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SOME DYNAMIC MIXED BOUNDARY VALUE PROBLEMS IN LINEAR VISCOELASTICITY.

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SOME DYNAMIC MIXED BOUNDARY VALUE PROBLEMS

IN

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by

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Dedicated to the memory of Rev. D. T. Faught
- a man who devoted his life to the teaching
of mathematics

Abstract

In this dissertation the following problems are considered:

(i) the dynamic impact of a rigid axisymmetric indentor with a viscoelastic half-space;

(ii) the dynamic problems of a penny-shaped crack within a viscoelastic solid in tension and under torsion;

(iii) toroidal crack problems in tension and under torsion in both viscoelastic and elastic solids.

The general method used was to transform the problems to the solution of dual (or triple) intergral equations. These were further reduced, to the solution of single integral equations. Approximate numerical solutions were then found for these single integral equations.

In the first problem mentioned, all stresses and displacements are given as functions of the normal pressure beneath the indentor. It was found that the dynamic normal pressure differed slightly from that in the static case. This result allows the calculation of the other stresses and displacements using the static normal pressure (a closed form expression) as a good approximation. Approximate values for the contact radius are also given.

In the problem of the dynamic penny-shaped crack in tension, the stress intensity factor and normal displacement within the crack region are given in the case of a constant growing crack and a normal pressure dependent on time alone. The dynamic effect on the stress intensity factor is shown to reduce this factor as the crack velocity increases. In was also determined that the normal displacement within the crack region was approximately 12% less than that of the static case.

In the problem of the dynamic penny-shaped crack under torsion, the stress intensity factor and the tangential displacement are determined. In the former, the dynamic effect was that of a reduction with an increase in velocity. However, the dynamic effect of the displacement was found to be far more complex. The results given indicate that more analysis must be carried out before the nature of the dynamic effect can be predicted with a high level of confidence.

In the elastic toroidal crack problems considered, all the stress intensity factors are given as well as the displacements in the crack region. It was found possible to obtain an exact expression for the stress intensity factor on the inner edge of a toroidal crack in tension. In the limit, the solutions given were found to agree with the known solutions for the cases of a penny-shaped crack in tension and under torsion and an external crack in tension. However, the closed form solution found in the limiting case for an external crack under torsion with axisymmetric loading is a new result.

The viscoelastic solutions to the toroidal crack problems were found by an application of an extension to the correspondence principle of viscoelasticity. As has been noted in the literature, the stress intensity factors were found to be the same in viscoelasticity as in elasticity while the viscoelastic displacements calculated differed from their counterparts in elasticity.

Throughout the dissertation the philosophy was to follow general procedures in solving the problems rather than ad hoc methods. These procedures attempted to give a systematic way of handling the complex mixed boundary value problems in dynamic viscoelasticity (or elasticity). It is hoped that these will be of some use in future research in these areas.

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INTRODUCTION

The purpose of this dissertation is to consider the solutions of the following problems: (i) the dynamic impact problem of a rigid axisymmetric indentor with a viscoelastic[1] half-space; (ii) the dynamic problems of a penny-shaped crack within a viscoelastic solid in tension and under torsion, and (iii) toroidal crack problems in tension and under torsion in both viscoelastic and elastic solids. The historical background[2] and the specific results found, for each of the problems listed above, are discussed in the following outline.

Chapter 1 deals with the standard mathematical definitions and results which are used throughout this work. Chapter 2 gives the formulation of boundary value problems in quasi-static and dynamic viscoelasticity. For both of these cases, general solutions to the field equations in terms of potential functions are presented. In dynamic viscoelastic problems we assume Poisson's ratio to be a real constant. As has been noted in Christensen [1], without this assumption problems in general become intractable. In the particular situation of axisymmetry the general solutions are discussed in more detail and, the equivalent elastic solutions are stated as well. The last section of Chapter 2 gives the pertinent results and equations for the axisymmetric torsion problem in both viscoelasticity and elasticity. Before continuing our description of the remaining chapters we briefly discuss methods of solution of boundary value problems in viscoelasticity.

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[1] In this work when using the terms viscoelastic or elastic we shall always be referring to the linear theories.

[2] The remarks that follow are mainly concerned with three-dimensional problems.

Problems in viscoelasticity are classified as either dynamic or quasi-static depending upon whether the inertia terms in the equations of motion are retained or neglected. The classical method of solving boundary value problems in quasi-static viscoelasticity is to apply an integral transform (with respect to time) to the time-dependent field equations and boundary conditions. The transformed field equations then have the same form as the field equations of elasticity and if a solution to these, which is compatible with the transformed boundary conditions, can be found then the solution to the original problem is reduced to a transform inversion. This method is referred to as the "correspondence principle" (for reference see Lee [2]). This method in general fails when the boundary regions are functions of time. Some work has been done either to modify the principle, by Ting [3] and Graham and Sabin [4] or give conditions under which it is still applicable even though the boundary regions are functions of time (see Graham [5]). For the dynamic case several general methods have been suggested for solving boundary value problems in viscoelasticity (e.g. Tao [6] and Bland [7]). The difficulty with these proposals is that they are either not applicable in the case of moving boundaries or they require a family of exact elastic dynamic solutions. The former is not very useful since most dynamic problems of interest have moving boundaries. In the latter case, the obvious criticism is that only a few exact elastic solutions exist. We shall now describe the method we have used in solving dynamic problems in viscoelasticity.

The approach we have taken involves finding the general solutions by means of integral transform techniques which satisfy the field equations, initial conditions and conditions at infinity. These solutions turn out to be in terms of one or more unknown functions. By applying the boundary conditions to the general solutions we then reduce the problem to a pair of dual integral equations. These in turn are transformed into a Volterra integral equation of the second kind. The free term in this latter integral equation is exactly the result that would be found for the quasi-static problems. These equations are of such a complex nature that it is necessary to solve them numerically. The advantage that one gains in this technique over that of a numerical approach ab initio (e.g. finite element analysis) is that one can obtain approximate analytic expressions which can be used to determine the nature of the solution without ever resorting to the computer. We shall now describe the outline of the remaining chapters.

In Chapter 3 we consider the dynamic problem of the impact of a rigid axisymmetric indentor on a viscoelastic half-space. The analogous problem in the static theory of elasticity was solved by Hertz [8]. A solution to this problem was given by Hunter [9] for the quasi-static theory of viscoelasticity in the particular case when the indentor is a sphere. Both of these solutions were incomplete in the sense that some numerical analysis had to be performed to determine the contact radius as a function of time. Papers by Deresiewicz [10] and Graham [11] presented the necessary calculations for the elastic problem while Calvit [12] did the same for quasi-static viscoelasticity. The solution of the impact problem in the case of dynamic elasticity was given by Tsai [13]. One particular result that Tsai obtained was that the normal contact stress in the dynamic case differed insignificantly from the static or Hertz result. Following the method outlined in the previous paragraph we have solved the impact problem for dynamic viscoelasticity. We perform numerical integrations comparable to [12] to first determine the contact radius. The results are given in Figure 3.2. We next calculated the normal stress beneath the indentor, (Table 3.2), and as in [13] find that there is no significant difference in the dynamic and quasi-static results. The importance of this result is that all other stress and displacement quantities are given in terms of this normal pressure for which we now have a closed form expression with insignificant error.

In Chapter 4 we present the dynamic solution of a growing penny-shaped crack in a viscoelastic solid opened by a normal pressure acting on its surface. This problem has been solved in several different ways in the case of static elasticity (see Sneddon and Lowengrub [14]). The quasi-static viscoelastic solution was given by Graham [15]. Craggs [16] considered the dynamic problem of a crack growing with a constant velocity in an elastic solid with constant normal pressure acting on it's surface. Atkinson [17] re-examined Craggs method and solution and linked it to the integral transform approach. A more general problem than that examined by Craggs and Atkinson was considered by Tsai [18]. Tsai reduced his problem to a single integral equation and solved this exactly in the case considered by Craggs. Tsai, however, made no comparison with Craggs solution. There are not many results concerning crack problems in dynamic viscoelasticity. Willis [19] has investigated a steady state dynamic viscoelastic crack problem. He considered the specific case of an extending crack in anti-plain strain. Recently Atkinson and List [20] solved a problem similar to that treated in [19] with a different type of loading. The problem we consider in this chapter also reduces to a single integral equation similar to that found by Tsai [18]. We also consider the specific case for which an exact solution was found in [18]. However in the viscoelastic problem an exact solution does not seem feasible. The normal displacement has been calculated numerically and the difference between the dynamic and static results is indicated, (Table 4.2). The other quantity of interest in crack problems, the stress intensity factor[3], is also determined by approximate means, (Figure 4.3). In calculation of the stress-intensity factor it is noticed that the expression for the normal stress contains two terms, one of which is the quasi-static part, and the other, in the form of an integral, representing the dynamic part. It is observed that the contribution to the singularity in the normal stress from the integral term is due to a derivative, in the integrand, of a function which possessed a jump discontinuity. Hence to determine these integral's contribution to the stress intensity factor it is only necessary to evaluate the jump discontinuity. Observations of this nature are not uncommon in viscoelasticity (c.f. Christensen [1], Chapter 4). We may state that without using arguments of this nature the determination of the stress intensity factor would have been more difficult.

The dynamic problem of a growing penny-shaped crack under torsion in a viscoelastic solid has been considered in Chapter 5. The history of this problem is rather less extensive. A brief discussion is given in [14] for the case of static elasticity. Quite recently, Kaloni and Smith [21] have presented the solution to the quasi-static problem in viscoelasticity. Sih and Embley [22] and Kennedy and Achenbach [23] have given solutions to the dynamic problem in elasticity. The former is more numerical in nature than the latter. Neither give exact solutions. The solution we present in this chapter is similar in form to that of Chapter 4. We specialize our results to the case of a constant growing crack when the loading is of the form $S(t) \cdot r$, where S is a function of time t and r is the curvilinear co-ordinate. We calculate the tangential displacement u_θ and indicate the

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(3) The stress intensity factor for this problem is essentially the coefficient of the singular part of the normal stress.

difference between the dynamic and static results, (Table 5.2). Using the methods developed in the last chapter we are again able to calculate the stress intensity factor, (Table 5.3). In the last section an exact dynamic elastic solution of a constant growing penny-shaped crack under torsion, when the loading is also constant, is presented.

In the final chapter, five distinct solutions are given, four of these involve toroidal cracks and the fifth, an external crack under torsion. We restrict our consideration in this chapter to the static theory of elasticity and quasi-static theory of viscoelasticity. Toroidal crack problems have received little attention in the literature due, most likely, to the difficult nature of the resulting boundary value problems. The toroidal crack is a flat annular crack with an inner radius which decreases with time and an outer radius which increases. The problem has several interesting features besides that of the obvious physical applications. It provides a link between two crack problems which have been considered in the literature previously, i.e. the penny-shaped crack problem and the external crack problem. Both of these problems are limiting cases of the toroidal crack problem. In all the solutions that we find it is possible to recover those known solutions for the limiting cases. One limiting case, that of an external crack in an elastic solid under torsion has not appeared in literature to this date. The solution we obtain in this case is in closed form and the displacement and the stress intensity factor are calculated for general loading.

The toroidal crack in tension is considered for both elasticity and quasi-static viscoelasticity. Solutions to this problem in elasticity have been given by Smetanin [24] and Moss and Kobayashi [25]. The former presents an approximate solution valid for extreme cases of the ratio of the two radii. The latter gives essentially an iterative method making use of the superposition principle of linear elasticity. Approximate values of the stress intensity factors are calculated but the normal displacement is not calculated which for this solution represents an estimate of the error. The solution which we give exploits a technique by Cooke [26]. As a specific example we consider the case of constant pressure on the crack surface. The normal displacement is calculated for this case, (Figure 6.1). In the calculation of the inner stress intensity factor, it is noticed that certain terms in the integral equation giving the normal stress can be ignored in the region of the crack edge. By dropping these terms, we are able to solve exactly the resulting integral equation and hence find the exact expression for the stress intensity factor on the inside of the crack. As was observed in [24] and [25], we found that this term becomes singular as the inner radius shrinks to zero. This result confirms what has been noted in practice [25], i.e. toroidal cracks in tension will always tend to become penny-shaped cracks, sometimes with catastrophic results. We next calculate an approximate expression for the outer stress intensity factor making use of our previous results. The expression we find agrees in the limiting case of a penny-shaped crack. Both intensity factors are given in Figure 6.2. To solve the corresponding viscoelasticity problem we make use of the principle stated by Graham [5] and are able to generalize our elastic results to the viscoelastic case. The stress intensity factors for this problem are the same as that found in the elastic case. The normal displacement is calculated and is given in Table 6.1.

In this chapter we also consider the problem of a toroidal crack under torsion for both the elastic and viscoelastic cases. This problem has only recently been considered in elasticity by Kanwal and Pasha [27]. They consider approximate solutions for the two extreme cases of the ratio of the two radii. There however appears to be some question about the nature of their solution. The solution that we find is similar in nature to that obtained in the tension case. For the specific loading of $S_o \cdot r$, S_o a constant, we calculate the displacement, (Figure 6.3) and the stress intensity factors (Figure 6.4). The viscoelastic generalization is achieved in the same manner as before. Table 6.2 gives the results of the calculations of the tangential displacement in the specific case outlined above.

CHAPTER I

MATHEMATICAL PRELIMINARIES

1. The Stieltjes Convolution and Laplace Transform.

Throughout all chapters $H(t)$ denotes the Heaviside unit step-function, of time alone, which is defined through

$$H(t) = 0, \quad -\infty < t < 0, \quad H(t) = 1, \quad 0 \leq t \leq \infty. \tag{1.1.1}$$

In this section we will assume that f, g and h are always sufficiently smooth functions of the position vector $\underline{x}$ and time t. Then the Stieltjes convolution $f*dg$ stands for the function defined by

$$[f*dg](\underline{x},t) = f(\underline{x},t)g(\underline{x},0) + \int_0^t f(\underline{x},t-\tau)\frac{\partial g}{\partial \tau}(\underline{x},\tau)d\tau, \quad 0 \leq t < \infty, \tag{1.1.2}$$

provided the integral is meaningful. Some properties of the convolution (1.1.2), which will be needed later, are listed below:

$$\begin{aligned} f*dg &= g*df \\ f*d(g*dh) &= (f*dg)*dh = f*dg*dh \\ f*d(g+h) &= f*dg + f*dh \\ f*dH &= f. \end{aligned} \tag{1.1.3}$$

If $f(\underline{x},0)$ does not vanish then f has a unique Stieltjes inverse, f^{-1}, such that

$$f*df^{-1} = f^{-1}*df = H. \tag{1.1.4}$$

Proofs of these properties, (1.1.3) and (1.1.4) are contained in Gurtin and Sternberg [28]. While the notation introduced with respect to the Stieltjes convolution is very convenient we must be careful of its use. For example let the function $a(t)$ be monotonically increasing in t, and let F be defined as

$$F(r,t) = \begin{cases} f_1(r,t) & , \quad r < a(t), \\ f_2(r,t) & , \quad r > a(t). \end{cases}$$

Further, let G be some function of t, then we have the following in general

$$[G*dF](r,t) \neq [G*df_1](r,t) , \quad r < a(t).$$

This follows from (1.1.2), the fact a is monotonically increasing and the definition of F. It is however true that

$$[G*dF](r,t) = [G*df_2](r,t), \quad r > a(t). \tag{1.1.5}$$

We shall indicate by the notation

$$\bar{f}(\underline{x},s) = L[f(\underline{x},t) \; ; \; t \to s] = \int_0^\infty f(\underline{x},t)e^{-st}dt, \qquad f(\underline{x},t) = L^{-1}[\bar{f}(\underline{x},s) \; ; \; s \to t] , \tag{1.1.6}$$

the Laplace transform with respect to time t, of a function $f(\underline{x},t)$, and its inverse Laplace transform, respectfully. Several relations which are consequences of (1.1.6) are stated now (for proofs see e.g. Sneddon [29]). The Convolution Theorem states

$$L^{-1}[\bar{f}(x,s) \; \bar{g}(x,s) \; ; \; s \to t] = \int_0^t f(x,t - \tau)g(x,\tau)d\tau \tag{1.1.7}$$

where the roles of f and g on the right hand side of (1.1.7) may be reversed.

The Laplace transform of the time derivative of a function is given by

$$\left(\overline{\frac{\partial f}{\partial t}}\right)(\underline{x},s) = s\overline{f}(\underline{x},s) - f(\underline{x},0). \qquad (1.1.8)$$

Further results are now given. If we take the Laplace transform of the convolution (1.1.2) we obtain

$$L[(f*dg)(\underline{x},t) \quad ; \quad t \to s] = \overline{f}(\underline{x},s)g(\underline{x},0) + \overline{f}(x,s)\left(\overline{\frac{\partial g}{\partial t}}\right)(\underline{x},s)$$

which, on using equation (1.1.8), becomes

$$L[(f*dg)(\underline{x},t) \quad ; \quad t \to s] = s\overline{f}(\underline{x},s)\overline{g}(\underline{x},s). \qquad (1.1.9)$$

By taking the inverse Laplace transform of (1.1.9) we get

$$[f*dg](x,t) = L^{-1}[s\overline{f}(\underline{x},s)\overline{g}(\underline{x},s) \quad ; \quad s \to t]. \qquad (1.1.10)$$

2. <u>The Hankel Transform.</u>

If $\sqrt{r}f(r)$ is continuous and absolutely integrable on the positive real line, and if $\nu > -1/2$, we indicate by the notation

$$f^*_\nu(\xi) = H_\nu[f \; ; \; \xi] = \int_0^\infty rf(r)J_\nu(r\xi)dr \qquad (1.2.1)$$

the Hankel transform of the function $f(r)$ of order ν. Here J_ν denotes the Bessel function of order ν. From the Hankel Inversion theorem we have

$$f(r) = \int_0^\infty \xi f^*_\nu(\xi)J_\nu(r\xi)d\xi \; . \qquad (1.2.2)$$

We now state a result involving the differential operator

$$B_\nu = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\nu^2}{r^2} \; , \qquad (1.2.3)$$

namely

$$H_\nu[B_\nu f\ ;\ \xi] = -\xi^2 f_\nu^*(\xi) \tag{1.2.4}$$

In the particular cases $\nu = 0$ and $\nu = 1$, combining (1.2.3) and (1.2.4) we find

$$H_o[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\ ;\ \xi] = -\xi^2 f_o^*(\xi), \tag{1.2.5}$$

$$H_1[\frac{\partial^2 f}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2}\ ;\ \xi] = -\xi^2 f_1^*(\xi). \tag{1.2.6}$$

Proofs of (1.2.2) - (1.2.6) may be found in [29].

Other useful results are that

$$H_\nu\left[r^{-\nu-1}\frac{\partial}{\partial r}[r^{\nu+1}f(r)]\ ;\ \xi\right] = \xi H_{\nu+1}[f(r)\ ;\ \xi] \tag{1.2.7}$$

which in the particular case $\nu = 0$ reduces to

$$H_o\left[\frac{\partial f}{\partial r} + \frac{f}{r}\ ;\ \xi\right] = \xi f_1^*(\xi). \tag{1.2.8}$$

We also note the recurrence relations

$$\frac{\partial}{\partial r}\left[r^{-\nu}J_\nu(\xi r)\right] = -\xi r^{-\nu}J_{\nu+1}(\xi r),$$
$$\frac{\partial}{\partial r}\left[r^{\nu}J_\nu(\xi r)\right] = \xi r^{\nu}J_{\nu-1}(\xi r) \tag{1.2.9}$$

which in the particular cases $\nu = 0$ and $\nu = 1$ reduce to

$$\frac{\partial}{\partial r}J_o(\xi r) = -\xi J_1(\xi r)$$

and (1.2.10)

$$\frac{\partial}{\partial r}\ rJ_1(\xi r) = \xi r J_o(\xi r),$$

respectively.

3. Abel Transform.

We define the Abel transforms A_1 and A_2 through the equations

$$\hat{f}_1(x) = A_1[f(t)\ ;\ x] = \int_0^x \frac{f(t)dt}{(x^2-t^2)^{\frac{1}{2}}} \ , \qquad (1.3.1)$$

$$\hat{f}_2(x) = A_2[f(t)\ ;\ x] = \int_x^a \frac{f(t)dt}{(t^2-x^2)^{\frac{1}{2}}} \ , \quad x < a.$$

The inverse transforms are then given as

$$f(t) = A_1^{-1}\ [\hat{f}_1(x)\ ;\ t] = \frac{2}{\pi}\ \frac{d}{dt} \int_0^t \frac{x\hat{f}_1(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \ , \qquad (1.3.2)$$

$$f(t) = A_2^{-1}\ [\hat{f}_2(x)\ ;\ t] = \frac{-2}{\pi}\ \frac{d}{dt} \int_t^a \frac{x\hat{f}_2(x)dx}{(x^2-t^2)^{\frac{1}{2}}} \ , \quad x < a\ .$$

These transform are essentially the solutions given by Abel of integral equations. They are stated as such and in more generality in Noble's paper [30]. In his book [29], Sneddon defines Abel transforms slightly different in that a is replaced by ∞. However for our purposes a finite constant is more appropriate.

4. Young's Approximate Product Integration.

A. Young [31] proposed a method of approximate product integration of the form

$$\int_a^b f(x)\phi(x)dx = \sum_{r=1}^{n} \alpha_r f(x_r) + R \ , \qquad (1.4.1)$$

where $x_1, x_2, \ldots, x_n (x_1 < x_2 < \ldots < x_n)$ are n abscissae with associated weights $\alpha_1, \alpha_2, \ldots, \alpha_n$ and R is a correction term. The main features of this method are that the weights α_r and the correction term R depend on certain standard matrices. Further the formula is exact when $f(x)$ is a polynomial of degree less than n. The following formulae are developed for a fixed $\phi(x)$ and any $f(x)$ which can be expanded in a Taylor series.

The function $\phi(x)$ may be discontinuous provided

$$\int_a^b (x - \eta)^s \phi(x) dx$$

exists for each $s \leq (n - 1)$.

Let

$$\int_a^b (x - \eta)^s \phi(x) dx = h^s \mu s \tag{1.4.2}$$

where

$$h = (x_n - x_1)/(n - 1) \tag{1.4.3}$$

and $s = 0,1, \ldots, (n - 1)$. Further if we denote

$$(x_r - \eta) = hX_r \tag{1.4.4}$$

then in vector notation

$$X\underline{\alpha} = \underline{\mu} \tag{1.4.5}$$

where X is an $n \times n$ matrix with element $X_{i,j} = X_j^{i-1}$ and $\underline{\mu}$ is a column vector whose i'th component is μ_{i-1} (c.f., (1.4.2)). Hence the weights, α_r, are given by

$$\underline{\alpha} = X^{-1}\underline{\mu} . \tag{1.4.6}$$

We are free to choose x_r and η to be any value within the range in which the expansion of $f(x)$ is valid. These values then are not restricted to lie within the range of integration. By (1.4.4) we observe that once a choice of x_r and η has been made we can calculate the inverse of X once and for all. As an example we consider equal interval end-point formulae. We let $\eta = x_1$ and $x_r = x_1 + (r - 1)h$. X^{-1} has been calculated by Young in this case for $n = 2, \ldots, 7$. We quote the result for $n = 5$:

$$X^{-1} = \frac{1}{4!}\begin{bmatrix} 24 & -50 & 35 & -10 & 1 \\ 0 & 96 & -104 & 36 & -4 \\ 0 & -72 & 114 & -48 & 6 \\ 0 & 32 & -56 & 28 & -4 \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix} \qquad (1.4.7)$$

Since we shall not make use of the correction factor R we will not discuss it here. The power of this method lies in its usefulness with integrals which have singularities within the region of integration. Its major criticism is that more analytical work must be done before one is able to carry out complete integration, i.e. μ_r must be calculated for each different $\phi(x)$.

CHAPTER II

THE FORMULATION OF BOUNDARY VALUE PROBLEMS IN VISCOELASTICITY AND ELASTICITY

1. Introduction.

Suppose a fixed region R with boundary B is occupied by a homogeneous and isotropic linear viscoelastic solid. Let u_i, e_{ij}, σ_{ij}, each of which is to represent a function of the position vector $\underline{x}$ and time t where $\underline{x}$ is a point in R and $0 \leq t \leq \infty$, denote the Cartesian components of displacement, strain, and stress respectively. Then, following the usual indicial notation we can record the fundamental system of field equations relevant to the linear theory of viscoelasticity (e.g. see Christensen [1]). The linearized displacement - strain relations take the form

$$2e_{ij}(\underline{x},t) = u_{i,j}(\underline{x},t) + u_{j,i}(\underline{x},t). \tag{2.1.1}$$

The basic field equations are given as

$$\sigma_{ij,j}(\underline{x},t) + F_i(\underline{x},t) = 0, \tag{2.1.2}$$

$$\sigma_{ij,j}(\underline{x},t) + F_i(\underline{x},t) = \rho\frac{\partial^2 u_i}{\partial t^2}(\underline{x},t), \tag{2.1.3}$$

$$\sigma_{ij}(\underline{x},t) = \sigma_{ji}(\underline{x},t) \tag{2.1.4}$$

where F_i denotes the components of the prescribed body force and ρ the density of the material. Equations (2.1.2) apply to quasi-static problems and (2.1.3) apply to dynamic problems. In this work we shall use the integral form of the stress-strain relations. We cite the relaxation integral law, (e.g. see [1] or [28])

$$\sigma_{ij}(\underline{x},t) = [G_1 * de_{ij}](\underline{x},t) + \delta_{ij}[\frac{(G_2 - G_1)}{3} * de_{KK}](\underline{x},t) \qquad (2.1.5)$$

where δ_{ij} is Kronecker's delta. Here G_1 and G_2, which are functions of time t, $0 \leq t < \infty$, denote the relaxation functions in shear and isotropic compression respectively. In equation (2.1.5) we have used the notation defined by (1.1.2).

To complete the formulation of any boundary value problem in viscoelasticity we need to specify certain boundary and initial conditions. If we prescribe the surface displacement and traction, respectively, on complementary subsets $B_1(t)$, $B_2(t)$ of the boundary B then the boundary conditions take the form

$$\begin{aligned} u_i(\underline{x},t) &= U_i(\underline{x},t) , \quad \underline{x} \text{ on } B_1(t), \\ \sigma_{ij}(\underline{x},t)n_j(\underline{x},t) &= T_i(\underline{x},t) , \quad \underline{x} \text{ on } B_2(t), \end{aligned} \qquad (2.1.6)$$

where $n_j(\underline{x},t)$ are the components of the outward drawn unit normal to B. The initial conditions are specified to be of the type

$$\begin{aligned} u_i(\underline{x},0) &= W_i(\underline{x}) , \quad \underline{x} \text{ in } R, \\ \frac{\partial u_i}{\partial t}(\underline{x},0) &= V_i(\underline{x}) , \quad \underline{x} \text{ in } R. \end{aligned} \qquad (2.1.7)$$

In the above $\underline{U}$, $\underline{T}$, $\underline{W}$, $\underline{V}$ are given vector valued functions, which are at least piece-wise continuous in the respective domains. Equations (2.1.1), (2.1.3) - (2.1.7), represent a complete formulation of a dynamic boundary value problem in viscoelasticity. Similarly equations (2.1.1), (2.1.2), (2.1.4) - (2.1.6) give a complete formulation for quasi-static problems in viscoelasticity.

2. The General Solution in the form of Potential Functions.

(a) Dynamic Problems

For this class of problems to simplify the subsequent analysis we assume similar behaviour in both shear and dilation. Then G_1 and G_2 will be related by a constant value ν of Poisson's ratio as follows

$$G_1(t) = \frac{(1 - 2\nu)}{(1 + \nu)} G_2(t) . \tag{2.2.1}$$

If we combine equations (2.1.1), (2.1.3), (2.1.5) and (2.2.1) we can write the field equations, in terms of displacements alone, as

$$\left[\frac{G_1}{2} *d \left(u_{i,jj} + \frac{1}{1 - 2\nu} u_{j,ji}\right)\right] (\underline{x},t) + F_i(\underline{x},t) = \rho\frac{\partial^2 u_i}{\partial t^2} (\underline{x},t).$$

Following the usual vector notation we can write the above in the equivalent form

$$\left[\frac{G_1}{2} *d \left(\nabla^2\underline{u} + \frac{1}{1 - 2\nu} \underline{\nabla}(\underline{\nabla} \cdot \underline{u})\right)\right] (\underline{x},t) + \underline{F}(\underline{x},t) = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} (\underline{x},t). \tag{2.2.2}$$

Equation (2.2.2) can be simplified by the following device. We decompose the body force $\underline{F}$ and the displacement $\underline{u}$ in the following manner

$$\underline{F} = \underline{\nabla}F_1 + \underline{\nabla} \times \underline{F}_2 \tag{2.2.3}$$

and

$$\underline{u} = \underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi} \tag{2.2.4}$$

where ϕ and $\underline{\psi}$ are unknown scalar and vector functions respectively. By elementary vector analysis we have

$$\underline{\nabla} \cdot \underline{u} = \nabla^2\phi . \tag{2.2.5}$$

If we substitute (2.2.3), (2.2.4), and (2.2.5) into (2.2.2) we obtain

$$\left[\frac{G_1}{2} * d\left(\nabla^2(\underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi}) + \frac{1}{(1-2\nu)}\,\underline{\nabla}(\nabla^2\phi)\right)\right](\underline{x},t)$$

$$+ \underline{\nabla}F_1 + \underline{\nabla} \times \underline{F}_2 = \rho\frac{\partial^2}{\partial t^2}(\underline{\nabla}\phi + \underline{\nabla} \times \underline{\psi})$$

which can also be written as

$$\underline{\nabla}\left\{\left(\frac{1-\nu}{1-2\nu}\right)[G_1 * d\nabla^2\phi](\underline{x},t) + F_1 - \rho\frac{\partial^2\phi}{\partial t^2}\right\}$$
$$+ \underline{\nabla} \times \left\{[\frac{G_1}{2} * d\nabla^2\underline{\psi}](\underline{x},t) + \underline{F}_2 - \rho\frac{\partial^2\underline{\psi}}{\partial t^2}\right\} = 0. \tag{2.2.6}$$

Thus (2.2.4) leads to a solution of (2.2.2) provided ϕ and $\underline{\psi}$ are chosen to satisfy the equations

$$\frac{1-\nu}{1-2\nu}[G_1 * d\nabla^2\phi](\underline{x},t) = \rho\frac{\partial^2\phi}{\partial t^2} - F_1 \quad , \tag{2.2.7}$$

$$[\frac{G_1}{2} * d\nabla^2\underline{\psi}](\underline{x},t) = \rho\frac{\partial^2\underline{\psi}}{\partial t^2} - \underline{F}_2 \, . \tag{2.2.8}$$

The general procedure of writing the displacement $\underline{u}$ in the form of (2.2.4) is discussed more extensively in Sneddon and Berry [32] for the dynamic elastic problem.

