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A geometric algebra approach ton-qubit systems

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A Geometric Algebra Approach to n-Qubit Systems

by

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Abstract

A geometric formalism for the treatment of n-qubit systems is presented in terms of the Clifford's Geometric algebra $C\ell_m$, as an alternative to the traditional matrix formulation. This objective is accomplished by generalizing the one-qubit system formulated in terms of the bivector space of $C\ell_3$. This formulation is based in the well known isomorphism between the $so(3)$ and $su(2)$ Lie algebras.

It is known that a quantum system with N orthogonal states (levels) is controllable with the $SU(N)$ group. However, a system with an even number of orthogonal states may also be state-controllable with the $USp(N)$ group, which is a subgroup of $SU(N)$ with a Lie algebra isomorphic with the Lie algebra of the symplectic group $Sp(N)$.

The isomorphism between the $sp(4)$ and $spin(5)$ Lie algebras allows the formulation of a two-qubit system in terms of the the bivector space of $C\ell_5$, as a natural instance of the $spin(5)$ Lie algebra. Another isomorphism between the $sp(4)$ Lie algebra and the anti-Hermitian space of $C\ell_4$ is revealed, therefore allowing the formulation of a two-qubit system in terms of $C\ell_4$ as well.

More isomorphisms are exposed between some subspaces of higher dimensional Clifford algebras with the Lie algebras of the $USp(N)$ and $SU(N)$ groups. The immediate consequence is the possibility to represent an n-qubit

system in terms of some anti-Hermitian space of some Clifford algebra.

Another threated problem is the calculation of the average fidelity and it is shown how the original continuous integral can be reduced to a finite series, leading to a formula easier to evaluate.

Finally, an equivalence between Fermion algebras and Clifford algebras is presented, revealing a completely different way to think about geometry.

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Contents

Chapter 1

Introduction

The main reason to pursue new formations of physics in terms of Clifford's Geometric algebra is to gain insight that may be hidden when expressed in terms of the standard formalism. Clifford algebras have already successfully described relativistic quantum mechanics of a single particle [1, 2]. Some attempts have been made to develop many-body quantum mechanics in terms of Clifford algebras, most of them using the direct product of single qubit systems [3, 4]. This work explores the possibility of using higher-dimensional spaces and avoiding the use of the direct product.

The formalism of a single qubit is introduced in this chapter, and at the same time a gentle introduction to the formalism of Clifford algebras is given. Some details of this task can be found in the Appendices.

1.1 Fundamental Axiom

Clifford algebras assume the existence of an associative product of vectors. The fundamental axiom, from which the complete formalism of Clifford algebras of Euclidean spaces can be obtained, states that the square of a

vector $\mathbf{v}$ is a scalar. In fact, the square of a vector is identified as the scalar product (dot product) with itself

$$\mathbf{v}^2 \equiv \mathbf{vv} = \mathbf{v} \cdot \mathbf{v}. \tag{1.1}$$

The generalized associative product between vectors is also called the *geometric product.* Additional insight is gained letting $\mathbf{v} \rightarrow \mathbf{u} + \mathbf{v}$ in the fundamental axiom

$$(\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}). \tag{1.2}$$

After a distribution of the products we obtain

$$\mathbf{u}^2 + \mathbf{vu} + \mathbf{uv} + \mathbf{v}^2 = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \tag{1.3}$$

from which the scalar product of the two vectors is identified as the symmetric part of the geometric product

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu}). \tag{1.4}$$

An important consequence of this expression is that two vectors anticommute

$$\mathbf{uv} + \mathbf{vu} = 0, \tag{1.5}$$

if $\mathbf{u} \cdot \mathbf{v} = 0$.

The antisymmetric part of the geometric product is defined as the *wedge* product

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu}). \tag{1.6}$$

With this definition in hand, the geometric product can be written as

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}, \tag{1.7}$$

showing a decomposition of the geometric product of two vectors in terms of two elements, one being scalar and the other being some kind of higher order vector defined as a *bivector.*

1.2 Orthonormal Basis

The formalism takes more concrete shape when an orthonormal basis is defined. The fundamental axiom leads to the following identity obeyed by the n basis vectors $\mathbf{e}_j$ of an n-dimensional Euclidean space

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2\delta_{jk}, \tag{1.8}$$

This condition implies that two different orthonormal vectors anti commute

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j, \quad j \neq k \tag{1.9}$$

and the square of any basis vector is one

$$\mathbf{e}_j\mathbf{e}_j = 1. \tag{1.10}$$

The product of vectors is well defined and allows us to extend the basis to all the possible multivector products in the Clifford algebra, which is denoted as $C\ell_n$ for an n-dimensional Euclidean space.

In order to optimize the notation, a multi-product of basis elements is usually represented with the help of multiple indexes. For example, the bivector $\mathbf{e}_1\mathbf{e}_2$ is represented as

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12}, \tag{1.11}$$

such that the geometric product is implicitly understood and for example

$$\mathbf{e}_{12} = -\mathbf{e}_{21}. \tag{1.12}$$

All the possible distinct biproducts of the basis vectors define the bivec-

tor subspace of $C\ell_n$

$$\{\mathbf{e}_{jk}\} = \left\{ \begin{array}{ccccc} \mathbf{e}_1\mathbf{e}_2 & & & & \\ \mathbf{e}_1\mathbf{e}_3 & \mathbf{e}_2\mathbf{e}_3 & & & \\ \mathbf{e}_1\mathbf{e}_4 & \mathbf{e}_2\mathbf{e}_4 & \mathbf{e}_3\mathbf{e}_4 & & \\ \vdots & & & & \\ \mathbf{e}_1\mathbf{e}_n & \mathbf{e}_2\mathbf{e}_n & \mathbf{e}_3\mathbf{e}_n & \ldots & \mathbf{e}_{n-1}\mathbf{e}_n. \end{array} \right\} \tag{1.13}$$

The complete Clifford algebra $C\ell_n$ contains all the possible multivectors such that the complete basis can be constructed as

$$\{C\ell_n\} = \{1, \mathbf{e}_{k_1}, \mathbf{e}_{k_1k_2}, \mathbf{e}_{k_1k_2k_3}, ..., \mathbf{e}_{k_1...k_n}\}, \quad k_1 < k_2 < k_3...k_n \tag{1.14}$$

Scalars are defined as grade 0 elements, vectors are defined as grade 1 elements, bivectors are defined as grade 2 elements and so on, depending on the number of vectors that participate in the product.

The number of distinct $C\ell_n$ bivector basis elements defines the dimension of the bivector subspace $C\ell_n$

$$Dimension(Bivector\ C\ell_n) = \binom{n}{2}. \tag{1.15}$$

Following with the trivectors in $C\ell_n$, we see that

$$Dimension(Trivector\ C\ell_n) = \binom{n}{3}, \tag{1.16}$$

and so on until a highest grade multivector is obtained. The highest-grade multivector basis is a single element and called the *volume element.*

The number of all the basis elements including all the different grades defines the dimension of the complete $C\ell_n$ algebra

$$Dimension(C\ell_n) = \sum_{m=0}^{n} \binom{n}{m} = 2^n \tag{1.17}$$

For example, the $C\ell_3$ algebra contains 4 different grades with total dimension 8 as shown in Table 1.1, where the trivector plays the role of the volume element.

Grade	Denomination	Basis element
0	Real Scalar	1
1	Vector	$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$
2	Bivector	$\mathbf{e}_{12}$, $\mathbf{e}_{23}$, $\mathbf{e}_{31}$
3	Trivector (Pseudoscalar)	$\mathbf{e}_{123}$

Table 1.1. The $C\ell_3$ algebra and its different elements.

An arbitrary element that belongs to the $C\ell_n$ algebra is called a Clifford number. Any Clifford number can be expressed as a linear superposition of a complete basis, and if the superposition is only composed of elements of the same grade, this Clifford number is called homogeneous.

1.3 Reversion Conjugation

The reversion conjugation † is an anti-automorphism that reverses the order of the products, such that

$$(AB)^\dagger = B^\dagger A^\dagger, \tag{1.18}$$

where real scalars and vectors are defined to be invariant

$$\mathbf{a}^\dagger = \mathbf{a} \tag{1.19}$$

$$1^\dagger = 1 \tag{1.20}$$

The reversion conjugation applied to a homogeneous multivector, which comprises elements of a single grade, either maintains the element invariant or changes its sign. The reversion conjugation plays the role of the

Hermitian conjugation in the standard matrix representation of $\mathcal{C}\ell_n$, so the homogeneous mutivectors that do not change sign are defined as Hermitian. On the other hand, the homogeneous multivectors that change sign are defined as anti-Hermitian. Hermitian Clifford numbers are made of elements of grades 0, 1 or whenever the grade is 0 or 1 mod 4. The remaining grades make anti-Hermitian Clifford numbers. The Hermitian or anti-Hermitian classification of some homogeneous multivectors is given in Table 1.2.

Grade	Classification
0	Hermitian
1	Hermitian
2	Anti-Hermitian
3	Anti-Hermitian
4	Hermitian
5	Hermitian
6	Anti-Hermitian
7	Anti-Hermitian
⋮	⋮

Table 1.2. Grades with their classification as Hermitian or anti-Hermitian.

The classification in terms of Hermitian and anti-Hermitian elements is important because the exponential function of anti-Hermitian multivectors is expanded in terms of trigonometric functions (sin and cos) and is thus bounded, whereas the exponential of Hermitian multivectors is expanded in terms of hyperbolic functions and is generally unbounded.

Another commonly used conjugation is the *Clifford conjugation* defined in Appendix A.

1.4 Matrix Representation

A multivector basis can be represented through the use of a suitable set of matrices. A matrix representation of the vector $C\ell_3$ algebra is given by the Pauli matrices shown in Table 1.3, where the identity was introduced with the subscript 0. A matrix representation is said to be faithful whenever each

$\mathbf{1} = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathbf{e}_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{e}_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\mathbf{e}_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Table 1.3. The Pauli matrices representing orthonormal vectors and the identity representing the unit real scalar. All of them producing a faithful representation of the $C\ell_3$ algebra

matrix represents a unique element of the basis. There are an infinite number of matrix representations, but we are mostly interested in the smallest ones that are still faithful.

It is possible to obtain matrix representations for higher dimensions in terms of the direct product of the Pauli matrices. Once such a vector matrix representation is given, it is straightforward to calculate the matrix representation of the complete $C\ell_n$ algebra. Representations that use quaternion numbers are not considered in this thesis.

A possible matrix representation of the vectors of $C\ell_5$ and $C\ell_7$ are given in Table 1.4. The rest of the algebra can be found by calculating the respective multiproducts, but this does not guarantee a faithful representation. It can be shown that faithful representations of $C\ell_4$ and $C\ell_7$ can be found in terms of 4×4 and 8×8 matrices, respectively. On the other hand,

Basis element	Representation
$\mathbf{e}_1$	$\sigma_3 \otimes \sigma_1$
$\mathbf{e}_2$	$\sigma_3 \otimes \sigma_2$
$\mathbf{e}_3$	$\sigma_3 \otimes \sigma_3$
$\mathbf{e}_4$	$-\sigma_2 \otimes \sigma_0$
$\mathbf{e}_5$	$-\sigma_1 \otimes \sigma_0$

Basis element	Representation
$\mathbf{e}_1$	$\sigma_0 \otimes \sigma_3 \otimes \sigma_1$
$\mathbf{e}_2$	$\sigma_0 \otimes \sigma_3 \otimes \sigma_2$
$\mathbf{e}_3$	$\sigma_0 \otimes \sigma_3 \otimes \sigma_3$
$\mathbf{e}_4$	$\sigma_0 \otimes \sigma_2 \otimes \sigma_0$
$\mathbf{e}_5$	$\sigma_3 \otimes \sigma_1 \otimes \sigma_0$
$\mathbf{e}_6$	$\sigma_1 \otimes \sigma_1 \otimes \sigma_0$
$\mathbf{e}_7$	$\sigma_2 \otimes \sigma_1 \otimes \sigma_0$

Table 1.4. Possible Matrix representation of the orthonormal vectors in $\mathcal{C}\ell_5$ and $\mathcal{C}\ell_7$. Even though the $\mathcal{C}\ell_5$ vectors allow a matrix representation in terms of 4×4 complex matrices, this does not generate a faithful representation of the universal $\mathcal{C}\ell_5$ algebra because the volume element is represented by the identity matrix. A representation in terms of 8×8 complex matrices is sufficient to provide a faithful representation of the complete $\mathcal{C}\ell_5$ algebra

a representation of $\mathcal{C}\ell_5$ in terms of 4×4 matrices is not faithful because the volume element is represented by the identity matrix. Table 1.5 shows that a given matrix represents both a vector and a four-vector in $\mathcal{C}\ell_5$. In similar way, both bivectors and trivectors share the same set of matrices and finally, the volume element shares the the same representation as the unit real. This means that in this representation, a basis element and its product with the volume element share the same matrix representation. A

Basis element	Representation	Basis element	Representation
$\mathbf{e}_1$	$\sigma_3 \otimes \sigma_1$	$\mathbf{e}_{1234}$	$-\sigma_1 \otimes \sigma_0$
$\mathbf{e}_2$	$\sigma_3 \otimes \sigma_2$	$\mathbf{e}_{1235}$	$\sigma_2 \otimes \sigma_0$
$\mathbf{e}_3$	$\sigma_3 \otimes \sigma_3$	$\mathbf{e}_{1245}$	$\sigma_3 \otimes \sigma_3$
$\mathbf{e}_4$	$-\sigma_2 \otimes \sigma_0$	$\mathbf{e}_{1345}$	$-\sigma_3 \otimes \sigma_2$
$\mathbf{e}_5$	$-\sigma_1 \otimes \sigma_0$	$\mathbf{e}_{2345}$	$\sigma_3 \otimes \sigma_1$

Table 1.5. Matrix representation of vectors and 4-vectors of $\mathcal{C}\ell_5$ in terms of 4×4 matrices, showing the use of the same set of matrices.

faithful representation of $\mathcal{C}\ell_5$ is possible in terms of 8×8 matrices within the matrix representation of $\mathcal{C}\ell_7$.

