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Second order effects in an elastic half-space acted upon by non-uniform loads.

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University of Windsor

SECOND ORDER EFFECTS IN AN ELASTIC HALF-SPACE ACTED UPON BY NON-UNIFORM LOADS

by

Jianlin Guo

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ABSTRACT

In this thesis we consider the second order elasticity problems in an isotropic compressible and incompressible elastic half-space which is acted upon by the non-uniformly distributed loads. The two cases of non-uniform normal and shear loading are considered separately. In both cases we obtain the closed form solutions. The method of integral transform is employed to determine the solutions for both linear and nonlinear cases.

The basic equations governing the finite elastic deformation are given in chapter I. For the purposes of reference the equilibrium equations of the classical elasticity and their general solutions are also written down in this chapter.

Chapter II is concerned with the normal load. By noticing the symmetry of the problem in the present case we employ the Papkovitch-Neuber displacement solution to both linear and second order problems. Several linear and a second order illustrations are presented. Some of these linear solutions also occur in the physical circumstances and the others are probably new. Solutions to the incompressible material are also considered. Some numerical results for the compressible and incompressible materials are given in the final section.

Chapter III discusses the shear load. Since the problem now is no longer symmetric the equations to be solved are much more complicated. The displacement vector is chosen to be the Garlerkin's solution plus an additional term. By this choice we are able to solve some non-homogeneous fourth order partial differential equations. Again some linear illustrations are presented and most of these appear to be new. A second order illustration is then discussed in the final section.

In the final chapter some conclusions are given.

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Dedicated to my parents for their care and encouragement

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CHAPTER I

BASIC EQUATIONS

1.1 Introduction.

In finite elasticity theory the mathematical equations governing the deformation of an isotropic compressible elastic material are highly nonlinear. As a result, the exact solutions of the boundary value problems have been possible in only some restricted cases. Most of these are axially symmetric problems for which the differential equations are effectively reduced to ordinary nonlinear equations and which can be integrated when material is incompressible. For two-dimensional problems other simplifications can be made and the complex variable method can often be used. However, when we are concerned with compressible materials or with general type of deformations recourse has to be taken to the approximate methods. The method of successive approximation is one such technique which has received considerable attention. In the method of successive approximation, the displacements,stresses, etc. are expanded in a power series in some suitable parameter, with non-zero radius of convergence. A general expansion scheme has been given by Green and Adkins(1970). Signorini(1949) and Stoppeli(1954,1955) have discussed the results on existence and uniqueness of series solution under suitable conditions. Signorini(1949) has shown that when the applied tractions are specified over the boundary, such that the total load is equipollent to zero but does not possess an axis of equilibrium, then a series expansion of the elastic equations is unique if it exists. Further, when the applied tractions and body forces both contain a multiplying parameter, Stoppeli(1954,1955) has given a proof of existence and uniqueness of the solution of the general elastic equations. Stoppeli(1954,1955)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX has also shown that the displacement can be expanded as an absolutely convergent power series in some parameter, with non-zero radius of convergence, provided the parameter is sufficiently small and provided sufficiently smooth solutions of the classical linear equations of elasticity exist. As Green and Adkins (1970) point out, there may, however, be solutions of the linear elastic equations which do not satisfy these conditions. Examples of such cases include the solutions of flat-ended punch problem and crack problems.

In the method of successive approximation, if we keep the first order terms in the parameter and neglect the terms of higher order than the first we get the classical or linear elasticity equations. If we keep the first and second order terms and neglect the terms of higher order than the second we get the second order elasticity equations. Similarly, if we keep all the terms lower or equal to the n-th order and neglect the terms of higher order than the n-th we obtain the n-th order elasticity equations. By series expansion, Signorini(1930) reduces a problem of the n-th order to n problems in linearized theory, for the same material. For the second order theory a method of the same kind was developed by Rivlin(1953). In this thesis we consider the second order elasticity equations. Second order solutions include terms which are quadratic in the displacement gradients. In general, obtaining a particular integral in explicit form is a formidable task in solving the second order problems. Rivlin(1953) and Green and Spratt(1954) were among the first to formulate second order theories. To find the solution of a second order problem, Rivlin(1953) gives following steps to be sufficient:

1. On the basis of the linearized theory, calculate the displacements arising from the given forces.

2. On the basis of the second order theory, calculate the additional forces needed to maintain the displacements found in step 1.

3. On the basis of the linearized theory, calculate the displacements corresponding to the additional forces just determined. These displacements, reversed, are the second order displacements arising from the given forces.

This approach reduces the second order problem to the solution of linear elasticity problem with body forces and surface displacements or tractions which are quadratic in the first order solutions. A comprehensive account of this method is given by Truesdell and Noll(1965), Green and Adkins(1970) and Spencer(1970). Earlier Spencer(1959) has also considered the approximation based on perturbation of the strain energy function. Goodman and Naghdi(1989) have presented the use of displacement potentials for the solution of compressible or incompressible second order elasticity problem, but this method is somewhat similar to Rivlin's method.

Several methods for solving special problems in the second order elasticity have also been developed. Shield(1967) has discussed inverse deformation results in finite elasticity and Choi and Shield(1981) have applied this approach to some axisymmetric problems. It is found that there are only two elastic constants that govern this special class of compressible material. Carlson and Shield(1965) have developed a method to find the second order solution for a special class of problems without solving the boundary value problem once the first order solution is known. For incompressible material a variety of techniques for the second order theories have been proposed. Chan and Carlson(1970) have developed a method and applied it to solve the second order torsion problem. The key to this method is that it reduces the second order problem to a classical problem of plane strain, without body force. Their results are expressed in terms of two elastic constants since the strain energy function in this situation takes the Mooney's form (Mooney(1940)). Chan and Carlson suggest that their method may be applicable to other problems, and their discussion is amplified by Hill(1973). Hill has shown that the Chan and Carlson's procedure takes on complete generality whenever the deformation of the material is such that the strain energy function is a symmetric function of the strain invariants $\bar{I}_1$ and $\bar{I}_2$. He has discussed this approach for a special class of material by using the results for the inverse deformation developed by Shield(1967). For axisymmetric deformation the general expansion expressions of isotropic incompressible elastic equations have been given by Selvadurai and Spencer(1972) and they have been applied by Selvadurai(1974) to torsion of a thick spherical shell. Carroll and Rooney(1984) have extended the Chan and Carlson's method and have shown that the induced body force field can be expressed as the sum of a conservative field and a residual field. The conservative field can be absorbed in the arbitrary pressure. The residual field is also conservative for several classes of problems, including torsion, plane strain, antiplane shearing and potential displacements. They discuss two such illustrative problems. The contact problems in second order elasticity theory have been considered by Choi and Shield(1981) and Sabin and Kaloni(1983, 1989). As stated earlier, Choi and Shield(1981) used the inverse deformation approach of Shield(1967) in their work, while Sabin and Kaloni(1989) employed the standard second order elasticity model in their calculations.

In the present thesis we follow Rivlin's approach to consider the second order problem in a compressible elastic half-space which is acted upon by non-uniformly distributed loads. In Chapter II we consider normal load case. In this case the problem is axisymmetric. We use Papkovitch-Neuber displacement solution and employ the method of integral transforms, as discussed by Sneddon(1972), in both the linear and second order solutions. Several special linear solutions are given in accordance with the classical results. In the final section we specialize the second order solution for isotropic incompressible material. Chapter III deals with the shear loads. The procedure of finding solutions is same as that in Chapter II.

However, the problem now is no longer axisymmetric and the equations to be solved are much more complicated. It turns out that by selecting the displacement vector to be the Garlerkin's solution plus the curl of an additional harmonic vector in the linear solution and the Garlerkin's solution plus an irrotational term in the second order solution and by employing the integral transformation technique we are able to obtain the linear and second order solutions. Several linear solutions are again documented in this case and two of these solutions appear to be new. The second order solution is then discussed for one particular situation.

1.2 Constitutive Equations.

Suppose that the elastic body $\mathcal{B}$ occupies the region $\mathcal{R}_0$ at time $t = 0$ and moves so that at a subsequent time t it occupies a region $\mathcal{R}$. We make the assumption(which is an essential feature of continuum mechanics) that we can identify individual particles of the body $\mathcal{B}$, that is, we assume that we can identify a point of $\mathcal{R}$ (denoted by P) with position vector $\mathbf{y}$ which is occupied at t by the particle which was at P_0 at the time $t = 0$. Then the motion of $\mathcal{B}$ can be described by specifying the dependence of the positions $\mathbf{y}$ of the particles of $\mathcal{B}$ at time t on their positions $\mathbf{x}$ at time $t = 0$, that is, by equations of the form

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t) \tag{1.1}$$

We assume that the Jacobian

$$J = det(\frac{\partial y_i}{\partial x_j}) > 0 \tag{1.2}$$

The physical significance of this assumption is that the material of the body cannot penetrate itself, and that material occupying a finite non-zero volume cannot be compressed to a point or expanded to infinite volume during the motion. Mathematically (1.2) implies that (1.1) has the unique inverse

$$\mathbf{x} = \mathbf{x}(\mathbf{y}, t) \tag{1.3}$$

The displacement vector $\mathbf{u}$ of a typical particle from its position $\mathbf{x}$ in the reference configuration to its position $\mathbf{y}$ at time t is

$$\mathbf{u} = \mathbf{y} - \mathbf{x} \tag{1.4}$$

In the material description $\mathbf{u}$ is regarded as a function of $\mathbf{x}$ and t, so that

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{y}(\mathbf{x}, t) - \mathbf{x} \tag{1.5}$$

and in the spatial description $\mathbf{u}$ is regarded as a function of $\mathbf{y}$ and t, so that

$$\mathbf{u}(\mathbf{y}, t) = \mathbf{y} - \mathbf{x}(\mathbf{y}, t) \tag{1.6}$$

The representation (1.5) determines the displacement at time t of the particle defined by the material coodinates x_j. The representation (1.6) determines the displacement which has been undergone by the particle which occupies the position $\mathbf{y}$ at time t.

For linear elasticity the constitutive equations can be written as

$$t_{ik} = \frac{\partial W}{\partial e_{ik}} \tag{1.7}$$

where W is the strain energy function which may be approximated by a quadratic function of the infinitesimal strain components e_{ik}. However, for finite elastic deformation the constitutive equations are much more complicated. The general form may be given by

$$t_{ik} = f_{ik}(F_{jp}) \tag{1.8}$$

where f_{ik} are the single-valued functions of $F_{jp} = \partial y_j / \partial x_p$ and satisfy $f_{ik} = f_{ki}$. When the material is hyper-elastic there exists strain energy function W which is an arbitrary function of the deformation gradient components F_{jp} and can be expressed in the form $W = W(\mathbf{C})$ such that(cf. Atkin and Fox(1980))

$$t_{ik} = \frac{\rho}{\rho_0} \frac{\partial y_i}{\partial x_k} \frac{\partial y_k}{\partial x_s} \left(\frac{\partial W}{\partial C_{ps}} + \frac{\partial W}{\partial C_{sp}} \right) \tag{1.9}$$

where $\mathbf{C} = \mathbf{F}^T \bullet \mathbf{F}$, ρ_0 and ρ are densities at time $t = 0$ and time t respectively. Equation (1.9) is the general form of the constitutive equation for a finite elastic solid.

If the material is isotropic, then W is an invariant of $\mathbf{C}$ and therefore it can be expressed as a function of the strain invariants $\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$, so that

$$W = W(\bar{I}_1, \bar{I}_2, \bar{I}_3) \tag{1.10}$$

and we have

$$\frac{\partial W}{\partial C_{ps}} = \frac{\partial W}{\partial \bar{I}_1} \frac{\partial \bar{I}_1}{\partial C_{ps}} + \frac{\partial W}{\partial \bar{I}_2} \frac{\partial \bar{I}_2}{\partial C_{ps}} + \frac{\partial W}{\partial \bar{I}_3} \frac{\partial \bar{I}_3}{\partial C_{ps}} \tag{1.11}$$

$$\frac{\partial \bar{I}_1}{\partial C_{ps}} = \frac{\partial C_{kk}}{\partial C_{ps}} = \delta_{kp} \delta_{ks} = \delta_{ps} \tag{1.12}$$

$$\frac{\partial \bar{I}_2}{\partial C_{ps}} = \frac{1}{2} \frac{\partial C_{ii} C_{kk} - C_{ik} C_{ik}}{\partial C_{ps}} = \bar{I}_1 \delta_{ps} - C_{ps} \tag{1.13}$$

Since $\bar{I}_3$ can be expressed as

$$\bar{I}_3 = \frac{1}{3} \{ tr\mathbf{C}^3 - \bar{I}_1 tr\mathbf{C}^2 + \bar{I}_2 tr\mathbf{C} \}$$

it follows that

$$\frac{\partial \bar{I}_3}{\partial C_{ps}} = \frac{1}{3}\{\bar{I}_2\delta_{ps} + \bar{I}_1^2\delta_{ps} - 3\bar{I}_1 C_{ps} - tr\mathbf{C}^2\delta_{ps} + 3C_{pk}C_{sK}\} = \bar{I}_2\delta_{ps} - \bar{I}_1 C_{ps} + C_{pk}C_{sk} \tag{1.14}$$

By substituting from (1.11) to (1.14) into (1.9) we obtain

$$t_{ik} = 2\frac{\rho}{\rho_0}\frac{\partial y_i}{\partial x_p}\frac{\partial y_k}{\partial x_s}\{(W_1 + \bar{I}_1 W_2 + \bar{I}_2 W_3)\delta_{ps} - (W_2 + \bar{I}_1 W_3)C_{ps} + W_3 C_{pj}C_{sj}\} \tag{1.15}$$

where

$$W_1 = \frac{\partial W}{\partial \bar{I}_1}, \quad W_2 = \frac{\partial W}{\partial \bar{I}_2}, \quad W_3 = \frac{\partial W}{\partial \bar{I}_3}$$

This is a general form of the constitutive equation for an isotropic finite elastic solid. It may be expressed more concisely by using compact notation and $\bar{I}_3 = (\rho_0/\rho)^2$ as

$$\mathbf{T} = 2\bar{I}_3^{-\frac{1}{2}}\mathbf{F} \bullet \{(W_1 + \bar{I}_1 W_2 + \bar{I}_2 W_3)\mathbf{I} - (W_2 + \bar{I}_1 W_3)\mathbf{C} + W_3\mathbf{C}^2\} \bullet \mathbf{F}^T \tag{1.16}$$

This equation can be further simplified by noting that

$$\mathbf{B} = \mathbf{F} \bullet \mathbf{F}^T, \quad \mathbf{B}^2 = \mathbf{F} \bullet \mathbf{C} \bullet \mathbf{F}^T, \quad \mathbf{B}^3 = \mathbf{F} \bullet \mathbf{C}^2 \bullet \mathbf{F}^T$$

and hence equation (1.16) may be rewritten as

$$\mathbf{T} = 2\bar{I}_3^{-\frac{1}{2}}\{(W_1 + \bar{I}_1 W_2 + \bar{I}_2 W_3)\mathbf{B} - (W_2 + \bar{I}_1 W_3)\mathbf{B}^2 + W_3\mathbf{B}^3\} \tag{1.17}$$

By the Cayley-Hamilton Theorem, **B** satisfies that

$$\mathbf{B}^3 - \bar{I}_1\mathbf{B}^2 + \bar{I}_2\mathbf{B} - \bar{I}_3\mathbf{I} = 0 \tag{1.18}$$

and therefore the constitutive equation can finally be written as

$$\mathbf{T} = 2\bar{I}_3^{-\frac{1}{2}}\{(\bar{I}_2 W_2 + \bar{I}_3 W_3)\mathbf{I} + W_1\mathbf{B} - \bar{I}_3 W_2\mathbf{B}^{-1}\} \tag{1.19}$$

This equation can also be written in the component form as

$$t_{ik} = 2\bar{I}_3^{-\frac{1}{2}}\{(\bar{I}_2 W_2 + \bar{I}_3 W_3)\delta_{ik} + W_1 B_{ik} - W_2 G_{ik}\} \tag{1.20}$$

where G_{ik} denotes the co-factor of B_{ik} in $\det B_{ik}$. Further simplification arises if the material is incompressible. In this case $\bar{I}_3 = 1$, and the constitutive equation can be expressed in the form

$$\mathbf{T} = -p\mathbf{I} + 2W_1\mathbf{B} - 2W_2\mathbf{B}^{-1} \tag{1.21}$$

where $-p$ is an arbitrary hydrostatic pressure and is not given by a constitutive equation but can only be determined by using equation of motion and boundary conditions.

There are many forms of strain energy function, which are the special cases of equation (1.10), that have been proposed in the literature for compressible and incompressible elastic solids. We mention some of these here now.

For incompressible materials we note that $\bar{I}_3 = 1$ and hence

$$W = W(\bar{I}_1, \bar{I}_2) \tag{1.22}$$

Since in the reference configuration $\mathbf{C} = \mathbf{I}$, the definitions of $\bar{I}_1$ and $\bar{I}_2$ imply $\bar{I}_1 = \bar{I}_2 = 3$. Accordingly W can be regarded as a function of $\bar{I}_1 - 3$ and $\bar{I}_2 - 3$ which will vanish in the reference configuration. For incompressible materials, we have the earliest equation proposed by Treloar(1948)

$$W = C_1(\bar{I}_1 - 3) \tag{1.23}$$

where C_1 is a constant. It is also called as neo-Hookean equation. The next in the sequence is Mooney-Rivlin form(see Spencer (1980)) given by

$$W = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_2 - 3) \tag{1.24)}$$

where C_1 and C_2 are again constants. Rivlin and Saunders(1952) suggested that an equation of the form

$$W = C_1(\bar{I}_1 - 3) + f(\bar{I}_2 - 3) \tag{1.25}$$

gives more accurate description of vulcanised rubber for some ranges of extension.

More recent development in this direction is due to Ogden(1972) who proposed

$$W = \sum_n (\mu_n/\alpha_n)(b_1^{\alpha_n} + b_2^{\alpha_n} + b_3^{\alpha_n} - 3) \tag{1.26}$$

where b_1, b_2, b_3 are the principal values of **B**, the μ_n are constants, and the α_n are not necessarily integers and may be positive or negative. Equation (1.26) includes neo-Hookean and Mooney-Rivlin forms as special cases.

Finally for compressible rubberlike materials, Blatz and Ko(1962) have suggested a strain energy function of the form

$$\begin{aligned} W = & \frac{1}{2}\mu f\{J_1 - 1 - \frac{1}{\nu} + \frac{1-2\nu}{\nu} J_3^{-2\nu/1-2\nu}\} \\ & + \frac{1}{2}\mu(1-f)\{J_2 - 1 - \frac{1}{\nu} + \frac{1-2\nu}{\nu} J_3^{2\nu/1-2\nu}\} \end{aligned} \tag{1.27}$$

where μ, f, ν are constants, and

$$J_1 = \bar{I}_1, \quad J_2 = \bar{I}_2/\bar{I}_3, \quad J_3 = \bar{I}_3^{\frac{1}{2}} \tag{1.28}$$

We note that when $\nu = 1/2$ and the material is incompressible so that $\bar{I}_3 = 1$, (1.28) reduces to the Mooney-Rivlin form.

We now return to the development of the equation for the second order elasticity theory. For small deformation, such that $\partial u_i/\partial x_k$ are all small compared with unity, $\bar{I}_1 - 3, \bar{I}_2 - 3$ and $\bar{I}_3 - 1$ are, in general, of the first order of smallness in the quantities $\partial u_i/\partial x_k$. We may construct three other mutually independent scalar invariants,

J_1, J_2 and J_3 which are respectively of the first, second and third orders of smallness in quantities $\partial u_i/\partial x_k$. Such scalar invariants may be defined by the relations

$$\begin{aligned} J_1 &= \bar{I}_1 - 3 \\ J_2 &= \bar{I}_2 - 2\bar{I}_1 + 3 \\ J_3 &= \bar{I}_3 - \bar{I}_2 + \bar{I}_1 - 1 \end{aligned} \tag{1.29}$$

or

$$\begin{aligned} \bar{I}_1 &= J_1 + 3 \\ \bar{I}_2 &= J_2 + 2J_1 + 3 \\ \bar{I}_3 &= J_3 + J_2 + J_1 + 1 \end{aligned} \tag{1.30}$$

Since W is a function of $\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$,it can be expressed as a function of J_1, J_2 and J_3. If we consider finite deformations of the elastic body which are sufficiently small so that terms of higher degree than the second in the quantities $\partial u_i/\partial x_k$ can be neglected in the expressions for the stress components, then we can, following Rivlin(1953), express W in the form

$$W = a_0 J_1 + a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3 \tag{1.31}$$

in which $a_0, a_1, \ldots, a_5$ are physical constants for the material considered. It has been shown by Murnaghan(1937) that if material is such that the stress is zero in the undeformed state, $a_0 = 0$, so that

$$W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3 \tag{1.32}$$

If J_1, J_2 and J_3 are regarded as functions of the displacement gradients and we neglect those terms in the displacement gradients occurring in (1.32), which are of higher degree than the third, we obtain

$$W = 2(a_1+2a_2)\left(\frac{\partial u_k}{\partial x_k}\right)^2 - a_1\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right) + 2(a_5-a_1)\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}$$
$$+ 2(a_1+2a_2-a_3-a_5)\frac{\partial u_j}{\partial x_j}\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k} + 4(a_3+2a_4+\frac{1}{3}a_5)\left(\frac{\partial u_k}{\partial x_k}\right)^3$$
$$- 2(a_3+a_5)\frac{\partial u_j}{\partial x_j}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i} + \frac{2}{3}a_5\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i} \tag{1.33}$$

From (1.30) and (1.32) we get

$$\frac{\partial W}{\partial \bar{I}_1} = (a_5-2a_1) + 2(a_2-a_3)J_1 + a_3J_2 + 3a_4J_1^2$$
$$\frac{\partial W}{\partial \bar{I}_2} = (a_1-a_5) + a_3J_1 \tag{1.34}$$
$$\frac{\partial W}{\partial \bar{I}_3} = a_5$$

Rivlin(1953) has shown that neglecting terms of higher degree than the second in $\partial u_i/\partial x_k$ in the expressions for $B_{ik}, G_{ik}, \bar{I}_1, \bar{I}_2, \bar{I}_3, J_1$ and J_2 we obtain

$$B_{ik} = \delta_{ik} + e_{ik} + \alpha_{ik}$$
$$G_{ik} = (1+2\Delta+\alpha)\delta_{ik} - e_{ik} - \alpha_{ik} + E_{ik}$$
$$\bar{I}_1 = 3 + 2\Delta + \alpha$$
$$\bar{I}_2 = 3 + 4\Delta + 2\alpha + E \tag{1.35}$$
$$\bar{I}_3 = 1 + 2\Delta + \alpha + E$$
$$J_1 = 2\Delta + \alpha$$
$$J_2 = E$$

with the notations

$$
\begin{aligned}
e_{ik} &= \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \\
\alpha_{ik} &= \frac{\partial u_i}{\partial x_s}\frac{\partial u_k}{\partial x_s} \\
e &= 2\Delta = e_{ss} \\
\alpha &= \alpha_{ss} \\
E &= E_{ss}
\end{aligned}
\tag{1.36}
$$

where E_{ik} =co-factor of e_{ik} in det e_{ik}.