(b) Quasi-Static Problems

For quasi-static problems we shall not make the assumption (2.2.1). Gurtin and Sternberg[28] have given a generalized Papkovitch Neuber solution of equations (2.1.1), (2.1.2), (2.1.4) and (2.1.5).
In particular if F_i is set zero then

$$\underline{u} = [(G_1 + 2G_2)*d\underline{\nabla}(\phi + \underline{x}\cdot\underline{\psi})](\underline{x},t) - 4[(2G_1 + G_2)*d\underline{\psi}](\underline{x},t), \tag{2.2.9}$$

where

$$\nabla^2\phi = \nabla^2\underline{\psi} = 0. \tag{2.2.10}$$

3. Axisymmetric Problems.

(a) Dynamic Problems

We choose the co-ordinate system (r,θ,z) and let

$$\begin{aligned} \phi &= \phi(r,z,t) \quad , \\ \underline{\psi} &= (0,\psi(r,z,t),0) \ . \end{aligned} \tag{2.3.1}$$

With the choice of (2.3.1) equations (2.2.7) and (2.2.8) take the form

$$\frac{(1-\nu)}{(1-2\nu)}\left[G_1*d\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}\right)\right](r,z,t) = \rho\frac{\partial^2\phi}{\partial t^2}(r,z,t) \ , \tag{2.3.2}$$

$$\left[\frac{G_1}{2}*d\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2\psi}{\partial z^2}\right)\right](r,z,t) = \rho\frac{\partial^2\psi}{\partial t^2}(r,z,t), \tag{2.3.3}$$

where we have set F_i zero. We also record, for future use, the displacement and stresses in the co-ordinate system in terms of ϕ and $\underline{\psi}$ as given by (2.3.1). If we combine (2.3.1) and (2.2.4) then we arrive at

$$\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z} \quad , \\ u_\theta &= 0 \quad , \\ u_z &= \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r} \ . \end{aligned} \tag{2.3.4}$$

When we substitute (2.3.4) into (2.1.1) and (2.1.5) we obtain

$$\sigma_{rr} = \left[G_1 * d\left(\frac{\partial^2\phi}{\partial r^2} - \frac{\partial^2\psi}{\partial z\partial r}\right)\right] + \frac{\nu}{1-2\nu}\,[G_1 * d(\nabla^2\phi)] \ ,$$

$$\sigma_{\theta\theta} = \frac{1}{r}\left[G_1 * d\left(\frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}\right)\right] + \frac{\nu}{1-2\nu}\,[G_1 * d(\nabla^2\phi)] \ , \qquad (2.3.5)$$

$$\sigma_{zz} = \left[G_1 * d\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\psi}{\partial z\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial z}\right)\right] + \frac{\nu}{1-2\nu}[G_1 * d(\nabla^2\phi)] \ ,$$

$$\sigma_{zr} = \frac{1}{2}\left[G_1 * d\left(\frac{2\partial^2\phi}{\partial z\partial r} - \frac{2\partial^2\psi}{\partial z^2} + \nabla^2\psi - \frac{\psi}{r^2}\right)\right] \ ,$$

$$\sigma_{\theta r} = \sigma_{\theta z} = 0 \ ,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \ .$$

(b) Quasi-Static Problems

In this case, for ϕ and ψ in (2.2.10) we let

$$\phi(r,z,t) = [(2G_1 + G_2) * d(G_1 + 2G_2)^{-1} * dx](r,z,t),$$

$$\psi_1 = \psi_2 = 0 \ , \quad \psi_3 = \frac{\partial x}{\partial z}\,(r,z,t). \qquad (2.3.6)$$

With the aid of (2.3.6), we can write the displacements and stresses as

$$u_r = [3G_1 * d(G_1 + 2G_2)^{-1} * d\,\frac{\partial x}{\partial r}] + z\frac{\partial^2 x}{\partial r\partial z} \ ,$$

$$u_z = -2[(2G_1 + G_2) * d(G_1 + 2G_2)^{-1} * \frac{\partial x}{\partial z}] + z\frac{\partial^2 x}{\partial z^2} \ ,$$

$$\sigma_{rr} = -G_1 * d[\frac{\partial^2 x}{\partial z^2} - z\frac{\partial^3 x}{\partial r^2\partial z} + \frac{3}{r}G_1 * d(G_1 + 2G_2)^{-1} * d\frac{\partial x}{\partial r}] \ ,$$

$$\sigma_{\theta\theta} = -G_1 * d[\frac{\partial^2 x}{\partial z^2} - \frac{z}{r}\frac{\partial^2 x}{\partial r\partial z} + 3G_1 * d(G_1 + 2G_2)^{-1} * d\frac{\partial^2 x}{\partial r^2}] \ , \qquad (2.3.7)$$

$$\sigma_{zz} = -G_1 * d[\frac{\partial^2 x}{\partial z^2} - z\frac{\partial^3 x}{\partial z^3}] \ ,$$

$$\sigma_{zr} = zG_1 * d\frac{\partial^3 x}{\partial r\partial z^2} \ ,$$

$$u_\theta = \sigma_{\theta z} = \sigma_{\theta r} = 0,$$

where

$$\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = 0. \tag{2.3.8}$$

In the elastic case, that is when

$$G_1(t) = 2\mu H(t) \quad \text{and} \quad G_2(t) = 3KH(t) \ , \tag{2.3.9}$$

equations (2.3.7) with the use of (1.1.3) reduce to

$$\begin{aligned}
u_r &= (1 - 2\nu)\frac{\partial \chi}{\partial r} + z\frac{\partial^2 \chi}{\partial r \partial z} \ , \\
u_z &= -2(1 - \nu)\frac{\partial \chi}{\partial z} + z\frac{\partial^2 \chi}{\partial z^2} \ , \\
\sigma_{rr} &= -2\mu\left(\frac{\partial^2 \chi}{\partial z^2} - z\frac{\partial^3 \chi}{\partial r^2 \partial z} + \frac{(1 - 2\nu)}{r}\frac{\partial \chi}{\partial r}\right) \ , \\
\sigma_{\theta\theta} &= -2\mu\left(\frac{\partial^2 \chi}{\partial z^2} - \frac{z}{r}\frac{\partial^2 \chi}{\partial r \partial z} + (1 - 2\nu)\frac{\partial^2 \chi}{\partial r^2}\right) \ , \\
\sigma_{zz} &= -2\mu\left(\frac{\partial^2 \chi}{\partial z^2} - z\frac{\partial^3 \chi}{\partial z^3}\right) \ , \\
\sigma_{zr} &= 2\mu z\frac{\partial^3 \chi}{\partial r \partial z^2} \ , \\
u_\theta &= \sigma_{\theta z} = \sigma_{\theta r} = 0.
\end{aligned} \tag{2.3.10}$$

Equations (2.3.10) represent a general solution to the field equations of elasticity in the case of axisymmetry. The field equations of linear elasticity are given by (2.1.1), (2.1.2), (2.1.4) and in place of (2.1.5) we have

$$\sigma_{ij}(\underline{x},t) = 2\mu e_{ij}(\underline{x},t) + \delta_{ij}\left(\frac{3K - 2\mu}{3}\right) e_{KK}(\underline{x},t). \tag{2.3.11}$$

4. Torsion Problems.

When considering torsion problems in linear viscoelasticity or elasticity we assume that $u_r = u_z = 0$ and u_θ does not depend on θ. This assumption leads to following stress-displacement relationships in viscoelasticity

$$\sigma_{z\theta}(r,z,t) = \left[\frac{G_1}{2} *d \frac{\partial u_\theta}{\partial z}\right](r,z,t) \quad ,$$
$$\sigma_{r\theta}(r,z,t) = \left[\frac{G_1}{2} *d \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right](r,z,t) \tag{2.4.1}$$

and

$$\sigma_{zz} = \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zr} = 0. \tag{2.4.2}$$

Similarly we find in elasticity that

$$\sigma_{z\theta}(r,z) = \mu \frac{\partial u_\theta}{\partial z}(r,z)$$
$$\sigma_{r\theta}(r,z) = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \quad , \tag{2.4.3}$$

and (2.4.2) hold for this problem.

The equation of motion, in this case, in viscoelasticity can be written as

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 2\rho\left[G_1^{-1} *d \left(\frac{\partial^2 u_\theta}{\partial t^2}\right)\right] \tag{2.4.4}$$

for dynamic problems and

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{2.4.5}$$

for quasi-static problems. For elastic problems equation (2.4.4) takes the form

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{\rho}{\mu}\frac{\partial^2 u_\theta}{\partial t^2} \tag{2.4.6}$$

whereas (2.4.5) remains the same.

CHAPTER III

THE IMPACT OF A RIGID AXISYMMETRIC INDENTOR ON A VISCOELASTIC HALF-SPACE

1. Statement of the Problem.

Suppose that the region R is the half-space $z \geq 0$ with the boundary B given by the plane $z = 0$. We consider the problem of determining the displacement and stress fields set up in a viscoelastic half-space when it is deformed by the impact of an axisymmetric indentor of smooth profile. It is assumed that over the contact area, $\Omega(t)$, the normal surface displacement must conform to the surface geometry of the indentor while outside $\Omega(t)$ the normal traction vanishes. In addition the boundary is assumed to be free of shear tractions. Thus in terms of circular cylindrical coordinates (r,θ,z), we are considering the problem governed by the following boundary conditions:

$$\begin{aligned} \sigma_{rz}(r,0,t) &= \sigma_{\theta z}(r,0,t) = 0 &&, \ r \geq 0, \\ u_z(r,0,t) &= D(t) - b(r)H(t) &&, \ 0 \leq r \leq a(t), \\ \sigma_{zz}(r,0,t) &= 0 &&, \ r > a(t), \end{aligned} \tag{3.1.1}$$

and the conditions at infinity

$$\sigma_{ij}(\underset{\sim}{x},t) \to 0 \quad \text{as} \quad \sqrt{(x_i x_i)} \to \infty \quad , \tag{3.1.2}$$

where the field quantities are independent of θ and $a(t)$, the contact radius, is initially an increasing function of time. We assume the body force F_i is zero. In (3.1.1) b is prescribed by the surface of the indentor and $D(t)$ is the depth of penetration, at time t, of its tip into the half-space. Note that we can only specify one of either the contact radius $a(t)$ or the depth of penetration $D(t)$. This is due to the condition that if the indentor has a smooth profile the normal stress σ_{zz} must remain finite around the edge of contact between the indentor and half-space (see Sneddon [33] or Graham [34]). However for the impact problem we have an additional equation (that arises from Newton's second law of motion) which determines $D(t)$ (or $a(t)$) in terms of the impact velocity V. The equation is

$$m \frac{d^2D(t)}{dt^2} = -P(t) \tag{3.1.3}$$

where m is the mass of the indentor, P is the total pressure over the contact area, and where we have used the fact that

$$\frac{d^2u}{dt^2}(0,0,t) = \frac{d^2D(t)}{dt^2} .$$

To complete the statement of the problem we prescribe that initially the half-space is undeformed, i.e.,

$$\sigma_{ij}(\underline{x},0) = u_i(\underline{x},0) = 0 \tag{3.1.4}$$

and in conjunction with (3.1.3)

$$D(0) = 0 \quad , \quad \frac{dD(0)}{dt} = V . \tag{3.1.5}$$

Equations (2.3.2) - (2.3.5) subject to the boundary and initial conditions (3.1.1) - (3.1.5) now determine a dynamic viscoelastic impact problem.

In order to solve the above problem we first consider an associated second problem given by the equations (2.3.2) - (2.3.5), (3.1.1) - (3.1.4) and

$$D(0) = 0 \quad , \quad \frac{dD(0)}{dt} = 0 . \tag{3.1.6}$$

If $(\hat{u}_i, \hat{e}_{ij}, \hat{\sigma}_{ij})$ is the solution to the second problem and $(u_i, e_{ij}, \sigma_{ij})$ is the solution to the first problem then it is simple to verify that the two solutions will be identical except for the normal displacement where we have

$$u_z(r,z,t) = \hat{u}_z(r,z,t) + Vt \quad . \tag{3.1.7}$$

Hence we can solve the second problem and then use (3.1.7) to obtain the complete solution to the original problem. The reason for this procedure is to eliminate complications that arise when we use the Laplace transform.

2. A General Solution of an Axisymmetric Shear Free Half-Space Problem in Dynamic Viscoelasticity.

We begin by finding general solutions to the equations (2.3.2) and (2.3.3) subject to the conditions (3.1.2), (3.1.4) and (3.1.6). Keeping the last two conditions in mind, we assume that

$$\phi(r,z,0) = \frac{\partial\phi}{\partial t}(r,z,0) = \psi(r,z,0) = \frac{\partial\psi}{\partial t}(r,z,0) = 0. \tag{3.2.1}$$

We now apply the Laplace transform to both (2.3.2) and (2.3.3) and with the aid of (1.1.9) and (1.1.8) find

$$\frac{\partial^2\overline{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\phi}}{\partial r} + \frac{\partial^2\overline{\phi}}{\partial z^2} = K_1^2\overline{\phi} \ , \tag{3.2.2}$$

$$\frac{\partial^2\overline{\psi}}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{\psi}}{\partial r} - \frac{\overline{\psi}}{r^2} + \frac{\partial^2\overline{\psi}}{\partial z^2} = K_2^2\overline{\psi} \ , \tag{3.2.3}$$

where we define K_1 and K_2 as

$$K_1^2 = \rho\frac{(1-2\nu)}{(1-\nu)}\frac{s}{\overline{G}_1(s)} \ ,$$
$$K_2^2 = \frac{2\rho s}{\overline{G}_1(s)} \ , \tag{3.2.4}$$

and note the relation

$$K^2 = K_2^2 \;/\; K_1^2 \;=\; \frac{2(1-\nu)}{1-2\nu} \quad . \tag{3.2.5}$$

At this point, we also introduce the elastic wave speeds c_1 and c_2 as

$$c_1^2 = \frac{G_1(0)}{\rho} \frac{(1-\nu)}{(1-2\nu)} \quad ,$$
$$c_2^2 = \frac{G_1(0)}{2\rho} \quad . \tag{3.2.6}$$

On taking the Hankel transforms of orders zero and one of equations (3.2.2) and (3.2.3) respectively, and following the notation of (1.2.1) and the results (1.2.5) and (1.2.6), we obtain

$$\frac{d^2\overline{\phi}_o^*}{dz^2} - (\xi^2 + K_1^2)\overline{\phi}_o^* = 0 \quad ,$$
$$\frac{d^2\overline{\psi}_1^*}{dz^2} - (\xi^2 + K_2^2)\overline{\psi}_1^* = 0 \quad . \tag{3.2.7}$$

Solutions of these equations can be written as

$$\overline{\phi}_o^* = A \; e^{-\alpha z} + C_1 \; e^{\alpha z} \; ,$$
$$\overline{\psi}_1^* = B \; e^{-\beta z} + C_2 \; e^{\beta z} \; , \tag{3.2.8}$$

where

$$\alpha^2 = \xi^2 + K_1^2 \quad ,$$
$$\beta^2 = \xi^2 + K_2^2 \quad , \tag{3.2.9}$$

and A, B, C_1 and C_2 are undetermined constants. Recalling the condition (3.1.2) we note that we must set $C_1 = C_2 = 0$ so that ϕ and ψ remain finite for large z. Hence equations (3.2.8) become

$$\bar{\phi}^*_o = A\, e^{-\alpha z} \quad ,$$
$$\bar{\psi}^*_o = B\, e^{-\beta z} \quad . \qquad (3.2.10)$$

We now determine a relationship between the unknown constants A and B by using the first boundary condition of (3.1.1). To do this we take the Laplace transform of σ_{zr} as given by (2.3.5) and after making use of (3.2.3) obtain

$$\bar{\sigma}_{zr} = \frac{s\bar{G}_1}{2}\left[\frac{\partial^2\bar{\phi}}{2\partial z\partial r} - 2\frac{\partial^2\bar{\psi}}{\partial z^2} + K_2^2\bar{\psi}\right] \qquad (3.2.11)$$

Next we apply Hankel transform of order one to the above equation and with the use of (1.2.10) find

$$\bar{\sigma}^*_{zr_1} = \frac{s\bar{G}_1}{2}\left[2\frac{d}{dz}\,(-\xi\bar{\phi}^*_o) - 2\frac{d^2}{dz^2}\;\bar{\psi}^*_1 + K_2^2\bar{\psi}^*_1\right] \quad . \qquad (3.2.12)$$

Substitution of (3.2.10) into (3.2.12), when $z = 0$, gives

$$\bar{\sigma}^*_{zr_1} = \frac{s\bar{G}_1}{2}\left[2\xi\alpha A - 2\beta^2 B + K_2^2 B\right] \quad . \qquad (3.2.13)$$

However, by (3.1.1), we have that when $z = 0$, $\bar{\sigma}^*_{zr_1} = 0$. Hence on combining this result with (3.2.13) we get

$$B = \frac{2\xi\alpha}{2\beta^2 - K_2^2}\, A \quad . \qquad (3.2.14)$$

On substituting (3.2.14) into (3.2.10) and applying the Hankel and Laplace inversion theorems we determine

$$\phi(r,z,t) = L^{-1}\left\{\int_o^\infty \xi A(\xi) e^{-\alpha z}\, J_o(\xi r) d\xi \quad ; \quad s \to t\right\} \quad ,$$
$$\psi(r,z,t) = L^{-1}\left\{\int_o^\infty \frac{2\xi^2\alpha A(\xi) e^{-\beta z}}{(2\xi^2 + K_2^2)}\, J_1(\xi r) d\xi \; ; \; s \to t\right\} . \qquad (3.2.15)$$

The functions ϕ and ψ defined above determine the displacements and stresses through equations (2.3.4), (2.3.5) within an arbitrary function, $A(\xi)$. These displacements and stresses are now stated for future use:

$$u_r = -L^{-1}\left\{\int_0^\infty \left[e^{-\alpha z} - \frac{2\alpha\beta e^{-\beta z}}{2\xi^2 + K_2^2}\right] \xi^2 A(\xi) J_1(\xi r) d\xi \quad ; \quad s \to t\right\} ,$$

$$u_z = -L^{-1}\left\{\int_0^\infty \left[e^{-\alpha z} - \frac{2\xi^2 e^{-\beta z}}{2\xi^2 + K_2^2}\right] \xi\alpha A(\xi) J_o(\xi r) d\xi \quad ; \quad s \to t\right\} ,$$

$$G_1^{-1} * d\sigma_{rr} = L^{-1}\left\{\int_0^\infty \left[e^{-\alpha z} - \frac{2\alpha\beta e^{-\beta z}}{2\xi^2 + K_2^2}\right] \xi^2 A(\xi) \left[\frac{J_1(\xi r)}{r} - \xi J_o(\xi r)\right] d\xi \right.$$

$$\left. + \frac{\nu K_1^2}{1 - 2\nu} \int_0^\infty \xi e^{-\alpha z} A(\xi) J_o(\xi r) d\xi \quad ; \quad s \to t \right\} ,$$

$$G_1^{-1} * d\sigma_{\theta\theta} = L^{-1}\left\{-\frac{1}{r} \int_0^\infty \left[e^{-\alpha z} - \frac{2\alpha\beta}{2\xi^2 + K_2^2}\right] \xi^2 A(\xi) J_1(\xi r) d\xi \right.$$

$$\left. + \frac{\nu K_1^2}{1 - 2\nu} \int_0^\infty \xi e^{-\alpha z} A(\xi) J_o(\xi r) d\xi \quad ; \quad s \to t \right\} , \qquad (3.2.16)$$

$$G_1^{-1} * d\sigma_{zz} = \frac{1}{2} L^{-1}\left\{\int_0^\infty \left[(2\xi^2 + K_2^2) e^{-\alpha z} - \frac{4\xi^2\alpha\beta e^{-\beta z}}{2\xi^2 + K_2^2}\right] \xi A(\xi) J_o(\xi r) d\xi \quad ; \quad s \to t\right\} ,$$

$$G_1^{-1} * d\sigma_{zr} = L^{-1}\left\{\int_0^\infty [e^{-\alpha z} - e^{-\beta z}] \xi^2 \alpha A(\xi) J_1(\xi r) d\xi \quad ; \quad s \to t\right\} .$$

We point out that in deriving equations (3.2.16) the following equations and results were used : (1.2.8), (1.2.9), (1.2.10), (3.2.3), (3.2.4), (3.2.5), (2.3.3) and (2.3.4). Equations (3.2.16) then represent a solution of a class of axisymmetric shear free half-space problems in dynamic viscoelasticity.

The unknown function A will be determined by the specification of further boundary conditions on B. The result of these boundary conditions when imposed will be the creation of one or more integral equations. The solution of these equations will be the unknown function A. Hence it follows that we can reduce a class of boundary value problems in the dynamic theory of viscoelasticity to the solution of one or more integral equations in terms of a single unknown function.

3. The Solution of Certain Dual Integral Equations.

In section 3.2 we have found a general solution to equations (2.3.2) - (2.3.5) subject to (3.1.2), (3.1.4), (3.1.6) and the first condition of (3.1.1). We shall now impose the remaining two conditions of (3.1.1) upon the solution given by equations (3.2.16). The result is a pair of dual integral equations:

$$-L^{-1}\left\{\int_0^\infty \left(1 - \frac{2\xi^2}{2\xi^2 + K_2^2}\right) \xi\alpha A(\xi) J_0(\xi r)d\xi \quad ; \quad s \to t\right\} = D(t) - b(r),\ 0 \le r \le a(t) \tag{3.3.1}$$

$$\frac{G_1}{2} * dL^{-1}\int_0^\infty \left[2\xi^2 + K_2^2 - \frac{4\xi^2\alpha\beta}{2\xi^2 + K_2^2}\right] \xi A(\xi) J_0(\xi r)d\xi \quad ; \quad s \to t\Bigg\} = 0,\ r > a(t)\ . \tag{3.3.2}$$

We can reduce the complexity of these equations by defining a new function $P(r,t)$ such that

$$-\overline{P}_o^* = \left[(2\xi^2 + K_2^2) - \frac{4\xi^2\alpha\beta}{(2\xi^2 + K_2^2)}\right] A(\xi). \tag{3.3.3}$$

If we substitute (3.3.3) into (3.3.1) and (3.3.2) we obtain

$$L^{-1}\left\{\int_0^\infty \frac{\xi\alpha K_2^2\ \overline{P}_o^*(\xi,s)}{[(2\xi^2 + K_2^2)^2 - 4\xi^2\alpha\beta]} J_0(\xi r)d\xi \ ;\ s \to t\right\} = D(t) - b(r),\ 0 \le r \le a(t) \tag{3.3.4}$$

$$[\frac{G_1}{2} * dP](r,t) = 0 \quad , \quad r > a(t)\ . \tag{3.3.5}$$

Since $a(t)$ is monotonically increasing, it follows from (1.1.5) and (3.3.5) that

$$P(r,t) = 0 \quad , \quad r > a(t) . \tag{3.3.6}$$

Furthermore we observe that

$$\sigma_{zz}(r,0,t) = - [\frac{G_1}{2} * dP](r,t) . \tag{3.3.7}$$

Before proceeding with the solution of (3.3.4) we note that we must find the Laplace inverse of the function $\overline{f}(\xi,s)$ defined as

$$\overline{f}(\xi,s) = \frac{\xi\alpha K_2}{[(2\xi^2 + K_2^2)^2 - 4\xi^2\alpha\beta]} \tag{3.3.8}$$

If we let $\eta = K_2/\xi$ then by using (3.2.5) and (3.2.9) we find

$$\overline{f}(\xi,s) = \frac{1}{\xi}\frac{\eta(1 + \eta^2/K^2)^{\frac{1}{2}}}{[(2 + \eta^2)^2 - 4(1 + \eta^2)^{\frac{1}{2}}(1 + \eta^2/K^2)^{\frac{1}{2}}]} \tag{3.3.9}$$

The above equation with the help of (A.1) and (A.17) can be rewritten as

$$\overline{f}(\xi,s) = \frac{(1 - \nu)}{K_2} + c_2\textstyle\sum_1 \displaystyle\int_0^\infty \cos(\xi y c_2\tau)e^{-K_2 c_2\tau}d\tau . \tag{3.3.10}$$

where $\sum_1$ is defined by (A.17). We point out that in this equation we have introduced the elastic wave speed c_2 (see (3.2.6)) for dimensionality purposes.

By (3.3.8) and (3.3.10) we now simplify (3.3.4) to get the following

$$(1 - \nu) \int_0^\infty P_0^*(\xi,t)J_0(\xi r)d\xi = D(t) - b(r) \tag{3.3.11}$$

$$- \textstyle\sum_1 \displaystyle\int_0^\infty J_0(\xi r)d\xi \int_0^\infty \cos(\xi y c_2 x)[u_2 * dP_0^*](x,\xi,t)dx, \quad 0 \le r \le a(t),$$

where

$$u_2(t,\tau) = L^{-1}\left[\frac{c_2 K_2(s)}{s} e^{-K_2 c_2\tau} \quad ; \quad s \to t\right] . \tag{3.3.12}$$

We observe that (3.3.11) has the form of (B.10) if we treat the right hand side as a known function $f(r)$. Also by (3.3.6), P satisfies (B.2). Hence we can make use of the result (B.13) in (3.3.11) to obtain

$$P(r,t) = -\frac{2}{\pi r(1-\nu)}\frac{d}{dr}\int_r^{a(t)}\frac{\xi d\xi}{(\xi^2-r^2)^{\frac{1}{2}}}\frac{d}{d\xi}\int_0^{\xi}\frac{x[D(t)-b(x)]}{(\xi^2-x^2)^{\frac{1}{2}}}dx$$

$$+\frac{2\sum_1}{\pi r(1-\nu)}\frac{\partial}{\partial r}\int_r^{a(t)}\frac{\rho d\rho}{(\rho^2-r^2)^{\frac{1}{2}}}\int_0^{\infty}\cos(\xi\rho)d\xi\int_0^{\infty}\cos(\xi y c_2 x)\,.$$

$$[u_2 * dP_o^*](x,\xi,t)dx \quad , \tag{3.3.13}$$

where we have also used the result (see [35]).

$$\int_0^{\rho}\frac{xJ_o(\xi x)}{(\rho^2-x^2)^{\frac{1}{2}}}dx = \frac{\sin(\xi\rho)}{\xi} \quad . \tag{3.3.14}$$

For notational convenience we denote P_I as

$$P_I(r,t) = \int_0^{\infty}\cos(\xi r)d\xi\int_0^{\infty}\cos(\xi y c_2 x)\,.$$

$$[u_2 * dP_o^*](x,\xi,t)dx \tag{3.3.15}$$

If we use (3.3.15) in (3.3.13) and integrate the right hand side by parts, we find

$$P(r,t) = \frac{2}{\pi(1-\nu)}\frac{D(t)}{(a^2-r^2)^{\frac{1}{2}}} - \frac{2}{\pi(1-\nu)}\frac{a(t)}{(a^2-r^2)^{\frac{1}{2}}}\int_0^{a(t)}\frac{b'(x)dx}{[a^2-x^2]^{\frac{1}{2}}}$$

$$+\frac{2}{\pi(1-\nu)}\int_r^{a(t)}\frac{1}{(\xi^2-r^2)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}\xi\int_0^{\xi}\frac{b'(x)dx}{[\xi^2-x^2]^{\frac{1}{2}}}$$

$$-\frac{2\sum_1}{\pi(1-\nu)}\frac{P_I(a(t),t)}{(a^2-r^2)^{\frac{1}{2}}} + \frac{2}{\pi}\frac{\sum_1}{(1-\nu)}\int_r^{a(t)}\frac{1}{(\xi^2-r^2)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}P_I(\xi,t)d\xi \,. \tag{3.3.16}$$

From (3.3.16) we deduce the following two equations

$$P(r,t) = + \frac{2}{\pi(1 - \nu)} \int_r^{a(t)} \frac{1}{(\xi^2 - r^2)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} \xi \int_0^{\xi} \frac{b'(x)dx}{[\xi^2 - x^2]^{\frac{1}{2}}} \quad ,$$

$$+ \frac{2\sum_1}{\pi(1 - \nu)} \int_r^{a(t)} \frac{1}{(\xi^2 - r^2)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} P_I(\xi,t)d\xi \quad , \qquad (3.3.17)$$

and

$$D(t) = a(t) \int_0^{a(t)} \frac{b'(x)dx}{[a^2 - x^2]^{\frac{1}{2}}} + \textstyle\sum_1 P_I(a(t),t). \qquad (3.3.18)$$

The solution of equations (3.3.17) and (3.3.18) gives the solution of the dual integral equations (3.3.1) and (3.3.2). When these equations are solved we then have a solution to equations (2.3.2) - (2.3.5) subject to (3.1.2), (3.1.4), (3.1.6) and (3.1.1). Furthermore if this solution also satisfies (3.1.3) we will then have the solution to the second problem posed in section 3.1. The addition of the term Vt to the normal displacement, u_z, will complete the solution to the original problem.

In view of (3.3.7) we note that we can rewrite (3.1.3) as

$$m \frac{d^2D}{dt^2}(t) = -P(t) = -2\pi \int_0^{a(t)} r\sigma_{zz}(r,0,t)dr \quad ,$$

$$m \frac{d^2D}{dt^2}(t) = -\pi \int_0^{a(t)} r[G_I * dP](r,t)dr. \qquad (3.3.19)$$

When the solution to the above equations is found the general displacements and stresses are determined by equations (3.2.16) and (3.3.3). In particular in the plane $z = 0$ the displacements and stresses can be integrated to give

$$u_r = -\frac{(1-2\nu)}{2r}\int_0^\infty P_o^*(\xi,t)J_1(\xi r)d\xi - \frac{\sum_2}{2}T_2 \quad ,$$

$$u_z = (1-\nu)\int_0^\infty P_o^*(\xi,t)J_o(\xi,r)d\xi + \sum_1 T_3 \quad ,$$

$$G_1^{-1}*d\sigma_{\theta\theta} = -\nu P(r,t) - \frac{(1-2\nu)}{2r}\int_0^\infty P_o^*(\xi,t)J_1(\xi r)d\xi$$

$$- \frac{\nu}{2(1-\nu)}\sum_2 T_1 - \frac{1}{2r}\sum_2 T_2 \quad ,$$

$$G_1^{-1}*d\sigma_{rr} = -\frac{P(r,t)}{2} + \frac{(1-2\nu)}{2r}\int_0^\infty P_o^*(\xi,t)J_1(\xi r)d\xi$$

$$- \frac{1}{2(1-\nu)}\sum_2 T_1 + \frac{1}{2r}\sum_2 T_2 \quad ,$$

$$G_1^{-1}*d\sigma_{zz} = -\frac{1}{2}P(r,t) \quad , \tag{3.3.20}$$

$$\sigma_{zr} = 0 \ ,$$

where $\sum_2$ is defined by (A.25) and

$$T_1 = \int_0^\infty \xi J_o(\xi r)\int_0^\infty \cos(c_2 y\xi\tau)[u_2*dP_o^*](\xi,t)d\tau d\xi \quad ,$$

$$T_2 = \int_0^\infty J_1(\xi r)\int_0^\infty \cos(c_2 y\xi\tau)[u_2*dP_o^*](\xi,t)d\tau d\xi \quad , \tag{3.3.21}$$

$$T_3 = \int_0^\infty J_o(\xi r)\int_0^\infty \cos(c_2 y\xi\tau)[u_2*dP_o^*](\xi,t)d\tau d\xi \quad .$$

We note that equations (3.3.20) were arrived at by using equations (3.2.16), (3.3.3), (A.20), (A.21), (A.26) and (A.27).

We emphasize that to complete the solution of the problem the surface geometry of the indentor must be specified and that the unknown functions $P(r,t)$, $D(t)$ and $a(t)$ are determined from the solution of (3.3.17) - (3.3.19) and (3.3.15).

4. The Case of a Spherical Indentor

In this section we shall consider the impact of a rigid sphere with a viscoelastic half-space. For this case the condition on u_z in (3.1.1) is given as

$$u_z(r,0,t) = D(t) - \frac{r^2}{2R} H(t) \quad , \quad 0 \leq r \leq a(t) \tag{3.4.1}$$

where R is the radius of the sphere. As a result equations (3.3.17) and (3.3.18) can be simplified to be

$$P(r,t) = \frac{4}{\pi R(1 - \nu)} (a^2(t) - r^2)^{\frac{1}{2}} + \frac{2\sum_1}{\pi(1 - \nu)} \int_r^{a(t)} \frac{1}{(\xi^2 - r^2)^{\frac{1}{2}}} \frac{\partial}{\partial \xi} P_I(\xi,t) d\xi \ , \tag{3.4.2}$$

and

$$D(t) = \frac{a^2(t)}{R} + \sum\nolimits_1 P_I(a(t),t) \ . \tag{3.4.3}$$

Equations (3.4.3) and (3.3.19) can be combined to give one equation by eliminating $D(t)$. We do this by integrating (3.3.19) twice and using (3.1.5). The resulting equation combines with (3.4.3) to give

$$a^2(t) = RVt - \frac{\pi R}{m} \int_0^t \int_0^{\theta_1} \int_0^{a(\theta_2)} r[G_1 * dP](r,\theta_2) dr d\theta_2 d\theta_1 - \sum\nolimits_1 P_I(a(t),t) \ . \tag{3.4.4}$$

Equations (3.4.2) and (3.4.4) differ from the results in the quasi-static case by the terms involving P_I. To gain an approximate solution we shall use the leading term in (3.4.2) as a first approximation for $P(r,t)$. This approximation will then be used to calculate P_I and $a(t)$. This being done, a second approximation can then be calculated for $P(r,t)$

from (3.4.2). We note that the Hankel transform of the first approximation of P is

$$P_o^*(\xi,t) = \frac{4}{\pi R(1-\nu)}\left[\frac{\sin(\xi a(t))}{\xi^3} - \frac{a(t)\cos(\xi a(t))}{\xi^2}\right] . \tag{3.4.5}$$

[For simplicity sake we shall throughout this work treat approximations and exact expressions notationally the same.] We first use (3.4.5) to simplify P_I and then reduce (3.4.4) to a form suitable for numerical computations. Substitution of (3.4.5) into (3.3.15) gives

$$P_I(r,t) = \frac{4}{\pi R(1-\nu)}\int_0^\infty \cos(\xi r)d\xi \int_0^\infty \cos(\xi y c_2 x)\int_0^t u_2(t-\tau,x)\frac{\sin(a\xi)}{\xi}a\dot{a}d\tau dx,$$

$$P_I(r,t) = \frac{4}{\pi R(1-\nu)}\int_0^t a(\tau)\dot{a}(\tau)d\tau\int_0^\infty u_2(t-\tau,x)dx\int_0^\infty \cos(\xi r)\cos(\xi y c_2 x)\frac{\sin(a\xi)}{\xi}d\xi,$$

$$P_I(r,t) = \frac{}{\pi R(1-\nu)}\int_0^t a(\tau)\dot{a}(\tau)d\tau\int_0^\infty u_2(t-\tau,x)dx\int_0^\infty \sum_{i=1}^{4}\sin(\xi\gamma_i)\frac{d\xi}{\xi} \tag{3.4.6}$$

where

$$\begin{aligned}\gamma_1 &= a(\tau) + r + c_2 yx \quad ,\\ \gamma_2 &= a(\tau) + r - c_2 yx \quad ,\\ \gamma_3 &= a(\tau) - r - c_2 yx \quad ,\\ \gamma_4 &= a(\tau) - r + c_2 yx \quad .\end{aligned} \tag{3.4.7}$$

Since we have the result that

$$\int_0^\infty \sin(ax)\frac{dx}{x} = \frac{\pi}{2}\,\text{sign}(a) \quad , \tag{3.4.8}$$

we must determine the sign of γ_i in the region of integration $R = \{(x,\tau)\,|\,0 \le \tau \le t,\ 0 \le x < \infty\}$ for $r \le a(t)$. Table 3.1 gives the sign of γ_i in the regions indicated in Figure 3.1 as well as the total contribution, C, from different regions.

Table 3.1 The sign of γ_i in the region A_K .