Non-faithful matrix representations may be well suited for some applications but they must always be used carefully. The $\mathcal{C}\ell_5$ algebra does not have a faithful representation in terms of 4×4 matrices but this is not important if only vectors or bivectors are used. The matrix representation of the bivector $\mathcal{C}\ell_5$ algebra, shown in Table 1.6, is good enough to represent the $spin(5)$ Lie algebra, and this is useful in describing 2-qubit systems.

The *complexification* of a Clifford algebra, $\mathcal{C}\ell_n \times \mathbb{C}$, is possible by involving the imaginary number i in the abstract algebra (before the introduction of the matrix representation). This doubles the dimension of the $\mathcal{C}\ell_n$ alge-

Bivector	Representation	Bivector	Representation
$\mathbf{e}_{12}$	$i\sigma_0 \otimes \sigma_3$	$\mathbf{e}_{24}$	$i\sigma_1 \otimes \sigma_2$
$\mathbf{e}_{13}$	$-i\sigma_0 \otimes \sigma_2$	$\mathbf{e}_{25}$	$-i\sigma_2 \otimes \sigma_2$
$\mathbf{e}_{14}$	$i\sigma_1 \otimes \sigma_1$	$\mathbf{e}_{34}$	$i\sigma_1 \otimes \sigma_3$
$\mathbf{e}_{15}$	$-i\sigma_2 \otimes \sigma_1$	$\mathbf{e}_{35}$	$-i\sigma_2 \otimes \sigma_3$
$\mathbf{e}_{23}$	$i\sigma_0 \otimes \sigma_1$	$\mathbf{e}_{45}$	$-i\sigma_3 \otimes \sigma_0$

Table 1.6. Matrix representation of the bivector algebra of $C\ell_5$ in terms of 4×4 matrices.

bra, because for each element $\mathbf{e}_{jkl\ldots}$ there is another $i\mathbf{e}_{jkl\ldots}$. An important application of a *complexified* Clifford algebra is the use of the anti-Hermitian $C\ell_4 \times \mathbb{C}$ algebra to represent the $su(4)$ Lie algebra.

Another accidental isomorphism exists between the anti-Hermitian $C\ell_4 \times \mathbb{C}$ space and the bivectors of $C\ell_6$, and it is explictly shown in Table 1.7.

Anti-Hermitian $\mathcal{C}\ell_4 \times \mathbb{C}$	Bivector $\mathcal{C}\ell_6$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$
$\mathbf{e}_{14}$	$\mathbf{e}_{14}$
$\mathbf{e}_{234}$	$\mathbf{e}_{15}$
$i\mathbf{e}_1$	$\mathbf{e}_{16}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$
$\mathbf{e}_{24}$	$\mathbf{e}_{24}$
$\mathbf{e}_{134}$	$-\mathbf{e}_{25}$
$i\mathbf{e}_2$	$\mathbf{e}_{26}$
$\mathbf{e}_{34}$	$\mathbf{e}_{34}$
$\mathbf{e}_{124}$	$\mathbf{e}_{35}$
$i\mathbf{e}_3$	$\mathbf{e}_{36}$
$\mathbf{e}_{123}$	$-\mathbf{e}_{45}$
$i\mathbf{e}_4$	$\mathbf{e}_{46}$
$i\mathbf{e}_{1234}$	$\mathbf{e}_{56}$

Table 1.7. Table of the isomorphic elements of the anti-Hermitian $\mathcal{C}\ell_4 \times \mathbb{C}$ algebra and the bivector $\mathcal{C}\ell_6$ algebra. Both of them generate a Lie algebra isomorphic to $su(4)$.

1.5 Embedded Compact Lie Algebras

It is known that Clifford algebras provide an important alternative to matrices to represent the generators of the classical Lie algebras (see Appendix B), as shown by Doran *et al.* [5]. This reference describes how to construct any classical Lie algebra within the bivector algebra of a certain Clifford algebra, not necessarily Euclidean. In this thesis we will explore the novel possibility of using the complete set of anti-Hermitian elements of certain Clifford algebras for similar purposes.

The study of the representation of n-qubit systems is important because they could ultimately help us to device better control schemes and more efficient simulations. Some example of these new possibilities were given by Rau [6] and Uskov [7].

The bivector space of $\mathcal{C}\ell_N$ directly gives an instance of the $spin(N)$ Lie algebra and the exponential of the bivector algebra gives an instance of the $Spin(N)$ group. The $Spin(N)$ group is known as the double cover of the $SO(N)$ group because for each element of $SO(N)$ there are two equivalent elements in $Spin(N)$. Given that $O^j{}_k$ is the matrix representation of an element of the $SO(N)$ group and that r^k are the components of a vector, the rotation is performed as

$$r^j \rightarrow r'_j = O^j{}_k r^k \tag{1.21}$$

On the other hand, if R is an element of the $Spin(N)$ group and the vector is represented as $\mathbf{r} = r^j \mathbf{e}_j$, the rotation is performed in spinorial form as a unitary transformation

$$\mathbf{r} \rightarrow \mathbf{r}' = R\mathbf{r}R^\dagger, \tag{1.22}$$

where we notice that both R and $-R$ can be used for the same purpose.

The bivector subspace is part of the larger anti-Hermitian subspace of the respective Clifford algebra. A fundamental observation provided in this thesis is that the anti-Hermitian subspace can be used to form *semisimple* Lie algebras.

We are mostly interested in semisimple Lie algebras, characterized by not containing invariant closed subalgebras. This means that any element that commutes with the rest of the algebra must be disregarded as a potential generator. This is a possibility for the volume element in some odd-dimensional Clifford algebras $\mathcal{C}\ell_n$ such as $n = 3, 7, 11, ...$ or $(n+1) \mod 4 = 0$ in general. The anti-Hermitian subspace without the volume element in these dimensions will be denoted with a star super-index as anti-Hermitian*. If the Clifford algebra is complexified, the i element is also automatically excluded as a potential generator.

Some of the Lie algebras that can be constructed within Clifford algebras are shown in Table 1.8. The proof that the anti-Hermitian $\mathcal{C}\ell_4$ forms a $sp(4)$ Lie algebra is shown in Appendix C. The proof of the rest of the isomorphisms is not provided in this thesis because the procedures are very similar and not particularly illuminating. This table shows symplectic Lie algebras for 2 and 3 qubits but not for 4 and 5 qubits. This does not mean that they do not exist. This only means the the $sp(16)$ and $sp(32)$ Lie algebras do not have simple representations in terms of Clifford algebras $\mathcal{C}\ell_n$. For example, the $sp(16)$ Lie algebra can be constructed with the union of the 120 elements of the $anti-Hermitian\, \mathcal{C}\ell_8$ and the 16 elements shown in Table 1.9

There is an accidental isomorphism between the Lie algebra of the anti-Hermitian $\mathcal{C}\ell_4$ algebra and the $spin(5)$ Lie algebra, and for this reason they are going to be used for similar purposes. Table 1.10 shows the elements of

$\mathcal{C}\ell_n$ subspace	Qubits	Generators	Lie algebra
Bivector $\mathcal{C}\ell_3$	1	3	$su(2)$
Anti-Hermitian $\mathcal{C}\ell_4$	2	10	$sp(4) \sim spin(5)$
Anti-Hermitian $\mathcal{C}\ell_4 \times \mathbb{C}$	2	15	$su(4) \sim spin(6)$
Anti-Hermitian $\mathcal{C}\ell_5$	3	20	$spin(5) \times spin(5)$
Anti-Hermitian $\mathcal{C}\ell_6$	3	36	$sp(8)$
Anti-Hermitian $\mathcal{C}\ell_6 \times \mathbb{C}$	3	63	$su(8)$
Anti-Hermitian* $\mathcal{C}\ell_7$	3	63	$su(8)$
Anti-Hermitian $\mathcal{C}\ell_8 \times \mathbb{C}$	4	255	$su(16)$
Anti-Hermitian $\mathcal{C}\ell_{10} \times \mathbb{C}$	5	1023	$su(32)$
Anti-Hermitian* $\mathcal{C}\ell_{11}$	5	1023	$su(32)$
Anti-Hermitian $\mathcal{C}\ell_{12}$	6	2080	$sp(64)$

Table 1.8. Some anti-Hermitian algebras and their Lie algebras. The * identifies the anti-Hermitian algebras from which the volume element has been removed.

$-ie_{1478} - ie_{1568}$
$ie_{2468} - ie_{1478}$
$ie_{2468} - ie_{2578}$
$\frac{i}{2}e_{4568} - \frac{i}{2}e_{12368}$
$\frac{i}{2}e_{4578} - \frac{i}{2}e_{123678}$
$\frac{i}{4}e_{3468} - \frac{i}{4}e_{3578} - \frac{i}{4}e_{12478} - \frac{i}{4}e_{12568}$
$\frac{i}{4}e_{3478} - \frac{i}{4}e_{3568} - \frac{i}{4}e_{12468} - \frac{i}{4}e_{12578}$
$ie_{13468} - ie_{13578}$
$\frac{i}{2}e_{2368} - \frac{i}{2}e_{14568}$
$\frac{i}{2}e_{2378} - \frac{i}{2}e_{14578}$
$-ie_{13578} - ie_{23478}$
$ie_{13468} - ie_{23568}$
$\frac{i}{2}e_{1368} + \frac{i}{2}e_{24568}$
$-\frac{i}{2}e_{1378} - \frac{i}{2}e_{24578}$
$\frac{i}{2}e_{34568} - \frac{i}{2}e_{1268}$
$\frac{i}{2}e_{34578} - \frac{i}{2}e_{1278}$

Table 1.9. The 16 elements that have to be added to the anti-Hermitian $C\ell_8$ algebra to form a basis for the $sp(16)$ Lie algebra.

both algebras and their equivalent elements.

Bivector $\mathcal{C}\ell_5$	Anti-Hermitian $\mathcal{C}\ell_4$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$
$\mathbf{e}_{14}$	$\mathbf{e}_{14}$
$\mathbf{e}_{15}$	$\mathbf{e}_{234}$
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$
$\mathbf{e}_{24}$	$\mathbf{e}_{24}$
$\mathbf{e}_{25}$	$-\mathbf{e}_{134}$
$\mathbf{e}_{34}$	$\mathbf{e}_{34}$
$\mathbf{e}_{35}$	$\mathbf{e}_{124}$
$\mathbf{e}_{45}$	$-\mathbf{e}_{123}$

Table 1.10. Equivalence table of the isomorphic Lie algebras *spin*(5)(bivectors in $\mathcal{C}\ell_5$) and the anti-Hermitian subspace of $\mathcal{C}\ell_4$ as a Lie algebra.

1.6 Single qubit systems

A single qubit system is characterized by having two orthogonal states that are linearly combined with complex coefficients to form the general state

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle. \tag{1.23}$$

These coefficients are normalized such that

$$|c_1|^2 + |c_2|^2 = 1, \tag{1.24}$$

so that the number of degrees of freedom is reduced to 3 (each complex coefficient contributes with two degrees of freedom). The notation indicates

the orthogonal states of a spin-1/2 system, but they could represent the orthogonal states of any 2-level system.

The general state can be characterized by the orientation of a rigid body in three dimensions, which can be defined in terms of the $SO(3)$ rotation group. However, $SU(2)$, which is the double cover of $SO(3)$, is the right choice, because it can be used to perform unitary transformations on the states. This isomorphism is the reason for the strong geometrical content of a single-qubit system, which was rediscovered in many occasions [8, 9]. The rest of this chapter follows the $C\ell_3$ approach [10]. However there are other alternatives in terms of the *space-time* algebra, developed with the relativistic case in mind [11, 3, 4, 12, 13].

Given that $\mathbf{r}$ is a vector in $C\ell_3$, it can be expanded in a basis as

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3. \tag{1.25}$$

The spinorial form of a rotation takes the form

$$\mathbf{r} \rightarrow \mathbf{r}' = R\mathbf{r}R^\dagger, \tag{1.26}$$

where the so called *rotor* R, is written as an exponential of the $C\ell_3$ bivector algebra (an instance of the $su(2)$ Lie algebra)

$$R = e^A, \quad A \in su(2). \tag{1.27}$$

A more explicit form of the rotor is

$$R = e^{\theta\mathbf{B}/2}, \tag{1.28}$$

where $\mathbf{B}$ is a $C\ell_3$ bivector such that $\mathbf{B}^2 = -1$. This bivector defines the plane of rotation and the real number θ defines the rotation angle. For example,

rotations in the three orthogonal planes are carried out by the following rotors

$$R_{12} = e^{-\phi \mathbf{e}_{12}/2} \tag{1.29}$$

$$R_{23} = e^{-\eta \mathbf{e}_{23}/2} \tag{1.30}$$

$$R_{31} = e^{-\theta \mathbf{e}_{31}/2}, \tag{1.31}$$

The general orientation of a rigid body many be given by the following rotor

$$R = e^{-\phi \mathbf{e}_{12}/2} e^{-\theta \mathbf{e}_{31}/2} e^{-\eta \mathbf{e}_{12}/2} \tag{1.32}$$

In the three dimensional space, described by $C\ell_3$, we usually associate a rotation with a certain axis. This is possible because in $C\ell_3$ there are three vectors and three bivectors, and in general it is possible to associate a vector with a bivector. Each vector is dual to the bivector of the orthogonal plane. This is not possible in higher dimensions, where the only consistent way to perform a rotation is by defining the plane in which the rotation takes place.