Substituting (1.34) and (1.35) into (1.20) and neglecting terms of higher degree than the second in $\partial u_i/\partial x_k$, we obtain

$$
\begin{aligned}
t_{ik} =& 2[\{-a_1 e_{ik} + 2(a_1+2a_2)\Delta\delta_{ik}\} + \{(4a_2 - 2a_3 + a_1)\Delta e_{ik} - a_1\alpha_{ik} - (a_1 - a_5)E_{ik}\} \\
&+ \{(a_1+2a_2)\alpha + (a_1+a_3)E + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta^2\}\delta_{ik}]
\end{aligned}
\tag{1.37}
$$

Here e_{ik} and Δ are homogeneous expressions of the first degree in $\partial u_i/\partial x_k$ and $\alpha_{ik}, E_{ik}, \alpha$ and E are homogeneous expressions of the second degree. If we neglect terms of the second degree in $\partial u_i/\partial x_k$ in the expressions for t_{ik}, we obtain the expressions for the stress components of linear elasticity theory

$$t_{ik} = 2[-a_1 e_{ik} + 2(a_1 + 2a_2)\Delta\delta_{ik}] \tag{1.38}$$

It is found that the Lame's constants λ and μ are given by

$$
\begin{aligned}
\lambda &= 4(a_1 + 2a_2) \\
\mu &= -2a_1
\end{aligned}
\tag{1.39}
$$

For incompressible material it has been shown by Mooney(1940) that the strain energy function may be written, to terms of the third order of smallness in the principal extensions, as

$$W = C_1(\bar{I}_1 - 3) + C_2(\bar{I}_2 - 3) \tag{1.40}$$

where C_1 and C_2 are constants. The stress components in this case may be written, to terms of the second order of smallness in the principal extensions, as

$$t_{ik} = 2[B'_{ik}C_1 - G'_{ik}C_2] + p \tag{1.41}$$

where $B'_{ik} = B_{ik} - \delta_{ik}$ and $G'_{ik} = G_{ik} - \delta_{ik}$

1.3 Equilibrium Equations and Boundary Conditions.

If an elastic body undergoes deformation by a system of body forces X_i per unit mass of the material and surface forces $X_{\nu i}$ per unit area of surface measured in the undeformed state of the material, then in the static state the equations for equilibrium are given by

$$\frac{\partial t_{ik}}{\partial y_k} + \rho X_i = 0 \tag{1.42}$$

and the boundary conditions may be written as

$$X_{\nu i}\frac{dS}{dS'} = t_{ik}l'_k \tag{1.43}$$

where dS and dS' are elements of area of the surface of the body measured in the undeformed and deformed states respectively, so that $X_{\nu i}\frac{dS}{dS'}$ is the surface traction per unit area of the surface measured in the deformed state of the body, and l'_k are the direction-cosines of the normal to the deformed surface of the body. From $y_i = x_i + u_i$ we get

$$\frac{\partial y_i}{\partial y_k} = \frac{\partial x_i}{\partial y_k} + \frac{\partial u_i}{\partial y_k}$$

or

$$\delta_{ik} = \frac{\partial x_i}{\partial y_k} + \frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial y_k} \qquad \text{i=1,2,3} \tag{1.44}$$

Solving for $\partial x_j/\partial y_k$, we obtain

$$\frac{\partial x_j}{\partial y_k} = \bar{I}_3^{-\frac{1}{2}}\frac{\partial \bar{I}_3^{\frac{1}{2}}}{\partial H_{kj}} \tag{1.45}$$

where $H_{ik} = \partial u_i/\partial x_k$ (i,k=1,2,3) and then

$$\frac{\partial}{\partial y_k} = \frac{\partial x_j}{\partial y_k}\frac{\partial}{\partial x_j} \tag{1.46}$$

Using (1.46) and $\rho_0 = \rho \bar{I}_3^{\frac{1}{2}}$ in (1.42), we get

$$\frac{\partial \bar{I}_3^{\frac{1}{2}}}{\partial H_{kj}}\frac{\partial t_{ik}}{\partial x_j} + \rho_0 X_i = 0 \tag{1.47}$$

From Spencer(1980) we note that

$$l'_k = \det \mathbf{F}\frac{dS}{dS'}\mathbf{l} \bullet \mathbf{F}^{-1} \tag{1.48}$$

or

$$l'_k = \frac{dS}{dS'} l_s \frac{\partial \bar{I}_3^{\frac{1}{2}}}{\partial H_{ks}} \tag{1.49}$$

where l_s are the direction-cosines of the normal to the undeformed surface. Introducing (1.49) into (1.43) gives

$$X_{\nu i} = \frac{\partial \bar{I}_3^{\frac{1}{2}}}{\partial H_{ks}} l_s t_{ik} \tag{1.50}$$

It has been shown by Rivlin(1953) that equations (1.47) and (1.50) can be written as

$$\left[(1+\Delta)\delta_{sk} - \frac{\partial u_s}{\partial x_k}\right]\frac{\partial t'_{ik}}{\partial x_s} + \frac{\partial t''_{ik}}{\partial x_k} + \rho_0 X_i = 0 \tag{1.51}$$

and

$$X_{\nu i} = \left[(1+\Delta)\delta_{sk} - \frac{\partial u_s}{\partial x_k}\right] l_s t'_{ik} + l_k t''_{ik} \tag{1.52}$$

where

$$t_{ik} = t'_{ik} + t''_{ik} \tag{1.53}$$

and

$$t'_{ik} = 2[-a_1 e_{ik} + 2(a_1 + 2a_2)\Delta\delta_{ik}] \tag{1.54}$$

$$
\begin{aligned}
t''_{ik} =& 2[\{(4a_2 - 2a_3 + a_1)\Delta e_{ik} - a_1\alpha_{ik} - (a_1 - a_5)E_{ik}\} \\
&+ \{(a_1 + 2a_2)\alpha + (a_1 + a_3)E + 2(6a_4 + 3a_3 - a_1 - 2a_2)\Delta^2\}\delta_{ik}]
\end{aligned} \tag{1.55}
$$

The expression for t'_{ik} contains only terms of the first order in the space derivatives of u_i and that for t''_{ik} contains only the second order terms. Now, the displacements u_i may be determined from equations (1.51) subject to the boundary conditions (1.52). Rivlin(1953) has proposed a general procedure to solve this boundary value problem in the second order theory of elasticity as following:

(I) Find the solution of the linear elastic problem represented by

$$\frac{\partial \tau_{ik}}{\partial x_k} + \rho_0 X_i = 0 \tag{1.56}$$

subject to

$$X_{\nu i} = l_k \tau_{ik} \tag{1.57}$$

where

$$
\begin{aligned}
\tau_{ik} &= 2\left[-a_1 e'_{ik} + 2(a_1 + 2a_2)\Delta' \delta_{ik}\right] \\
e'_{ik} &= \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \\
\Delta' &= \frac{1}{2} e'_{ss}
\end{aligned} \tag{1.58}
$$

(II) Obtain the solution to the second order elastic problem given by

$$\frac{\partial \tau''_{ik}}{\partial x_k} + \rho_0 X'_i = 0 \tag{1.59}$$

subject to

$$X'_{\nu i} = l_k \tau''_{ik} \tag{1.60}$$

where

$$
\begin{aligned}
\tau''_{ik} &= 2\left[-a_1 e''_{ik} + 2(a_1 + 2a_2)\Delta'' \delta_{ik}\right] \\
e''_{ik} &= \frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i} \\
\Delta'' &= \frac{1}{2} e''_{ik}
\end{aligned} \tag{1.61}
$$

and

$$X'_{\nu i} = -\left[\Delta'\delta_{ik} - \frac{\partial v_s}{\partial x_k}\right] l_s \tau_{ik} - l_k \tau'_{ik} \tag{1.62}$$

$$\rho_0 X'_i = \left(\Delta'\delta_{ik} - \frac{\partial v_s}{\partial x_k}\right)\frac{\partial \tau_{ik}}{\partial x_s} + \frac{\partial \tau'_{ik}}{\partial x_k} \tag{1.63}$$

where

$$\begin{aligned}\tau'_{ik} =& 2[\{(4a_2 - 2a_3 + a_1)\Delta' e'_{ik} - a_1\alpha'_{ik} - (a_1 - a_5)E'_{ik}\} \\ &+ \{(a_1 + 2a_2)\alpha' + (a_1 + a_3)E' + 2(6a_4 + 2a_3 - a_1 - 2a_2)\Delta'^2\}\delta_{ik}]\end{aligned} \tag{1.64}$$

with notations $\alpha'_{ik} = (\partial v_k/\partial x_s)(\partial v_k/\partial x_s), \alpha' = \alpha'_{ss}, E' = E'_{ss}$ and E'_{ik} = co-factor of e'_{ik} in det e'_{ik}. The displacements u_i are now given by

$$u_i = v_i + w_i \tag{1.65}$$

1.4 Equations of Linear Elasticity Theory.

Since in order to solve the second order elasticity problems we need to solve first the corresponding problems in linear elasticity, for completeness, we now write down the basic equations of the linear elasticity theory. In the static state the equations of equilibrium take the form

$$\frac{\partial t_{ik}}{\partial x_k} + \rho X_i = 0 \tag{1.66}$$

For the isotropic medium the stress-strain relation takes the form

$$t_{ik} = \lambda\Delta\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{1.67}$$

where Δ denotes the dilatation

$$\Delta = \frac{\partial u_i}{\partial x_i} \tag{1.68}$$

If we replace Lame's constants λ and μ by the Young's modulus E and the Poisson's ratio η equation (1.67) becomes

$$t_{ik} = \frac{E}{2(1+\eta)}[\frac{2\eta}{1-2\eta}\Delta\delta_{ik} + \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}] \tag{1.69}$$

If we substitute from (1.69) into (1.66) we find that displacements u_i satisfy the equation

$$\nabla^2 u_i + \frac{1}{1-2\eta}\frac{\partial \Delta}{\partial x_i} + F_i = 0 \tag{1.70}$$

where $F_i = \frac{2(1+\eta)}{E}\rho X_i$

In the absence of body forces equation (1.70) reduces to

$$\nabla^2 u_i + \frac{1}{1-2\eta}\frac{\partial \Delta}{\partial x_i} = 0 \tag{1.71}$$

The first general solution of the equilibrium equation (1.64) would appear to be due to Galerkin(1930). If we express the displacements u_i in terms of a vector G_i through the equation

$$u_i = 2(1-\eta)\,\nabla^2\, G_i - \frac{\partial^2 G_k}{\partial x_k \partial x_i} \tag{1.72}$$

or in vector form

$$\mathbf{u} = 2(1-\eta)\nabla^2\mathbf{G} - \nabla(\nabla \bullet \mathbf{G}) \tag{1.73}$$

then equation (1.71) is equivalent to the biharmonic equation

$$\nabla^4 G_i = 0 \tag{1.74}$$

Corresponding to the displacement field (1.72) we have the stress field

$$t_{ik} = \frac{E}{1+\eta}\left[\eta\delta_{ik}\frac{\partial(\nabla^2 G_j)}{\partial x_j} + (1-\eta)\{\frac{\partial(\nabla^2 G_i)}{\partial x_k} + \frac{\partial(\nabla^2 G_k)}{\partial x_i}\} - \frac{\partial^3 G_j}{\partial x_i \partial x_k \partial x_j}\right] \tag{1.75}$$

Another solution in terms of four scalar potential functions was given by Papkovitch(1932) and Neuber(1934). If we write

$$u_i = \frac{\partial[\Phi + x_j\psi_j]}{\partial x_i} - 4(1-\eta)\psi_i \tag{1.76}$$

or in vector form

$$\mathbf{u} = \nabla(\Phi + \mathbf{r} \bullet \Psi) - 4(1-\eta)\Psi \tag{1.77}$$

then the equations of elastic equilibrium (1.71) are equivalent to the equations

$$\frac{2(1-\eta)}{1-2\eta}\left[\frac{\partial(\nabla^2\Phi)}{\partial x_i} + x_j\frac{\partial(\nabla^2\psi_j)}{\partial x_i} - (1-4\eta)\nabla^2\psi_i\right] = 0 \tag{1.78}$$

It follows immediately that if Φ and ψ_i are harmonic functions, so that

$$\nabla^2\Phi = 0, \qquad \nabla^2\psi_i = 0 \tag{1.79}$$

equation (1.71) is satisfied. The stress field corresponding to this displacement field is given by the equations

$$t_{ik} = \frac{E}{1+\eta}\left[\frac{\partial^2\Phi}{\partial x_i \partial x_k} - (1-2\eta)\left(\frac{\partial\psi_k}{\partial x_i} + \frac{\partial\psi_i}{\partial x_k}\right) + x_j\frac{\partial^2\psi_j}{\partial x_i\partial x_k} - 2\eta\frac{\partial\psi_j}{\partial x_j}\delta_{ik}\right] \tag{1.80}$$

The connection between Galerkin's solution and the Papkovitch-Neuber solution was pointed out by Mindlin(1936). If we put

$$\begin{aligned} \Phi &= \frac{1}{2}x_i(\nabla^2 G_i) - \frac{\partial G_i}{\partial x_i} \\ \psi_i &= -\frac{1}{2}(\nabla^2 G_i) \end{aligned} \tag{1.81}$$

into the Papkovitch-Neuber solution we get Galerkin's solution.

CHAPTER II

SECOND ORDER EFFECTS IN AN ELASTIC HALF-SPACE ACTED UPON BY A NON-UNIFORM NORMAL LOAD

2.1 Statement of the Problem.

We consider a compressible elastic half-space in which a non-uniform normal load, of total amount P, is acting over a circle of radius a(see Fig.1). We choose cylindrical polar coordinates (r, θ, z) such that the load is acting in the plane $z = 0$ in the z-direction. The boundary conditions are

$$X_{\nu r} = 0, \quad X_{\nu z} = -2\mu f(r) \tag{2.1}$$

where

$$f(r) = \frac{(1+\delta)(a^2 - r^2)^{\delta} H(a-r) P}{2\pi\mu a^{2(1+\delta)}} \tag{2.2}$$

and $\delta > -1$ is a constant. $X_{\nu i}$ are the surface tractions and H is the Heaviside unit function.We assume that there are no body forces. According to the Rivlin's procedure the problem to be solved can be split into following two subproblems:

(I) The Linear Solution: solve

$$\begin{aligned} \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \tag{2.3}$$

subiect to

$$\begin{aligned} \tau_{rz}(r,0) &= 0 \\ \tau_{zz}(r,0) &= -\frac{(1+\delta)(a^2 - r^2)^{\delta} H(a-r) P}{\pi a^{2(1+\delta)}} \end{aligned} \tag{2.4}$$

(II) The Second Order Solution: solve

$$\begin{aligned} \frac{\partial \tau''_{rr}}{\partial r} + \frac{\partial \tau''_{rz}}{\partial z} + \frac{\tau''_{rr} - \tau''_{\theta\theta}}{r} + \rho_0 X'_r &= 0 \\ \frac{\partial \tau''_{rz}}{\partial r} + \frac{\partial \tau''_{zz}}{\partial z} + \frac{\tau''_{rz}}{r} + \rho_0 X'_z &= 0 \end{aligned} \tag{2.5}$$

subject to

$$\tau''_{rz}(r,0) = -X'_{\nu r}, \quad \tau''_{zz}(r,0) = -X'_{\nu z} \tag{2.6}$$

where (cf. Appendix A_1)

$$\begin{aligned}
\rho_0 X'_r = &-[\frac{\partial v_r}{\partial r}\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial v_z}{\partial r}\frac{\partial \tau_{rr}}{\partial z} + \frac{v_r}{r^2}(\tau_{rr} - \tau_{\theta\theta}) + \frac{\partial v_r}{\partial z}\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial v_z}{\partial z}\frac{\partial \tau_{rz}}{\partial z}] \\
&+ \frac{\tau'_{rr}}{\partial r} + \frac{\partial \tau'_{rz}}{\partial z} + \frac{\tau'_{rr} - \tau'_{\theta\theta}}{r} \\
\rho_0 X'_z = &-[\frac{\partial v_r}{\partial r}\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial v_z}{\partial r}\frac{\partial \tau_{rz}}{\partial z} + \frac{v_r \tau_{rz}}{r^2} + \frac{\partial v_r}{\partial z}\frac{\partial \tau_{zz}}{\partial r} + \frac{\partial v_z}{\partial z}\frac{\partial \tau_{zz}}{\partial z}] \\
&+ \frac{\partial \tau'_{rz}}{\partial r} + \frac{\partial \tau'_{zz}}{\partial z} + \frac{\tau_{rz}}{r}
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
X'_{\nu r} &= -\frac{\partial v_z(r,0)}{\partial r}\tau_{rr}(r,0) + \tau'_{rz}(r,0) \\
X'_{\nu z} &= [\frac{\partial v_r(r,0)}{\partial r} + \frac{v_r(r,0)}{r}]\tau_{zz}(r,0) + \tau'_{zz}(r,0)
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\tau'_{rr} &= 2[(4a_2 - 2a_3 + a_1)\Delta' e'_{rr} - a_1\alpha'_{rr} - (a_1 - a_5)E'_{rr} + \Sigma] \\
\tau'_{\theta\theta} &= 2[(4a_2 - 2a_3 + a_1)\Delta' e'_{\theta\theta} - a_1\alpha'_{\theta\theta} - (a_1 - a_5)E'_{\theta\theta} + \Sigma] \\
\tau'_{zz} &= 2[(4a_2 - 2a_3 + a_1)\Delta' e'_{zz} - a_1\alpha'_{zz} - (a_1 - a_5)E'_{zz} + \Sigma] \\
\tau'_{rz} &= 2[(4a_2 - 2a_3 + a_1)\Delta' e'_{rz} - a_1\alpha'_{rz} - (a_1 - a_5)E'_{rz}] \\
\Sigma &= (a_1 + 2a_2)\alpha' + (a_1 + a_3)E' + 2(6a_4 + 2a_3 - 2a_2 - a_1)\Delta'^2
\end{aligned} \tag{2.9}$$

2.2 The Linear Solution.

For the linear solution we are required to solve the subproblem (I)

We employ Papkovitch-Neuber displacement solutions

$$v_i = \frac{\partial(\Phi + x_j\psi_j)}{\partial x_i} - 4(1-\eta)\psi_i \tag{2.10}$$

together with

$$\Phi = (1-2\eta)\phi(r,z), \quad \psi_1 = \psi_2 = 0, \quad \psi_3 = \frac{\partial \phi(r,z)}{\partial z} \tag{2.11}$$

where ϕ satisfies

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{2.12}$$

The displacement components in cylindrical polar coordinates are given by

$$\begin{aligned} v_r(r,z) &= (1-2\eta)\frac{\partial\phi}{\partial r} + z\frac{\partial^2\phi}{\partial r\partial z} \\ v_\theta(r,z) &= 0 \\ v_z(r,z) &= -2(1-\eta)\frac{\partial\phi}{\partial z} + z\frac{\partial^2\phi}{\partial z^2} \end{aligned} \tag{2.13}$$

By using the constitutive equations we find that

$$\begin{aligned} \tau_{rz} &= 2\mu z\frac{\partial^3\phi}{\partial z\partial z^2} \\ \tau_{zz} &= -2\mu[\frac{\partial^2\phi}{\partial z^2} - z\frac{\partial^3\phi}{\partial z^3}] \\ \tau_{r\theta} &= \tau_{z\theta} = 0 \\ \tau_{rr} &= 2\mu[(1-2\eta)\frac{\partial^2}{\partial r^2} + z\frac{\partial^3\phi}{\partial r^2\partial z}] - 4\mu\eta\frac{\partial^2\phi}{\partial z^2} \\ \tau_{\theta\theta} &= 2\mu[\frac{1-2\eta}{r}\frac{\partial\phi}{\partial r} + \frac{z}{r}\frac{\partial^2\phi}{\partial r\partial z}] - 4\mu\eta\frac{\partial^2\phi}{\partial z^2} \end{aligned} \tag{2.14}$$

If we let

$$\bar{\phi} = \int_0^\infty rJ_0(\xi r)\phi(r,z)\,dr \tag{2.15}$$

it then follows that (2.12) reduces to

$$\frac{\partial^2\bar{\phi}}{\partial z^2} - \xi^2\bar{\phi} = 0 \tag{2.16}$$

The appropriate solution to (2.16) is $\bar{\phi} = Ae^{-\xi z}$, where A is an arbitrary function of ξ. On using the boundary condition (2.4) we find

$$A(\xi) = \frac{QJ_{(1+\delta)}(a\xi)}{\xi^{3+\delta}} \tag{2.17}$$

where

$$Q = \frac{(1+\delta)\Gamma(1+\delta)P}{2^{1-\delta}\pi\mu a^{(1+\delta)}} \tag{2.18}$$

Using (2.15) and taking following Hankel transforms: $H_1[v_r]$, $H_0[v_z]$, $H_0[\tau_{zz}]$, $H_1[\tau_{rz}]$, $H_0[\tau_{rr}+\tau_{\theta\theta}]$, $H_0[v_r+\tau_{rr}/2\mu]$ and then taking the inverse transforms we find

$$\begin{aligned}
v_r(r,z) =& Q\left[zK(r,z,-\delta)-(1-2\eta)K(r,z,-(1+\delta))\right] \\
v_z(r,z) =& Q\left[2(1-\eta)I(r,z,-(1+\delta))+zI(r,z,-\delta)\right] \\
\tau_{rr}(r,z) =& -2\mu Q\left[I(r,z,-\delta)-zI(r,z,(1-\delta))\right] \\
&+\frac{2\mu Q}{r}\left[(1-2\eta)K(r,z,-(1+\delta))-zK(r,z,-\delta)\right] \\
\tau_{\theta\theta}(r,z) =& -4\mu\eta QI(r,z,-\delta) \\
&-\frac{2\mu Q}{r}\left[(1-2\eta)K(r,z,-(1+\delta))-zK(r,z,-\delta)\right] \\
\tau_{rz}(r,z) =& -2\mu QzK(r,z,(1-\delta)) \\
\tau_{zz}(r,z) =& -2\mu Q\left[I(r,z,-\delta)+zI(r,z,(1-\delta))\right]
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
I(r,z,s) &= \int_0^\infty \xi^s J_0(\xi r)J_{(1+\delta)}(\xi a)e^{-\xi z}\,d\xi \\
K(r,z,s) &= \int_0^\infty \xi^s J_1(\xi r)J_{(1+\delta)}(\xi a)e^{-\xi z}\,d\xi
\end{aligned} \tag{2.20}$$

Equations (2.19) and (2.20) thus give the non-zero displacement and stress components for the linear elasticity problem. However, in many circumstances, the values of displacements and stresses,which are of most interet, are on the surface of the half-space. We shall now give the solutions on the surface $z=0$. From Gradshteyn and Ryzhik(1965) we note that

$$I(r,0,s) = \begin{cases} \frac{2^s\Gamma((2+\delta+s)/2)}{\Gamma((2+\delta-s)/2)a^{s+1}}F_1(\frac{2+\delta+s}{2},\frac{s-\delta}{2},1,\frac{r^2}{a^2}), & r\le a \\ \frac{2^s\Gamma((2+\delta+s)/2)a^{1+\delta}}{\Gamma((\delta-s)/2)\Gamma(2+\delta)r^{2+\delta+s}}F_1(\frac{2+\delta+s}{2},\frac{2+\delta+s}{2},2+\delta,\frac{a^2}{r^2}), & r>a \end{cases}$$

$$K(r,0,s) = \begin{cases} \frac{2^s r\Gamma((3+\delta+s)/2)}{\Gamma((1+\delta-s)/2)a^{2+s}}F_1(\frac{3+\delta+s}{2},\frac{1-\delta+s}{2},2,\frac{r^2}{a^2}), & r\le a \\ \frac{2^s\Gamma((3+\delta+s)/2)a^{1+\delta}}{\Gamma((1-\delta-s)/2)\Gamma(2+\delta)r^{2+\delta+s}}F_1(\frac{3+\delta+s}{2},\frac{1+\delta+s}{2},2+\delta,\frac{a^2}{r^2}), & r>a \end{cases}$$

where F_1 is the hypergeometric function. Some typical components of the surface solutions are written below.

$$v_r(r,0) = -(1-2\eta)QK(r,0-(1+\delta))$$

$$v_z(r,0) = 2(1-\eta)QI(r,0,-(1+\delta))$$

$$\tau_{rz}(r,0) = 0$$

$$\tau_{zz}(r,0) = -2\mu QI(r,0,-\delta)$$

$$\tau_{rr}(r,0) = -2\mu QI(r,0,-\delta) + \frac{2\mu(1-2\eta)}{r}QK(r,0,-(1+\delta))$$

$$\tau_{\theta\theta}(r,0) = -4\mu\eta QI(r,0,-\delta) - \frac{2\mu(1-2\eta)}{r}QK(r,0,-(1+\delta))$$

2.3 The Second Order Solution.

In order to solve the second order problem we note that the boundary value problem to be solved is now subproblem (II). We again use Papkovitch-Neuber displacement solutions with

$$\Phi = \phi(r,z), \quad \psi_1 = \psi_2 = 0, \quad \psi_3 = \psi(r,z)$$

The displacement and stress components are now given as

$$\begin{aligned} w_r(r,z) &= \frac{\partial\phi}{\partial r} + z\frac{\partial\psi}{\partial r} \\ w_\theta(r,z) &= 0 \\ w_z(r,z) &= \frac{\partial\phi}{\partial z} + z\frac{\partial\psi}{\partial z} - (3-4\eta)\psi \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \tau''_{rz}(r,z) &= 2\mu\left[\frac{\partial^2\phi}{\partial r\partial z} + z\frac{\partial^2\psi}{\partial r\partial z} - (1-2\eta)\frac{\partial\psi}{\partial r}\right] \\ \tau''_{zz}(r,z) &= 2\mu\left[\frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\psi}{\partial z^2} - z(1-\eta)\frac{\partial\psi}{\partial z} + \frac{\eta}{(1-2\eta)}(\nabla^2\phi + z\nabla^2\psi)\right] \\ \tau''_{rr}(r,z) &= 2\mu\left[\frac{\partial^2\phi}{\partial r^2} + z\frac{\partial^2\psi}{\partial r^2} - 2\eta\frac{\partial\psi}{\partial z} + \frac{\eta}{(1-2\eta)}(\nabla^2\phi + \nabla^2\psi)\right] \\ \tau''_{\theta\theta}(r,z) &= 2\mu\left[\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{z}{r}\frac{\partial\psi}{\partial r} - 2\eta\frac{\partial\psi}{\partial z} + \frac{\eta}{(1-2\eta)}(\nabla^2\phi + \nabla^2\psi)\right] \end{aligned} \tag{2.22}$$

On employing (2.22) into (2.3) we find

$$\nabla^2\phi = \phi_0, \quad \nabla^2\psi = \psi_0 \tag{2.23}$$

where

$$\begin{aligned}\phi_0 &= \frac{\rho_0}{4\eta(1-\eta)}[z\frac{\partial}{\partial z}\int_r^\infty X_r'(x,z)\,dx + 2(1-2\eta)\int_r^\infty X_r'(x,z)\,dx - zX_z'(r,z)]\\ \psi_0 &= \frac{\rho_0}{4\eta(1-\eta)}[X_z'(r,z) - \frac{\partial}{\partial z}\int_r^\infty X_r'(r,z)\,dx]\end{aligned} \tag{2.24}$$

We again denote $\bar{\phi}$ and $\bar{\psi}$ as

$$\bar{\phi} = \int_0^\infty rJ_0(\xi r)\phi(r,z)\,dr, \quad \bar{\phi}_0 = \int_0^\infty rJ_0(\xi r)\phi_0(r,z)\,dr$$

and then from (2.39) we find

$$\bar{\phi} = Ce^{-\xi z} + e^{-\xi z}\int_0^z e^{2\xi z_2}\int_0^{z_2}\bar{\phi}_0(\xi,z_1)e^{-\xi z_1}\,dz_1\,dz_2$$

where C is an arbitrary function of ξ. It then follows that

$$\phi = \int_0^\infty \xi J_0(\xi r)(C + \phi_0^*)e^{-\xi z}\,d\xi \tag{2.25}$$

where

$$\phi_0^*(\xi,z) = \int_0^z e^{2\xi z_2}\int_0^{z_2}\bar{\phi}_0(\xi,z_1)e^{-\xi z_1}\,dz_1\,dz_2$$