	A_1	A_2	A_3	A_4
γ_1	+	+	+	+
γ_2	+	+	+	-
γ_3	-	+	-	-
γ_4	-	+	+	+
C	0	2π	π	0

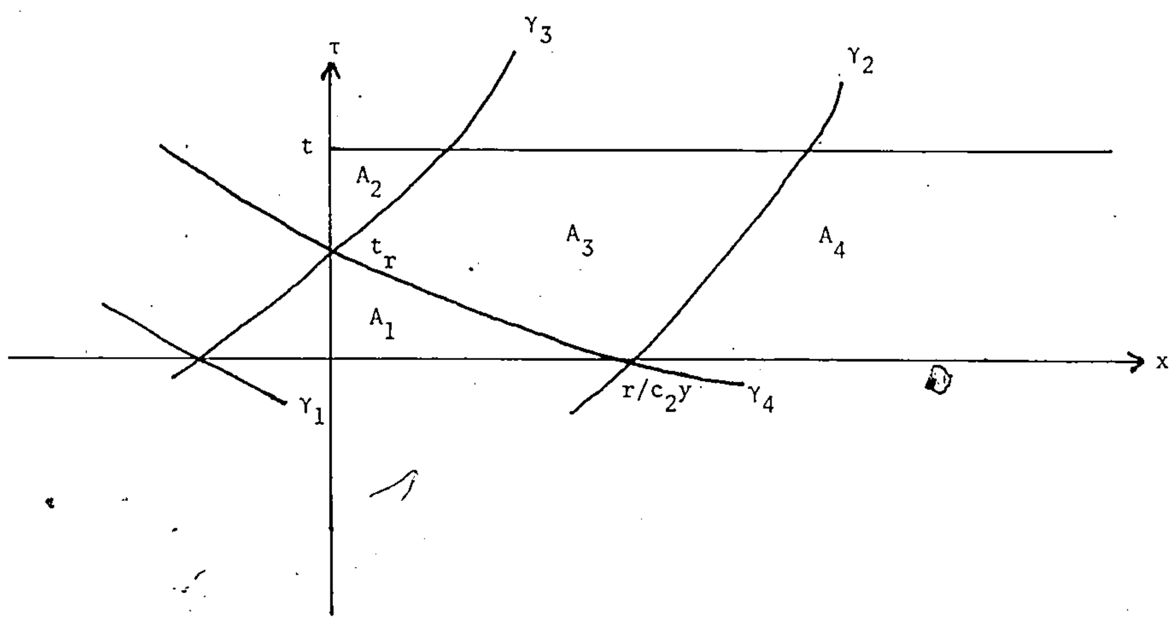


Figure 3.1 $r < a(t)$, $R = A_1 \cup A_2 \cup A_3 \cup A_4$, $a(t_r) = r$.

We can see that only two regions contribute in the integration of P_I. Using these results we can simplify (3.4.6) and obtain

$$P_I(r,t) = \frac{1}{\pi R(1-\nu)}\left\{\int_o^{t_r} a(\tau)\dot{a}(\tau)\int_{\frac{r-a(\tau)}{c_2 y}}^{\frac{r+a(\tau)}{c_2 y}} u_2(t-\tau,x)\pi dx d\tau\right.$$

$$+\int_{t_r}^{t} a(\tau)\dot{a}(\tau)\int_{\frac{a(\tau)-r}{c_2 y}}^{\frac{r+a(\tau)}{c_2 y}} u_2(t-\tau,x)\pi dx d\tau$$

$$\left.+\int_{t_r}^{t} a(\tau)\dot{a}(\tau)\int_{o}^{\frac{a(\tau)-r}{c_2 y}} u_2(t-\tau,x)2\pi dx d\tau\right\}. \qquad (3.4.9)$$

If we differentiate $P_I(r,t)$ with respect to r then the result is

$$\frac{\partial}{\partial r}P_I(r,t) = \frac{1}{R(1-\nu)c_2 y}\left\{\int_o^t a(\tau)\dot{a}(\tau)u_2\left(t-\tau, \frac{r+a(\tau)}{c_2 y}\right)d\tau\right.$$

$$-\int_o^{t_r} a(\tau)\dot{a}(\tau)u_2\left(t-\tau,\frac{r-a(\tau)}{c_2 y}\right)d\tau$$

$$\left.-\int_{t_r}^{t} a(\tau)\dot{a}(\tau)u_2\left(t-\tau,\frac{a(\tau)-r}{c_2 y}\right)d\tau\right\}. \qquad (3.4.10)$$

Let us now return to the consideration of equation (3.4.4). If we substitute our approximation for P in this equation and drop the last term we find

$$a^2(t) = RVt - \frac{4}{3(1-\nu)m}\int_o^t\int_o^{\theta_1}\int_o^{\theta_2} G_1(\theta_2-\theta_3)\frac{d}{d\theta_3}a^3(\theta_3)d\theta_3 d\theta_2 d\theta_1 .$$

We switch the order of integration of the first two integrals and after integrating the inner integral obtain

$$a^2(t) = RVt - \frac{4}{3(1-\nu)m}\int_o^t(t-\theta_2)\int_o^{\theta_2} G_1(\theta_2-\theta_3)\frac{da^3}{d\theta_3}(\theta_3)d\theta_3 d\theta_2 .$$

The orders of integration can be changed once more to give

$$a^2(t) = RVt - \frac{4}{3(1-\nu)m}\int_0^t \frac{da^3}{d\theta}(\theta)\int_\theta^t (t-\theta_2)G_1(\theta_2-\theta)d\theta_2 d\theta \quad . \tag{3.4.11}$$

To solve the integral equation (3.4.11) we use the technique suggested by Lee and Rogers [36]. To begin with we let $t = s\delta$, s is an integer and δ will be the step-size. With this change (3.4.11) can be written as

$$0 = a^2(s\delta) - RVs\delta + \frac{4}{3(1-\nu)m}\sum_{i=1}^{s}\int_{(i-1)\delta}^{i\delta}\frac{da^3}{d\theta}(\theta)\int_\theta^{s\delta}(s\delta-\theta_1)G_1(\theta_1-\theta)d\theta_1 d\theta \; . \tag{3.4.12}$$

If δ is small enough we may make the approximation,

$$\frac{da^3}{d\theta}(\theta) \doteq \frac{a^3(i\delta) - a^3((i-1)\delta)}{\delta} \quad \text{for} \quad (i-1)\delta \le \theta \le i\delta \tag{3.4.13}$$

This approximation transforms equation (3.4.12) into the form

$$0 = a^2(s\delta) - RVs\delta + \frac{4}{3(1-\nu)m}\sum_{i=1}^{s}\int_{(i-1)\delta}^{i\delta}\frac{a^3(i\delta)-a^3((i-1)\delta)}{\delta}\int_0^{s\delta}(s\delta-\theta_1)G_1(\theta_1-\theta)d\theta_1 d\theta \; . \tag{3.4.14}$$

Following [36] we write (3.4.14) as

$$0 = a^2(s\delta) - RVs\delta + \frac{4}{3(1-\nu)m}\sum_{i=1}^{s}[a^3(i\delta) - a^3((i-1)\delta)] \; .$$

$$\cdot \; \frac{1}{2}\left\{\int_{i\delta}^{s\delta}(s\delta-\theta_1)G_1(\theta_1-i\delta)d\theta_1 + \int_{(i-1)}^{s\delta}(s\delta-\theta_1)G_1(\theta_1-(i-1)\delta)d\theta_1\right\} . \tag{3.4.15}$$

By making the change of variables $\theta = \theta_1 - i\delta$ we can transform the two integrals above so that we have

$$a^2(s\delta) - RVs\delta + \frac{2}{3(1-\nu)m}\sum_{i=1}^{s}[a^3(i\delta) - a^3((i-1)\delta)].$$

$$[F(s-i) + F(s-i+1)] = 0 \; , \tag{3.4.16}$$

where

$$F(s) = \int_0^{s\delta}(s\delta - \theta)G_1(\theta)d\theta \; . \tag{3.4.17}$$

The series in (3.4.16) can be simplified if we introduce the notation that $a(i\delta) = a_{i+1}$. Using this notation, we write the series, A, in (3.4.16) as

$$A = \sum_{i=1}^{s} (a_{i+1}^3 - a_i^3)[F(s-i) + F(s-i+1)] \quad ,$$

$$A = \sum_{i=1}^{s-1} a_{i+1}^3[F(s-i) + F(s-i+1)] + a_{s+1}^3[F(0) + F(1)]$$

$$- \sum_{i=2}^{s} a_i^3[F(s-i) + F(s-i+1)] - a_1^3[F(s-1) + F(s)].$$

However, from (3.4.17), we have $F(0) = 0$ and also that $a_1^3 = a^3(0) = 0$. In the second series above we let $j = i - 1$ and find

$$\sum_{j=1}^{s-1} a_{j+1}^3[F(s-j-1) + F(s-j)].$$

Using these results we reduce A to

$$A = \sum_{i=1}^{s-1} a_{i+1}^3[F(s-i+1) - F(s-i-1)] + F(1)a_{s+1}^3 \; . \tag{3.4.18}$$

The substitution of (3.4.18) into (3.4.16) gives

$$\frac{2F(1)}{3(1-\nu)m} a_{s+1}^3 + a_{s+1}^2 - RVs\delta + \frac{2}{3(1-\nu)m} \sum_{i=1}^{s-1} a_{i+1}^3[F(s-i+1) - F(s-i-1)] = 0, \tag{3.4.19}$$

where we are still using the convention

$$a(i\delta) = a_{i+1} \; . \tag{3.4.20}$$

5. The Numerical Solution for a Maxwell Material.

In this section we specialize the above results for a Maxwell material, in which case

$$G_1(t) = G_o e^{-\eta_o t} . \tag{3.5.1}$$

The constant η_o is the relaxation time. From (3.2.4), (3.2.6), (3.5.1) we can simplify (3.3.12) as follows

$$u_2(t,\tau) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{s+\eta_o}{s}\right)^{\frac{1}{2}} e^{-\tau(s^2+s\eta_o)^{\frac{1}{2}}} e^{st} ds . \tag{3.5.2}$$

If we let $s + \eta_o/2 = p$ then (3.5.2) becomes

$$u_2(t,\tau) = \frac{e^{-t\eta_o/2}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}(p+\eta_o/2)\frac{e^{-\tau(p^2-\eta_o^2/4)^{\frac{1}{2}}}}{(p^2-\eta_o^2/4)^{\frac{1}{2}}} e^{pt} dp . \tag{3.5.3}$$

We observe that (see [37])

$$L^{-1}\left\{\frac{e^{-b(s^2-a^2)^{\frac{1}{2}}}}{(s^2-a^2)^{\frac{1}{2}}} ; s \to t\right\} = I_o[a(t^2-b^2)^{\frac{1}{2}}]H(t-b), \tag{3.5.4}$$

where I_o is the modified Bessel's function of zero order [38].

With the help of (3.5.4) we can write (3.5.3) as

$$u_2(t,\tau) = \left(\frac{G_1}{G_o}\right)^{\frac{1}{2}}(t)\left\{\frac{\partial}{\partial t}\left[I_o[\eta_o/2(t^2-\tau^2)^{\frac{1}{2}}]H(t-\tau)\right]\right.$$

$$\left. + \frac{\eta_o}{2}\left[I_o[\eta_o/2(t^2-\tau^2)^{\frac{1}{2}}]H(t-\tau)\right]\right\} . \tag{3.5.5}$$

The result (3.5.5) when substituted into (3.4.10) yields

$$\frac{\partial P_I}{\partial r}(r,t) = \frac{1}{R(1-\nu)}\left[\frac{a(t_1)\mathring{a}(t_1)}{(c_2y+\mathring{a}(t_1))}\left(\frac{G_1}{G_o}\right)^{\frac{1}{2}}(t-t_1) - \frac{a(t_2)\mathring{a}(t_2)}{(c_2y+\mathring{a}(t_2))}\left(\frac{G_1}{G_o}\right)^{\frac{1}{2}}(t-t_2)\right]$$

$$+ \frac{1}{R(1-\nu)c_2y}\left\{\int_o^t a(\tau)\mathring{a}(\tau)\,\Pi(t-\tau, \frac{r+a(\tau)}{c_2y})d\tau\right.$$

$$- \int_o^{t_r} a(\tau)\mathring{a}(\tau)\,\Pi(t-\tau, \frac{r-a(\tau)}{c_2y})d\tau$$

$$\left. - \int_{t_r}^t a(\tau)\mathring{a}(\tau)\,\Pi(t-\tau, \frac{a(\tau)-r}{c_2y})d\tau\right\} , \tag{3.5.6}$$

where

$$t_1 = t - \left(\frac{a(t_1) + r}{c_2 y}\right) \quad , \quad t_2 = t - \left(\frac{a(t_2) - r}{c_2 y}\right) \quad , \qquad (3.5.7)$$

$$\Pi(t,\tau) = \frac{\eta_o}{2}\left(\frac{G_1}{G}\right)^{\frac{1}{2}}(t)\left\{\frac{tI_1[\eta_o/2(t^2 - \tau^2)^{\frac{1}{2}}]}{[t^2 - \tau^2)^{\frac{1}{2}}} + I_o[\eta_o/2(t^2 - \tau^2)^{\frac{1}{2}}]\right\} \cdot H(t-\tau). \qquad (3.5.8)$$

Now if (3.5.1) is substituted into (3.4.17) we determine

$$F(s) = \frac{G_o}{\eta_o^2}\,[e^{-\eta_o s\delta} - 1 + \eta_o s\delta] \quad . \qquad (3.5.9)$$

In practise it turns out that the numbers dealt with are so small that computer round off gives $e^{-\eta_o s\delta} = 1$. Hence to gain better results we expand $e^{-\eta_o s\delta}$ and find

$$e^{-\eta_o s\delta} = 1 - \eta s\delta + \frac{(\eta_o s\delta)^2}{2!} - \frac{(\eta_o s\delta)^3}{3!} + \frac{(\eta_o s\delta)^4}{4!} - \ldots\ldots$$

When this expansion is substituted into (3.5.9) we obtain

$$F(r) = \frac{G_o\delta^2}{2}\left[s^2 - \frac{s^3\eta_o\delta}{3} + \frac{s^4\eta_o^2\delta^2}{4.3} - \ldots\right] .$$

Since $\eta\delta << 1$, we will truncate this series after the third term giving

$$F(s) \doteq \frac{G_o\delta^2}{2}\left[r^2 - \frac{r^3\eta_o\delta}{3} + \frac{r^4\eta_o\delta^2}{12}\right] \quad . \qquad (3.5.10)$$

Equations (3.4.19) and (3.5.10), after considerable algebraic simplification give the following result

$$\text{cof}_4\; a_{s+1}^3 + \text{cof}_3\; a_{s+1}^2 + \text{cof}_1 = 0, \quad s \geq 1 \; , \qquad (3.5.11)$$

where

$$\text{cof}_4 = G_o\delta^2 \left[1 - \frac{\eta_o\delta}{3} + \frac{\eta^2{}_o\delta^2}{12}\right] ,$$

$$\text{cof}_3 = 3m(1 - \nu) ,$$

$$\text{cof}_1 = 4G_o\delta^2 \sum_{i=1}^{s-1} a^3_{i+1} \tilde{F}(s - i) - \text{cof}_3 RVs\delta , \tag{3.5.12}$$

$$\tilde{F}(r) = r - \frac{\eta_o\delta}{6}[3r^2 + 1] + \frac{\eta_o^2\delta^2}{6}[r^3 + r].$$

Equation (3.5.11) is a cubic equation in a_{s+1} and all the cof_i , $i = 1,2,3,$ are independent of a_{s+1} .

For $s = 1$ we have

$$\text{cof}_4\, a_2^3 + \text{cof}_3\, a_2^2 - \text{cof}_3\, RV = 0 . \tag{3.5.13}$$

This equation can be solved numerically for a_2 and the result is used to solve for a_3 etc.

It is also necessary to calculate values for $a(t)\dot{a}(t)$. To derive an expression for this term we first differentiate (3.4.11) and find

$$2a(t)\dot{a}(t) - RV + \frac{4}{3(1 - \nu)m}\int_o^t \frac{da^3}{d\theta}(\theta)\int_\theta^t G_1(\theta,-\theta)d\theta_1 d\theta = 0. \tag{3.5.14}$$

We next consider equation (3.4.11) again and this time integrate the inner integral by parts to obtain

$$a^2(t) - RVt + \frac{4G_o}{3(1 - \nu)m\eta_o}\int_o^t (t - \theta)\frac{da^3}{d\theta}(\theta)d\theta$$

$$- \frac{4}{3(1 - \nu)m\eta_o}\int_o^t \frac{da^3}{d\theta}(\theta)\int_\theta^t G_1(\theta_1 - \theta)d\theta_1 d\theta = 0, \tag{3.5.15}$$

where we have used (3.5.1). Equations (3.5.14) and (3.5.15) combine to give

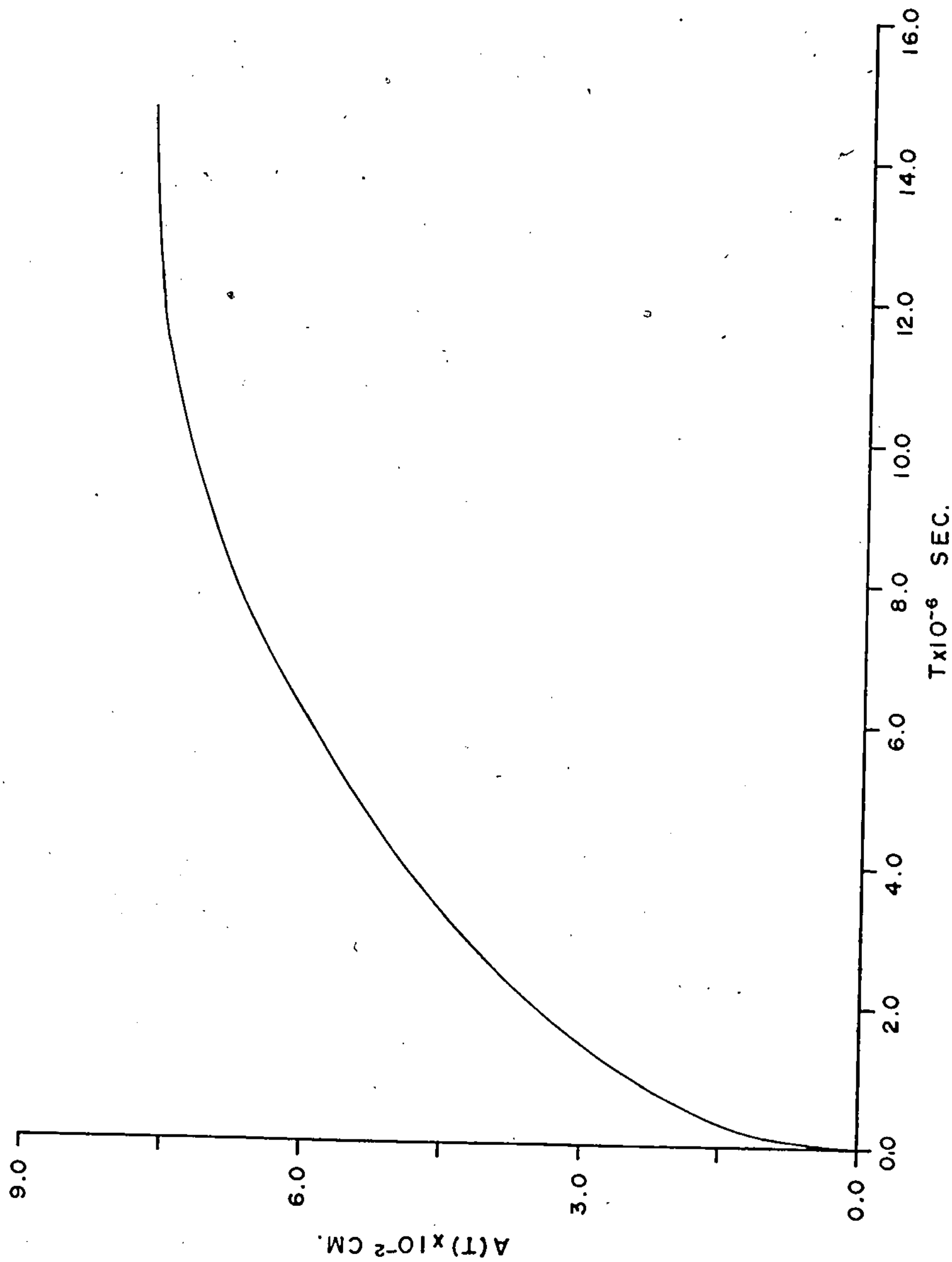


FIGURE 3.2

Table 3.2 Percentage difference between the first and second approximations of $P(r, t)$ for varying r and t. In the table below r is expressed as a fraction of the contact radius $a(t)$ and t as a fraction of T, the time taken to reach a maximum penetration.

$r/a(t)$ \ t/T	.1	.5	.9
.1	-.00029	.0021	.0037
.5	-.00029	.0021	.0037
.9	-.00029	.0021	.0037

$$2a(t)\dot{a}(t) - RV + \eta_o a^2(t) - RVt\eta_o + \frac{4G_o}{3(1-\nu)m}\int_o^t (t-\theta)\frac{da^3}{d\theta}(\theta)d\theta = 0.$$

Integrating this equation by parts and solving for $a(t)\dot{a}(t)$ we find

$$a(t)\dot{a}(t) = \frac{1}{2}\left\{RV(1+t\eta_o) - \eta_o a^2(t) - \frac{4G_o}{3(1-\nu)m}\int_o^t a^3(\theta)d\theta\right\}. \qquad (3.5.16)$$

By making the same approximation that gave (3.4.15) we are able to finally rewrite (3.5.16) as

$$a_{s+1}\dot{a}_{s+1} = \frac{1}{2}\left\{RV(1+s\eta_o\delta) - \eta_o a_{s+1}^2 - \frac{2G_o\delta}{3(1-\nu)m}\left[a_{s+1}^3 + 2\sum_{i=1}^{s-1} a_{i+1}^3\right]\right\}. \qquad (3.5.17)$$

Equations (3.5.11) and (3.5.17) were solved numerically and the results are given in Figure 3.2. The values of the constants used are listed below.

$$\begin{aligned} G_o &= 1.659 \times 10^{10} \text{ dyne/cm}^2. \;, \\ \nu &= .35 \;, \\ R &= .87313 \text{ cm.} \;, \\ m &= \tfrac{4}{3}\pi R^3 \rho_I \text{ gm.} \;, \\ \rho_I &= 7.8 \text{ gm./cm}^3. \;, \\ V &= 70.6 \text{ cm./sec.} \;, \\ \delta &= 10^{-6} \;, \; \eta_o = 1 \;. \end{aligned} \qquad (3.5.18)$$

Having calculated $a(t)$ and $\dot{a}(t)$ we return to equation (3.4.2) in order to find a second approximation for $P(r,t)$. Since the integrand of the integral in (3.4.2) has a singularity we must take special precautions. We use Young's method of approximate product integration to integrate (3.4.2). This method is outlined in section 1.4. Following the theory stated there we write

$$P(r,t) = \frac{4}{\pi R(1 - \nu)} (a^2(t) - r^2)^{\frac{1}{2}} + \frac{2\sum_1}{\pi(1 - \nu)} \left[\sum_{i=1}^{5} \alpha_i \frac{\partial P_I}{\partial x}(x_i, t) \right], \qquad (3.5.19)$$

where α_i are given by (1.4.6), (1.4.7), (C.3), and the x_i by (C.2). The integrations in (3.5.6) and for the operator $\sum_1$ were carried out by Gaussian quadrature formula (see Kopal [39] for a discussion of this method).

6. Discussion.

The results of the second approximation of $P(r,t)$ appear in Table 3.2 for various values of r and t. There was in general a difference of less than 1% between the first and second approximations. This suggests that we use the first approximation when we calculate other stress or displacement quantities. This procedure represents a large saving in computer time since to calculate the second approximation of $P(r,t)$ for one value takes approximately 40 sec. of C.P.U. time, while the first approximation can be determined by hand once $a(t)$ is known.

CHAPTER IV

THE DYNAMIC SOLUTION OF A GROWING PENNY-SHAPED CRACK IN TENSION IN AN INFINITE VISCOELASTIC SOLID

1. Statement of the Problem.

In this chapter we will give a solution to the problem of a plane circular crack in an infinite viscoelastic medium which is opened by a normal pressure acting on its surface. In terms of circular cylindrical co-ordinates (r,θ,z), the distribution of stress and displacement for this problem is the same as that in a semi-infinite body $z \geq 0$, when its surface, B, is subject to the boundary conditions:

$$\begin{aligned} \sigma_{rz}(r,0,t) &= \sigma_{z\theta}(r,o,t) = 0 \quad , \quad r \geq 0 \\ \sigma_{zz}(r,0,t) &= -p(r,t) \quad , \quad 0 \leq r \leq a(t) \\ u_z(r,0,t) &= 0 \quad , \quad r > a(t) \ , \end{aligned} \tag{4.1.1}$$

and the conditions at infinity (3.1.2). Here $a(t)$, which gives the radius of the crack at time t, is monotonically increasing and we assume that the body force F_i is zero. The problem thus posed will be considered solved when we find the solution to equations (2.3.2) - (2.3.5) subject to the boundary and initial conditions (4.1.1), (3.1.2), (3.1.4) and (3.1.6). However, apart from the last two conditions of (4.1.1), we note that the stresses and displacements given by (3.2.16) satisfy all the above remaining conditions, in terms of an unknown function A. When (4.1.1) is employed in (3.2.16) these equations result in a pair of dual integral equations which determine the function A.

2. The Solution of Certain Dual Integral Equations.

As stated above we shall impose the last two conditions of (4.1.1) upon the solution given by equations (3.2.16). The result is the following pair of dual integral equations:

$$-p(r,t) = \frac{G_1}{2} *d\, L^{-1}\left\{\int_0^\infty \left[(2\xi^2+K_2^2) - \frac{4\xi^2\alpha\beta}{2\xi^2+K_2^2}\right] \xi A(\xi) J_o(\xi r)d\xi;\ s \to t\right\}, \quad 0 \le r \le a(t), \tag{4.2.2}$$

$$0 = L^{-1}\left\{\int_0^\infty \left[1 - \frac{2\xi^2}{2\xi^2+K_2^2}\right] \xi\alpha A(\xi)J_o(\xi r)d\xi\ ;\ s \to t\right\}, \quad r > a(t). \tag{4.2.3}$$

We simplify these dual integral equations by defining a new function $W(r,t)$ such that

$$A(\xi) = -\overline{W}_o^* \Big/ \left[\alpha - \frac{2\xi^2\alpha}{2\xi^2+K_2^2}\right] . \tag{4.2.4}$$

If we substitute (4.2.4) into (4.2.2) and (4.2.3) these equations reduce to

$$p(r,t) = \frac{G_1}{2} *d\, L^{-1}\left\{\int_0^\infty \frac{[(2\xi^2+K_2^2)^2 - 4\xi^2\alpha\beta]}{\alpha K_2^2} \xi\overline{W}_o^* J_o(\xi r)d\xi\ ;\ s \to t\right\},\ 0 \le r \le a(t), \tag{4.2.5}$$

$$0 = W(r,t) \quad , \quad r > a(t) . \tag{4.2.6}$$

From (4.2.6) we note that the function $W(r,t)$ is actually the normal displacement $u_z(r,0,t)$. We further simplify (4.2.5) by defining

$$W(r,t) = 2[G_1^{-1} * dw](r,t) . \tag{4.2.7}$$

Then, if we use results (1.1.8), (1.1.9), (1.1.4), and (4.2.7) we find that (4.2.5) transforms to

$$p(r,t) = \int_0^\infty L^{-1}\left\{\left[\frac{(2\xi^2+K_2^2)^2 - 4\xi^2\alpha\beta}{\alpha K_2^2}\right] w_o^*;\ s \to t\right\} \xi J_o(\xi r)d\xi, \quad 0 \le r \le a(t). \tag{4.2.8}$$

Let us denote the function $F(\xi,t)$ to be the function whose Laplace transform is given as

$$\overline{F}(\xi,s) = \left[\frac{(2\xi^2+K_2^2)^2 - 4\xi^2\alpha\beta}{\alpha K_2^2}\right]\overline{w}_o^* . \tag{4.2.9}$$

To determine $F(\xi,t)$ we first make the change of variable $\eta = K_2/\xi$ in (4.2.9) and find

$$\overline{F}(\xi,s) = K_2\overline{w}_o^* \left[\frac{\eta}{(1+\eta^2/K^2)^{\frac{1}{2}}} + 4\left(\frac{1+\eta^2-(1+\eta^2/K^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{1}{2}}}{(1+\eta^2/K^2)^{\frac{1}{2}}\eta^3}\right)\right] \tag{4.2.10}$$

where we have used (3.2.5) and (3.2.9). The above equation, with the help of (A.28), (A.36) and (A.37) can be rewritten as

$$\overline{F}(\xi,s) = \frac{\overline{w}_o^* K_2^2}{\xi}\int_o^\infty KJ_o(Kt')e^{-\eta t'}dt'$$
$$+ \overline{w}_o^* K_2 \int_o^\infty \left[\frac{1}{1-\nu} + 4\sum\nolimits_3 \cos(yt')\right] e^{-\eta t'}dt' . \tag{4.2.11}$$

By a slight change of variables in (4.2.11) and integrating one term we find

$$\overline{F}(\xi,s) = \frac{\xi\overline{w}_o^*}{1-\nu} + \int_o^\infty J_o(c_1\xi\tau)\overline{w}_o^* K_2^2 c_1 e^{-K_1c_1\tau}d\tau$$
$$+ 4\xi\sum\nolimits_3 \int_o^\infty \cos(c_2y\xi\tau)\overline{w}_o^* K_2c_2e^{-K_2c_2\tau}d\tau . \tag{4.2.12}$$

We now take the Laplace inverse of (4.2.12) and obtain

$$F(\xi,t) = \frac{\xi w_o^*(\xi,t)}{1-\nu} + \int_o^\infty J_o(c_1\xi\tau)\frac{\partial}{\partial t}[u_1*dw_o^*](\xi,t)d\tau \tag{4.2.12}$$
$$+ 4\xi\sum\nolimits_3 \int_o^\infty \cos(c_2y\xi\tau)[u_2*dw_o^*](\xi,t)d\tau$$

where

$$u_1(t,\tau) = L^{-1}\left[\frac{K_2^2 c_1}{s^2} e^{-K_1 c_1 \tau} \; ; \; s \to t\right] \tag{4.2.13}$$

and u_2 is given by (3.3.12). We use the result (4.2.12) to simplify (4.2.8) and get the following

$$\frac{1}{1-\nu}\int_0^\infty \xi^2 w_o^*(\xi,t)J_o(\xi r)d\xi = p(r,t)$$

$$-\int_0^\infty \xi J_o(\xi r)\int_0^\infty J_o(c_1\xi\tau)\frac{\partial}{\partial t}[u_1 * dw_o^*](\xi,t)d\tau d\xi$$

$$-4\sum\nolimits_3 \int_0^\infty \xi^2 J_o(\xi r)\int_0^\infty \cos(c_2 y\xi\tau)[u_2 * dw_o^*](\xi,t)d\tau d\xi \quad , \; 0 \le r \le a(t). \tag{4.2.14}$$

Equation (4.2.14) has the form (B.1) and also from (4.2.6) and (4.2.7) we can see that $w(r,t)$ satisfies the condition (B.2). Hence using the solution (B.9) we obtain for $0 \le r \le a(t)$,

$$w(r,t) = \frac{2(1-\nu)}{\pi}\int_r^{a(t)} \frac{1}{(x^2-r^2)^{1/2}}\int_0^x \frac{\xi p(\xi,t)d\xi}{(x^2-\xi^2)^{1/2}}\,dx$$

$$-\frac{2(1-\nu)}{\pi}\int_r^{a(t)} \frac{1}{(x^2-r^2)^{1/2}}\left\{4\sum\nolimits_3 \int_0^\infty \xi\sin(\xi x)\int_0^\infty \cos(c_2 y\xi\tau)[u_2 * dw_o^*](\xi,t)d\tau d\xi\right.$$

$$\left. + \int_0^\infty \sin(\xi x)\int_0^\infty J_o(c_1\xi\tau)\frac{\partial}{\partial t}[u_1 * dw_o^*](\xi,t)d\tau d\xi\right\}dx, \tag{4.2.15}$$

where we have used the result (3.3.14).

