\gamma}{\theta+\gamma}\mu$. That is, the arrival rate is less than the average service rate.

If $\mu_i = \mu$, and $n \to \infty$, this is a self-service queueing system. Then the right hand side of (13) is

$$\frac{\sum_{i=0}^{\infty} \frac{i}{i!} (\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{\infty} \frac{1}{i!} (\frac{\gamma}{\theta})^i} = \frac{\frac{\gamma}{\theta}\mu e^{\frac{\gamma}{\theta}}}{e^{\frac{\gamma}{\theta}}} = \frac{\gamma}{\theta}\mu .$$

Then the stability condition is $\lambda < \frac{\gamma}{\theta}\mu$.

If there are n repair persons in the system, the stability condition is as follows.

Let $A = A_0 + A_1 + A_2$ and $\pi A = 0$, $\pi e = 1$, where $\pi = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \cdots \quad \pi_n]$ and e is the column vector with all entries 1.

Then

$$A = \begin{bmatrix} -n\gamma & n\gamma & & & & & \\ \theta & -(\theta + (n-1)\gamma) & (n-1)\gamma & & & & \\ & & \cdots & & & & \\ & & & i\theta & -(i\theta + (n-i)\gamma) & & \\ & & & & \cdots & & \\ & & & & & -((n-1)\theta + \gamma) & \gamma \\ & & & & & n\theta & -n\theta \end{bmatrix}$$

.

The balance equations are

$$\pi_0(-n\gamma) + \pi_1(\theta) = 0, \tag{21}$$

$$\pi_0(n\gamma) + \pi_1(-\theta - (n-1)\gamma) + \pi_2(2\theta) = 0, \tag{22}$$

$$\pi_1(n-1)\gamma + \pi_2(-2\theta - (n-2)\gamma) + \pi_3(3\theta) = 0, \tag{23}$$

$$\ldots,$$

$$\pi_{n-1}(\gamma) + \pi_n(-n\theta) = 0. \tag{24}$$

From (21), we can get:

$$\pi_1 = \pi_0(n\gamma)(\theta)^{-1} = \tfrac{n\gamma}{\theta}\pi_0. \tag{25}$$

Substituting (25) into (22), we can get:

$$\pi_2 = \frac{n(n-1)\gamma^2}{2\theta^2}\pi_0 = \binom{n}{2}\frac{\gamma^2}{\theta^2}\pi_0. \tag{26}$$

Substituting (25), (26) to (23),

$$\pi_3 = \binom{n}{3}\frac{\gamma^2}{\theta^2}\pi_0. \tag{27}$$

From equations (25), (26) and (27), we find that

$$\pi_i = \binom{n}{i}(\frac{\gamma}{\theta})^i \pi_0 . \tag{28}$$

We know that

$$\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_n = 1 . \tag{29}$$

Substituting (25), (26),(27) and (28) into (29), gives us

$$\sum_{i=0}^{n}\binom{n}{i}(\frac{\gamma}{\theta})^i \pi_0 = 1 , \tag{30}$$

$$\pi_0 = [\sum_{i=0}^{n}\binom{n}{i}(\frac{\gamma}{\theta})^i]^{-1} .$$

Now, $\pi A_0 e = \pi(\lambda I)e = \lambda \pi e = \lambda$, and (31)

$$\pi A_2 e = \pi \begin{pmatrix} o \\ \mu_1 \\ \vdots \\ n\mu_n \end{pmatrix} = (\pi_0 \quad \pi_1 \quad \cdots \quad \pi_n)\begin{pmatrix} o \\ \mu_1 \\ \vdots \\ n\mu_n \end{pmatrix}$$

$$= \pi_1\mu_1 + 2\pi_2\mu_2 + \cdots + n\pi_n\mu_n \tag{32}$$

$$= \frac{\sum_{i=0}^{n} i\binom{n}{i}(\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{n}\binom{n}{i}(\frac{\gamma}{\theta})^i} .$$

So, the stability condition for the M/M/n queueing system with server failures and n repair persons is

$$\lambda < \frac{\sum_{i=0}^{n} i\binom{n}{i}(\frac{\gamma}{\theta})^i \mu_i}{\sum_{i=0}^{n}\binom{n}{i}(\frac{\gamma}{\theta})^i} . \tag{33}$$

3.3 STATIONARY DISTRIBUTION

From part 3.1, we know that the infinitesimal matrix of the M/M/n queueing system with server failures leading to possible change of service rate is as follows:

$$\begin{bmatrix} B_{00} & A_0 & & & & & & & \\ B_{10} & B_{11} & A_0 & & & & & & \\ & & \cdots & & & & & & \\ & & & B_{n-1,n-2} & B_{n-1,n-1} & A_0 & & & \\ & & & & B_{n,n-1} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \cdots & \end{bmatrix}.$$

From the structure of the infinitesimal matrix, we know that this queueing system is a Quasi-Birth-and-death (QBD) process. We can use the matrix geometric methods (page 83 of reference [30]) to obtain the stationary distribution. The method is as shown in table 3.1.

Table 3.1 Matrix geometric method for QBD processes

Step 1. Generate the infinitesimal matrix Q.
Step 2. Stability condition analysis.
Step 3. Calculate matrix R from $0 = A_0 + RA_1 + R^2 A_2$.
Step 4. Calculate stationary distribution vector X from $Xe=1$, boundary matrix $B[R]$, and the relationships of X_i $(i>=n)$.
Step 5. Calculate the performance measurements such as server utilization, mean queue length and mean waiting time.
Step 6. Analysis and discussion.

We have discussed the infinitesimal matrix Q and stability condition already. Here we start to discuss the method to obtain matrix R and stationary distribution vector X. From reference [30], we know that R is the minimal nonnegative solution of the matrix-quadratic equation $0 = A_0 + RA_1 + R^2 A_2$. Here we evaluate the matrix R numerically. To facilitate the computation, first we let R_1 be a zero matrix of order n+*1*. Then we compute $R_{k+1} = -(A_0 + R_k{}^2 A_2)A_1^{-1}$, where k is the notation of iteration. After every iteration, we compare R_{k+1} and R_k, until each element of the difference matrix is less than a very small number ε. Then we let $R = R_{k+1}$.

We can get the boundary matrix *B[R]* by calculating the block matrices B_{ij} in the infinitesimal matrix of the M/M/n queueing system in part 3.1 for i = 0, 1, 2, ..., n, and j = 0, 1, 2, ..., n. Be careful that $B_{nn} = A_1 + RA_2$, not A_1.

Since $X_{i+1} = X_i R$ for $i \geq n$,

$$\begin{aligned}\sum X_i e &= (X_0 + X_1 + X_2 + \cdots + X_n)e + (X_{n+1} + X_{n+2} + \cdots)e \\ &= (X_0 + X_1 + X_2 + \cdots + X_n)e + (X_n R + X_n R^2 + \cdots)e \\ &= (X_0 + X_1 + X_2 + \cdots + X_n)e + X_n (I - R)^{-1} e = 1,\end{aligned}$$

where e is the column vector with all entries 1.

We can combine ***0***=*XB[R]* and $(X_0 + X_1 + X_2 + \cdots + X_n)e + X_n(I - R)^{-1}e = 1$ to obtain the first *(n+1)(n+1)* dimensions of the stationary distribution vector *X*. Then obtain the other dimensions of X by the relationship of $X_{i+1} = X_i R$.

Then we can get the mean queue length

$$
\begin{aligned}
L &= \sum iX_i e \\
&= \sum_{1}^{n-1} iX_i e + nX_n e + (n+1)X_n \mathrm{Re} + (n+2)X_n R^2 e + \cdots \\
&= \sum_{1}^{n-1} iX_i e + nX_n (I + R + R^2 + \cdots)e + X_n (R + 2R^2 + \cdots)e \\
&= \sum_{1}^{n-1} iX_i e + nX_n (I - R)^{-1} e + X_n R[(I - R)^{-1}]^2 e,
\end{aligned}
$$

the mean waiting time $W = L / \lambda$ and the server utilization $1 - X_0 e$.

3.4 NUMERICAL EXAMPLES

Computer programs (MATLAB) have been written to implement the calculation to check the stability condition, to obtain the stationary distribution, the mean queue length and the mean waiting time. Here are some results obtained from the computer programs.

The initial parameters:

Customer arrival rate $\lambda = 0.01$;

Service rates $\mu_i = \mu = 0.05$;

Number of servers n;

Number of repair persons k $(1<=k<=n)$;

Server failure rate θ;

Server repair rate γ.

Here are the sample results from MATLAB programs for n = 3 and k = 2. Table 3.2 illustrates some results obtained for the M/M/n queueing system with unreliable servers.

This system is a stable system, continue!

X =(0.0009 0.0184 0.1843 0.6146 0.0002 0.0037 0.0369 0.1231

0.0000 0.0004 0.0037 0.0124 0.0000 0.0000 0.0003 0.0009...)

meanlength = 0.2012

meanwaitingtime= 20.1194

We can find that when the server number is the same, the mean queue length and the mean waiting time decrease when the number of repair persons increases.

Table3.2 MATLAB program results of M/M/n queueing system with unreliable servers

Number of servers n	Number of repair persons k	Failure rate θ	Repair rate γ	Mean queue length L	Mean waiting time W
3	1	0.01	0.10	0.2032	20.3176
			0.15	0.2013	20.1308
			0.20	0.2008	20.0754
			0.25	0.2005	20.0518
		0.05	0.10	0.2799	27.9891
			0.15	0.2337	23.3682
			0.20	0.2178	21.7801
			0.25	0.2108	21.0796
		0.10	0.10	0.4423	44.2316
			0.15	0.3158	31.5825
			0.20	0.2666	26.6618
			0.25	0.2426	24.2560
		0.01	0.10	0.2013	20.1301
			0.15	0.2007	20.0675
			0.20	0.2005	20.0454
			0.25	0.2003	20.0347
			0.10	0.2237	22.3705

Table 3.2 (Cont'd)

3	2	0.05	0.15	0.2105	21.0492
			0.20	0.2059	20.5920
			0.25	0.2038	20.3839
		0.10	0.10	0.2774	27.7384
			0.15	0.2367	23.6705
			0.20	0.2212	22.1241
			0.25	0.2138	21.3807
	3	0.01	0.10	0.2012	20.1185
			0.15	0.2006	20.0642
			0.20	0.2004	20.0440
			0.25	0.2003	20.0340
		0.05	0.10	0.2175	21.7465
			0.15	0.2082	20.8245
			0.20	0.2049	20.4865
			0.25	0.2033	20.3261
		0.10	0.10	0.2537	25.3670
			0.15	0.2267	22.6724
			0.20	0.2161	21.6054
			0.25	0.2108	21.0755
	1	0.01	0.10	0.2011	20.1113
			0.15	0.2003	20.0319
			0.20	0.2001	20.0141
			0.25	0.2001	20.0078
		0.05	0.10	0.2714	27.1429
			0.15	0.2263	22.6299
			0.20	0.2120	21.2031
			0.25	0.2064	20.6355
		0.10	0.10	0.4361	43.6099
			0.15	0.3085	30.8475
			0.20	0.2590	25.9005
			0.25	0.2353	23.5295
		0.01	0.10	0.2002	20.0237
			0.15	0.2001	20.0089
			0.20	0.2000	20.0048

Table 3.2 (Cont'd)

			0.25	0.2000	20.0032
	2	0.05	0.10	0.2144	21.4357
			0.15	0.2049	20.4905
			0.20	0.2022	20.2242
			0.25	0.2012	20.1221
4		0.10	0.10	0.2626	26.2560
			0.15	0.2254	22.5433
			0.20	0.2127	21.2704
			0.25	0.2072	20.7222
	3	0.01	0.10	0.2002	20.0168
			0.15	0.2001	20.0071
			0.20	0.2000	20.0041
			0.25	0.2000	20.0028
		0.05	0.10	0.2072	20.7222
			0.15	0.2026	20.2649
			0.20	0.2013	20.1292
			0.25	0.2007	20.0745
		0.10	0.10	0.2314	23.1411
			0.15	0.2129	21.2929
			0.20	0.2066	20.6630
			0.25	0.2039	20.3885
	4	0.01	0.10	0.2002	20.0162
			0.15	0.2001	20.0070
			0.20	0.2000	20.0041
			0.25	0.2000	20.0028
		0.05	0.10	0.2060	20.6032
			0.15	0.2023	20.2316
			0.20	0.2012	20.1164
			0.25	0.2007	20.0685
		0.10	0.10	0.2249	22.4879
			0.15	0.2106	21.0630
			0.20	0.2056	20.5606
			0.25	0.2034	20.3358
			0.10	0.2005	20.0485

Table 3.2 (Cont'd)

5	1	0.01	0.15	0.2001	20.0098
			0.20	0.2000	20.0033
			0.25	0.2000	20.0015
		0.05	0.10	0.2687	26.8657
			0.15	0.2233	22.3316
			0.20	0.2096	20.9590
			0.25	0.2045	20.4520
		0.10	0.10	0.4350	43.5028
			0.15	0.3066	30.6595
			0.20	0.2566	25.6553
			0.25	0.2326	23.2616
	2	0.01	0.10	0.2001	20.0054
			0.15	0.2000	20.0014
			0.20	0.2000	20.0006
			0.25	0.2000	20.0003
		0.05	0.10	0.2107	21.0672
			0.15	0.2029	20.2894
			0.20	0.2011	20.1081
			0.25	0.2005	20.0496
		0.10	0.10	0.2577	25.7749
			0.15	0.2211	22.1147
			0.20	0.2094	20.9384
			0.25	0.2047	20.4742
	3	0.01	0.10	0.2000	20.0027
			0.15	0.2000	20.0008
			0.20	0.2000	20.0004
			0.25	0.2000	20.0002
		0.05	0.10	0.2039	20.3853
			0.15	0.2011	20.1075
			0.20	0.2004	20.0424
			0.25	0.2002	20.0205
		0.10	0.10	0.2236	22.3637
			0.15	0.2081	20.8091
			0.20	0.2035	20.3520

Table 3.2 (Cont'd)

			0.25	0.2018	20.1786
	4	0.01	0.10	0.2000	20.0023
			0.15	0.2000	20.0008
			0.20	0.2000	20.0004
			0.25	0.2000	20.0002
		0.05	0.10	0.2025	20.2452
			0.15	0.2007	20.0732
			0.20	0.2003	20.0304
			0.25	0.2002	20.0154
		0.10	0.10	0.2145	21.4480
			0.15	0.2051	20.5091
			0.20	0.2023	20.2287
			0.25	0.2012	20.1197
	5	0.01	0.10	0.2000	20.0022
			0.15	0.2000	20.0008
			0.20	0.2000	20.0004
			0.25	0.2000	20.0002
		0.05	0.10	0.2022	20.2180
			0.15	0.2007	20.0673
			0.20	0.2003	20.0286
			0.25	0.2001	20.0147
		0.10	0.10	0.2123	21.2289
			0.15	0.2045	20.4454
			0.20	0.2020	20.2046
			0.25	0.2011	20.1090

From the results, we also have the following figures. In each figure, the other parameters do not change.