Similarly the solution of $(2.23)_2$ leads to

$$\psi = \int_0^\infty \xi J_0(\xi,r)(D + \psi_0^*)e^{-\xi z}\,d\xi \tag{2.26}$$

where

$$\psi_0^*(\xi,z) = \int_0^z e^{2\xi z_2}\int_0^{z_2}\psi_0(\xi,z)e^{-\xi z_1}\,dz_1\,dz_2$$

and D is an arbitrary function of ξ. On employing (2.25) and (2.26) into (2.22) we obtain

$$\tau''_{rz}(r,0) = 2\mu \int_0^\infty \xi J_1(\xi r)[C\xi^2 + (1-2\eta)D\xi]\, d\xi$$
$$\tau''_{zz}(r,0) = 2\mu \int_0^\infty \xi J_0(\xi r)[C\xi^2 + 2(1-\eta)D\xi]\, d\xi + \frac{2\mu(1-\eta)}{(1-2\eta)}\phi_0(r,0)$$

On applying the boundary condition (2.6) we find

$$C = \frac{1}{2\mu\xi^2}[(1-2\eta)h_1(\xi) - 2(1-\eta)h_2(\xi)]$$
$$D = \frac{1}{2\mu\xi}[h_2(\xi) - h_1(\xi)] \tag{2.27}$$

where

$$h_1(\xi) = \int_0^\infty rJ_0(\xi r)[X'_{\nu z}(r,0) + \int_0^\infty \rho_0 X'_r(x,0)\, dx]\, dr$$
$$h_2(\xi) = \int_0^\infty rJ_1(\xi r)X'_{\nu r}(r,0)\, dr \tag{2.28}$$

The displacements and stresses for the second order problem are thus given by

$$w_r(r,z) = -\int_0^\infty \xi^2 J_1(\xi r)[C + Dz + \xi\phi_0^* + \xi z\psi_0^*]e^{-\xi z}\, d\xi$$
$$w_z(r,z) = \int_0^\infty \xi J_0(\xi r)\{\int_0^z [\bar{\phi}_0(\xi,z_1) + z\psi_0(\xi,z_1)]e^{-\xi z_1}\, dz_1\}e^{\xi z}\, d\xi$$
$$- \int_0^\infty \xi J_0(\xi r)[C\xi + (3-4\eta)D + Dz\xi + \xi\phi_0^* + (3-4\eta+z\xi)\psi_0^*]e^{-\xi z}\, dz \tag{2.29}$$

and

$$\tau''_{rz}(r,z) = 2\mu \int_0^\infty \xi^2 J_1(\xi r)[(1-2\eta)D + C\xi + Dz\xi + \xi\phi_0^* + (1-2\eta+z\xi)\psi_0^*]e^{-\xi z}\, d\xi$$
$$- 2\mu \int_0^\infty \xi^2 J_1(\xi r)\{\int_0^z [\bar{\phi}_0^*(\xi,z_1) + z\psi_0(\xi,z_1)]e^{-\xi z_1}\, dz_1\}e^{\xi z}\, d\xi$$
$$\tau''_{zz}(r,z) = \frac{2\mu(1-\eta)[\phi_0(r,z) + z\psi_0(r,z)]}{(1-2\eta)}$$
$$+ 2\mu \int_0^\infty \xi^2 J_0(\xi r)[2(1-\eta)D + C\xi + Dz\xi + \xi\phi_0^* + (2-2\eta+\xi z)\psi_0^*]e^{-\xi z}\, d\xi$$
$$- 4\mu(1-\eta) \int_0^\infty \xi J_0(\xi r)[\int_0^z \bar{\psi}_0(\xi,z_1)e^{-\xi z_1}\, dz_1]e^{\xi z}\, d\xi$$

$$
\begin{aligned}
\tau''_{rr}(r,z) =& \frac{2\mu\eta}{(1-2\eta)}[\phi_0(r,z) + z\psi_0(r,z)] \\
&+ \frac{2\mu}{r}\int_0^\infty \xi^2 J_1(\xi r)[C + Dz + \phi_0^* + z\psi_0^*]e^{-\xi z}\,d\xi \\
&+ 2\mu\int_0^\infty \xi^2 J_0(\xi r)[2\eta D - C\xi - Dz\xi - \xi\phi_0^* + (2\eta - z\xi)\psi_0^*]e^{-\xi z}\,d\xi \\
&- 4\mu\eta\int_0^\infty \xi J_0(\xi r)[\int_0^z \bar{\psi}_0(\xi, z_1)e^{-\xi z_1}\,dz_1]e^{\xi z}\,d\xi \\
\tau''_{\theta\theta}(r,z) =& \frac{2\mu\eta}{(1-2\eta)}[\phi_0(r,z) + z\psi_0(r,z)] \\
&- \frac{2\mu}{r}\int_0^\infty \xi^2 J_1(\xi r)[C + Dz + \phi_0^* + z\psi_0^*]e^{-\xi z}\,d\xi \\
&+ 4\mu\eta\int_0^\infty \xi^2 J_0(\xi r)[D\xi + \xi\psi_0^*]e^{-\xi z}\,d\xi \\
&- 4\mu\eta\int_0^\infty \xi J_0(\xi r)[\int_0^z \bar{\psi}_0(\xi, z_1)e^{-\xi z_1}\,dz_1]e^{\xi z}\,d\xi
\end{aligned} \tag{2.30}
$$

The expressions for τ'_{zz}, τ'_{rz}, etc. can be written as

$$
\begin{aligned}
\frac{\tau'_{zz}(r,z)}{2Q^2} =& 4(1-2\eta)(4a_2 - 2a_3 + a_1)I(r,z,-\delta)[(1-2\eta)I(r,z,1-\delta) + zI(r,z,-\delta)] \\
&- 4(a_1 - a_5)[\frac{z}{r}K(r,z,-\delta) - \frac{(1-2\eta)}{r}K(r,z,-(1+\delta))][zI(r,z,1-\delta) \\
&- \frac{z}{r}K(r,z,-\delta) - (1-2\eta)I(r,z,-\delta) + \frac{(1-2\eta)}{r}K(r,z,-(1+\delta))] \\
&- a_1[2(1-\eta)K(r,z,-\delta) + zK(r,z,1-\delta)]^2 \\
&- a_1[I(r,z,-\delta) + zI(r,z,1-\delta)]^2 + \frac{\Sigma}{Q^2}
\end{aligned} \tag{2.31}
$$

$$
\begin{aligned}
\frac{\tau'_{rz}(r,z)}{2Q^2} =& 4(1-2\eta)z(4a_2 - 2a_3 + a_1)I(r,z,-\delta)K(r,z,1-\delta) \\
&+ 2a_1(1-\eta)K(r,z,-\delta)[2zI(r,z,1-\delta) - \frac{z}{r}K(r,z,-\delta) \\
&+ \frac{(1-2\eta)}{r}K(r,z,-(1+\delta))] + a_1 zK(r,z,-\delta)[-\frac{z}{r}K(r,z,-\delta) \\
&- 2(1-2\eta)I(r,z,-\delta) + \frac{(1-2\eta)}{r}K(r,z,-(1+\delta))] \\
&- 4(a_1 - a_5)zK(r,z,1-\delta)[\frac{z}{r}K(r,z,-\delta) - \frac{(1-2\eta)}{r}K(r,z,-(1+\delta))]
\end{aligned} \tag{2.32}
$$

with similar expressions for $\tau'_{\theta\theta}$ and τ'_{rr} and where

$$
\begin{aligned}
\frac{\Sigma}{Q^2} =&(a_1+2a_2)\{[zI(r,z,1-\delta)-\frac{z}{r}K(r,z,-\delta)-(1-2\eta)I(r,z,-\delta)\\
&+\frac{(1-2\eta)}{r}K(r,z,-(1+\delta))]^2+[-zK(r,z,1-\delta)+2(1-\eta)K(r,z,-\delta)]^2\\
&+[\frac{z}{r}K(r,z,-\delta)-\frac{(1-2\eta)}{r}K(r,z,-(1+\delta))]^2\\
&+[zI(r,z,1-\delta)+(1-2\eta)I(r,z,-\delta)]^2\\
&+[zK(r,z,1-\delta)+2(1-\eta)K(r,z,-\delta)]^2\}\\
&+(a_1+a_3)\{-4[(1-2\eta)I(r,z,-\delta)+zI(r,z,1-\delta)][zI(r,z,1-\delta)\\
&-(1-2\eta)I(r,z,-\delta)]-4z^2K^2(r,z,1-\delta)\\
&+4[\frac{z}{r}K(r,z,-\delta)-\frac{(1-2\eta)}{r}K(r,z,-(1+\delta))][zI(r,z,1-\delta)\\
&-\frac{z}{r}K(r,z,-\delta)-(1-2\eta)I(r,z,-\delta)+\frac{(1-2\eta)}{r}K(r,z,-(1+\delta))]\}\\
&+8(1-2\eta)^2(6a_4+2a_3-a_1-2a_2)I^2(r,z,-\delta)
\end{aligned}
\tag{2.33}
$$

Equations (2.29), (2.30), (2.19) and (2.20) together with the expressions for τ'_{ij} constitute the solutions of the second order problem. On the surface $z=0$ these solutions can be written as

$$
\begin{aligned}
w_r(r,0) =& -\frac{1}{2\mu}[(1-2\eta)\int_0^\infty xX'_{\nu z}(x,0)+\int_0^\infty \rho_0 X'_r(y,0)\,dyK_1(x)\,dx\\
&-2(1-\eta)\int_0^\infty xX'_{\nu r}(x,0)K_3(x)\,dx]\\
w_z(r,0) =& -\frac{1}{2\mu}[(1-2\eta)\int_0^\infty xX'_{\nu r}(x,0)K_2(x)\,dx\\
&-2(1-\eta)\int_0^\infty xX'_{\nu z}(x,0)+\int_0^\infty \rho_0 X'_r(y,0)\,dyK_4(x)\,dx]\\
\tau''_{rz}(r,0) =& -X'_{\nu r}(r,0)=-[c_6I(r,0,-\delta)K(r,0,-\delta)\\
&+\frac{c_7}{r}K(r,0,-\delta)K(r,0,-(1+\delta))]
\end{aligned}
$$

$$\tau''_{zz}(r,0) = -X'_{\nu z}(r,0) = -[c_8 I^2(r,0,-\delta) - \frac{c_9}{r} I(r,0,-\delta)K(r,0,-(1+\delta))$$
$$+ \frac{c_9}{r^2} K^2(r,0,-(1+\delta)) + c_{10} K^2(r,0,-\delta)]$$
$$\tau'_{rz}(r,0) = \frac{4(1-\eta)(1-2\eta)a_1 Q^2}{r} K(r,0,-\delta)K(r,0,-(1+\delta))$$
$$\tau'_{zz}(r,0) = 2(1-2\eta)^2 Q^2 \{(a_1 + 4a_2 + 12a_3 + 48a_4) I^2(r,0,-\delta)$$
$$- \frac{a_1 + 2a_2 - 2a_3 - 2a_5}{r} I(r,0,-\delta)K(r,0,-(1+\delta))$$
$$+ \frac{a_1 + 2a_2 - 2a_3 - 2a_5}{r^2} K^2(r,0,-(1+\delta))\}$$
$$+ 8(1-\eta)^2 (a_1 + 4a_2) Q^2 K^2(r,0,-\delta)$$

with similar expressions for τ'_{rr} and $\tau'_{\theta\theta}$. Also

$$\rho_0 X'_r(r,0) = \frac{c_1}{r}[I(r,0,-\delta) - \frac{2}{r}K(r,0,-(1+\delta))]^2$$
$$+ \frac{c_2}{r} K(r,0,1-\delta)K(r,0,-(1+\delta)) + c_3 I(r,0,-\delta)K(r,0,1-\delta)$$
$$+ c_4 I(r,0,1-\delta)K(r,0,-\delta) + \frac{c_5}{r} K^2(r,0,-\delta)$$

where c_{ij} are constants and listed in the appendix A_4 and the kernel functions are given by

$$K_1(x) = \begin{cases} \frac{1}{r}, & x < r \\ 0, & x > r \end{cases}$$
$$K_2(x) = \begin{cases} \frac{1}{\pi x}[F(\frac{x}{r}) - E(\frac{x}{r})], & x < r \\ \frac{1}{\pi r}[F(\frac{r}{x}) - E(\frac{r}{x})], & x > r \end{cases}$$
$$K_3(x) = \begin{cases} 0, & x < r \\ \frac{1}{x}, & x > r \end{cases}$$
$$K_4(x) = \begin{cases} \frac{2}{\pi r}F(\frac{x}{r}), & x < r \\ \frac{2}{\pi x}F(\frac{r}{x}), & x > r \end{cases}$$

where $F(x) = \int_0^{\frac{\pi}{2}} (1 - x^2 \sin^2 \tau)^{-\frac{1}{2}}\, d\tau$ and $E(x) = \int_0^{\frac{\pi}{2}} (1 - x^2 \sin^2 \tau)^{\frac{1}{2}}\, d\tau$ are the complete elliptic integrals of the first and second kind, respectively.

2.4 Illustrations.

It is now of some interest to write down the displacement and stress components for specific values of δ.

We present first some cases for linear elasticity theory.

(a) Linear Case.

(i) $\delta = -\frac{1}{2}$, this case is equivalent to the flat-ended punch problem. The solutions on the surface $z = 0$ are in agreement with Sneddon(1965).

For $r \leq a$

$$\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu a}\frac{r}{a+\sqrt{a^2-r^2}} \\
v_z(r,0) &= \frac{(1-\eta)P}{2\mu a} \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= -\frac{P}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} \\
\tau_{rr}(r,0) &= -\frac{P}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} + \frac{(1-2\eta)P}{2\pi r}\frac{1}{a+\sqrt{a^2-r^2}} \\
\tau_{\theta\theta}(r,0) &= -\frac{\eta P}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} - \frac{(1-2\eta)P}{2\pi r}\frac{1}{a+\sqrt{a^2-r^2}}
\end{aligned} \tag{2.34a}$$

For $r > a$

$$\begin{aligned}
v_r(r,0) &= \frac{(1-2\eta)P}{4\pi\mu a} \\
v_z(r,0) &= \frac{(1-\eta)P}{2\pi\mu a}\arcsin(\frac{a}{r}) \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= 0 \\
\tau_{rr}(r,0) &= \frac{(1-2\eta)P}{2\pi r} \\
\tau_{\theta\theta}(r,0) &= -\frac{(1-2\eta)P}{2\pi r}
\end{aligned} \tag{2.34b}$$

(ii) $\delta = 0$, corresponds to uniformly distributed load. The solutions on the surface $z = 0$ are

For $r \leq a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu}\frac{r}{a^2} \\
v_z(r,0) &= \frac{2(1-\eta)P}{\mu a \pi^2}E(\frac{r}{a}) \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= -\frac{P}{\pi a^2} \\
\tau_{rr}(r,0) &= -\frac{(1+2\eta)P}{2\pi a^2} \\
\tau_{\theta\theta}(r,0) &= -\frac{(1+2\eta)P}{2\pi a^2}
\end{aligned}
\tag{2.35a}
$$

For $r > a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu r} \\
v_z(r,0) &= \frac{2(1-\eta)P}{\pi^2 \mu a}[\frac{r}{a}E(\frac{r}{a}) - \frac{r^2-a^2}{ar}F(\frac{a}{r})] \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= 0 \\
\tau_{rr}(r,0) &= \frac{(1-2\eta)P}{2\pi r^2} \\
\tau_{\theta\theta}(r,0) &= \frac{(1-2\eta)P}{2\pi r^2}
\end{aligned}
\tag{2.35b}
$$

(*iii*) $\delta = \frac{1}{2}$,this case corresponds to the punch in the form of a paraboloid of revolution. The solutions on the surface $z = 0$ are (cf. Sneddon(1965)):

For $r \leq a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu}\frac{[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}]}{r} \\
v_z(r,0) &= \frac{3(1-\eta)P}{8\mu a}[1-\frac{r^2}{2a^2}]
\end{aligned}
$$

$$
\begin{aligned}
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= -\frac{3P\sqrt{a^2-r^2}}{2\pi a^3} \\
\tau_{rr}(r,0) &= -\frac{3P\sqrt{a^2-r^2}}{2\pi a^3} + \frac{(1-2\eta)P}{2\pi r^2}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}] \\
\tau_{\theta\theta}(r,0) &= -\frac{3\eta P\sqrt{a^2-r^2}}{\pi a^3} - \frac{(1-2\eta)P}{2\pi r^2}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}]
\end{aligned}
\tag{2.36a}
$$

For $r > a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu r} \\
v_z(r,0) &= \frac{3(1-\eta)P}{8\pi\mu a}[(2-\frac{r^2}{a^2})\arcsin(\frac{a}{r}) + \frac{1}{a}\sqrt{r^2-a^2}] \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= 0 \\
\tau_{rr}(r,0) &= \frac{(1-2\eta)P}{2\pi r^2} \\
\tau_{\theta\theta}(r,0) &= -\frac{(1-2\eta)P}{2\pi r^2}
\end{aligned}
\tag{2.36b}
$$

(*vi*) $\delta = \frac{3}{2}$ In this case we find

For $r \leq a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu r}[1-(1-\frac{r^2}{a^2})^{\frac{5}{2}}] \\
v_z(r,0) &= \frac{15(1-\eta)P}{32\mu a^2\sqrt{\pi}}[1-\frac{r^2}{a^2}+\frac{3}{8}\frac{r^4}{a^4}] \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= -\frac{5P}{2\pi a^2}(1-\frac{r^2}{a^2})^{\frac{3}{2}} \\
\tau_{rr}(r,0) &= -\frac{5P}{2\pi a^2}(1-\frac{r^2}{a^2})^{\frac{3}{2}} + \frac{(1-2\eta)P}{2\pi r^2}[1-(1-\frac{r^2}{a^2})^{\frac{5}{2}}] \\
\tau_{\theta\theta}(r,0) &= -\frac{5P}{2\pi a^2}(1-\frac{r^2}{a^2})^{\frac{3}{2}} - \frac{(1-2\eta)P}{2\pi r^2}[1-(1-\frac{r^2}{a^2})^{\frac{5}{2}}]
\end{aligned}
\tag{2.37a}
$$

For $r > a$

$$
\begin{aligned}
v_r(r,0) &= -\frac{(1-2\eta)P}{4\pi\mu r} \\
v_z(r,0) &= \frac{15(1-\eta)P}{16\pi\mu a^2 r}\left[\frac{r^2-a^2}{2} + \frac{3(\frac{r^2}{2}-a^2)^2}{2a^2}\frac{r}{a}\arcsin\frac{a}{r}\right. \\
&\quad \left. - \frac{3(\frac{r^2}{2}-a^2)r\sqrt{r^2-a^2}}{4a^2}\right] \\
\tau_{rz}(r,0) &= 0 \\
\tau_{zz}(r,0) &= 0 \\
\tau_{rr}(r,0) &= \frac{(1-2\eta)P}{2\pi r^2} \\
\tau_{\theta\theta}(r,0) &= -\frac{(1-2\eta)P}{2\pi r^2}
\end{aligned}
\tag{2.37b}
$$

Similarly, for $\delta = 5/2, 7/2, 9/2, \cdots$, we can get the other exact solutions.

(v) Point Load: By letting a tend to zero we obtain the solutions for the case of the point load. On noting that:

$$\lim_{a\to 0} QI(r,z,s) = \frac{P}{4\pi\mu}\int_0^\infty \xi^{(1+\delta+s)} J_0(\xi r)e^{-\xi z}\, d\xi$$

$$\lim_{a\to 0} QK(r,z,s) = \frac{P}{4\pi\mu}\int_0^\infty \xi^{(1+\delta+s)} J_1(\xi r)e^{-\xi z}\, d\xi$$

and

$$\int_0^\infty J_n(\xi r)e^{-\xi z}\, d\xi = \frac{[\sqrt{r^2+z^2}-z]^n}{r^n\sqrt{r^2+z^2}}$$

we obtain

$$
\begin{aligned}
v_r(r,z) &= -\frac{P}{4\pi\mu}\left[\frac{(1-2\eta)(\sqrt{r^2+z^2}-z)}{r\sqrt{r^2+z^2}} - \frac{r^2}{\sqrt[3]{r^2+z^2}}\right] \\
v_z(r,z) &= -\frac{P}{4\pi\mu}\left[\frac{2(1-\eta)}{\sqrt{r^2+z^2}} + \frac{r^2}{(r^2+z^2)^{\frac{3}{2}}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\tau_{rz}(r,z) &= -\frac{3Pz^2r}{2\pi(r^2+z^2)^{\frac{5}{2}}} \\
\tau_{zz}(r,z) &= -\frac{P}{2\pi}\frac{3z^3}{(r^2+z^2)^{\frac{5}{2}}} \\
\tau_{rr}(r,z) &= -\frac{P}{2\pi}[\frac{1-2\eta}{r^2}(1+\frac{z}{\sqrt{r^2+z^2}})-\frac{3r^2z}{(r^2+z^2)^{\frac{5}{2}}}] \\
\tau_{\theta\theta}(r,z) &= \frac{(1-2\eta)P}{2\pi}[\frac{z}{(r^2+z^2)^{\frac{5}{2}}}-\frac{1}{r^2}(1-\frac{z}{\sqrt{r^2+z^2}})]
\end{aligned}
\tag{2.38}
$$

(b) Second Order Case.

In this case it suffices to give solutions for one value of δ, since computations become quite involved, and we select $\delta = \frac{1}{2}$. We first need to calculate the expressions for τ'_{ij} and these are: for $r \le a$

$$
\begin{aligned}
\tau'_{rz}(r,0) &= \frac{2(1-\eta)(1-2\eta)a_1}{3r}Q^2[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}] \\
\tau'_{zz}(r,0) &= 2(1-2\eta)^2Q^2\{\frac{2(a_1+4a_2+12a_3+48a_4)}{\pi a^3}(a^2-r^2) \\
&\quad -\frac{2(a_1+2a_2+2a_3+2a_5)}{3\pi r^2}a[(1-\frac{r^2}{a^2})^{\frac{1}{2}}-(1-\frac{r^2}{a^2})^2] \\
&\quad +\frac{2(a_1+2a_2+2a_3+2a_5)}{9\pi r^4}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}]^2\} \\
&\quad +\frac{(1-\eta)^2(a_1+4a_2)}{a^3}Q^2r^2
\end{aligned}
\tag{2.39}
$$

with similar expressions for τ'_{rr} and $\tau'_{\theta\theta}$ and

for $r > a$

$$
\begin{aligned}
\tau'_{rz}(r,0) &= \frac{4(1-\eta)(1-2\eta)a_1aQ^2}{3\pi r^2}[\frac{r}{a}\arcsin(\frac{a}{r})-(1-\frac{a^2}{r^2})^{\frac{1}{2}}] \\
\tau'_{zz}(r,0) &= \frac{4(1-2\eta)^2(a_1+2a_2-2a_3-2a_5)}{9\pi r^4}a^3Q^2 \\
&\quad +\frac{4(1-\eta)^2(a_1+4a_2)Q^2}{\pi a}[\frac{r}{a}\arcsin(\frac{a}{r})-(1-\frac{a^2}{r^2})^{\frac{1}{2}}]^2
\end{aligned}
\tag{2.40}
$$

with similar expressions for $\tau'_{rr}(r,0)$ and $\tau_{\theta\theta}(r,0)$, where $Q = 3P/(4\sqrt{2\pi}\mu a^{3/2})$.