The *spinor* wave function is introduced as the projection of the rotor

$$\Psi = RP_3, \tag{1.33}$$

where the projector, P_3, is a *paravector* (see Appendix D) defined as

$$P_3 = \frac{1}{2}(1 + \mathbf{e}_3), \tag{1.34}$$

where the unitary vector along $\mathbf{e}_3$ was chosen arbitrarily. This projector as any projector, is *idempotent*

$$P_3 = P_3 P_3 = P_3 P_3 P_3 \ldots . \tag{1.35}$$

This can be proved either though direct computation or by observing its matrix representation

$$P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.36}$$

which also shows that P_3 does not have inverse. On the other hand, the complementary projector is

$$\overline{P}_3 = \frac{1}{2}(1 - \mathbf{e}_3), \tag{1.37}$$

with matrix representation

$$\overline{P}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.38}$$

The resolution of the identity is easily seen as

$$1 = P_3 + \overline{P}_3. \tag{1.39}$$

These complementary projectors are the eigenfunctions of $\mathbf{e}_3$ because the following identities are verified by using the pacwoman property (Appendix E) or direct computation

$$\mathbf{e}_3 P_3 = (+1)P_3 \tag{1.40}$$

$$\mathbf{e}_3 \overline{P}_3 = (-1)\overline{P}_3. \tag{1.41}$$

The two orthogonal states can be identified as

$$|\uparrow\rangle = P_3 \tag{1.42}$$

$$|\downarrow\rangle = \mathbf{e}_1 P_3, \tag{1.43}$$

where the second state can be reached from the first one by the following unitary transformation

$$|\downarrow\rangle = e^{\pi \mathbf{e}_{13}/2} P_3. \tag{1.44}$$

In this way, the matrix representation of the states are

$$|\uparrow\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.45}$$

$$|\downarrow\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.46}$$

Using the pacwoman property, the spinor wave function can always be written in terms of the following null basis of paravectors (Appendix D)

$$\{P_3, \mathbf{e}_1 P_3\} \tag{1.47}$$

so that, in general, the following expansion is possible

$$\Psi = \psi_1 P_3 + \psi_2 \mathbf{e}_1 P_3, \tag{1.48}$$

where the coefficients, ψ_1 and ψ_2, belong to the combined scalar-pseudoscalar $\mathcal{C}\ell_3$ algebra. This algebra is isomorphic to the standard complex algebra, where the volume element $\mathbf{e}_{123}$ plays the role of the imaginary number i.

Once the matrix representation is used, the spinor wave function has the general form

$$\Psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}, \tag{1.49}$$

where the components, ψ_1 and ψ_2, are now just two standard complex numbers.

The formulation of density matrices in terms of Clifford algebras is straightforward. If R is a rotor and P is the projector, the expression that plays the role of the density matrix for the pure state, ΨP, is

$$\rho = \Psi\Psi^\dagger = RPR^\dagger. \tag{1.50}$$

Using the explicit projector (1.34), this becomes

$$\rho = \frac{1}{2}(1 + R\mathbf{e}_3 R^\dagger), \tag{1.51}$$

but the effect of a general rotor directs the initial vector along an arbitrary direction denoted by a unit vector $\hat{\mathbf{p}}$

$$\rho = \frac{1}{2}(1 + \hat{\mathbf{p}}). \tag{1.52}$$

This expression in the traditional notation that uses the Pauli matrices is

$$\rho = \frac{1}{2}(1 + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}). \tag{1.53}$$

However, this notation is very limiting because of many reasons including the fact that can be used only in three dimensions.

The action of a unitary transformation, carried out by the rotor, can be understood as a change in the spin polarization of the pure state. In fact, the direction of the polarization, $\hat{\mathbf{p}}$, determines the state of the wave function up to a global phase. The sphere defined by the range of the unit vector, $\hat{\mathbf{p}}$, is called Bloch sphere [8].

In the most general case of mixed states, the density matrix is formulated as a superposition of pure density matrices as

$$\rho = p_1 R_1 P R_1^\dagger + p_2 R_1 P R_1^\dagger + ... + p_m R_m P R_m^\dagger, \tag{1.54}$$

where the coefficients are normalized as probabilities

$$\sum_{j=1}^{j=m} p_j = 1. \tag{1.55}$$

This general density matrix can also be written in terms of a unit polarization vector as

$$\rho = \frac{1}{2}(1 + \alpha\, \hat{\mathbf{p}}), \tag{1.56}$$

with $\alpha = 1$ for pure states and $0 \leq \alpha < 1$ for mixed states, where the 0 value corresponds to the fully mixed (unpolarized) state.

The Von Neumann entropy is usually defined in terms of the trace of the matrix representation as

$$S = -Tr(\rho \ln(\rho)), \tag{1.57}$$

or as the scalar part in Clifford algebras (Appendixes F and G)

$$S(\alpha) = -2\langle \rho \ln(\rho) \rangle_S, \tag{1.58}$$

resulting in

$$S(\alpha) = -\left(\frac{(1+\alpha)}{2} \ln(\frac{1+\alpha}{2}) + \frac{(1-\alpha)}{2} \ln(\frac{1-\alpha}{2}) \right), \tag{1.59}$$

as shown in Appendix H. The entropy of a pure state is calculated in the following limit as

$$S(\alpha \rightarrow 1) = 0, \tag{1.60}$$

whereas the fully mixed state gives

$$S(\alpha \rightarrow 0) = \ln 2. \tag{1.61}$$

1.7 Interactions

A single-qubit system is in general a system with two levels. A spin-1/2 system with magnetic interaction, as studied in Nuclear Magnetic Resonance (NMR), and a two-level system, driven by an electromagnetic field, are two examples. Let us consider the following Hamiltonian driven by an electric field $E(t)$, where $\hbar$ is set to one for convenience

$$H = -\frac{\omega_0}{2}|1\rangle\langle 1| + \frac{\omega_0}{2}|2\rangle\langle 2| + E(t)(\mu_{12}|1\rangle\langle 2| + \mu_{12}^*|2\rangle\langle 1|). \tag{1.62}$$

This Hamiltonian can be written in terms of the Pauli matrices as

$$H = \frac{\omega_0}{2}\sigma_z + E(t)|\mu_{12}|\sigma_x e^{i\phi\sigma_z}, \tag{1.63}$$

where $\mu_{12} = |\mu_{12}|e^{i\phi}$. The electric field induces a transition of populations between the levels, but in general that process is inefficient if the magnitude

of the electric field times μ_{12} is small with respect to ω_0. However, there is a resonant method that allows very efficient transitions by tuning the frequency of the electric field. Let us define a harmonic electric field with frequency ν and write it as

$$E(t) = \epsilon \cos(\nu t) = \frac{\epsilon}{2}(e^{i\nu t\sigma_z} + e^{-i\nu t\sigma_z}). \tag{1.64}$$

Furthermore, let us define the Rabi frequency as $\Omega = \epsilon|\mu_{12}|$, so that the Hamiltonian becomes

$$H = \frac{\omega_0}{2}\sigma_z + \frac{\Omega}{2}e^{i(\nu t-\phi)\sigma_z}\sigma_x + \frac{\Omega}{2}e^{-i(\nu t+\phi)\sigma_z}\sigma_x. \tag{1.65}$$

Going further, the Hamiltonian can be written as

$$H = e^{-i(\nu t+\phi)/2\sigma_z}(\frac{\omega_0}{2}\sigma_z + \frac{\Omega}{2}e^{2i\nu t\sigma_z}\sigma_x + \frac{\Omega}{2}\sigma_x)e^{i(\nu t+\phi)/2\sigma_z}. \tag{1.66}$$

This form suggests the unitary transformation

$$\psi = U\Psi = e^{-i(\nu t+\phi)/2\sigma_z}\Psi \tag{1.67}$$

such that the Schrödinger equation becomes

$$(U^\dagger HU - iU^\dagger\dot{U})\Psi = i\dot{\Psi} \tag{1.68}$$

or explicitly

$$\left(\frac{1}{2}(\omega_0 - \nu)\sigma_z + \frac{\Omega}{2}\sigma_x + \frac{\Omega}{2}e^{2i\nu t\sigma_z}\sigma_x\right)\Psi = i\dot{\Psi}. \tag{1.69}$$

The term with the exponential is highly oscillating, so it can be neglected

$$\left(\frac{1}{2}(\omega_0 - \nu)\sigma_z + \frac{\Omega}{2}\sigma_x\right)\Psi = i\dot{\Psi}. \tag{1.70}$$

The solution of this equation is straightforward

$$\Psi = e^{-i\frac{1}{2}(\omega_0-\nu)t\sigma_z - i\frac{\Omega}{2}t\sigma_x}\Psi_0. \tag{1.71}$$

It is convenient to define a unit vector as

$$\hat{u} = \frac{(\omega_0 - \nu)\sigma_z + \Omega\sigma_x}{\sqrt{(\omega_0 - \nu)^2 + \Omega^2}} \tag{1.72}$$

so that the solution is conveniently written as

$$\Psi = \left(\cos(\frac{\sqrt{(\omega_0 - \nu)^2 + \Omega^2}}{2} t) - i\hat{u} \sin(\frac{\sqrt{(\omega_0 - \nu)^2 + \Omega^2}}{2} t) \right) \Psi_0. \tag{1.73}$$

The resonant case appears when $\omega_0 = \nu$ resulting in the most efficient condition for the transfer of populations.

Chapter 2

Controllability Condition

2.1 Notions of Controllability of Quantum Systems

An n-qubit system can be seen as a composition of n single-qubit systems. A simple form of a n-qubit state is

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes .. \otimes |\psi_n\rangle, \tag{2.1}$$

which is an example of a pure separable state. A complete basis for the n-qubit Hilbert space can be made of $N = 2^n$ states as

$$\begin{pmatrix} |\uparrow\rangle \otimes .. \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \\ |\uparrow\rangle \otimes .. \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \\ |\uparrow\rangle \otimes .. \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \\ |\uparrow\rangle \otimes .. \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \\ |\uparrow\rangle \otimes .. \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \\ \vdots \\ |\downarrow\rangle \otimes .. \otimes |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \end{pmatrix}. \tag{2.2}$$

These n-qubit systems are a subset of the more general quantum systems of finite discrete levels.

Each qubit can be represented by the projection of a rotor in $C\ell_3$ as explained in Chapter 1. This means that an n-qubit system could be represented in terms of the direct product of rotors in $C\ell_3$. For example, the state (2.1) could be represented as

$$|\Psi\rangle = (R_1 \otimes R_2 \otimes .. \otimes R_n)(P_3 \otimes P_3 \otimes .. \otimes P_3). \tag{2.3}$$

As an alternative, these systems can be also represented in terms of rotors in higher dimensions. Choosing a suitable Clifford algebra it is possible to avoid the use of the direct product. This approach is explicitly described in Chapters 2 and 3 for two and three qubits and is mainly developed in the pursuit of gaining more insight. Besides theoretical objectives, this can be also useful for designing computer programs that would rely only on the implementation of the geometric product and would avoid calculations in the matrix representation entirely.

The ability to represent n-qubit states within a certain Clifford algebra means that we are able to represent an arbitrary state of the Hilbert space. This ability is equivalent to being able to represent the unitary operators needed to transform any given state to any other state in the Hilbert space. The state of an n-qubit system can be viewed as lying on a $2N$ sphere, and the Lie group that enables one to access any state on this sphere is $SO(2N)$. However, there are also certain subgroups that are able to access the complete Hilbert space. These groups were found by Montgomery and Samelson [14] and listed in general applications to control theory by Brockett [15]. Specific applications to quantum mechanics were made by [16] and more recently by [17], who summarized in a theorem the conditions for the

controllability of states. This theorem says that given the following unitary transformation of pure states

$$\psi_0 \rightarrow \psi = e^{\mathcal{L}}\psi_0, \tag{2.4}$$

the states are completely controllable if and only if the Lie algebra is isomorphic to $su(N)$ or $sp(N)$ for even N and $su(N)$ for odd N. The symplectic algebras of interest $sp(N)$ generate unitary groups denoted as $USp(N)$ that belong to the intersection of the unitary and symplectic groups. State controllability ensures the controllability of pure-state density matrices, but not the controllability of mixed-state density matrices.

2.2 Infinitessimal Controllability Condition

Even though the controllability of a scheme can be established by identifying the Lie algebra and its dimension, this section introduces a sufficient control condition based on an infinitesimal expansion of the exponential form. This sufficient condition is useful for establishing the controllability of a scheme given a set of generators, especially when the complete Lie algebra of the generators is not known.

A control scheme can be defined in terms of a sequence of unitary operators, each parametrized by a single variable. Theorem 2.3 is going to be used to choose such a sequence of unitary operators, but first the following preliminary work is needed.

The primitive projector P, whose matrix representation is

$$P = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \\ 0 & 0 & \dots \end{pmatrix}, \tag{2.5}$$

represents a state of the system that can be taken as a reference state of a pass state (or pivot state). According to the theorem of Albertini *et al.* [17], it is possible to transform the system from the pass state, P, to an arbitrary state by means of a unitary operator of the form $e^{\mathcal{L}_{su}}$, where $\mathcal{L}_{su}$ is an element of $su(N)$. We therefore say that $su(N)$ is a state-control algebra relative to the pass state P. (A stronger statement can be made, but is not needed at this point.) In particular, a transformation to an arbitrary state in the infinitesimal neighborhood of P is realized by $e^{\delta\mathcal{L}_{su}}$, where $\delta\mathcal{L}_{su}$ is some infinitesimal element of $su(N)$. As we show below by construction, there are also subalgebras of $su(N)$ able to exercise control in the infinitesimal neighborhood of P.

Definition 2.1 *A Lie algebra, $\mathcal{A} \subset su(N)$, is said to be a state-control algebra for the system in the infinitesimal neighborhood of the pass state P if for any state*

$$\psi = \begin{pmatrix} 1 + i\delta y_1 & 0 & \dots \\ \delta x_2 + i\delta y_2 & 0 & \dots \\ \vdots & \vdots & \\ \delta x_N + i\delta_N & 0 & \dots \end{pmatrix} \tag{2.6}$$

in the infinitesimal neighborhood of P there exists an infinitesimal element of the form

$$\delta\mathcal{L} = \delta\theta^j \mathcal{L}_j \in \mathcal{A},$$

with nonzero real parameters $\delta\theta^j \in \mathbb{R}$, *where* $\{\mathcal{L}_j\}$ *is a basis of* $\mathcal{A}$, *that takes* P *into* ψ :

$$e^{\delta\theta^j \mathcal{L}_j} P = (1 + \delta\theta^j \mathcal{L}_j) P = \psi$$

with real displacements $\delta x_j = \delta x_j(\delta\theta^1, \delta\theta^2, ..) \in \mathbb{R}$, $\delta y_j = \delta y_j(\delta\theta^1, \delta\theta^2, ..) \in \mathbb{R}$.

From the definition, the following theorem follows immediately:

Theorem 2.1 *Let* $\mathcal{A} \subset su(N)$ *be a state-control algebra for the system in the infinitesimal neighborhood of the pass state* P. *It is then equivalent to* $su(N)$ *in its infinitesimal action around* P *(2.6). That is, for any infinitesimal element* $\delta\mathcal{L}_{su}$ *of* $su(N)$, *there exists an infinitesimal element* $\delta\mathcal{L} \in \mathcal{A}$ *such that*

$$e^{\delta\mathcal{L}_{su}} P = e^{\delta\mathcal{L}} P.$$

The next theorem extends this result.