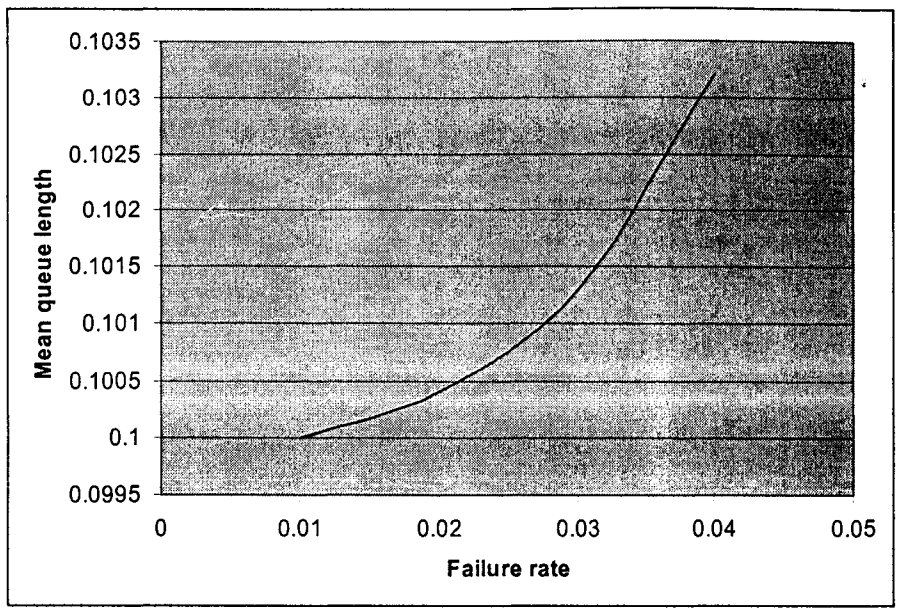

Figure 3.1 Mean queue length increases with the failure rate

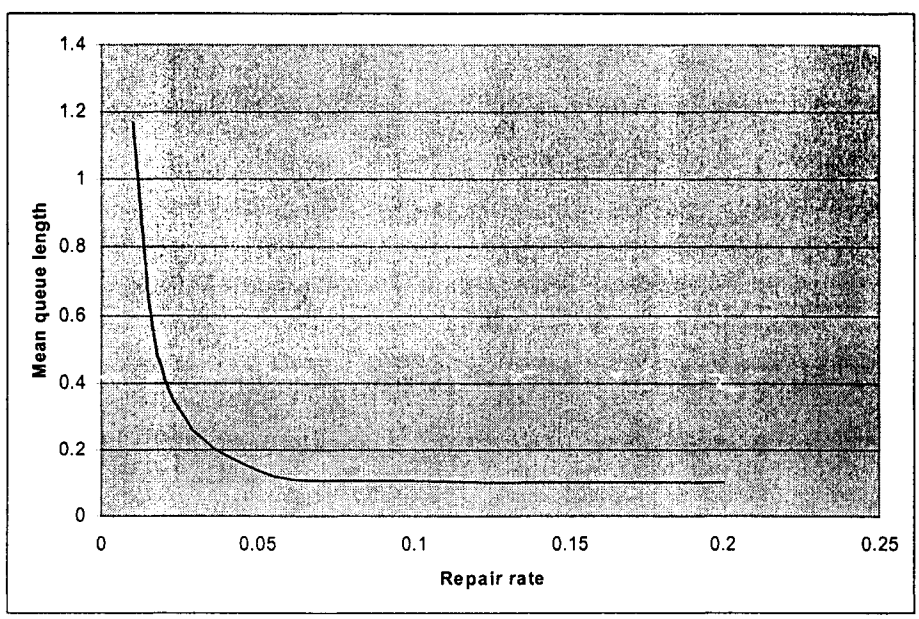

Figure 3.2 Mean queue length decreases with the repair rate

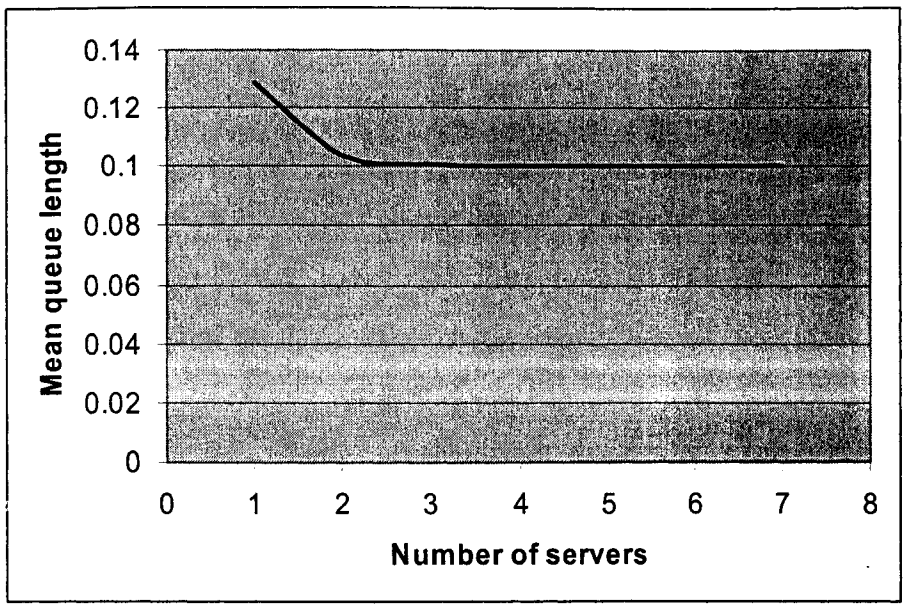

Figure 3.3 Mean queue length decreases with the number of servers

These figures tell us that the number of servers, failure rate, and repair rate affect the performance of a queueing system. Thus, we should take all these besides the arrival rate and service rate into account when we design or evaluate a queueing system. When n>4, the mean length and the mean waiting time do not change much. This is consistent with the common knowledge that we do not need to provide many servers when we have a small number of customers.

3.5 APPLICATION – FLEXIBLE MANUFACTURING CELL

There are 3 processing workstations, CNC (Computer Numerical Control) machining centers, at a flexible manufacturing cell (FMC) in a plant. The parts are loaded to this cell according to a Poisson arrival process of which the rate is 54 parts per hour. The

processing times of the workstations are exponentially distributed with different rates; that is, the rate changes with the number of the available workstations. When only 1 workstation is available, the processing rate is 16 parts per hour; 2 workstations are available, the rate is 18 parts per hour per workstation; while all the 3 workstations are available, the rate is 20 parts per hour per workstation, since the number of fixture changes is much less. Parts are unloaded from this cell right after they are processed. When one workstation is working, the probability that it breaks down in the next minute is exponentially distributed with the rate of 0.0001. Repair on the workstation begins immediately after it breaks down. There is only one repair group in this FMC. While one workstation is under repair, the probability that the repair is finished and the workstation returns to a working state in the next minute is exponentially distributed with a rate of 0.05.

This problem can be modeled as an M/M (M, M) /n queueing system with server failures (n=3). While the rate of the Poisson arrival process λ is 0.9, the service rate is $\mu_1 =$ 0.26667, $\mu_2 =$ 0.3, and $\mu_3 =$ 0.33333 per minute per workstation when 1, 2 or 3 workstations are working, respectively. The server failure time is exponentially distributed with the rate of $\theta = 0.0001$. The server repair time is exponentially distributed with the rate of $\gamma = 0.05$. The number of repair persons k is 1.

Our computer programs (group mmsu) can be used to solve this model. The parameters are as follows:

$n = 3;$

$\lambda = 0.9;$

$\mu = [0.266667, 0.3, 0.333333];$ // μ_1, μ_2 and μ_3

$\gamma = 0.05;$

$\theta = 0.0001;$

$k = 1;$

After running the program, we get the first 16 components of the steady state distribution:

$X = ($ 0.0000 0.0000 0.0000 0.0177 0.0000 0.0000 0.0001 0.0598
0.0000 0.0000 0.0002 0.0898 0.0000 0.0000 0.0002 0.0808 ...)

and

meanlength = 10.5638

meanwaitingtime = 11.7376

This means that there are about 10 parts in this FMC on the average, and the average time of one part staying in this cell is about 12 minutes. The probability that there is a workstation waiting for a part is 16.76%.

We consider the case 2 in which the loading rate is increased to 58 parts per hour. Then the arrival rate becomes 0.9667. If the service rates keep the same as the first scenario, the parameters will be:

$n = 3;$

$\lambda = 0.9667;$

$\mu = [0.266667, 0.3, 0.333333]$; // μ_1, μ_2 and μ_3

$\gamma = 0.05$;

$\theta = 0.0001$;

$k = 1$;

with the program, we can get the first 16 components of the steady state distribution.

$X = (0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0051 \quad 0.0000 \quad 0.0000 \quad 0.0000 \quad 0.0187$

$0.0000 \quad 0.0000 \quad 0.0001 \quad 0.0301 \quad 0.0000 \quad 0.0000 \quad 0.0001 \quad 0.0291\ldots)$

and

meanlength = 33.0959

meanwaitingtime = 34.2360

Compared with the first scenario, the loading rate is only increased by 7.4%, but the mean queueing length and mean waiting time are increased by about 200 percent. If the space in the cell is enough, it is nice to have the load speed increased a little bit since we can get a higher utilization of the workstations.

Table3.3 MATLAB program results of a flexible manufacturing cell

Case	Arrival rate λ	Failure rate θ	Repair rate γ	Mean queue length L	Mean waiting time W
1	0.9	0.0001	0.05	10.5638	11.7376
2	0.9667	0.0001	0.05	33.0959	34.2360
3	0.9	0.0001	0.025	11.2837	12.5375
4	0.9	0.0002	0.05	10.9664	12.1849

We also consider the case 3 that the repair rate is 0.025, which means the average repair time is 40 minutes. Other conditions keep the same as the first scenario. Then the parameters, mean queueing length and mean waiting time are as case 3 in table 3.3.

We consider case 4 in which the failure rate is 0.0002. Other conditions keep the same as the first scenario. Then the parameters, mean queueing length and mean waiting time are as case 4 in table 3.3.

From these cases, we know that we can improve the performance of a FMC through increasing the repair speed, prolonging the working period between the failure states, setting a good load rate, and so on. We also know that the analysis of a queueing system with server failures may give decision-makers some important information when they make decisions.

3.6 DISCUSSION

The queueing system M/M/n with different service rates and server failures was discussed in this chapter. The infinitesimal matrix was built, the stability condition was analyzed, the method to obtain the stationary distribution was presented, and some performance evaluation was discussed. Also, the computer programs were developed. An example was also provided to show the different results of the performance measurements with the change of the different number of servers and repair persons. This could give managers useful information to take into consideration when a system is evaluated.

CHAPTER 4 M/PH (M,M) /n QUEUEING SYSTEM WITH SERVER FAILURES

In many realistic cases the service time is generally distributed. We know that phase type distributions capture most features of general distributions. So we next talk about the M/PH(M,M)/n and M/PH(M,M)/n/c (later abbreviated as M/PH/n and M/PH/n/c) queueing systems with server failures in this chapter.

Notation in this chapter:

λ Intensity or rate of the Poisson arrival process

θ Parameter of the exponential distribution of the server failure time

γ Parameter of the exponential distribution of the server repair time

L Mean queue length

W Mean of the waiting time

n Number of the servers

k Number of the repair persons $(1<=k<=n)$

(β,S) The representation of the phase type distribution

μ Expected service rate

π Limiting or equilibrium distribution of queue length

Assumptions:

- A server keeps its service phase after breakdown and repair.
- Servers cannot help one another; even some of them are idle.

- A server can fail when it is idle.
- *There are k(1<=k<=n)* repair persons.

4.1 M/PH/n QUEUEING SYSTEM WITH SERVER FAILURES

The M/PH/1 and M/PH/2 queueing systems are discussed first. Then the M/PH/n queueing system is discussed which actually includes the former two queueing systems. In the matrices, an element is omitted if it is 0.

4.1.1 M/PH/1 Queueing System with Server Failures

We consider the M/PH/1 queueing system with server failures first. Only one repair person is needed in this system. We assume that the failure rate of the server is θ and the repair rate is γ. The representation of the phase type distributed service time is (β, S) of order v, which means the number of phases of a server is v and $S^0 = \boldsymbol{0} - Se$, where $\boldsymbol{0}$ is a column vector of all entries 0 and e is a column vector of all entries 1. So, the state space of the system is $E=\{(i, j, m);\ i>=0,\ j=0,1,\ 1<=m<=v\}$, where i denotes the number of customers in the system, j denotes the number of available servers, and m denotes the phase of the server. For X, either a row or a column vector, we define

$$diag(X) = \begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & X_3 & \\ & & & \ddots \end{bmatrix}.$$

The infinitesimal matrix has the following format:

$$\begin{bmatrix} B_{00} & B_{01} & & & \\ B_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & & & \ddots \\ & & & & \end{bmatrix},$$

where

$$B_{00} = \begin{bmatrix} -\lambda I - \gamma I & \gamma I \\ \theta I & -\theta I - \lambda diag(\beta) \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} \lambda I & \\ & \lambda diag(\beta) \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} & \\ & diag(S^0) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda I & \\ & \lambda I \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda I - \gamma I & \gamma I \\ \theta I & S - \theta I - \lambda I \end{bmatrix},$$

$$A_2 = \begin{bmatrix} & \\ & S^0 \beta \end{bmatrix}.$$

4.1.2 M/PH/2 Queueing System with Server Failures

We consider the M/PH/2 queueing system with server failures, one repair person. The failure rate is θ and the repair rate is γ. The representation of the phase type distributed service time is (β, S) of order v. So, the state space of the server is $E=\{(i, j, m);\ i>=0,\ j=0,1,2,\ 1<=m<=v\}$. There are two servers in this queueing system.

The generator matrix has the following format:

$$\begin{bmatrix} B_{00} & B_{01} & & & \\ B_{10} & B_{11} & B_{12} & & \\ & B_{21} & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \cdots & \end{bmatrix},$$

where

$$B_{00} = \begin{bmatrix} -\gamma I - \lambda I & \gamma I & \\ \theta I & -\theta I - \gamma I - \lambda diag(\beta) & \gamma I \\ & 2\theta I & -2\theta I - \lambda diag(\beta) \end{bmatrix},$$

$$B_{01} = \begin{bmatrix} \lambda I & & \\ & \lambda diag(\beta) & \\ & & \lambda diag(\beta) \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} o & & \\ & diag(S^0) & \\ & & diag(S^0) \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -\gamma I - \lambda I & \gamma I & \\ \theta I & S - \theta I - \gamma I - \lambda I & \gamma I \\ & 2\theta I & S - 2\theta I - \lambda diag(\beta) \end{bmatrix},$$

$$B_{12} = \begin{bmatrix} \lambda I & & \\ & \lambda I & \\ & & \lambda diag(\beta) \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} o & & \\ & S^0 \beta & \\ & & 2diag(S^0) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\gamma I - \lambda I & \gamma I & \\ \theta I & S - \theta I - \gamma I - \lambda I & \gamma I \\ & 2\theta I & 2S - 2\theta I - \lambda I \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda I & & \\ & \lambda I & \\ & & \lambda I \end{bmatrix},$$

$$A_2 = \begin{bmatrix} o & & \\ & S^0\beta & \\ & & 2S^0\beta \end{bmatrix}.$$

I is an identity matrix of order v.