The solutions for the second order elastic problem turn out to be: for $r \leq a$

$$
\begin{aligned}
w_r(r,0) = &- \frac{1-2\eta}{2\mu}\{(c_{12} + \frac{4-\pi^2}{2}c_{11})\frac{r}{a} + (c_{13} + \frac{c_{14}}{4})\frac{r^3}{a^3} \\
&+ \frac{c_1+c_2}{3\pi\mu a}[r^2 \ln\frac{a+\sqrt{a^2-r^2}}{a} + a^2 - a\sqrt{a^2-r^2}] \\
&- \frac{2(c_1+c_9)}{9\pi r}[\frac{a^3-(a^2-r^2)^{\frac{3}{2}}}{a^2} + \frac{a^3}{r^2} - \frac{(a^2-r^2)^{\frac{5}{2}}}{a^2r^2} - \frac{5a}{2}]\} \\
&+ \frac{1-\eta}{\mu}\{\frac{c_6 I_3(r)}{2\pi a^3} - \frac{c_7 I_4(r)}{6\pi a^3} + \frac{c_6 a r^2 I_5(r)}{2\pi} \\
&+ \frac{c_7 a I_6(r)}{6\pi} + \frac{c_7 a I_7(r)}{3\pi r^2}\} \\
w_z(r,0) = &\frac{1-2\eta}{2\mu}\{\frac{(4I_1+\pi)c_7}{12\pi} + \frac{3c_6+c_7}{18}\frac{(a^2-r^2)^{\frac{3}{2}}}{a^3} \\
&+ \frac{c_7}{6}[\frac{\sqrt{a^2-r^2}}{a} + \ln\frac{a}{a+\sqrt{a^2-r^2}}]\} \\
&+ \frac{1-\eta}{\mu}\{\frac{2(c_{15}+4c_{11}-\pi^2c_{11})}{\pi}E(\frac{r}{a}) + \frac{7(c_{14}+c_{16})}{3\pi}\frac{r^3}{a^3} \\
&+ \frac{4(3c_1-c_9)}{9\pi^2 r}[I_9(r)+I_{12}(r)] + \frac{4(c_1+c_2)}{3\pi^2}[\frac{I_{10}(r)}{ar} + rI_{13}(r)] \\
&- \frac{8(9c_1+c_9)}{27\pi^2 r}[aI_{11}(r) + I_{14}(r)] \\
&+ \frac{2(c_{14}+c_{16})}{9\pi a^2}[(a^2+4r^2)E(\frac{r}{a}) + 2(a^2-r^2)F(\frac{r}{a})] \\
&+ \frac{4(c_1+c_9)I_{15}(r)}{9\pi^2 r^3} - \frac{8c_{14}rI_{16}(r)}{\pi^3} - \frac{8c_{11}rI_{17}(r)}{\pi}\}
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
\tau''_{rz}(r,0) = &\frac{c_6 r\sqrt{a^2-r^2}}{2a^3} + \frac{c_7[a^3-(a^2-r^2)^{\frac{3}{2}}]}{6a^3 r} \\
\tau''_{zz}(r,0) = &-\frac{2c_8(a^2-r^2)}{\pi a^3} - \frac{2c_9}{9\pi}\frac{[a^3-(a^2-r^2)^{\frac{3}{2}}]^2}{a^3 r^4} \\
&+ \frac{2c_9}{3\pi}\frac{a^3\sqrt{a^2-r^2}-(a^2-r^2)^2}{a^3 r^2} - \frac{\pi c_{10} r^2}{8a^3}
\end{aligned}
\tag{2.42}
$$

with similar expressions for $\tau''_{rr}(r,0)$ and $\tau''_{\theta\theta}(r,0)$.

for $r > a$:

$$
\begin{aligned}
w_r(r,0) = &-\frac{1-2\eta}{2\mu}\{c_{10}\frac{a}{r} - \frac{c_1+c_9}{9\pi}\frac{a^3}{r^3} + \frac{c_{14}}{\pi^2}[\frac{r}{a} + \frac{r^3}{a^3}\arcsin^2\frac{a}{r}] \\
&+ \frac{2c_{14}}{3\pi^2}\frac{r\sqrt{r^2-a^2}\arcsin\frac{a}{r}}{a^2} - \frac{8c_{14}}{3\pi^2}\frac{(r^2-a^2)^{\frac{3}{2}}\arcsin\frac{a}{r}}{a^2 r} \\
&+ \frac{2c_{10}+4c_{14}+5c_5}{12\pi}\frac{\sqrt{r^2-a^2}\arcsin\frac{a}{r}}{r} - 2c_{11}\frac{r}{a}\arcsin^2\frac{a}{r}\} \\
&+ \frac{1-\eta}{\mu}\{\frac{c_6 I_3(r)}{2\pi a^3} + \frac{c_7 I_4(r)}{6\pi a^3} + \frac{a c_7 I_8(r)}{3\pi^2} + \frac{a c_7 I_7(r)}{3\pi r^2}\} \\
w_z(r,0) = &-\frac{(1-2\eta)ac_7}{6\pi\mu}[\frac{\sqrt{r^2-a^2}}{2r^2} + \frac{\arcsin\frac{a}{r}}{2a} + I_2(r)] \\
&+ \frac{1-\eta}{\mu}\{\frac{2(c_{15}+4c_{11}-\pi^2 c_{11})}{\pi a r}[r^2 E(\frac{a}{r}) + (a^2-r^2)F(\frac{a}{r})] \\
&+ \frac{2(c_{14}+c_{16})}{9\pi a r^3}[(a^2+4r^2)(a^2-r^2)F(\frac{a}{r}) + (4r^4+a^2r^2)E(\frac{a}{r})] \\
&+ \frac{4(3c_1-c_9)}{9\pi^2 r}I_9(a) + \frac{4(c_1+c_2)}{3\pi^2 a r}I_{10}(a) - \frac{8(9c_1+c_9)a}{27\pi^2 r}I_{11}(a) \\
&+ \frac{4(c_1+c_9)a^3}{9\pi^2 r}I_{18}(r) - \frac{8c_{14}}{\pi^3 a r}I_{19}(r) - \frac{8c_{11}}{\pi a r}I_{20}(r) \\
&+ \frac{4(c_1+c_9)}{9\pi^2 r^3}I_{15}(a) - \frac{8c_{14}r}{\pi^3}I_{16}(a) - \frac{8c_{11}r}{\pi}I_{17}(a)\} \\
\tau''_{rz}(r,0) = &-\frac{c_7}{3\pi}\frac{r^2\arcsin\frac{a}{r} - a\sqrt{r^2-a^2}}{r^3} \\
\tau''_{zz}(r,0) = &-\frac{2c_9 a^3}{9\pi r^4} - \frac{c_{10}}{2\pi a}[\frac{r}{a}\arcsin\frac{a}{r} - \frac{\sqrt{r^2-a^2}}{r}]^2
\end{aligned}
\tag{2.43}
$$

with similar expressions for $\tau''_{rr}(r,0)$ and $\tau''_{\theta\theta}(r,0)$ and where I_j are listed in the Appendix A_3 and c_{ij} are listed in the appendix A_4.

In comparision to the linear solution given by (2.37) we note that expressions for displacement and stress in the second order theory are very complicated. In particular, the simple paraboloidal shape of linear elasticity, $(2.37)_3$ is completely changed to a new form as given by $(2.42)_1$ and $(2.43)_1$. Similarly, the shape of the deformed boundary, on $z = 0$, as compared to the linear theory, $(2.37)_2$, again is considerably changed in the second order theory (cf.$(2.41)_2$ and $(2.43)_2$).

2.5 Reduction to the Incompressible Case.

We now follow the limit process introduced by Rivlin(1953) to obtain results for isotropic incompressible materials. First we require a_2 and a_3 tend to infinity in such a manner that $(a_3 - 2a_2)$ remains finite. Moreover if we set

$$a_1 = -(C_1 + C_2)$$
$$a_5 = -(C_1 + 2C_2) \tag{2.44}$$

then the strain energy function W takes the Mooney's form

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{2.45}$$

where C_1 and C_2 are constants. On employing the above limiting process and setting $\eta = \frac{1}{2}$ we find that complete second order solution, for this particular case, simplifies to

For $r \leq a$

$$\begin{aligned} u_r(r,0) &= -\frac{Q^2}{2\pi}[\frac{I_3(r)}{a^3} + ar^2 I_5(r)] \\ u_z(r,0) &= -\frac{3P}{32a_1 a}[1 - \frac{r^2}{2a^2}] - \frac{Q^2}{4}\{\frac{3(4-\pi^2)}{\pi^2}E(\frac{r}{a}) \\ &\quad - \frac{13rI_{17}(r)}{\pi}\} \end{aligned}$$

For $r > a$

$$\begin{aligned} u_r(r,0) &= -\frac{I_3(a)Q^2}{2\pi a^3} \\ u_z(r,0) &= -\frac{3P}{32a_1 a\pi}[(2 - \frac{r^2}{a^2})\arcsin\frac{a}{r} + \frac{\sqrt{r^2 - a^2}}{a}] \\ &\quad - \frac{Q^2}{4}\{\frac{3(4-\pi^2)}{\pi^2 ar}[r^2 E(\frac{a}{r}) + (a^2 - r^2)F(\frac{a}{r})] \\ &\quad - \frac{12I_{20}(r)}{\pi^2 ar} - \frac{12rI_{17}(a)}{\pi^2}\} \end{aligned} \tag{2.46}$$

The stresses are given by

For $r \le a$

$$
\begin{aligned}
t_{rr}(r,0) &= -\frac{3P\sqrt{a^2-r^2}}{2\pi a^3} - \frac{2a_1 a Q^2 r I_5(r)}{\pi} - \frac{8a_1 Q^2 I_{22}(r)}{\pi a r} \\
&+ \frac{4a_1 Q^2(3I_{23}(r) - I_{24}(r))r}{\pi a} - \frac{2a_1 Q^2(I_3(r) + 6I_{21}(r))}{\pi a^3 r} \\
t_{zz}(r,0) &= -\frac{3P\sqrt{a^2-r^2}}{2\pi a^3} \\
t_{rz}(r,0) &= \frac{2a_1 Q^2 r\sqrt{a^2-r^2}}{a^3} \\
t_{\theta\theta}(r,0) &= -\frac{3P\sqrt{a^2-r^2}}{2\pi a^3} + \frac{\pi a_1 Q^2 r^2}{4a^3} + \frac{2a_1 Q^2(I_3(r) + 6I_{21}(r))}{\pi a^3 r} \\
&+ \frac{4a_1 Q^2 r(3I_{23}(r) - I_{24}(r))}{\pi a} + \frac{2a_1 Q^2 r I_5(r)}{\pi} - \frac{4a_1 Q^2 I_{22}(r)}{\pi a r}
\end{aligned}
$$

For $r > a$

$$
\begin{aligned}
t_{rr}(r,0) &= \frac{2a_1 Q^2(12I_{21}(a) - I_{24}(a))}{\pi a^3 r} - \frac{8a_1 Q^2 I_{22}(a)}{\pi a r} \\
t_{zz}(r,0) &= 0 \\
t_{rz}(r,0) &= 0 \qquad (2.47) \\
t_{\theta\theta}(r,0) &= \frac{a_1 Q^2}{4\pi r}\left[\frac{r}{a}\arcsin\frac{a}{r} - \frac{\sqrt{r^2-a^2}}{r}\right]^2 + \frac{2a_1 Q^2(I_3(a) + 6I_{21}(a))}{\pi a^3 r} \\
&- \frac{4a_1 Q^2 I_{22}(r)}{\pi a r}
\end{aligned}
$$

It is apparent that the expressions in the case of incompressible material are much simpler as compared to those for compressible material. In particular we note that while there is significant change in the displacement components, the second order solution has no effect on the normal stress t_{zz}, on $z = 0$, in the incompressible case. It should be remarked that Rivlin's method cannot be applied, as used in this thesis, by starting with (2.45). Known solutions for compressible material, however, can be specialized for incompressible material by the appropriate limiting process as illustrated above.

2.6 Numerical Results.

In order to show the second order effect, we now present some numerical solutions. In the following numerical calculations, the leading term is the solution to be found in linear elasticity and the remaining term represents the second order solution. We are interested in the z-direction displacement and stress.

For compressible material,the strain invariants, $\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$, can be written as

$$\begin{aligned} \bar{I}_1 &= 3 + 2e_{rr} \\ \bar{I}_2 &= 3 + 4e_{rr} + 2(e_{rr}e_{ss} - e_{rs}e_{rs}) \\ \bar{I}_3 &= \det(\delta_{rs} + 2e_{rs}) \\ &= 1 + 2e_{rr} + 2(e_{rr}e_{ss} - e_{rs}e_{rs}) + 8\det(e_{rs}) \end{aligned} \tag{2.48}$$

where

$$e_{rs} = \frac{1}{2}(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r} + \frac{\partial u_k}{\partial x_r}\frac{\partial u_k}{\partial x_s})$$

Using three other strain invariants, I_1^*, I_2^* and I_3^*, as constructed by Murnaghan(1937) we can rewrite $\bar{I}_1, \bar{I}_2, \bar{I}_3$ as

$$\begin{aligned} \bar{I}_1 &= 3 + 2I_1^* \\ \bar{I}_2 &= 3 + 4I_1^* + 4I_2^* \\ \bar{I}_3 &= 1 + 2I_1^* + 4I_2^* + 8I_3^* \end{aligned} \tag{2.49}$$

where

$$I_1^* = e_{rr}, \quad I_2^* = \frac{e_{rr}e_{ss} - e_{rs}e_{rs}}{2}, \quad I_3^* = \det(e_{rs})$$

The five elastic coefficients used by Murnaghan(1937) are λ, μ, l, m, n. The relationship between Murnaghan's and Rivlin's coefficients is given by Truesdell and Noll(1965) as

$$\begin{aligned} &a_1 = -\mu/2, \quad a_2 = (\lambda + 2\mu)/8 \\ &a_3 = m + \mu, \quad a_4 = -\mu/3 + l, \quad a_5 = n - \mu \end{aligned} \tag{2.50}$$

Foux(1962) gives following experimental data for iron

$$\mu = 8.26 \times 10^3 kg/mm^2$$
$$K = \lambda + 2\mu/3 = 17.0 \times 10^3 kg/mm^2 \qquad (2.51)$$

$$l/\mu = -1.6, \quad m/\mu = -10.1, \quad n/\mu = -22.7 \qquad (2.52)$$

Using (2.50), (2.51) and (2.52) we find

$$a_1/\mu = -0.5, \quad a_2/\mu = \frac{1}{6} + \frac{17}{8.26}, \quad a_3/\mu = -9.1$$
$$a_4/\mu = -(1.6 + \frac{1}{3}), \quad a_5/\mu = -23.7 \qquad (2.53)$$

and

$$\eta = \frac{862}{2963}$$

Using above values and denoting $\bar{r} = r/a$ we get following numerical results for the displacement and the normal stress in the z-direction:

$\bar{r}$	0.0	0.2	0.4	0.5
u_z/a	$0.2659\epsilon + 0.9217\epsilon^2$	$0.2606\epsilon + 1.4730\epsilon^2$	$0.2446\epsilon + 1.5632\epsilon^2$	$0.2327\epsilon + 1.6842\epsilon^2$
t_{zz}/μ	$-0.4775\epsilon + 1.1453\epsilon^2$	$-0.4678\epsilon + 0.9072\epsilon^2$	$-0.4376\epsilon + 0.7695\epsilon^2$	$-0.4135\epsilon + 0.6647\epsilon^2$

$\bar{r}$	0.8	0.85	$1.0 - 0$	$1.0 + 0$
u_z/a	$0.1808\epsilon + 2.4798\epsilon^2$	$0.1698\epsilon + 2.6887\epsilon^2$	$0.1330\epsilon + 5.0624\epsilon^2$	$0.1330\epsilon + 1.1170\epsilon^2$
t_{zz}/μ	$-0.2865\epsilon + 0.1805\epsilon^2$	$-0.2515\epsilon + 0.0658\epsilon^2$	$-0.6250\epsilon^2$	$0.7832\epsilon^2$

$\bar{r}$	2.0	8.0	20.0	100
u_z/a	$0.0580\epsilon + 0.7289\epsilon^2$	$0.0141\epsilon + 0.1772\epsilon^2$	$0.0056\epsilon + 0.0716\epsilon^2$	$0.0011\epsilon + 0.0149\epsilon^2$
t_{zz}/μ	$0.0783\epsilon^2$	$0.0003\epsilon^2$	0.0	0.0

where $\epsilon = P/(\mu a^2)$

From these tables we find that for compressible materials the second order effect is to enlarge z-direction displacement. The second order stress has,however, its direction opposite to the direction of the linear stress, and therefore it makes the total stress smaller in magnitude than the linear stress. We also find that the second order displacement and stress possess discontinuity at $r = a$.

The same calculations have been made for an incompressible material, such as a rubber-like material. In this case, we have

$$a_1 = -(C_1 + C_2), \qquad a_2 = -(C_1 + 2C_2)$$

and from the experiments of Haines and Wilson(1979), we have $C_1 = 0.179$ and $C_2 = 0.009$. These values give the following tables

$\bar{r}$	0.0	0.2	0.4	0.5
u_z/a	$0.1875\epsilon + 0.0975\epsilon^2$	$0.1837\epsilon + 0.0886\epsilon^2$	$0.1725\epsilon + 0.0872\epsilon^2$	$0.1641\epsilon + 0.0862\epsilon^2$
t_{zz}/μ	-0.4775ϵ	-0.4678ϵ	-0.4376ϵ	-0.4135ϵ

$\bar{r}$	0.8	0.85	$1.0 - 0$	$1.0 + 0$
u_z/a	$0.1275\epsilon + 0.0815\epsilon^2$	$0.1198\epsilon + 0.0805\epsilon^2$	$0.0938\epsilon + 0.0787\epsilon^2$	$0.0938\epsilon + 0.0442\epsilon^2$
t_{zz}/μ	-0.2865ϵ	-0.2515ϵ	0.0	0.0

$\bar{r}$	2.0	8.0	20.0	100.0
u_z/a	$0.0409\epsilon + 0.0183\epsilon^2$	$0.0100\epsilon + 0.0051\epsilon^2$	$0.0040\epsilon + 0.0021\epsilon^2$	$0.0008\epsilon + 0.0004\epsilon^2$
t_{zz}/μ	0.0	0.0	0.0	0.0

where $\epsilon = P/(\mu a^2)$

From the above tables we note that for incompressible material the second order effect also increases the z-direction displacement. The magnitude of increase is, however, much smaller as compared to the compressible case. In the incompressible case, there is no effect of the second order elasticity in the z-direction normal stress, but it affects the t_{rr} and $t_{\theta\theta}$ stress components. Also, displacement is not continuous at $r = a$.

Finally, we remark that for both compressible and incompressible materials, the parameter ϵ determines the magnitude of the second order elastic effect, that is, the more the total appplied force P the larger the second order effect and the greater the elastic constant μ the smaller the second order effect.

CHAPTER III

SECOND ORDER EFFECTS IN AN ELASTIC HALF-SPACE ACTED UPON BY A NON-UNIFORM SHEAR LOAD

3.1 Statement of the Problem.

In this Chapter we consider an elastic half-space in which a non-uniform shear load, of total magnitude P, is acting over a circle of radius a in the x-direction(see Fig.2). In classical elasticity, this problem of stress distribution within an elastic half-space when it is deformed by the uniform tangential force to the surface seems to have been considered first by Cerruti(1882). An alternative solution to this problem, using Hankel transform method, was also given by Muki(1960). Here we consider the second order problem with non-uniform tangential load. Again, we choose cylindrical polar coordinates (r, θ, z) such that the load is acting in the plane $z = 0$. The boundary conditions are

$$\begin{aligned} t_{zz} &= 0 \\ t_{rz} &= \frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2 - r^2)^\delta H(a-r)\cos\theta \\ t_{\theta z} &= -\frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2 - r^2)^\delta H(a-r)\sin\theta \end{aligned} \tag{3.1}$$

where constant $\delta > -1$. We assume that there are no body forces. For the linear solutions and second order solutions the problem can also be split into two subproblems:

(I) The Linear Solution: solve

$$\begin{aligned} \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{1}{r}\tau_{rz} &= 0 \end{aligned} \tag{3.2}$$

subject to

$$\begin{aligned}
\tau_{zz}|_{z=0} &= 0 \\
\tau_{rz}|_{z=0} &= \frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^{\delta}H(a-r)\cos\theta \\
\tau_{\theta z}|_{z=0} &= -\frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^{\delta}H(a-r)\sin\theta
\end{aligned} \tag{3.3}$$

(II) The Second Order Solution: solve

$$\begin{aligned}
\frac{\partial\tau''_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{r\theta}}{\partial\theta}+\frac{\partial\tau''_{rz}}{\partial z}+\frac{\tau_{rr}-\tau''_{\theta\theta}}{r}+\rho_0X'_r=0 \\
\frac{\partial\tau''_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{\theta\theta}}{\partial\theta}+\frac{\partial\tau''_{\theta z}}{\partial z}+\frac{2}{r}\tau''_{r\theta}+\rho_0x'_{\theta}=0 \\
\frac{\partial\tau''_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau''_{\theta z}}{\partial\theta}+\frac{\partial\tau''_{zz}}{\partial z}+\frac{1}{r}\tau''_{rz}+\rho_0X'_z=0
\end{aligned} \tag{3.4}$$

subject to

$$\tau''_{zz}|_{z=o}=-\bar{X}''_z,\quad \tau''_{rz}|_{z=0}=-\bar{X}''_r,\quad \tau''_{\theta z}|_{z=0}=-\bar{X}''_{\theta} \tag{3.5}$$

where body forces and surface tractions are listed in the Appendix A_1.

3.2 The Linear Solution.

For solving the subproblem (I) we use Muki's displacement solution

$$\mathbf{v}=\frac{1}{2\mu}\{2(1-\eta)\nabla^2\mathbf{G}-\nabla(\nabla\bullet\mathbf{G})+\nabla\times\mathbf{A}\} \tag{3.6}$$

where **G** is a biharmonic vector and **A** is a harmonic vector. Muki proposed single z components for both **G** and **A**

$$\mathbf{G}=(0,0,G_z(r,\theta,z)),\quad \mathbf{A}=(0,0,A_z(r,\theta,z)) \tag{3.7}$$

We select $G_z(r,\theta,z)=\phi(r,z)\cos\theta$ and $A_z(r,\theta,z)=\psi(r,z)\sin\theta$. Then the displacement components (v_r,v_θ,v_z) bocome

$$\begin{aligned}
v_r &= \frac{1}{2\mu}[-\frac{\partial^2\phi}{\partial r\partial z}+\frac{2\psi}{r}]\cos\theta \\
v_\theta &= \frac{1}{2\mu}[\frac{1}{r}\frac{\partial\phi}{\partial z}-2\frac{\partial\psi}{\partial r}]\sin\theta \\
v_z &= \frac{1}{2\mu}[2(1-\eta)\nabla_1^2\phi-\frac{\partial^2\psi}{\partial z^2}]\cos\theta
\end{aligned} \tag{3.8}$$

which satisfy (3.2) ,provided $\phi(r,z)$ and $\psi(r,z)$ satisfy, respectively,

$$\nabla_1^4\phi = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2})^2\phi = 0 \tag{3.9}$$

$$\nabla_1^2\psi = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2})\psi = 0 \tag{3.10}$$

The stress field corresponding to displacement field (3.8) can be written as

$$\begin{aligned}
\tau_{rr} &= [\frac{\partial(\eta\nabla_1^2\phi - \frac{\partial^2\phi}{\partial r^2})}{\partial z} + (\frac{2}{r}\frac{\partial\phi}{\partial r} - \frac{2\psi}{r^2})]\cos\theta \\
\tau_{\theta\theta} &= [\frac{\partial(\eta\nabla_1^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\phi}{r^2})}{\partial z} - (\frac{2}{r}\frac{\partial\psi}{\partial r} - \frac{2\psi}{r^2})]\cos\theta \\
\tau_{zz} &= [\frac{\partial((2-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2})}{\partial z}]\cos\theta \\
\tau_{\theta z} &= [-\frac{1}{r}((1-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2}) - \frac{\partial^2\psi}{\partial z\partial r}]\sin\theta \\
\tau_{rz} &= [\frac{\partial((1-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2})}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial z}]\cos\theta \\
\tau_{r\theta} &= [\frac{\partial^2\frac{\phi}{r}}{\partial z\partial r} - (2\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2})]\sin\theta
\end{aligned} \tag{3.11}$$

Use of (3.3) leads to

$$\begin{aligned}
&[\frac{\partial((2-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2})}{\partial z}]|_{z=0} = 0 \\
&[-\frac{1}{r}((1-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2}) - \frac{\partial^2\psi}{\partial z\partial r}]|_{z=0} = -\frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^\delta H(a-r) \\
&[\frac{\partial((1-\eta)\nabla_1^2\phi - \frac{\partial^2\phi}{\partial z^2})}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial z}]|_{z=0} = \frac{(1+\delta)P}{\pi a^{2(1+\delta)}}(a^2-r^2)^\delta H(a-r)
\end{aligned} \tag{3.12}$$

We now denote by

$$\bar{\phi} = \int_0^\infty rJ_1(\xi r)\phi(r,z)\,dr$$

$$\bar{\psi} = \int_0^\infty rJ_1(\xi r)\psi(r,z)\,dr$$

and take the first order Hankel transforms of (3.9) and (3.10), respectively, to obtain ordinary differntial equations for $\bar{\phi}(\xi,z)$ and $\bar{\psi}(\xi,z)$. Useful solution of these differential equations for our purpose are

$$\bar{\phi}(\xi,z) = (A_2 + A_3\xi z)e^{-\xi z}$$
$$\bar{\psi}(\xi,z) = A_1 e^{-\xi z} \tag{3.13}$$

where A_1, A_2, A_3 are arbitrary functions of ξ. On taking the first order Hankel transform of $(3.12)_1$, the second order Hankel transform of $(3.12)_2 + (3.12)_3$ and the zero order Hankel transform of $(3.12)_2 - (3.12)_3$ and solving the resulting equations for A_1, A_2, A_3 we find

$$A_1 = -\frac{T}{2}\frac{J_{1+\delta}(a\xi)}{\xi^{3+\delta}}$$
$$A_2 = -\frac{T}{2}\frac{(1-2\eta)J_{1+\delta}(a\xi)}{\xi^{4+\delta}} \tag{3.14}$$
$$A_3 = \frac{T}{2}\frac{J_{1+\delta})a\xi)}{\xi^{4+\delta}}$$

where $T = 2^{1+\delta}(1+\delta)\Gamma(1+\delta)P/(\pi a^{1+\delta})$.

We now take the following Hankel transforms of the stress and displacement functions: $H_1[\tau_{zz}/\cos\theta]$, $H_2[\tau_{rz}/\cos\theta + \tau_{\theta z}/\sin\theta]$, $H_0[\tau_{rz}/\cos\theta - \tau_{\theta z}/\sin\theta]$, $H_2[v_r/\cos\theta + v_\theta/\sin\theta]$, $H_0[v_r/\cos\theta - v_\theta/\sin\theta]$, $H_1[\tau_{rr}/\cos\theta + \tau_{\theta\theta}/\sin\theta]$,$H_1[\tau_{rr}/\cos\theta + 2\mu v_r/(r\cos\theta) + 2\mu v_\theta/(r\sin\theta)]$ and then using (3.13) and (3.14) on inverting the resulting equations we obtain

$$v_r = \frac{T}{4\mu}[-(2-\eta)L(0,-(1+\delta),z) - \eta L(2,-(1+\delta),z) + \frac{z}{2}L(0,-\delta,z)$$
$$- \frac{z}{2}L(2,-\delta,z)]\cos\theta$$
$$v_\theta = \frac{T}{4\mu}[(2-\eta)L(0,-(1+\delta),z) - \eta L(2,-(1+\delta),z) - \frac{z}{2}L(0,-\delta,z)$$
$$- \frac{z}{2}L(2,-\delta,z)]\sin\theta$$
$$v_z = -\frac{T}{4\mu}[(1-2\eta)L(1,-(1+\delta),z) + zL(1,-\delta,z)]\cos\theta$$

$$\tau_{rr} = T[\frac{\eta}{r}L(2,-(1+\delta),z) + \frac{z}{2r}L(2,-\delta,z) + L(1,-\delta,z) - \frac{z}{2}L(0,-\delta,z)$$
$$- \frac{z}{2}L(1,1-\delta,z)]\cos\theta$$
$$\tau_{\theta\theta} = T[-\frac{\eta}{r}L(2,-(1+\delta),z) - \frac{z}{2r}L(2,-\delta,z) - 3\eta L(1,-\delta,z)]\cos\theta$$
$$\tau_{zz} = T[\frac{z}{2}L(1,1-\delta,z)]\cos\theta \tag{3.15}$$
$$\tau_{rz} = T[-\frac{1}{2}L(0,-\delta,z) + \frac{z}{4}L(0,1-\delta,z) - \frac{z}{4}L(2,1-\delta,z)]\cos\theta$$
$$\tau_{\theta z} = T[\frac{1}{2}L(0,-\delta,z) - \frac{z}{4}L(0,1-\delta,z) - \frac{z}{4}L(2,1-\delta,z)]\sin\theta$$
$$\tau_{r\theta} = T[\frac{\eta}{r}L(2,-(1+\delta),z) - \frac{z}{2r}L(2,-\delta,z) - \frac{1}{2}L(1,-\delta,z)]\sin\theta$$

where $L(n,s,z)$ is defined as

$$L(n,s,z) = \int_0^\infty \xi^s J_n(\xi r) J_{1+\delta}(\xi a) e^{-\xi z}\, dr \tag{3.16}$$

Equations (3.15) and (3.16) give the displacement and stress components for the linear elasticity problem.