Theorem 2.2 *Given that* $\delta\mathcal{L}_{su}$ *is an arbitrary infinitesimal instance of* $su(N)$ *and* $\mathcal{A} \subset su(N)$ *is a state-control algebra for the system in the infinitesimal neighborhood of the pass state* P, *an infinitesimal instance* $\delta\mathcal{L}$ *of* $\mathcal{A}$ *can be found such that*

$$e^{\delta\mathcal{L}_{su}} e^{\mathcal{L}_0} P = e^{\delta\mathcal{L}} e^{\mathcal{L}_0} P, \tag{2.7}$$

where $\mathcal{L}_0$ *is any finite element of* $\mathcal{A}$ *(not necessarily infinitesimal).*

Proof. Defining the infinitesimal element $\delta\mathcal{L}'_{su}$ by the unitary transformation

$$\delta\mathcal{L}'_{su} = e^{-\mathcal{L}_0}\delta\mathcal{L}_{su}e^{\mathcal{L}_0}, \tag{2.8}$$

it is clear that that $\delta\mathcal{L}'_{su}$ and $\delta\mathcal{L}_{su}$ are elements of the same Lie algebra, namely $su(N)$. By the theorem above, we know that an infinitesimal element $\delta\mathcal{L}' \in \mathcal{A}$ exists that is equivalent to $\delta\mathcal{L}'_{su}$ at P :

$$e^{\delta\mathcal{L}'_{su}}P = e^{-\mathcal{L}_0}e^{\delta\mathcal{L}_{su}}e^{\mathcal{L}_0} = e^{\delta\mathcal{L}'}P.$$

It follows that

$$e^{\delta\mathcal{L}_{su}}e^{\mathcal{L}_0}P = e^{\mathcal{L}_0}e^{\delta\mathcal{L}'}P = e^{\delta\mathcal{L}}e^{\mathcal{L}_0}P, \tag{2.9}$$

where $\delta\mathcal{L} = e^{\mathcal{L}_0}\delta\mathcal{L}'e^{-\mathcal{L}_0}$ is clearly infinitesimal and belongs to the same Lie algebra, namely $\mathcal{A}$, as $\mathcal{L}_0$ and $\delta\mathcal{L}'$. Thus, the theorem is proved. ■

Theorem 2.3 *Let $\mathcal{A} \subset su(N)$ be a state-control algebra for the system in the infinitesimal neighborhood of the pass state P. It is also a state-control algebra relative to the pass state, P, not just in the infinitesimal neighborhood of P.*

Proof. As noted above, the pass state, P, can be transformed into any state ψ by a unitary operator $e^{\mathcal{L}_{su}}$, where $\mathcal{L}_{su}$ is an element of $su(N)$. The exponential can be written as a limit:

$$\psi = e^{\mathcal{L}_{su}}P = \lim_{n\to\infty}\left(1 + \frac{\mathcal{L}_{su}}{n}\right)^n P, \tag{2.10}$$

which can be decomposed as

$$\psi = \lim_{n\to\infty}\left(1 + \frac{\mathcal{L}_{su}}{n}\right)^{n-1} e^{\frac{\mathcal{L}_{su}}{n}}P. \tag{2.11}$$

Theorems (2.1) establishes that in the limit $n \to \infty$, the action of the infinitesimal transformation $e^{\frac{\mathcal{L}_{su}}{n}}$ can be replaced by the action of $e^{\frac{\mathcal{L}_1}{n}}$, where

$\frac{\mathcal{L}_1}{n}$ is an element of the Lie algebra $\mathcal{A}$ that satisfies the full infinitesimal expansion (2.6):

$$\psi = \lim_{n\to\infty} \left(1 + \frac{\mathcal{L}_{su}}{n}\right)^{n-1} e^{\frac{\mathcal{L}_1}{n}} P. \tag{2.12}$$

By theorem (2.2), such a substitution can be reproduced for each of the factors of the infinite sequence so that

$$\psi = \lim_{n\to\infty} e^{\frac{\mathcal{L}_n}{n}} ... e^{\frac{\mathcal{L}_j}{n}} ... e^{\frac{\mathcal{L}_1}{n}} P, \tag{2.13}$$

which reduces the action of the $SU(N)$ group to the group generated by the Lie algebra $\mathcal{A}$. Thus, the Lie algebra $\mathcal{A}$ can be used to define the state-control scheme. ■

This theorem implies that it is only necessary to test the infinitesimal expansion (2.6) around the state P to ensure the infinitesimal accessibility around any other state by using the Lie algebra defined by the generators of the infinitesimal expansion.

The practical value of theorem (2.3) lies in the potential to optimize the computational work to establish the controllability of a certain control scheme, given an initial set of generators. Current methods require the calculation of the complete Lie algebra through the recursive application of the Lie products, but this is an expensive procedure in computational terms when the dimension of the matrices is large. By using this new sufficient condition, one is allowed to stop the recursion as soon as the infinitesimal expansion (2.6) has been achieved.

Before going to the next subsection, we could state another related theorem that may clarify the concept of complete accessibility of the neighborhood around the state P

Theorem 2.4 *The Jacobian of the transformation of variables around the*

state P defined in infinitesimal expansion (2.6) is different from zero

$$det(\frac{\partial q_i}{\partial \theta_j})|_P \neq 0, \tag{2.14}$$

where $q_1 = \delta y_1, q_2 = \delta x_2, q_3 = \delta y_2, \ldots, q_{2N-1} = \delta y_N$.

Proof. This theorem is proved by realizing that in the infinitesimal limit, the q_k are linear functions of $\delta\theta^j$ but never constant. ■

2.3 Explicit Control Scheme

Now we present an explicit control scheme, to show that an arbitrary state of a quantum system, with an even number of energy levels that are symmetrically distributed about an offset, can be produced from another arbitrary state using a set of fields. Moreover the Lie algebra generated by these field-couplings are seen to be of dimension $N(N+1)/2$. This scheme is based on the subspace controllability theorem [18] that describes the method of transferring any superposition of states to any other superposition through a pivot state (pass state). If the implemented interactions accomplish the objective to completely transfer all the amplitudes from an arbitrary superposition of states to the pivot state, the state controllability is implied because the inverse transformation can be carried out to restore the original arbitrary superposition. This builds on the work done by Eberly and coworkers on the control of harmonic oscillator states [19, 20].

This scheme can be implemented on an even N-level system with symmetric energies and the sequence of interactions results in a set of generators that ultimately form a symplectic Lie algebra. The two-qubit case with symmetric energy levels as shown in figure (2.1) is described in detail below.

Assuming that the two-qubit system is in a superposition of the four basis states

$$|\Psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \tag{2.15}$$

the path to the ground (pass) state is accomplished as

- Applying a certain pulse of frequency Δ_{21} in order to transfer the complete amplitude in state $|3\rangle$ to state $|2\rangle$. This transfer, as all the others referred in this sequence, involves two generators in order to take into account the phase of the states. The symmetry of the energy levels implies that there is also a simultaneous transfer of amplitudes between the states $|0\rangle$ and $|1\rangle$, but this is irrelevant for our purposes.
- Applying a pulse of frequency ω_1 to transfer the amplitude from state $|2\rangle$ to state $|1\rangle$.
- Applying a pulse of frequency Δ_{21} to transfer the amplitude of state $|1\rangle$ to the ground state $|0\rangle$ (this is the same interaction applied in the first step).

The generators of the interaction Δ_{12} (see Figure 2.1) are

$$G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad G_1' = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{2.16}$$

and the generator of the interaction ω_1 are

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_2' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.17}$$

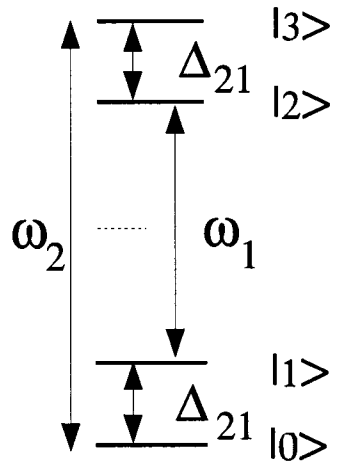

Figure 2.1. Representation of a four-level system with symmetric energy levels.

This initial set of generators is equivalent to the matrix representation of the following elements

$$G_1 = -\mathbf{e}_{13} \tag{2.18}$$

$$G_1' = \mathbf{e}_{23} \tag{2.19}$$

$$G_2 = -(\mathbf{e}_{15} + \mathbf{e}_{24})/2 \tag{2.20}$$

$$G_2' = (\mathbf{e}_{14} - \mathbf{e}_{25})/2 \tag{2.21}$$

The complete algebra is then found from all the new possible independent commutators calculated recursively until the linear space is exhausted [21]. The dimension of this Lie algebra is seen to be 10 as is the dimension of $sp(4)$. This fact leads us to conclude that this Lie algebra is isomorphic to $sp(4)$ because the other alternative would be $su(4)$ with dimension 15.

Two particular generators that can be used to define the Cartan subal-

gebra are directly calculated from the initial generators as

$$C_1 = [G_1, G_1'] = \begin{pmatrix} 2i & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix} \tag{2.22}$$

$$C_2 = [G_2, G_2'] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.23}$$

The Hamiltonian must be defined within the Cartan subalgebra because a basis of the Cartan subalgebra provides the maximum number of independent generators that can be simultaneously diagonalized. Adding any other independent diagonal matrix would automatically open the original Lie algebra. The Cartan subalgebra with the basis elements C_1 and C_2 corresponds to a system with symmetric energy levels as can be seen explicitly by writing the Hamiltonian without loss of generality as

$$H = \frac{\omega_2}{2i} C_1 + \frac{\omega_1 + \omega_2}{2i} C_2 = \begin{pmatrix} \omega_2 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & -\omega_2 \end{pmatrix} \tag{2.24}$$

For the general even N-level case, the interactions give $N/2$ real plus

$N/2$ complex generators as follows

$$G_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \dots$$

$$G_1' = \begin{pmatrix} 0 & i & 0 & \dots & 0 & 0 & 0 \\ i & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & i \\ 0 & 0 & 0 & \dots & 0 & i & 0 \end{pmatrix} \quad G_2' = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & i & & 0 & 0 & 0 \\ 0 & i & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & i & 0 \\ 0 & 0 & 0 & & i & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \dots, \tag{2.25}$$

from which, it can be verified that a Lie algebra of dimension $N(N+1)/2$ can be found after an exhaustive computation of all the possible commutators.

The Cartan subalgebra of dimension $N/2$ can be calculated directly from

the initial set as

$$
\begin{aligned}
C_1 \;=\; [G_1, G_1'] \;=\; & \begin{pmatrix} 2i & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -2i & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 2i & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -2i \end{pmatrix} \\
C_2 \;=\; [G_2, G_2'] \;=\; & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2i & 0 & & 0 & 0 & 0 \\ 0 & 0 & -2i & & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & & 2i & 0 & 0 \\ 0 & 0 & 0 & & 0 & -2i & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \\
\vdots & \\
C_{N/2} \;=\; [G_{N/2}, G_{N/2}'] \;=\; & \dots,
\end{aligned}
\tag{2.26}
$$

which ultimately are able to define the most general Hamiltonian with symmetric energy levels.

These results are expected because $N(N+1)/2$ is the dimension of the Lie algebra $sp(N)$, and $N/2$, the dimension of its Cartan subalgebra. Similar results are going to be presented in the next two chapters, where the representation is given in terms of Clifford algebras.

Chapter 3

Two-qubit System

3.1 Two-qubit Control with $SU(4)$

The purpose of this chapter is to express the Lie algebras of interest, $su(4)$ and $sp(4)$, in terms of certain Clifford algebras and to study the control schemes based on them. The last part of the chapter deals with the description of the entanglement of pure states.

The control of pure two-qubit states can be carried out by $SU(4)$. The accidental isomorphism between $su(4)$ and $spin(6)$, was exploited by Havel and Doran [22] in order to develop a geometrical formulation of a two-qubit system. However there are more possibilities by using other Lie algebras presented in Table 1.8. The following statements summarize some important facts that are used in the rest of this chapter

- There is an instance of the $su(4)$ Lie algebra in terms of the anti-Hermitian $\mathcal{C}\ell_4 \times \mathbb{C}$.
- The anti-Hermitian $\mathcal{C}\ell_4$ (real) is seen to form a Lie algebra isomorphic to $sp(4)$.

- The $spin(5)$ lie algebra, which is accidentally isomorphic to the $sp(4)$ Lie algebra, is naturally obtained in terms of the bivectors in $C\ell_5$.

- The $su(2) \times su(2)$ Lie algebra is a subalgebra of $su(4)$ but not a subalgebra of $sp(4)$

The explicit calculation of the Cartan matrix of the anti-Hermitian $C\ell_4$ algebra is shown in Appendix C. The Cartan matrix contains the essential information of the Lie algebra and therefore two isomorphic Lie algebras share the same Cartan matrix.

A basis of the complexified anti-Hermitian $C\ell_4 \times \mathbb{C}$ algebra is given in Table 3.1.

$\mathbf{e}_{12}$	$=$	$i\sigma_0 \otimes \sigma_3$	$\mathbf{e}_{123}$	$=$	$i\sigma_3 \otimes \sigma_0$
$\mathbf{e}_{13}$	$=$	$-i\sigma_0 \otimes \sigma_2$	$i\mathbf{e}_4$	$=$	$-i\sigma_2 \otimes \sigma_0$
$\mathbf{e}_{23}$	$=$	$i\sigma_0 \otimes \sigma_1$	$i\mathbf{e}_{1234}$	$=$	$-i\sigma_1 \otimes \sigma_0$
$\mathbf{e}_{14}$	$=$	$i\sigma_1 \otimes \sigma_1$	$\mathbf{e}_{24}$	$=$	$i\sigma_1 \otimes \sigma_2$
$\mathbf{e}_{34}$	$=$	$i\sigma_1 \otimes \sigma_3$	$\mathbf{e}_{124}$	$=$	$-i\sigma_2 \otimes \sigma_3$
$\mathbf{e}_{134}$	$=$	$i\sigma_2 \otimes \sigma_2$	$\mathbf{e}_{234}$	$=$	$-i\sigma_2 \otimes \sigma_1$
$i\mathbf{e}_1$	$=$	$i\sigma_3 \otimes \sigma_1$	$i\mathbf{e}_2$	$=$	$i\sigma_3 \otimes \sigma_2$
$i\mathbf{e}_3$	$=$	$i\sigma_3 \otimes \sigma_3$			

Table 3.1. Matrix representation of the complexified anti-Hermitian $C\ell_4 \times \mathbb{C}$ space.