If there are two repair persons in the system, only the matrices corresponding to the B_{00}, B_{11} and A_1 change. They will become the following matrices:

$$B_{00} = \begin{bmatrix} -2\gamma I - \lambda I & 2\gamma I & \\ \theta I & -\theta I - \gamma I - \lambda diag(\beta) & \gamma I \\ & 2\theta I & -2\theta I - \lambda diag(\beta) \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -2\gamma I - \lambda I & 2\gamma I & \\ \theta I & S - \theta I - \gamma I - \lambda I & \gamma I \\ & 2\theta I & S - 2\theta I - \lambda diag(\beta) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -2\gamma I - \lambda I & 2\gamma I & \\ \theta I & S - \theta I - \gamma I - \lambda I & \gamma I \\ & 2\theta I & 2S - 2\theta I - \lambda I \end{bmatrix}.$$

Other block matrices will be the same as those with only one repair person.

4.1.3 M/PH/n Queueing System with Server Failures

4.1.3.1 The modeling of the M/PH/n queue with server failures

We consider the M/PH/n queue with server failures, $k(1<=k<=n)$ repair persons. For one server, the failure rate is θ and the repair rate is γ. The representation of the phase type

distributed service time is (β, S) of order v. So, the state space of the system is $=\{(i, j, m);\ i>=0,\ 0<=j<=n,\ 1<=m<=v\}$. The infinitesimal matrix has the following format.

$$\begin{bmatrix} B_{00} & B_{01} & & & & & & & \\ B_{10} & B_{11} & B_{12} & & & & & & \\ & & \cdots & & & & & & \\ & & & B_{n-1,n-2} & B_{n-1,n-1} & B_{n-1,n} & & & \\ & & & & B_{n,n-1} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \cdots & \end{bmatrix},$$

where

$$B_{i,i-1} = \begin{bmatrix} S^0\beta & & & & & & \\ & 2S^0\beta & & & & & \\ & & \cdots & & & & \\ & & & (i-1)S^0\beta & & & \\ & & & & i*diag(S^0) & & \\ & & & & & \cdots & \\ & & & & & & i*diag(S^0) \end{bmatrix},$$

$$B_{i,j} = \begin{bmatrix} -(k\gamma+\lambda)I & k\gamma I & & & & & \\ \theta I & S-\theta I-\min(n-1,k)\gamma I-\lambda I & \min(n-1,k)\gamma I & & & & \\ & & \cdots & & & & \\ & & & i\theta I & iS-i\theta I-\min(n-i,k)\gamma I-\lambda I & & \\ & & & & \cdots & & \\ & & & & & iS-(n-1)\theta I-\gamma I-\lambda diag(\beta) & \gamma I \\ & & & & & n\theta I & iS-n\theta I-\lambda diag(\beta) \end{bmatrix},$$

$$B_{i,i+1} = \begin{bmatrix} \lambda I & & & & & & & \\ & \lambda I & & & & & & \\ & & \lambda I & & & & & \\ & & & \cdots & & & & \\ & & & & \lambda I & & & \\ & & & & & \lambda diag(\beta) & & \\ & & & & & & \cdots & \\ & & & & & & & \lambda diag(\beta) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda I & & & \\ & \lambda I & & \\ & & \lambda I & \\ & & & \cdots \\ & & & & \lambda I \end{bmatrix},$$

$$A_1 = B_{nn},$$

$$A_2 = \begin{bmatrix} & S^0\beta & & & & & & \\ & & 2S^0\beta & & & & & \\ & & & \cdots & & & & \\ & & & & iS^0\beta & & & \\ & & & & & (i+1)S^0\beta & & \\ & & & & & & \cdots & \\ & & & & & & & nS^0\beta \end{bmatrix}.$$

where I is of order v. The matrices $B_{i,i-1}$, $B_{i,i+1}$, A_2 and A_0 are square matrices of order $(n+1)v$ and all their entries are non-negative. All B_{ii} and A_1 are square matrices of order $(n+1)v$ too. The diagonal elements are strictly negative, all other elements are non-negative. The row sums of the generator matrix are equal to 0.

We explain the matrix of B_{ii} with an example. There are i customers in the system and no service finishes or starts at this moment. If the number of the available servers l is equal

to or less than i, then the servers may fail and the failure matrix is $l\theta I$. The servers may also be repaired and the repair matrix is $\min(n-l,k)\gamma I$ if there are k $(1<=k<=n)$ repair persons in the system. The diagonal matrix would be $lS - l\theta I - \min(n-l,k)\gamma I - \lambda I$ since there are l servers available to serve customers. If the number of the available servers l is greater than i, then the failure matrix is $l\theta I$, and the repair matrix is $\min(n-l,k)\gamma I$. The diagonal matrices would be $iS - l\theta I - \min(n-l,k)\gamma I - \lambda diag(\beta)$ since there are only i customers in the system. All other elements of matrices B_{ii} are 0.

4.1.3.2 The stability conditions of M/PH/n queue with server failures

From reference [30], we know that the stability condition for QDB processes is $\pi A_0 e < \pi A_2 e$. We discover a property that the stability condition for the M/PH/n queueing system is $\lambda < N_{average} \times \mu$, where $N_{average}$ is the average number of servers working. We will prove this property in subsequent sections.

4.1.3.2.1 M/PH/1 system with server failures

The stability condition of the M/PH/1 queueing system according to reference [30] is $\pi A_0 e < \pi A_2 e$.

The block structure of A $(= A_0 + A_1 + A_2)$ is

$$\begin{bmatrix} -\gamma I & \gamma I \\ \theta I & S + S^0\beta - \theta I \end{bmatrix}$$

The balance equations are

$$\pi_0(-\gamma I) + \pi_1(\theta I) = 0 \qquad (1)$$

$$\pi_0(\gamma I) + \pi_1(S + S^0\beta - \theta I) = 0. \tag{2}$$

From (1), we can get:

$$\pi_1 = \pi_0(\gamma I)(\theta I)^{-1} = \pi_0 \tfrac{\gamma}{\theta} I. \tag{3}$$

Substituting (3) into (2), we get:

$$\pi_0 \tfrac{\gamma}{\theta}(S + S^0\beta) = 0. \tag{4}$$

We know that

$$(\pi_0 + \pi_1)e = 1. \tag{5}$$

Substituting (3) into (5), gives us

$$\frac{\theta+\gamma}{\theta}\pi_0 e = 1, \tag{6}$$

$$\pi A_0 e = \pi(\lambda I)e = \lambda \pi e = \lambda. \tag{7}$$

Because $\mu = -\beta S^{-1}e$, from equation (4), we know that

$$\pi_0(S + S^0\beta) = 0,$$

and so

$$\pi_0 = -\pi_0 S^0 \beta S^{-1}.$$

Substituting into equation (6),

$$\begin{aligned}
&\frac{\theta+\gamma}{\theta}\pi_0 e \\
&= \frac{\theta+\gamma}{\theta}[-\pi_0 S^0 \beta S^{-1}]e \\
&= \frac{\theta+\gamma}{\theta}[\pi_0 S^0][-\beta S^{-1}e] \\
&= \frac{\theta+\gamma}{\theta}\pi_0 S^0 \frac{1}{\mu} \\
&= 1.
\end{aligned}$$

Then,

$$\mu = \frac{\theta+\gamma}{\theta}\pi_0 S^0 .$$

Hence,

$$\begin{aligned} \pi A_2 e &= \pi\begin{pmatrix} 0 \\ S^0 \end{pmatrix} = \begin{pmatrix} \pi_0 & \frac{\gamma}{\theta}\pi_0 \end{pmatrix}\begin{pmatrix} 0 \\ S^0 \end{pmatrix} = \frac{\gamma}{\theta}\pi_0 S^0 \\ &= \frac{\gamma}{\theta}\frac{\theta}{\theta+\gamma}\mu = \frac{\gamma}{\theta+\gamma}\mu. \end{aligned} \tag{8}$$

So, the stability condition for the M/PH/1 queueing system is

$$\lambda < \tfrac{\gamma}{\theta+\gamma}\mu . \tag{9}$$

Here we try to find the average number of servers working, $N_{average}$, and get the stability condition from our property. We can find the probabilities of 0 server and 1 server working by the following method. Suppose the probabilities are $W=\begin{pmatrix} w_0 & w_1 \end{pmatrix}$. Then we can calculate them from the following matrix:

$$V=\begin{bmatrix} -\gamma & \gamma \\ \theta & -\theta \end{bmatrix}.$$

From $WV = 0$, we have

$$-w_0\gamma + w_1\theta = 0$$

with the equation

$$w_0 + w_1 = 1.$$

We can get

$$w_1 = \frac{\gamma}{\theta+\gamma},$$
$$w_0 = \frac{\theta}{\theta+\gamma}.$$

So, the average number of server working $N_{average}$ is

$$N_{average} = 0 \times w_0 + 1 \times w_1 = \frac{\gamma}{\theta + \gamma}.$$

We can see that the stability condition of the M/PH/1 queueing system has the form of

$$\lambda < N_{average} \times \mu .$$

This is exactly the same as the conclusion (9), which can be obtained from the method of reference [30].

4.1.3.2.2 M/PH/2 queueing system with server failures (1 repair person)

The stability condition according to reference [30] is $\pi A_0 e < \pi A_2 e$.

The block structure of $A\ (= A_0 + A_1 + A_2)$ is

$$\begin{bmatrix} -\gamma I & \gamma I & \\ \theta I & S + S^0\beta - \theta I - \gamma I & \gamma I \\ & 2\theta I & 2S + 2S^0\beta - 2\theta I \end{bmatrix}.$$

The balance equations are

$$\pi_0(-\gamma I) + \pi_1(\theta I) = 0 \tag{11}$$

$$\pi_0(\gamma I) + \pi_1(S + S^0\beta - \theta I - \gamma I) + \pi_2(2\theta I) = 0 \tag{12}$$

$$\pi_1(\gamma I) + \pi_2(2S + 2S^0\beta - 2\theta I) = 0 . \tag{13}$$

From (11), we can get:

$$\pi_1 = \pi_0(\gamma I)(\theta I)^{-1} = \pi_0 \tfrac{\gamma}{\theta} I = \tfrac{\gamma}{\theta}\pi_0 . \tag{14}$$

Substituting (14) into (12), we can get:

$$\pi_2 = \pi_0[-\tfrac{\gamma}{2\theta^2}(S + S^0\beta) + \frac{\gamma^2}{2\theta^2} I] . \tag{15}$$

Substituting (14), (15) to (13),

$$\pi_0[-\tfrac{\gamma}{\theta^2}(S+S^0\beta)^2+(\tfrac{\gamma}{\theta}+\tfrac{\gamma^2}{\theta^2})(S+S^0\beta)]=0. \tag{16}$$

We know that

$$(\pi_0+\pi_1+\pi_2)e=1. \tag{17}$$

Substituting (14), (15) into (17), gives us

$$(1+\frac{\gamma}{\theta}+\frac{\gamma^2}{2\theta^2})\pi_0 e=1. \tag{18}$$

Substituting $\pi_0(S+S^0\beta)=0$ into equation (18) gives

$$\begin{aligned}
&(1+\frac{\gamma}{\theta}+\frac{\gamma^2}{2\theta^2})\pi_0 e\\
&=\frac{2\theta^2+2\gamma\theta+\gamma^2}{2\theta^2}[-\pi_0 S^0\beta S^{-1}]e\\
&=\frac{2\theta^2+2\gamma\theta+\gamma^2}{2\theta^2}(\pi_0 S^0)(-\beta S^{-1}e)\\
&=\frac{2\theta^2+2\gamma\theta+\gamma^2}{2\theta^2}(\pi_0 S^0)\frac{1}{\mu}\\
&=1,
\end{aligned}$$

so,

$$\pi_0 S^0=\frac{2\theta^2}{2\theta^2+2\theta\gamma+\gamma^2}\mu.$$

Then,

$$\pi_0 A_2 e = \pi \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_0 & \frac{\gamma}{\theta}\pi_0 & \pi_0[-\frac{\gamma}{2\theta^2}(S + S^0\beta) + \frac{\gamma^2}{2\theta^2} I] \end{pmatrix} \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix}$$

$$= \pi_0[-\tfrac{\gamma}{\theta^2}(S + S^0\beta) + (\tfrac{\gamma}{\theta} + \tfrac{\gamma^2}{\theta^2})I]S^0$$

$$= \frac{\theta\gamma + \gamma^2}{\theta^2}\pi_0 S^0$$

$$= \frac{\theta\gamma + \gamma^2}{\theta^2} \frac{2\theta^2}{2\theta^2 + 2\theta\gamma + \gamma^2}\mu$$

$$= \frac{2\theta\gamma + 2\gamma^2}{2\theta^2 + 2\theta\gamma + \gamma^2}\mu.$$

Because

$$\pi A_0 e = \pi(\lambda I)e = \lambda \pi e = \lambda, \tag{19}$$

so, the stability condition of the M/PH/2 queueing system is

$$\lambda < \frac{2\theta\gamma + 2\gamma^2}{2\theta^2 + 2\theta\gamma + \gamma^2}\mu. \tag{20}$$

Next we try to find $N_{average}$ and get the conclusion from our property. We find the probabilities of 0 servers, 1 server and 2 servers working by the following method. Suppose the probabilities are $W = \begin{pmatrix} w_0 & w_1 & w_2 \end{pmatrix}$. Then we can calculate them from the following matrix.

$$V = \begin{bmatrix} -\gamma & \gamma & \\ \theta & -\gamma-\theta & \gamma \\ & 2\theta & -2\theta \end{bmatrix}.$$

From $WV=0$, we have

$$
\begin{aligned}
&-w_0\gamma + w_1\theta = 0 \\
&w_0\gamma - (\gamma+\theta)w_1 + 2\theta w_2 = 0 \\
&\gamma w_1 - 2\theta w_2 = 0.
\end{aligned}
\tag{21}
$$

From (21), we get

$$
\begin{aligned}
w_0 &= \frac{2\theta^2}{2\theta^2 + 2\theta\gamma + \gamma^2} \\
w_1 &= \frac{2\theta\gamma}{2\theta^2 + 2\theta\gamma + \gamma^2} \\
w_2 &= \frac{\gamma^2}{2\theta^2 + 2\theta\gamma + \gamma^2}.
\end{aligned}
\tag{22}
$$

So, the average number of server working is

$$
0 \times w_0 + 1 \times w_1 + 2 \times w_2 = \frac{2\theta\gamma + 2\gamma^2}{2\theta^2 + 2\theta\gamma + \gamma^2}. \tag{23}
$$

According to the conjecture, the stability condition of the M/PH/2 with one repair person is

$$
\lambda < \frac{2\theta\gamma + 2\gamma^2}{2\theta^2 + 2\theta\gamma + \gamma^2}\mu.
$$

This conclusion is exactly the same as (20).

4.1.3.2.3 M/PH/2 queueing system with server failures (2 repair persons)

The stability condition according to reference [30] is $\pi A_0 e < \pi A_2 e$.