We here give the surface solutions. By denoting $L(n,s)$ for $L(n,s,0)$ we can write the linear displacement and stress components as:

$$v_r = -\frac{T}{4\mu}[(2-\eta)L(0,-(1+\delta)) + \eta L(2,-(1+\delta))]\cos\theta$$
$$v_\theta = \frac{T}{4\mu}[(2-\eta)L(0,-(1+\delta)) - \eta L(2,-(1+\delta))]\sin\theta$$
$$v_z = -\frac{T}{4\mu}(1-2\eta)L(1,-(1+\delta))\cos\theta$$
$$\tau_{rr} = T[\frac{\eta}{r}L(2,-(1+\delta)) + L(1,-\delta)]\cos\theta$$
$$\tau_{\theta\theta} = -T[\frac{\eta}{r}rL(2,-(1+\delta)) + 3\eta L(1,-\delta)]\cos\theta \tag{3.17}$$
$$\tau_{zz} = 0$$
$$\tau_{r\theta} = T[\frac{\eta}{r}L(2,-(1+\delta)) - \frac{1}{2}L(1,-\delta)]\sin\theta$$
$$\tau_{rz} = -\frac{T}{2}L(0,-\delta)\cos\theta$$
$$\tau_{\theta z} = \frac{T}{2}L(0,-\delta)\sin\theta$$

3.3 The Second Order Solution.

In order to solve the second order problem we are required to solve the subproblem (II). For the present, the additional forces and surface tractions may be written as

$$\begin{aligned}\rho_0 X'_r &= f_r^1(r,z) + f_r^2(r,z)\cos 2\theta \\ \rho_0 X'_\theta &= f_\theta(r,z)\cos\theta\sin\theta \\ \rho_0 X'_z &= f_z^1(r,z) + f_z^2(r,z)\cos 2\theta\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\bar{X}''_r &= -X_{\nu r}^1(r,z) - X_{\nu r}^2(r,z)\cos 2\theta \\ \bar{X}''_\theta &= -X_{\nu\theta}(r,z)\cos\theta\sin\theta \\ \bar{X}''_z &= -X_{\nu z}^1(r,z) - X_{\nu z}^2(r,z)\cos 2\theta\end{aligned} \tag{3.19}$$

We select displacement vector to be Garlerkin's solution plus an irrotational term

$$\mathbf{w} = \frac{1}{2\mu}\{2(1-\eta)\nabla^2\mathbf{G} - \nabla(\nabla \bullet \mathbf{G}) + \nabla\Psi\}$$

where

$$\begin{aligned}\mathbf{G} &= \{G_1(r,z)\cos\theta, G_2(r,z)\sin\theta, G_3(r,z)\cos 2\theta + G_4(r,z)\} \\ \Psi &= (1-2\eta)\mu\Phi(r,z)\cos 2\theta\end{aligned} \tag{3.20}$$

With this choice, the displacement components become

$$\begin{aligned}2\mu w_r &= 2(1-\eta)[\nabla_1^2 G_1\cos^2\theta + \nabla_1^2 G_2\sin^2\theta] - \frac{\partial G_0}{\partial r} - \frac{\partial^2 G_4}{\partial r\partial z} - \frac{\partial^2 G_3}{\partial r\partial z}\cos 2\theta \\ &+ (1-2\eta)\mu\frac{\partial\Phi}{\partial r}\cos 2\theta \\ 2\mu w_\theta &= 2(1-\eta)[\nabla_1^2 G_2 - \nabla_1^2 G_1]\cos\theta\sin\theta - \frac{1}{r}\frac{\partial G_0}{\partial\theta} + \frac{2}{r}\frac{\partial G_3}{\partial z}\sin 2\theta \\ &- (1-2\eta)\mu\frac{2\Phi}{r}\sin 2\theta \\ 2\mu w_z &= -\frac{\partial G_0}{\partial z} + [2(1-\eta)\nabla_0^2 G_4 - \frac{\partial^2 G_4}{\partial z^2}] + [2(1-\eta)\nabla_2^2 G_3 - \frac{\partial^2 G_3}{\partial z^2}]\cos 2\theta \\ &+ (1-2\eta)\mu\frac{\partial\Phi}{\partial z}\cos 2\theta\end{aligned} \tag{3.21}$$

where

$$G_0 = \frac{1}{2}[\frac{\partial G_1}{\partial r} + \frac{G_1}{r} + \frac{\partial G_2}{\partial r} + \frac{G_2}{r}] + \frac{1}{2}[\frac{\partial G_1}{\partial r} - \frac{G_1}{r} - \frac{\partial G_2}{\partial r} + \frac{G_2}{r}]\cos 2\theta$$

$$\nabla_n^2 = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} + \frac{\partial^2}{\partial z^2}) \tag{3.22}$$

The stresses are given by

$$\tau''_{rr} = \frac{\eta}{2}g_0 + (1+\eta)\left[\frac{\partial[\nabla_1^2 G_1 + \nabla_2^2 G_2]}{\partial r} + \frac{\partial[\nabla_1^2 - \nabla_1^2 G_2]}{\partial r}\cos 2\theta\right] - \frac{\partial^2 G_0}{\partial r^2}$$
$$+ \frac{\partial[\eta\nabla_0^2 G_4 - \frac{\partial^2 G_4}{\partial r^2}]}{\partial z} + \frac{\partial[\eta\nabla_2^2 G_3 - \frac{\partial^2 G_3}{\partial r^2}]}{\partial z}\cos 2\theta$$
$$+ [\mu\eta\nabla_2^2\Phi + (1-2\eta)\mu\frac{\partial^2\Phi}{\partial r^2}]\cos 2\theta$$

$$\tau''_{\theta\theta} = \frac{\eta}{2}g_0 + (1+\eta)\left[\frac{\nabla_1^2 G_1 + \nabla_2^2 G_2}{r} + \frac{\nabla_1^2 G_2 - \nabla_1^2 G_1}{r}\cos 2\theta\right] - \left[\frac{1}{r}\frac{\partial G_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G_0}{\partial \theta^2}\right]$$
$$+ \frac{\partial[\eta\nabla_0^2 G_4 - \frac{1}{r}\frac{\partial G_4}{\partial r}]}{\partial z} + \frac{\partial[\eta\nabla_2^2 G_3 - \frac{1}{r}\frac{\partial G_3}{\partial r} + \frac{4}{r^2}G_3]}{\partial z}\cos 2\theta$$
$$+ \left[\mu\eta\nabla_2^2\Phi + \frac{(1-2\eta)\mu}{r}(\frac{\partial\Phi}{\partial r} - \frac{4\Phi}{r})\right]\cos 2\theta$$

$$\tau''_{zz} = \frac{\eta}{2}g_0 - \frac{\partial^2 G_0}{\partial z^2} + \frac{\partial[(2-\eta)\nabla_0^2 G_4 - \frac{\partial^2 G_4}{\partial z^2}]}{\partial z} + \frac{\partial[(2-\eta)\nabla_2^2 G_3 - \frac{\partial^2 G_3}{\partial z^2}]}{\partial z}\cos 2\theta$$
$$+ \left[\mu\eta\nabla_2^2\Phi + (1-2\eta)\mu\frac{\partial^2\Phi}{\partial z^2}\right]\cos 2\theta$$

$$\tau''_{rz} = \frac{1-\eta}{2}\left[\frac{\partial[\nabla_1^2 G_1 + \nabla_1^2 G_2]}{\partial z} + \frac{\partial[\nabla_1^2 G_1 - \nabla_1^2 G_2]}{\partial z}\cos 2\theta\right] - \frac{\partial^2 G_0}{\partial r\partial z}$$
$$+ \frac{\partial[(1-\eta)\nabla_0^2 G_4 - \frac{\partial^2 G_4}{\partial z^2}]}{\partial r} + \frac{\partial[(1-\eta)\nabla_2^2 G_3 - \frac{\partial^2 G_3}{\partial z^2}]}{\partial r}\cos 2\theta$$
$$+ (1-2\eta)\mu\frac{\partial^2\Phi}{\partial r\partial z}\cos 2\theta$$

$$\tau''_{\theta z} = (1-\eta)\frac{\partial[\nabla_1^2 G_2 + \nabla_1^2 G_1]}{\partial z}\cos\theta\sin\theta + \frac{2}{r}\frac{\partial[\frac{\partial G_1}{\partial r} - G_1 - \frac{\partial G_2}{\partial r} + G_2]}{\partial z}\cos\theta\sin\theta$$
$$- \frac{4}{r}\left[(1-\eta)\nabla_0^2 G_3 - \frac{\partial^2 G_3}{\partial z^2}\right]\cos\theta\sin\theta - \frac{4(1-2\eta)\mu}{r}\frac{\partial\Phi}{\partial r}\cos\theta\sin\theta$$

$$\tau''_{r\theta} = (1-\eta)\left[\frac{\nabla_1^2 G_2 - \nabla_1^2 G_1}{r} + \frac{\partial[\nabla_1^2 G_1 - \nabla_1^2 G_2]}{\partial r}\right]\cos\theta\sin\theta - \frac{1}{r}\frac{\partial^2 G_0}{\partial r\partial\theta} + \frac{1}{2r^2}\frac{\partial G_0}{\partial\theta}$$
$$- \frac{4}{r}\frac{\partial[\frac{G_3}{r} - \frac{\partial G_3}{\partial r}]}{\partial z}\cos\theta\sin\theta + \frac{4(1-2\eta)\mu}{r}\left[\frac{\Phi}{r} - \frac{\partial\Phi}{\partial r}\right]\cos\theta\sin\theta \tag{3.23}$$

where

$$g_0 = \frac{\partial[\nabla_1^2 G_1 + \nabla_1^2 G_2]}{\partial r} + \frac{\nabla_1^2 G_1 - \nabla_1^2 G_2}{r} + \{\frac{\partial[\nabla_1^2 G_1 - \nabla_1^2 G_2]}{\partial r} + \frac{\nabla_1^2 G_2 - \nabla_1^2 G_1}{r}\}\cos 2\theta \tag{3.24}$$

On substituting (3.18) and (3.23) into (3.4) and rearranging the terms we find

$$\nabla_1^4(G_1 + G_2) = -2f_r^1 \tag{3-25}$$

$$\frac{\partial \nabla_2^2 \Phi}{\partial r} - \frac{2}{r}\nabla_2^2 \Phi = -\frac{2f_r^2 + f_\theta}{2} \tag{3.26}$$

$$\nabla_1^4 G_1 = \frac{f_\theta - 2f_r^1 - (4/r)\nabla_2^2\Phi}{2} \tag{3.27}$$

$$\nabla_0^4 G_4 = -f_z^1 \tag{3.28}$$

$$\nabla_2^4 G_3 + \frac{\partial \nabla_2^2 \Phi}{\partial z} = -f_z^2 \tag{3.29}$$

We now take the third order Hankel transform of both sides of equation (3.26) and obtain

$$(\frac{d^2}{dz^2} - \xi^2)\bar{\Phi} = \frac{1}{\xi}\int_0^\infty rJ_3(\xi r)\frac{2f_r^2 + f_\theta}{2}\,dr \triangleq \phi(\xi, z) \tag{3.30}$$

where

$$\bar{\Phi} = \int_0^\infty rJ_2(\xi r)\Phi(\xi, r)\,dr$$

From (3.30) we find that appropriate solution of (3.26) is

$$\Phi = H_2[(A + \phi^*)e^{-\xi z}; \xi \to r] \tag{3.31}$$

where

$$\phi^* = \int_0^z e^{2\xi z}\int_0^{z_2} \phi(\xi, z_1)e^{-\xi z_1}\,dz_1\,dz_2$$

and A is an arbitrary function of ξ. From equation (3.27), we find

$$(\frac{d^2}{dz^2} - \xi^2)^2 H_1[G_1] = \int_0^\infty rJ_1(\xi r)\frac{f_\theta - 2f_r^1 - (4/r)\nabla_2^2\Phi}{2}\,dr \triangleq g_1(\xi, z) \tag{3.32}$$

The appropriate solution of (3.27) is given as

$$G_1 = H_1[(A_1\xi z + g_1^*)e^{-\xi z}; \xi \to r] \tag{3.33}$$

where

$$g_1^*(\xi, z) = \frac{1}{2\xi}\int_0^z e^{2\xi z_2}\int_0^{z_2}(2z_2 - z - z_1)g_1(\xi, z_1)e^{-\xi z_1}\,dz_1\,dz_2 \tag{3.34}$$

and A_1 is arbitrary function of ξ.

In a similar manner it can be shown that

$$\begin{aligned} G_2 &= H_1[(A_2\xi z + g_2^*)e^{-\xi z}; \xi \to r] \\ G_3 &= H_2[(A_3\xi z + g_3^*)e^{-\xi z}; \xi \to r] \\ G_4 &= H_0[(A_4\xi z + g_4^*)e^{-\xi z}; \xi \to r] \end{aligned} \tag{3.35}$$

where A_2, A_3, A_4 are arbitrary functions of ξ, and

$$g_i^*(\xi, z) = \frac{1}{2\xi}\int_0^z e^{2\xi z_2}\int_0^{z_2}(2z_2 - z - z_1)g_i(\xi, z_1)e^{-\xi z_1}\,dz_1\,dz_2 \quad i = 2,3,4 \tag{3.36}$$

$$\begin{aligned} g_2(\xi, z) &= \int_0^\infty rJ_1(\xi r)\frac{-2f_r^1 - f_\theta + (4/r)\nabla_2^2\Phi}{2}\,dr \\ g_3(\xi, z) &= -\int_0^\infty rJ_2(\xi r)[f_z^2 + \frac{\partial(\nabla_2^2\Phi)}{\partial z}]\,dr \\ g_4(\xi, z) &= -\int_0^\infty rJ_0(\xi r)f_z^1\,dr \end{aligned} \tag{3.37}$$

After having determined the solutions for Φ, G_1 to G_4, we now need to determine the arbitrary functions A, A_1 to A_4. This is accomplished by substituting the displacement components in the stress components and then using boundary condition

(3.5). After considerable algebraic manipulations we get

$$
\begin{aligned}
A &= \frac{1}{(1-2\eta)\mu\xi^2}\left[\frac{(9\eta-8\eta^2)h_5}{4-5\eta+2\eta^2}+\frac{(3-4\eta)h_2+(1+2\eta-4\eta^2)h_3}{2(4-5\eta+2\eta^2)}\right] \\
A_1 &= \frac{1}{\xi^3}\left[\frac{(1-2\eta)h_1+2\eta h_4}{3-4\eta}+\frac{(1-\eta)h_2+h_3+2\eta h_5}{2(4-5\eta+2\eta^2)}\right] \\
A_2 &= \frac{1}{\xi^3}\left[\frac{(1-2\eta)h_1+2\eta h_4}{3-4\eta}-\frac{(1-\eta)h_2+h_3+2\eta h_5}{2(4-5\eta+2\eta^2)}\right] \qquad (3.38) \\
A_3 &= \frac{1}{\xi^3}\left[\frac{4(1-\eta)h_5}{4-5\eta+2\eta^2}+\frac{(1-2\eta)h_3-(3-2\eta)h_2}{2(4-5\eta+2\eta^2)}\right] \\
A_4 &= \frac{(3-2\eta)h_4-2(1-\eta)h_1}{(3-4\eta)\xi^3}
\end{aligned}
$$

where

$$
\begin{aligned}
h_1 &= \int_0^\infty rJ_1(\xi r)X^1_{\nu r}\,dr \\
h_2 &= \int_0^\infty rJ_3(\xi r)(2X^2_{\nu r}+X_{\nu\theta})\,dr \\
h_3 &= \int_0^\infty rJ_1(\xi r)(2X^2_{\nu r}-X_{\nu\theta})\,dr \qquad (3.39) \\
h_4 &= \int_0^\infty rJ_0(\xi r)X^1_{\nu z}\,dr \\
h_5 &= \int_0^\infty rJ_2(\xi r)X^2_{\nu z}\,dr-\mu(1-\eta)\phi(\xi,0)
\end{aligned}
$$

With the solutions for **G** and Ψ known we can write down the complete second order solutions from equations (3.21) to (3.24).

On the surface of the half-space,the second order displacement and stress components can be written as

$$
\begin{aligned}
2\mu w_r = & -2(1-\eta)\int_0^\infty xX^1_{\nu r}K_{11}(0,x)\,dx + (1-2\eta)\int_0^\infty xX^1_{\nu z}K_{10}(0,x)\,dx + \\
& \frac{\cos 2\theta}{4-5\eta+2\eta^2}\left\{\int_0^\infty x(2X^2_{\nu r}+X_{\nu\theta})[(1+\eta-2\eta^2)K_{13}(0,x)+\frac{2\eta}{r}K_{33}(-1,x)]\,dx\right. \\
& -\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})[2(1-\eta)^2K_{11}(0,x)+\frac{2(1-2\eta)^2}{r}K_{21}(-1,x)]\,dx \\
& -\int_0^\infty xX^2_{\nu z}[(4+9\eta+4\eta^2)K_{12}(0,x)+\frac{2(4+5\eta-8\eta^2)}{r}K_{22}(-1,x)]\,dx \\
& +\mu(1-\eta)\int_0^\infty x\frac{2f_r^2+f_\theta}{2}[(4+9\eta+\eta^2)K_{13}(-1,x) \\
& \left.+\frac{2(4+5\eta-8\eta^2)}{r}K_{23}(-2,x)]\,dx\right\}
\end{aligned}
$$

$$
\begin{aligned}
2\mu w_\theta = & \frac{2\sin\theta}{4-5\eta+2\eta^2}\left\{\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})[(1-\eta)^2K_{13}(0,x)-\frac{3(1-\eta)}{r}K_{23}(-1,x)]\,dx\right. \\
& +\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})[(1-\eta)K_{11}(0,x)\frac{2\eta(1-\eta)}{r}K_{21}(-1,x)]\,dx \\
& +\int_0^\infty xX^2_{\nu z}[2\eta(1-\eta)K_{12}(0,x)+\frac{4-13\eta+8\eta^2}{r}K_{22}(-1,x)]\,dx \\
& \left.-\mu(1-\eta)\int_0^\infty x\frac{f_r^2-f_\theta}{2}[2\eta(1-\eta)K_{13}(-1,x)+\frac{4-13\eta+8\eta^2}{r}K_{23}(-2,x)]\,dx\right\}
\end{aligned}
$$

$$
\begin{aligned}
2\mu w_z = & (1-2\eta)\int_0^\infty xX^1_{\nu r}K_{01}(0,x)\,dx - 2(1-\eta)\int_0^\infty xX^1_{\nu z}K_{00}(0,x)\,dx \\
& +\frac{\cos 2\theta}{4-5\eta+2\eta^2}\left\{\frac{10-21\eta+8\eta^2}{2}\int_0^\infty xX^2_{\nu r}+X_{\nu\theta})K_{23}(0,x)\,dx\right\} \\
& +(5\eta-6\eta^2)\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})K_{12}(0,x)\,dx \qquad (3.40) \\
& -(8-34\eta+24\eta^2)[\int_0^\infty xX^2_{\nu z}K_{22}(0,x)\,dx \\
& -\mu(1-\eta)\int_0^\infty x\frac{f_r^2+f_\theta}{2}K_{23}(-1,x)\,dx]
\end{aligned}
$$

$$
\begin{aligned}
\tau''_{rr} = & -\frac{3X^1_{\nu z}}{3-4\eta} - \frac{2(1-2\eta)}{3-4\eta}\int_0^\infty \frac{d(xX^1_{\nu r})}{dx}K_{00}(0,x)\,dx + \\
& \frac{1}{r}[\frac{3+2\eta-4\eta^2}{3-4\eta}\int_0^\infty xX^1_{\nu z}K_{10}(0,x)\,dx - \frac{4(1-\eta)}{3-4\eta}\int_0^\infty xX^1_{\nu r}K_{11}(0,x)\,dx] \\
& + \frac{\cos 2\theta}{4-5\eta+2\eta^2}\Big\{(1+5\eta-3\eta^2)[4\int_0^\infty (2X^2_{\nu r}-X_{\nu\theta})K_{02}(0,x)\,dx \\
& - \int_0^\infty \frac{d(x(2X^2_{\nu r}+X_{\nu\theta}))}{dx}K_{00}(0,x)\,dx] \\
& - 2(1-\eta)\int_0^\infty \frac{d(x(2X^2_{\nu r}-X_{\nu\theta}))}{dx}K_{00}(0,x)\,dx + (-4+\eta+6\eta^2)\left[-X^2_{\nu z}\right. \\
& + 2\int_0^\infty X^2_{\nu z}K_{01}(0,x)\,dx - \mu(1-\eta)\int_0^\infty x\frac{2f_r^2+f_\theta}{2}K_{03}(0,x)\,dx] \\
& + \frac{1}{r}\int_0^\infty x(2X^2_{\nu r}+X_{\nu\theta})[(7-3\eta+8\eta^2)K_{13}(0,x) + \frac{18(1-\eta)}{r}K_{23}(-1,x)]\,dx \\
& + \frac{1}{r}\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})[2(1-4\eta+\eta^2)K_{11}(0,x) + \frac{12(1-\eta)}{r}K_{21}(-1,x)]\,dx \\
& + \frac{1}{r}\int_0^\infty xX^2_{\nu z}[(16-31\eta+8\eta^2)K_{12}(0,x) - \frac{6(4-13\eta+8\eta^2)}{r}K_{22}(-1,x)]\,dx \\
& - \frac{\mu(1-\eta)}{r}\int_0^\infty x\frac{2f_r^2+f_\theta}{2}[(16-31\eta+8\eta^2)K_{13}(-1,x) \\
& - \frac{6(4-13\eta+8\eta^2)}{r}K_{23}(-1,x)]\,dx + \mu\eta r^2\int_r^\infty \frac{2f_r^2+f_\theta}{2x^2}\,dx\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\tau''_{r\theta} = & \frac{\cos\theta\sin\theta}{4-5\eta+2\eta^2}\Big\{2(1-\eta)^2\int_0^\infty [3(2X^2_{\nu r}+X_{\nu\theta}) + x\frac{d(2X^2_{\nu r}+X_{\nu\theta})}{dx}]K_{22}(0,x)\,dx \\
& + 2(1-\eta)\int_0^\infty [2X^2_{\nu r}-X_{\nu\theta} - x\frac{d(2X^2_{\nu r}-X_{\nu\theta})}{dx}]K_{22}(0,x)\,dx + 4\eta(1-\eta)X^2_{\nu z} \\
& + \frac{4}{r}\int_0^\infty x(2X^2_{\nu r}+X_{\nu\theta})[-2(1-\eta)K_{13}(0,x) + \frac{9(1-\eta)}{r}K_{23}(-1,x)]\,dx \\
& + \frac{4}{r}\int_0^\infty x(2X^2_{\nu r}-X_{\nu\theta})[-2\eta(1-\eta)K_{11}(0,x) + \frac{6\eta(1-\eta)}{r}K_{21}(-1,x)]\,dx \\
& + \frac{4}{r}\int_0^\infty xX^2_{\nu z}[(4-13\eta+8\eta^2)K_{12}(0,x) - \frac{3(4-13\eta+8\eta^2)}{r}K_{23}(-1,x)]\,dx \\
& - \frac{4\mu(1-\eta)}{r}\int_0^\infty x\frac{2f_r^2+f_\theta}{2}[(4-143\eta+8\eta^2)K_{13}(-1,x) \\
& - \frac{3(4-13\eta+8\eta^2)}{r}K_{23}(-2,x)]\,dx - 4\mu\eta(1-\eta)^2\int_0^\infty x\frac{2f_r^2+f_\theta}{2}K_{23}(0,x)\,dx\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\tau''_{\theta\theta} &= \frac{2\eta}{3-4\eta}\left[X^1_{\nu z} - \int_0^\infty \frac{d(xX^1_{\nu r})}{dx} K_{00}(0,x)\,dx\right] \\
&+ \frac{1}{r}\left[\frac{3-10\eta+8\eta^2}{3-4\eta}\int_0^\infty xX^1_{\nu z}K_{10}(0,x)\,dx - 2(1-\eta)\int_0^\infty xX^1_{\nu r}K_{11}(0,x)\,dx\right] + \\
&\frac{\cos 2\theta}{4-5\eta+2\eta^2}\left\{(-4\eta+3\eta^2)\int_0^\infty [3(2X^2_{\nu r}+X_{\nu\theta}) + x\frac{d(2X^2_{\nu r}+X_{\nu\theta})}{dx}]K_{22}(0,x)\,dx\right. \\
&- 2\eta^2\int_0^\infty [2X^2_{\nu r} - X_{\nu\theta} - x\frac{d(2X^2_{\nu r}-X_{\nu\theta})}{dx}]K_{22}(0,x)\,dx \\
&+ (8\eta-9\eta^2)X^2_{\nu z} - \mu(1-\eta)(8\eta-9\eta^2)\int_0^\infty x\frac{2f^2_r+f_\theta}{2}K_{23}(0,x)\,dx \\
&+ \frac{1}{r}\int_0^\infty x(2X^2_{\nu r}+X_{\nu\theta})[(5-7\eta+2\eta^2)K_{13}(0,x) - \frac{18(1-\eta)}{r}K_{23}(-1,x)]\,dx \\
&+ \frac{1}{r}\int_0^\infty x(2X^2_{\nu r}+X\nu\theta)[2(1-\eta^2)K_{11}(0,x) - \frac{12\eta(1-\eta)}{r}K_{21}(-1,x)]\,dx \\
&+ \frac{1}{r}\int_0^\infty xX^2_{\nu z}[(-4+17\eta-12\eta^2)K_{12}(0,x) + \frac{6(4-13\eta+8\eta^2)}{r}K_{22}(-1,x)]\,dx \\
&- \frac{\nu(1-\eta)}{r}\int_0^\infty x\frac{2f^2_r+f_\theta}{2}[(-4+17\eta-12\eta^2)K_{13}(-1,x) \\
&\left.+\frac{6(4-13\eta+8\eta^2)}{r}K_{23}(-2,x)]\,dx + \mu\eta r^2\int_r^\infty \frac{2f^2_r+f_\theta}{2x^2}\,dx\right\}
\end{aligned}
$$