A basis of the anti-Hermitian $C\ell_4$ space that defines an instance of the $sp(4)$ algebra is given in Table 3.2. The respective explicit matrix representation is given in Table 3.4. A basis of the $su(2) \times su(2)$ Lie algebra is given in Table 3.3.

Generalizing the single-qubit case, we write the spinor wave-function as

$\mathbf{e}_{12}$	=	$i\sigma_0 \otimes \sigma_3$
$\mathbf{e}_{13}$	=	$-i\sigma_0 \otimes \sigma_2$
$\mathbf{e}_{23}$	=	$i\sigma_0 \otimes \sigma_1$
$\mathbf{e}_{14}$	=	$i\sigma_1 \otimes \sigma_1$
$\mathbf{e}_{24}$	=	$i\sigma_1 \otimes \sigma_2$
$\mathbf{e}_{34}$	=	$i\sigma_1 \otimes \sigma_3$
$\mathbf{e}_{123}$	=	$i\sigma_3 \otimes \sigma_0$
$\mathbf{e}_{124}$	=	$-i\sigma_2 \otimes \sigma_3$
$\mathbf{e}_{134}$	=	$i\sigma_2 \otimes \sigma_2$
$\mathbf{e}_{234}$	=	$-i\sigma_2 \otimes \sigma_1$

Table 3.2. Matrix representation of the anti-Hermitian $\mathcal{C}\ell_4$ algebra.

$\mathbf{e}_{12}$	=	$i\sigma_0 \otimes \sigma_3$
$\mathbf{e}_{31}$	=	$i\sigma_0 \otimes \sigma_2$
$\mathbf{e}_{23}$	=	$i\sigma_0 \otimes \sigma_1$
$\mathbf{e}_{123}$	=	$i\sigma_3 \otimes \sigma_0$
$i\mathbf{e}_{1234}$	=	$-i\sigma_1 \otimes \sigma_0$
$i\mathbf{e}_4$	=	$-i\sigma_2 \otimes \sigma_0$

Table 3.3. Representation of the $su(2) \times su(2)$ Lie algebra

$\mathbf{e}_{12} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$	$\mathbf{e}_{123} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$
$\mathbf{e}_{13} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\mathbf{e}_{23} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$
$\mathbf{e}_{14} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$	$\mathbf{e}_{24} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
$\mathbf{e}_{34} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$	$\mathbf{e}_{124} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
$\mathbf{e}_{134} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$	$\mathbf{e}_{234} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Table 3.4. Explicit matrix representation of the generators of the anti-Hermitian $\mathcal{C}\ell_4$ algebra. The elements $\mathbf{e}_{12}$ and $\mathbf{e}_{123}$ can be naturally chosen as a basis of the Cartan subalgebra.

the projection of the exponential form of the Lie algebra as

$$\Psi = e^{\mathcal{L}} P, \tag{3.1}$$

where P in matrix representation is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

An infinitesimal expression of the $su(4)$ Lie algebra around zero can be constructed as follows

$$\mathcal{L} = \sum_{0<j\leq N} i\delta\theta_j \mathbf{e}_j + \sum_{0<j<k\leq 4} \delta\theta_{jk}\mathbf{e}_{jk} + \sum_{0<j<k<m\leq 4} \delta\theta_{jkm}\mathbf{e}_{jkm} + i\delta\theta_{1234}\mathbf{e}_{1234}, \tag{3.3}$$

where the infinitesimal angles $\delta\theta_{jk\ldots}$ are real numbers. The first-order expansion of the exponential form is

$$\Psi = e^{\mathcal{L}} P = \begin{pmatrix} 1 + i\delta\theta_3 + i\delta\theta_{12} + i\delta\theta_{123} & 0 & 0 & 0 \\ i\delta\theta_1 - \delta\theta_2 + \delta\theta_{13} + i\delta\theta_{23} & 0 & 0 & 0 \\ \delta\theta_4 - i\delta\theta_{1234} + \delta\theta_{124} + i\delta\theta_{34} & 0 & 0 & 0 \\ \delta\theta_{234} - \delta\theta_{24} + i\delta\theta_{14} - i\delta\theta_{134} & 0 & 0 & 0 \end{pmatrix}, \tag{3.4}$$

and thus, it satisfies the infinitesimal expansion (2.6).

This allows us to define control schemes to access the complete Hilbert space. For example, it is possible to define the following control scheme in an infinitesimal neighborhood of P

$$\Psi = e^{-\theta_{234}\mathbf{e}_{234}/2} e^{\theta_{14}\mathbf{e}_{14}/2} e^{-i\theta_4\mathbf{e}_4/2} e^{-i\theta_{1234}\mathbf{e}_{1234}/2} e^{-\theta_{13}\mathbf{e}_{13}/2} e^{\theta_{23}\mathbf{e}_{23}/2} e^{\theta_{12}\mathbf{e}_{12}/2} P, \tag{3.5}$$

because the infinitesimal expansion gives

$$\Psi = \begin{pmatrix} 1 + i\delta\theta_{12}/2 & 0 & 0 & 0 \\ i\delta\theta_{23}/2 - \delta\theta_{13}/2 & 0 & 0 & 0 \\ i\delta\theta_{1234}/2 - \delta\theta_{4}/2 & 0 & 0 & 0 \\ i\delta\theta_{14}/2 - \delta\theta_{234}/2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

This expansion is made around the state P but an explicit computational calculation shows that the complete infinitesimal expansion also exists everywhere except at isolated points. Theorem 2.3 of chapter 2 guarantees the complete infinitesimal expansion around any point when all the Lie algebra is used.

The fundamental primitive projector P is defined as

$$P = \frac{1}{4}(1 - i\mathbf{e}_{123})(1 - i\mathbf{e}_{12}) \tag{3.7}$$

such that its matrix representation is

$$P = \frac{1}{4}(\mathbf{1} + \sigma_3 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \sigma_3). \tag{3.8}$$

In this form, the primitive projector P is seen to be the product of two independent commuting projectors that are defined as

$$P_{12} = \frac{1}{2}(1 - i\mathbf{e}_{12}) = \frac{1}{2}(\mathbf{1} + 1 \otimes \sigma_3) \tag{3.9}$$

$$P_{123} = \frac{1}{2}(1 - i\mathbf{e}_{123}) = \frac{1}{2}(\mathbf{1} + \sigma_3 \otimes 1), \tag{3.10}$$

such that P_{12} is associated with one qubit and P_{123} is associated with the other qubit.

The primitive projector P is part of four primitive projectors that provide a resolution of the identity. It is convenient to introduce a new label to denote the projector P as

$$Q_1 = P, \tag{3.11}$$

so that the other three primitive projectors are defined as

$$Q_2 = P_{123}P_{-12} = \frac{1}{4}(1 - i\mathbf{e}_{123})(1 + i\mathbf{e}_{12}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.12)$$

$$Q_3 = P_{-123}P_{12} = \frac{1}{4}(1 + i\mathbf{e}_{123})(1 - i\mathbf{e}_{12}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.13)$$

$$Q_4 = P_{-123}P_{-12} = \frac{1}{4}(1 + i\mathbf{e}_{123})(1 + i\mathbf{e}_{12}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.14)$$

Thus, the identity can be resolved as

$$1 = Q_1 + Q_2 + Q_3 + Q_4. \quad (3.15)$$

The Cartan subalgebra of $su(4)$ contains three elements. A natural choice of members of the Cartan subalgebra are those elements with diagonal matrix representation

$$C = \{\mathbf{e}_{12}, \mathbf{e}_{123}, i\mathbf{e}_3\}. \quad (3.16)$$

The free-evolution Hamiltonian is constructed with the Cartan subalgebra. There are three basis elements so we can always form a Hamiltonian with arbitrary spacing between the energy levels.

3.2 Control with $USp(4)$

The anti-Hermitian $C\ell_4$ space is an instance of the $sp(4)$ Lie algebra. The argument of the exponential form constructed with the infinitesimal real numbers $\delta\theta_{jk}$ and $\delta\theta_{jkm}$ is

$$\begin{aligned}\mathcal{L} &= \delta\theta_{12}\mathbf{e}_{12} + \delta\theta_{13}\mathbf{e}_{13} + \delta\theta_{14}\mathbf{e}_{14} + \delta\theta_{23}\mathbf{e}_{23} + \delta\theta_{24}\mathbf{e}_{24} + \\ &\quad \delta\theta_{34}\mathbf{e}_{34} + \delta\theta_{123}\mathbf{e}_{123} + \delta\theta_{124}\mathbf{e}_{124} + \delta\theta_{134}\mathbf{e}_{134} + \delta\theta_{234}\mathbf{e}_{234},\end{aligned}$$

so that the expansion up to first order is

$$\Psi P = RP = e^{\mathcal{L}}P = \begin{pmatrix} 1 + i\delta\theta_{12} + i\delta\theta_{123} & 0 & 0 & 0 \\ \delta\theta_{13} + i\delta\theta_{23} & 0 & 0 & 0 \\ \delta\theta_{124} + i\delta\theta_{34} & 0 & 0 & 0 \\ \delta\theta_{234} - \delta\theta_{24} + i\delta\theta_{14} - i\delta\theta_{134} & 0 & 0 & 0 \end{pmatrix}, \tag{3.17}$$

and therefore we conclude that the Lie algebra satisfies the infinitesimal expansion (2.6).

A possible control scheme that satisfies the infinitesimal expansion (2.6) could be defined by applying the following minimal sequence of seven unitary transformations with seven parameters

$$\Psi = RP = e^{v\mathbf{e}_{234}/2} e^{u\mathbf{e}_{14}/2} e^{\theta_{23}\mathbf{e}_{23}/2} e^{\theta_{13}\mathbf{e}_{13}/2} e^{\theta_{124}\mathbf{e}_{124}/2} e^{\theta_{34}\mathbf{e}_{34}/2} e^{\theta_{12}\mathbf{e}_{12}/2} P \tag{3.18}$$

The Hamiltonian, written in terms of the Cartan subalgebra is given by

$$iH = \frac{\omega_1 - \omega_2}{2}\mathbf{e}_{12} + \frac{\omega_1 + \omega_2}{2}\mathbf{e}_{123}, \tag{3.19}$$

which has the following matrix representation

$$H = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & -\omega_2 & 0 \\ 0 & 0 & 0 & -\omega_1 \end{pmatrix}. \tag{3.20}$$

The matrix representation of the Hamiltonian shows a system with a symmetric distribution of energy levels. A more general system cannot be represented without breaking the $sp(4)$ Lie algebra and extending it to $su(4)$.

3.3 Control With $Spin(5)$

The $spin(5)$ Lie algebra is isomorphic to $sp(4)$. This can be proved by calculating the Cartan matrices and then verifying that are the same. The $spin(5)$ Lie algebra is constructed by using the $C\ell_5$ bivectors. Evidently, the first-order expansion around the identity fulfills the pure-state controllability condition

$$RP = e^A P = \begin{pmatrix} i\delta\theta_{12} - i\delta\theta_{45} & 0 & 0 & 0 \\ \delta\theta_{13} + i\delta\theta_{23} & 0 & 0 & 0 \\ i\delta\theta_{34} + \delta\theta_{35} & 0 & 0 & 0 \\ i\delta\theta_{14} + \delta\theta_{15} - \delta\theta_{24} + i\delta\theta_{25} & 0 & 0 & 0 \end{pmatrix}, \tag{3.21}$$

where the Lie algebra is given by the linear space of all the bivectors

$$A = \delta\theta_{12}\mathbf{e}_{12} + \delta\theta_{13}\mathbf{e}_{13} + \delta\theta_{14}\mathbf{e}_{14} + \delta\theta_{15}\mathbf{e}_{15}... \tag{3.22}$$

The fundamental primitive projector P is now defined as

$$P = \frac{1}{4}(1 + i\mathbf{e}_{45})(1 - i\mathbf{e}_{12}) \tag{3.23}$$

and the unitary group is the $Spin(5)$ group. The advantage of this representation is that the unitary transformations preserve the grade of the multivectors but this cannot be exploited easily because the 4×4 matrix representation is not faithful.

3.4 Entanglement of Pure States

For pure states with spinor $\Psi = RP$, the algebraic object that plays the role of the density matrix is

$$P \rightarrow \rho = RPR^{\dagger} = (RP_{123}R^{\dagger})(RP_{12}R^{\dagger}) = (RP_{12}R^{\dagger})(RP_{123}R^{\dagger}). \quad (3.24)$$

If the initial state is chosen as P, the non-entangled pure states can be generated by using the Lie algebra $su(2) \times su(2)$, which transforms P as

$$P \rightarrow \frac{1}{4}(\mathbf{1} + (\xi_1\sigma_1 \otimes \sigma_0 + \xi_2\sigma_2 \otimes \sigma_0 + \xi_3\sigma_3 \otimes \sigma_0)) \quad (3.25)$$

$$(\mathbf{1} + (\eta_1\sigma_0 \otimes \sigma_1 + \eta_2\sigma_0 \otimes \sigma_2 + \eta_3\sigma_0 \otimes \sigma_3)) \quad (3.26)$$

where the coefficients η_j and ξ_j are normalized to unity $\sum \xi_j^2 = 1$, $\sum \eta_j^2 = 1$. A more efficient notation of this transformation reads

$$P \rightarrow \rho = \frac{1}{4}(\mathbf{1} + \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \hat{\mathbf{p}}_1), \quad (3.27)$$

where $\hat{\mathbf{p}}_1\hat{\mathbf{p}}_1 = \sigma_0$ and $\hat{\mathbf{p}}_2\hat{\mathbf{p}}_2 = \sigma_0$, such that $|\hat{\mathbf{p}}_1| = |\hat{\mathbf{p}}_2| = 1$.