The block structure of $A\ (= A_0 + A_1 + A_2)$ is

$$
\begin{bmatrix}
-2\gamma I & 2\gamma I & \\
\theta I & S + S^0\beta - \theta I - \gamma I & \gamma I \\
 & 2\theta I & 2S + 2S^0\beta - 2\theta I
\end{bmatrix}.
$$

The equations are

$$\pi_0(-2\gamma I)+\pi_1(\theta I)=0, \tag{25}$$

$$\pi_0(2\gamma I)+\pi_1(S+S^0\beta-\theta I-\gamma I)+\pi_2(2\theta I)=0, \tag{26}$$

$$\pi_1(\gamma I)+\pi_2(2S+2S^0\beta-2\theta I)=0. \tag{27}$$

From (25), we can get:

$$\pi_1=\pi_0(2\gamma I)(\theta I)^{-1}=\pi_0\tfrac{2\gamma}{\theta}I=\tfrac{2\gamma}{\theta}\pi_0. \tag{28}$$

Substituting (28) into (26), we can get:

$$\pi_2=\pi_0[-\tfrac{\gamma}{\theta^2}(S+S^0\beta)+\frac{\gamma^2}{\theta^2}I]. \tag{29}$$

Substituting (28), (29) into (27),

$$\pi_0[-\tfrac{2\gamma}{\theta^2}(S+S^0\beta)^2+2(\tfrac{\gamma}{\theta}+\tfrac{\gamma^2}{\theta^2})(S+S^0\beta)]=0. \tag{30}$$

We know that

$$(\pi_0+\pi_1+\pi_2)e=1. \tag{31}$$

Substituting (28), (29) into (31), gives us

$$(1+\frac{2\gamma}{\theta}+\frac{\gamma^2}{\theta^2})\pi_0 e=1. \tag{32}$$

Substituting $\pi_0(S+S^0\beta)=0$ into equation (32),

$$\begin{aligned}
&(1+\frac{2\gamma}{\theta}+\frac{\gamma^2}{\theta^2})\pi_0 e\\
&=\frac{\theta^2+2\gamma\theta+\gamma^2}{\theta^2}[-\pi_0 S^0\beta S^{-1}]e\\
&=\frac{\theta^2+2\gamma\theta+\gamma^2}{\theta^2}(\pi_0 S^0)(-\beta S^{-1}e)\\
&=\frac{\theta^2+2\gamma\theta+\gamma^2}{\theta^2}(\pi_0 S^0)\frac{1}{\mu}\\
&=1,
\end{aligned}$$

so,

$$\pi_0 S^0 = \frac{\theta^2}{\theta^2 + 2\theta\gamma + \gamma^2}\mu$$

$$\pi_0 A_2 e = \pi \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_0 & \frac{2\gamma}{\theta}\pi_0 & \pi_0[-\frac{\gamma}{\theta^2}(S + S^0\beta) + \frac{\gamma^2}{\theta^2}I] \end{pmatrix} \begin{pmatrix} o \\ S^0 \\ 2S^0 \end{pmatrix}$$

$$= \pi_0[-\tfrac{2\gamma}{\theta^2}(S + S^0\beta) + (\tfrac{2\gamma}{\theta} + \tfrac{2\gamma^2}{\theta^2})I]S^0 = \pi_0[-\tfrac{2\gamma}{\theta^2}(S + S^0\beta) + (\tfrac{2\gamma}{\theta} + \tfrac{2\gamma^2}{\theta^2})I]S^0$$

$$= \frac{2\theta\gamma + 2\gamma^2}{\theta^2}\pi_0 S^0 = \frac{2\theta\gamma + 2\gamma^2}{\theta^2}\frac{\theta^2}{\theta^2 + 2\theta\gamma + \gamma^2}\mu$$

$$= \frac{2\theta\gamma + 2\gamma^2}{\theta^2 + 2\theta\gamma + \gamma^2}\mu = \frac{2\gamma(\theta + \gamma)}{(\theta + \gamma)^2}\mu$$

$$= \frac{2\gamma}{\theta + \gamma}\mu,$$

because

$$\pi A_0 e = \pi(\lambda I)e = \lambda\pi e = \lambda .$$

Then, the stability condition of the M/PH/2 queueing system with two repair persons is

$$\lambda < \frac{2\gamma}{\theta + \gamma}\mu . \tag{33}$$

Next we try to find $N_{average}$ and get the conclusion from our property. We can find the average number of servers working for the M/PH/2 with 2 repair persons.

$$V = \begin{bmatrix} -2\gamma & 2\gamma & \\ \theta & -\gamma - \theta & \gamma \\ & 2\theta & -2\theta \end{bmatrix}$$

We can get the following expressions from $W \times V = 0$:

$$w_0 = \frac{\theta^2}{\theta^2 + 2\theta\gamma + \gamma^2}$$

$$w_1 = \frac{2\theta\gamma}{\theta^2 + 2\theta\gamma + \gamma^2}$$

$$w_2 = \frac{\gamma^2}{\theta^2 + 2\theta\gamma + \gamma^2}.$$

So, the average number of servers working is

$$0 \times w_0 + 1 \times w_1 + 2 \times w_2 = \frac{2\theta\gamma + 2\gamma^2}{\theta^2 + 2\theta\gamma + \gamma^2} = \frac{2\gamma(\theta + \gamma)}{(\theta + \gamma)^2} = \frac{2\gamma}{\theta + \gamma}.$$

Then, the stability condition of M/PH/2 with two repair persons is

$$\lambda < \frac{2\gamma}{\theta + \gamma}\mu.$$

This conclusion is exactly the same as (33).

4.1.3.2.4 M/PH/ n queueing system with server failures (1 repair person)

The stability condition according to Neuts [30] is $\pi A_0 e < \pi A_2 e$.

$$A_0 = \begin{bmatrix} \lambda I & & & & \\ & \lambda I & & & \\ & & \lambda I & & \\ & & & \cdots & \\ & & & & \lambda I \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \gamma I - \lambda I & \gamma I & & & \\ \theta I & S - \theta I - \gamma I - \lambda I & \gamma I & & \\ & 2\theta I & 2S - 2\theta I - \gamma I - \lambda I & & \\ & & & \cdots & \\ & & & n\theta I & nS - n\theta I - \lambda I \end{bmatrix},$$

$$A_2 = \begin{bmatrix} o & & & & \\ & S^0\beta & & & \\ & & 2S^0\beta & & \\ & & & \cdots & \\ & & & & nS^0\beta \end{bmatrix}.$$

The block structure of A $(= A_0 + A_1 + A_2)$ is

$$\begin{bmatrix} -\gamma I & \gamma I & & & \\ \theta I & S + S^0\beta - \theta I - \gamma I & \gamma I & & \\ & 2\theta I & 2S + 2S^0\beta - 2\theta I - \gamma I & \gamma I & \\ & & & \cdots & \\ & & & n\theta I & nS + nS^0\beta - n\theta I \end{bmatrix}.$$

The equations are

$$\pi_0(-\gamma I) + \pi_1(\theta I) = 0, \tag{34}$$

$$\pi_0(\gamma I) + \pi_1(S + S^0\beta - \theta I - \gamma I) + \pi_2(2\theta I) = 0, \tag{35}$$

$$\pi_1(\gamma I) + \pi_2(2S + 2S^0\beta - 2\theta I - \gamma I) + \pi_3(3\theta I) = 0, \tag{36}$$

$$\cdots$$

$$\pi_{n-1}(\gamma I) + \pi_n(nS + nS^0\beta - n\theta I) = 0. \tag{37}$$

From (34), we can get:

$$\pi_1 = \pi_0(\gamma I)(\theta I)^{-1} = \pi_0 \tfrac{\gamma}{\theta} I = \tfrac{\gamma}{\theta}\pi_0. \tag{38}$$

Substituting (38) into (35), we can get:

$$\pi_2 = \pi_0[-\tfrac{\gamma}{2\theta^2}(S + S^0\beta) + \frac{\gamma^2}{2\theta^2} I]. \tag{39}$$

Substituting (38), (39) into (36),

$$\pi_3 = \pi_0[-\tfrac{\gamma}{3\theta^3}(S + S^0\beta)^2 - (\frac{\gamma}{3\theta^2} + \frac{\gamma^2}{2\theta^3})(S + S^0\beta) + \frac{\gamma^3}{6\theta^3} I]. \tag{40}$$

From equations (38), (39) and (40), we know that

$$\pi_i = \pi_0[a(S+S^0\beta)^{i-1} + b(S+S^0\beta)^{i-2} + \ldots + c(S+S^0\beta) + \frac{1}{i!}(\frac{\gamma}{\theta})^i I]. \tag{41}$$

We know that

$$(\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_n)e = 1. \tag{42}$$

Substituting (38), (39) into (41), gives us

$$\sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i \pi_0 e = 1. \tag{43}$$

Substituting $\mu = -\beta S^{-1}e$ and $\pi_0(S+S^0\beta) = 0$ into equation (43)

$$\begin{aligned}
&\sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i \pi_0 e \\
&= \sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i [-\pi_0 S^0 \beta S^{-1}]e \\
&= \sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i (\pi_0 S^0)(-\beta S^{-1}e) \\
&= \sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i (\pi_0 S^0)\frac{1}{\mu} \\
&= 1.
\end{aligned}$$

So,

$$\pi_0 S^0 = \frac{1}{\sum_{i-0}^{n} \frac{1}{i!}(\frac{\gamma}{\theta})^i}\mu.$$

Then

$$\pi A_2 e = \pi \begin{pmatrix} o \\ S^0 \\ \vdots \\ nS^0 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 & \cdots & \pi_n \end{pmatrix} \begin{pmatrix} o \\ S^0 \\ \vdots \\ nS^0 \end{pmatrix}$$

$$= \pi_0 [a'(S+S^0\beta)^{n-1} + b'(S+S^0\beta)^{n-2} + \cdots + c'(S+S^0\beta) + \sum_{i=0}^{n} \frac{i}{i!} (\tfrac{\gamma}{\theta})^i I] S^0 \tag{44}$$

$$= \sum_{i=0}^{n} \frac{i}{i!} (\tfrac{\gamma}{\theta})^i \pi_0 S^0$$

$$= \frac{\sum_{i=0}^{n} \frac{i1}{i!} (\tfrac{\gamma}{\theta})^i}{\sum_{i=0}^{n} \frac{1}{i!} (\tfrac{\gamma}{\theta})^i} \mu.$$

and $\quad \pi A_0 e = \pi(\lambda I)e = \lambda \pi e = \lambda$.

That is, the stability condition of the M/PH/n queueing system with one repair person is

$$\lambda < \frac{\sum_{i=0}^{n} i \times \frac{\gamma^i}{i!\theta^i}}{\sum_{i=0}^{n} \frac{\gamma^i}{i!\theta^i}} \mu. \tag{45}$$

Next we try to find $N_{average}$ and get the conclusion from our property. We find the probabilities of 0 servers, 1 server, 2 servers, ... , n servers working by the following method. Suppose the probabilities are $W = \begin{pmatrix} w_0 & w_1 & \cdots & w_n \end{pmatrix}$. Then we can calculate them from the following matrix.

$$V = \begin{bmatrix} -\gamma & \gamma & & & & & \\ \theta & -\gamma-\theta & \gamma & & & & \\ & & \cdots & & & & \\ & & & i\theta & -\gamma-i\theta & & \\ & & & & & \cdots & \\ & & & & & -\gamma-(n-1)\theta & \gamma \\ & & & & & n\theta & -n\theta \end{bmatrix}$$

From $WV=0$, we have

$$
\begin{aligned}
&-w_0\gamma + w_1\theta = 0 \\
&w_0\gamma - (\gamma+\theta)w_1 + 2\theta w_2 = 0 \\
&w_1\gamma - (\gamma+2\theta)w_2 + 3\theta w_3 = 0 \\
&\cdots \\
&\gamma w_{n-1} - n\theta w_n = 0.
\end{aligned}
$$

From the equations, we get

$$
\begin{aligned}
&w_1 = \frac{\gamma}{\theta} w_0 \\
&w_2 = \frac{\gamma^2}{2\theta^2} w_0 \\
&\cdots \\
&w_i = \frac{\gamma^i}{i!\theta^i} w_0 \\
&\cdots \\
&w_n = \frac{\gamma^n}{n!\theta^n} w_0.
\end{aligned}
$$

We know that

$$\sum_{i=0}^{n} w_i = w_0 \sum_{i=0}^{n} \frac{\gamma^i}{i!\theta^i} = 1.$$

That means,

$$w_0 = \left(\sum_{i=0}^{n} \frac{\gamma^i}{i!\theta^i}\right)^{-1}.$$

The average number of servers working is

$$\sum_{i=0}^{n} i \times \frac{\gamma^i}{i!\theta^i} w_0 = w_0 \sum_{i=0}^{n} i \times \frac{\gamma^i}{i!\theta^i} = \frac{\sum_{i=0}^{n} i \times \frac{\gamma^i}{i!\theta^i}}{\sum_{i=0}^{n} \frac{\gamma^i}{i!\theta^i}}.$$

Then, the stability condition of the M/PH/n queueing system with one repair person is

$$\lambda < \frac{\sum_{i=0}^{n} i \times \frac{\gamma^i}{i!\theta^i}}{\sum_{i=0}^{n} \frac{\gamma^i}{i!\theta^i}} \mu .$$

This conclusion is exactly the same as (45).

4.1.3.2.5 M/PH/n queueing system with server failures (n repair persons)

The stability condition according to reference [30] is $\pi A_0 e < \pi A_2 e$.

$$A_0 = \begin{bmatrix} \lambda I & & & & \\ & \lambda I & & & \\ & & \lambda I & & \\ & & & \cdots & \\ & & & & \lambda I \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \gamma I - n\lambda I & n\gamma I & & & \\ \theta I & S - \theta I - (n-1)\gamma I - \lambda I & (n-1)\gamma I & & \\ & 2\theta I & 2S - 2\theta I - (n-2)\gamma I - \lambda I & & \\ & & & \cdots & \\ & & & n\theta I & nS - n\theta I - \lambda I \end{bmatrix}$$

$$A_2 = \begin{bmatrix} S^0\beta & & & & \\ & 2S^0\beta & & & \\ & & \cdots & & \\ & & & nS^0\beta \end{bmatrix}$$

The block structure of $A\ (= A_0 + A_1 + A_2)$ is

$$\begin{bmatrix} -n\gamma I & n\gamma I & & & \\ \theta I & S + S^0\beta - \theta I - (n-1)\gamma I & (n-1)\gamma I & & \\ & 2\theta I & 2S + 2S^0\beta - 2\theta I - (n-2)\gamma I & (n-2)\gamma I & \\ & & & \cdots & \\ & & & n\theta I & nS + nS^0\beta - n\theta I \end{bmatrix}$$

The equations are

$$\pi_0(-n\gamma I)+\pi_1(\theta I)=0 \tag{46}$$

$$\pi_0(n\gamma I)+\pi_1(S+S^0\beta-\theta I-(n-1)\gamma I)+\pi_2(2\theta I)=0 \tag{47}$$

$$\pi_1((n-1)\gamma I)+\pi_2(2S+2S^0\beta-2\theta I-(n-2)\gamma I)+\pi_3(3\theta I)=0 \tag{48}$$

$$\cdots$$

$$\pi_{n-1}(\gamma I)+\pi_n(nS+nS^0\beta-n\theta I)=0. \tag{49}$$

From (46), we get:

$$\pi_1=\pi_0(n\gamma I)(\theta I)^{-1}=\pi_0\tfrac{n\gamma}{\theta}I=\tfrac{n\gamma}{\theta}\pi_0. \tag{50}$$

Substituting (50) into (47), we get:

$$\pi_2=\pi_0[-\tfrac{n\gamma}{2\theta^2}(S+S^0\beta)+\frac{n(n-1)\gamma^2}{2\theta^2}I]. \tag{51}$$

Substituting (50), (51) into (48),

$$\pi_3=\pi_0[-\tfrac{n\gamma}{3\theta^3}(S+S^0\beta)^2-(\frac{n\gamma}{3\theta^2}+\frac{n(3n-4)\gamma^2}{6\theta^3})(S+S^0\beta)+\frac{n(n-1)(n-2)\gamma^3}{6\theta^3}I]. \tag{52}$$