$$
\begin{aligned}
\tau''_{zz} &= X^1_{\nu z} + X^2_{\nu z}\cos 2\theta \\
\tau''_{rz} &= X^1_{\nu r} + X^2_{\nu r}\cos 2\theta \qquad (3.41) \\
\tau''_{\theta z} &= X_{\nu\theta}\cos\theta\sin\theta
\end{aligned}
$$

where

$$
\begin{aligned}
X^1_{\nu r} &= \frac{T^2}{8\mu}[\frac{2\eta(1-2\eta)}{r^2}L(1,-(1+\delta))L(2,-(1+\delta)) \\
&- \frac{\eta(1-2\eta)}{r}L(0,-\delta)L(2,-(1+\delta)) + \frac{1-2\eta}{r}L(1,-\delta)L(1,-(1+\delta)) \\
&+ \eta L(1,-\delta)L(0,-\delta)] - \tau^1_{rz} \\
X^2_{\nu r} &= \frac{T^2}{8\mu}[\frac{3(1-2\eta)}{2r}L(1,-\delta)L(1,-(1+\delta)) - \frac{\eta(1-2\eta)}{r}L(0,\delta)L(2,-(1+\delta)) \\
&- (2-3\eta)L(1,-\delta)L(0,-\delta)] - \tau^2_{rz}
\end{aligned}
$$

$$
\begin{aligned}
X^1_{\nu z} &= \frac{(1-2\eta)T^2}{16\mu} I^2(1,-\delta) - \tau^1_{zz} \\
X^2_{\nu z} &= \frac{T^2}{8\mu}[\frac{1-2\eta}{2} I^2(0,-\delta) - \frac{1-2\eta}{2} L(0,-\delta)L(1,-(1+\delta))] - \tau^2_{zz} \\
X_{\nu\theta} &= \frac{T^2}{4\mu}[\frac{3-4\eta}{2} L(0,-\delta)L(1,-\delta) - \frac{\eta(1-\eta)}{r} L(0,-\delta)L(2,-(1+\delta)) \\
&\quad - \frac{(1-2\eta)(1+6\eta)}{2r} L(1,-\delta)L(1,-(1+\delta))] - \tau^1_{\theta z} \\
\tau^1_{rz} &= \frac{T^2}{16\mu^2}[b_1 L(0,-\delta)L(1,-\delta) + b_2 L(1,-\delta)L(2,-\delta) \\
&\quad + \frac{b_3}{r} L(1,-\delta)L(1,-(1+\delta)) + \frac{b_4}{r^2} L(1,-(1+\delta))L(2,-(1+\delta)) \\
&\quad + \frac{b_5}{r} L(0,-\delta)L(2,-(1+\delta)) + \frac{b_6}{r} L(2,-\delta)L(2,-(1+\delta))] \\
\tau^2_{rz} &= \frac{T^2}{16\mu^2}[b_7 L(1,-\delta)L(0,-\delta) + b_8 L(1,-\delta)L(2,-\delta) \\
&\quad + \frac{b_9}{r} L(1,-\delta)L(1,-(1+\delta)) + \frac{b_{10}}{r} L(0,-\delta)L(2,-(1+\delta))] \qquad (3.42) \\
\tau^1_{zz} &= \frac{T^2}{16\mu^2}[b_{11} I^2(1,-\delta) b_{12} I^2(0,-\delta) + \frac{b_{13}}{r^2} I^2(1,-(1+\delta)) \\
&\quad + \frac{b_{14}}{r^2} I^2(2,-(1+\delta)) + \frac{b_{15}}{r} L(0,-\delta)L(1,-(1+\delta)) \\
&\quad + \frac{b_{16}}{r} L(1,-\delta)L(2,-(1+\delta)) + b_{17} I^2(2,-\delta) \\
&\quad + \frac{b_{18}}{r} L(2,-\delta)L(1,-(1+\delta)) b_{19} L(0,-\delta)L(2,-\delta)] \\
\tau^2_{zz} &= \frac{T^2}{16\mu^2}[b_{20} I^2(1,-\delta) + b_{21} I^2(0,-\delta) + \frac{b_{22}}{r} L(0,-\delta)L(1,-(1+\delta)) \\
&\quad + b_{23} L(1,-\delta)L(2,-(1+\delta)) + b_{24} L(0,-\delta)L(2,-\delta)] \\
\tau^1_{\theta z} &= \frac{T^2}{8\mu^2}[\frac{b_{25}}{r} L(1,-\delta)L(1,-(1+\delta)) + b_{26} L(1,-\delta)L(0,-\delta) \\
&\quad + b_{27} L(1,-\delta)L(2,-\delta) + \frac{b_{28}}{r} L(0,-\delta)L(2,-(1+\delta)) \\
&\quad + \frac{b_{29}}{r^2} L(1,-(1+\delta))L(2,-(1+\delta))]
\end{aligned}
$$

$$f_r^2 = \frac{T^2}{16\mu^2}[b_{30}L(0,-\delta)L(0,-\delta) + \frac{b_{32}}{r^2}L(1,-\delta)L(2,-(1+\delta))$$
$$+ \frac{b_{33}}{r}L(0,1-\delta)L(2,-(1+\delta)) + b_{34}L(0,-\delta)L(1,1-\delta)$$
$$+ b_{35}L(2,-\delta)L(1,1-\delta) + \frac{b_{36}}{r}L(1,-(1+\delta))L(1,1-\delta)$$
$$+ \frac{b_{37}}{r}L(0,-\delta)L(2,-\delta) + \frac{b_{38}}{r^2}L(0,-\delta)L(1,-(1+\delta))$$
$$+ b_{39}L(1,-\delta)L(2,1-\delta) + \frac{b_{40}}{r^2}L(2,-\delta)L(1,-(1+\delta)) + \frac{b_{41}}{r}I^2(0,-\delta)]$$
$$f_\theta = \frac{T^2}{8\mu^2}[b_{42}L(1,-\delta)L(0,1-\delta) + \frac{b_{43}}{r}I^2(0,-\delta) + b_{44}L(0,-\delta)L(1,1-\delta)$$
$$+ b_{45}L(2,-\delta)L(1,1-\delta) + \frac{b_{46}}{r}I^2(2,-\delta) + \frac{b_{47}}{r}L(0,1-\delta)L(2,-(1+\delta))$$
$$+ \frac{b_{48}}{r^2}L(1,-\delta)L(2,-(1+\delta)) + \frac{b_{49}}{r}L(1,1-\delta)L(1,-(1+\delta))$$
$$+ b_{50}L(1,-\delta)L(2,-\delta) + \frac{b_{51}}{r}L(0,-\delta)L(2,-\delta)$$
$$+ \frac{b_{52}}{r^2}L(2,-\delta)L(1,-(1+\delta)) + \frac{b_{53}}{r}I^2(0,-\delta) + \frac{b_{54}}{r^2}L(0,-\delta)L(1,-(1+\delta))] \tag{3.43}$$

where we have following relations

$$\tau'_{rz} = \tau^1_{rz} + \tau^2_{rz}\cos 2\theta, \quad \tau'_{\theta z} = \tau^1_{\theta z} + \tau^2_{\theta z}\cos\theta\sin\theta, \quad \tau'_{zz} = \tau^1_{zz} + \tau^2_{zz}\cos 2\theta$$

With similar expressions for $\tau'_{rr}, \tau'_{\theta\theta}, \tau'_{r\theta}, f^1_r, f^1_z$ and f^2_z. We remark that the quantities $K_{ij}(s,x)$ and b_{ij} are listed in Appendix A_2 and Appendix A_6 respectively.

3.4 Illustration.

The solutions presented above are applicable for all value of $\delta > -1$,but are very complicated. As illustrations of the method we give below the linear and second order solutions for the specific values of δ.

Linear Case.

(i) Point Force: We first check our results for Cerruti problem. On recalling that

$$\lim_{a\to 0} TL(n,s,z) = \frac{P}{\pi}\int_0^\infty J_n(\xi r)\xi^{(s+\delta+1)}e^{-\xi z}\,d\xi$$

we get

$$v_r = -\frac{P}{4\pi\mu R}[\frac{r^2}{R^2} + \frac{r^2}{(R+z)^2} + \frac{(3-2\eta)z}{(R+z)^2}]\cos\theta$$

$$v_\theta = \frac{P}{4\pi\mu R}[\frac{2(1-\eta)r^2}{(R+z)^2} + \frac{3(2-\eta)z}{(R+z)^2}]\sin\theta$$

$$v_z = -\frac{P}{4\pi\mu R}[\frac{rz}{R^2} + \frac{(1-2\eta)r}{(R+z)}]\cos\theta$$

$$\tau_{rr} = \frac{P}{2\pi R^3}[\frac{2\eta r^3 - (1-2\eta)rz}{(R+z)^2} + \frac{4rz+2Rr}{R+z} - \frac{3rz^2}{R^2}]\cos\theta$$

$$\tau_{\theta\theta} = \frac{P}{2\pi R^3}[\frac{(1-2\eta)rz^2 - 2\eta r^3}{(R+z)^2} - \frac{2rz}{R+z} - 6\eta r]\cos\theta$$

$$\tau_{zz} = \frac{P}{2\pi}\frac{3rz^2}{R^5}\cos\theta$$

$$\tau_{rz} = -\frac{P}{2\pi}\frac{3zr^2}{R^5}\cos\theta$$

$$\tau_{\theta z} = 0$$

$$\tau_{r\theta} = \frac{P}{2\pi R}\frac{(1-2\eta)r}{(R+z)^2}\sin\theta$$

where

$$R^2 = r^2 + z^2$$

(ii) we now consider the case $\delta = -\frac{1}{2}$. In this case we find that

For $r \le a$

$$v_r = -\frac{T}{4\mu}\frac{(2-\eta)\sqrt{\pi}}{\sqrt{2a}}\cos\theta$$

$$v_\theta = \frac{T}{4\mu}\frac{(2-\eta)\sqrt{\pi}}{\sqrt{2a}}\sin\theta$$

$$v_z = -\frac{T}{4\mu}\frac{(1-2\eta)r\cos\theta}{\sqrt{2\pi}a^{\frac{3}{2}}(1+(1-\frac{r^2}{a^2})^{\frac{1}{2}})}$$

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = \tau_{r\theta} = 0$$

$$\tau_{rz} = -\frac{T}{\sqrt{2\pi a^3}}(1-\frac{r^2}{a^2})^{-\frac{1}{2}}\cos\theta$$

$$\tau_{\theta z} = \frac{T}{\sqrt{2\pi a^3}}(1-\frac{r^2}{a^2})^{-\frac{1}{2}}\sin\theta$$

For $r > a$

$$v_r = -\frac{T}{4\mu}[(2-\eta)\sqrt{\frac{2}{\pi a}}\arcsin\frac{a}{r} + \eta\sqrt{\frac{2}{\pi a}}\frac{\sqrt{r^2-a^2}}{r^2}]\cos\theta$$
$$v_\theta = \frac{T}{4\mu}[(2-\eta)\sqrt{\frac{2}{\pi a}}\arcsin\frac{a}{r} - \eta\sqrt{\frac{2}{\pi a}}\frac{\sqrt{r^2-a^2}}{r^2}]\sin\theta$$
$$v_z = -\frac{T}{4\mu}\frac{(1-2\eta)\sqrt{2a}\cos\theta}{\sqrt{\pi r}}$$
$$\tau_{rr} = T\sqrt{\frac{2a}{\pi}}[\frac{\eta}{r^2}(1-\frac{a^2}{r^2})^{\frac{1}{2}} + \frac{1}{r^2}(1-\frac{a^2}{r^2})^{-\frac{1}{2}}]\cos\theta$$
$$\tau_{\theta\theta} = -T\frac{\eta\sqrt{2a}}{\sqrt{\pi}r^2}[(1-\frac{a^2}{r^2})^{\frac{1}{2}} + 3(1-\frac{a^2}{r^2})^{-\frac{1}{2}}]\cos\theta$$
$$\tau_{r\theta} = T\frac{\eta\sqrt{2a}}{\sqrt{\pi}r^2}[(1-\frac{a^2}{r^2})^{\frac{1}{2}} - \frac{1}{2}(1-\frac{a^2}{r^2})^{-\frac{1}{2}}]\sin\theta$$
$$\tau_{zz} = \tau_{rz} = \tau_{\theta z} = 0$$

(*iii*) We next consider $\delta = 0$. This case corresponds to a uniform shearing force. The results are

For $r \le a$

$$v_r = -\frac{T}{4\mu}[\frac{4\eta}{3\pi}F(\frac{r}{a}) + \frac{4(3-2\eta)}{3\pi}E(\frac{r}{a}) + \frac{4\eta a^2}{3\pi r^2}(E(\frac{r}{a}) - F(\frac{r}{a}))]\cos\theta$$
$$v_\theta = \frac{T}{4\mu}[-\frac{4\eta}{3\pi}F(\frac{r}{a}) - \frac{4(3-2\eta)}{3\pi}E(\frac{r}{a}) + \frac{4\eta a^2}{3\pi r^2}(E(\frac{r}{a}) - F(\frac{r}{a}))]\sin\theta$$
$$v_z = -\frac{T}{4\mu}\frac{(1-2\eta)r}{2a}\cos\theta$$
$$\tau_{rr} = T[\frac{2\eta}{3\pi r}(2F(\frac{r}{a}) - E(\frac{r}{a})) + \frac{4\eta a^2 - 3r^2}{3\pi r^3}(E(\frac{r}{a}) - F(\frac{r}{a}))]\cos\theta$$
$$\tau_{\theta\theta} = -T\eta[\frac{2}{3\pi r}(2F(\frac{r}{a}) - E(\frac{r}{a})) + \frac{4a^2 - 9r^2}{3\pi r^3}(E(\frac{r}{a}) - F(\frac{r}{a}))]\cos\theta$$
$$\tau_{zz} = 0$$
$$\tau_{r\theta} = T[\frac{2\eta}{3\pi r}(2F(\frac{r}{a}) - E(\frac{r}{a})) + \frac{8\eta a^2 + 3r^2}{6\pi r^3}(E(\frac{r}{a}) - F(\frac{r}{a}))]\sin\theta$$
$$\tau_{rz} = -\frac{T}{2a}\cos\theta$$
$$\tau_{\theta z} = \frac{T}{2a}\sin\theta$$

For $r > a$

$$
\begin{aligned}
v_r &= -\frac{T}{4\mu}\Big[\frac{4(3-\eta)}{3\pi r}F(\frac{a}{r}) + \frac{4a\eta}{3\pi r}E(\frac{a}{r}) + \frac{4(3-\eta)r}{3\pi a}(E(\frac{a}{r}) - F(\frac{a}{r}))\Big]\cos\theta \\
v_\theta &= \frac{T}{4\mu}\Big[\frac{4(3-2\eta)}{3\pi r}F(\frac{a}{r}) - \frac{4a\eta}{3\pi r}E(\frac{a}{r}) + \frac{4(3-2\eta)r}{3\pi a}(E(\frac{a}{r}) - F(\frac{a}{r}))\Big]\sin\theta \\
v_z &= -\frac{T}{4\mu}\frac{(1-2\eta)a}{2r}\cos\theta \\
\tau_{rr} &= T\Big[\frac{2a\eta}{3\pi r^2}(F(\frac{a}{r}) + 2E(\frac{a}{r})) - \frac{(3-2\eta)}{3\pi a}(E(\frac{a}{r}) - F(\frac{a}{r}))\Big]\cos\theta \\
\tau_{\theta\theta} &= -T\eta\Big[\frac{2a}{3\pi r^2}(F(\frac{a}{r}) + 2E(\frac{a}{r})) - \frac{7}{3\pi a}(E(\frac{a}{r}) - F(\frac{a}{r}))\Big]\cos\theta \\
\tau_{zz} &= \tau_{rz} = \tau_{\theta z} = 0 \\
\tau_{r\theta} &= T\Big[\frac{2a\eta}{3\pi r^2}(F(\frac{a}{r}) + 2E(\frac{a}{r})) + \frac{(3+4\eta)}{6\pi a}(E(\frac{a}{r}) - F(\frac{a}{r}))\Big]\sin\theta
\end{aligned}
$$

(vi) Finally we consider $\delta = \frac{1}{2}$. In this case we get

For $r \le a$

$$
\begin{aligned}
v_r &= -\frac{T}{4\mu}\Big[\frac{(2-\eta)\sqrt{2\pi a}}{4} - \frac{(4-3\eta)\sqrt{\pi}r^2}{8\sqrt{2a^3}}\Big]\cos\theta \\
v_\theta &= \frac{T}{4\mu}\Big[\frac{(2-\eta)\sqrt{2\pi a}}{4} - \frac{(4-\eta)\sqrt{\pi}r^2}{8\sqrt{2a^3}}\Big]\sin\theta \\
v_z &= -\frac{(1-2\eta)T}{4\mu}\frac{\sqrt{2a^3}}{3\sqrt{\pi}r}\Big[1 - (1 - \frac{r^2}{a^2})^{\frac{3}{2}}\Big]\cos\theta \\
\tau_{rr} &= T\frac{(4+\eta)\sqrt{\pi}r}{8\sqrt{2a^3}}\cos\theta \\
\tau_{\theta\theta} &= -T\frac{13\eta\sqrt{\pi}r}{8\sqrt{2a^3}}\cos\theta \\
\tau_{zz} &= 0 \\
\tau_{r\theta} &= -T\frac{(2-\eta)\sqrt{\pi}r}{8\sqrt{2a^3}}\sin\theta \\
\tau_{rz} &= -\frac{T}{\sqrt{2\pi a}}(1 - \frac{r^2}{a^2})^{\frac{1}{2}}\cos\theta \\
\tau_{\theta z} &= \frac{T}{\sqrt{2\pi a}}(1 - \frac{r^2}{a^2})^{\frac{1}{2}}\sin\theta
\end{aligned}
\qquad (3.44)
$$

For $r > a$

$$v_r = -\frac{T\sqrt{a}}{16\mu\sqrt{2\pi}}[\frac{(4-\eta)r}{a}\sqrt{1-\frac{a^2}{r^2}} + (8-4\eta-(4-3\eta)\frac{r^2}{a^2})\arcsin\frac{a}{r}$$
$$-\frac{2\eta r}{a}(1-\frac{a^2}{r^2})^{\frac{3}{2}}]\cos\theta$$
$$v_\theta = \frac{T\sqrt{a}}{16\mu\sqrt{2\pi}}[\frac{(4-3\eta)r}{a}(1-\frac{a^2}{r^2})^{\frac{1}{2}} + (8-4\eta) - (4-\eta)\frac{r^2}{a^2}\arcsin\frac{a}{r}$$
$$+\frac{2\eta r}{r}(1-\frac{a^2}{r^2})^{\frac{3}{2}}]\sin\theta$$
$$v_z = -\frac{(1-2\eta)T}{4\mu}\frac{\sqrt{2a^3}}{3\sqrt{\pi r}}\cos\theta$$
$$\tau_{rr} = \frac{T}{\sqrt{2\pi a}}[\frac{4+\eta}{4}\frac{r}{a}\arcsin\frac{a}{r} - \frac{4-\eta}{4}(1-\frac{a^2}{r^2})^{\frac{1}{2}}$$
$$-\frac{\eta}{2}(1-\frac{a^2}{r^2})^{\frac{3}{2}}]\cos\theta$$

$$\tau_{\theta\theta} = -\frac{T\eta}{\sqrt{2\pi a}}[\frac{5r}{4a}\arcsin\frac{a}{r}$$
$$-\frac{3}{4}(1-\frac{a^2}{r^2})^{\frac{1}{2}} - \frac{1}{2}(1-\frac{a^2}{r^2})^{\frac{3}{2}}]\cos\theta$$
$$\tau_{zz} = 0$$
$$\tau_{r\theta} = \frac{T}{\sqrt{2\pi a}}[\frac{2+\eta}{4}(1-\frac{a^2}{r^2})^{\frac{1}{2}} - \frac{2-\eta}{4}\frac{r}{a}\arcsin\frac{a}{r} \qquad (3.45)$$
$$-\frac{\eta}{2}(1-\frac{a^2}{r^2})^{\frac{3}{2}}]\sin\theta$$
$$\tau_{rz} = 0$$
$$\tau_{\theta z} = 0$$

We point out that in the foregoing expressions the symbols $F(x)$ and $E(x)$ represent the complete elliptic integrals of the first and second kind,respectively.

The expressions when $\delta = 1$,can be computed easily. We find that these again involve the elliptic integrals.

We remark that, for $\delta = 3/2, 5/2, 7/2, 9/2, \cdots$,we can again find the exact solutions.

The Second Order Case.

In this case, since the calculation is very complicated we only select $\delta = \frac{1}{2}$.

In order to find the second order solutions we first need to find $L(n,s)$. The remaining calculations involve integration and algebraic manipulations. First we list $L(n,s)$, as required for our purposes:

$$
\begin{aligned}
L(0,-\frac{3}{2}) &= \begin{cases} (\frac{\pi a}{8})^{\frac{1}{2}}(1-\frac{1}{2}\frac{r^2}{a^2}), & r \le a \\ (\frac{a}{8\pi})^{\frac{1}{2}}[\frac{(r^2-a^2)^{\frac{1}{2}}}{a} + (2-\frac{r^2}{a^2})\arcsin\frac{a}{r}], & r > a \end{cases} \\
L(0,-\frac{1}{2}) &= \begin{cases} (\frac{2}{\pi a})^{\frac{1}{2}}(1-\frac{r^2}{a^2})^{\frac{1}{2}}, & r \le a \\ 0, & r > a \end{cases} \\
L(0,\frac{1}{2}) &= \begin{cases} (\frac{\pi}{2a^3})^{\frac{1}{2}}, & r \le a \\ (\frac{2}{\pi a})^{\frac{1}{2}}[\frac{1}{a}\arcsin\frac{a}{r} - (\frac{1}{r^2-a^2})^{\frac{1}{2}}], & r > a \end{cases} \\
L(1,-\frac{3}{2}) &= \begin{cases} (\frac{2a^3}{9\pi r^2})^{\frac{1}{2}}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}], & r \le a \\ (\frac{2a^3}{9\pi r^2})^{\frac{1}{2}}, & r > a \end{cases} \\
L(1,-\frac{1}{2}) &= \begin{cases} (\frac{\pi r^2}{8a^3})^{\frac{1}{2}}, & r \le a \\ (\frac{1}{2\pi a})^{\frac{1}{2}}[\frac{r}{a}\arcsin\frac{a}{r} - (1-\frac{a^2}{r^2})^{\frac{1}{2}}], & r > a \end{cases} \\
L(1,\frac{1}{2}) &= \begin{cases} (\frac{2r^2}{\pi a^5})^{\frac{1}{2}}(1-\frac{r^2}{a^2})^{-\frac{1}{2}}, & r < a \\ 0, & r > a \end{cases} \\
L(2,-\frac{3}{2}) &= \begin{cases} (\frac{\pi r^4}{128a^3})^{\frac{1}{2}}, & r \le a \\ (\frac{r}{32\pi a})^{\frac{1}{2}}[(1-\frac{a^2}{r^2})^{\frac{1}{2}} + \frac{r}{a}\arcsin\frac{a}{r} - 2(1-\frac{a^2}{r^2})^{\frac{3}{2}}], & r > a \end{cases} \\
L(2,-\frac{1}{2}) &= \begin{cases} (\frac{2}{9\pi a})^{\frac{1}{2}}[2\frac{a^2}{r^2}(1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}) - (1-\frac{r^2}{a^2})^{\frac{3}{2}}], & r \le a \\ (\frac{8a^3}{9\pi r^4})^{\frac{1}{2}}, & r > a \end{cases} \\
L(2,\frac{1}{2}) &= \begin{cases} 0, & r < a \\ (\frac{2a^3}{\pi r^6})^{\frac{1}{2}}(1-\frac{a^2}{r^2})^{-\frac{1}{2}}, & r > a \end{cases}
\end{aligned}
\tag{3.46}
$$