It is possible to parametrize the general spinor with a completely controllable scheme with six parameters excluding the global phase such that only two of the Euler-like angles induce entanglement, say u and v

$$\Psi = RP = e^{v\mathbf{e}_{234}/2}e^{u\mathbf{e}_{14}/2}e^{\theta_{23}\mathbf{e}_{23}/2}e^{\theta_{13}\mathbf{e}_{13}/2}e^{\theta_{124}\mathbf{e}_{124}/2}e^{\theta_{34}\mathbf{e}_{34}/2}P, \quad (3.28)$$

where the generators $\{\mathbf{e}_{23}, \mathbf{e}_{13}\}$ control the orientation of the polarization $\mathbf{p}_1$ and the generators $\{\mathbf{e}_{124}, \mathbf{e}_{34}\}$ control the orientation of the polarization

$\mathbf{p}_2$. A unitary transformation that induces entanglement is called *tanglor*. The tanglor operator in the current case is

$$U_E(u,v) = e^{v\,\mathbf{e}_{234}/2} e^{u\,\mathbf{e}_{14}/2}. \tag{3.29}$$

The spinor decomposition provided by [22] does not allow infinitesimal control around the P state as with the most common parametrization of the single qubit case. This seems to be the reason why it only indicates only one parameter for the entanglement of two qubits. More work should be done to clarify this point.

The density matrix of a pure state, entangled or not, can be expanded as

$$\rho = \frac{1}{4}(\mathbf{1} - i\hat{\mathbf{q}}_2)(\mathbf{1} - i\hat{\mathbf{q}}_1), \tag{3.30}$$

where $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_2$ belong to the anti-Hermitian $C\ell_4 \times \mathbb{C}$ space, (isomorphic to $su(4)$) and are only constrained to obey the following conditions

$$\hat{\mathbf{q}}_1\hat{\mathbf{q}}_1 = \mathbf{1} \tag{3.31}$$

$$\hat{\mathbf{q}}_2\hat{\mathbf{q}}_2 = \mathbf{1} \tag{3.32}$$

$$[\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_1] = 0 \tag{3.33}$$

A measurement of the entanglement is given by the entropy of entanglement that is calculated from the partial trace over one of the particles. The action of the partial trace on a given qubit can be understood as an effective average. The partial traces in terms of the partial scalar part (Appendix E) are

$$\rho_1 = 2\langle\rho\rangle_{S\otimes 1} \tag{3.34}$$

$$\rho_2 = 2\langle\rho\rangle_{1\otimes S}, \tag{3.35}$$

or in terms of the partial traces

$$\rho_1 = [Tr \otimes 1](\rho) \tag{3.36}$$

$$\rho_2 = [1 \otimes Tr](\rho), \tag{3.37}$$

so that the entropy of entanglement is arbitrarily written either in terms of scalar parts of ρ_1 or ρ_2 as

$$E = -2\langle \rho_1 \ln(\rho_1) \rangle_S = -2\langle \rho_2 \ln(\rho_2) \rangle_S, \tag{3.38}$$

or in terms of the trace

$$E = -Tr(\rho_1 \ln(\rho_1)) = -Tr(\rho_2 \ln(\rho_2)). \tag{3.39}$$

However, a slightly different definition is needed in order to extend the entanglement measurement to n qubits. The new entanglement measure is defined as follows in term of the sum of the two possible partial scalar parts

$$E = -2\langle \rho_1 \ln(\rho_1) \rangle_S - 2\langle \rho_2 \ln(\rho_2) \rangle_S, \tag{3.40}$$

or equivalently

$$E = -Tr(\rho_1 \ln(\rho_1)) - Tr(\rho_2 \ln(\rho_2)). \tag{3.41}$$

In the two-qubit case, the two possible partial scalar parts give the same result, hence, the new definition of entanglement simply leads to an extra factor of two. The entanglement of a pure two-qubit system is then given by the following formula

$$E = -4\langle \rho_1 \ln(\rho_1) \rangle_S, \tag{3.42}$$

or equivalently in terms of the trace as

$$E = -2Tr(\rho_1 \ln(\rho_1)). \tag{3.43}$$

The values of the entropy of entanglement ranges from 0 for non-entangled states, where the reduced states ρ_1 and ρ_2 are pure states, to $2\ln(2)$ for maximally entangled states. In this latter case the reduced states are fully mixed $\rho_1 = \rho_2 = \sigma_0/2$.

This new definition of the entanglement is adopted because its extension to pure n-qubit states leads to a maximum entanglement measure of $n\ln(2)$. The three-qubit case is studied in the next chapter where the maximum entanglement measurement gives $3\ln(2)$.

If we apply this formula to the spinor state given by $\Psi = U_E P$ (see equation3.29) the entropy of entanglement is explicitly calculated as

$$E(u,v) = -2(\frac{1+\alpha}{2})\ln(\frac{1+\alpha}{2}) - 2(\frac{1-\alpha}{2})\ln(\frac{1-\alpha}{2}), \tag{3.44}$$

where $\alpha = \cos u \cos v$.

The entropy of entanglement is directly related with the remaining polarization after the application of the partial trace. The polarization remaining on the qubit on the right slot is

$$\mathbf{p}_1 = 2\langle \rho - \frac{1}{2} \rangle_{S\otimes 1}, \tag{3.45}$$

Applying this formula to the state $\Psi = U_E P$ we obtain

$$|\mathbf{p}_1| = |\cos(u)\cos(v)|, \tag{3.46}$$

which is nothing else than $|\alpha|$. The remaining polarization on the other qubit is calculated in the same way, giving the same value. The plot of the entropy of entanglement as a function of the variables u and v is shown in figures (3.1) and (3.2), showing a maximum value of $2\ln(2)$.

A given tanglor does not induce the same amount of entanglement on all pure non entangled states. For example, the following non entangled state

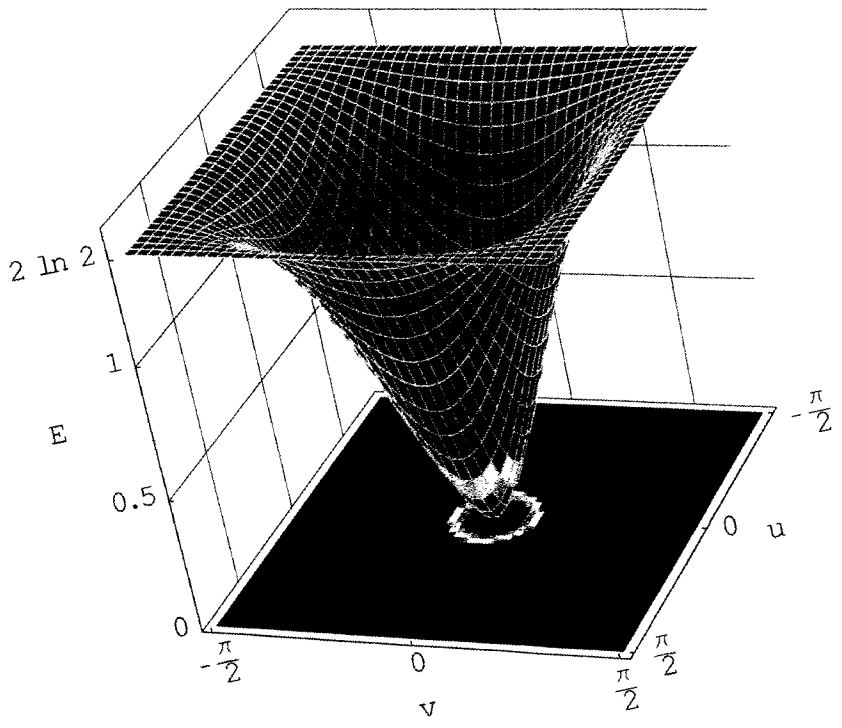

Figure 3.1. Entropy of entanglement of two qubits for the pure state $\rho = U_E(u,v)PU_E(u,v)^\dagger$.

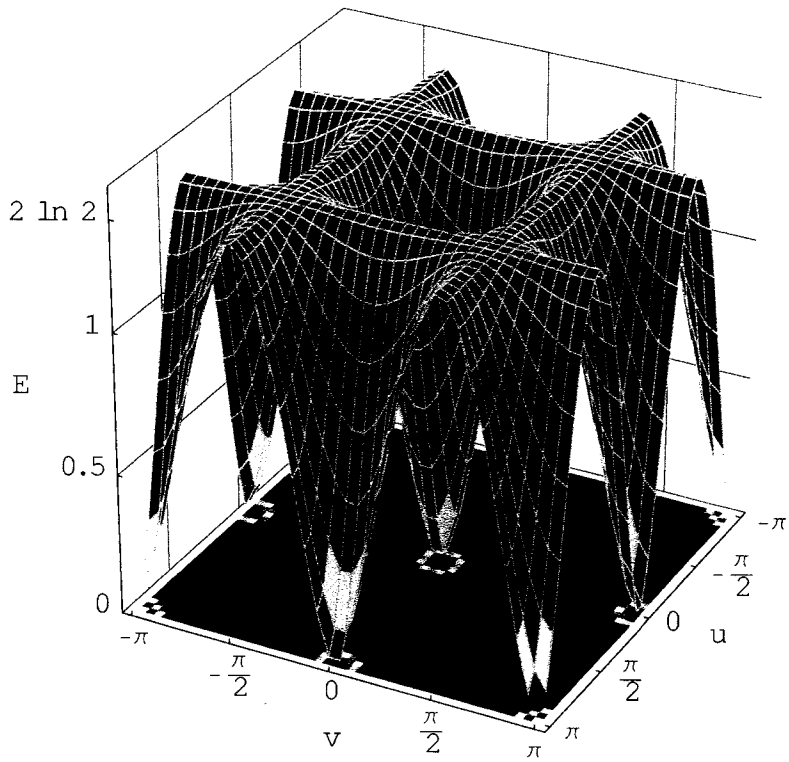

Figure 3.2. Entropy of entanglement of two qubits for the pure state $\rho = U_E(u,v)PU_E(u,v)^\dagger$ for a broader range.

is completely immune to the tanglor (3.29) if $v = 0$

$$\rho = \frac{1}{4}(1 + \sigma_1 \otimes \sigma_0)(1 + \sigma_0 \otimes \sigma_1). \tag{3.47}$$

The four orthogonal states can be identified as

$$|\uparrow\uparrow\rangle = P \tag{3.48}$$

$$|\uparrow\downarrow\rangle = e^{\pi/2\,\mathbf{e}_{13}}P = \mathbf{e}_{13}P \tag{3.49}$$

$$|\downarrow\uparrow\rangle = e^{\pi/2\,\mathbf{e}_{124}}P = \mathbf{e}_{124}P \tag{3.50}$$

$$|\downarrow\downarrow\rangle = e^{\pi/2\,\mathbf{e}_{234}}P = \mathbf{e}_{234}P, \tag{3.51}$$

or in terms of the Pauli matrices

$$|\uparrow\uparrow\rangle = P \tag{3.52}$$

$$|\uparrow\downarrow\rangle = -i\sigma_0 \otimes \sigma_2 P \tag{3.53}$$

$$|\downarrow\uparrow\rangle = -i\sigma_2 \otimes \sigma_3 P = -i\sigma_2 \otimes \sigma_0 P \tag{3.54}$$

$$|\downarrow\downarrow\rangle = -i\sigma_2 \otimes \sigma_1 P. \tag{3.55}$$

Applying the (3.29) unitary operator on $|\uparrow\uparrow\rangle$ we obtain the following state

$$\begin{aligned} U_E(u,v)|\uparrow\uparrow\rangle = {} & (\cos(u/2)\cos(v/2) - i\sin(u/2)\sin(v/2))|\uparrow\uparrow\rangle + \\ & (\cos(u/2)\sin(v/2) + i\cos(v/2)\sin(u/2))|\downarrow\downarrow\rangle. \end{aligned}$$

The following Bell states are found and labeled as

$$\psi^2 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) = U_E(0,\pi/2)|\uparrow\uparrow\rangle \tag{3.56}$$

$$\psi^1 = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) = U_E(0,-\pi/2)|\uparrow\uparrow\rangle \tag{3.57}$$

$$\tag{3.58}$$

On the other hand, we have the following expression with an additional prefactor

$$\begin{aligned} U_E(u,v)e^{\pi/2\mathbf{e}_{13}}|\uparrow\uparrow\rangle &= (\cos(u/2)\cos(v/2) - i\sin(u/2)\sin(v/2))|\uparrow\downarrow\rangle + \\ &\quad (\cos(u/2)\sin(v/2) + i\cos(v/2)\sin(u/2))|\downarrow\uparrow\rangle \end{aligned}$$

so that the other two Bell states are given by

$$\psi^3 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = U_E(0,\pi/2)e^{\pi/2\mathbf{e}_{13}}|\uparrow\uparrow\rangle \tag{3.59}$$

$$\psi^0 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = U_E(0,-\pi/2)e^{\pi/2\mathbf{e}_{13}}|\uparrow\uparrow\rangle. \tag{3.60}$$

The Bell states can be classified into two groups. The first group contains only the singlet Bell state with the following density matrix

$$\rho^0 = \frac{1}{4}(\mathbf{1} - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3) \tag{3.61}$$

This state obeys the properties of the singlet zero angular momentum and therefore is invariant under rotations of the kind

$$R = (e^{\phi\sigma_3\otimes\sigma_0}e^{\theta\sigma_2\otimes\sigma_0}) \otimes (e^{\phi\sigma_0\otimes\sigma_3}e^{\theta\sigma_0\otimes\sigma_3}) \tag{3.62}$$

The second group of the Bell states forms a triplet that can be accommodated in the following form

$$\rho^{\mathbf{n}} = \frac{1}{4}(\mathbf{1} + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) - \frac{1}{2}\mathbf{n} \otimes \mathbf{n}, \tag{3.63}$$

such that the triplet Bell states are

$$\rho^1 = \frac{1}{4}(\mathbf{1} - \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) \tag{3.64}$$

$$\rho^2 = \frac{1}{4}(\mathbf{1} + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) \tag{3.65}$$

$$\rho^3 = \frac{1}{4}(\mathbf{1} + \sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3) \tag{3.66}$$

These states are not invariant under general rotations but each of them is invariant under rotations in a plane perpendicular to their respective $\mathbf{n}$.