For notational convenience we introduce the terms T_{11} and T_{22} by the equations:

$$\begin{aligned} T_{11}(x,t,y) &= \int_0^\infty \cos(\xi x)\int_0^\infty \cos(c_2 y\xi\tau)[u_2 * dw_o^*](\xi,t)d\tau d\xi \\ T_{22}(x,t) &= \int_0^\infty \sin(\xi x)\int_0^\infty J_o(c_1\xi\tau)\frac{\partial}{\partial t}[u_1 * dw_o^*](\xi,t)d\tau d\xi . \end{aligned} \tag{4.2.16}$$

Using (4.2.16) we can rewrite (4.2.15) with a little manipulation as

$$w(r,t) = \frac{2(1-\nu)}{\pi}\int_r^{a(t)} \frac{1}{(x^2-r^2)^{1/2}} \int_0^x \frac{\xi p(\xi,t)\,d\xi}{(x^2-\xi^2)^{1/2}}\,dx$$

$$- \frac{2(1-\nu)}{\pi}\int_r^{a(t)} \frac{1}{(x^2-r^2)^{1/2}} \left\{-4 \sum_3 \frac{\partial}{\partial x}\; T_{11} + T_{22}\right\} dx \,. \tag{4.2.17}$$

At this point we write the stress and displacement fields for this problem in terms of $w(r,t)$ which must be determined from (4.2.17). On substituting (4.2.4) and (4.2.7) into (3.2.16) we obtain:

$$G_1 * du_r = 2L^{-1}\left\{\int_0^\infty [\gamma^2 e^{-\alpha z} - 2\alpha\beta e^{-\beta z}]\, \frac{\xi^2}{\alpha K_2^2}\, \overline{w}_o^*\, J_1(\xi r)\,d\xi \;;\; s \to t\right\},$$

$$G_1 * du_z = 2L^{-1}\left\{\int_0^\infty [\gamma^2 e^{-\alpha z} - 2\xi^2 e^{-\beta z}]\, \frac{\xi}{K_2^2}\, \overline{w}_o^*\, J_o(\xi r)\,d\xi \;;\; s \to t\right\},$$

$$\sigma_{rr} = -2L^{-1}\left\{\int_0^\infty [\gamma^2 e^{-\alpha z} - 2\alpha\beta e^{-\beta z}]\, \frac{\xi^2}{\alpha K_2^2}\, \overline{w}_o^* \left[\frac{J_1(\xi r)}{r} - \xi J_o(\xi r)\right] d\xi \right.$$
$$\left. + \frac{\nu}{2(1-\nu)}\int_0^\infty \frac{\gamma^2 \xi}{\alpha}\, e^{-\alpha z}\, \overline{w}_o^*\, J_o(\xi r)\,d\xi \;;\; s \to t\right\},$$

$$\sigma_{\theta\theta} = 2L^{-1}\left\{\frac{1}{r}\int_0^\infty [\gamma^2 e^{-\alpha z} - 2\alpha\beta e^{-\beta z}]\, \frac{\xi^2}{\alpha K_2^2}\, \overline{w}_o^*\, J_1(\xi r)\,d\xi \right.$$
$$\left. - \frac{\nu}{2(1-\nu)}\int_0^\infty \frac{\gamma^2 \xi}{\alpha}\, e^{-\alpha z}\, \overline{w}_o^*\, J_o(\xi r)\,d\xi \;;\; s \to t\right\},$$

$$\sigma_{zz} = -L^{-1}\left\{\int_0^\infty [\gamma^4 e^{-\alpha z} - 4\xi^2\alpha\beta e^{-\beta z}]\, \frac{\xi}{\alpha K_2^2}\, \overline{w}_o^*\, J_o(\xi r)\,d\xi \;;\; s \to t\right\},$$

$$\sigma_{zr} = -2L^{-1}\left\{\int_0^\infty [e^{-\alpha z} - e^{-\beta z}]\, \frac{\gamma^2 \xi}{K_2^2}\, \overline{w}_o^*\, J_1(\xi r)\,d\xi \;;\; s \to t\right\},$$

where

$$\gamma^2 = 2\xi^2 + K_2^2 \;,\quad \alpha^2 = \xi^2 + K_1^2 \;,\quad \beta^2 = \xi^2 + K_2^2 \,. \tag{4.2.18}$$

It is possible to integrate these equations further for the special case $z = 0$. We find in this case:

$$G_1 * du_r = -\frac{2}{K^2}\int_0^\infty \xi w_o^*(\xi,t) J_1(\xi r) d\xi + 2\textstyle\sum_5 T_3 \quad ,$$

$$G_1 * du_z = w(r,t) \quad ,$$

$$\sigma_{\theta\theta} = -\int_0^\infty \left[\frac{2}{K^2}\frac{J_1(\xi r)}{r} + \frac{2\nu\xi}{1-\nu} J_o(\xi r)\right] \xi w_o^*(\xi,t) d\xi \; .$$

$$+ \frac{2}{r}\textstyle\sum_5 T_6 + \frac{2\nu}{1-\nu}\sum_4 T_4 - \frac{\nu}{1-\nu} T_5 \quad ,$$

$$\sigma_{rr} = \int_0^\infty \left[\frac{2}{K^2}\frac{J_1(\xi r)}{r} - \frac{\xi}{1-\nu} J_o(\xi r)\right] \xi w_o^*(\xi,t) d\xi$$

$$- \frac{2}{r}\textstyle\sum_5 T_6 - \frac{\nu}{1-\nu} T_5 + 2\sum_6 T_4 \quad ,$$

$$\sigma_{zz} = -\frac{1}{1-\nu}\int_0^\infty \xi^2 w_o^*(\xi,t) J_o(\xi r) d\xi - 4\textstyle\sum_3 T_4 - T_5 \quad ,$$

$$\sigma_{zr} = 0 \quad , \tag{4.2.19}$$

where

$$T_4 = \int_0^\infty \xi^2 J_o(\xi r) \int_0^\infty \cos(c_2 y\xi\tau)[u_2 * dw_o^*](\xi,t) d\tau d\xi \quad ,$$

$$T_5 = \int_0^\infty \xi J_o(\xi r) \int_0^\infty J_o(\xi c_1 \tau)\frac{\partial}{\partial t}[u_1 * dw_o^*](\xi,t) d\tau d\xi \quad ,$$

$$T_6 = \int_0^\infty \xi J_1(\xi r) \int_0^\infty \cos(y\xi\tau)[u_2 * dw_o^*](\xi,t) d\tau d\xi \; . \tag{4.2.20}$$

The $\sum$ operators are defined in Appendix A by equations (A.40). It is important to note that in (4.2.19) if all terms involving T's were dropped then we would have the result found in the quasi-static theory. Hence these terms represent the dynamic part of the solution.

If we make use of (B.3) the terms T_i can be rewritten in a simpler form. For example if we denote by Q the integral in the integrand for T_4 then we find

$$T_4 = \int_0^\infty \xi^2 \frac{2}{\pi} \int_0^r \frac{\cos(\xi s)ds}{(r^2 - s^2)^{\frac{1}{2}}} Q d\xi \quad .$$

Interchanging the orders of integration and carrying out some minor manipulations we arrive at

$$T_4 = -\frac{2}{\pi} \int_0^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}} \frac{\partial^2}{\partial s^2} \int_0^\infty \cos(\xi s) Q d\xi ds.$$

Similar results can be found for T_5 and T_6. If we make use of (3.3.14) then we can write

$$T_4 = -\frac{2}{\pi} \int_0^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}} \frac{\partial^2}{\partial s^2} T_{11}(s,t,y)ds \quad ,$$

$$T_5 = \frac{2}{\pi} \int_0^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}} \frac{\partial}{\partial s} T_{22}(s,t,y)ds \quad , \tag{4.2.21}$$

$$T_6 = -\frac{2}{\pi} \int_0^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}} T_{11}(s,t,y)ds \quad .$$

In order to determine completely the stresses and displacements, given by (4.2.11) and (4.2.21), we note that we should specify $a(t)$ and $P(r,t)$.

3. The Case When $a(t) = Vt$ and $P(r,t) = P(t)$.

In this case equation (4.2.15) becomes

$$w(r,t) = \frac{2}{\pi}(1 - \nu)P(t)(a^2(t) - r^2)^{\frac{1}{2}} \tag{4.3.1}$$

$$+ \frac{2(1 - \nu)}{\pi} \int_r^{a(t)} \frac{1}{(\rho^2 - r^2)^{\frac{1}{2}}} \left[4\sum_3 \frac{\partial}{\partial \rho} T_{11} - T_{22}\right] d\rho \ , \ 0 \leq r \leq a(t) \ .$$

The above equation must be solved numerically. When this is done the results can be substituted into (5.1.16) or (5.1.19) to determine the stresses and displacements in general. However this approach fails in the vicinity of singularities, which the stress field possesses near the crack tip as r

approaches $a(t)^+$. The singularity will be of the form $\frac{K}{(r^2 - a^2(t))^{\frac{1}{2}}}$. To determine K we have to consider an approximate solution of (5.2.1) and integrate analytically rather than numerically.

We shall take as an approximate solution to (4.3.1)

$$w(r,t) = D(t)(a^2(t) - r^2)^{\frac{1}{2}} \quad , \tag{4.3.2}$$

$$D(t) = \frac{2}{\pi}(1 - \nu)P(t)D_o \quad .$$

Here D_o can be considered as a dynamic correction factor determined from the numerical solution of (4.3.1). This is analogous to the elastic dynamic problem for which (4.3.2) represents an exact solution in the case $P(t) = P_o$,[18]. The Hankel transform of order zero of w is given as

$$w_o^*(\xi,t) = D(t)\left[\frac{\sin(\xi a)}{\xi^3} - a\,\frac{\cos(\xi a)}{\xi^2}\right] \tag{4.3.3}$$

We now consider the expression for the normal stress component in the crack plane since it is directly related to the stress intensity factor. If we substitute (4.3.3) into σ_{zz} as given in (4.2.19) we find for $r > a(t)$

$$\begin{aligned}\sigma_{zz} &= -\frac{D}{1-\nu}\sin^{-1}(a(t)/r) + \frac{Da(t)}{(1-\nu)}\frac{1}{(r^2 - a^2(t))^{\frac{1}{2}}} \\ &+ \frac{8}{\pi}\sum_3 \int_o^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}}\frac{\partial}{\partial s}\frac{\partial T_{11}}{\partial s}\,ds \\ &- \frac{2}{\pi}\int_o^r \frac{1}{(r^2 - s^2)^{\frac{1}{2}}}\frac{\partial}{\partial s}T_{22}\,ds \quad , \quad r > a(t) \quad ,\end{aligned} \tag{4.3.4}$$

where the first term has been simplified by using (B.4) and the result, [38],

$$\int_o^\infty \sin(at)J_o(rt)\,\frac{dt}{t} = \sin^{-1}(a/r) \quad , \quad 0 \le a \le r \quad . \tag{4.3.5}$$

In order that the last two integrals contribute to the stress singularity at $a(t)$ the terms $\frac{\partial T_{11}}{\partial s}$ and T_{22} should have jump discontinuities at $a(t)$. We introduce the notation

$$[f]_a = \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) \quad . \tag{4.3.6}$$

We can then write (4.3.4) as

$$\begin{aligned}\sigma_{zz} = &\left[\frac{Da(t)}{1-\nu} + \frac{8}{\pi} \sum_3 \left[\frac{\partial T_{11}}{\partial s}\right]_a - \frac{2}{\pi}\left[T_{22}\right]_a \right] \frac{1}{(r^2 - a^2(t))^{1/2}} \\ &- \frac{D}{1-\nu} \sin^{-1}(a(t)/r) + \frac{8}{\pi}\sum_3 \int_o^{a(t)} \frac{1}{(r^2 - s^2)^{1/2}} \frac{\partial^2 T_{11}}{\partial s^2} ds \\ &+ \frac{8}{\pi}\sum_3 \int_{a(t)}^{r} \frac{1}{(r^2 - s^2)^{1/2}} \frac{\partial^2 T_{11}}{\partial s^2} ds - \frac{2}{\pi}\int_o^{a(t)} \frac{1}{(r^2 - s^2)^{1/2}} \frac{\partial T_{22}}{\partial s} ds \\ &- \frac{2}{\pi}\int_{a(t)}^{r} \frac{1}{(r^2 - s^2)^{1/2}} \frac{\partial T_{22}}{\partial s} ds \quad .\end{aligned} \tag{4.3.7}$$

The last terms can now be integrated numerically since the singular part has been removed. We can now calculate the stress intensity factor which is defined by the relation

$$N(t) = \lim_{r\to a^+(t)} \{[r - a(t)]^{1/2}\sigma_{zz}(r,0,t)\} \quad . \tag{4.3.8}$$

In the present case N is given as

$$N(t) = \frac{1}{[2a(t)]^{1/2}} \left[\frac{Da(t)}{1-\nu} + \frac{8}{\pi}\sum_3 \left[\frac{\partial T_{11}}{\partial s}\right]_a - \frac{2}{\pi}[T_{22}]_a\right] \tag{4.3.9}$$

To find $N(t)$ we now must calculate $\left[\frac{\partial T_{11}}{\partial s}\right]_a$ and $[T_{22}]_a$. We observe that the time derivative of $w_o^*(\xi,t)$ appears in (4.2.16). From (4.3.3) we see that time derivative of w_o^* is

$$\frac{\partial w_o^*}{\partial t}(\xi,t) = D(t)a(t)\dot{a}(t)\frac{\sin(\xi a(t))}{\xi} + \dot{D}(t)\left[\frac{\sin(\xi a(t))}{\xi^3} - \frac{a(t)\cos(\xi a(t))}{\xi^2}\right]. \quad (4.3.10)$$

It has been examined in some detail and determined that only the first term contributes to any jump discontinuity in $\frac{\partial T_{11}}{\partial s}$ or T_{22}. Thus we shall only substitute the first term in (4.3.10) for $\frac{\partial w_o^*}{\partial t}(\xi,t)$ when we calculate $\left[\frac{\partial T_{11}}{\partial s}\right]_a$ and $[T_{22}]_a$. With this in mind we first consider $\frac{\partial T_{11}}{\partial s}$. From (4.2.16) and the first term in (4.3.10) we write

$$\frac{\partial T_{11}}{\partial s} = \frac{\partial}{\partial s}\int_0^\infty \cos(\xi s)d\xi \int_0^\infty \cos(c_2 yx\xi)\int_0^t D(\tau)u_2(t-\tau,x)\frac{\sin(\xi a)}{\xi}a\dot{a}d\tau dx \quad ,$$

$$\frac{\partial T_{11}}{\partial s} = \frac{\partial}{\partial s}\int_0^t D(\tau)\dot{a}(\tau)a(\tau)d\tau \int_0^\infty u_2(t-\tau,x)dx \int_0^\infty \cos(\xi s)\cos(\xi c_2 yx)\sin(\xi a)\frac{d\xi}{\xi} \quad ,$$

$$\frac{\partial T_{11}}{\partial s} = \frac{1}{4}\frac{\partial}{\partial s}\int_0^t D(\tau)\dot{a}(\tau)a(\tau)d\tau \int_0^\infty u_2(t-\tau,x)dx \int_0^\infty \sum_{i=1}^4 \sin(\xi\gamma_i)\frac{d\xi}{\xi} \quad , \qquad (4.3.11)$$

where the γ_i are given by (3.4.7). In view of the result (3.4.8) we must determine the sign of γ_i in the region of integration $R = \{(x,\tau)\,|\,0 \le \tau \le t,\ 0 \le x < \infty\}$, for $s < a(t)$ and $s > a(t)$. Table 4.1 gives the sign of γ_i in the regions indicated in Fig. 4.1 and 4.2 as well as the

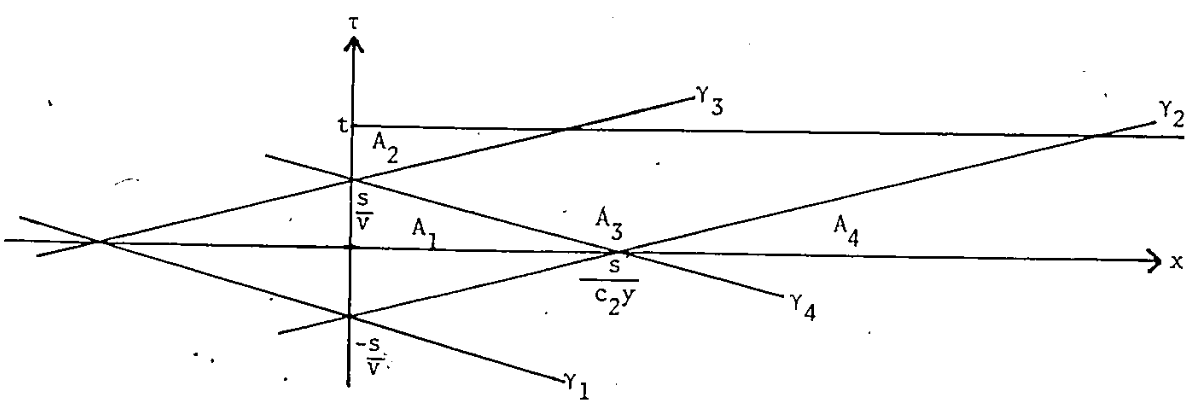


Figure 4.1 $\quad s < a(t) \qquad R = A_1 \cup A_2 \cup A_3 \cup A_4$.

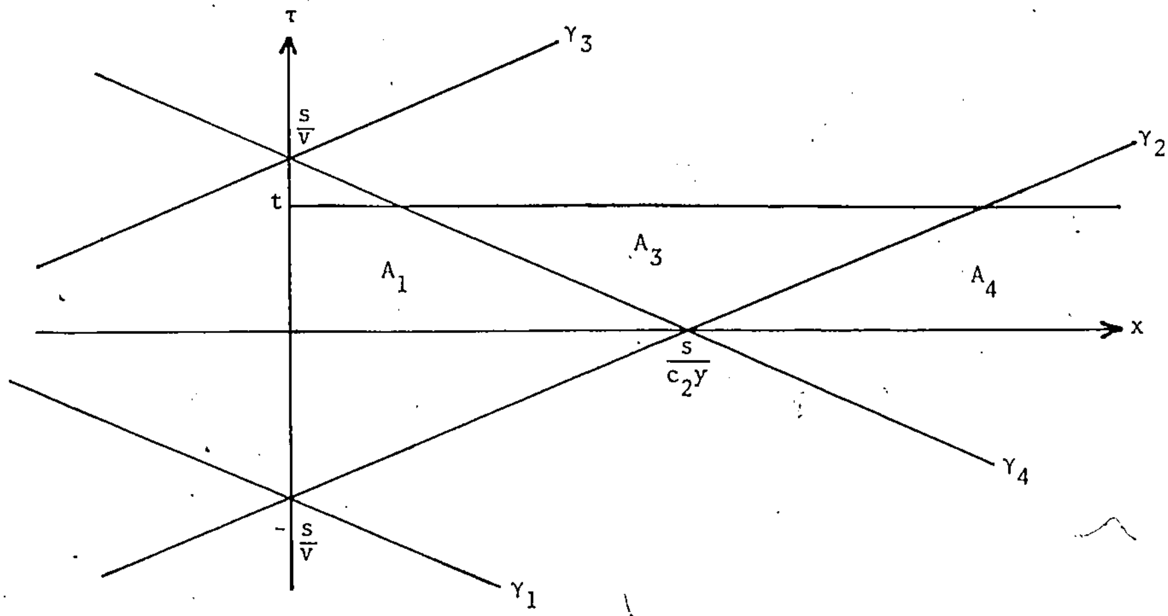


Figure 4.2 $s > a(t)$ $R = A_1 \cup A_3 \cup A_4$

Table 4.1 The sign of γ_i in the region A_K.

	A_1	A_2	A_3	A_4
γ_1	+	+	+	+
γ_2	+	+	+	-
γ_3	-	+	-	-
γ_4	-	+	+	+
C	0	2π	π	0

total contribution from that region. We note that for $s < a(t)$ two regions contribute while for $s > a(t)$ only one contributes. Using these results we can simplify (4.3.11) for the two cases. In the first case we have

$$\frac{\partial T_{11}}{\partial s} = \frac{V^2}{4} \frac{\partial}{\partial s} \left\{ \int_0^{s/V} D(\tau)\tau d\tau \int_{\frac{s-V\tau}{c_2 y}}^{\frac{s+V\tau}{c_2 y}} u_2(t-\tau,x)\pi \, dx \right.$$

$$+ \int_{s/V}^{t} D(\tau)\tau d\tau \int_{\frac{V\tau-s}{c_2 y}}^{\frac{s+V\tau}{c_2 y}} u_2(t-\tau,x)\pi \; dx$$

$$+ \int_{s/V}^{t} D(\tau)\tau d\tau \int_{o}^{\frac{V\tau-s}{c_2 y}} u_2(t-\tau,x)2\pi \; dx \Bigg\} \quad , \quad s < a(t) \; ,$$

and in the second case

$$\frac{\partial T_{11}}{\partial s} = \frac{V^2}{4}\frac{\partial}{\partial s}\int_o^t D(\tau)\tau d\tau \int_{\frac{s-V\tau}{c_2 y}}^{\frac{s+V\tau}{c_2 y}} u_2(t-\tau,x)\pi \; dx \quad , \quad s > a(t) \quad .$$

If we carry out the differentiation with respect to s then we find that there will be no contribution from the limits of the outer integrals. Hence after some minor simplification we obtain

$$\frac{\partial T_{11}}{\partial s} = \frac{V^2\pi}{4c_2 y}\Bigg\{\int_o^t D(\tau)\tau u_2(t-\tau,\frac{s+V\tau}{c_2 y})d\tau - \int_o^{s/V} \tau D(\tau)u_2(t-\tau,\frac{s-V\tau}{c_2 y})d\tau$$
$$- \int_{s/V}^{t} D(\tau)\tau u_2(t-\tau,\frac{V\tau-s}{c_2 y})d\tau\Bigg\} \quad , \quad s < a(t) \; , \tag{4.3.12}$$

and

$$\frac{\partial T_{11}}{\partial s} = \frac{V^2\pi}{4c_2 y}\int_o^t D(\tau)\tau[u_2(t-\tau,\frac{s+V\tau}{c_2 y}) - u_2(t-\tau,\frac{s-V\tau}{c_2 y})]d\tau \quad , \quad s > a(t). \tag{4.3.13}$$

If we combine (4.3.12), (4.3.13) and (4.3.6) we can now write

$$\left[\frac{\partial T_{11}}{\partial s}\right]_a = \frac{V^2\pi}{4c_2 y}\lim_{\substack{s\to a(t)^+ \\ s'\to a(t)^-}}\Bigg\{\int_o^t \tau D(\tau)[u_2(t-\tau,\frac{s+V\tau}{c_2 y}) - u_2(t-\tau,\frac{s'+V\tau}{c_2 y})]d\tau$$
$$- \int_o^{s'/V} D(\tau)\tau[u_2(t-\tau,\frac{s-V\tau}{c_2 y}) - u_2(t-\tau,\frac{s'-V\tau}{c_2 y})]d\tau$$
$$- \int_{s'/V}^{t} D(\tau)\tau[u_2(t-\tau,\frac{s-V\tau}{c_2 y}) - u_2(t-\tau,\frac{V\tau-s'}{c_2 y})]d\tau\Bigg\} .$$

The first two integrals in the above expression go to zero when the indicated limits are taken; however, the last gives a contribution. To see what this is we make the following change of variables, $s = Vt + V\varepsilon$, $s' = Vt - V\varepsilon$ and $\varepsilon z = t - \tau$, $\varepsilon > 0$. With these changes the above becomes

$$\left[\frac{\partial T_{11}}{\partial s}\right]_a = - \frac{V^2 \pi}{4c_2 y} \lim_{\varepsilon \to 0^+} \int_0^1 (t-\varepsilon z) D(t-\varepsilon z) [\varepsilon u_2(\varepsilon z, \frac{\varepsilon(V+Vz)}{c_2 y}) - \varepsilon u_2(\varepsilon z, \frac{\varepsilon(V-Vz)}{c_2 y})] dz \quad . \tag{4.3.14}$$

In order to simplify this further we consider the following

$$E = \varepsilon u_2(\varepsilon t, \varepsilon \tau) \quad , \quad \varepsilon > 0 \quad .$$

From (3.3.12) we can write E as

$$E = L^{-1}\left[\varepsilon c_2 \frac{K_2}{s} e^{-c_2 K_2 \varepsilon \tau} \quad ; \quad s \to \varepsilon t\right] \tag{4.3.15}$$

where from (3.2.4) and (3.2.6)

$$c_2^2 K_2^2 = \frac{G_1(0) s^2}{[s\bar{G}_1(s)]} \quad . \tag{4.3.16}$$

If we make the change of variables $x = s\varepsilon$ then

$$s\bar{G}_1(s) = g(x,\varepsilon) = \frac{x}{\varepsilon} \int_0^\infty G_1(t) e^{-xt/\varepsilon} dt \ ,$$

$$g(x,\varepsilon) = x \int_0^\infty G_1(t'\varepsilon) e^{-xt'} dt' \ .$$

The limit of $g(x,\varepsilon)$ as $\varepsilon \to 0^+$ is

$$\lim_{\varepsilon \to 0^+} g(x,\varepsilon) = G_1(0) \ . \tag{4.3.17}$$

We can then make the change of variables $x = s\varepsilon$ in (4.3.15) and using (4.3.16, (4.3.17) obtain the following result

$$\lim_{\varepsilon \to 0^+} \varepsilon u_2(\varepsilon t, \varepsilon\tau) = L^{-1}[e^{-x\tau} \; ; \; x \to t] = \delta(t - \tau) . \tag{4.3.18}$$

Applying this result we can reduce (4.3.14) to

$$\left[\frac{\partial T_{11}}{\partial s}\right]_a = - \frac{V^2 \pi t D(t)}{4c_2 y} \int_0^1 \left\{ \delta\left[z - \frac{(V+Vz)}{c_2 y}\right] - \delta\left[z - \frac{(V-Vz)}{c_2 y}\right]\right\} d\tau$$

If we break up the integral and make a change of variables we can make use of the properties of the Dirac delta function to finally obtain

$$\left[\frac{\partial T_{11}}{\partial s}\right]_a = \frac{V^2 \pi t D(t)}{4c_2 y} \left[\frac{c_2 y}{c_2 y - V} - \frac{c_2 y}{c_2 y + V}\right]$$

$$\left[\frac{\partial T_{11}}{\partial s}\right]_a = - \frac{\pi v_2^2 a(t) D(t)}{2(y^2 - v_2^2)} , \tag{4.3.19}$$

where $v_2 = V/c_2$.

To find $[T_{22}]_a$ we first combine T_{22} from (4.2.16) and the first term of (4.3.10) to get

$$T_{22} = \int_0^\infty \sin(\xi s) d\xi \int_0^\infty J_0(c_1 \xi x) \frac{\partial}{\partial t} \int_0^t D(\tau) u_1(t-\tau, x) \frac{\sin(\xi a)}{\xi} a\dot{a} d\tau dx .$$

This expression can be rearranged as follows

$$T_{22} = \frac{\partial}{\partial t} \int_0^s d\eta \int_0^t D(\tau) a\dot{a} d\tau \int_0^\infty u_1(t-\tau, x) dx \int_0^\infty \cos(\xi\eta) \sin(\xi a) J_0(\xi c_1 x) d\xi ,$$

$$T_{22} = \frac{V^2}{2} \frac{\partial}{\partial t} \int_0^s d\eta \int_0^t D(\tau) \tau d\tau \int_0^\infty u_1(t-\tau, x) dx \cdot$$

$$\cdot \int_0^\infty [\sin(\xi(\eta + a(\tau))) + \sin(\xi(a(\tau) - \eta))] J_0(\xi c_1 x) d\xi .$$

If we use the identity

$$\int_0^\infty \sin(bt)J_o(at)dt = \frac{H(b-a)}{\sqrt{b^2-a^2}} \tag{4.3.20}$$

then T_{22} becomes

$$T_{22} = \frac{V^2}{2}\frac{\partial}{\partial t}\int_o^s d\eta \int_o^t D(\tau)\tau d\tau \int_o^\infty u_1(t-\tau,x)\left\{\frac{H[a(\tau)+\eta-c_1x]}{[(a(\tau)+\eta)^2-c_1^2x^2]^{\frac{1}{2}}} + \frac{H[|a(\tau)-\eta|-c_1x]}{[(a(\tau)-\eta)^2-c_1^2x^2]^{\frac{1}{2}}}\right\}dx .$$

We now write T_{22} for the two cases, $s < a(t)$

$$T_{22} = \frac{V^2}{2}\frac{\partial}{\partial t}\left\{\int_o^s d\eta \int_o^t D(\tau)\tau d\tau \int_o^{\frac{V\tau+\eta}{c_1}} \frac{u_1(t-\tau,x)dx}{[(V\tau+\eta)^2-c_1^2x^2]^{\frac{1}{2}}} + \int_o^s d\eta \int_{\eta/V}^t D(\tau)\tau d\tau \int_o^{\frac{V\tau-\eta}{c_1}} \frac{u_1(t-\tau,x)dx}{[(V\tau-\eta)^2-c_1^2x^2]^{\frac{1}{2}}} - \int_o^s d\eta \int_o^{\eta/V} D(\tau)\tau d\tau \int_o^{\frac{\eta-V\tau}{c_1}} \frac{u_1(t-\tau,x)dx}{[(\eta-V\tau)^2-c_1^2x^2]^{\frac{1}{2}}}\right\} \tag{4.3.21}$$

and $s > a(t)$

$$T_{22} = \frac{V^2}{2}\frac{\partial}{\partial t}\left\{\int_o^s d\eta \int_o^t D(\tau)\tau d\tau \int_o^{\frac{V\tau+\eta}{c_1}} \frac{u_1(t-\tau,x)dx}{[(V\tau+\eta)^2-c_1^2x^2]^{\frac{1}{2}}} + \int_o^{a(t)} d\eta \int_{\eta/V}^t D(\tau)\tau d\tau \int_o^{\frac{V\tau-\eta}{c_1}} \frac{u_1(t-\tau,x)dx}{[(V\tau-\eta)^2-c_1^2x^2]^{\frac{1}{2}}} - \int_o^{a(t)} d\eta \int_o^{\eta/V} D(\tau)\tau d\tau \int_o^{\frac{\eta-V\tau}{c_1}} \frac{u_1(t-\tau,x)dx}{[(\eta-V\tau)^2-c_1^2x^2]^{\frac{1}{2}}} - \int_{a(t)}^s d\eta \int_o^t D(\tau)\tau d\tau \int_o^{\frac{\eta-V\tau}{c_1}} \frac{u_1(t-\tau,x)dx}{[(\eta-V\tau)^2-c_1^2x^2]^{\frac{1}{2}}}\right\} . \tag{4.3.22}$$

When we calculate $[T_{22}]_a$ it can be seen that only contribution comes from the derivatives of limits of the last three integrals for T_{22}, $s > a(t)$.

Thus we find,

$$[T_{22}]_a = \frac{V^3}{2}\left\{\lim_{\eta'\to a^-(t)} \int_{\eta'/V}^{t} D(\tau)\tau d\tau \int_0^{\frac{V\tau-\eta'}{c_1}} \frac{u_1(t-\tau,x)dx}{[(V\tau-\eta')^2 - c_1^2x^2]^{\frac{1}{2}}}\right.$$

$$- \lim_{\eta'\to a^-(t)} \int_0^{\eta'/V} D(\tau)\tau d\tau \int_0^{\frac{\eta'-V\tau}{c_1}} \frac{u_1(t-\tau,x)dx}{[(\eta'-V\tau)^2 - c_1^2x^2]^{\frac{1}{2}}}$$

$$\left. + \lim_{\eta\to a^+(t)} \int_0^{t} D(\tau)\tau d\tau \int_0^{\frac{\eta-V\tau}{c_1}} \frac{u_1(t-\tau,x)dx}{[(\eta-V\tau)^2 - c_1^2x^2]^{\frac{1}{2}}}\right\} . \qquad (4.3.23)$$

We point out that there is a certain amount of cancellation between the last two integrals and care must be taken to insure the correct results. To evaluate the above integrals we let $\eta = Vt + V\varepsilon$, $\eta' = Vt - V\varepsilon$, and $y = t - \tau$.

The result is

$$[T_{22}]_a = \frac{DV^3}{2} \lim_{\varepsilon\to 0^+}\left\{\int_0^{\varepsilon}(t-y)D(t-y)dy \int_0^{v_1(\varepsilon-y)} \frac{u_1(y,x)dx}{[V^2(\varepsilon-y)^2 - c_1^2x^2]^{\frac{1}{2}}}\right. \qquad (4.3.24)$$

$$- \int_{\varepsilon}^{t}(t-y)D(t-y)dy \int_0^{v_1(y-\varepsilon)} \frac{u_1(y,x)dx}{[V^2(y-\varepsilon)^2 - c_1^2x^2]^{\frac{1}{2}}}$$

$$\left. + \int_0^{t}(t-y)D(t-y)dy \int_0^{v_1(y+\varepsilon)} \frac{u_1(y,x)dx}{[V^2(y+\varepsilon)^2 - c_1^2x^2]^{\frac{1}{2}}}\right\},$$ where $v_1 = V/c_1$.

To simplify this further we let $y = \varepsilon z$ and $x = \varepsilon r$ and take the limit for the first integral. The other two integrals partially cancel in the limit and for the remainder we make the same changes as for the first. In both cases we make use of a similar result as (4.3.18), namely

$$\lim_{\varepsilon\to 0^+} \varepsilon u_1(\varepsilon t,\varepsilon\tau) = \frac{c_1}{c_2^2}\,\delta(t-\tau). \tag{4.3.25}$$

The expression for $[T_{22}]_a$ now becomes

$$[T_{22}]_a = \frac{c_1 V^3}{2c_2^2} D(t) \cdot t \left\{ \int_0^{\frac{v_1}{1+v_1}} \frac{dy}{[V^2(1-y)^2 - c_1^2y^2]^{\frac{1}{2}}} + \int_0^{\frac{v_1}{1-v_1}} \frac{dy}{[V^2(1+y)^2 - c_1^2y^2]^{\frac{1}{2}}} \right\}$$

which can be completely integrated to give

$$[T_{22}]_a = \frac{\pi}{2} \frac{K^2v_1^2a(t)D(t)}{[1 - v_1^2]^{\frac{1}{2}}}. \tag{4.3.26}$$

Collecting the results (4.3.19) and (4.3.26) with (4.3.2) and (4.3.9) we can write N as

$$N(t) = \frac{[2a(t)]^{\frac{1}{2}}}{\pi} P(t)D_o \left[1 - 4(1+\nu) \textstyle\sum_3 \frac{v_2^2}{y^2 - v_2^2} - \frac{2(1-\nu)^2v_1^2}{(1-2\nu)(1-v_1^2)^{\frac{1}{2}}}\right] \tag{4.3.27}$$

This result agrees with those results given for this problem in elasticity in the case $P(t) = P_o$. If we let $D_o = (1+\varepsilon)^{-1}$ then (4.3.27) is exactly the result found in [18]. Note ε is defined in [18]. We can simplify (4.3.27) by fully evaluating the integrals in the operator $\sum_3$. By integrating and performing some algebraic manipulation we reduce (4.3.27) to

$$N(t) = \frac{[2a(t)]^{\frac{1}{2}}}{\pi} P(t)D_o \frac{(1-\nu)}{v_2^2} \left[4(1-v_2^2)^{\frac{1}{2}} - \frac{K(2-v_2^2)^2}{(K^2-v_2^2)^{\frac{1}{2}}}\right], \tag{4.3.28}$$

where we have used (3.2.5) and (3.2.6). The form of (4.3.28) is exactly that given in [16] for the elastic problem. We note that the above comments indicate that the results in [16] and [18] are in agreement with each other. This point has not been made in the past.

We point out that the above analysis which has given (4.3.27) can most likely be generalized to the case when $a(t) = Vt^2$ i.e. for an accelerating crack. We have, however, not explored this idea any further.

4. Numerical Calculations for a Maxwell Material.

The two quantities of primary interest are $N(t)$ given by (4.3.28) and the normal displacement u_z. To show the dynamic effect on these terms we calculate the following normalized quantities

$$\frac{N(t)}{[2a(t)]^{\frac{1}{2}}P(t)} = \frac{D_o(1 - \nu)}{\pi v_2^2} \left[4(1 - v_2^2)^{\frac{1}{2}} - \frac{K(2 - v_2^2)^2}{(K^2 - v_2^2)^{\frac{1}{2}}} \right] \qquad (4.4.1)$$

and w/w_s

where $w_s(r,t) = \frac{2}{\pi}(1 - \nu)P_o(a^2(t) - r^2)^{\frac{1}{2}}$ (4.4.2)

and w is given by (4.3.1). We note that by (4.2.7) w is related to u_z in such a way that the ratio w/w_s indicates the dynamic effect on u_z. We also point out that for numerical computations we have set $P(t) = P_o$, a constant.