On substituting thses values and the other values from (3.41) to (3.43) into (3.40) we find the second order displacements to be:

$$\frac{16\mu^2 w_r}{T^2} = -2(1-\eta)M_1(r) + (1-2\eta)M_2(r) + \frac{\cos 2\theta}{4-5\eta+2\eta^2}[(1+\eta-2\eta^2)M_3(r) + \frac{2\eta}{r}M_4(r) - 2(1-\eta)^2 - \frac{1(1-2\eta)^2}{r}M_6(r) - (4+9\eta+4\eta^2)M_7(r) + \frac{2(4+5\eta-8\eta^2)}{r}M_8(r) + \frac{\mu(1-\eta)(4+9\eta+4\eta^2)}{2}M_9(r) + \frac{\mu(1-\eta)(4+5\eta-8\eta^2)}{r}M_{10}(r)]$$

$$\frac{16\mu^2 w_\theta}{T^2} = \frac{2\sin 2\theta}{4-5\eta+2\eta^2}[(1-\eta)^2 M_3(r) - \frac{3(1-\eta)}{r}M_4(r) + (1-\eta)M_5(r) - \frac{2\eta(1-\eta)}{r}M_6(r) + 2\eta(1-\eta)M_7(r) + \frac{4-13\eta+8\eta^2}{r}M_8(r) - \mu\eta(1-\eta)^2 M_9(r) - \frac{\mu(1-\eta)(4-13\eta+8\eta^2)}{2r}M_{10}(r)]$$

$$\frac{16\mu^2 w_z}{T^2} = (1-2\eta)M_{11}(r) - 2(1-\eta)M_{12}(r) + \frac{\cos 2\theta}{4-5\eta+2\eta^2}[\frac{10-21\eta+8\eta^2}{2}M_{13}(r) + (5\eta-6\eta^2)M_{14}(r) - (8-34\eta+24\eta^2)M_{15}(r) + \mu(1-\eta)(8-34\eta+24\eta^2)M_{16}(r)] \tag{3.47}$$

where

$$M_1(r) = B_1\int_0^a x^2(1-\frac{x^2}{a^2})^{\frac{1}{2}}K_{11}(0,x)\,dx + B_2\int_0^a [1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}]K_{11}(0,x)\,dx + B_3\int_a^\infty P(x)K_{11}(0,x)\frac{dx}{x} - \frac{4b_2+b_6}{24}\int_0^a [1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{11}(0,x)\,dx + B_4\int_a^\infty (1-\frac{a^2}{x^2})^{\frac{1}{2}}K_{11}(0,x)\frac{dx}{x^3}$$

$$M_2(r) = B_5\int_0^a x(1-\frac{x^2}{a^2})K_{10}(0,x)\,dx - B_6\int_0^a x^3K_{10}(0,x)\,dx$$
$$-\frac{a^3b_{13}}{9\pi}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]^2K_{10}(0,x)\frac{dx}{x^3}$$
$$-\frac{2a^3b_{18}}{9\pi}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}][1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{10}(0,x)\frac{dx}{x^3}$$
$$-\frac{4a^3b_{17}}{9\pi}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]^2K_{10}(0,x)\frac{dx}{x^3}$$
$$-\frac{a(3b_{15}-b_{18})}{9\pi}\int_0^a(1-\frac{x^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}]K_{10}(0,x)\frac{dx}{x}$$
$$-\frac{a(6b_{19}-4b_{17})}{9\pi}\int_0^a(1-\frac{x^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{10}(0,x)\frac{dx}{x}$$
$$-B_7\int_a^\infty K_{10}(0,x)\frac{dx}{x^3} - B_8\int_a^\infty P^2(x)K_{10}(0,x)\,dx$$
$$-\frac{a(b_{14}+2b_{16})}{16\pi}\int_a^\infty(1-\frac{a^2}{x^2})^{\frac{1}{2}}P(x)K_{10}(0,x)\frac{dx}{x}$$
$$-\frac{b_{14}a^3}{16\pi}\int_a^\infty(1-\frac{a^2}{x^2})K_{10}(x)\frac{dx}{x^3}$$

$$M_3(r) = B_9\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}]K_{13}(0,x)\,dx$$
$$-\frac{b_8+b_{27}}{8}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{13}(0,x)\,dx$$
$$+B_{10}\int_0^a x^2(1-\frac{x^2}{a^2})^{\frac{1}{2}}K_{13}(0,x)\,dx + B_{11}\int_a^\infty P(x)K_{13}(0,x)\frac{dx}{x}$$
$$-\frac{a^3b_{29}}{6\pi}\int_a^\infty(1-\frac{a^2}{x^2})^{\frac{1}{2}}K_{13}(0,x)\frac{dx}{x^3}$$

$$M_5(r) = B_{12}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}]K_{11}(0,x)\,dx$$
$$+\frac{b_{27}-b_8}{3}\int_0^a[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{11}(0,x)\,dx$$
$$-B_{13}\int_0^a x^2(1-\frac{x^2}{a^2})^{\frac{1}{2}}K_{11}(0,x)\,dx + B_{14}\int_a^\infty P(x)K_{11}(0,x)\frac{dx}{x}$$
$$+\frac{a^3b_{29}}{6\pi}\int_a^\infty(1-\frac{a^2}{x^2})^{\frac{1}{2}}K_{11}(0,x)\frac{dx}{x^3}$$

$$
\begin{aligned}
M_7(r) = &-\frac{\pi(4b_{20}+b_{23})}{64a^3}\int_0^a x^3 K_{12}(0,x)\,dx \\
&+ B_{15}\int_0^a x(1-\frac{x^2}{a^2})K_{12}(0,x)\,dx- \\
&\frac{a(2\mu(1-2\eta)+b_{22})}{3\pi}\int_0^a [(1-\frac{x^2}{a^2})^{\frac{1}{2}}-(1-\frac{x^2}{a^2})^2]K_{12}(0,x)\frac{dx}{x} \\
&-\frac{2ab_{24}}{3\pi}\int_0^a (1-\frac{x^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{12}(0,x)\frac{dx}{x} \\
&-\frac{4b_{20}+b_{23}}{16\pi a}\int_a^\infty P^2(x)K_{12}(0,x)x\,dx \\
&-\frac{ab_{23}}{8\pi}\int_a^\infty (1-\frac{a^2}{x^2})^{\frac{1}{2}}P(x)K_{12}(0,x)\frac{dx}{x}
\end{aligned}
$$

$$
\begin{aligned}
M_9(r) = &B_{16}\int_0^a x^2 K_{13}(-1,x)\,dx-\frac{4(b_{35}+b_{45})}{3\pi a}\int_0^a [1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{13}(-1,x)\,dx+ \\
&B_{17}\int_0^a (1-\frac{x^2}{a^2})K_{13}(-1,x)\,dx+\frac{2b_{36}}{3\pi a}\int_0^a [(1-\frac{x^2}{a^2})^{-\frac{1}{2}}-(1-\frac{x^2}{a^2})]K_{13}(-1,x)\,dx \\
&+\frac{4a(b_{27}+b_{51})}{3\pi}\int_0^a [(1-\frac{x^2}{a^2})^{\frac{1}{2}}-(1-\frac{x^2}{a^2})]K_{13}(-1,x)\frac{dx}{x^2} \\
&+B_{18}\int_0^a [(1-\frac{x^2}{a^2})^{\frac{1}{2}}-(1-\frac{x^2}{a^2})^2]K_{13}(-1,x)\frac{dx}{x^2} \\
&+\frac{4a^3(b_{40}+b_{42})}{9\pi}\int_0^a [1-(1-\frac{x^2}{a^2})^{\frac{3}{2}}][1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}]K_{13}(-1,x)\frac{dx}{x^4} \\
&+\frac{2b_{46}}{9\pi a}\int_0^a R(x)K_{13}(-1,x)\,dx+\frac{b_{50}}{6a^2}\int_0^a x^2R(x)K_{13}(-1,x)\,dx \\
&+B_{19}\int_a^\infty P(x)Q(x)K_{13}(-1,x)\frac{dx}{x^4}+B_{20}\int_a^\infty P^2(x)K_{13}(-1,x)\,dx \\
&+\frac{a(b_{33}+b_{48})}{4\pi}\int_a^\infty (1-\frac{a^2}{x^2})^{\frac{1}{2}}P(x)K_{13}(-1,x)\frac{dx}{x^2} \\
&+B_{21}\int_a^\infty \frac{K_{13}(-1,x)}{x^4}\,dx+\frac{a(b_{33}+b_{47})}{2\pi}\int_a^\infty (1-\frac{a^2}{x^2})^{\frac{1}{2}}Q(x)K_{13}(-1,x)\,dx \\
&+\frac{ab_{39}}{\pi}\int_a^\infty (1-\frac{a^2}{x^2})^{-\frac{1}{2}}K_{13}(-1,x)\frac{dx}{x^2}+\frac{2ab_{50}}{3\pi}\int_a^\infty P(x)K_{13}(-1,x)\frac{dx}{x}
\end{aligned}
\qquad (3.48)
$$

and where

$$P(x) = \frac{x}{a}\arcsin\frac{a}{x} - (1-\frac{a^2}{x^2})^{\frac{1}{2}}$$
$$Q(x) = \frac{x}{a}\arcsin\frac{a}{x} - (1-\frac{a^2}{x^2})^{-\frac{1}{2}} \quad (3.49)$$
$$R(x) = \frac{2a^2}{x^2}[1-(1-\frac{x^2}{a^2})^{\frac{1}{2}}] - (1-\frac{x^2}{a^2})^{\frac{1}{2}}$$

We remark that by replacing $K_{13}(0,x)$ with $K_{23}(-1,x)$ in M_3 we get M_4,by replacing $K_{11}(0,x)$ with $K_{21}(-1,x)$ in M_5 we get M_6. Similarly we get M_8 and M_{10} by replacing $K_{12}(0,x), K_{13}(-1,x)$ with $K_{22}(-1,x), K_{23}(-2,x)$ in M_7 and M_9, respectively.Again, when we replace $K_{11}(0,x), K_{10}(0,x), K_{13}(0,x), K_{11}(0,x), K_{12}(0,x)$, $K_{13}(0,x)$ by $K_{01}(0,x), K_{00}(0,x), K_{23}(0,x), K_{21}(0,x), K_{22}(0,x), K_{23}(-1,x)$ in M_1, M_2, M_3, M_5, M_7, M_9 respectively, we obtain $M_{11}, M_{12}, M_{13}, M_{14}, M_{15}, M_{16}$. For the stress components we only give the components τ''_{zz} and τ''_{zr} and the other components can be computed in a similar manner. We find

(i) For $r \leq a$

$$\frac{8\mu^2\tau''_{zz}}{T^2} = B_5(1-\frac{r^2}{a^2}) - B_6 r^2 - \frac{a^3 b_{13}}{9\pi r^4}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}]^2 - \frac{4a^2 b_{17}}{9\pi r^4}[1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}]^2 -$$
$$\frac{2a^3 b_{18}}{9\pi r^4}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}][1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}] - \frac{a(3b_{15}-b_{18})}{9\pi r^2}(1-\frac{r^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}] -$$
$$\frac{a(6b_{19}-4b_{17})}{9\pi r^2}(1-\frac{r^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}] + \cos 2\theta \left\{ B_{15}(1-\frac{r^2}{a^2}) - \frac{4(b_{20}+b_{23})\pi r^2}{64a^3} \right.$$
$$\left. -\frac{a(2\mu(1-2\eta)+b_{22})}{3\pi r^2}[(1-\frac{r^2}{a^2})^{\frac{1}{2}} - (1-\frac{r^2}{a^2})^2] - \frac{2ab_{24}}{3\pi r^2}(1-\frac{r^2}{a^2})^{\frac{1}{2}}[1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}] \right\}$$

$$\frac{8\mu^2\tau''_{rz}}{T^2} = B_1 r(1-\frac{r^2}{a^2})^{\frac{1}{2}} + \frac{B_2}{r}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}] - \frac{4b_2+b_6}{24r}[1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}]$$
$$+\cos 2\theta \left\{ -B_{22} r(1-\frac{r^2}{a^2})^{\frac{1}{2}} + \frac{3\mu(1-2\eta)-b_9}{12r}[1-(1-\frac{r^2}{a^2})^{\frac{3}{2}}] \right.$$
$$\left. -\frac{b_8}{6r}[1-(1-\frac{r^2}{a^2})^{\frac{1}{2}}] \right\}$$

(*ii*) For $r > a$

$$\frac{8\mu^2\tau''_{zz}}{T^2} = -\frac{B_7}{r^4} - B_8 P^2(r) - \frac{a^3 b_{14}}{16\pi r^4}(1 - \frac{a^2}{r^2}) - \frac{a(b_{14} + 2b_{16})}{16\pi r^2} P(r)(1 - \frac{a^2}{r^2})^{\frac{1}{2}}$$
$$- \cos 2\theta \left[\frac{4b_{20} + b_{23}}{16\pi r^2} P^2(r) + \frac{ab_{23}}{8\pi r^2} P(r)(1 - \frac{a^2}{r^2})^{\frac{1}{2}} \right]$$

$$\frac{8\mu^2\tau''_{rz}}{T^2} = \frac{B_3 P(r)}{r^2} + \frac{B_4}{r^2} \left[P(r) + \frac{2a^2}{r^2}(1 - \frac{a^2}{r^2})^{\frac{1}{2}} \right]$$
$$+ \left[\frac{a\mu(1 - 2\eta)}{\pi} - \frac{a(2b_8 + b_9)}{6\pi} \right] \frac{\cos 2\theta P(r)}{r^2}$$

In the above B_{ij} are given in the Appendix A_5, $P(r)$ is defined in (3.49), and T is given by (3.14).

CHAPTER IV

SUMMARY AND FUTURE DIRECTIONS

In this thesis, after reviewing the development of the compressible finite elasticity equations we have given solutions to two traction boundary value problems for the second order elastic materials.

In the first problem we have found an analytical solution for the problem in a compressible elastic half-space which is acted upon by a non-uniform normal distributed load for any value of $\delta > -1$. The integral transform method is employed to determine both linear and second order solutions. These solutions are then specialized for particular value of δ. In the linear case we consider:

(i) $\delta = -\frac{1}{2}$, a solution which corresponds to the flat-ended punch problem. Our solution agrees with that given by Sneddon(1965).

(ii) $\delta = 0$, corresponds to uniformly distrinuted load. This solution agrees with Boussinesq's solution as given in Sneddon(1972).

(iii) $\delta = \frac{1}{2}$, corresponds to the punch with form of a paraboloid of revolution. The solution again agrees with Sneddon(1965).

(iv) By letting $a \to 0$ we get the solution for a point load. The soluton again agrees with Sneddon(1972).

(v) The solutions for $\delta = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$ are all new. Hopefully these will soon find applications in other practical situations.

For the second order elastic case, general expressions for the displacement and stress components are given when $\delta = \frac{1}{2}$. Some numerical calculations are carried out and it is noted that the effect of the consideration of the second order elasticity is to increase the displacement in the z-direction and to decrease the overall value of the normal stress in the same direction. The solutions are then specialized for an incompressible elastic material and the corresponding numerical solutions are also presented and discussed.

In the second problem, an analytical solution, again, is found for the problem when the compressible elastic half-space is acted upon by a non-uniform shear load. Even though, because of the non-symmetrical nature of the problem, the mathematical analysis is much more difficult in this case, we have succeeded in obtaining exact solution. The method of integral transform is again employed for both linear and second order solutions. In the linear case we again specialized δ for different possible values and found that the solution when $\delta = 0$, corresponding to uniform shear force, and when $a \to 0$, the point force solution, again match with the existing solutions. The solutions in other cases, when $\delta = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ are, to our knowledge, all new. For the second order elastic case the general expressions for the displacement and stress components are given when $\delta = \frac{1}{2}$. Numerical solution pertaining to this case are being carried out.

All the above solutions,both for linear and second order elastic cases, apart of being new solutions, are also useful as preparatory material for contact or crack problems. We recall that in the case of contact problems we assume shearing stresses vanish on the boundary and prescribe normal component of the displacement vector. In the case of crack problems we prescribe normal stress within the crack region and assume shearing stresses to vanish on the boundary plane. In the case of contact problems we thus have solution known outside the contact region but have to match it with the displacement solution in the contact region. By considering different values for δ we can identify different kind of punch shapes and then determine the corresponding solutions both for the second order and new linear cases. Similar remarks apply to crack problems. We hope to carry out such calculations in the near future.

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For completeness we here give expressions for some quantities in cylindrical polar coordinate by assuming $v_r = v_r(r, z, \theta), v_\theta = v_\theta(r, z, \theta)$ and $v_z = v_z(r, z, \theta)$.

$$e'_{rr} = 2\frac{\partial v_r}{\partial r}, \quad e'_{\theta\theta} = 2(\frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}), \quad e'_{zz} = 2\frac{\partial v_z}{\partial z}$$

$$e'_{r\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}, \quad e'_{rz} = \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}, \quad e'_{\theta z} = \frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}$$

$$\alpha'_{rr} = (\frac{\partial v_r}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r})^2 + (\frac{\partial v_r}{\partial z})^2$$

$$\alpha'_{r\theta} = \frac{\partial v_r}{\partial r}\frac{\partial v_\theta}{\partial r} + \frac{\partial v_r}{\partial z}\frac{\partial v_\theta}{\partial z} + \frac{1}{r^2}(\frac{\partial v_r}{\partial \theta}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_r}{\partial \theta}v_r - \frac{\partial v_\theta}{\partial \theta}v_\theta - v_r v_\theta)$$

$$\alpha'_{\theta\theta} = (\frac{\partial v_\theta}{\partial r})^2 + (\frac{\partial v_\theta}{\partial z})^2 + (\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r})^2$$

$$\alpha'_{rz} = \frac{\partial v_r}{\partial r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial v_z}{\partial \theta}(\frac{\partial v_r}{\partial \theta} - v_\theta) + \frac{\partial v_z}{\partial z}\frac{\partial v_r}{\partial z}$$

$$\alpha'_{zz} = (\frac{\partial v_z}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_z}{\partial \theta})^2 + (\frac{\partial v_z}{\partial z})^2$$

$$\alpha'_{\theta z} = \frac{\partial v_z}{\partial r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial v_z}{\partial \theta}(\frac{\partial v_\theta}{\partial \theta} + v_r) + \frac{\partial v_z}{\partial z}\frac{\partial v_\theta}{\partial z}$$

On recognizing that the boundary is $z = 0$ and the elastic body occupies the half-space $z \geq 0$, we have $(l_1, l_2, l_3) = (0, 0, -1)$, expressions for the tractions on $z = 0$ take the form

$$X'_{\nu r} = -\frac{\partial v_z}{\partial r}\tau_{rr} - \frac{\tau_{r\theta}}{r}\frac{\partial v_z}{\partial \theta} + (\Delta' - \frac{\partial v_z}{\partial z})\tau_{rz} + \tau'_{rz}$$

$$X'_{\nu\theta} = -\frac{\partial v_z}{\partial r}\tau_{r\theta} - \frac{\tau_{\theta\theta}}{r}\frac{\partial v_z}{\partial \theta} + (\Delta' - \frac{\partial v_z}{\partial z})\tau_{\theta z} + \tau'_{\theta z}$$

$$X'_{\nu z} = -\frac{\partial v_z}{\partial r}\tau_{rz} - \frac{\tau_{\theta z}}{r}\frac{\partial v_z}{\partial \theta} + (\Delta' - \frac{\partial v_z}{\partial z})\tau_{zz} + \tau'_{zz}$$

Here

$$\Delta' = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

The body forces are given by

$$
\begin{aligned}
\rho_0 X_r' = & -\frac{\partial v_r}{\partial r}\frac{\partial \tau_{rr}}{\partial r} + \frac{2\tau_{r\theta}}{r}\frac{\partial v_\theta}{\partial r} - \frac{\partial v_z}{\partial r}\frac{\partial \tau_{rr}}{\partial z} + \frac{v_\theta}{r}\frac{\partial \tau_{r\theta}}{\partial r} \\
& - \frac{v_r}{r^2}(\tau_{rr} - \tau_{\theta\theta}) - \frac{\partial v_r}{\partial z}\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{\theta z}}{r}\frac{\partial v_\theta}{\partial z} - \frac{\partial v_z}{\partial z}\frac{\partial \tau_{rz}}{\partial z} \\
& - \frac{1}{r}\left[\frac{\partial v_r}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{rr}}{\partial \theta}\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial \theta}\right] \\
& + \left[\frac{v_r}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial v_\theta}{\partial \theta}\frac{\tau_{rr} - \tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial \theta}\frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta}\frac{\partial \tau_{r\theta}}{\partial z}\right] \\
& + \frac{\partial \tau_{rr}'}{\partial r} + \frac{1}{r}\frac{\partial \tau_{rr}'}{\partial \theta} + \frac{\partial \tau_{rz}'}{\partial z} + \frac{\tau_{rr}' - \tau_{\theta\theta}'}{r}
\end{aligned}
$$

$$
\begin{aligned}
\rho_0 X_\theta' = & -\frac{\partial v_r}{\partial r}\frac{\partial \tau_{r\theta}}{\partial r} - \frac{\partial v_\theta}{\partial r}\frac{\tau_{rr} - \tau_{\theta\theta}}{r} - \frac{\partial v_z}{\partial r}\frac{\partial \tau_{r\theta}}{\partial z} + \frac{v_\theta}{r}\frac{\partial \tau_{\theta\theta}}{\partial r} \\
& - \frac{2v_r}{r^2}\tau_{r\theta} - \frac{\partial v_r}{\partial z}\frac{\partial \tau_{\theta z}}{\partial r} - \frac{\tau_{rz}}{r}\frac{\partial v_\theta}{\partial z} - \frac{\partial v_z}{\partial z}\frac{\partial \tau_{\theta z}}{\partial z} \\
& - \frac{1}{r}\left[\frac{\partial v_r}{\partial \theta}\frac{\partial \tau_{\theta\theta}}{\partial r} + \frac{\partial v_\theta}{\partial r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\frac{\partial \tau_{\theta\theta}}{\partial \theta}\right] \\
& + \left[\frac{v_r}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial v_z}{\partial \theta}\frac{\partial \tau_{\theta\theta}}{\partial z}\right] \\
& + \frac{\partial \tau_{r\theta}'}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}'}{\partial \theta} + \frac{2\tau_{r\theta}'}{r} + \frac{\partial \tau_{\theta z}'}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
\rho_0 X_z' = & -\frac{\partial v_r}{\partial r}\frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{\theta z}}{r}\frac{\partial v_\theta}{\partial r} - \frac{\partial v_z}{\partial r}\frac{\partial \tau_{rz}}{\partial r} \\
& + \frac{v_\theta}{r}\frac{\partial \tau_{\theta z}}{\partial r} - \frac{v_r \tau_{rz}}{r^2} - \frac{\partial v_r}{\partial z}\frac{\partial \tau_{zz}}{\partial r} - \frac{\partial v_z}{\partial z}\frac{\partial \tau_{zz}}{\partial z} \\
& - \frac{1}{r}\left[\frac{\partial v_r}{\partial \theta}\frac{\partial \tau_{\theta z}}{\partial r} + \frac{\partial v_z}{\partial r}\frac{\partial \tau_{rz}}{\partial \theta} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}\frac{\partial \tau_{\theta z}}{\partial \theta}\right] \\
& + \left[\frac{\tau_{rz}}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\frac{\partial \tau_{zz}}{\partial \theta} + \frac{\partial v_z}{\partial \theta}\frac{\partial \tau_{\theta z}}{\partial z}\right] \\
& + \frac{\partial \tau_{rz}'}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}'}{\partial \theta} + \frac{\tau_{rz}'}{r} + \frac{\partial \tau_{zz}'}{\partial z}
\end{aligned}
$$

$$K_{00}(0,x)=\begin{cases}\frac{2}{\pi r}F(\frac{x}{r}), & x<r\\ \frac{2}{\pi x}F(\frac{r}{x}), & x>r\end{cases}$$

$$K_{11}(0,x)=\begin{cases}\frac{2}{\pi r}[F(\frac{x}{r})-E(\frac{x}{r})], & x<r\\ \frac{2}{\pi x}[F(\frac{r}{x})-E(\frac{r}{x})], & x>r\end{cases}$$

$$K_{10}(0,x)=\begin{cases}\frac{1}{r}, & x<r\\ 0, & x>r\end{cases}$$

$$K_{01}(0,x)=\begin{cases}0, & x<r\\ \frac{1}{x}, & x>r\end{cases}$$

$$K_{02}(0,x)=\begin{cases}\frac{2}{\pi r}F(\frac{x}{r})+\frac{4r}{\pi x^2}[E(\frac{x}{r})-F(\frac{x}{r})], & x<r\\ \frac{4}{\pi x}E(\frac{r}{x})-\frac{2}{\pi x}F(\frac{r}{x}), & x>r\end{cases}$$

$$K_{03}(0,x)=\begin{cases}0, & x<r\\ \frac{1}{x}(1-2\frac{r^2}{x^2}), & x>r\end{cases}$$

$$K_{12}(0,x)=\begin{cases}0, & x<r\\ \frac{r}{x^2}, & x>r\end{cases}$$

$$K_{21}(0,x)=\begin{cases}\frac{x}{r^2}, & x<r\\ 0, & x>r\end{cases}$$

$$K_{13}(0,x)=\begin{cases}\frac{13F(\frac{x}{r})-5E(\frac{x}{r})}{3\pi x}+\frac{16r^2}{3\pi x^3}[E(\frac{x}{r})-F(\frac{x}{r})], & x<r\\ \frac{11}{3\pi r}[F(\frac{r}{x})-E(\frac{r}{x})]+\frac{8r}{3\pi x^2}[2E(\frac{r}{x})-F(\frac{r}{x})], & x>r\end{cases}$$

$$K_{22}(0,x)=\begin{cases}\frac{2}{3\pi r}[F(\frac{x}{r})-2E(\frac{x}{r})]-\frac{4r}{3\pi x^2}[E(\frac{x}{r})-F(\frac{x}{r})], & x<r\\ \frac{2}{3\pi x}[F(\frac{r}{x})-2E(\frac{r}{x})]-\frac{4x}{3\pi r^2}[E(\frac{r}{x})-F(\frac{r}{x})], & x>r\end{cases}$$

$$K_{23}(0,x)=\begin{cases}0, & x<r\\ \frac{r^2}{x^3}, & x>r\end{cases}$$

$$K_{13}(-1,x)=\begin{cases}0, & x<r\\ \frac{r}{2x}(1-\frac{r^2}{x^2}), & x>r\end{cases}$$

$$K_{21}(-1,x) = \begin{cases} \frac{2x}{3\pi r}[F(\frac{x}{r}) + 2E(\frac{x}{r})] + \frac{2r}{3\pi x}[E(\frac{x}{r}) - F(\frac{x}{r})], & x < r \\ \frac{2}{3\pi}[2F(\frac{r}{x}) - E(\frac{r}{x})] + \frac{4x^2}{3\pi r^2}[E(\frac{r}{x}) - F(\frac{r}{x})], & x > r \end{cases}$$

$$K_{22}(-1,x) = \begin{cases} \frac{x^2}{4r^2}, & x < r \\ \frac{r^2}{4x^2}, & x > r \end{cases}$$

$$K_{23}(-2,x) = \begin{cases} \frac{x^3}{24r^2}, & x < r \\ \frac{r^2}{8x}(1 - \frac{2r^2}{3x^3}), & x > r \end{cases}$$

$$K_{23}(-1,x) = -\frac{2x}{15\pi r}[9F(\frac{x}{r}) + 2E(\frac{x}{r})] + \frac{8r}{15\pi x}F(\frac{x}{r}) + (\frac{16r^3}{15\pi x^3} - \frac{24r}{15\pi x})[E(\frac{x}{r}) - F(\frac{x}{r})], \quad x < r$$

$$K_{23}(-1,x) = \frac{8r^2}{15\pi x^2}[2F(\frac{r}{x}) - E(\frac{r}{x})] + \frac{2}{15\pi}[2F(\frac{r}{x}) - 3E(\frac{r}{x})] - \frac{4x^2}{15\pi r^2}[E(\frac{r}{x}) - F(\frac{r}{x})], \quad x > r$$

where $E(r)$ and $F(r)$ are complete elliptic integrals of the first and second kind, respectively.