A fundamental property of pure entangled states is that they cannot be expressed in the form (3.27). This means that the polarizations cannot be expressed as single-body terms. The polarization of the two qubits for the ρ^1 Bell state are calculated from their initial polarizations as

$$U_E(0,-\pi/2)(-i\mathbf{e}_{12})U_E(0,-\pi/2)^\dagger \rightarrow i\mathbf{e}_{14} = -\sigma_1 \otimes \sigma_1 \quad (3.67)$$

$$U_E(0,-\pi/2)(-i\mathbf{e}_{123})U_E(0,-\pi/2)^\dagger \rightarrow -i\mathbf{e}_{134} = \sigma_2 \otimes \sigma_2, \quad (3.68)$$

showing that the polarizations lack of single-body terms at all. This is also the case for the rest of the Bell states. This suggests that the entanglement can be measured by the content of multi-body terms in the polarizations.

3.5 Mixed States

The density matrix of a mixed state is characterized for a weaker polarization, so, it is natural to investigate the role of the following form if the state is separable

$$\rho = \frac{1}{4}(\mathbf{1} + \mathbf{p}_2 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \mathbf{p}_1), \quad (3.69)$$

but now $0 \leq \beta = |\mathbf{p}_1| < 1$ and $0 \leq \alpha = |\mathbf{p}_2| < 1$. This form of the mixed state suggests a decomposition in terms of four projectors that may be considered as pure non-entangled density matrices, namely

$$P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} = \frac{1}{4}(\mathbf{1} + \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \hat{\mathbf{p}}_1) \quad (3.70)$$

$$P_{\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} = \frac{1}{4}(\mathbf{1} + \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} - \sigma_0 \otimes \hat{\mathbf{p}}_1) \quad (3.71)$$

$$P_{-\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} = \frac{1}{4}(\mathbf{1} - \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \hat{\mathbf{p}}_1) \quad (3.72)$$

$$P_{-\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} = \frac{1}{4}(\mathbf{1} - \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} - \sigma_0 \otimes \hat{\mathbf{p}}_1), \quad (3.73)$$

where $\hat{\mathbf{p}}_1$ and $\hat{\mathbf{p}}_2$ are respectively parallel to $\mathbf{p}_1$ and $\mathbf{p}_2$.

From the pacwoman property, and the fact that the sum of these states gives a resolution of the identity

$$\mathbf{1} = P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} + P_{\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} + P_{-\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} + P_{-\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1}, \tag{3.74}$$

the mixed state (3.69) is decomposed as

$$\begin{aligned} \rho \;=\; & \frac{1}{4}(1+\alpha)(1+\beta)P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} + \frac{1}{4}(1+\alpha)(1-\beta)P_{\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} + & (3.75)\\ & \frac{1}{4}(1-\alpha)(1+\beta)P_{-\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} + \frac{1}{4}(1-\alpha)(1-\beta)P_{-\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1}. & (3.76)\end{aligned}$$

The four projectors act as four eigenvectors of ρ because

$$\rho P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} = \frac{1}{4}(1+\alpha)(1+\beta)P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} \tag{3.77}$$

$$\rho P_{\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} = \frac{1}{4}(1+\alpha)(1-\beta)P_{\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} \tag{3.78}$$

$$\rho P_{-\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} = \frac{1}{4}(1-\alpha)(1+\beta)P_{-\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1} \tag{3.79}$$

$$\rho P_{-\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1} = \frac{1}{4}(1-\alpha)(1-\beta)P_{-\hat{\mathbf{p}}_2,-\hat{\mathbf{p}}_1}. \tag{3.80}$$

The four eigenvalues lie in a space of two degrees of freedom, whereas the eigenvalues of a general separable mixed state have three degrees of freedom. This means that the most general separable state cannot be written as (3.69). A generalization that solves this problem is

$$\rho = \gamma\frac{1}{4}(\mathbf{1}+\mathbf{p}_2\otimes\sigma_0)(\mathbf{1}+\sigma_0\otimes\mathbf{p}_1) + (1-\gamma)\rho_0, \tag{3.81}$$

where ρ_0 is some pure separable state that can be chosen as $P_{\hat{\mathbf{p}}_2,\,\hat{\mathbf{p}}_1}$ so that

the respective eigenvalues can be written as

$$p_1 = \gamma\frac{1}{4}(1+\alpha)(1+\beta)+(1-\gamma) \quad (3.82)$$

$$p_2 = \gamma\frac{1}{4}(1+\alpha)(1-\beta) \quad (3.83)$$

$$p_3 = \gamma\frac{1}{4}(1-\alpha)(1+\beta) \quad (3.84)$$

$$p_4 = \gamma\frac{1}{4}(1-\alpha)(1-\beta). \quad (3.85)$$

We therefore conclude that the most general separable two-qubit mixed state has cardinality 2 [23], because we were able to write it as the sum of two density matrices, each of them as the direct product of the two partial density matrices.

The two possible partial traces of a two-qubit system are

$$\rho_1 = [Tr \otimes 1](\rho) \quad (3.86)$$

$$\rho_2 = [1 \otimes Tr](\rho) \quad (3.87)$$

which give

$$\rho_1 = \gamma\frac{1}{2}(\sigma_0+\mathbf{p}_1)+(1-\gamma)\frac{1}{2}(\sigma_0+\hat{\mathbf{p}}_1) \quad (3.88)$$

$$\rho_2 = \gamma\frac{1}{2}(\sigma_0+\mathbf{p}_2)+(1-\gamma)\frac{1}{2}(\sigma_0+\hat{\mathbf{p}}_2) \quad (3.89)$$

that after collecting terms gives

$$\rho_1 = \frac{1}{2}\left(\sigma_0+(1-\gamma(1-|\mathbf{p}_1|))\hat{\mathbf{p}}_1\right) \quad (3.90)$$

$$\rho_2 = \frac{1}{2}\left(\sigma_0+(1-\gamma(1-|\mathbf{p}_2|))\hat{\mathbf{p}}_2\right). \quad (3.91)$$

The magnitude of the polarizations left are are extracted as

$$|2\rho_1-\sigma_0| = 1-\gamma(1-|\mathbf{p}_1|) \quad (3.92)$$

$$|2\rho_2-\sigma_0| = 1-\gamma(1-|\mathbf{p}_2|). \quad (3.93)$$

These expressions show that there is no loss of polarization if the initial state is pure (and separable) because $|\mathbf{p}_1| = |\mathbf{p}_2| = 1$ in this case. If this system is subject to a unitary transformation that induces entanglement, the remaining polarizations are reduced because the polarization vectors enter a larger space that comprises multi-body terms.

3.6 Separability Conditions

There are many other entanglement measures that have been proposed along with many separability conditions. An important separability condition known as the partial positive transpose (PPT), was given by [24, 25]. It is easily seen that the transpose of a single qubit can be understood as an operation that changes the sign of σ_2 only, as applied in [26] . Denoting this operation by Φ_T, the transpose of a single qubit is defined as

$$\Phi_T(\sigma_2) = -\sigma_2 \tag{3.94}$$

$$\Phi_T(\sigma_{j:0,1,3}) = \sigma_{j:0,1,3} \tag{3.95}$$

The partial transpose on the particle of the right slot of the tensor product, $(1 \otimes \Phi_T)$, can defined such that

$$(1 \otimes \Phi_T)(\sigma_j \otimes \sigma_2) = -\sigma_j \otimes \sigma_2 \tag{3.96}$$

$$(1 \otimes \Phi_T)(\sigma_j \otimes \sigma_{k:0,1,3}) = \sigma_j \otimes \sigma_k, \tag{3.97}$$

with the partial transpose on the left slot defined in similar way.

The Peres-Horodecki [24, 25] criterion applied to 2-qubit systems establishes that a sufficient and necessary condition for the system to be separable is that the partial transpose maintains the spectrum as positive. This means that the partial transpose of a non-separable 2-qubit state does not lead to

a valid density matrix at all. For example, the eigenvalues of the partial transpose of the pure density matrix coming from the unitary transformation (3.29)

$$\rho = U_E P U_E^\dagger \tag{3.98}$$

gives

$$p_1 = \frac{1}{2}(1 - \cos(u)\,\cos(v)) \tag{3.99}$$

$$p_2 = \frac{1}{2}(1 + \cos(u)\,\cos(v)) \tag{3.100}$$

$$p_3 = -\frac{1}{2}\sqrt{\sin(u)^2 + \cos(u)^2 \sin(v)^2} \tag{3.101}$$

$$p_4 = \frac{1}{2}\sqrt{\sin(u)^2 + \cos(u)^2 \sin(v)^2}, \tag{3.102}$$

The third eigenvector is negative, so the partial transposition of the pure density matrix is not a density matrix, unless the entanglement is zero.

Chapter 4

Three-qubit System

4.1 Control with $USp(8)$

A three-qubit system as an eight-state system is state-controllable either by $su(8)$, with dimension 63, or by $sp(8)$, with dimension 36.

Accoding to Table 1.8, there are two instances of $su(8)$ in terms of the anti-Hermitian $C\ell_7$ space and the anti-Hermitian $C\ell_6 \times \mathbb{C}$ space. On the other hand, this table also shows that there is an instance of the $sp(8)$ Lie algebra in terms of the anti-Hermitian $C\ell_6$ space. The purpose of the rest of this section is to study the control schemes defined by the anti-Hermitian $C\ell_6$ space.

Given the following expression of the anti-Hermitian $C\ell_6$ space

$$\mathcal{L} = \sum \delta\theta_{jk}\mathbf{e}_{jk} + \sum \delta\theta_{jkm}\mathbf{e}_{jkm} + \delta\theta_{123456}\mathbf{e}_{123456}, \tag{4.1}$$

the first order expansion of the exponential form is

$$e^{\mathcal{L}}P = \begin{pmatrix} i\theta_{12} - i\theta_{45} + i\theta_{123} - i\theta_{345} & 0... \\ \theta_{13} + i\theta_{23} - i\theta_{145} + \theta_{245} & 0... \\ -i\theta_{34} - \theta_{35} - \theta_{124} + i\theta_{125} & 0... \\ -i\theta_{14} - \theta_{15} + \theta_{24} - i\theta_{25} + i\theta_{134} + \theta_{135} + \theta_{234} + i\theta_{235} & 0... \\ -i\theta_{46} - \theta_{56} - i\theta_{346} - \theta_{356} & 0... \\ -i\theta_{146} - \theta_{156} + \theta_{246} - i\theta_{256} & 0... \\ -\theta_{36} + i\theta_{126} - i\theta_{456} - \theta_{123456} & 0... \\ -\theta_{16} - i\theta_{26} + \theta_{136} + i\theta_{236} & 0... \end{pmatrix}. \tag{4.2}$$

Thus, the anti-Hermitian $\mathcal{C}\ell_6$ space is a state-control algebra for the system in the infinitesimal neighborhood of the pass state P.

A possible parametrization with 15 parameters can be designed by choosing a set of 15 generators that fulfill the infinitesimal control condition (2.6). A table of a possible generators within the anti-Hermitian $\mathcal{C}\ell_6$ space is give in Table 4.1. Numerical results indicate that this parametrization allows us to access the complete Hilbert space, except at isolated points.

The Cartan subalgebra has dimension 4, which is sufficient to define a Hamiltonian with symmetric energy levels. A natural basis of the Cartan subalgebra can be taken as

$$\{\mathbf{e}_{12}, \mathbf{e}_{45}, \mathbf{e}_{123}, \mathbf{e}_{345}\}. \tag{4.3}$$

The symmetric nature of the Hamiltonian with the Cartan subalgebra is

Global phase	$\mathbf{e}_{12}$	$i\sigma_0 \otimes \sigma_0 \otimes \sigma_3$
Polarization	$\mathbf{e}_{56}$	$i\sigma_2 \otimes \sigma_0 \otimes \sigma_0$
	$\mathbf{e}_{46}$	$-i\sigma_1 \otimes \sigma_3 \otimes \sigma_0$
	$\mathbf{e}_{124}$	$i\sigma_0 \otimes \sigma_2 \otimes \sigma_3$
	$\mathbf{e}_{34}$	$-i\sigma_0 \otimes \sigma_1 \otimes \sigma_3$
	$\mathbf{e}_{13}$	$-i\sigma_0 \otimes \sigma_0 \otimes \sigma_2$
	$\mathbf{e}_{23}$	$i\sigma_0 \otimes \sigma_0 \otimes \sigma_1$
Bipartite Entanglement	$\mathbf{e}_{24}$	$-i\sigma_0 \otimes \sigma_1 \otimes \sigma_2$
	$\mathbf{e}_{14}$	$-i\sigma_0 \otimes \sigma_1 \otimes \sigma_1$
	$\mathbf{e}_{246}$	$-i\sigma_1 \otimes \sigma_0 \otimes \sigma_2$
	$\mathbf{e}_{146}$	$-i\sigma_1 \otimes \sigma_0 \otimes \sigma_1$
	$\mathbf{e}_{456}$	$i\sigma_2 \otimes \sigma_2 \otimes \sigma_0$
	$\mathbf{e}_{123456}$	$i\sigma_2 \otimes \sigma_1 \otimes \sigma_0$
Tripartite Entanglement	$\mathbf{e}_{16}$	$i\sigma_1 \otimes \sigma_2 \otimes \sigma_1$
	$\mathbf{e}_{26}$	$i\sigma_1 \otimes \sigma_2 \otimes \sigma_2$

Table 4.1. List of 15 generators, that satisfy the controllability condition. The first generator controls the global phase, the next six generators control the orientation of the three polarizations, and the rest induce entanglement.

evident in the following form

$$H = i\frac{1}{4}(-\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)\mathbf{e}_{12} \tag{4.4}$$

$$i\frac{1}{4}(-\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)\mathbf{e}_{45} \tag{4.5}$$

$$i\frac{1}{4}(-\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\mathbf{e}_{123} \tag{4.6}$$

$$i\frac{1}{4}(-\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)\mathbf{e}_{345}) \tag{4.7}$$

because it gives the following matrix representation

$$H = \begin{pmatrix} \epsilon_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 \end{pmatrix}. \tag{4.8}$$