In carrying out the calculations we specify G_1 to be given by (3.5.1) and make use of (3.5.5).

We take as a first approximation of w, (4.4.2) and substitute this into the second term of (4.3.1). In calculating this term we arrive at an expression similar to (3.5.6). The results of these calculations appear in Table 4.2 which gives the ratio w/w_s. We use these computations to determine an average correction factor D_o which is used in calculating the normalized stress intensity factor. This ratio is then calculated and the results appear in Figure 4.3.

Table 4.2 (a) Calculated values of $w(r, t)/w_s(r, t)$ for $v_2 = .5$.

r/Vt \ t	.001	.01	.1	1
0	.8665	.866	.861	.36
.2	.86611	.8654	.8585	.364
.4	.86578	.865	.85834	.391
.6	.86531	.8646	.85838	.437
.8	.86477	.864	.85838	.489

Table 4.2 (b) Calculated values of $w(r, t)/w_s(r, t)$ for $r/Vt = .5$ and $t = .01$.

η_o \ ν_2	.1	.3	.7	.9
.1	0.99248	0.9423	0.7692	0.663
10	0.99253	0.9424	0.7672	0.660
100	0.9896	0.915	0.619	0.432

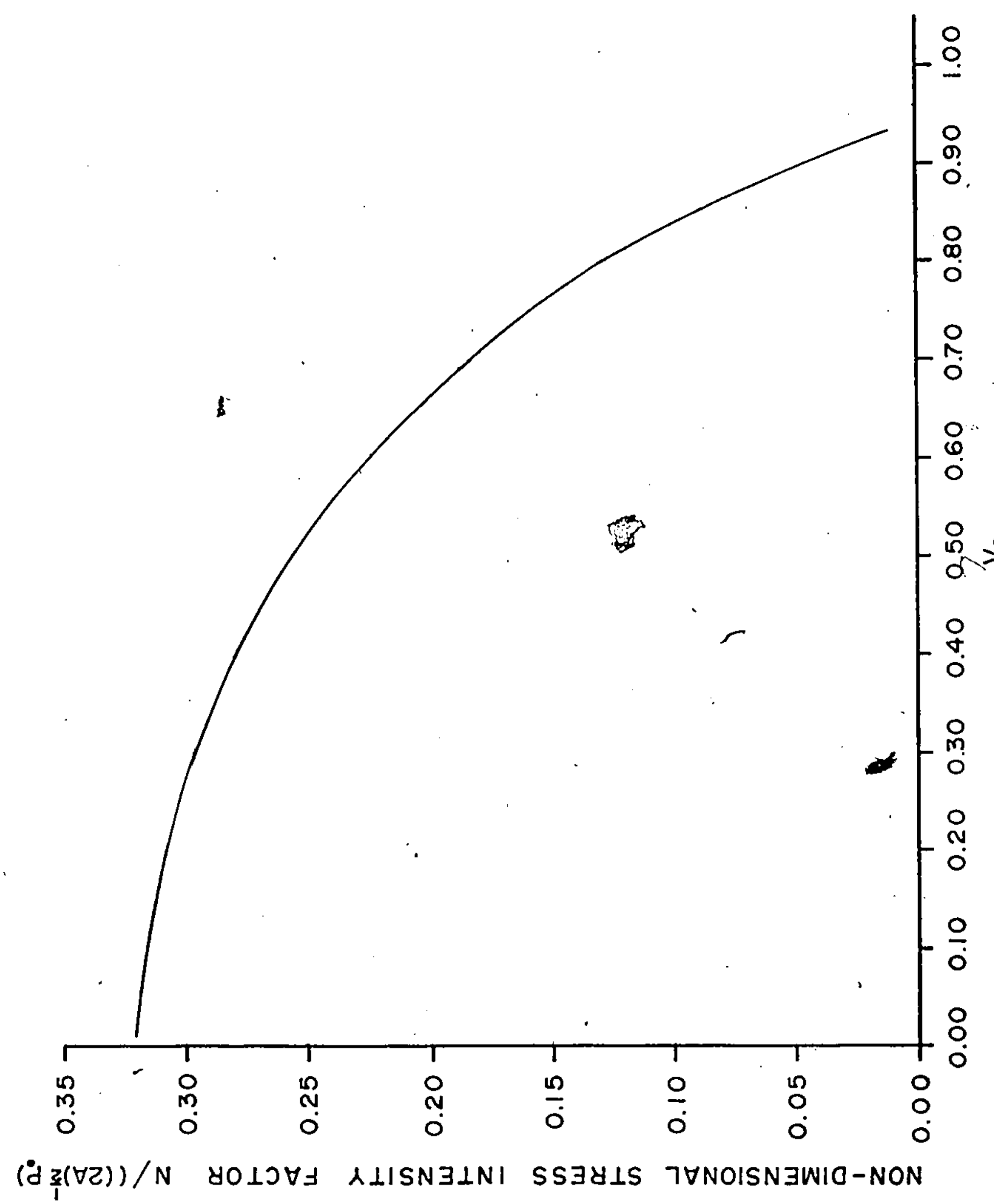

NON-DIMENSIONAL STRESS INTENSITY FACTOR N/((2A)½P)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
V2

FIGURE 4.3

5. Discussion

In calculating values of $w(r, t)$ from equation (4.3.1) we made the assumption that $\eta_o t < 1$. This assumption is justified by the fact that for all but the smallest values of v_2 the time of crack growth will be very small.

We observe from Table 4.2 (a) that there is little variation in the fractional difference between the first and second approximations for $t < .1$ and $r < Vt$. If we make the approximation that the difference over this range of r and t is constant and equal to the average value -.13679 then the solution of (4.3.1) is of the form

$$w(r, t) = D_o w_s(r, t) \tag{4.5.1}$$

where D_o is a constant. When we substitute (4.5.1) into equation (4.3.1), we find the identity

$$D_o w_s = w_s - .13679 D_o w_s$$

from which we obtain the following value of D_o,

$$D_o = \frac{1}{1.13679} = .87967$$

Hence the above approximation, that the difference is a constant, yields the approximate solution

$$w(r, t) = .87967 w_s(r, t) \text{ for } r < Vt, t < .1 \tag{4.5.2}$$

This means that for $v_2 = .5$ the dynamic displacement u_z will be approximately 12% less than the corresponding static displacement. The trend indicated by Table 4.2 implies that this percentage will grow much larger if $\eta_o >> 1$ or $v_2 \rightarrow 1$.

We note that the range of the expression on the right hand side of (4.4.1) is from D_o/π to 0, for v_2 = 0 to .93501 respectively. In Figure 4.3 the value used for D_o was 1. If the correct value had been used the general form would not be changed but only the scale.

CHAPTER V

THE DYNAMIC SOLUTION OF A GROWING PENNY-SHAPED CRACK UNDER TORSION IN AN INFINITE VISCOELASTIC SOLID

1. Statement and Formulation of the Problem.

We consider an infinite viscoelastic solid containing a circular crack which is under torsion. We choose cylindrical polar coordinates (r,θ,z) such that the crack lies in the plane $z = 0$ and has radius $a(t)$ where a is a monotonic increasing function of time. The distribution of stress and displacement for this problem is the same as that in a semi-infinite body $z \geq 0$, when its surface, B, is subject to the boundary conditions:

$$\begin{aligned} \sigma_{\theta z}(r,0,t) &= -S(r,t) \quad , \quad 0 \leq r \leq a(t) \\ u_\theta(r,0,t) &= 0 \quad , \quad r > a(t) \end{aligned} \tag{5.1.1}$$

and the conditions at infinity

$$u_\theta(r,z,t) \to 0 \quad \text{as} \quad \sqrt{r^2 + z^2} \to \infty \quad . \tag{5.1.2}$$

In addition we specify the initial conditions

$$u_\theta(r,z,0) = \frac{\partial u_\theta}{\partial t}(r,z,0) = 0 \; . \tag{5.1.3}$$

The solution of this problem reduces to solving (2.4.4) subject to (2.4.1), (5.1.1), (5.1.2) and (5.1.3). A general solution to (2.4.4) subject to (5.1.2) and (5.1.3) is determined as follows. If we take the Laplace transform of (2.4.4) we find, using (4.1.3), that

$$\frac{\partial^2 \bar{u}_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}_\theta}{\partial r} - \frac{\bar{u}_\theta}{r^2} + \frac{\partial^2 \bar{u}_\theta}{\partial z^2} = K_2^2 \bar{u}_\theta \tag{5.1.4}$$

where

$$K_2^2 = \frac{2\rho s}{\bar{G}_1(s)} \quad .$$

We now apply the Hankel transform of order one to (5.1.4) and employ the result (1.2.6) to obtain

$$\frac{d^2\bar{u}^*_{\theta 1}}{dz^2} - (\xi^2 + K_2^2)\bar{u}^*_{\theta 1} = 0. \tag{5.1.5}$$

The solution to (5.1.5) is

$$\bar{u}^*_{\theta 1} = Ae^{-\beta z} \;, \quad \text{where} \quad \beta^2 = \xi^2 + K_2^2 \;. \tag{5.1.6}$$

We point out that we have made use of (5.1.2) in arriving at equation(5.1.6). Here A is an unknown function of ξ and s, the transform variables. On applying the inverse Laplace and Hankel transforms to (5.1.6) we obtain the general solution of (2.4.4) subject to (5.1.2) and (5.1.3) as

$$u_\theta(r,z,t) = L^{-1}\left\{\int_0^\infty A(\xi,s)e^{-\beta z}\, J_1(\xi r)d\xi \quad ; \quad s \to t\right\}. \tag{5.1.7}$$

We can now write $\sigma_{\theta z}$ and $\sigma_{r\theta}$ in terms of the unknown function A as follows:

$$\begin{aligned}\sigma_{z\theta}(r,z,t) &= -\frac{G_1}{2}*dL^{-1}\left\{\int_0^\infty \xi A(\xi,s)\beta e^{-\beta z}\, J_1(\xi r)d\xi \quad ; \quad s \to t\right\} \\ \sigma_{r\theta}(r,z,t) &= \frac{G_1}{2}*dL^{-1}\left\{\int_0^\infty \xi A(\xi s)e^{-\beta z} \left[\xi J_0(\xi r) - \frac{2J_1(\xi r)}{r}\right] d\xi \;;\; s \to t\right\}.\end{aligned} \tag{5.1.8}$$

The problem remains to find a function $A(\xi,s)$ such that u_θ and $\sigma_{\theta z}$ as defined by (5.1.7) and (5.1.8) meet the boundary conditions (5.1.1). Explicitly, we must solve the dual integral equations:

$$\begin{aligned}&L^{-1}\left\{\int_0^\infty \xi A(\xi,s)\;\; J_1(\xi r)d\xi \quad ; \quad s \to t\right\} = 0, \quad r > a(t) \\ &\frac{G_1}{2}*dL^{-1}\left\{\int_0^\infty \xi A(\xi,s)\;\; \beta J_1(\xi r)d\xi \;;\; s \to t\right\} = S(r,t), \; 0 < r < a(t) \;.\end{aligned} \tag{5.1.9}$$

2. Solution of the Dual Integral Equations.

We first note that from the first equation of (5.1.9) we can deduce that

$$A(\xi,s) = \bar{u}^*_{\theta 1}(\xi,s) . \tag{5.2.1}$$

We next define a new function $e(r,t)$ such that

$$\xi A(\xi,s) = 2s\ \bar{G}_1^{\ -1}(s)\bar{e}^*_1(\xi,s) \quad . \tag{5.2.2}$$

If we substitute (5.2.2) into (5.1.9) we obtain

$$\frac{G_1^{-1}}{2} *d \left[\int_o^\infty e^*_1\ J_1(\xi r)d\xi \right] (t) = 0 \quad , \quad r > a(t) \tag{5.2.3}$$

and

$$\int_o^\infty L^{-1}[\beta\bar{e}^*_1(\xi,s)\ ;\ s \to t]\ J_1(\xi r)d\xi \ = \ S(r,t), \quad 0 \le r \le a(t) . \tag{5.2.4}$$

Since $a(t)$ is monotonically increasing, it follows from (1.1.5) that we can replace (5.2.3) by the simpler equation

$$\int_o^\infty e^*_1(\xi,t)\ J_1(\xi r)d\xi = 0 \quad , \quad r > a(t) \ . \tag{5.2.5}$$

Before proceeding further we note that we must find the Laplace inverse of β. To do this we rewrite β in the following manner:

$$\beta = (\xi^2 + K_2^2)^{\frac{1}{2}} = K_2 \left[\frac{\xi^2}{K_2(\xi^2+K_2^2)^{\frac{1}{2}}} \right] + K_2^2 \left[\frac{1}{(\xi^2+K_2^2)^{\frac{1}{2}}} \right] . \tag{5.2.6}$$

If we let $\eta = K_2/\xi$ in the first term in (5.2.6) we can make use of (A.41) and (A.42) to conclude

$$K_2\left[\frac{\xi^2}{K_2(\xi^2+K_2^2)^{1/2}}\right] = \xi - K_2\xi c_2 \sum_7 \int_0^\infty \cos(c_2\xi\tau y)e^{-K_2c_2\tau}\,d\tau \ . \tag{5.2.7}$$

Similarly if we let $\eta = K_2$ in the second term we can identify the result as the Laplace transform with respect to η of $J_o(\xi t)$. We thus obtain

$$\frac{1}{(\xi^2+K_2^2)^{1/2}} = c_2\int_0^\infty J_o(\xi c_2\tau)e^{-K_2c_2\tau}\,d\tau \ . \tag{5.2.8}$$

Combining the results (5.2.6) and (5.2.7) we write (5.2.4) as

$$\int_0^\infty \xi e_1^*(\xi,t)\ J_1(\xi,r)d\xi = S_o(r,t) \quad , \quad 0 \le r \le a(t) \tag{5.2.9}$$

where

$$S_o(r,t) = S(r,t) + \sum_7 \int_0^\infty \xi J_1(\xi r)d\xi \left[\int_0^\infty \cos(\xi c_2\tau y) \cdot [u_2 * de_1^*](\xi,t)d\tau\right] - \int_0^\infty J_1(\xi r)d\xi \cdot \int_0^\infty J_o(\xi c_2\tau) \cdot \frac{\partial}{\partial t}[u_3 * de_1^*](\xi,t)d\tau . \tag{5.2.10}$$

where u_2 is given by (3.3.12) and

$$u_3(t,\tau) = K\,u_1(t,\tau) \ . \tag{5.2.11}$$

The solution of the dual integral equations represented by (5.2.9) and (5.2.5) can be found by using a general result of Noble [30]. We find that

$$e_1^*(\xi,t) = \frac{(2\xi)^{1/2}}{\pi^{1/2}}\int_0^a s^{-1/2} J_{3/2}(\xi s)ds \int_0^s \frac{S_o(x,t)x^2dx}{(s^2-x^2)^{1/2}} \quad ,$$

and

$$E(r,t) = \int_0^\infty J_1(\xi r)\frac{(2\xi)^{1/2}}{\pi^{1/2}}\int_0^a s^{-1/2} J_{3/2}(\xi s)ds \cdot \int_0^s \frac{S_o(x,t)x^2dx}{(s^2-x^2)^{1/2}}\,d\xi \quad , \tag{5.2.13}$$

(5.2.12)

where

$$\xi E_1^*(\xi,t) = e_1^*(\xi,t) \tag{5.2.14}$$

We observe by (5.2.1), (5.2.2) and (5.2.14), that

$$E(r,t) = \left[\frac{G_1}{2} * du_\theta\right](r,t) \ . \tag{5.2.15}$$

We obtain by reversing the orders of integration of the first two integrals and employing the result (c.f.[29], page 314)

$$\int_0^\infty x^{1-\mu+\nu} J_\mu(ax)J_\nu(\xi x)dx = \frac{\xi^\nu(a^2-\xi^2)^{\mu-\nu+1}}{2^{\mu-\nu-1}\Gamma(\mu-\nu)a^\mu} H(a-\xi) \tag{5.2.16}$$

that

$$E(r,t) = \frac{2r}{\pi}\int_r^a \frac{ds}{s^2(s^2-r^2)^{\frac{1}{2}}} \int_0^s \frac{S_o(x,t)x^2dx}{(s^2-x^2)^{\frac{1}{2}}} \ . \tag{5.2.17}$$

We shall now substitute equation (5.2.10) into (5.2.17) and make use of the result [38]

$$\int_0^s \frac{J_1(\xi x)x^2}{(s^2-x^2)^{\frac{1}{2}}}dx = \frac{\sin(\xi s)}{\xi^2} - \frac{s\cos(\xi s)}{\xi} \ , \tag{5.2.18}$$

to get

$$\begin{aligned} E(r,t) = {} & \frac{2r}{\pi}\int_r^a \frac{ds}{s^2(s^2-r^2)^{\frac{1}{2}}} \int_0^s \frac{S(x,t)x^2}{(s^2-x^2)^{\frac{1}{2}}}dx \\ & + \frac{2r}{\pi}\int_r^a \frac{ds}{s^2(s^2-r^2)^{\frac{1}{2}}} \Bigg\{ \Sigma_7 \int_0^\infty [\sin(\xi s) - s\xi\cos(\xi s)]d\xi \ . \\ & \cdot \int_0^\infty \cos(\xi c_2\tau y)[u_2 * dE_1^*](\xi,t)d\tau \\ & - \int_0^\infty \left[\frac{\sin(\xi s)}{\xi} - s\cos(\xi s)\right]d\xi \int_0^\infty J_o(\xi c_2\tau)\frac{\partial}{\partial t}[u_3 * dE_1^*](\xi,t)d\tau \Bigg\} \end{aligned} \tag{5.2.19}$$

The integral equation (5.2.19) must be solved to complete the solution of the problem. At this point we record $\sigma_{\theta z}$ and $\sigma_{\theta r}$ in terms of the unknown function E when $z = 0$,

$$\sigma_{\theta z}(r,0,t) = -\int_0^\infty \xi^2 E_1^*(\xi,t) J_1(\xi r) d\xi$$

$$+ \textstyle\sum_5 \displaystyle\int_0^\infty \xi^2 J_1(\xi r) d\xi \int_0^\infty \cos(\xi c_2 \tau y)[u_2 * dE_1^*](\xi,t) d\tau$$

$$- \int_0^\infty \xi J_1(\xi r) d\xi \int_0^\infty J_o(\xi c_2 \tau) \frac{\partial}{\partial t} [u_3 * dE_1^*](\xi,t) d\tau \quad ,$$

$$\sigma_{\theta r}(r,0,t) = \int_0^\infty \xi E_1^*(\xi,t) \left[\xi J_o(\xi r) - \frac{2J_1(\xi r)}{r} \right] d\xi \ . \tag{5.2.20}$$

We can simplify the expression for $\sigma_{\theta z}$ by first noting the identity

$$J_1(\xi r) = \frac{1}{r} \frac{2}{\pi} \int_0^r \frac{s \sin(\xi s)}{(r^2-s^2)^{\frac{1}{2}}} ds \quad . \tag{5.2.21}$$

This can be obtained by differentiating (B.3) with respect to ξ . If we substitute this result for J_1 into (5.2.20) we find

$$\sigma_{\theta z}(r,0,t) = -\int_0^\infty \xi^2 E_1^*(\xi,t) J_1(\xi r) d\xi$$

$$- \frac{1}{r} \frac{2}{\pi} \int_0^r \frac{s}{(r^2-s^2)^{\frac{1}{2}}} \frac{\partial}{\partial s} \left[\textstyle\sum_7 Q_1 - Q_2 \right] ds \tag{5.2.22}$$

where

$$Q_1 = \frac{\partial}{\partial s} \int_0^\infty \sin(\xi s) \int_0^\infty \cos(\xi c_2 \tau y)[u_2 * dE_1^*](\xi,t) d\tau \tag{5.2.23}$$

and

$$Q_2 = \int_0^\infty \cos(\xi s) \int_0^\infty J_o(\xi c_2 \tau) \frac{\partial}{\partial t} [u_3 * dE_1^*](\xi,t) d\tau \ . \tag{5.2.24}$$

When $S(r,t)$ and $a(t)$ are specified, the integral equation (5.2.19) can be solved numerically for $E(r,t)$.

We shall now consider the particular case when

$$S(r,t) = S(t)r \tag{5.2.25}$$

and

$$a(t) = Vt . \tag{5.2.26}$$

To find the unknown function E we must resort to numerical methods. Once this is done, the result can be substituted into (5.2.20) and (5.2.22) for $\sigma_{\theta r}$ and $\sigma_{\theta z}$ respectively. However, the nature of $\sigma_{\theta z}$ at the crack tip, i.e. $r \to a^+$, is again difficult to determine numerically because of the singularity there. Hence for this value we will derive an approximate analytic expression by taking the first term in (5.2.19) as the value of $E(r,t)$. This term, upon taking the Hankel transform of order one, becomes

$$\underset{\sim}{E}_1^*(\xi,t) = \frac{4S(t)}{3\pi}\int_0^{a(t)} s\left[\frac{\sin(\xi s)}{\xi^2} - \frac{s\cos(\xi s)}{\xi}\right] ds . \tag{5.2.27}$$

If we differentiate this by t we find

$$\frac{d}{dt}\underset{\sim}{E}_1^*(\xi,t) = -\frac{4\dot{S}(t)}{3\pi}\left[\frac{a^2}{\xi^2}\sin(\xi a) + \frac{3a}{\xi^3}\cos(\xi a) - \frac{3\sin(\xi a)}{\xi^4}\right]$$
$$+ \frac{4S(t)}{3\pi} a\dot{a}\left[\frac{\sin(\xi a)}{\xi^2} - \frac{a\cos(\xi a)}{\xi}\right] . \tag{5.2.28}$$

When these terms are substituted into (5.2.22) we get for $r > a(t)$,

$$\sigma_{\theta z}(r,0,t) = \frac{4a^3(t)S(t)}{3\pi r(r^2-a^2)^{\frac{1}{2}}} + \frac{2a(t)S(t)(r^2-a^2)^{\frac{1}{2}}}{\pi r} - \frac{2}{\pi} S(t)r\sin^{-1}(a/r)$$
$$- \frac{8}{3\pi^2}\frac{1}{r}\int_0^r \frac{s}{(r^2-s^2)^{\frac{1}{2}}}\frac{\partial}{\partial s}\left\{\sum_7 \frac{\partial}{\partial s}\int_0^\infty \sin(\xi s)d\xi\right. .$$
$$\int_0^\infty \cos(\xi c_2 xy)\int_0^t u_2(t-\tau,x)\left[\dot{S}(\tau)\int_0^{a(t)} \eta\left[\frac{\sin(\xi\eta)}{\xi^2} - \frac{\eta\cos(\xi\eta)}{\xi}\right] d\eta\right.$$

$$+ S(\tau)a\dot{a}\left[\frac{\sin(\xi a)}{\xi^2} - \frac{a\cos(\xi a)}{\xi}\right]\Bigg] d\tau dx$$

$$- \int_0^\infty \cos(\xi s)d\xi \int_0^\infty J_o(\xi c_2 x)dx \frac{\partial}{\partial t}\int_0^t u_3(t-\tau,x)\Bigg[\dot{S}(\tau)\cdot \tag{5.2.29}$$

$$\int_0^{a(\tau)} \eta\left[\frac{\sin(\xi\eta)}{\xi^2} - \frac{\eta\cos(\xi\eta)}{\xi}\right]d\eta + S(\tau)a\dot{a}\left[\frac{\sin(\xi a)}{\xi^2} - \frac{a\cos(\xi a)}{\xi}\right]\Bigg] d\tau\Bigg\} .$$

The first three terms in (5.2.29) represent the quasi-static solution. The integral in (5.2.29) will not give a singularity as $r \to a^+$ unless the term in brackets, in the integrand, has a jump discontinuity at $r = a(t)$. As before, we employ the notation that

$$[f]_a = \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) \quad . \tag{5.2.30}$$

We assume that the term in question does have such a discontinuity and write (5.2.29) as

$$\sigma_{\theta z}(r,0,t) = \left[a^2(t)S(t) - \frac{2}{\pi}\sum_7[Q_1]_a + \frac{2}{\pi}[Q_2]_a\right]$$

$$\cdot \frac{4a(t)}{3\pi r(r^2-a^2)^{\frac{1}{2}}} + \frac{2a(t)S(t)(r^2-a^2)^{\frac{1}{2}}}{\pi r} - \frac{2}{\pi}S(t)r\sin^{-1}(a/r) \tag{5.2.31}$$

$$- \frac{8}{3\pi^2 r}\int_0^a \frac{s}{(r^2-s^2)^{\frac{1}{2}}}\frac{\partial}{\partial s}\left[\sum_7 Q_1 - Q_2\right]ds - \frac{8}{3\pi^2 r}\int_a^r \frac{s}{(r^2-s^2)^{\frac{1}{2}}}ds\frac{\partial}{\partial s}\left[\sum_7 Q_1 - Q_2\right]$$

The last two integrals can now be handled numerically. The task remaining is to verify our assumption that there does exist jump discontinuities and find them. From our experience with this type of problem we can determine that only the terms containing the power ξ^{-1} will contribute to jump discontinuities. Hence we need only to consider the following two terms from Q_1 and Q_2, which we shall denote as

$$Q_1' = -\frac{\partial}{\partial s}\int_0^\infty \sin(\xi s)d\xi \int_0^\infty \cos(\xi c_2 xy)\int_0^t u_2(t-\tau,x)S(\tau)\cdot$$

$$\cdot\ a^2(\tau)\dot{a}(\tau)\frac{\cos(\xi a)}{\xi}d\tau dx \tag{5.2.32}$$

and

$$Q_2' = -\int_0^\infty \cos(\xi s)d\xi \int_0^\infty J_0(\xi c_2 x)dx \frac{\partial}{\partial t}\int_0^t u_3(t-\tau,x)S(\tau)\cdot$$

$$a^2(\tau)\dot{a}(\tau)\frac{\cos(\xi a)}{\xi}d\tau . \tag{5.2.33}$$

To calculate the jump in Q_1' we follow the same analysis that led to equation (4.3.11). Accordingly we find

$$Q_1' = -\frac{1}{4}\frac{\partial}{\partial s}\int_0^t S(\tau)a^2(\tau)\dot{a}(\tau)d\tau \int_0^\infty u_2(t-\tau,x)dx \int_0^\infty \sum_{i=1}^4 \sin(\xi\gamma_i)\frac{d\xi}{\xi}$$

where

$$\begin{aligned} \gamma_1 &= s + a(\tau) + c_2yx, \\ \gamma_2 &= s + a(\tau) - c_2yx, \\ \gamma_3 &= s - a(\tau) + c_2yx, \\ \gamma_4 &= s - a(\tau) - c_2yx, \end{aligned} \tag{5.2.34}$$

and their graphs are indicated in Figure 4.1 and 4.2. The table of signs for γ_i for the present problem is given by Table 5.1.

Table 5.1 The sign of γ_i in the region A_K

	A_1	A_2	A_3	A_4
γ_1	+	+	+	+
γ_2	+	+	+	-
γ_3	+	-	+	+
γ_4	+	-	-	-
C	2π	0	π	0

Following the same procedure, by which we arrived at (4.3.12) and (4.3.13), we find in the present case

$$Q_1' = -\frac{\pi V^3}{4c_2 y}\left\{\int_o^t S(\tau)\tau^2 u_2(t-\tau,\frac{s+V\tau}{c_2 y})d\tau + \int_{s/V}^t S(\tau)\tau^2 u_2(t-\tau,\frac{V\tau-s}{c_2 y})d\tau + \int_o^{s/V} S(\tau)\tau^2 u_2(t-\tau,\frac{s-V\tau}{c_2 y})d\tau\right\}, \quad s < a(t) \tag{5.2.35}$$

and

$$Q_1' = -\frac{\pi V^3}{4c_2 y}\int_o^t S(\tau)\tau^2[u_2(t-\tau,\frac{s+V\tau}{c_2 y}) + u_2(t-\tau,\frac{s-V\tau}{c_2 y})]d\tau, \quad s > a(t) \tag{5.2.36}$$

We now combine (5.2.30), (5.2.35) and (5.2.36) to get

$$[Q_1']_a = -\frac{\pi V^3}{4c_2 y}\lim_{\substack{s\to a^+(t)\\ s\to a^-(t)}}\left\{\int_o^t S(\tau)\tau^2[u_2(t-\tau,\frac{s+V\tau}{c_2 y}) - u_2(t-\tau,\frac{s'+V\tau}{c_2 y})]d\tau + \int_o^{s'/V} S(\tau)\tau^2[u_2(t-\tau,\frac{s-V\tau}{c_2 y}) - u_2(t-\tau,\frac{s'V\tau}{c_2 y})]d\tau + \int_{s'/V}^t S(\tau)\tau^2[u_2(t-\tau,\frac{s-V\tau}{c_2 y}) - u_2(t-\tau,\frac{V\tau-s'}{c_2 y})]d\tau\right\}. \tag{5.2.37}$$

When the limits are taken in (5.2.37) the first two integrals go to zero. The last integral however, gives a non-zero contribution. It's value is determined in the same manner as (4.3.19) and we record the final result

$$[Q_1']_a = -\frac{\pi}{2}\frac{S(t)a^2(t)v_2^2}{(y^2-v_2^2)}. \tag{5.2.38}$$

We return now to the determination of $[Q_2']_a$. We note that equation (5.2.33) can be written as

$$Q_2' = + \frac{\partial}{\partial t} \int_0^s d\eta \int_0^t S(\tau)a^2(\tau)\dot{a}(\tau)d\tau \int_0^\infty u_3(t-\tau,x)dx \cdot$$

$$\cdot \int_0^\infty \sin(\xi\eta)\cos(\xi a)\ J_o(\xi c_2 x)d\xi \quad . \tag{5.2.39}$$

In (5.2.39) we have added a term independent of s which when differentiated will disappear. This term is represented by the lower limit of the outer integral. We now follow the analysis which led to calculation of $[T_{22}]_a$ in the previous chapter. We make use of a trigonometric identity and (4.3.20) to find that

$$Q_2' = + \frac{\partial}{\partial t} \frac{V}{2} \int_0^s d\eta \int_0^t S(\tau)a^2(\tau)d\tau \int_0^\infty u_3(t-\tau,x) \cdot$$

$$\cdot \left[\frac{H(\eta+a(\tau)-c_2x)}{[(\eta+a(\tau))^2-c_2^2x^2]^{\frac{1}{2}}} + \frac{H[|\eta-a(\tau)| - c_2x]}{[(\eta-a(\tau))^2-c_2^2x^2]^{\frac{1}{2}}} \right] dx \tag{5.2.40}$$

We can expand (5.2.40) for the two cases, $s < a(t)$ and $s > a(t)$ and arrive at parallel equations to (4.3.21) and (4.3.22). We now use (5.2.30) and calculate $[Q_2']_a$ to be

$$[Q_2']_a = -\frac{V^2}{2} \left\{ + \lim_{\eta\to a^-(t)} \int_{\eta'/V}^t S(\tau)a^2(\tau)d\tau \int_0^{\frac{V\tau-\eta'}{c_2}} \frac{u_3(t-\tau,x)}{[(V\tau-\eta')^2-c_2^2x^2]^{\frac{1}{2}}} dx \right.$$

$$- \lim_{\eta\to a^-(t)} \int_0^{\eta'/V} S(\tau)a^2(\tau)d\tau \int^{\frac{\eta'-V\tau}{c_2}} \frac{u_3(t-\tau,x)}{[(\eta'-V\tau)^2-c_2^2x^2]^{\frac{1}{2}}} dx$$

$$\left. + \lim_{\eta\to a^+(t)} \int_0^t S(\tau)a^2(\tau)d\tau \int^{\frac{\eta-V\tau}{c_2}} \frac{u_3(t-\tau,x)}{[(\eta-V\tau)^2-c_2^2x^2]^{\frac{1}{2}}} dx \right\} \tag{5.2.41}$$

Equation (5.2.41) is similar to (4.3.23). Hence, since

$$\lim_{\varepsilon\to 0^+} \varepsilon u_3(\varepsilon t,\varepsilon\tau) = \frac{1}{c_2}\delta(t-\tau), \tag{5.2.42}$$

we deduce that

$$[Q_2'] = -\frac{\pi}{2}\ \frac{S(t)a^2(t)v_2^2}{[1-v_2^2]^{\frac{1}{2}}}\ . \tag{5.2.43}$$

We point out that the result (5.2.42) can be found by the same method which determined (4.3.18). Similarly (5.2.43) was obtained using the same arguments that led to (4.3.26).

We summarize our results now by collecting (5.2.38), (5.2.43) and (5.2.31) and writing the singular part of $\sigma^s_{\theta z}$ as

$$\sigma^s_{\theta z} = \frac{4a^3(t)S(t)}{3\pi r(r^2-a^2)^{\frac{1}{2}}} \cdot \left[1 + \sum_7 \frac{v_2^2}{y^2-v_2^2} - \frac{v_2^2}{[1-v_2^2]^{\frac{1}{2}}} \right] . \tag{5.2.44}$$

The term involving $\sum_7$ can be integrated to give

$$\sigma^s_{\theta z} = \frac{4a^3(t)S(t)}{3\pi(2a(t))^{\frac{1}{2}}} \cdot \sqrt{1-v_2^2}\ . \tag{5.2.45}$$

With this term we are thus able to determine $\sigma_{\theta z}$ completely when the remainder of (5.2.31) is evaluated numerically.