From equations (50), (51) and (52), we know

$$\pi_i=\pi_0[d(S+S^0\beta)^{i-1}+e(S+S^0\beta)^{i-2}+\ldots+f(S+S^0\beta)+\binom{n}{i}(\frac{\gamma}{\theta})^iI]. \tag{53}$$

We know that

$$(\pi_0+\pi_1+\pi_2+\cdots+\pi_n)e=1. \tag{54}$$

Substituting (50), (51) into (53), gives us

$$\sum_{i=0}^{n}\binom{n}{i}(\frac{\gamma}{\theta})^i\pi_0e=(1+\frac{\gamma}{\theta})^n\pi_0e=1. \tag{55}$$

substituting $\mu=-\beta S^{-1}e$ and $\pi_0(S+S^0\beta)=0$ into equation (55) gives

$$
\begin{aligned}
&(1+\frac{\gamma}{\theta})^n \pi_0 e \\
&= (1+\frac{\gamma}{\theta})^n [-\pi_0 S^0 \beta S^{-1}] e \\
&= (1+\frac{\gamma}{\theta})^n (\pi_0 S^0)(-\beta S^{-1} e) \\
&= (1+\frac{\gamma}{\theta})^n (\pi_0 S^0) \frac{1}{\mu} \\
&= 1.
\end{aligned}
$$

so,

$$
\pi_0 S^0 = \frac{1}{(1+\frac{\gamma}{\theta})^n} \mu \cdot
$$

Then

$$
\begin{aligned}
&\pi A_2 e = \pi \begin{pmatrix} o \\ S^0 \\ \vdots \\ nS^0 \end{pmatrix} = (\pi_0 \quad \pi_1 \quad \cdots \quad \pi_n) \begin{pmatrix} o \\ S^0 \\ \vdots \\ nS^0 \end{pmatrix} \\
&= (\pi_1 + 2\pi_2 + \cdots + n\pi_n) S^0 \\
&= \pi_0 [d'(S+S^0\beta)^{n-1} + e'(S+S^0\beta)^{n-2} + \cdots + f'(S+S^0\beta) + \sum_{i=0}^{n} i \binom{n}{i} (\tfrac{\gamma}{\theta})^i I] S^0 \\
&= \pi_0 [d'(S+S^0\beta)^{n-1} + e'(S+S^0\beta)^{n-2} + \cdots + f'(S+S^0\beta) + \frac{n\gamma}{\theta}(1+\frac{\gamma}{\theta})^{n-1} I] S^0 \\
&= \frac{n\gamma}{\theta}(1+\frac{\gamma}{\theta})^{n-1} \pi_0 S^0 = \frac{\frac{n\gamma}{\theta}(1+\frac{\gamma}{\theta})^{n-1}}{(1+\frac{\gamma}{\theta})^n} \mu \\
&= \frac{n\gamma}{\theta+\gamma} \mu.
\end{aligned}
$$

And $\pi A_0 e = \pi(\lambda I) e = \lambda \pi e = \lambda$.

So the stability condition for the M/PH/n queueing system with n repair persons is

$$
\lambda < \frac{n\gamma}{\theta+\gamma} \mu . \tag{56}
$$

Next we try to find $N_{average}$ and get the conclusion from our property. We find the probabilities of 0 servers, 1 server, 2 servers, ... , n servers working by the following method. Suppose the probabilities are $W=\begin{pmatrix} w_0 & w_1 & \cdots & w_n \end{pmatrix}$. Then we can calculate them from the following matrix:

$$V=\begin{bmatrix} -n\gamma & n\gamma & & & & & \\ \theta & -(n-1)\gamma-\theta & (n-1)\gamma & & & & \\ & & \cdots & & & & \\ & & & i\theta & -(n-i)\gamma-i\theta & & \\ & & & & & \cdots & \\ & & & & & -\gamma-(n-1)\theta & \gamma \\ & & & & & n\theta & -n\theta \end{bmatrix}.$$

From $WV=0$, we have

$$\begin{aligned} & -w_0 n\gamma + w_1\theta = 0 \\ & w_0 n\gamma - [(n-1)\gamma+\theta]w_1 + 2\theta w_2 = 0 \\ & w_1(n-1)\gamma - [(n-2)\gamma+2\theta]w_2 + 3\theta w_3 = 0 \\ & \cdots \\ & \gamma w_{n-1} - n\theta w_n = 0. \end{aligned}$$

From the equations, we get

$$\begin{aligned} & w_1 = \frac{n\gamma}{\theta} w_0 \\ & w_2 = \frac{n(n-1)\gamma^2}{2\theta^2} w_0 \\ & \cdots \\ & w_i = \binom{i}{n} (\frac{\gamma}{\theta})^i w_0 \\ & \cdots \\ & w_n = (\frac{\gamma}{\theta})^n w_0. \end{aligned}$$

We know that

$$\sum_{i=0}^{n} w_i = w_0 \sum_{i=0}^{n} \binom{n}{i} (\frac{\gamma}{\theta})^i = 1.$$

That means,

$$w_0 = [\sum_{i=0}^{n} \binom{n}{i} (\frac{\gamma}{\theta})^i]^{-1} = [(1+\tfrac{\gamma}{\theta})^n]^{-1} = (1+\tfrac{\gamma}{\theta})^{-n}.$$

The average number of servers working is

$$\sum_{i=0}^{n} i \times \binom{n}{i} (\frac{\gamma}{\theta})^i w_0 = w_0 \sum_{i=0}^{n} i \times \binom{n}{i} (\frac{\gamma}{\theta})^i = w_0 \frac{n\gamma}{\theta} (1+\frac{\lambda}{\theta})^{n-1}$$
$$= \frac{n\gamma}{\theta} (1+\frac{\lambda}{\theta})^{n-1} (1+\frac{\lambda}{\theta})^{-n} = \frac{n\gamma}{\theta} (1+\frac{\lambda}{\theta})^{-1} = \frac{n\gamma}{\theta+\gamma}.$$

Then, the stability condition of the M/PH/n queueing system with n repair persons is

$$\lambda < \frac{n\gamma}{\theta+\gamma} \mu.$$

This conclusion is exactly the same as (56).

4.1.3.3 Method to solve the stationary distribution

From the structure of the generator matrix in part 4.1.3.1, we know that the queueing system M/PH/n is a QBD process. We can use matrix R method to obtain the stationary distributions. The method is also as in table 3.1. Only the substructure of the infinitesimal generator matrix changes to fit the service time of phase type distribution.

4.1.3.4 Results of programs for M/PH/n systems

In order to test the generator matrices in part 4.1, some MATLAB programs are written. Some of the results are as follows.

The initial parameters are:

Customer arrival rate $\lambda = 0.1$;

Server failure rate $\theta = 0.2$;

Server repair rate $\gamma = 0.4$;

Number of servers n;

Number of the repair persons k $(1<=k<=n)$;

The representation of the phase distribution of the service rate (β,S):

$$S = \begin{bmatrix} -0.8 & 0.1 & 0.1 & 0.2 \\ 0.3 & -0.7 & 0.1 & 0.2 \\ 0.1 & 0.1 & -0.5 & 0.3 \\ 0.1 & 0.2 & 0.3 & -0.9 \end{bmatrix};$$

$\beta = [0.3, 0.2, 0.2, 0.3]$.

With the computer programs (group mphn), we can get the performance measurements with different server numbers and repair persons. The following data are the sample results from MATLAB programs for n = 4 and k = 1. Here are presented only the first $(n+1)\times(n+1)\times v$ dimensions. We can get other dimensions after this by the relationship of $X_{i+1} = X_i R$.

difference1 = 0.2370

Conjecture: this system is a stable system, continue!

difference2 = 0.2370

Real conclusion: this system is a stable system, continue!

x =(0.0435 0.0090 0 0.0328 0.1088 0.0225 0 0.0820

0.1149 0.0239 0 0.0869 0.0784 0.0164 0 0.0594

0.0396	0.0083	0	0.0300	0.0138	0.0050	0.0042	0.0116
0.0129	0.0079	0.0104	0.0127	0.0120	0.0077	0.0106	0.0122
0.0078	0.0051	0.0071	0.0080	0.0038	0.0025	0.0035	0.0039
0.0044	0.0023	0.0027	0.0041	0.0042	0.0032	0.0048	0.0045
0.0022	0.0019	0.0030	0.0026	0.0012	0.0010	0.0016	0.0014
0.0005	0.0005	0.0007	0.0006	0.0015	0.0010	0.0014	0.0016
0.0016	0.0014	0.0021	0.0018	0.0009	0.0009	0.0014	0.0011
0.0004	0.0003	0.0006	0.0004	0.0002	0.0001	0.0002	0.0002
0.0006	0.0004	0.0006	0.0006	0.0006	0.0006	0.0009	0.0008
0.0004	0.0004	0.0006	0.0005	0.0002	0.0002	0.0003	0.0002
0.0001	0.0001	0.0001	0.0001 ...)				

meanlength = 0.3822

meanwaitingtime = 3.8223

Where the *difference1* $= \dfrac{\sum_{i=0}^{n} i \times \dfrac{\gamma^i}{i!\theta^i}}{\sum_{i=0}^{n} \dfrac{\gamma^i}{i!\theta^i}} \mu - \lambda$, and *difference2* $= \pi A_2 e - \pi A_0 e$. We can see that we get the same results for *difference1* and *difference2*. We also get the same results for queueing system with other parameters (the number of repair persons is 1 or n) in Table 4.1. This proves our property in 4.1.3.2 numerically. Table 4.1 illustrates some results obtained for M/PH/n queueing system with unreliable servers.

Table 4.1 MATLAB program results of M/PH/n queueing system with unreliable servers

Number of servers n	Number of repair persons k	Failure rate θ	Repair rate γ	Mean queue length L	Mean waiting time W
3	1	0.01	0.2	0.1372	1.3717
			0.3	0.1353	1.3528
			0.4	0.1347	1.3471
			0.5	0.1345	1.3446
		0.10	0.2	0.5323	5.3233
			0.3	0.2779	2.7786
			0.4	0.2037	2.0367
			0.5	0.1739	1.7389
		0.20	0.2	1.8144	18.1436
			0.3	0.7140	7.1395
			0.4	0.4229	4.2293
			0.5	0.3034	3.0343
	2	0.01	0.2	0.1355	1.3552
			0.3	0.1347	1.3475
			0.4	0.1345	1.3446
			0.5	0.1343	1.3432
		0.10	0.2	0.2308	2.3083
			0.3	0.1730	1.7298
			0.4	0.1549	1.5489
			0.5	0.1471	1.4709
		0.20	0.2	0.4894	4.8943
			0.3	0.2794	2.7943
			0.4	0.2121	2.1208
			0.5	0.1825	1.8251
	3	0.01	0.2	0.1355	1.3545
			0.3	0.1347	1.3473
			0.4	0.1345	1.3446
			0.5	0.1343	1.3431
		0.10	0.2	0.2040	2.0401
			0.3	0.1646	1.6457

Table 4.1 (Cont'd)

			0.4	0.1513	1.5126
			0.5	0.1452	1.4522
		0.20	0.2	0.3698	3.6978
			0.3	0.2382	2.3824
			0.4	0.1929	1.9292
			0.5	0.1720	1.7203
4	1	0.01	0.2	0.1343	1.3429
			0.3	0.1338	1.3384
			0.4	0.1337	1.3374
			0.5	0.1337	1.3370
		0.10	0.2	0.4836	4.8359
			0.3	0.2440	2.4403
			0.4	0.1799	1.7989
			0.5	0.1567	1.5671
		0.20	0.2	1.7126	17.1264
			0.3	0.6608	6.6078
			0.4	0.3822	3.8223
			0.5	0.2701	2.7008
	2	0.01	0.2	0.1338	1.3382
			0.3	0.1337	1.3372
			0.4	0.1337	1.3369
			0.5	0.1337	1.3368
		0.10	0.2	0.1893	1.8932
			0.3	0.1509	1.5093
			0.4	0.1412	1.4116
			0.5	0.1376	1.3762
		0.20	0.2	0.4006	4.0064
			0.3	0.2276	2.2759
			0.4	0.1772	1.7721
			0.5	0.1573	1.5733
		0.01	0.2	0.1338	1.3379
			0.3	0.1337	1.3371
			0.4	0.1337	1.3369

Table 4.1 (Cont'd)

	3		0.5	0.1337	1.3367
		0.10	0.2	0.1599	1.5986
			0.3	0.1426	1.4265
			0.4	0.1379	1.3790
			0.5	0.1360	1.3605
		0.20	0.2	0.2541	2.5408
			0.3	0.1784	1.7844
			0.4	0.1555	1.5550
			0.5	0.1461	1.4609
	4	0.01	0.2	0.1338	1.3379
			0.3	0.1337	1.3371
			0.4	0.1337	1.3369
			0.5	0.1337	1.3367
		0.10	0.2	0.1555	1.5549
			0.3	0.1416	1.4156
			0.4	0.1375	1.3751
			0.5	0.1359	1.3587
		0.20	0.2	0.2279	2.2787
			0.3	0.1705	1.7050
			0.4	0.1523	1.5225
			0.5	0.1445	1.4452
	1	0.01	0.2	0.1338	1.3379
			0.3	0.1337	1.3367
			0.4	0.1336	1.3365
			0.5	0.1336	1.3364
		0.10	0.2	0.4692	4.6922
			0.3	0.2313	2.3127
			0.4	0.1704	1.7039
			0.5	0.1500	1.4998
		0.20	0.2	1.6968	16.9683
			0.3	0.6491	6.4905
			0.4	0.3707	3.7067
			0.5	0.2590	2.5903
			0.2	0.1337	1.3366

Table 4.1 (Cont'd)

5	2	0.01	0.3	0.1336	1.3364
			0.4	0.1336	1.3364
			0.5	0.1336	1.3364
		0.10	0.2	0.1746	1.7459
			0.3	0.1437	1.4370
			0.4	0.1372	1.3721
			0.5	0.1352	1.3522
		0.20	0.2	0.3763	3.7625
			0.3	0.2105	2.1047
			0.4	0.1653	1.6525
			0.5	0.1489	1.4890
	3	0.01	0.2	0.1337	1.3365
			0.3	0.1336	1.3364
			0.4	0.1336	1.3364
			0.5	0.1336	1.3364
		0.10	0.2	0.1471	1.4714
			0.3	0.1372	1.3716
			0.4	0.1350	1.3498
			0.5	0.1343	1.3427
		0.20	0.2	0.2203	2.2034
			0.3	0.1605	1.6048
			0.4	0.1448	1.4475
			0.5	0.1391	1.3912
	4	0.01	0.2	0.1337	1.3365
			0.3	0.1336	1.3364
			0.4	0.1336	1.3364
			0.5	0.1336	1.3364
		0.10	0.2	0.1420	1.4198
			0.3	0.1360	1.3600
			0.4	0.1346	1.3460
			0.5	0.1341	1.3412
		0.20	0.2	0.1842	1.8423
			0.3	0.1501	1.5011
			0.4	0.1408	1.4077

Table 4.1 (Cont'd)

			0.5	0.1373	1.3730
	5	0.01	0.2	0.1337	1.3365
			0.3	0.1336	1.3364
			0.4	0.1336	1.3364
			0.5	0.1336	1.3364
		0.10	0.2	0.1411	1.4107
			0.3	0.1358	1.3583
			0.4	0.1345	1.3454
			0.5	0.1341	1.3410
		0.20	0.2	0.1764	1.7644
			0.3	0.1481	1.4811
			0.4	0.1401	1.4007
			0.5	0.1370	1.3700