Appendix A_3

$$I_1 = \int_0^1 \frac{\arcsin t}{t}\,dt$$

$$I_2 = \int_0^1 \frac{\arcsin \frac{at}{r}}{at}\,dt$$

$$I_3 = \int_0^x t\sqrt{a^2 - t^2}[F(\frac{t}{r}) - E(\frac{t}{r})]\,dt$$

$$I_4 = \int_0^x [a^3 - (a^2 - t^2)^{\frac{3}{2}}][F(\frac{t}{r}) - E(\frac{t}{r})]\,dt$$

$$I_5 = \int_r^a \sqrt{t^2 - r^2}[F(\frac{t}{a}) - E(\frac{t}{a})]\frac{dt}{t^5}$$

$$I_6 = \int_r^a [t^3 - (t^2 - r^2)^{\frac{3}{2}}][F(\frac{t}{a}) - E(\frac{t}{a})]\frac{dt}{t^5}$$

$$I_7 = \int_0^x [r^2 \arcsin\frac{t}{r} - t\sqrt{r^2 - t^2}][F(\frac{t}{a}) - E(\frac{t}{a})]\frac{dt}{t^2}$$

$$I_8 = \int_a^r [\frac{t}{a}\arcsin\frac{a}{t} - \frac{\sqrt{t^2 - a^2}}{t}][F(\frac{t}{a}) - E(\frac{t}{a})]\frac{dt}{t^2}$$

$$I_9 = \int_0^x [a - \sqrt{a^2 - t^2}]F(\frac{t}{r})\frac{dt}{t}$$

$$I_{10} = \int_0^x t\ln\frac{a + \sqrt{a^2 + t^2}}{a}F(\frac{t}{r})\,dt$$

$$I_{11} = \int_0^x [\frac{1}{2} - \frac{a^3 - (a^2 - t^2)^{\frac{3}{2}}}{3at^2}]F(\frac{t}{a})\frac{dt}{t}$$

$$I_{12} = \int_r^a [t - \sqrt{t^2 - r^2}]F(\frac{t}{a})\frac{dt}{t}$$

$$I_{13} = \int_r^a \ln\frac{t + \sqrt{t^2 - r^2}}{t}F(\frac{t}{a})\frac{dt}{t^2}$$

$$I_{14} = \int_r^a [\frac{t}{2} - \frac{t^3 - (t^2 - r^2)^{\frac{3}{2}}}{3r^2}]F(\frac{t}{a})\frac{dt}{t}$$

$$I_{15} = \int_0^x t^2 F(\frac{t}{a})\,dt$$

$$I_{16} = \int_0^x [\frac{r}{t}\arcsin\frac{t}{r} - \frac{\sqrt{r^2 - t^2}}{r}]^2 F(\frac{t}{a})\frac{dt}{t^2}$$

$$I_{17} = \int_0^x [\arcsin^2(\frac{t}{r}) - \frac{t^2}{r^2}]F(\frac{t}{a})\frac{dt}{t^2}$$

$$I_{18} = \int_a^r F(\frac{t}{r})\frac{dt}{t^3}$$

$$I_{19} = \int_a^r t[\frac{t}{a}\arcsin\frac{a}{t} - \frac{\sqrt{t^2 - a^2}}{t}]F(\frac{t}{r})\,dt$$

$$I_{20} = \int_a^r t[\arcsin^2\frac{a}{t} - \frac{a^2}{t^2}]F(\frac{t}{r})\,dt$$

$$I_{21} = \int_0^x t^2\sqrt{a^2 - t^2}F(\frac{t}{r})\,dt$$

$$I_{22} = \int_0^x \frac{t^2}{\sqrt{a^2 - t^2}}F(\frac{t}{r})\,dt$$

$$I_{23} = \int_r^a \sqrt{t^2 - r^2}F(\frac{t}{r})\,dt$$

$$I_{24} = \int_r^a \frac{1}{\sqrt{t^2 - r^2}}F(\frac{t}{a})\frac{dt}{t}$$

Appendix A_4

$$c_1 = 2(1-2\eta)^2(a_1 - 4a_2 + 4a_3)Q^2$$

$$c_2 = 2(1-2\eta)[(1-2\eta)(3a_1 + 20a_2 - 12a_3 - 4a_5) + 2(3a_1 - 4a_5)]Q^2$$

$$c_3 = 2(1-2\eta)[(1-2\eta)(a_1 + 4a_2 + 12a_3 + 48a_4) + 2(a_1 + 8a_2 - 4a_3)]Q^2$$

$$c_4 = 8(1-\eta)[2(1-\eta)(a_1 + 4a_2) - a_1]Q^2$$

$$c_5 = -16(1-\eta)^2(3a_1 + 4a_2)Q^2$$

$$c_6 = 8(1-\eta)a_1 Q^2$$

$$c_7 = -4(1-\eta)(1-2\eta)a_1 Q^2$$

$$c_8 = 2(1-2\eta)[(1-2\eta)(a_1 + 4a_2 + 12a_3 + 48a_4) - 2a_1]Q^2$$

$$c_9 = 4(1-2\eta)^2(a_1 + 2a_2 - 2a_3 - 2a_5)Q^2$$

$$c_{10} = 8(1-\eta)^2(a_1 + 2a_2)Q^2$$

$$c_{11} = (1-\eta)(5-4\eta)a_1 Q^2/\pi$$

$$c_{12} = (1-2\eta)Q^2[(1-2\eta)(3a_1 + 36a_2 + 10a_3 + 43a_4 + 8a_5) + 24(2a_2 - a_3)]/\pi$$

$$c_{13} = (1-2\eta)Q^2[12(a_5 + 6a_3 - 12a_2) - (1-2\eta)(27a_1 + 156a_2 + 300a_3 + 129a_4 + 4a_5)]/\pi$$

$$c_{14} = (1-2\eta)(3-2\eta)\pi a_1 Q^2$$

$$c_{15} = 2(1-2\eta)Q^2[(1-2\eta)(53a_1 + 220a_2 + 236a_3 + 1296a_4 - 52a_5) + 6(-a_1 + 16a_2 - 8a_3 - 2a_5)]/(9\pi)$$

$$c_{16} = 2(1-2\eta)Q^2[12(a_5 - 12a_2 + 6a_3) - (1-2\eta)(43a_1 + 188a_2 + 268a_3 + 1296a_4 - 28a_5)]/(8\pi)$$

$$c_{17} = 2(1-2\eta)Q^2[2(6a_1 + 24a_2 - 12a_3 - 2a_5) + (1-2\eta)(5a_1 + 36a_2 + 209a_3 + 144a_4 - 4a_5)]/(3\pi)$$

$$c_{18} = 2(1-2\eta)Q^2[12(-3a_1 - 12a_2 + 6a_3 + a_5)$$

$$7 - (1-2\eta)(17a_1 + 100a_2 + 698a_3 + 432a_4 - 12a_5)]/(9\pi)$$

$$c_{19} = 2(1-2\eta)Q^2[(1-2\eta)(67a_1 + 260a_2 + 244a_3 + 1296a_4 - 44a_5)$$

$$+ 36(4a_2 - 2a_3 - a_5)]/(36\pi) + (1-\eta)a_1 Q^2/\pi$$

Appendix A_5

$$B_1 = \frac{\eta\mu(1-2\eta)}{8a^2} - \frac{2b_1 - 4b_2 + 3b_5 - b_6}{48a^2}$$

$$B_2 = \frac{\mu(1-2\eta)(2+\eta)}{12} - \frac{4b_3 + b_4}{48}$$

$$B_3 = \frac{a}{8\pi}[\mu(2+\eta)(1-2\eta) - (2b_2 + b_3 + \frac{b_4 + 2b_6}{4})]$$

$$B_4 = \frac{a^3}{3\pi}[\mu\eta(1-2\eta) - \frac{b_4 + 2b_6}{4}]$$

$$B_5 = \frac{1}{\pi a}[\mu(1-2\eta) - b_{12} - \frac{b_{17}}{9} + \frac{b_{19}}{3}]$$

$$B_6 = \frac{\pi}{16a^3}[b_{11} + \frac{b_{14}}{16} + \frac{b_{16}}{4}]$$

$$B_7 = \frac{a^3}{16\pi}[2b_{13} + 8b_{17} + 4b_{18}]$$

$$B_8 = \frac{16b_{11} + b_{14} + 4b_{16}}{64\pi a}$$

$$B_9 = \frac{\mu(1-3\eta)(1-2\eta)}{4} - \frac{4b_{25} + b_{29} + 4b_9}{32}$$

$$B_{10} = -\frac{\mu(1-2\eta)}{2a^2} + \frac{12b_{26} + 4b_{27} + 3b_{28} - 12b_7 + 4b_8 - 3b_{10}}{24a^2}$$

$$B_{11} = \frac{a}{6\pi}[(8-24\eta)(1-2\eta)\mu - (4b_8 + 2b_9 + 4b_{25} + 8b_{27} + b_{29})]$$

$$B_{12} = \frac{(2+3\eta)(1-2\eta)}{3} + \frac{4b_{25} + b_{29} - 4b_9}{24}$$

$$B_{13} = \frac{(7-19\eta)\mu + \mu\eta(1-2\eta)}{2a^2} + \frac{12b_{26} - 4b_{27} + 3b_{28} + 12b_7 - 4b_8 + 3b_{10}}{24a^2}$$

$$B_{14} = \frac{a}{6\pi}[(16+24\eta)(1-2\eta)\mu - 4b_8 - 2b_9 + 4b_{25} + 8b_{27} + b_{29}]$$

$$B_{15} = \frac{3\mu(1-2\eta) - 3b_{21} + b_{24}}{3\pi a}$$

$$B_{16} = \frac{\pi(16b_{30} + 8b_{31} + 2b_{32} + 4b_{33} + 8b_{42} + 4b_{43} + 2b_{27} + b_{48})}{16a^3} + \frac{6b_{34} - 2b_{35} + 6b_{44} - 2b_{45}}{3\pi a^3}$$

$$B_{17} = \frac{6b_{41} - 2b_{37} + 6b_{53} - 2b_{51}}{3\pi a}$$

$$B_{18} = \frac{2a(3b_{38} - b_{40} + 3b_{54} - b_{52})}{9\pi}$$

$$B_{19} = \frac{4b_{30} + b_{33} + 4b_{42} + b_{47}}{4\pi a}$$

$$B_{20} = \frac{4b_{31} + b_{32} + 4b_{42} + b_{47}}{4\pi a}$$

$$B_{21} = \frac{4a^3(b_{40} + b_{25} + 2b_{45})}{9\pi}$$

$$B_{22} = \frac{\mu\eta(1-2\eta) + 4\mu(2-3\eta)}{8a^2} + \frac{12b_7 - 4b_8 + 3b_{10}}{48a^2}$$

Appendix A_6

$$b_1 = -(4 - 9\eta + 10\eta^2)a_1 + (1-2\eta)(6+4\eta)(2a_2 - a_3) + (5-2\eta)a_5$$

$$b_2 = -(1-2\eta)(2-\eta)a_1 - 2(1-2\eta)^2(2a_2 - a_3) + (1-2\eta)a_5$$

$$b_3 = 6(1-2\eta)(1-\eta)a_1 + 4(1-2\eta)^2(2a_2 - a_3) - 2(1-2\eta)a_5$$

$$b_4 = -12\eta(1-2\eta)a_1 + 8\eta(1-2\eta)a_5$$

$$b_5 = (14 - 4\eta)^2 a_1 - 16\eta a_5$$

$$b_6 = 4\eta(1-2\eta)(a_1 - a_5)$$

$$b_7 = (6 + 5\eta - 10\eta^2)a_1 + (1-2\eta)(6+4\eta)(2a_2 - a_3) + (5-2\eta)a_5$$

$$b_8 = \eta(1-2\eta)a_1 - 2(1-2\eta)^2(2a_2 - a_3) - (1-2\eta)a_5$$

$$b_9 = (1-2\eta)(2-6\eta)a_1 + 4(1-2\eta)^2(2a_2 - a_3) + 2(1-2\eta)a_5$$

$$b_{10} = -6(1-2\eta)a_1 + 4\eta(1-2\eta)a_5$$

$$b_{11} = 4\eta^2 a_1 + 16(\eta + \eta^2)a_2 + (12 - 32\eta - 16\eta^2)a_3 + 48(1 - 2\eta)^2 a_4 - 4a_5$$

$$b_{12} = \frac{7 - 16\eta}{2} a_1 + (27 - 28\eta + 12\eta^2)a_2 - (9 - 2\eta + 2\eta^2)a_3$$

$$b_{13} = -(1 - 2\eta)^2(a_1 - 4a_2 + 2a_3)$$

$$b_{14} = 16\eta^2(a_1 + 2a_2 - 2a_3 - 2a_5)$$

$$b_{15} = 2(1 - 2\eta)^2(a_1 - 2a_2 + 2a_3)$$

$$b_{16} = 4\eta^2(7a_1 - 2a_2 + 4a_3 - 4a_5)$$

$$b_{17} = \frac{(1 - 2\eta)^2}{2}(2a_2 - a_3)$$

$$b_{18} = 2(1 - 2\eta)^2(a_1 + a_3)$$

$$b_{19} = -(1 - 2\eta)^2(a_1 + a_3)$$

$$b_{20} = -(8 + 4\eta^2)^2 a_1 - 16(1 - \eta)a_2 + (1 - 2\eta)(20 - 24\eta)a_3 + 48(1 - 2\eta)^2 a_4 + 4a_5$$

$$b_{21} = 2(1 - 2\eta)^2 a_2 - 4(1 - 2\eta)(a_1 + a_3)$$

$$b_{22} = -4(1 - 2\eta)^2 a_2 - 8(1 - 2\eta)(a_1 + a_3)$$

$$b_{23} = -16\eta(2 - \eta)(a_1 + a_3)$$

$$b_{24} = 4(1 - 2\eta)(a_1 + a_3)$$

$$b_{25} = -6(1 - 2\eta)a_1 + 4(1 - 2\eta)^2(2a_2 + a_3) + (1 - 2\eta)(6 - 4\eta)a_5$$

$$b_{26} = -(10 - 19\eta + 6\eta^2)a_1 - (1 - 2\eta)(10 - 4\eta)(2a_2 - a_3) + (7 - 6\eta + 4\eta^2)a_5$$

$$b_{27} = -(1 - 2\eta)(11 - 4\eta)a_1 - 2(1 - 2\eta)^2(2a_2 - a_3) + (1 - 2\eta)(3 - 2\eta)a_5$$

$$b_{28} = -(18 + 4\eta^2)a_1 + 16\eta a_5$$

$$b_{29} = 4\eta(1 - 2\eta)a_1$$

$$b_{30} = (8 - 32\eta + 2\eta^2)a_1 + (8 - 56\eta + 16\eta^2)a_2 + (1 - 2\eta)(20 - 44\eta)a_3$$
$$+ 96(1 - 2\eta)^2 a_4 - (6 - 2\eta)a_5$$

$$b_{31} = (8 + 4\eta + 40\eta^2)a_1 - 16(2 - 7\eta + 2\eta^2)a_2 + 8(1 + 3\eta - 4\eta^2)a_3$$
$$- 96(1 - 2\eta)^2 a_4 - 4\eta a_5$$

$$b_{32} = -8(19\eta - 7\eta^2)a_1 - 64(2\eta - \eta^2)a_3$$

$$b_{33} = (66\eta - 44\eta^2)a_1 + 32\eta(1-2\eta)a_2 + 32\eta^2 a_3 - 8\eta(3-2\eta)a_5$$

$$b_{34} = \frac{17 + 16\eta - 4\eta^2}{2}a_1 - 16(1-2\eta)a_2 + 2(1-4\eta^2)a_3 - 3(5-2\eta)a_5$$

$$b_{35} = \frac{5 - 16\eta + 12\eta^2}{2}a_1 + 4(1-2\eta)^2 a_2 - (1-2\eta)(6-4\eta)a_3 - 3(1-2\eta)a_5$$

$$b_{36} = -\frac{(1-2\eta)(13+2\eta)}{2}a_1 + (1-2\eta)(10-4\eta)a_3 + 6(1-2\eta)a_5$$

$$b_{37} = -2(1-2\eta)(9-5\eta)a_1 + 2(1-2\eta)(5-2\eta)a_5$$

$$b_{38} = 2 - (1-2\eta)^2(a_1 - a_5)$$

$$b_{39} = (2 - \eta + \eta^2)a_1 + (1-2\eta)(2-2\eta)(2a_2 - a_3) + (1-\eta)a_5$$

$$b_{40} = 2(1-2\eta)^2(2a_2 - a_3)$$

$$b_{41} = -4(2-\eta)^2(a_1 - a_5)$$

$$b_{42} = (12 - 2\eta + 6\eta^2)a_1 - (1-2\eta)(4+4\eta)(2a_2 - a_3) - (6 - 10\eta + 4\eta^2)a_5$$

$$b_{43} = -4(12\eta - 11\eta^2)a_1 + 8(3-2\eta)a_2 - 8(1-2\eta)(5-7\eta)a_3$$
$$- 96(1-2\eta)^2 a_4 + 20\eta(1-2\eta)a_5$$

$$b_{44} = (9 - 17\eta + 6\eta^2)a_1 + 2(1-2\eta)^2(2a_2 - a_3) + (1-2\eta)(2-\eta)a_5$$

$$b_{45} = (4 - 9\eta + 2\eta^2)a_1 + 2(1-2\eta)^2(2a_2 - a_3) - (1-2\eta)(2-\eta)a_5$$

$$b_{46} = -(1-2\eta)^2(a_1 - a_5)$$

$$b_{47} = (32\eta - 24\eta^2)a_1 + 16\eta(1-2\eta)(2a_2 - a_3) - 8\eta(3+2\eta)a_5$$

$$b_{48} = (16 + 24\eta - 108\eta^2)a_1 + 32(2\eta - 2\eta^2)a_3 + 128\eta^2 a_5$$

$$b_{49} = -(2 - 34\eta + 40\eta^2)a_1 - 4(1-2\eta)^2(2a_2 - a_3) + (1-2\eta)(8-4\eta)a_5$$

$$b_{50} = (8 - 12\eta + 6\eta^2)a_1 + (1-2\eta)(4-4\eta)(2a_2 - a_3) - (6 - 10\eta + 4\eta^2)a_5$$

$$b_{51} = -(1-2\eta)(30-12\eta)a_1 - 8(1-2\eta)^2 a_3 + 12(1-2\eta)a_5$$

$$b_{52} = -2(1-2\eta)^2 a_1$$

$$b_{53} = -3(1-2\eta)^2 a_1 - 4(1-2\eta)^2 a_2 - 8(1-2\eta)a_3 - (1-2\eta)(3+2\eta)a_5$$

$$b_{54} = (1-2\eta)(17-10\eta)a_1 + 8(1-2\eta)^2 a_2 + 16(1-2\eta)a_3$$

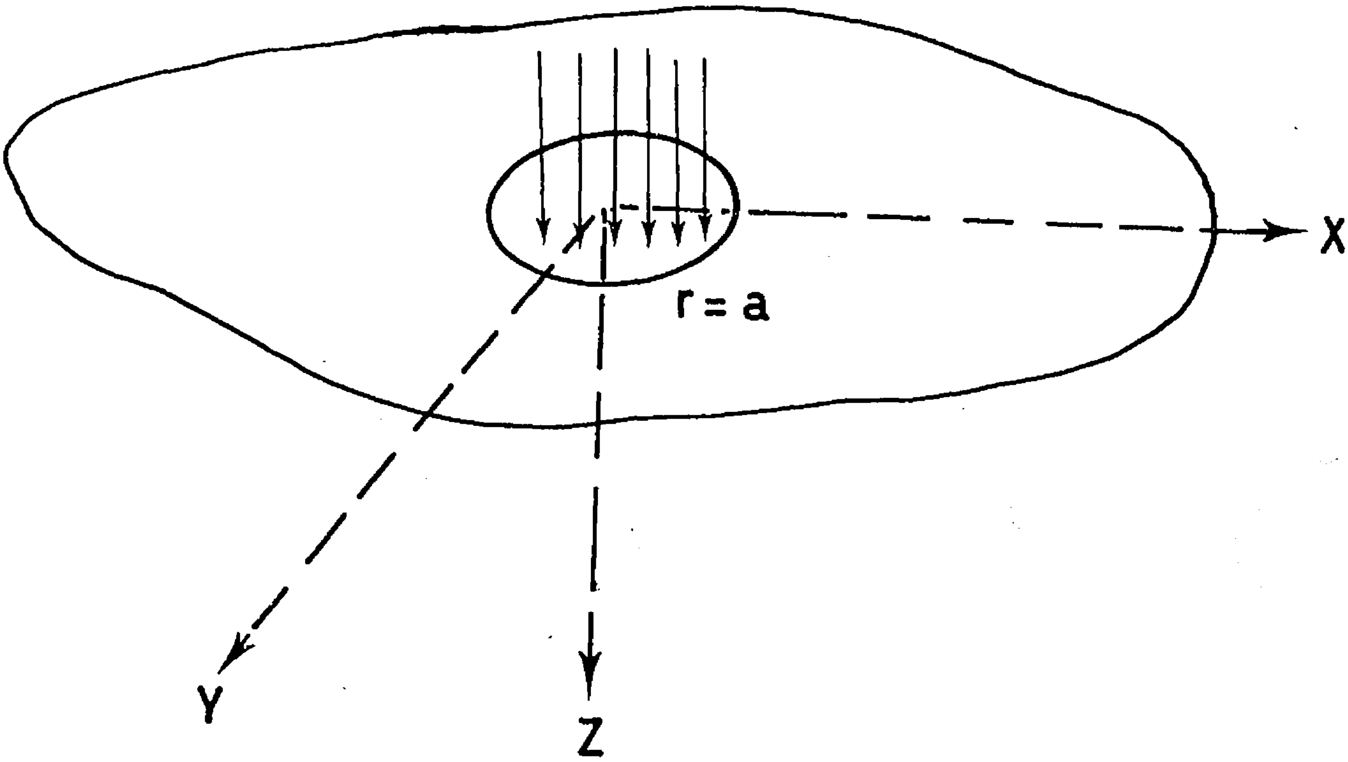

Fig. 1. Normal Loading

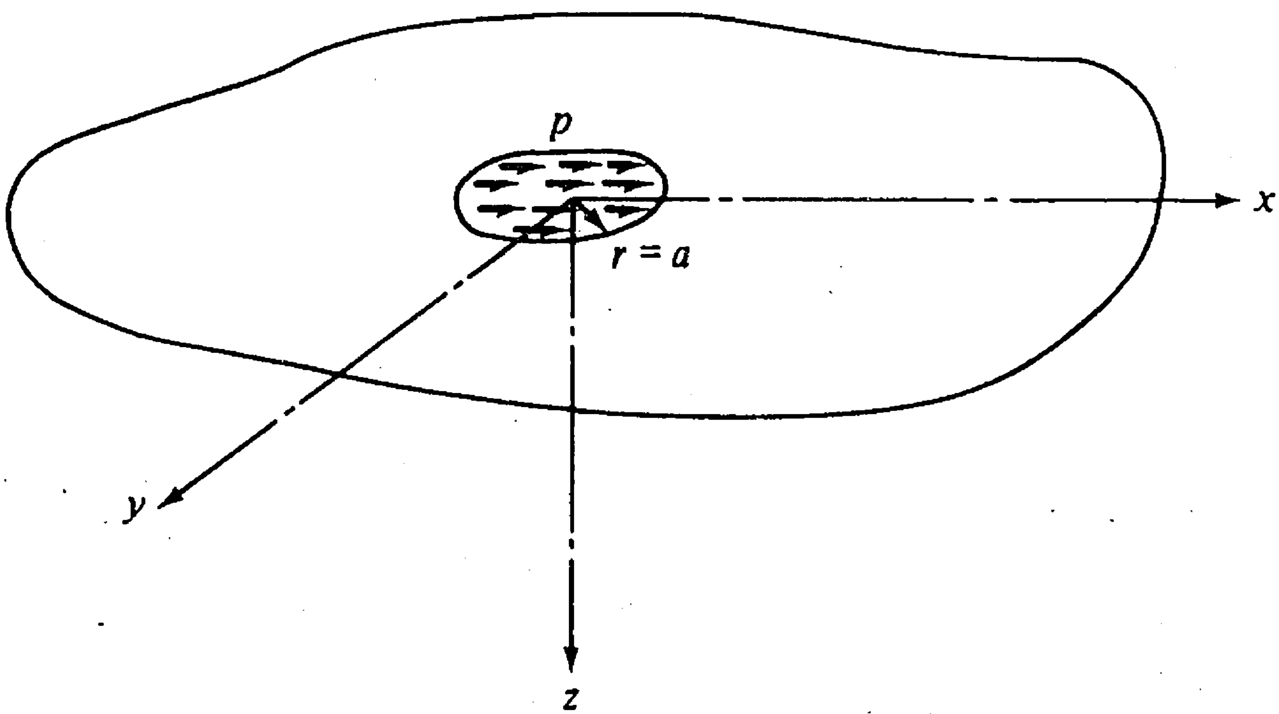

Fig. 2. Shear Loading

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