We now define the stress intensity factor for this problem to be

$$N(t) = \lim_{r\to a^+} \{[r - a(t)]^{\frac{1}{2}}\ \sigma_{\theta z}(r,0,t)\}\ . \tag{5.2.46}$$

From equations (5.2.31), (5.2.45) and (5.2.46) we find $N(t)$ to be

$$N(t) = \frac{4a^2(t)S(t)}{3\pi(2a(t))^{\frac{1}{2}}} \cdot \sqrt{1-v_2^2} \tag{5.2.47}$$

From this equation we observe that the term $\sqrt{1-v_2^2}$ represents a correction factor on the quasi-static result.

The above technique can also be used to find an approximate expression in the case when $S(r,t) \equiv S(r)$. We state the final result

Table 5.2 Calculated values of $E(r,t)/E_s(r,t)$ for $\dot{v}_2 = .5$

r/Vt \ t	.000001	.00001	.0001	.001	.01
.2	.828	.874	.869	.869	.87
.4	.88	.928	.939	.941	.941
.6	.896	.981	1.011	1.015	1.015
.8	.881	.903	.904	.904	.905

Table 5.3 Stress intensity factor for varying v_2.

v_2	$\frac{N(t)\ (2a(t))^{\frac{1}{2}}}{a^2(t)\ S(t)}$
0	.42441
.1	.42228
.2	.41583
.3	.40486
.4	.38898
.5	.36755
.6	.33953
.7	.30309
.8	.25465
.9	.18499
1	0

$$N(t) = \frac{2\sqrt{1-v_2^2}}{\pi a(t)(2a(t))^{\frac{1}{2}}} \int_0^{a(t)} \frac{S(x)x^2dx}{(a^2-x^2)^{\frac{1}{2}}} \qquad (5.2.48)$$

As a final remark we point out that it is also likely possible to replace $S(x)$ in (5.2.48) by $S(x,t)$ but the analysis then becomes more complicated and has not been considered.

3. Numerical Calculations for a Maxwell Material.

The quantities of interest are $N(t)$ given by (5.2.48) and the tangential displacement u_θ. To indicate the dynamic effect on u_θ we have calculated the quantity E/E_s where E is given by (5.2.19) and (5.2.25) and E_s is defined as

$$E_s(r,t) = \frac{4r}{3\pi}(a^2(t) - r^2)^{\frac{1}{2}} S_o \qquad (5.3.1)$$

The results of these calculations are in Table 5.2. We note that for numerical computations we have set $S(t) = S_o$.

The stress intensity factor is calculated in a normalized form and the results are given in Table 5.3.

4. An Exact Solution of a Dynamic Elastic Crack Under Torsion.

In the course of the present investigation an exact dynamic solution was found in the case of a growing penny shaped crack in an elastic solid under torsion when $a(t) = Vt$ and $S(r,t) = S_o$. The general analysis that led to the integral equations (5.1.9) still holds when the material is elastic. Equations (5.1.9) however, take the simple form,

$$\mu \int_0^\infty u_{\theta 1}^*(\xi,t)\ J_1(\xi r)d\xi = 0 \quad , \qquad r > a(t) \ , \tag{5.4.1}$$

$$\int_0^\infty \xi L^{-1}[\sqrt{\xi^2+s^2/c_2^2}\ \ \bar{u}_{\theta 1}^*\ (\xi,s);\ s \to t]\ J_1(\xi r)d\xi = S_o \ , \quad 0 < r < a(t) \tag{5.4.2}$$

Equation (5.4.1) is identical to that found in static elasticity. Hence if we let u_θ be equal to the result found in the static problem then (5.4.1) will be automatically met. The question however, remains whether this choice also satisfies equation (5.4.2). It turns out that the answer to this question is in the affirmative and the general expression for $\sigma_{\theta z}$ with the use of (5.4.2) can be integrated to give

$$\sigma_{\theta z}(r,0,t) = S_o \int_0^\infty \left\{ \frac{-1}{\xi} + S_1 \frac{\cos(a\xi)}{\xi} + S_2 \frac{a}{2} \sin(a\xi) - \sum\nolimits_8 \frac{\cos(ya\xi)}{\xi} \right\} J_1(\xi r)d\xi \tag{5.4.3}$$

where

$$u_\theta(r,0;t) = \frac{r}{2\mu} \ln \left[\frac{a(t) + (a^2(t) - r^2)^{\frac{1}{2}}}{r} \right] , \quad 0 < r < a(t) ,$$

$$S_1 = \frac{2 - v_2^2}{2(1-v_2^2)^{\frac{1}{2}}} \ , \quad S_2 = (1 - v_2^2)^{\frac{1}{2}} \ , \quad \text{and} \tag{5.4.4}$$

$$\sum\nolimits_8 f(y) = \frac{2}{\pi} \int_1^\infty \frac{(y^2-1)^{\frac{1}{2}}\ f(y)}{y(y^2-v_2^2)^2}\ dy \ . \tag{5.4.5}$$

The integral in (5.4.3) can be simplified by making use of the results

$$\int_0^\infty \cos(a\xi)\ \frac{J_1(\xi r)}{\xi}\ d\xi = H(r - a)\ \frac{\sqrt{r^2-a^2}}{r} \ , \tag{5.4.6}$$

$$\int_0^\infty \sin(a\xi)\ J_1(\xi r)d\xi = \frac{aH(r-a)}{r\sqrt{r^2-a^2}} \ . \tag{5.4.7}$$

We find that

$$\sigma_{\theta z}(r,0,t) = - S_o + S_o H(r - a) \frac{\sqrt{r^2-a^2}}{r} S_1 + \frac{S_o H(r-a) a^2 S_2}{2r \sqrt{r^2-a^2}}$$

$$- S_o \sum_8 \frac{\sqrt{r^2-a^2y}}{r} H(r - ay) . \qquad (5.4.8)$$

This result satisfies (5.4.2) and hence verifies our original assumption. The stress intensity factor can be calculated to be

$$N(t) = \frac{a(t) S_o}{2(2a(r))^{1/2}} \sqrt{1 - v_2^2} \qquad (5.4.9)$$

This result has not appeared in the literature to this date. The term $\sigma_{\theta r}$ on the plane $z = 0$ will be the same as that for the static case. Away from the plane all terms however, differ from the static results.

5. Discussion

The results listed in Table 5.2 indicate that the dynamic effect on the tangential displacement is of a complex nature. It was observed in calculating u_θ that the solution seemed less stable for larger times. The exact explanation of these observations must await further numerical analysis. For the present, the results displayed in Table 5.2 should only be considered in a qualitative manner.

CHAPTER VI

THE TOROIDAL CRACK PROBLEM IN ELASTICITY

1. A Toridal Crack in a Infinite Elastic Solid in Tension.

In this section we will give a solution to the problem of a plane toroidal crack in an infinite elastic medium which is opened by a normal pressure acting on its surface. In terms of circular cylindrical coordinates, (r,θ,z), the stresses and displacements for this problem are the same as that in a semi-infinite body $z \geq 0$ when its surface B, is subject to the boundary conditions:

$$\begin{aligned} &\sigma_{rz}(r,0) = \sigma_{z\theta}(r,0) = 0 \quad , \quad r \geq 0 , \\ &\sigma_{zz}(r,0) = -P(r) \quad , \quad a \leq r \leq b, \\ &u_z(r,0) = 0 \quad , \quad 0 \leq r \leq a, \ \ b \leq r, \end{aligned} \tag{6.1.1}$$

and the conditions at infinity (3.1.2).

The problem posed above requires the solution of equations (2.3.10), (2.3.8), (6.1.1) and (3.1.2). We observe that the stresses given by (2.3.10) satisfy the first condition of (6.1.1) immediately. We now find a general solution to equation (2.3.8). If we take the Hankel transform of order zero we obtain using (1.2.5)

$$\frac{d^2 x_0^*}{dz^2}(\xi,z) - \xi^2 x_0^*(\xi,z) = 0. \tag{6.1.2}$$

The solution of (6.1.2) which also satisfies (3.1.2) is

$$x_0^*(\xi,z) = e^{-\xi z} A(\xi) \quad , \tag{6.1.3}$$

and hence we have

$$x(r,z) = \int_0^\infty \xi A(\xi) e^{-\xi z} J_0(\xi r) d\xi \; . \qquad (6.1.4)$$

By substituting (6.1.4) into (2.3.10) we get the stresses and displacements in terms of an unknown function $\psi(\xi)$ defined as $\xi A(\xi) = \psi(\xi)$:

$$u_r = -\int_0^\infty [1 - 2\nu - \xi z] \xi \psi(\xi) e^{-\xi z} J_1(\xi r) d\xi \; ,$$

$$u_z = \int_0^\infty [2(1-\nu) + \xi z] \xi \psi(\xi) e^{-\xi z} J_0(\xi r) d\xi \; ,$$

$$\sigma_{rr} = -2\mu \int_0^\infty (1 - \xi z) \xi^2 \psi(\xi) e^{-\xi z} J_0(\xi r) d\xi + \frac{2\mu}{r} \int_0^\infty [(1-2\nu) - \xi z] \xi \psi(\xi) e^{-\xi z} J_1(\xi r) d\xi \; ,$$

$$\sigma_{\theta\theta} = -4\mu\nu \int_0^\infty \xi^2 \psi(\xi) e^{-\xi z} J_0(\xi r) d\xi - \frac{2\mu}{r} \int_0^\infty [(1-2\nu) - \xi z] \xi \psi(\xi) e^{-\xi z} d\xi \; ,$$

$$\sigma_{zz} = -2\mu \int_0^\infty (1 + \xi z) \xi^2 \psi(\xi) e^{-\xi z} J_0(\xi r) d\xi \; ,$$

$$\sigma_{zr} = -2\mu z \int_0^\infty \xi^3 \psi(\xi) e^{-\xi z} J_1(\xi r) d\xi \; ,$$

$$u_\theta = \sigma_{\theta z} = \sigma_{\theta r} = 0. \qquad (6.1.5)$$

If we employ the last two conditions of (6.1.1) on (6.1.5) the unknown function $\psi(\xi)$ will be determined as the solution of a set of triple integral equations.

2. Solution of a Set of Triple Integral Equations.

By combining the remaining two conditions of (6.1.1) and (6.1.5) we find:

$$\int_0^\infty \xi\psi(\xi)J_o(\xi r)d\xi = 0 \quad , \quad 0 \le r \le a \quad ,$$

$$\int_0^\infty \xi^2\psi(\xi)J_o(\xi r)d\xi = -P(r) \quad , \quad a \le r \le b \quad , \tag{6.2.1}$$

$$\int_0^\infty \xi\psi(\xi)J_o(\xi r)d\xi = 0 \quad ; \quad r \ge b.$$

The solution of equations of this type has been considered by Cooke [26]. His approach transforms the problem to that of a Fredholm integral equation of the second kind in either the unknown normal stress on the inside $(0 \le r \le a)$, σ', or outside $(r > b)$, σ^2. In terms of σ', σ^2 and the prescribed function $P(r)$ we can write the solution of (6.1.2) as

$$\xi\psi(\xi) = \int_0^a \lambda\sigma'(\lambda)J_o(\xi\lambda)d\lambda - \int_a^b \lambda P(\lambda)J_o(\xi\lambda)d\lambda$$
$$+ \int_b^\infty \lambda\sigma^2(\lambda)J_o(\xi\lambda)d\lambda \quad , \tag{6.2.2}$$

where from Cooke we have

$$\sigma^2(\lambda) = + \frac{2}{\pi(\lambda^2-b^2)^{\frac{1}{2}}} \int_a^b \frac{t(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-t^2)} P(t)dt$$
$$- \frac{4}{\pi^2(\lambda^2-b^2)^{\frac{1}{2}}} \int_a^b P(t)K_2(t,\lambda)dt \tag{6.2.3}$$
$$+ \frac{4}{\pi^2(\lambda^2-b^2)^{\frac{1}{2}}} \int_b^\infty \sigma^2(t)K_2(t,\lambda)dt \quad , \quad b < \lambda < \infty$$

and

$$\sigma'(\lambda) = \frac{2}{\pi(a^2-\lambda^2)^{\frac{1}{2}}} \int_a^b \frac{t(t^2-a^2)^{\frac{1}{2}}P(t)dt}{(t^2-\lambda^2)} - \frac{4}{\pi^2(a^2-\lambda^2)^{\frac{1}{2}}} \int_a^b P(t)K_1(t,\lambda)dt$$
$$+ \frac{4}{\pi^2(a^2-\lambda^2)^{\frac{1}{2}}} \int_0^a \sigma'(t)K_1(t,\lambda)dt \quad , \quad 0 \le \lambda < a, \tag{6.2.4}$$

$$K_2(y,\lambda) = y(y^2 - a^2)^{\frac{1}{2}} \int_0^a \frac{t(b^2-t^2)^{\frac{1}{2}}dt}{(a^2-t^2)^{\frac{1}{2}}(\lambda^2-t^2)(y^2-t^2)} \quad , \tag{6.2.5}$$

$$K_1(y,\lambda) = y(b^2 - y^2)^{\frac{1}{2}} \int_b^\infty \frac{t(t^2-a^2)^{\frac{1}{2}}dt}{(t^2-b^2)^{\frac{1}{2}}(t^2-\lambda^2)(t^2-y^2)} \quad . \tag{6.2.6}$$

It is only necessary to solve one of either (6.2.3) or (6.2.4) since the two functions are related by the equation,

$$\sigma^2(\lambda) = \frac{2}{\pi(\lambda^2-b^2)^{\frac{1}{2}}} \int_a^b \frac{t(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-t^2)} P(t)dt$$

$$- \frac{2}{\pi(\lambda^2-b^2)^{\frac{1}{2}}} \int_0^a \frac{t(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-t^2)} \sigma'(t)dt \quad , \quad b < \lambda < \infty. \tag{6.2.7}$$

To write the normal displacement we note that when $z = 0$,

$$u^*_{zo}(\xi,0) = 2(1 - \nu)\psi(\xi) \tag{6.2.8}$$

and we can relate u_z and σ_{zz} in the following manner

$$\sigma_{zz}(r,0) = - \frac{\mu}{(1 - \nu)} \int_0^\infty \xi^2 u^*_{zo}(\xi,0) J_o(\xi r) d\xi \quad .$$

From (B.1), (B.9) and (6.1.1) we may conclude that

$$u_z(r,0) = - \frac{2(1-\nu)}{\pi\mu} \int_r^b \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_0^s \frac{\lambda\sigma_{zz}(\lambda,0)d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}} \quad , \quad a \leq r \leq b \; , \tag{6.2.9}$$

which can be expanded as follows

$$u_z(r,0) = \frac{2(1-\nu)}{\pi\mu} \int_r^b \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_a^s \frac{\lambda P(\lambda)d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}}$$

$$- \frac{2(1-\nu)}{\pi\mu} \int_r^b \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_0^a \frac{\lambda\sigma'(\lambda)d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}} \quad , \quad a \leq r \leq b. \tag{6.2.10}$$

An alternate form to (6.2.10) is the following,

$$u_z(r,0) = +\frac{2(1-\nu)}{\pi\ \mu}\int_a^r \frac{ds}{(x^2-s^2)^{\frac{1}{2}}}\int_s^b \frac{\lambda P(\lambda)d\lambda}{(\lambda^2-s^2)^{\frac{1}{2}}}$$
$$-\frac{2(1-\nu)}{\pi\ \mu}\int_a^r \frac{ds}{(x^2-s^2)^{\frac{1}{2}}}\int_b^\infty \frac{\lambda\sigma^2(\lambda)d\lambda}{(\lambda^2-s^2)^{\frac{1}{2}}}\ ,\quad a \le \rho \le b. \tag{6.2.11}$$

It is interesting to consider two limiting cases of this problem, namely the penny shaped crack $(a \to 0)$ and the external crack $(b \to \infty)$. In the first case if we let $a \to 0$, then $K_2 \to 0$ and (6.2.3) reduces to

$$\sigma_{zz}(\lambda,0) = \frac{2}{\pi}\int_o^b \frac{tP(t)}{(\lambda^2-t^2)}\ \frac{(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-b^2)^{\frac{1}{2}}}\, dt\ .$$

On using the result that

$$\int_t^b \frac{sds}{(s^2-t^2)^{\frac{1}{2}}(\lambda^2-s^2)^{3/2}} = \frac{1}{\lambda^2 - t^2}\ \frac{(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-b^2)^{\frac{1}{2}}} \tag{6.2.12}$$

and integrating by parts, after interchanging the order of integration, we find

$$\sigma_{zz}(r,0) = \frac{g_1(b)}{\sqrt{r^2-b^2}} - \int_o^b \frac{g_1'(t)dt}{\sqrt{r^2-t^2}}\ ,\quad r > b.$$

where

$$g_1(t) = \frac{2}{\pi}\int_o^t \frac{sP(s)ds}{(t^2-s^2)^{\frac{1}{2}}}\ .$$

This is the result given in [14], page 136. As for the displacement, if we let $a \to 0$ in (6.2.10), we obtain

$$u_z(r,0) = \frac{1-\nu}{\mu}\int_r^b \frac{g_1(s)ds}{(s^2-r^2)^{\frac{1}{2}}}\ ,\quad 0 \le r \le b\ .$$

Again this is the same result as in [13].

In the other limiting case as $b \to \infty$, then $K_1 \to 0$ and (6.2.4) becomes

$$\sigma_{zz}(\lambda,0) = \frac{2}{\pi}\int_a^\infty \frac{tP(t)}{(t^2-\lambda^2)}\frac{(t^2-a^2)^{1/2}}{(a^2-\lambda^2)^{1/2}}dt \ .$$

This time we use the result

$$\int_a^t \frac{sds}{(s^2-\lambda^2)^{3/2}(t^2-s^2)^{1/2}} = \frac{1}{(t^2-\lambda^2)}\frac{(t^2-a^2)^{1/2}}{(a^2-\lambda^2)^{1/2}} \quad , \tag{6.2.13}$$

and perform the same operations as before to arrive at

$$\sigma_{zz}(r,0) = \int_a^\infty \frac{g'_2(t)}{\sqrt{t^2-r^2}}\ dt + \frac{g_2(a)}{\sqrt{a^2-r^2}} \quad , \quad 0 \leq r < a \ , \tag{6.2.14}$$

where

$$g_2(t) = \frac{2}{\pi}\int_t^\infty \frac{sP(s)ds}{(s^2-t^2)^{1/2}} \ .$$

Similarly the normal displacement is given by (6.1.11) in the limiting case as

$$u_z(r,0) = \frac{(1-\nu)}{\mu}\int_a^r \frac{g_2(s)ds}{\sqrt{r^2-s^2}} \quad , \quad r > a.$$

Both of these results agree with those in [14], page 183.

Returning to the original problem we now consider the specific case when $P(r) = P_o$. The free terms in the integral equations (6.2.3) and (6.2.4) can now be integrated to a point. We first list, for references, some integrals we shall need to integrate these terms.

$$\int_a^b t\,\frac{(b^2-t^2)^{1/2}}{(\lambda^2-t^2)}\,dt = c - \sqrt{\lambda^2-b^2}\,\sin^{-1}\left[\frac{c}{\sqrt{\lambda^2-a^2}}\right], \quad \lambda > b \ ,$$

$$\int_a^b t\,\frac{(t^2-a^2)^{1/2}}{(t^2-\lambda^2)}\,dt = c - \sqrt{a^2-\lambda^2}\,\sin^{-1}\left[\frac{c}{\sqrt{b^2-\lambda^2}}\right], \quad \lambda < a \ ,$$

$$\int_a^b t\,\frac{(b^2-t^2)^{1/2}}{(t^2-\lambda^2)}\,dt = -c - \sqrt{b^2-\lambda^2}\,\ln\left[\frac{\sqrt{a^2-\lambda^2}}{c+\sqrt{b^2-\lambda^2}}\right], \quad \lambda < a \ ,$$

$$\int_a^b t\,\frac{(t^2-a^2)^{1/2}}{\lambda^2-t^2}\,dt = -c - \sqrt{\lambda^2-a^2}\,\ln\left[\frac{\sqrt{\lambda^2-b^2}}{c+\sqrt{\lambda^2-a^2}}\right], \quad \lambda > b \ ,$$

$$\int_0^a \frac{t(b^2-t^2)^{\frac{1}{2}}dt}{(a^2-t^2)^{\frac{1}{2}}(\lambda^2-t^2)(y^2-t^2)} = \sqrt{\frac{\lambda^2-b^2}{\lambda^2-a^2}}\,\frac{1}{\lambda^2-y^2}\ln\left[\frac{b\sqrt{\lambda^2-a^2}+a\sqrt{\lambda^2-b^2}}{\lambda c}\right]$$

$$+\frac{1}{2}\sqrt{\frac{b^2-y^2}{y^2-a^2}}\,\frac{1}{\lambda^2-y^2}\left[\sin^{-1}\left[\frac{a^2(b^2-y^2)-b^2(y^2-a^2)}{y^2c^2}\right]+\frac{\pi}{2}\right],\ \lambda > b,\ b > y > a\ ,$$

$$\int_b^\infty \frac{t(t^2-a^2)^{\frac{1}{2}}dt}{(t^2-b^2)^{\frac{1}{2}}(t^2-\lambda^2)(t^2-y^2)} = \sqrt{\frac{a^2-\lambda^2}{b^2-\lambda^2}}\,\frac{1}{y^2-\lambda^2}\ln\left[\frac{\sqrt{a^2-\lambda^2}+\sqrt{b^2-\lambda^2}}{c}\right]$$

$$+\frac{1}{2}\sqrt{\frac{y^2-a^2}{b^2-y^2}}\,\frac{1}{y^2-\lambda^2}\left[\sin^{-1}\left[\frac{(y^2-a^2)-(b^2-y^2)}{c^2}\right]+\frac{\pi}{2}\right],\ \lambda < a,\ b > y > a\ ,$$

$$\frac{1}{2}\int_a^b \frac{y(b^2-y^2)^{\frac{1}{2}}}{(\lambda^2-y^2)}\left[\sin^{-1}\left[\frac{a^2(b^2-y^2)-b^2(y^2-a^2)}{y^2c^2}\right]+\frac{\pi}{2}\right]dy$$

$$=\frac{c\pi}{2}-b\cos^{-1}\left(\frac{a}{b}\right)-a^2bc(\lambda^2-b^2)\int_0^1\frac{\sin^{-1}(y)\,dy}{[a^2+c^2y^2]^{\frac{1}{2}}[a^2(\lambda^2-b^2)+\lambda^2c^2y^2]},\ \lambda > b,$$

$$\frac{1}{2}\int_a^b \frac{y(y^2-a^2)^{\frac{1}{2}}}{y^2-\lambda^2}\left[\sin^{-1}\left[\frac{(y^2-a^2)-(b^2-y^2)}{c^2}\right]+\frac{\pi}{2}\right]dy$$

$$= c\left[\frac{\pi}{2}-1\right]-c(a^2-\lambda^2)\int_0^1\frac{\sin^{-1}(y)\,dy}{[a^2-\lambda^2+c^2y^2]}\ ,$$

where $c^2 = b^2-a^2$.

(6.2.15)

Collecting equations (6.2.3), (6.2.4), (6.2.5), (6.2.6) and (6.2.15) we find, in case $P(r) = P_o$, that

$$\sigma^2(\lambda) = \frac{4\,P_o b\cos^{-1}(a/b)}{\pi^2(\lambda^2-b^2)^{\frac{1}{2}}} - \frac{2}{\pi}P_o\sin^{-1}[c/\sqrt{\lambda^2-a^2}]$$

$$+\frac{4\,P_o}{\pi^2(\lambda^2-a^2)^{\frac{1}{2}}}\ln\left[\frac{b\sqrt{\lambda^2-a^2}+a\sqrt{\lambda^2-b^2}}{\lambda c}\right]\left\{c+\sqrt{\lambda^2-a^2}\cdot\ln\frac{\sqrt{\lambda^2-b^2}}{c+\sqrt{\lambda^2-a^2}}\right\}$$

$$+\frac{4\,P_o a^2bc(\lambda^2-b^2)^{\frac{1}{2}}}{\pi^2}\int_0^1\frac{\sin^{-1}(v)\,dv}{(a^2+c^2y^2)^{\frac{1}{2}}[a^2(\lambda^2-b^2)+\lambda^2c^2y^2]}$$

$$+\frac{4}{\pi^2}\int_b^\infty\sigma^2(y)K_2'(y,\lambda)\,dy\ ,\quad \lambda > b\ , \qquad (6.2.16)$$

where

$$K_2'(y,\lambda) = \frac{1}{(\lambda^2-a^2)^{\frac{1}{2}}} \ln\left[\frac{b\sqrt{\lambda^2-a^2} + a\sqrt{\lambda^2-b^2}}{\lambda c}\right] \cdot \frac{y(y^2-a^2)^{\frac{1}{2}}}{(\lambda^2-y^2)}$$

$$- \frac{1}{(\lambda^2-b^2)^{\frac{1}{2}}} \cdot \frac{y(y^2-b^2)^{\frac{1}{2}}}{(\lambda^2-y^2)} \ln\left[\frac{b\sqrt{y^2-a^2} + a\sqrt{y^2-b^2}}{yc}\right], \lambda > b, \tag{6.2.17}$$

and

$$\sigma'(\lambda) = \frac{4 P_o c}{\pi^2(a^2-\lambda^2)^{\frac{1}{2}}} - \frac{2}{\pi} P_o \sin^{-1}[c/\sqrt{b^2-\lambda^2}]$$

$$+ \frac{4 P_o}{\pi^2(b^2-\lambda^2)^{\frac{1}{2}}} \ln\left[\frac{\sqrt{a^2-\lambda^2} + \sqrt{b^2-\lambda^2}}{c}\right]\left\{c + \sqrt{b^2-\lambda^2} \ln\left[\frac{\sqrt{a^2-\lambda^2}}{c + \sqrt{b^2-\lambda^2}}\right]\right\}$$

$$+ \frac{4 P_o c(a^2-\lambda^2)^{\frac{1}{2}}}{\pi^2} \int_0^1 \frac{\sin^{-1}(v)dv}{a^2-\lambda^2 + c^2v^2}$$

$$+ \frac{4}{\pi^2} \int_\rho^a \sigma'(y) K_1'(y,\lambda) dy, \quad \lambda < a, \tag{6.2.18}$$

where

$$K_1'(y,\lambda) = \frac{1}{(b^2-\lambda^2)^{\frac{1}{2}}} \ln\left[\frac{\sqrt{a^2-\lambda^2} + \sqrt{b^2-\lambda^2}}{c}\right] \frac{y(b^2-y^2)^{\frac{1}{2}}}{(y^2-\lambda^2)}$$

$$- \frac{1}{(a^2-\lambda^2)^{\frac{1}{2}}} \frac{y(a^2-y^2)^{\frac{1}{2}}}{(y^2-\lambda^2)} \ln\left[\frac{\sqrt{a^2-y^2} + \sqrt{b^2-y^2}}{c}\right], \quad \lambda < a. \tag{6.2.19}$$

From (6.2.10) we can write the normal displacement as

$$u_z(\rho,0) = \frac{2(1-\nu)P_o}{\pi \mu} \left[\frac{b\sqrt{b^2-\rho^2}}{c} + \frac{(\rho^2-a^2)F(\phi,a/\rho) - \rho^2 E(\phi,a/\rho)}{\rho}\right]$$

$$\frac{-2(1-\nu)}{\pi \mu} \int_\rho^b \frac{ds}{(s^2-\rho^2)^{\frac{1}{2}}} \int_0^a \frac{\lambda\sigma'(\lambda)d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}}, \quad a \le \rho \le b, \tag{6.2.20}$$

where F and E are elliptic integrals of the first and second kind respectively [40] and

$$\sin^2\phi = (b^2-\rho^2)/c^2 . \tag{6.2.21}$$

Of primary interest in σ' and σ^2 is the singularity at crack edges $\rho = a$ and $\rho = b$. Knowledge of the terms containing the singularities is necessary to calculate the stress intensity factors. To this end we consider σ' and note that it is possible to find it exactly in the immediate neighbourhood of the crack edge. Accordingly, we first make the following transformations in the independent variable ρ and the function σ' as follows

$$\rho = \sqrt{a^2-c^2y^2} \tag{6.2.22}$$

and

$$S(y) = y\,\sigma'(\sqrt{a^2-c^2y^2})/P_o \ . \tag{6.2.23}$$

If we use (6.2.22) and (6.2.23) in (6.2.18) we obtain

$$\begin{aligned} S(y) &= \frac{4}{\pi^2} - y\,\frac{2}{\pi}\cdot\sin^{-1}(1/\sqrt{1+y^2}) \\ &+ \frac{4}{\pi^2}\,\frac{y}{(1+y^2)^{\frac{1}{2}}}\ \ln[y+\sqrt{1+y^2}]\,[1+(1+y^2)^{\frac{1}{2}}\ln[y/(1+\sqrt{1+y^2})]] \\ &+ \frac{4\,y^2}{\pi^2}\int_o^1 \frac{\sin^{-1}(v)\,dv}{y^2+v^2} \\ &- \frac{4\,y\,\ln\,[y+\sqrt{1+y^2}]}{\pi^2\,(1+y^2)^{\frac{1}{2}}}\int_o^{a/c}\frac{x(1+x^2)^{\frac{1}{2}}}{x^2-y^2}\ S(x)\,dx \\ &+ \frac{4}{\pi^2}\int_o^{a/c}\frac{x\,\ln[x+\sqrt{1+x^2}]}{x^2-y^2}\ S(x)\,dx \quad , \qquad 0 < y \le a/c \ . \end{aligned} \tag{6.2.24}$$

Now when $\lambda \to a^-$ then $y \to 0^+$. Hence if we let y be arbitrarily small the only significant terms on the right hand side of (6.2.24) will be the first and last terms. This suggests that we consider the following equation

$$S_o(y) = \frac{4}{\pi^2} + \frac{4}{\pi^2}\int_o^{a/c}\frac{x\,\ln[x+\sqrt{1+x^2}]}{x^2-y^2}\,S_o(x)\,dx \ . \tag{6.2.25}$$

Let us now set

$$y^2 = \frac{1}{2}\left[\frac{a^2}{c^2}s + \frac{a^2}{c^2}\right] \quad ; \quad g(x) = \ln\left[x + \sqrt{1+x^2}\right] \quad ;$$

$$\psi(s) = G(s)\, S_o\left(\sqrt{(a^2s/c^2 + a^2/c^2)/2}\right) \quad ;$$

$$G(s) = g\left(\sqrt{(a^2s/c^2 + a^2/c^2)/2}\right) \tag{6.2.26}$$

then (6.2.25) becomes

$$\frac{\psi(s)}{G(s)} = \frac{4}{\pi^2} + \frac{2}{\pi^2}\int_{-1}^{1} \frac{\psi(x)}{x-s}\,dx \quad , \quad -1 < s \leq 1. \tag{6.2.27}$$

Equations which have the form of (6.2.27) are known as Carleman type [41]. The solution of (6.2.27) can be found by the use of a result from [41]; namely

$$\alpha \int_{-1}^{1} \frac{A(x)dx}{x-y} = f(y)A(y) - \operatorname{sign}\alpha \tag{6.2.28}$$

where

$$A(x) = \frac{e^{\beta(x)}}{\sqrt{f^2(x) + \alpha^2\pi^2}} \tag{6.2.29}$$

and

$$\beta(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\tan^{-1}[\alpha\pi/f(y)]}{y-x}\,dy \tag{6.2.30}$$

The function f must be continuous in the open interval $(-1, 1)$.

If we let

$$\psi(x) = \frac{4}{\pi^2}A(x) \quad , \quad f(x) = \frac{1}{G(x)} \tag{6.2.31}$$

and

$$\alpha = \frac{2}{\pi^2} \tag{6.2.32}$$

then by the result (6.2.28) ψ satisfies (6.2.27). We now transform back to the original function S_o and we have

$$S_o(y) = \frac{4}{\pi^2}\left[\frac{\pi e^{\beta(2c^2y^2/a^2-1)}}{[\pi^2 + 4g^2(y)]^{\frac{1}{2}}}\right]. \quad (6.2.33)$$

The expression for S_o given by (6.2.33) represents a solution to (6.2.25). By the nature of the two equations (6.2.24) and (6.2.25) we have that

$$\lim_{y\to 0^+} S(y) = \lim_{y\to 0^+} S_o(y) \quad (6.2.34)$$

Hence from (6.2.34), (6.2.33), (6.2.30), (6.2.22) and (6.2.23) we can deduce that

$$\sigma'(\rho) = \frac{4\ P_o c}{\pi^2\sqrt{a^2-\rho^2}}\left[\frac{\Pi e^{B(\rho)}}{[\pi^2 + 4F^2(\rho/c)]^{\frac{1}{2}}}\right] \quad \text{for} \quad (a-\rho) < \varepsilon\ , \quad (6.2.35)$$

$$B(\rho) = \frac{2}{\pi}\int_0^{a/c} y\ \frac{\tan^{-1}[(2/\pi)F(y)]}{\rho^2/c^2 - y^2}\, dy\ , \quad (6.2.36)$$

$$F(y) = \ln\left[\frac{\sqrt{a^2-c^2y^2} + \sqrt{b^2-c^2y^2}}{c}\right]\ , \quad (6.2.37)$$

where $\varepsilon << 1$.

We can now calculate the stress intensity factor (see equation (4.3.8) for definition) on the inside, N_i, to be

$$N_i = \frac{4\ P_o c}{\pi^2(2a)^{\frac{1}{2}}}\ e^{B(a)} \quad (6.2.38)$$

where

$$B(a) = \frac{2}{\pi}\tan^{-1}[2/\pi F(0)]\ \ln[a/c] - \frac{2}{\pi^2}\ \frac{(a/b)[\log(a/c) - 2]}{[1 + 4/\pi^2F^2(0)]}$$

$$- \frac{2}{\pi^2}\int_0^{a/c} y\ \frac{[\log(a^2/c^2-y^2) - 2]\sqrt{a^2/c^2-y^2}}{[1 + 4/\pi^2F^2(y)](b^2/c^2-y^2)^{3/2}}\, dy$$

$$- \frac{16}{\pi^4}\int_0^{a/c} y\ \frac{F(y)[\log(a^2/c^2-y^2) - 2]\, dy}{[1 + 4/\pi^2F^2(y)]^2(b^2/c^2-y^2)}\ . \quad (6.2.39)$$

and F is given by (6.2.37). To calculate the stress intensity factor on the outer edge, N_o, we make use of the relation (6.2.7).