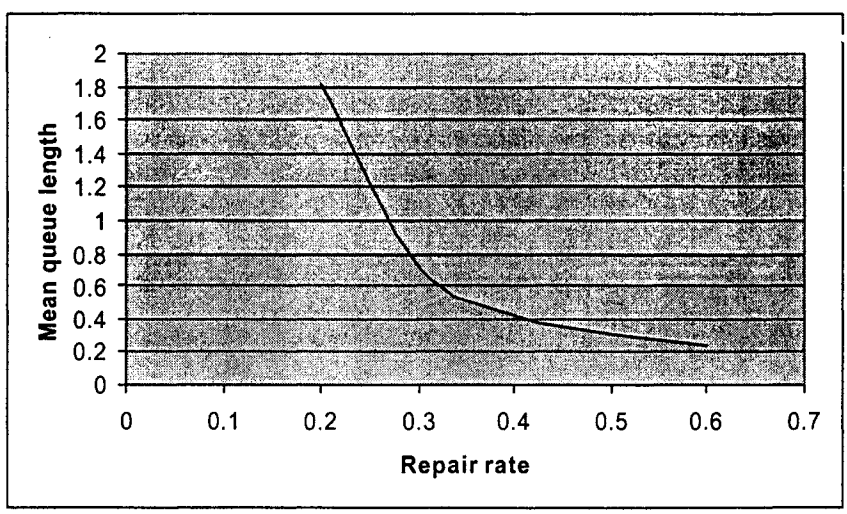

Figure 4.1 Mean queue length (M/PH/n) decreases with the repair rate

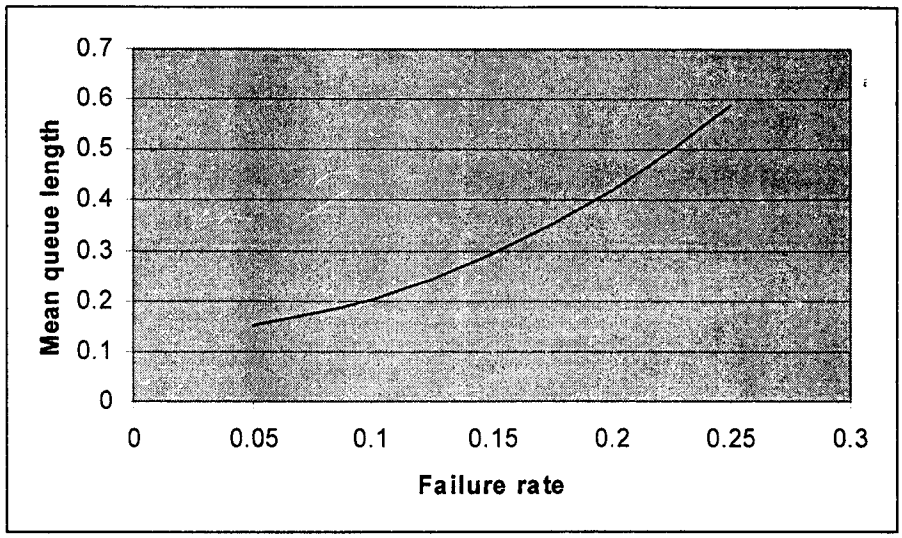

Figure 4.2 Mean queue length (M/PH/n) increases with the failure rate

From figures 4.1 and 4.2, we know that if we lower the failure rate of the queueing system, then the mean queue length and the mean waiting time will be less. Also, if we improve the repair rate, then the mean queue length and the mean waiting time will be less too.

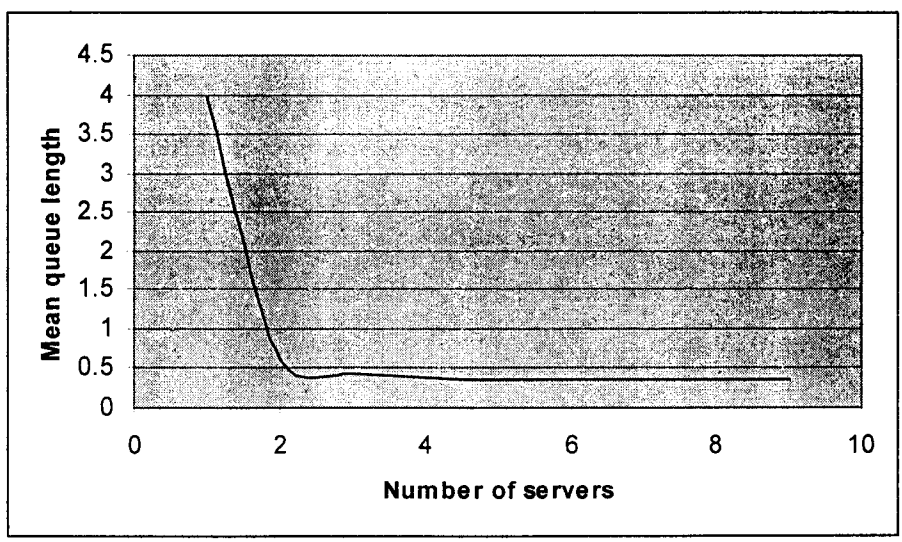

Figure 4.3 Mean queue length (M/PH/n) decreases with the number of servers

We find that when the server number is the same, the mean queue length and the mean waiting time decrease when the number of repair persons increases. We also find that the mean queue length and the mean waiting time do not change much when $n>6$. This is consistent with the common knowledge that we do not need to provide more servers when we already have enough servers to serve our customers. When the failure rate is very low, for example, 0.01 in the above example, the mean queue length and the mean waiting time do not change much with the increase of the repair rate.

4.2 M/PH/n/c QUEUEING SYSTEM WITH SERVER FAILURES

For this part, c is a new notation which means the capacity of the queueing systems.

4.2.1 The Modeling of M/PH/n/c Queueing System

The representation of the phase type distribution of the service time is (β, S) of order v. The state space $E=\{(i,m);\ 0<=i=<c,\ 1<m<=v\}$. The infinitesimal matrix for M/PH/n/c queue without server failures is

$$\begin{bmatrix} -\lambda diag(\beta) & \lambda diag(\beta) & & & & & & & \\ diag(S^0) & S-\lambda diag(\beta) & \lambda diag(\beta) & & & & & & \\ & 2diag(S^0) & 2S-\lambda diag(\beta) & \lambda diag(\beta) & & & & & \\ & & & \cdots & & & & & \\ & & & & (n-1)diag(S^0) & (n-1)S-\lambda diag(\beta) & \lambda diag(\beta) & & \\ & & & & & nS^0B^0 & nS-\lambda I & \lambda I & \\ & & & & & & nS^0B^0 & nS-\lambda I & \\ & & & & & & & \cdots & \\ & & & & & & & nS^0B^0 & nS \end{bmatrix},$$

where I is of order v.

4.2.2 M/PH/n/c Queueing System with Server Failures: Generator Matrix

We consider the M/PH/n/c queue with server failures, with $k(1<=k<=n)$ repair persons.

The state space of this system is $E=\{(i, j, m);\ 0<=i=<c,\ 0<=j<=n,\ 1<=m<=v\}$,

For the servers, the failure rate is θ and the repair rate is γ.

The generator matrix has the following format.

$$\begin{bmatrix} B_{00} & B_{01} & & & & & & & \\ B_{10} & B_{11} & B_{12} & & & & & & \\ & & \cdots & & & & & & \\ & & & B_{n-1,n-2} & B_{n-1,n-1} & B_{n-1,n} & & & \\ & & & & B_{n,n-1} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & & \cdots & A_1 & A_0 \\ & & & & & & & & A_2 & A_1+A_0 \end{bmatrix},$$

where

$$B_{i,i-1}=\begin{bmatrix} S^0\beta & & & & & \\ & 2S^0\beta & & & & \\ & & \cdots & & & \\ & & & (i-1)S^0\beta & & \\ & & & & i*diag(S^0) & \\ & & & & & \cdots \\ & & & & & i*diag(S^0) \end{bmatrix},$$

$$B_{i,i}=\begin{bmatrix} -(k\gamma+\lambda)I & k\gamma I & & & & & \\ \theta I & S-\theta I-\min(n-1,k)\gamma I-\lambda I & \min(n-1,k)\gamma I & & & & \\ & & \cdots & & & & \\ & & i\theta I & iS-i\theta I-\min(n-i,k)\gamma I-\lambda I & & & \\ & & & & \cdots & & \\ & & & & iS-(n-1)\theta I-\gamma I-\lambda diag(\beta) & \gamma I & \\ & & & & n\theta I & iS-n\theta I-\lambda diag(\beta) & \end{bmatrix},$$

$$B_{i,i+1} = \begin{bmatrix} \lambda I & & & & & & & \\ & \lambda I & & & & & & \\ & & \lambda I & & & & & \\ & & & \cdots & & & & \\ & & & & \lambda I & & & \\ & & & & & \lambda diag\,(\beta) & & \\ & & & & & & \cdots & \\ & & & & & & & \lambda diag\,(\beta) \end{bmatrix},$$

$$A_0 = \begin{bmatrix} \lambda I & & & \\ & \lambda I & & \\ & & \lambda I & \\ & & & \cdots \\ & & & & \lambda I \end{bmatrix},$$

$$A_1 = B_{nn},$$

$$A_2 = \begin{bmatrix} & S^0\beta & & & & & & \\ & & 2S^0\beta & & & & & \\ & & & \cdots & & & & \\ & & & & iS^0\beta & & & \\ & & & & & (i+1)S^0\beta & & \\ & & & & & & \cdots & \\ & & & & & & & nS^0\beta \end{bmatrix}.$$

4.2.3 M/PH/n/c Queueing system with Server Failures: Method to Get Stationary Distribution

Since this kind of queue is a finite state-space birth-and-death process in a Markovian environment, we can use the method in Gaver [11] to calculate the stationary distribution.

Let W_i denote the matrices on the diagonal of the generator matrix, that is

$W_i = B_{ii}$, for 0<=i<n,

$W_i = A_1$, for n<=i<=c.

Let U_i denote the matrices under the diagonal of the generator matrix, that is

$U_i = B_{i,i-1}$, for $1<=i<=n$,

$U_i = A_2$, for $n<i<=c$.

Let V_i denote the matrices above the diagonal of the generator matrix, that is

$V_i = B_{i,i+1}$, for $0<=i<n$,

$V_i = A_0$, for $n<=i<=c$.

Then, according to the Lemma 2 in Gaver [11],

$C_0 = W_0$,

$C_i = W_i + U_i(-C_{i-1}^{-1})V_{i-1}$, that is

$C_i = B_{ii} + B_{i,i-1}(-C_{i-1}^{-1})B_{i-1,i}$, for $0<i<=n$,

$C_i = A_1 + A_2(-C_{i-1}^{-1})A_0$, for $n<i<=c$.

According to the Theorem 1 in Gaver [11], the stationary distribution X_i , 0<=i<=c, are determined by the equations:

$X_c C_c = 0$

$\sum_{i=0}^{c} X_i e = 1$

$X_i = X_{i+1} U_{i+1}(-C_i^{-1})$, for $0<= i<n$

$X_i = X_{i+1} A_2(-C_i^{-1})$, for $n<=i<=c$.

Then, the algorithm to get X_i is shown in the table 4.2.

Table 4.2 Method to get the stationary distribution for finite birth-and-death processes

Step 1. Determine recursively the matrices C_i, $0<=i<=c$.
Step 2. Solve the system $P_c C_c = 0$, $P_c e = 1$.
Step 3. Compute recursively the vectors X_i, i=c-1, …, 0, using P_i instead of X_i.
Step 4. Re-normalize the vector X so obtained.

Computer programs are written to implement this algorithm. The queue performance evaluation is accomplished in this computer group too. Please see next part for the example.

4.3 APPLICATION - MOBILE WIRELESS COMMUNICATION

We considered a cell, the coverage area of a base station, in a homogeneous personal communications services (PCS) network, definition from reference [2]. The following are some data from experience. This cell consists of 64 channels. The calls, new calls or handoff calls, enter this cell with a Poisson arrival process with the rate of 600 per hour, and each call has a general phase type cell residence time with representation (0.2, 0.6, 0.2) and

$$\begin{bmatrix} -0.5 & 0.2 & 0.1 \\ 0.02 & -0.5 & 0.3 \\ 0.01 & 0.2 & -0.23 \end{bmatrix}$$

The channels may be noisy or not clear and hence lose connection, also called channel failure. The failure of the channels is exponentially distributed with an average of twice a year for all channels. If one channel does not work, the time that it returns to working

condition is also exponentially distributed with an average time of 3 minutes. Only one repair group is provided for this system. Phone calls can not be made through this system whenever all channels are busy.

This problem can be modeled as an M/PH (M, M) /n/c queueing system with server failures (n=64) with the following parameters.

Customer coming rate $\lambda = 10$; //600/60

Server failure rate $\theta = 0.00000006$; // $2/64/(60\times24\times365)$

Server repair rate $\gamma = 0.3333$;

Capacity of the system $c = 64$;

Number of servers $n = 64$;

Number of the repair groups $k = 1$;

The representation of the phase distribution of the service rate *(β, S):*

$$S = \begin{bmatrix} -0.5 & 0.2 & 0.1 \\ 0.02 & -0.5 & 0.3 \\ 0.01 & 0.2 & -0.23 \end{bmatrix},$$

$\beta = [0.2, 0.6, 0.2]$.

With the computer programs (group mphnc) we can get the steady state distribution. (the distribution shown here is only the distribution with the number of calls in this cell. The dimension of the original distribution, separate distribution for every server phase with the same number of calls in the system, is 12675):

```
lengthx = (0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
           0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
           0.0001  0.0002  0.0004  0.0006  0.0011  0.0018  0.0028  0.0043  0.0063
           0.0090  0.0124  0.0166  0.0215  0.0270  0.0330  0.0391  0.0450  0.0505
           0.0552  0.0588  0.0610  0.0618  0.0611  0.0590  0.0558  0.0515  0.0465
           0.0411  0.0356  0.0302  0.0251  0.0205  0.0163  0.0128  0.0099  0.0075
           0.0055  0.0040  0.0029  0.0020  0.0014  0.0010  0.0006  0.0004  0.0003
           0.0002  0.0001)
meanlength =  39.5212
meanwaitingtime =  3.9521
```

From the above result, we know that the load of this cell is light and there is 0.01% of calls lost.

Assume that the rate of the calls entering this cell would increase to 1000 calls per hour. Then we can get the steady state distribution.

```
lengthx =(0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0001  0.0001  0.0002  0.0003  0.0005  0.0008  0.0013  0.0019  0.0027
          0.0039  0.0054  0.0075  0.0100  0.0132  0.0171  0.0217  0.0271  0.0332
          0.0400  0.0474  0.0552  0.0633  0.0713  0.0791  0.0864  0.0930  0.0987
          0.1039  0.1146)
meanlength =  58.4953
meanwaitingtime =  3.5097
```

We know that there would be 11.46% of calls lost in this system.

Assume that the rate of the calls entering this cell would increase by 100%, that is, the rate would be 1200 calls per hour. Then we can get the steady state distribution.

```
lengthx =(0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
          0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0001  0.0001  0.0002
          0.0003  0.0005  0.0009  0.0014  0.0021  0.0032  0.0048  0.0071  0.0103
          0.0147  0.0205  0.0283  0.0384  0.0513  0.0674  0.0872  0.1113  0.1403
          0.1760  0.2337)
meanlength =   60.9911
meanwaitingtime =   3.0496
```

There will be 23.37% of calls lost in this system. So if we do have calls coming at this rate into this cell, we should try to increase the capacity of this cell to improve the service quality.