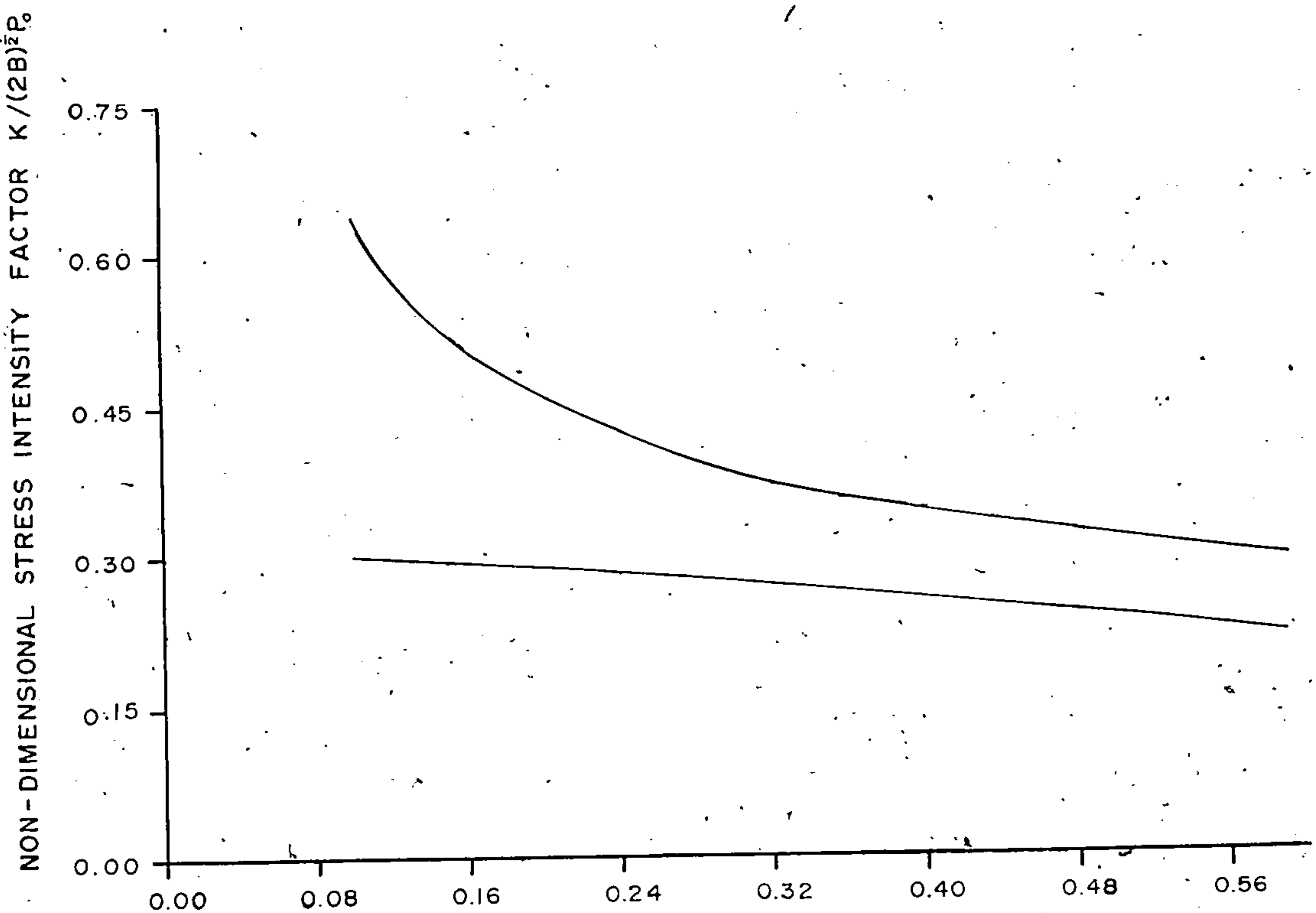

NON-DIMENSIONAL STRESS INTENSITY FACTOR $K/(2B)^{\frac{1}{2}}P_o$
0.75
0.60
0.45
0.30
0.15
0.00
0.00
0.08
0.16
0.24
0.32
0.40
0.48
0.56
RATIO OF OUTER AND INNER RADII A/B

FIGURE 6.1

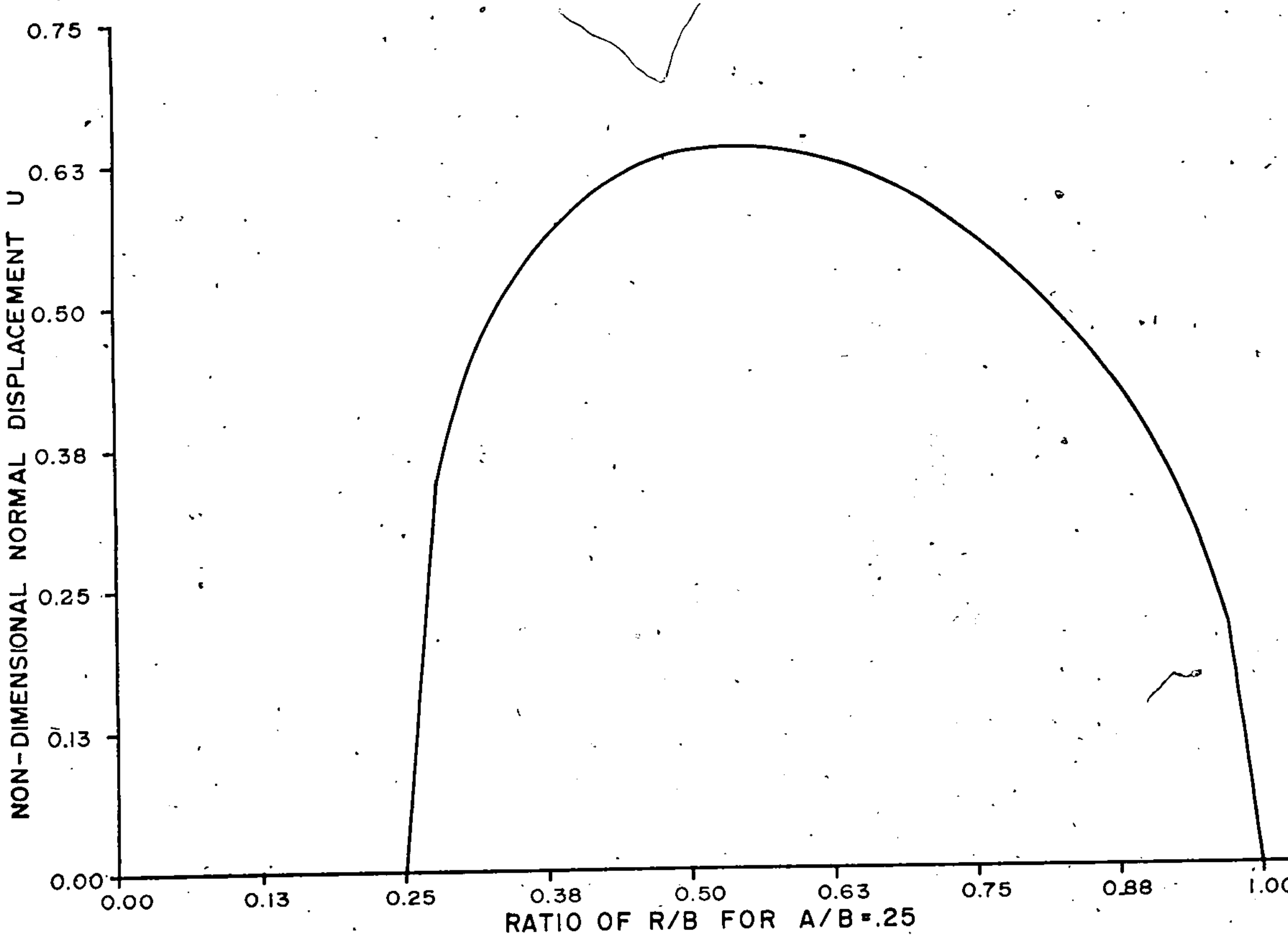

0.75
0.63
0.50
0.38
0.25
0.13
0.00
NON-DIMENSIONAL NORMAL DISPLACEMENT U
0.00
0.13
0.25
0.38
0.50
0.63
0.75
0.88
1.00
RATIO OF R/B FOR A/B=.25

FIGURE 6.2

By substituting (6.2.18) into (6.2.7) we find that

$$N_o = \frac{4\ P_o}{\pi^2(2b)^{\frac{1}{2}}} \left[b \cos^{-1}(a/b) + c \log[(b + a)/c] \cdot \left[1 - \frac{2}{\pi}\right] \right] . \qquad (6.2.40)$$

Figure 6.1 gives a graph of N_i and N_o.

Using the free term of (6.2.18) as an approximation for σ' and taking account of (6.2.35) we can calculate approximate values for $u_z(\rho,0)$. Figure 6.2 gives normalized values of $u_z(\rho,0)$ for various values of the ratios ρ/b and a/b.

3. A Toroidal Crack in an Infinite Elastic Solid Under Torsion.

We consider the problem of a plane toroidal crack in an infinite elastic medium which is under torsion. We follow the same procedure as in Section 6.2 and consider the half-space $z \geq 0$. In the present problem we have the boundary conditions

$$\begin{aligned} \sigma_{\theta z}(r,0) &= -S(r) \quad , \quad a \leq r \leq b \\ u_\theta(r,0) &= 0 \quad , \quad 0 \leq r \leq a,\ b \leq r. \end{aligned} \qquad (6.3.1)$$

We shall make use of the general solution (5.1.7) in the particular case of static elasticity. With these restrictions (5.1.7) can be rewritten as

$$u_\theta(r,z) = \int_0^\infty \xi\psi(\xi)e^{-\xi z}J_1(\xi r)d\xi \quad . \qquad (6.3.2)$$

If we combine (6.3.2), (2.4.3) and (6.3.1) the result is the following set of triple integral equations:

$$\begin{aligned} \int_0^\infty \xi\psi(\xi)J_1(\xi r)d\xi &= 0 \quad , \quad 0 \leq r \leq a, \\ \int_0^\infty \xi^2\psi(\xi)J_1(\xi r)d\xi &= -S(r) \quad , \quad a \leq r \leq b \quad , \end{aligned} \qquad (6.3.3)$$

$$\int_0^\infty \xi\psi(\xi)J_1(\xi r)d\xi = 0 \quad , \quad r \geq b.$$

The solution of this set of integral equations has been considered in [26]. The solution of the problem again reduces to that of a Fredholm integral equation of the second kind in either τ' or τ^2, where τ' is the unknown stress $\sigma_{\theta z}(\rho,0)$ for $0 \leq \rho < a$ and τ^2 represents $\sigma_{\theta z}$ for $\rho > b$. We can give the solution of (6.3.2) as

$$\xi\psi(\xi) = \int_0^a \lambda\tau'(\lambda)J_1(\xi\lambda)d\lambda - \int_a^b \lambda S(\lambda)J_1(\xi\lambda)d\lambda$$

$$+ \int_b^\infty \lambda\tau^2(\lambda)J_1(\xi\tau)d\lambda \quad , \tag{6.3.4}$$

where we have

$$\tau'(\lambda) = \frac{2}{\pi} \frac{\lambda}{(a^2-\lambda^2)^{\frac{1}{2}}} \int_a^b \frac{(t^2-a^2)^{\frac{1}{2}}}{(t^2-\lambda^2)} S(t)dt$$

$$- \frac{4}{\pi^2} \frac{\lambda}{(a^2-\lambda^2)^{\frac{1}{2}}} \int_a^b S(y) \; M_1(y,\lambda)dy \tag{6.3.5}$$

$$+ \frac{4}{\pi^2} \frac{\lambda}{(a^2-\lambda^2)^{\frac{1}{2}}} \int_0^a \tau'(y) \; M_1(y,\lambda)dy \quad , \quad 0 \leq \lambda < a$$

and

$$\tau^2(\lambda) = \frac{2}{\pi(\lambda^2-b^2)^{\frac{1}{2}}\lambda} \int_a^b \frac{t^2(b^2-t^2)^{\frac{1}{2}}}{(\lambda^2-t^2)} S(t)dt$$

$$- \frac{4}{\pi^2(\lambda^2-b^2)^{\frac{1}{2}}\lambda} \int_a^b S(y) \; M_2(y,\lambda)dy \tag{6.3.6}$$

$$+ \frac{4}{\pi^2(\lambda^2-b^2)^{\frac{1}{2}}\lambda} \int_b^\infty \tau^2(\lambda) \; M_2(y,\lambda)dy \quad , \quad b < \lambda < \infty,$$

$$M_1(y,\lambda) = y^2(b^2-y^2)^{\frac{1}{2}} \int_b^\infty \frac{(t^2-a^2)^{\frac{1}{2}} \, dt}{t(t^2-\lambda^2)(t^2-b^2)^{\frac{1}{2}}(t^2-y^2)} \quad , \tag{6.3.7}$$

$$M_2(y,\lambda) = (y^2-a^2)^{\frac{1}{2}} \int_0^a \frac{t^3(b^2-t^2)^{\frac{1}{2}} \, dt}{(\lambda^2-t^2)(a^2-t^2)^{\frac{1}{2}}(y^2-t^2)} \quad . \tag{6.3.8}$$

Furthermore τ^1 and τ^2 are related by the following equation,

$$\tau^2(\lambda) = \frac{2}{\pi(\lambda^2-b^2)^{1/2}\lambda}\int_a^b \frac{t^2(b^2-t^2)^{1/2}\,S(t)}{\lambda^2-t^2}\,dt$$

$$-\frac{2}{\pi(\lambda^2-b^2)^{1/2}\lambda}\int_o^a \frac{t^2(b^2-t^2)^{1/2}}{\lambda^2-t^2}\,\tau^1(t)dt \quad , \quad b < \lambda < \infty. \qquad (6.3.9)$$

The component of displacement $u_\theta(r,0)$ can be written as

$$u_\theta(r,0) = \frac{2r}{\pi\mu}\int_r^b \frac{1}{x^2(x^2-r^2)^{1/2}}\int_a^x \frac{S(\lambda)\lambda^2 d\lambda}{(x^2-\lambda^2)^{1/2}}\,dx$$

$$-\frac{2r}{\pi\mu}\int_r^b \frac{1}{x^2(x^2-r^2)^{1/2}}\int_o^a \frac{\tau^1(\lambda)\lambda^2 d\lambda}{(x^2-\lambda^2)^{1/2}}dx \quad , \quad a \le r \le b. \qquad (6.3.10)$$

If we again look at limiting cases of this problem we find, in the case

$a \to 0$, that

$$\sigma_{\theta z}(r,0) = \frac{2}{\pi(r^2-b^2)^{1/2}r}\int_o^b \frac{t^2(b^2-t^2)^{1/2}S(t)}{r^2-t^2}\,dt$$

and

$$u_\theta(r,0) = \frac{2r}{\pi\mu}\int_r^b \frac{1}{x^2(x^2-r^2)^{1/2}}\int_o^x \frac{S(\lambda)\lambda^2 d\lambda}{(x^2-\lambda^2)^{1/2}}\,dx\;. \qquad (6.3.11)$$

Both these results agree with [14] page 158 in the simple case $S(r) = S_o$, a constant. On the other hand, by letting $b \to \infty$ we find

$$\sigma_{\theta z}(r,0) = r\int_a^\infty \frac{1}{\sqrt{t^2-r^2}}\,\frac{dg_3}{dt}(t)dt + \frac{rg_3(a)}{\sqrt{a^2-r^2}} \quad , \quad 0 \le r < a \qquad (6.3.12)$$

and

$$u_\theta(r,0) = \frac{1}{\mu r}\int_a^r \frac{x^2}{(r^2-x^2)}\,g_3(x)dx \quad , \quad r > a\;, \qquad (6.3.13)$$

where

$$g_3(x) = \frac{2}{\pi}\int_x^\infty \frac{S(t)}{(t^2-x^2)^{1/2}}\,dt\;. \qquad (6.3.14)$$

These equations were arrived at by the same type of manipulations that gave

(6.2.14). To the best of our knowledge the solution to the external crack under torsion with axisymmetric loading represented by equations (6.3.12) - (6.3.14) has not been given in the literature previously. If the function g_3 is differentiable in the neighbourhood of $\rho = a$ then the first term of (6.3.12) will not have a singularity in it and we can write the stress intensity factor for this problem as

$$N = \lim_{\rho \to a^-} \sqrt{a - \rho} \quad \sigma_{z\theta}(\rho, 0)$$

$$N = \left(\frac{a}{2}\right)^{\frac{1}{2}} g_3(a) \qquad (6.3.15)$$

Let us now return to the original problem and consider the specific example when

$$S(r) = S_o r \quad . \qquad (6.3.16)$$

We record for reference the following integrals:

$$\int_b^\infty \frac{(t^2-a^2)^{\frac{1}{2}} \, dt}{t(t^2-b^2)^{\frac{1}{2}}(t^2-\rho^2)(t^2-y^2)} = -\frac{a}{b\rho^2 y^2} \ln[(b+a)/c]$$

$$+ \frac{1}{\rho^2(y^2-\rho^2)} \frac{\sqrt{a^2-\rho^2}}{\sqrt{b^2-\rho^2}} \ln\left[\frac{\sqrt{b^2-\rho^2} + \sqrt{a^2-\rho^2}}{c}\right]$$

$$+ \frac{1}{2y^2(y^2-\rho^2)} \frac{\sqrt{y^2-a^2}}{\sqrt{b^2-y^2}} \cos^{-1}\left[\frac{(b^2-y^2) - (y^2-a^2)}{c^2}\right], \quad a < y < b, \ \rho < a,$$

$$\int_a^b \frac{y^3(b^2-y^2)^{\frac{1}{2}}}{y^2-\rho^2} dy = \frac{c^3}{3} - \rho^2 c - \rho^2(b^2-\rho^2)^{\frac{1}{2}} \ln\left[\frac{(a^2-\rho^2)^{\frac{1}{2}}}{c + (b^2-\rho^2)^{\frac{1}{2}}}\right], \ \rho < a,$$

$$\int_a^b \frac{y^3(b^2-y^2)^{\frac{1}{2}}}{\rho^2-y^2} dy = \frac{-c^3}{3} + \rho^2 c - \rho^2(\rho^2-b^2)^{\frac{1}{2}} \sin^{-1}(c/\sqrt{\rho^2-a^2}) \ , \ \rho > b \ ,$$

$$\int_0^a \frac{y^3(b^2-y^2)^{1/2}}{(\rho^2-y^2)(a^2-y^2)^{1/2}}\,dy = \frac{-ab}{2} + (\rho^2-c^2/2)\ln[(b+a)/c]$$

$$+ \rho^2\frac{\sqrt{\rho^2-b^2}}{\sqrt{\rho^2-a^2}} \ln\left[\frac{b\sqrt{\rho^2-a^2} + a\sqrt{\rho^2-b^2}}{\rho c}\right] , \quad \rho > b ,$$

$$\int_0^a \frac{y(b^2-y^2)^{1/2}}{(\rho^2-y^2)(a^2-y^2)^{1/2}}\,dy = \ln\,[(b+a)/c]$$

$$+ \frac{\sqrt{\rho^2-b^2}}{\sqrt{\rho^2-a^2}} \ln\left[\frac{b\sqrt{\rho^2-a^2} + a\sqrt{\rho^2-b^2}}{\rho c}\right] , \quad \rho > b,$$

$$\int_0^a \frac{y^3(b^2-y^2)^{1/2}}{\rho^2-t^2} \sin^{-1}\left[\frac{c}{\sqrt{b^2-y^2}}\right] dy = (\rho^2 b - \tfrac{b^3}{3})\sin^{-1}[c/b]$$

$$- [\rho^2 c - c^3/3]\,\pi/2 - cab/6 + c[\rho^2 - c^2/6]\ln[(b+a)/c]$$

$$- \rho^2(\rho^2 - b^2)\int_c^b \frac{\sin^{-1}(c/y)dy}{y^2+\rho^2-b^2} , \quad \rho > b . \qquad (6.3.17)$$

Collecting together equations (6.3.5), (6.3.7), (6.3.16), (6.3.17) and (6.3.9) we find, in the case $S(r) = S_o r$, that

$$\tau'(\rho) = \frac{4\ S_o c\rho}{\pi^2(a^2-\rho^2)^{1/2}}\left[1 + \frac{ac^2}{3b\rho^2}\ln\left[\frac{b+a}{c}\right]\right]$$

$$- \frac{2\ S_o\rho}{\pi}\sin^{-1}\left[\frac{c}{(b^2-\rho^2)^{1/2}}\right] + \frac{4\ S_o\rho c(a^2-\rho^2)^{1/2}}{\pi^2}\int_0^1 \frac{\sin^{-1}(v)dv}{c^2v^2 + (a^2-\rho^2)}$$

$$+ \frac{4\ S_o\rho}{\pi^2(b^2-\rho^2)^{1/2}}\ln\left[\frac{(b^2-\rho^2)^{1/2} + (a^2-\rho^2)^{1/2}}{c}\right]\left[c - \frac{c^3}{3\rho^2} + (b^2-\rho^2)^{1/2}\ln\left[\frac{(a^2-\rho^2)^{1/2}}{c + (b^2-\rho^2)^{1/2}}\right]\right]$$

$$+ \frac{4\ \rho}{\pi^2(a^2-\rho^2)^{1/2}}\int_0^a M'(y,\rho)\ \tau'(y)dy , \quad 0 \le \rho < a , \qquad (6.3.18)$$

and

$$\tau^2(\rho) = \frac{4\ S_o}{\pi^2(\rho^2-b^2)^{1/2}\rho}\left\{\left[\frac{2b^3}{3}\sin^{-1}(c/b)\right.\right.$$

$$(1/\pi - 1/6)abc + [\rho^2 c(1 - 2/\pi) + c^3(1/\pi - 1/6) +$$

$$- \frac{2ac^3}{3\pi b} \cdot \ln[(b+a)/c]]\ln[(b+a)/c] \quad - \frac{\pi}{2}\rho^2\sqrt{\rho^2-b^2}\,\sin^{-1}\left[\frac{c}{\sqrt{\rho^2-a^2}}\right]$$

$$- \frac{2c}{\pi}\frac{\sqrt{\rho^2-b^2}}{\sqrt{\rho^2-a^2}}\ \ln\left[\frac{b\sqrt{\rho^2-a^2}+a\sqrt{\rho^2-a^2}}{\rho c}\right]\left[\lambda^2+\frac{c^2a}{3b}\ln\left[\frac{b+a}{c}\right]\right]$$

$$- \rho^2(\rho^2-b^2)^{\frac{1}{2}}\int_c^b \frac{\sin^{-1}(c/y)dy}{y^2+\rho^2-b^2} + b(\rho^2-b^2)\sin^{-1}(c/b)\Bigg\}$$

$$- \frac{2}{\pi(\rho^2-b^2)^{\frac{1}{2}}\rho}\int_0^a \frac{t^2(b^2-t^2)^{\frac{1}{2}}}{\rho^2-t^2}\ \tau'_R(t)dt \quad , \quad b<\rho<\infty\ , \qquad (6.3.19)$$

where

$$M'(y,\rho) = \frac{1}{\rho^2}\ \frac{\sqrt{a^2-\rho^2}}{\sqrt{b^2-\rho^2}}\ \ln\left[\frac{\sqrt{b^2-\rho^2}+\sqrt{a^2-\rho^2}}{c}\right]\frac{y^2\sqrt{b^2-y^2}}{y^2-\rho^2}$$

$$+ \frac{\sqrt{a^2-y^2}}{(\rho^2-y^2)}\ \ln\left[\frac{\sqrt{b^2-y^2}+\sqrt{a^2-y^2}}{c}\right] - \frac{a}{b\rho^2}\ln\left[\frac{b+a}{c}\right]\cdot\sqrt{b^2-y^2} \qquad (6.3.20)$$

and τ'_R represents τ' as given by (6.3.18) minus the first two terms. The displacement $u_\theta(\rho,0)$ becomes in this case

$$u_\theta(\rho,0) = \frac{2S_o}{\pi\mu}\left\{\frac{2}{3}(\rho^2-a^2)F(\phi,a/\rho) - [2\rho^2-a^2]\,E(\phi,a/\rho)\right.$$

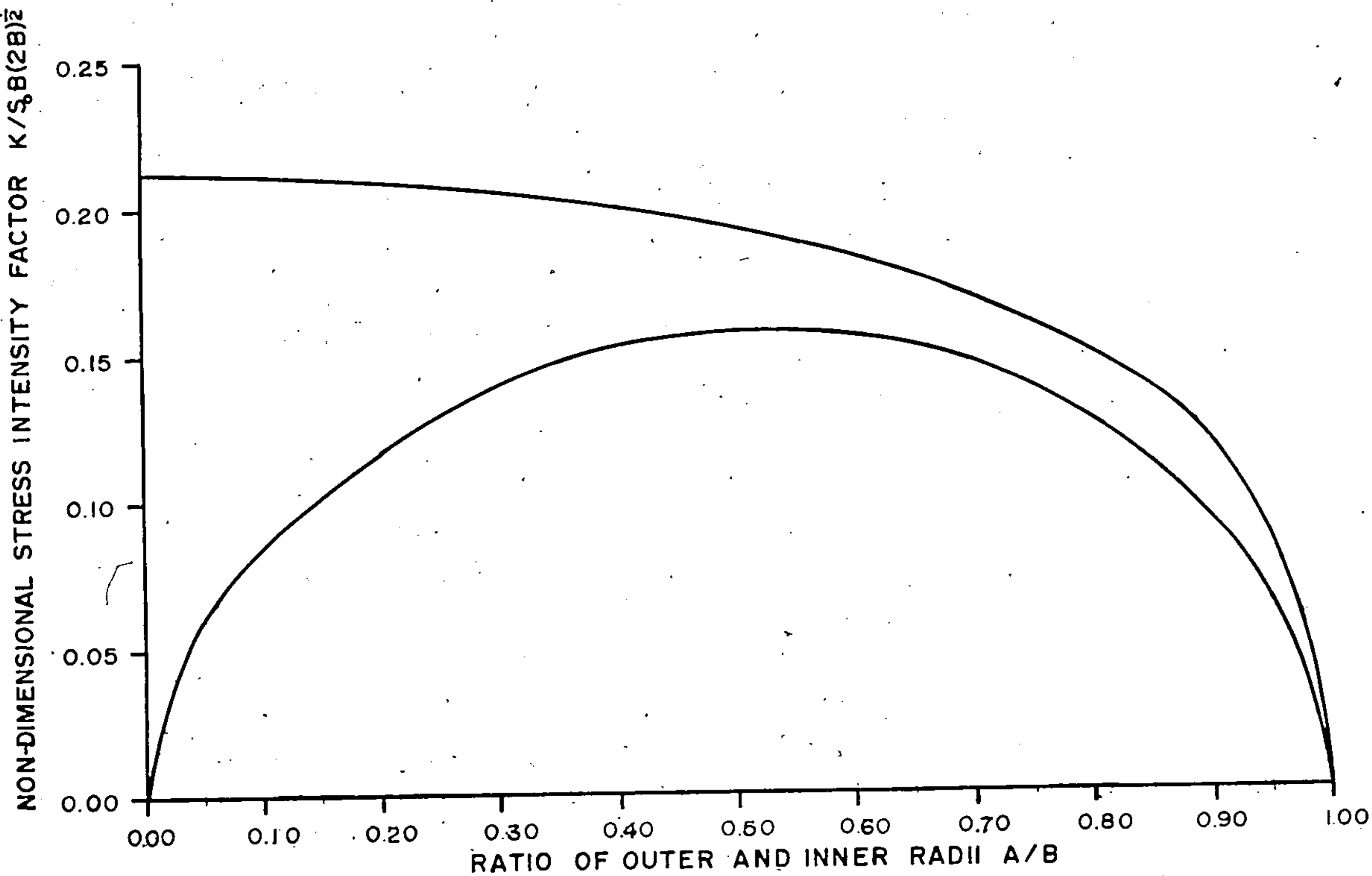
$$\left. + \frac{b\rho}{3c}\ \sqrt{b^2-\rho^2}\left[2-\frac{a^4}{\rho^2b^2}\right]\right\} \qquad (6.3.21)$$

$$- \frac{2\rho}{\pi\mu}\int_\rho^b \frac{1}{r^2(r^2-\rho^2)^{\frac{1}{2}}}\int_0^a \frac{\tau'(t)t^2dt}{(r^2-t^2)^{\frac{1}{2}}}\,dr \quad ,$$

where ϕ is defined by (6.2.21). The stress intensity factors for this problem can be approximated from (6.3.18) and (6.3.19) if we take the singular part of the free term in each equation. We find by this method that

$$N_o = \frac{4S_o}{\pi^2(2b)^{\frac{1}{2}}b}\left[\frac{2}{3}b^3\sin^{-1}(c/b) + (1/\pi - 1/6)abc\right. \qquad (6.3.22)$$

$$\left. + [b^2c(1-2/\pi) + c^3(1/\pi - 1/6) - 2ac^3/3\pi b]\ln[(a+b)/c]\right]$$



NON-DIMENSIONAL STRESS INTENSITY FACTOR K/S₀B(2B)^-1/2
0.25
0.20
0.15
0.10
0.05
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
RATIO OF OUTER AND INNER RADII A/B

FIGURE 6.3

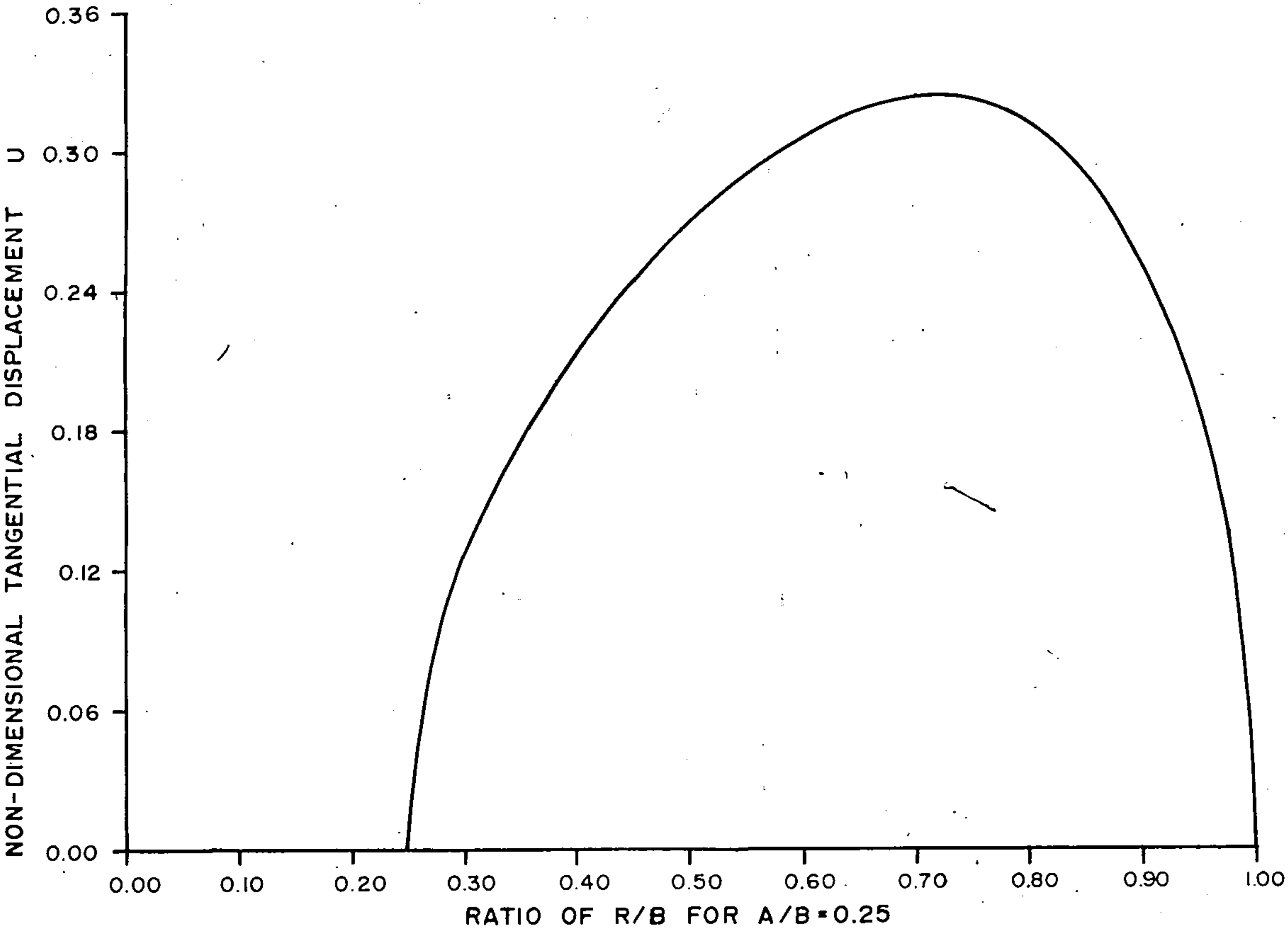

NON-DIMENSIONAL TANGENTIAL DISPLACEMENT U
0.36
0.30
0.24
0.18
0.12
0.06
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
RATIO OF R/B FOR A/B = 0.25

FIGURE 6.4

and

$$N_I = \frac{4\ S_o ca}{\pi^2 (2a)^{\frac{1}{2}}} \left[1 + \frac{c^2}{3ba} \ln\ [(b+a)/c] \right] . \qquad (6.3.23)$$

Figure 6.3 gives a graph of N_i and N_o.

Using the free term of (6.3.21) as an approximation for τ' we can calculate approximate values for $u_\theta(r,0)$. Figure 6.4 gives normalized values for $u_\theta(r,0)$ for various values of the ratios r/b and a/b.

4. Two Toroidal Crack Problems in Viscoelasticity.

In this section we shall give the solutions to the two problems outlined in sections (6.2) and (6.3) when the solid is viscoelastic. For the viscoelastic case we will assume that both a and b, the inner and outer radii, are functions of time. Further we assume that $a(t)$ is monotically decreasing with time while $b(t)$ is monotically increasing. If we denote by $\Omega(t)$ the crack surface area then $\Omega(t)$ must be a monotonic increasing function of time. The normal method used to solve boundary value problems in linear viscoelasticity is the classical correspondence principle (see [2] for reference). This method in general fails when the boundary regions are functions of time. Some work has been done either to modify the principle [3] and [4] or give conditions under which it is still applicable even though the boundary region is a function of time [5]. We shall make use of the latter reference since our problems meet the conditions set forth therein. The conditions, as they pertain to our problems, are that elastic constants are absent from the expression for $\sigma^e_{zz}(\rho,0,t)$ (or $\sigma^e_{\theta z}(\rho,0,t)$) for $\rho \in B - \Omega(t)$ and appear as a separate factor in the expression for $u^e_z(\rho,0,t)$ (or $u^e_\theta(\rho,0,t)$), for $\rho \in \Omega(t)$. The superscript e denotes the elastic solution to the problem at hand. Let us first consider a toroidal crack in tension in a viscoelastic solid. Since the elastic solutions given by (6.2.3), (6.2.4) and (6.2.10) for a toroidal crack in tension meet the above conditions we can immediately write the viscoelastic solution to the same problem as

$$u_z(\rho,0,t) = \frac{2}{\pi} K(0) v_1(\rho,0,t) + \frac{2}{\pi}\int_0^t K'(\theta) v_1(\rho,0,t-\theta)d\theta , \quad a(t) \le \rho \le b(t), \tag{6.4.1}$$

and $\sigma_{zz}(\rho,0,t)$ is given by σ' and σ^2 as defined by (6.2.3) and (6.2.4). The function v_1 and K are defined as

$$v_1(\rho,0,t) = u_z^e(\rho,0,t) \quad \frac{\mu}{1-\nu} \quad \frac{\pi}{2} , \tag{6.4.2}$$

$$K(t) = 2[(2G_1 + G_2) * d(G_1 + 2G_2)^{-1} * dG_1^{-1}](t).$$

The fact the stress field is the same for both problems implies that Figure 6.1 for the stress intensity factors is applicable in this case as well. The displacement however must be obtained from (6.4.1) in terms of the elastic solution. If we consider the particular case of a Maxwell material then

$$G_1(t) = G_o e^{-t/\tau_o} , \quad G_1^{-1}(t) = (1/G_o)(1 + t/\tau_o)$$

and

$$G_1(t) = \frac{1-2\nu}{1+\nu} \; G_2(t)$$

where ν , Poisson's ratio, is a constant. In this case (6.4.1) becomes

$$u_z(\rho,0,t) = u_z^e(\rho,0,t) + \frac{1}{\tau_o}\int_0^t u_z^e(\rho,0,\theta)d\theta . \tag{6.4.3}$$

Table 6.1 Calculated values of the normalized viscoelastic displacement U_z for V_o/V_i = .5 and where R = a(t)/b(t). For comparison, the last column contains values for the U_z that results in the case of a penny shaped crack.

$\rho/b(t)$ \ $V_i t$	0	.25	.5	.75	1.	1.
R	0.0	0.0	0.0	0.0	1	1.79
$\frac{(1+4R)}{5}$	0.0	.286	.507	.754	1.16	1.75
$\frac{(2+3R)}{5}$	0.0	.33	.612	.905	1.26	1.61
$\frac{(3+2R)}{5}$	0.0	.321	.606	.894	1.20	1.35
$\frac{(4+R)}{5}$	0.0	.243	.442	.629	.81	.83
1	0.0	0.0	0.0	0.0	0.0	0.0

If we specify $\tau_o = 1$ and $P(r,t) = P_o$

$$a(t) = 1 - V_i t, \qquad b(t) = 1 + V_o t \tag{6.4.4}$$

then (6.4.3) can be integrated. The results are given in Table 6.1 for various values of ρ/b and $V_i t$ when $V_o/V_i = .5$.

If we now consider the problem of a viscoelastic solid containing a toroidal crack under torsion, we find, by the same method as before

$$u_\theta(\rho,0,t) = \frac{4}{\pi} G_1^{-1}(0) v_2(\rho,0,t) + \frac{4}{\pi} \int_0^t \frac{dG_1^{-1}}{d\theta}(\theta) v_2(\rho,0,t-\theta)\, d\theta, \tag{6.4.5}$$

and $\sigma_{\theta z}(\rho,0,t)$ is given by τ' and τ^2 as defined by (6.3.5) and (6.3.6). The function v_2 is defined as

$$v_2(\rho,0,t) = u_\theta^e(\rho,0,t) \cdot \frac{\mu\pi}{2} . \tag{6.4.6}$$

The stress intensity factors for this problem are given by Figure 6.2. If the material is Maxwell's then following the same procedure as before we find

$$u_\theta(\rho,0,t) = u_\theta^e(\rho,0,t) + \frac{1}{\tau_o} \int_0^t u_\theta^e(\rho,0,\theta)\, d\theta .$$

We specify this time that $\tau_o = 1$ and $S(r,t) = rS_o$ and (6.4.4). These specifications result in Table 6.2 which gives u_θ for various values of ρ/b and $V_i t$ when $V_o/V_i = 2$.

Table 6.2 Calculated values of the normalized viscoelastic displacement U_θ for $V_o/V_i = 2$ and where $R = a(t)/b(t)$. For comparison the last column contained values for the normalized viscoelastic displacement U_θ that results in the case of a penny shaped crack.

$\rho/b(t)$ \ $V_i t$	0	.25	.5	.75	1	1
R	0	0	0	0	0	0
$\frac{1+4R}{5}$	0	.231	.259	.234	.189	.215
$\frac{2+3R}{5}$	0	.304	.382	.406	.416	.378
$\frac{3+2R}{5}$	0	.306	.395	.435	.464	.441
$\frac{4+R}{5}$	0	.249	.325	.362	.387	.382
1	0	0	0	0	0	0

5. Discussion

In determining the results contained in Figures 1-4 we have used the Gaussian quadrature formulae for any integrations needed except in a few special cases.

(a) Toroidal crack in tension

From Figure 6.1 we can see that the inner stress intensity factor is larger than the outer. We may conclude if the crack grows that it will grow on the inside first. Hence toroidal cracks in tension will tend to become penny shaped cracks. Further, as the ratio of inner and outer radii approaches zero, the inner stress intensity factor becomes unbounded. This leads to the possibility of large and sudden failure for solids in tension which have toroidal cracks with a small ratio.

We note that from Figure 6.2 that the normal displacement is not symmetric. The displacement rises very sharply on the inside and falls more gently on the outside. This is due to the larger stress intensity factor on the inside. In Table 6.2 we have the corresponding viscoelastic displacement. The same trend continues that was observed in the elastic case. An interesting point is to compare the displacements of a viscoelastic toroidal crack that has become a penny shaped one and a penny shaped crack which has always been one. The last two columns in Table 6.2 give this comparison. The reason for the higher values on the inside is because the penny shaped crack has always had a history at those points which is not the case for the toroidal crack.

(b) Toroidal crack under torsion

We observe from Figure 6.3 that the behaviour of the inner and outer stress intensity factors is quite different from those in the tension case. In this case the inner stress intensity factor is always smaller than the outer. The tangential displacement u_θ as well has a different form from that of u_z in the tension case. It rises steeply on the outside and slopes gently on the inside. As before we have made, in the viscoelastic case, a comparison between the toroidal crack which becomes a penny shaped one and the crack which has always been penny shaped. The last two columns of Table 6.2 show this difference.

APPENDIX A

1. A contour integration resulting from Chapter 3.

Here we shall determine the Laplace inverse of $\overline{h}(\eta)$ which is given as

$$\overline{h}(\eta) = \frac{\eta(1 + \eta^2/K^2)^{\frac{1}{2}}}{g(\eta)} \tag{A.1}$$

where $g(\eta) = [(2+\eta^2)^2 - 4\,(1+\eta^2)^{\frac{1}{2}}(1+\eta^2/K^2)^{\frac{1}{2}}]$.

The Laplace inverse of (A.1) is given as

$$h(t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{\eta(1+\eta^2/K^2)^{\frac{1}{2}}\, e^{\eta t} d\eta}{[(2+\eta^2)^2-4(1+\eta^2)^{\frac{1}{2}}(1+\eta^2/K^2)^{\frac{1}{2}}]} \tag{A.2}$$

To evaluate $h(t)$ we consider the contour C in connection with Figure A.1. We have that

$$\int_C I(\eta)d\eta = \lim_{\substack{R\to\infty \\ r\to\infty}} \left\{ \int_{\delta-iR}^{\delta+iR} + \sum_{i=1}^{6}\int_{Gr_i} + \sum_{i=1}^{8}\int_{L_i} + \sum_{i=1}^{3}\int_{CR_i} \right\} . I(\eta)d\eta \tag{A.3}$$

where I represents the integrand of (A.2).

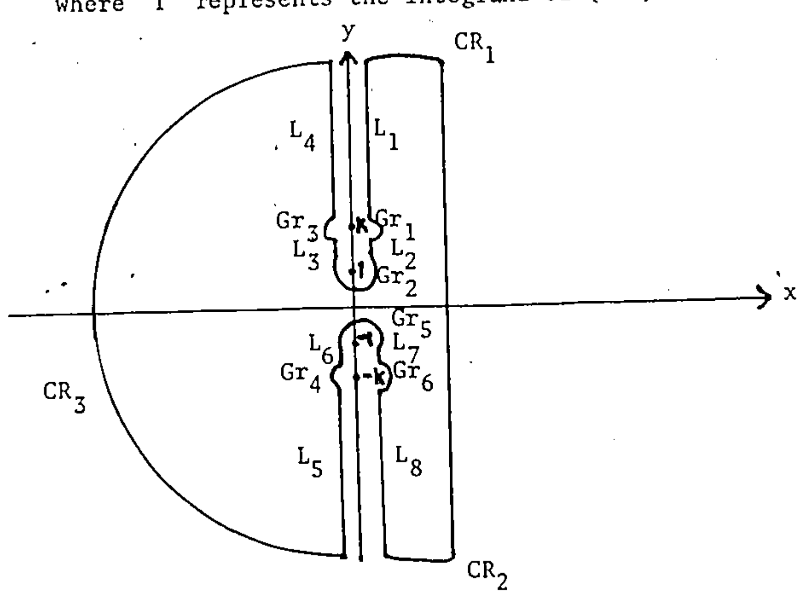


Figure A.1.

To simplify the contour C we first note that if r is sufficiently small, R sufficiently large and segments L_i, $i = 1, .., 8$ sufficiently close to the y-axis, then we find

$$\left|\int_{L_1} I d\eta - \int_R^{K+r} I' i dy\right| < \frac{\varepsilon}{15}, \quad \left|\int_{L_2} I d\eta - \int_{K-r}^{1+r} I' i dy\right| < \frac{\varepsilon}{15},$$
$$\left|\int_{L_3} I d\eta - \int_{1+r}^{K-r} -I' i dy\right| < \frac{\varepsilon}{15}, \quad \left|\int_{L_4} I d\eta - \int_{K+r}^{R} -I' i dy\right| < \frac{\varepsilon}{15},$$
$$\left|\int_{L_5} I d\eta - \int_{-R}^{-K-r} I' i dy\right| < \frac{\varepsilon}{15}, \quad \left|\int_{L_6} I d\eta - \int_{-K+r}^{-1-r} I' i dy\right| < \frac{\varepsilon}{15},$$
$$\left|\int_{L_7} I d\eta - \int_{-1+r}^{-K+r} -I' i dy\right| < \frac{\varepsilon}{15}, \quad \left|\int_{L_8} I d\eta - \int_{-K-r}^{-R} -I' i dy\right| < \frac{\varepsilon}{15},$$
$$\left|\int_{Gri} I d\eta\right| < \frac{\varepsilon}{15} \quad \text{for } i = 1, 6 \quad . \qquad \text{(A.4)}$$

In the above we denote the principal part of I by I'. On the contours CR_i we let $\eta = Re^{i\theta}$. For sufficiently large R we have

$$\left| \frac{\eta(1+\eta^2/K^2)^{\frac{1}{2}}}{[(2+\eta^2)^2 - 4(1+\eta^2)^{\frac{1}{2}}(1+\eta^2/K^2)]^{\frac{1}{2}}} \right| < \frac{K_o}{R^2}, \; R > R_o \quad ,$$

where K_o and R_o are finite constants. We now write

$$\left| \int_{CR_1} I(\eta) d\eta \right| \leq \int_{CR_1} |I(\eta)| \quad |d\eta| \quad ,$$
$$\leq \frac{K_o}{R^2} e^{\delta t} R \int_{\gamma}^{\pi/2} d\theta \quad ,$$
$$\leq \frac{K_o}{R} e^{\delta t} \sin^{-1}[\delta/R] \quad , \qquad \text{(A.5)}$$

where $\gamma = \cos^{-1}(\delta/R)$ and we have approximated $e^{R\cos\theta}$, $\alpha \leq \theta \leq \pi/2$, by $e^{\delta t}$. The integral CR_3 can be bounded in exactly the same manner. For the contour integral CR_2 we have

$$\left| \int_{CR_2} I(\eta) d\eta \right| \leq \frac{K_o}{R} \left| \int_{\pi/2}^{-\pi/2} e^{R\cos\theta} d\theta \right| \quad .$$

But in this region of integration $\cos\theta < 0$ and $e^{R\cos\theta} \leq 1$. Thus we obtain the bound

$$\left| \int_{CR_2} I(\eta)d\eta \right| \leq \frac{K_o \pi}{R} . \tag{A.6}$$

Using the results of (A.5) and (A.6) we conclude that if R is sufficiently large

$$\left| \sum_{i=1}^{3} \int_{CR_i} I(\eta)d\eta \right| < \frac{\varepsilon}{15} . \tag{A.7}$$

Before going further let us collect the results found so far. From (A.4), (A.7) and (A.3) we find for sufficiently large R and small r that

$$\left| \int_C I(\eta)d\eta + \int_{1+r}^{K-r} 2iI'(iy)dy + \int_{K+r}^{R} 2iI'(iy)dy + \int_{-(K+r)}^{-R} 2iI'(iy)dy \right.$$
$$\left. + \int_{-(1+r)}^{-(K-r)} 2iI'(iy)dy - \int_{\delta-iR}^{\delta+iR} I(\eta)d\eta \right| < \varepsilon . \tag{A.8}$$

We take the limit as $R \to \infty$ and $r \to 0$ and obtain

$$\int_{\delta-i\infty}^{\delta+i\infty} I(\eta)d\eta = \int_C I(\eta)d\eta + 2i \int_1^K [I'(iy) - I'(-iy)]dy$$
$$+ 2i \int_K^{\infty} [I'(iy) - I'(-iy)]dy . \tag{A.9}$$

The first integral on right hand side of (A.9) can be evaluated by the Residue Theorem of complex variables. We observe that $g(\eta)$ has eight roots. Two of these are of the form $\pm i\gamma$, $0 < \gamma < 1$ and there is a double root of zero. The nature of the four remaining roots is dependent on the value of ν. It has been determined that for $\nu \leq \nu_o = .263082...$ the four roots are of the form $\pm i\gamma_1$ and $\pm i\gamma_2$ where $\gamma_1 > \gamma_2 > K$. If $\nu_o < \nu$ then the roots have the form $\pm a \pm ib$. It has been determined that only the first four roots described give a real contribution by the Residue Theorem. The residue at zero is given as

$$\lim_{\eta\to 0} \eta I(\eta) = \lim_{\eta\to 0} \frac{\eta^2(1 + \eta^2/K^2)^{\frac{1}{2}} e^{\eta t}}{g(\eta)}$$

$$= \lim_{\eta\to 0} \frac{2}{g''(\eta)} = (1 - \nu) . \qquad \text{(A.10)}$$

The residues at $\pm i\gamma$ are

$$\lim_{\eta\to\pm i\gamma} (\eta \pm i\gamma) I(\eta) = \lim_{\eta\to\pm i\gamma} \frac{(\eta \pm i\gamma)\eta(1 + \eta^2/K^2)^{\frac{1}{2}} e^{\eta t}}{g(\eta)}$$

$$= \pm(1 - \gamma^2/K^2)^{\frac{1}{2}} e^{\pm it\gamma} \gamma/g'(\pm\gamma) , \qquad \text{(A.11)}$$

where

$$g'(\pm\gamma) = - i \frac{dg}{d\eta}(\eta)\Big|_{\eta=\pm i/\gamma} .$$

Since $g'(-\gamma) = -g'(\gamma)$ the residues at $\pm i\gamma$ combine to give

$$\lim_{\eta\to i\gamma} (\eta - i\gamma) I(\eta) + \lim_{\eta\to -i\gamma} (\eta + i\gamma) I(\eta) = \frac{2\gamma(1 - \gamma^2/K^2)^{\frac{1}{2}}\cos(\gamma t)}{g'(\gamma)} . \qquad \text{(A.12)}$$

Now by the Residue Theorem we write from (A.10) and (A.12) that

$$\int_C I(\eta) d\eta = 2\pi i \left[(1 - \nu) + \frac{2\gamma(1 - \gamma^2/K^2)^{\frac{1}{2}}\cos(\gamma t)}{g'(\gamma)} \right] . \qquad \text{(A.13)}$$

To simplify the last two integrals of (A.9) we consider $I(iy)$ as follows

$$I(iy) = \frac{iye^{iyt}(1 - y^2/K^2)^{\frac{1}{2}}(2 - y^2)^2}{(2 - \eta^2)^4 - 16(1 - \eta^2/K^2)(1 - \eta^2)}$$

$$+ \frac{4iye^{iyt}(1 - y^2/K^2)(1 - y^2)^{\frac{1}{2}}}{(2 - \eta^2)^4 - 16(1 - \eta^2/K^2)(1 - \eta^2)} . \qquad \text{(A.14)}$$

The first term of (A.14) has a branch point at $y = K$ and in any contour integration about it we must delete that part of the y-axis where $y \geq K$.

Similarly the second term has a branch point at $y = 1$ and we must delete $y \geq 1$. As a result of these comments only the second term in (A.14) is involved in the integral from 1 to K of (A.9). We find for this integral that

$$\int_1^K [I'(iy)-I'(-iy)]dy = +4i \int_1^K \frac{y(1-y^2/K^2)(1-y^2)^{\frac{1}{2}}[e^{iyt}+e^{-iyt}]}{(2-\eta^2)^4 - 16(1-\eta^2/K^2)(1-\eta^2)} dy$$

$$= -8 \int_1^K \frac{y(1-y^2/K^2)(y^2-1)^{\frac{1}{2}}\cos(yt)dy}{(2-\eta^2)^4 + 16(1-\eta^2/K^2)(\eta^2-1)} \ . \qquad (A.15)$$

By the same method we simplify the integral from K to ∞ in (A.9). Taking this result with (A.15) and (A.13) we rewrite (A.9) as

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} I(\eta)d\eta = (1-\nu) + \sum\nolimits_1 \cos(yt) \qquad (A.16)$$

where for notational convenience we have introduced the operator $\sum_1$ defined as

$$\begin{aligned}\sum\nolimits_1 f(yt) \;=\; & \frac{2\gamma(1-\gamma^2/K^2)^{\frac{1}{2}} f(\gamma t)}{g'(\gamma)} \\ & - \frac{8}{\pi}\int_1^K \frac{y(y^2-1)^{\frac{1}{2}}(1-y^2/K^2)f(yt)}{(2-\eta^2)^4 + 16(1-\eta^2/K^2)(\eta^2-1)}dy \\ & - \frac{2}{\pi}\int_K^\infty \frac{y(y^2/K^2-1)^{\frac{1}{2}}f(yt)dy}{(2-\eta^2)^2 + 4(\eta^2/K^2-1)^{\frac{1}{2}}(\eta^2-1)^{\frac{1}{2}}} \ . \end{aligned} \qquad (A.17)$$

Comparing (A.16) and (A.2) we observe that we have found $h(t)$. We now rewrite (A.1) as

$$\bar{h}(\eta) = \frac{(1-\nu)}{\eta} + \sum\nolimits_1 \int_0^\infty \cos(yt)e^{-\eta t}dt \ . \qquad (A.18)$$

As our last observation on this problem we note that γ must be determined by numerical methods for particular values of ν. It has been found for instance that if $\nu = .35$ then $\gamma = .935$. (A.19)

2. Results Needed in the Integration of Equations (3.3.20).

Equations (3.3.20) are found as a result of equations (3.2.16) and (3.3.3). In order to integrate (3.3.20) we must first find the following Laplace inverses:

$$\overline{E}_1(\xi,s) = 2\xi^2\left[\frac{2\xi^2 + K_2^2 - 2\alpha\beta}{(2\xi^2 + K_2^2)^2 - 4\xi^2\alpha\beta}\right] \overline{P}_o^* , \tag{A.20}$$

$$\overline{E}_2(\xi,s) = \frac{K_2^2(2\xi^2 + K_2^2)}{(2\xi^2 + K_2^2)^2 - 4\xi^2\alpha\beta} \overline{P}_o^* . \tag{A.21}$$

We note (A.20) can be written as

$$\overline{E}_1(\xi,s) = \overline{P}_o^* - \frac{K_2^2(2\xi^2 + K_2^2)\overline{P}_o^*}{(2\xi^2 + K_2^2)^2 - 4\xi^2\alpha\beta} . \tag{A.22}$$

Hence we need only find the Laplace inverse of $\overline{E}_2$. To find this inverse we find the inverse of

$$\overline{h}_1(\eta) = \frac{\eta(2 + \eta^2)}{g(\eta)} , \tag{A.23}$$

where g is given by (A.1). The analysis that gave $\overline{h}(\eta)$ can be used to find $\overline{h}_1(\eta)$. We record the final result,

$$\overline{h}_1(\eta) = \frac{2(1 - \nu)}{\eta} + \sum\nolimits_2 \int_0^\infty \cos(yt)e^{-\eta t}dt , \tag{A.24}$$

where

$$\sum\nolimits_2 f(yt) = \frac{2\gamma(2 - \gamma^2)f(\gamma t)}{g'(\gamma)}$$

$$- \frac{8}{\pi}\int_1^K \frac{(2-y^2)y(y^2-1)^{\frac{1}{2}}(1-y^2/K^2)^{\frac{1}{2}}}{(2-y^2)^4 + 16(y^2-1)(1-y^2/K^2)} f(yt)dt. \tag{A.25}$$

We make use of (A.24) and make a slight change of variables and find

$$E_1(\xi,t) = -(1 - 2\nu)P_o^*(\xi,t) - \sum\nolimits_2 \int_0^\infty \cos(\xi y c_2 x)[u_2 * dP_o^*]dx , \tag{A.26}$$

$$E_2(\xi,t) = 2(1-\nu)P_o^*(\xi,t) + \sum_2 \int_0^\infty \cos(\xi y c_2 x)[u_2 * dP_o^*]dx \ , \qquad (A.27)$$

where u_2 is given by (3.3.12).

3. The Laplace Inverses of Two Functions From (4.2.10).

We consider first the Laplace inverse of $\overline{g}(\eta)$ where

$$\overline{g}(\eta) = \frac{1+\eta^2-(1+\eta^2/K^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{1}{2}}}{(1+\eta^2/K^2)^{\frac{1}{2}}\eta^3} \ . \qquad (A.28)$$

The Laplace inverse can formally be written as

$$g(t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\left[\frac{1+\eta^2-(1+\eta^2/K^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{1}{2}}}{(1+\eta^2/K^2)^{\frac{1}{2}}\eta^3}\right]e^{\eta t}d\eta \ . \qquad (A.29)$$

The analysis that gave equation (A.9) can be repeated in this case and hence our problem has reduced to finding the residues of (A.29) and evaluating the branch cuts. We observe that the integrand of (A.29) has only a simple pole at the origin within the contour C. By the Residue Theorem we may write

$$\begin{aligned}\int_C I(\eta)d\eta &= 2\pi i \lim_{\eta\to 0} \eta \left[\frac{1+\eta^2-(1+\eta^2/K^2)^{\frac{1}{2}}(1+\eta^2)^{\frac{1}{2}}}{(1+\eta^2/K^2)^{\frac{1}{2}}\eta^3}\right]e^{\eta t} \\ &= \frac{2\pi i}{4(1-\nu)} \ ,\end{aligned} \qquad (A.30)$$

where we have used (3.3.8). To simplify the last two integrals of (A.9) we note that

$$I(iy) = \frac{(1-y^2)e^{iyt}}{(y^2/K^2-1)^{\frac{1}{2}}y^3} + \frac{(y^2-1)^{\frac{1}{2}}e^{iyt}}{y^3} \ . \qquad (A.31)$$

The first term has a branch point at $y = K$ and in any contour integration about it we must delete that part of the y-axis where $y \geq K$. Similarly, the second term has a branch point at $y = 1$ and we must delete $y \geq 1$ for this term. As a result of these comments we can write

$$\int_1^K [I'(iy) - I'(-iy)]dy = \int_1^K \frac{(y^2-1)^{\frac{1}{2}}}{y^3} [e^{iyt} + e^{-iyt}]dy$$

$$= 2 \int_1^K \frac{(y^2-1)^{\frac{1}{2}}}{y^3} \cos(yt)dy \ , \qquad \text{(A.32)}$$

and

$$\int_K^\infty [I'(iy) - I'(-iy)]dy = 2 \int_K^\infty \frac{[1-y^2+ (y^2/K^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}]}{(y^2/K^2-1)^{\frac{1}{2}}\, y^3} \cos(yt)dy \ . \qquad \text{(A.33)}$$

If we combine (A.30), (A.32), (A.33), (A.9) and (A.29) we find

$$g(t) = \frac{H(t)}{4(1 - \nu)} + \sum_3 \cos(yt) \qquad \text{(A.34)}$$

where

$$\sum_3 f(y) = \frac{2}{\pi} \int_1^\infty \frac{(y^2-1)^{\frac{1}{2}}}{y^3} f(y)dy - \frac{2}{\pi} \int_K^\infty \frac{(y^2-1)f(y)}{(y^2/K^2-1)^{\frac{1}{2}}\, y^3} dy \ . \qquad \text{(A.35)}$$

We can then rewrite (A.28) as

$$\overline{g}(\eta) = \frac{1}{4(1 - \nu)\eta} + \sum_3 \int_0^\infty \cos(yt)e^{-\eta t}dt \ , \qquad \text{(A.36)}$$

where we have simply to take the Laplace transform of (A.34) with respect to η.

The second result is elementary in that from tables on Laplace transforms we have

$$\frac{1}{(1 + \eta^2/K^2)^{\frac{1}{2}}} = K \int_0^\infty J_o(Kt)e^{-\eta t}dt \ . \qquad \text{(A.37)}$$

4. Results Needed to Integrate Equations (4.2.19).

Following the methods outlined in previous parts of this appendix we rewrite the terms

$$\overline{E}_1(\xi,s) = \frac{2\xi^2 + K_2^2}{(\xi^2 + K_1^2)^{\frac{1}{2}}}$$

and

(A.38)

$$\overline{E}_2(\xi,s) = \frac{(2\xi^2 + K_2^2) - 2(\xi^2 + K_1^2)^{\frac{1}{2}}(\xi^2 + K_2^2)^{\frac{1}{2}}}{(\xi^2 + K_1^2)^{\frac{1}{2}} K_2^2}$$

as

$$\overline{E}_2(\xi,s) = -\frac{1}{K_2^2\xi} + \frac{K_2}{\xi}\int_0^\infty \textstyle\sum_5 \cos(y\xi\tau)e^{-K_2\tau}\,d\tau , \tag{A.39}$$

$$\overline{E}_1(\xi,s) = 2\xi - 2K_2\xi\int_0^\infty \textstyle\sum_4 \cos(y\xi\tau)e^{-K_2\tau}\,d\tau$$

$$+ \frac{K_2^2 c_1}{\xi}\int_0^\infty J_0(c_1\xi\tau)e^{-K_1c_1\tau}\,d\tau .$$

The $\sum$ operators used here and in (4.1.14) are defined to be

$$\textstyle\sum_4 f(y) = \frac{2}{\pi}\int_K^\infty \frac{f(y)\,dy}{(y^2/K^2-1)^{\frac{1}{2}}\,y} ,$$

$$\textstyle\sum_5 f(y) = 2\sum_3 f(y) + \sum_4 f(y) , \tag{A.40}$$

$$\textstyle\sum_6 f(y) = 2\sum_3 f(y) + \frac{1}{1-\nu}\sum_4 f(y) .$$

5. A Result Needed for Equation (5.2.6).

We record the result that

$$\overline{Q}(\eta) = \frac{1}{\eta(1+\eta^2)^{\frac{1}{2}}} \tag{A.41}$$

can be rewritten as

$$\overline{Q}(\eta) = \frac{1}{\eta} - \textstyle\sum_7 \int_0^\infty \cos(ty)e^{-\eta t}\,dt , \tag{A.42}$$

where

$$\textstyle\sum_7 f(y) = \frac{2}{\pi}\int_1^\infty \frac{f(y)\,dy}{y(y^2-1)^{\frac{1}{2}}} . \tag{A.43}$$

This result was obtained in exactly the same manner as those before.

APPENDIX B

Solution of Two Integral Equations

1. The solution of the following integral equation is sought

$$\int_0^\infty \xi^2 \phi_0^*(\xi) J_0(\xi r) d\xi = f(r) \quad , \quad 0 \le r \le a \tag{B.1}$$

for the unknown function $\phi(r)$ given that

$$\phi(r) = 0 \quad , \quad r \ge a . \tag{B.2}$$

If we make use of the results (see [35])

$$J_0(\xi r) = \frac{2}{\pi} \int_0^r \frac{\cos(\xi t)}{(r^2-t^2)^{1/2}} dt \quad , \tag{B.3}$$

$$\int_0^\infty \cos(\xi t) J_0(\xi\lambda) d\xi = \frac{H(\lambda - t)}{(\lambda^2-t^2)^{1/2}} \tag{B.4}$$

and (B.2), then with some manipulation we can arrive at the result

$$-\frac{2}{\pi r} \frac{d}{dr} \int_0^r \frac{t}{(r^2-t^2)^{1/2}} \frac{d}{dt} \int_t^a \frac{\lambda\phi(\lambda)}{(\lambda^2-t^2)^{1/2}} d\lambda dt = f(r). \tag{B.5}$$

But the integrals on the left hand side are just the inverse Abel transforms, equations (1.3.2), so we can write (B.5) as

$$\frac{\pi}{2} \cdot \frac{1}{r} A_1^{-1}[A_2^{-1}[\phi(\lambda) \; ; \; t] \; ; \; r] = f(r) . \tag{B.6}$$

The solution of (B.6) can be written immediately

$$\phi(r) = \frac{2}{\pi} A_2[A_1[xf(x) \; ; \; t] \; ; \; r] \tag{B.7}$$

or written out fully

$$\phi(r) = \frac{2}{\pi} \int_r^a \frac{1}{(t^2-r^2)^{1/2}} \int_0^t \frac{xf(x)\, dx}{(t^2-x^2)^{1/2}} dt . \tag{B.8}$$

2. Here we wish to find the solution of the following integral equation

$$\int_0^\infty \phi_0^*(\xi)J_0(\xi r)d\xi = f(r) \quad , \quad 0 \le r \le a \tag{B.9}$$

where (B.2) still holds. We again use (B.3) and (B.4) to transform (B.9) to

$$\frac{2}{\pi}\int_0^r \frac{1}{(r^2-t^2)^{\frac{1}{2}}} \int_t^a \frac{\lambda\phi(\lambda)d\lambda}{(\lambda^2-t^2)^{\frac{1}{2}}} dt = f(r) \quad , \quad 0 \le r \le a. \tag{B.10}$$

The left hand side of (B.10) can again be written in terms of Abel transforms as follows

$$\frac{2}{\pi} A_1[A_2[\lambda\phi(\lambda) \ ; \ t]; \ r] = f(r) \ . \tag{B.11}$$

The solution of (B.11) can be written as

$$\frac{2}{\pi} r\phi(r) = A_2^{-1}[A_1^{-1}[f(x) \ ; \ t] \ ; \ r] \tag{B.12}$$

or

$$\phi(r) = -\frac{2}{\pi r}\frac{d}{dr}\int_r^a \frac{t\ dt}{(t^2-r^2)^{\frac{1}{2}}}\frac{d}{dt}\int_0^t \frac{xf(x)dx}{(t^2-x^2)^{\frac{1}{2}}} \ . \tag{B.13}$$

APPENDIX C

In this appendix we calculate μ_r from equation (1.4.2) in the particular case

$$\phi(x) = \frac{1}{\sqrt{x^2 - r^2}} , \qquad (C.1)$$

$$x_1 = r, \quad x_r = x_1 + (r - 1)h , \quad x_n = a. \qquad (C.2)$$

From (C.1), (C.2) and (1.4.2) we find

$$\mu_o = I_1$$
$$\mu_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} p^i I_{n+1-i} , \quad n \geq 1 , \qquad (C.3)$$

where $\binom{n}{i}$ are the binomial coefficients and

$$p = r/h , \qquad (C.4)$$

$$I_{n+1} = \frac{1}{h^n} \int_r^a \frac{x^n dx}{\sqrt{x^2-r^2}} . \qquad (C.5)$$

The integrals of (C.5) have been evaluated and the results are

$$I_1 = \ln \left[\frac{m + s}{p}\right] ,$$
$$I_2 = s , \qquad (C.6)$$
$$I_n = \frac{1}{n - 1} (m^{n-2}s + (n - 2)p^2 I_{n-2}) , \quad n \geq 3 ,$$

where

$$s = \sqrt{m^2 - p^2} ,$$
$$m = a/h . \qquad (C.7)$$

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