We consider the case that people hang up when there is noise while they make phone calls. If this situation happens once an hour for all channels and the noisy channel can return to good work state itself in 3 minutes. This situation can also be modeled as an

M/PH (M, M) /n/c queueing system with server failures (n = 64) with the following parameters.

Customer coming rate $\lambda = 10$; //600/60

Server failure rate $\theta = 0.00026048$; // 2/64/(60×24×365)+1/64/60

Server repair rate $\gamma = 0.3333$;

Capacity of the system $c = 64$;

Number of servers $n = 64$;

Number of the repair groups $k = 64$;

The representation of the phase distribution of the service rate *(β, S)*:

$$S = \begin{bmatrix} -0.5 & 0.2 & 0.1 \\ 0.02 & -0.5 & 0.3 \\ 0.01 & 0.2 & -0.23 \end{bmatrix},$$

$\beta = [0.2, 0.6, 0.2]$.

With the computer programs (group mphnc) we can get the steady state distribution:

lengthx = (0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
0.0000	0.0000	0.0001	0.0002	0.0004	0.0006	0.0011	0.0018	
0.0028	0.0043	0.0063	0.0090	0.0124	0.0166	0.0215	0.0270	
0.0330	0.0391	0.0450	0.0505	0.0552	0.0588	0.0610	0.0618	
0.0611	0.0590	0.0558	0.0515	0.0465	0.0411	0.0356	0.0302	
0.0251	0.0205	0.0163	0.0128	0.0099	0.0075	0.0055	0.0040	
0.0029	0.0020	0.0014	0.0010	0.0006	0.0004	0.0003	0.0002	

0.0001)

meanlength = 39.5214

meanwaitingtime = 3.9526

We can also get the mean queue length and mean waiting time when the noise occurs more frequently or the arrival rate increases. The results are in table 4.3.

Table 4.3 MATLAB program results for a communication cell

Arrival rate per hour	Average times of noise per hour	Mean queue length	Mean waiting time	Loss rate
600	0	39.5212	3.9526	0.01%
	1	39.5214	3.9526	0.01%
	5	39.5224	3.9529	0.02%
	10	39.5242	3.9533	0.02%
	15	39.5269	3.9539	0.03%
	20	39.5307	3.9547	0.04%
1000	0	58.4953	3.9639	11.46%
	1	58.5266	3.9921	12.04%
	5	58.6662	4.1134	14.43%
	10	58.8693	4.2840	17.55%
	15	59.0984	4.4760	20.78%
	20	59.3474	4.6896	24.07%
1200	0	60.9911	3.9794	23.37%
	1	61.0226	4.0298	24.29%
	5	61.1548	4.2415	27.91%
	10	61.3297	4.5280	32.28%
	15	61.5096	4.8374	36.42%
	20	61.6894	5.1680	40.32%

For a finite state-space birth-and-death process, the loss rate is a very important performance measurement. From table 4.3, we know that the loss rate increases with the increase of the occurring times of the noise. So we should pay attention to the frequency of the noise in the telecommunication system.

4.4 DISCUSSION

This chapter discussed the modeling, stability condition analysis, stationary distribution calculation, and computer program results analysis of M/PH/n and M/PH/n/c queueing systems. An example was also provided to show the application of the results.

CHAPTER 5 CONCLUSIONS AND FUTURE WORK

5.1 CONCLUSIONS

This research presented the modeling and analysis of three queueing systems with server failures, which are the M/M/n queueing system with different service rates, the M/PH/n and M/PH/n/c queueing systems. For all infinite queueing systems in this thesis, the *R* Matrix method was used to calculate the stationary distributions since the queueing systems are QBD processes.

The research of the M/M/n queueing system presented queueing systems with different service rates that were modeled and analyzed. The analysis of the M/PH/n and M/PH/n/c queueing systems with server failures is a good approximation for the M/G/n and M/G/n/c queueing systems.

5.2 CONTRIBUTIONS

This research has made the following contributions to the field of queueing systems with server failures:

(1) The M/M/n queueing system with server failures and different service rates was modeled and analyzed. Computer programs and application examples were also provided to demonstrate the analysis and calculation for this queueing system.

(2) The M/PH/n and M/PH/n/c queueing systems were modeled and analyzed too. Mathematical models were built, the stability condition for the M/PH/n queueing system was analyzed, and a method to gain stationary distribution and some performance

measurements for every queueing system were selected and implemented by computer programs.

5.3 FUTURE WORK

There are a number of issues which provide future research opportunities related to the research of this thesis.

5.3.1 The Proof of $\pi_0(S+S^0\beta)=0$ for the M/PH(M,M)/*n* Queueing System

In chapter 4, the conclusion of $\pi_0(S+S^0\beta)=0$ was proved mathematically for the M/PH(M,M)/1 queueing system. For the M/PH(M,M)/n queueing system, we only proved our conjectures numerically. Almost 300 groups of data were given and every data group verified our property numerically with computer programs (group mphn). We know that a similar conclusion has been proven for discrete queueing systems.

5.3.2 M/ PH (PH, PH)/n Queueing System

This queueing system is more general than the M/PH(M,M)/n queueing system which we discussed in Chapter 4. Sometimes we can not model a realistic system as the M/PH(M,M)/n queueing system because the server up and down time is not exponentially distributed. Then we might resort to phase type distribution up and down time since the phase type distribution could be a good approximation of general distribution.

5.3.3 Introduction to MAP (Markov Arrival Process)

The Markov Arrival Process is more general than the Poisson arrival process which we discussed a lot in this thesis. If we can model and analyze the MAP/M(M,M)/n queueing system, it would allow us to analyze more realistic systems than the M/M(M,M)/n. Similarly we can also analyze more realistic systems if we extend M/PH/n and M/PH/n/c to MAP/PH/n and MAP/PH/n/c respectively.

5.3.4 Introduction to MSP (Markov Service Process)

After the Markov Arrival Process, we will introduce the Markov Service Process to our research of queueing systems. A queueing system we intend to consider is the MAP/MSP(SMP, SMP)/n where SMP means Semi-Markov Process. This will extend the application of our research to a wider area.

REFERENCES

1. Al-Jararha, Jehad, Madan, K., 2002, Steady state analysis of an M/D/1 queue with coxian-2 server vacations and a single vacation policy, International Journal of Information and Management Sciences, Vol. 13, No. 4, December, pp. 69-81.

2. Alfa, Attahiru Sule, and Li, Wei, 2002, A Homogeneous PCS network with Markov Call Arrival Process and Phase Type Cell Residence Time, Wireless Networks, Vol. 8, No. 6, November, pp.597–605.

3. Avi-Itzhak, B. and Naor, P., 1961, On a problem of preemptive priority queuing, Operations Research, Vol. 9, pp. 664-672.

4. Avi-Itzhak, B. and Naor, P., 1962, Some problems with the service station subject to breakdown, Operations Research, Vol. 10, pp. 303-320.

5. Chao, Xiuli, Zhao, Yiqiang Q., 1998, Analysis of multi-server queues with station and server vacations, European Journal of Operational Research, Vol. 110, No. 2, October 16, pp. 392-406.

6. Choi, Bong Dae, Kim, Bara, Choi, Sung Ho, 2003, An M/G/1 queue with multiple types of feedback, gated vacations and FCFS policy, Computers and Operations Research, Vol. 30, No. 9, August, pp. 1289-1309.

7. Federgruen, A. and Green, L., 1986, Queueing systems with service interruptions, Operations Research, Vol. 34, pp. 752-768.

8. Federgruen, A. and Green, L., 1988, Queueing systems with service interruptions II, Naval Research Logistics Quarterly, Vol. 35, pp. 345-358.

9. Federgruen, A. and So, K. C., 1989, Optimal time to repair a broken server, Advanced Applied Probability, Vol. 21, pp. 376-397.

10. Federgruen, A and So, K.C., 1990, Optimal maintenance policies for single-server queueing systems subject to breakdowns, Operations Research, Vol. 38, pp. 330-343.

11. Gaver, D.P., Jacobs, P.A., Latouche, G., 1984, Finite birth-and-death models in randomly changing environments, Advanced Applied Probability, Vol. 16, pp. 715-731.

12. Gupur, Geni, 2002, Well-posedness of M/G/1 queueing model with single vacations, Computers and Mathematics with Applications, Vol. 44, No. 8-9, October/November, pp. 1041-1056.

13. Hsieh, Y. and Andersland, M.S., 1995, Repairable single server systems with multiple breakdown modes, Microelectronics and Reliability, Vol. 35, pp. 309-318.

14. Keilson, J. and Kooharian, A., 1960, On time dependent queuing processes, Annals of Mathematical Statistica, Vol. 31, pp. 104-112.

15. Keilson, J. and Kooharian, A., 1962, On the general time dependent queue with a single server, Annals of Mathematical Statistics, Vol. 33, pp. 767-791.

16. Keilson, J., 1962, Queues subject to service interruption, Annals of Mathematical Statistica, Vol. 33, pp. 1314-1322.

17. Ke, J.-C., Wang, K.-H., 1999, Cost analysis of the M/M/R machine repair problem with balking, reneging, and server breakdowns, Journal of the Operational Research Society, Vol. 50, No. 3, March, pp. 275-282.

18. Ke, Jau-Chuan, 2001, The control policy of an M[x]/G/1 queueing system with server startup and two vacation types, Mathematical Methods of Operations Research, Vol. 54, No. 3, February, pp. 471-490.

19. Ke, Jau-Chuan, 2002, Two-threshold policy for M[x]/G/1 queueing system with two vacation types, startup time and an un-reliable server, International Journal of Information and Management Sciences, Vol. 13, No. 2, June, pp. 51-67.

20. Ke, Jau-Chuan, 2003, The optimal control of an M/G/1 queueing system with server vacations, startup and breakdowns, Computers and Industrial Engineering, Vol. 44, No. 4, April, pp. 567-579.

21. Ke, Jau-Chuan, 2003, The optimal control of an M/G/1 queueing system with server startup and two vacation types, Applied Mathematical Modelling, Vol. 27, No. 6, June, pp. 437-450.

22. Li, Q., Xu, D. and Cao, J., 1997, Reliability approximation of a Markov Queueing system with server breakdown and repair, Microelectronics and Reliability, Vol. 37, pp. 1203-1212.

23. Madan, Kailash C., Saleh, Mohammad F., 2001, On M/D/1 queue with general server vacations, International Journal of Information and Management Sciences, Vol. 12, No. 2, June, pp. 25-37.

24. Madan, Kailash C., Saleh, Mohammad F., 2001, On M/D/1 queue with deterministic server vacations, Systems Science, Vol. 27, No.2, pp. 107-108.

25. Madan, Kailash C., 2002, On a single server queue with two-stage heterogeneous service and deterministic server vacations, International Journal of Systems Science, Vol. 32, No. 7, July, pp. 837-844.

26. Madan, Kailash C., Abu-Dayyeh, Walid, 2002, Restricted admissibility of batches into an M/G/1 type bulk queue with modified Bernoulli schedule server vacations, ESAIM - Probability and Statistics, Vol. 6, pp. 113-125.

27. Madan, Kailash C., Abu-Dayyeh, Walid, Taiyyan, Firas, 2003, A two server queue with Bernoulli schedules and a single vacation policy, Applied Mathematics and Computation (New York), Vol. 145, No. 1, December 20, pp. 59-71.

28. Mitrany, I.L. and Avi-Itzhak, B., 1968, A many server queue with service interruptions, Operations Research, Vol. 16, pp. 628-638.

29. Neuts, M.F. and Lucantoni, D.M., 1979, A markovian queue with N servers subject to breakdowns and repairs, Management Science, Vol. 15, pp. 849-861.

30. Neuts, M. F., Matrix-Geometric solutions in Stochastic Models: An Algorithmic Approach, The Johns Hopkins University Press, Baltimore, MD, 1981.

31. Sengupta, B., 1990, A queue with service interruptions in an alternating random environment, Operations Research, Vol. 38, pp. 308-318.

32. Tang, Y., 1995, Some reliability problems arising in GI/G/1 queueing system with repairable service station, Microelectronics and Reliability, Vol. 35, pp. 707-712.

33. Tang, Y., 1997, A single-server M/G/1 queueing system subject to breakdowns-some reliability and queueing problems, Microelectronics and Reliability, Vol. 37, pp. 315-321.

34. Tang, Yinghui, Tang, Xiaowo, 1999, Some reliability problems arising in M/G/1 queueing system with repairable service station and server vacations, Chinese Journal of Electronics, Vol. 8, No. 4, October, pp. 385-390.

35. Thiruvengdan, K., 1963, Queueing with breakdown, Operations Research, Vol. 11, pp. 62-71.

36. Vinod, B., 1985, Unreliable queueing systems. Computers and Operations Research, Vol. 12, pp. 323-340.

37. Wartenhorst, Pieter, 1995, N parallel queueing systems with server breakdown and repair, European journal of operational research, Vol. 82, pp. 302-322.

38. White, H.C. and Christie, L.S., 1958, Queuing with preemptive priorities or with breakdown, Operations Research, Vol. 6, pp. 79-95.

39. Winston, Wayne L., Operations Research Applications and Algorithms, Third edition. Duxbury Press, An imprint of Wadsworth Publishing Company, 1994.

40. Yue, D. and Cao, J., 1997, Reliability analysis of a $M_1^{x_1}$, $M_2^{x_2}/G_1$, $G_2/1$ queueing system with a repairable service station, Microelectronics and Reliability, Vol. 37, pp. 1225-1231.

41. Zhang, Zhe G., Vickson, Raymond G., van Eenige, Michel J.A., 1997, Optimal two-threshold policies in an M/G/1 queue with two vacation types, Performance Evaluation, Vol. 29, No. 1, February, pp. 63-80.

42. Zhang, Zhe G., Love, 1998, Charles Ernest, Threshold policy in an M/G/1 queue with an exceptional first vacation, INFOR Journal, Vol. 36, No. 4, November, pp. 193-204.

43. Zhang, Zhe George, Vickson, Raymond, Love, Ernie, 2001, The optimal service policies in an M/G/1 queueing system with multiple vacation types, INFOR Journal, Vol. 39, No. 4, November, pp. 357-366.

44. Zhang, Zhe George, Tian, Naishuo, 2003, Analysis on queueing systems with synchronous vacations of partial servers, Performance Evaluation, Vol. 52, No. 4, May, pp. 269-282.

APPENDIX A: PHASE TYPE DISTRIBUTION[30]

Definition:

We consider a Markov process on the states $\{1, \ldots, v+1\}$ with infinitesimal generator

$$Q = \begin{vmatrix} S & S^0 \\ 0 & 0 \end{vmatrix}, \qquad (1)$$

where the $v \times v$ matrix S satisfies $S_{ii} < 0$, for $1<=i<=v$, and $S_{ij} >=0$, for $i \neq j$. Also $Se+S^0=0$, and the initial probability vector of Q is given by (β, β_{m+1}), with $\beta e + \beta_{m+1} = 1$. We assume that the states $1, \ldots, v$ are all transient, so that absorption into the state $v+1$, from any initial state, is certain.

A probability distribution $F(\bullet)$ on $[0, \infty)$ is a distribution of phase type (*PH*-distribution) if and only if it is the distribution of the time until absorption in a finite Markov process of the type defined in equation (1). The pair (β, S) is called a representation of $F(\bullet)$.

APPENDIX B: PROGRAMS FOR M/M/n QUEUEING SYSTEM WITH SERVER FAILURES LEADING TO DIFFERENT SERVICE RATES

```
% M/M/n QUEUE WITH SERVER FAILURES LEADING TO DIFFERENT SERVICE RATES


% The initial values for the parameters
s=5;  %n, number of servers
m=0.5;
%u=0.1;

u=[0.166667,0.2,0.25,0.333333,0.4];

y=0.2;
b=0.02;
repair=min(5,s);


% calculate R
[r,finishflag]=calculater(s,m,u,y,b,repair)
if finishflag==0
   return;
end
%r=calculater(s,m,u,y,b,repair)


%solve vector x;
x=calculatex(s,m,u,y,b,r,repair)

% compute the mean queue length
meanlength=calmeanqlen(s,x,r)

% mean customer waiting time
meanwaitingtime=meanlength/m


% function calbii(s,i,m,u,y,b)
% this function is to calculate bii of the m/m/s queue with server failure

function [bii]=calbii(s,i,m,u,y,b,p)
mat=zeros(s+1, s+1);
for k=1:s+1
   for j=1:s+1
      if (k==1) & (j==1)
         mat(k,j)=-m-p*y;
      elseif (k==j) & (k==s+1)
         if(i==0)
            mat(k,j)=-s*b-m;
         else
            mat(k,j)=-i*u(i)-s*b-m;
         end
      elseif (k==j)&(k<=s)
         switch (k<=i)
         case 1
```

```
            mat(k,j)=-(k-1)*u(k-1)-(k-1)*b-min(p,s+1-k)*y-m;
        case 0
            if (i==0)
                mat(k,j)=-(k-1)*b-min(p,s+1-k)*y-m;
            else
                mat(k,j)=-i*u(i)-(k-1)*b-min(p,s+1-k)*y-m;
            end
        end
    else
        switch k-j
        case 1
            mat(k,j)=(k-1)*b;
        case -1
            mat(k,j)=min(p,s+1-k)*y;
        end
    end
  end
end
bii=mat;


% function calbii_1(s,u,i)
% This function is to calculate bi,i-1 of the m/m/s queue with server failure

function [bii_1]=calbii_1(s,u,i)
mat=zeros(s+1, s+1);
for k=1:s+1
  for j=1:s+1
    if (k==j)&(j==1)
        mat(k,j)=0;
    elseif (k==j)&(k<=i)
        mat(k,j)=(k-1)*u(k-1);
    elseif (k==j)&(k>i)
        mat(k,j)=i*u(i);
    end
  end
end
bii_1=mat;


% Function calculater
% this function is to calculate r

function [r,finishflag]=calculater(s,m,u,y,b,repair)
% the establishment of the basic matrics
a0=m*eye(s+1,s+1);
a1=calbii(s,s,m,u,y,b,repair);
a2=calbii_1(s,u,s);

%test if this is a stable system
a=a0+a1+a2;
testvector=a(:,1);
modifieda=a;
modifieda(:,1)=1;
rightvector=zeros(1, s+1);
rightvector(1)=1;
```

```
pai=rightvector*inv(modifieda);

%test the result of pai with the cut equation
paitest=pai*testvector;

%test pai with sum
paisum=sum(pai);

e=ones(s+1,1);
difference2=pai*a2*e-pai*a0*e;
if difference2>0
   disp('This system is a stable system, continue!');
else
   disp('This system is not stable, stop running!');
   finishflag=0;
   r=zeros(s+1,s+1);
   return
end

%try to get r using matrix r method;
iteration=0;
e=1e-050;
% the start R is r1
r1=zeros(s+1,s+1);
r2=-(a0+r1*r1*a2)*inv(a1);
difference=r2;
while difference(:)>e
   r1=r2;
   r2=-(a0+r1*r1*a2)*inv(a1);
   iteration=iteration+1;
   difference=r2-r1;
end
r=r2;

% Test the result of matrix r method
matrixRtest=a0+r*r*a2+r*a1;
finishflag=1;


% Function calculatex
% this function is to calculate the limiting distribution

function [x]=calculatex(s,m,u,y,b,r,repair)

I=eye(s+1);
e=ones(s+1,1);
smatrix=calsmatrix(s,m,u,y,b,repair,r);
testvector=smatrix(:,1);
modifiedsmatrix=smatrix;
for i=1:s*(s+1)
   modifiedsmatrix(i,1)=1;
end
modifiedsmatrix(s*(s+1)+1:(s+1)^2,1)=inv(I-r)*e;
rightvector=zeros(1,(s+1)^2);
rightvector(1)=1;
x=rightvector*inv(modifiedsmatrix);
```

```
% test the result of x with the cut equation
solutionXtest=x*testvector;

% test the result of x with the sum
xs=x(s*(s+1)+1:(s+1)^2);
sum(x)-sum(xs)+sum(xs*inv(I-r));


% Function calmeanqlen(s,x,r)
% this function is to calculate the mean queue length for m/m/s queue with server failures

function [meanlength]=calmeanqlen(s,x,r)
meanlength=0;
for i=0:s-1
  meanlength=meanlength+i*sum(x((s+1)*i+1:(s+1)*(i+1)));
end
I=eye(s+1);
e=ones(s+1,1);
xs=x(s*(s+1)+1:(s+1)^2);
meanlength=meanlength+s*xs*inv(I-r)*e+xs*r*(inv(I-r))^2*e;


% Function calsmatrix
% This function is to calculate the first (s+1)^2 rows and columns of the q matrix

function [tempmatrix]=calsmatrix(s,m,u,y,b,repair,r)
mat=zeros((s+1)^2,(s+1)^2);
for i=1:s+1
  for j=1:s+1
    switch i-j
    case 1
       mat((i-1)*(s+1)+1:i*(s+1),(j-1)*(s+1)+1:j*(s+1))=calbii_1(s,u,i-1);
    case -1
       mat((i-1)*(s+1)+1:i*(s+1),(j-1)*(s+1)+1:j*(s+1))=m*eye(s+1,s+1);
    case 0
      if i<(s+1)
        mat((i-1)*(s+1)+1:i*(s+1),(j-1)*(s+1)+1:j*(s+1))=calbii(s,i-1,m,u,y,b,repair);
      else
        mat((i-1)*(s+1)+1:i*(s+1),(j-1)*(s+1)+1:j*(s+1))=calbii(s,s,m,u,y,b,repair)+r*calbii_1(s,u,s);
      end
    end
  end
end
```

APPENDIX C: PROGRAMS FOR M/PH/n/c QUEUEING SYSTEM WITH SREVER FAILURES

```
% M/PH/n/c QUEUEING SYSTEM WITH SERVER FAILURES

% The initial values for the parameters
s=64; % 2; %number of servers
m=10;%10; %lamda
y=0.3333;%0.4;% gama
b=(2/64/(60*24*365))+(1/64/60);%0.00000006; %0.002; %sita
T=[-0.5 0.2 0.1; 0.02 -0.5 0.3; 0.01 0.2 -0.23];%[-0.85 0.01; 0.03 -0.92 ];
ph=size(T,1);
alpha=[0.2 0.6 0.2];%[0.3, 0.7];
testt=sum(T');
i=1;
%e=ones(ph,1);
%u=1/(-alpha*inv(T)*e)
while i<=ph
  if testt(i)>0
    disp('The sum of row')
    i
    disp('of phase distribution is greater than 0');
    return
  end
  i=i+1;
end
if sum(alpha)~=1
  disp('sum of alpha is not 1!');
  return;
end
if size(alpha)~=ph
  disp('dimension of alpha is not consistent with that of T!');
  return;
end
repair=min(64,s);
c=64;
if c<s
  disp('the given capacity is less than the number of the servers.');
  return;
end

% calculate Ci
matrixc=calculatec(s,m,T,y,b,repair,alpha,c);

%solve vector x (include x0 to xc);
x=calculatex(s,m,T,y,b,matrixc,repair,alpha,c);
%simplify x by add the probabilities with the same number of customers
lengthx=zeros(1,c+1);
for i=1:c+1
  lengthx(i)=sum(x((s+1)*ph*(i-1)+1:(s+1)*ph*i));
end
lengthx

% compute the mean queue length
meanlength=calmeanqlen(s,x,c,ph)
```

```
% customer mean waiting time
meanwaitingtime=meanlength/m


% Function calculatec
% this function is to calculate matrix Ci

function [matrixc]=calculatec(s,m,T,y,b,repair,alpha,c)
% the establishment of the basic matrices
ph=size(T,1);
a0=m*eye((s+1)*ph,(s+1)*ph);
a1=calbii(s,s,m,T,y,b,repair,alpha);
a2=calbii_1(s,T,s+1,alpha);

%calculate matrixC
mat=zeros((s+1)*ph, (s+1)*ph*(c+1));
tempc=zeros((s+1)*ph, (s+1)*ph);
for i=1:c+1
  if i==1
    mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i)=calbii(s,i-1,m,T,y,b,repair,alpha);
  elseif (i<=s+1)
    if i==c+1
      mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i)=a1+a0+calbii_1(s,T,s,alpha)*(-
inv(tempc))*calbii_2(s,s-1,m,T,alpha);
    else
      mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i)=calbii(s,i-1,m,T,y,b,repair,alpha)+calbii_1(s,T,i-
1,alpha)*(-inv(tempc))*calbii_2(s,i-2,m,T,alpha);
    end
  elseif (i<c+1)
    mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i)=a1+a2*(-inv(tempc))*a0;
  elseif i==c+1
    mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i)=a1+a0+a2*(-inv(tempc))*a0;
  end
  tempc=mat(1:(s+1)*ph, (s+1)*ph*(i-1)+1:(s+1)*ph*i);
end
matrixc=mat;


% function calbii(s,i,m,u,y,b)
% this function is to calculate bii of the M/PH/n/c queueing system with server failures

function [bii]=calbii(s,i,m,T,y,b,p,alpha)
ph=size(T,1);
mat=zeros((s+1)*ph, (s+1)*ph);
for k=1:s+1
  for j=1:s+1
    switch k-j
      case 1
        mat((k-1)*ph+1:k*ph, (j-1)*ph+1:j*ph)=(k-1)*b*eye(ph,ph);
      case -1
        mat((k-1)*ph+1:k*ph, (j-1)*ph+1:j*ph)=min(p,s+1-k)*y*eye(ph,ph);
      case 0
        mat((k-1)*ph+1:k*ph, (j-1)*ph+1:j*ph)=calnewT(s,i,m,T,y,b,p,k,alpha);
    end
  end
```

```
end
bii=mat;


% function calbii_1(s,T,i, alpha)
% This function is to calculate b_{i,i-1} of the M/PH/n/c queueing system with server failures

function [bii_1]=calbii_1(s,T,i,alpha)
ph=size(T,1);
mat=zeros((s+1)*ph, (s+1)*ph);
e=ones(ph,1);
T0=-T*e;
for k=1:s+1
   for j=1:s+1
      if k==j
         mat((k-1)*ph+1:k*ph,(j-1)*ph+1:j*ph)=calnewT0(i,k,T0,alpha);
      end
   end
end
bii_1=mat;


% function calbii_2(s,T,i, alpha)
% This function is to calculate b_{i,i+1} of the M/PH/n/c queueing system with server failures

function [bii_2]=calbii_2(s,i,m,T,alpha)

ph=size(T,1);
mat=m*eye((s+1)*ph);
for k=1:s+1
   for j=1:s+1
      if (k>=(i+2))&(k==j)
         mat((k-1)*ph+1:k*ph,(j-1)*ph+1:j*ph)=calT0B(m,ph,alpha);
      end
   end
end
bii_2=mat;

% function calnewT0(i,k,T0,alpha)
% this function is to calculate T0 for bii_1

function [newT0]=calnewT0(i,k,T0,alpha)
ph=size(T0,1);
newT0=eye(ph);
if k<i+1
   newT0=(k-1)*T0*alpha;
else
   for row=1:ph
      for column=1:ph
         if row-column==0
            newT0(row,column)=i*T0(row);
         end
      end
   end
end
```

```
% function calnewT(s,i,m,T,y,b,p,k,alpha)
% this function is to calculate T to calculate the T matrics in bii

function [newT]=calnewT(s,i,m,T,y,b,p,k,alpha)
ph=size(T,1);
newT=T;
for row=1:ph
  for column=1:ph
    if row-column==0
      if k<=i+1
        if k==(s+1)
          newT(row,column)=min(i,k-1)*T(row,column)-(k-1)*b-m;
        else
          newT(row,column)=min(i,k-1)*T(row,column)-(k-1)*b-min(p,s+1-k)*y-m;
        end
      else
        if k==(s+1)
          newT(row,column)=min(i,k-1)*T(row,column)-(k-1)*b-m*alpha(row);
        else
          newT(row,column)=min(i,k-1)*T(row,column)-(k-1)*b-min(p,s+1-k)*y-m*alpha(row);
        end
      end
    else
      newT(row,column)=min(i,k-1)*T(row,column);
    end
  end
end


% function calT0B(m,ph, alpha)
% This function is to calculate T0B in Bii_2

function [T0B]=calT0B(m,ph,alpha)

T0B=eye(ph);
for row=1:ph
  for column=1:ph
    if (row==column)
      T0B(row,column)=m*alpha(row);
    end
  end
end


% Function calculatex
% this function is to calculate the limiting distribution

function [x]=calculatex(s,m,T,y,b,matrixc,repair,alpha,c)

ph=size(T,1);
x=zeros(1,(s+1)*ph*(c+1));

%calculate the vector Xc(tempx, sum is 1)
temprightvector=zeros(1, (s+1)*ph);
tempc=matrixc(1:(s+1)*ph, (s+1)*ph*c+1:(s+1)*ph*(c+1));
```

```
%sum(tempc')
%testvector=tempc(:,1);
modifiedtempc=tempc;
for i=1:(s+1)*ph
   modifiedtempc(i,1)=1;
end
temprightvector(1)=1;
tempx=temprightvector*inv(modifiedtempc);

%test the result of Xc
%testxc=tempx*testvector

%modify vector x
x(1,(s+1)*ph*c+1:(s+1)*ph*(c+1))=tempx;

%calculate vectors X(c-1) to X(s)
for i=c-1:-1:s
   tempc=matrixc(1:(s+1)*ph, (s+1)*ph*i+1:(s+1)*ph*(i+1));
   a2=calbii_1(s,T,s+1,alpha);
   x(1,(s+1)*ph*i+1:(s+1)*ph*(i+1))=tempx*a2*(-inv(tempc));
   tempx=x(1,(s+1)*ph*i+1:(s+1)*ph*(i+1));
end
for i=s-1:-1:0
   tempc=matrixc(1:(s+1)*ph, (s+1)*ph*i+1:(s+1)*ph*(i+1));
   x(1,(s+1)*ph*i+1:(s+1)*ph*(i+1))=tempx*calbii_1(s,T,i+1,alpha)*(-inv(tempc));
   tempx=x(1,(s+1)*ph*i+1:(s+1)*ph*(i+1));
end

%normalization
sumx=sum(x);
x=x/sumx;


% Function calmeanqlen(s,x,c,ph)
% this function is to calculate the mean queue length for the M/PH/n/c queueing system with server %
failures

function [meanlength]=calmeanqlen(s,x,c,ph)
meanlength=0;
for i=0:c
   meanlength=meanlength+i*sum(x((s+1)*ph*i+1:(s+1)*ph*(i+1)));
end
```

VITA AUCTORIS

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