The primitive projector P can be explicitly defined as

$$P = \frac{1}{8}(1 + \mathbf{e}_{12345})(1 - i\mathbf{e}_{123})(1 - i\mathbf{e}_{12}), \tag{4.9}$$

which takes the following form in terms of the Pauli matrices

$$P = \frac{1}{8}(\mathbf{1} + \sigma_3 \otimes \sigma_0 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \sigma_3 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \sigma_0 \otimes \sigma_3). \tag{4.10}$$

As occurs with the two-qubit case, it is convenient to relabel the primitive projector as $Q_1 = P$, so that the complete set of projectors that resolve the

identity is

$$Q_1 = \frac{1}{8}(1+\mathbf{e}_{12345})(1-i\mathbf{e}_{123})(1-i\mathbf{e}_{12}) \tag{4.11}$$

$$Q_2 = \frac{1}{8}(1+\mathbf{e}_{12345})(1-i\mathbf{e}_{123})(1+i\mathbf{e}_{12}) \tag{4.12}$$

$$Q_3 = \frac{1}{8}(1+\mathbf{e}_{12345})(1+i\mathbf{e}_{123})(1-i\mathbf{e}_{12}) \tag{4.13}$$

$$Q_4 = \frac{1}{8}(1+\mathbf{e}_{12345})(1+i\mathbf{e}_{123})(1+i\mathbf{e}_{12}) \tag{4.14}$$

$$Q_5 = \frac{1}{8}(1-\mathbf{e}_{12345})(1-i\mathbf{e}_{123})(1-i\mathbf{e}_{12}) \tag{4.15}$$

$$Q_6 = \frac{1}{8}(1-\mathbf{e}_{12345})(1-i\mathbf{e}_{123})(1+i\mathbf{e}_{12}) \tag{4.16}$$

$$Q_7 = \frac{1}{8}(1-\mathbf{e}_{12345})(1+i\mathbf{e}_{123})(1-i\mathbf{e}_{12}) \tag{4.17}$$

$$Q_8 = \frac{1}{8}(1-\mathbf{e}_{12345})(1+i\mathbf{e}_{123})(1+i\mathbf{e}_{12}). \tag{4.18}$$

with

$$1 = \sum_{j=1}^{8} Q_j. \tag{4.19}$$

The eight orthogonal states can be identified as

$$|\uparrow\uparrow\uparrow\rangle = P \tag{4.20}$$

$$|\uparrow\uparrow\downarrow\rangle = e^{\pi \mathbf{e}_{13}/2} P \tag{4.21}$$

$$|\uparrow\downarrow\uparrow\rangle = e^{-\pi \mathbf{e}_{124}/2} P \tag{4.22}$$

$$|\uparrow\downarrow\downarrow\rangle = e^{\pi \mathbf{e}_{24}/2} P \tag{4.23}$$

$$|\downarrow\uparrow\uparrow\rangle = e^{-\pi \mathbf{e}_{56}/2} P \tag{4.24}$$

$$|\downarrow\uparrow\downarrow\rangle = e^{\pi \mathbf{e}_{246}/2} P \tag{4.25}$$

$$|\downarrow\downarrow\uparrow\rangle = e^{-\pi \mathbf{e}_{123456}/2} P \tag{4.26}$$

$$|\downarrow\downarrow\downarrow\rangle = e^{-\pi \mathbf{e}_{16}/2} P. \tag{4.27}$$

4.2 Entanglement of Pure States

The primitive projector transforms to the following form whenever no entanglement is present

$$\rho = \frac{1}{8}(\mathbf{1} + \hat{\mathbf{p}}_3 \otimes \sigma_0 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \hat{\mathbf{p}}_2 \otimes \sigma_0)(\mathbf{1} + \sigma_0 \otimes \sigma_0 \otimes \hat{\mathbf{p}}_1), \quad (4.28)$$

where each of the polarization vectors $\hat{\mathbf{p}}_j$ is a unit vector such that $\hat{\mathbf{p}}_j \hat{\mathbf{p}}_j = \sigma_0$. This form of the projector could be obtained, for example by involving the unitary operators constructed with the six polarization generators specified in Table 4.1.

The generators that induce entanglement in Table 4.1 can be classified in four pairs: three of those pairs induce bipartite entanglement and a single pair induces tripartite entanglement.

A measurement of entropy of pure states can be defined in terms of the partial scalar parts of the density matrix as

$$\rho_3 = 4\langle\rho\rangle_{1\otimes S\otimes S} \quad (4.29)$$

$$\rho_2 = 4\langle\rho\rangle_{S\otimes 1\otimes S} \quad (4.30)$$

$$\rho_1 = 4\langle\rho\rangle_{S\otimes S\otimes 1}, \quad (4.31)$$

or in terms of the partial traces

$$\rho_3 = [1 \otimes Tr \otimes Tr](\rho) \quad (4.32)$$

$$\rho_2 = [Tr \otimes 1 \otimes Tr](\rho) \quad (4.33)$$

$$\rho_1 = [Tr \otimes Tr \otimes 1](\rho). \quad (4.34)$$

The measure of the entanglement is proposed as

$$E_{pure} = -2\langle\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2 + \rho_3 \ln \rho_3\rangle_S, \quad (4.35)$$

because each term is a measure of the loss of the polarization of the respective qubit. The greater the loss of polarization, the higher is the measurement of the entanglement.

The reduced density matrices calculated from the pure separable state (4.28) are

$$\rho_3 = \frac{1}{2}(\mathbf{1} + \hat{\mathbf{p}}_3) \tag{4.36}$$

$$\rho_2 = \frac{1}{2}(\mathbf{1} + \hat{\mathbf{p}}_2) \tag{4.37}$$

$$\rho_1 = \frac{1}{2}(\mathbf{1} + \hat{\mathbf{p}}_1), \tag{4.38}$$

and as there is no loss of polarization, the measure of the entanglement (4.35) gives 0.

At the opposite limit of maximum entanglement, the reduced density matrices have no polarization at all

$$\rho_3 = \frac{1}{2}(\mathbf{1}) \tag{4.39}$$

$$\rho_2 = \frac{1}{2}(\mathbf{1}) \tag{4.40}$$

$$\rho_1 = \frac{1}{2}(\mathbf{1}), \tag{4.41}$$

and the measure of the entanglement (4.35) gives $3\ln(2)$.

The unitary operator that induces tripartite entanglement is

$$U_{E(123)} = e^{u\mathbf{e}_{16}/2} e^{v\mathbf{e}_{26}/2}, \tag{4.42}$$

which produces the following spinor state

$$\begin{aligned} \Psi(u,v) = {} & (\cos(u/2)\cos(v/2) + i\sin(u/2)\cos(v/2))|\uparrow\uparrow\uparrow\rangle - \\ & (\cos(v/2)\sin(u/2) + i\cos(u/2)\cos(v/2))|\downarrow\downarrow\downarrow\rangle. \end{aligned} \tag{4.43}$$

The plot of the measure of the three-qubit entanglement for this state is shown in figure (4.1), where the maximum value is

$$E_{max} = 3\ln(2) \tag{4.44}$$

The measurement of a three-qubit entanglement induced by a bipartite entanglement may reach a maximum value of $3\ln(2)$. The plot of the three-qubit entanglement for the following state, made from a bipartite tanglor multiplied by a tripartite tanglor is shown in figure (4.2)

$$U = e^{\theta_{16}\, e_{16}/2} e^{\theta_{123456}\, e_{123456}/2}, \tag{4.45}$$

The application of two bipartite tanglors is still able to produce some isolated states with the maximum three-qubit entanglement as shown in figure (4.3) for the tanglor

$$U = e^{\theta_{123456}\, e_{123456}/2} e^{\theta_{146} e_{146}/2}. \tag{4.46}$$

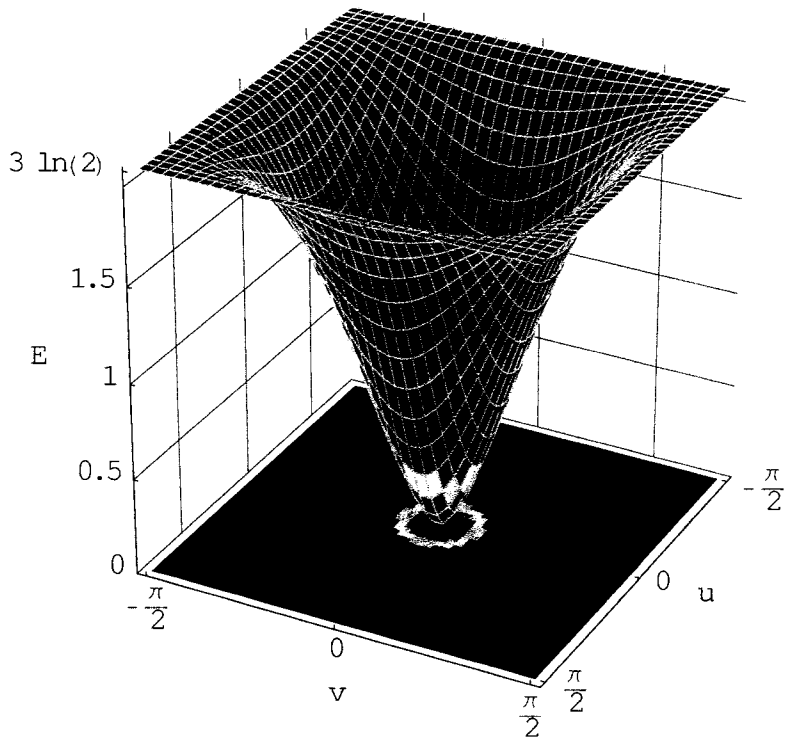

Figure 4.1. Entropy of entanglement of three qubits for the pure state $\rho = U_{E(123)}(u,v)PU^{\dagger}_{E(123)}(u,v)$.

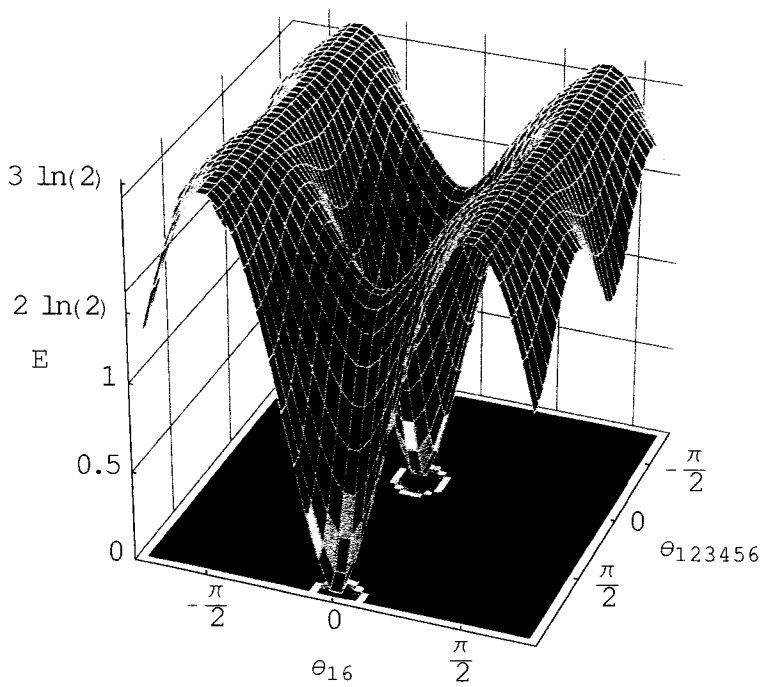

Figure 4.2. Entropy of entanglement of three qubits for the pure state $\rho = e^{\theta_{16}\, e_{16}/2} e^{\theta_{123456}\, e_{123456}/2} P e^{-\theta_{123456}\, e_{123456}/2} e^{-\theta_{16}\, e_{16}/2}$.

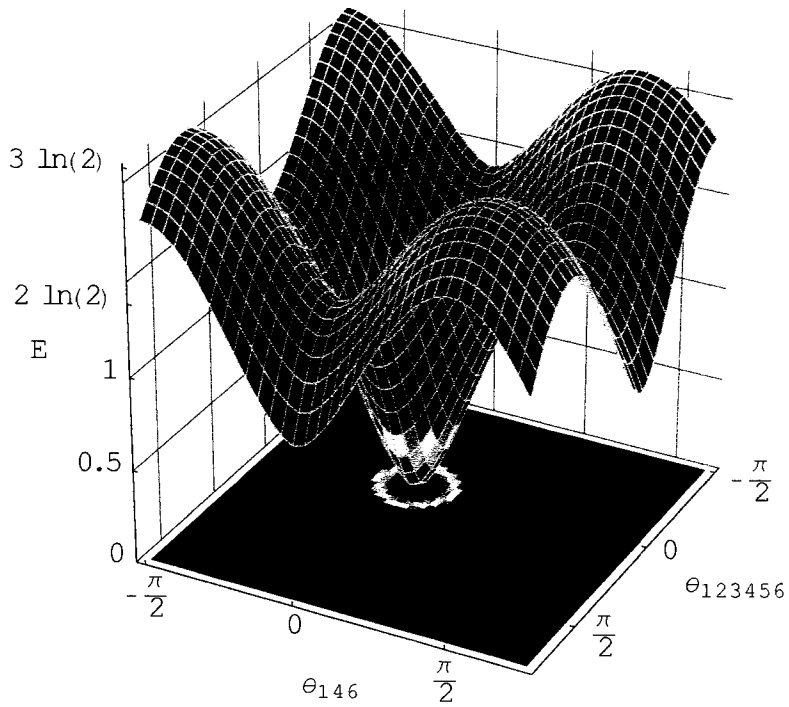

Figure 4.3. Entropy of entanglement of three qubits for the pure state $\rho = e^{\theta_{123456}\, e_{123456}/2} e^{\theta_{146}\, e_{146}/2} P e^{-\theta_{146}\, e_{146}/2} e^{-\theta_{123456}\, e_{123456}/2}$.

Chapter 5

Average Fidelity

This chapter treats the problem of the calculation of the average fidelity of n-qubit systems [27], which is difficult to evaluate in its original integral form. The fidelity measures the correlation between two states represented by their density matrices with a maximum value for two identical states. A well known measure of the fidelity is given by

$$F(\rho,\rho') = \left(Tr\sqrt{\rho^{1/2}\rho'\rho^{1/2}}\right)^2 \tag{5.1}$$

which is the square of the original formula given by Uhlmann [28]. This formula is simplified if one of the density matrices represents a pure state. Defining ρ as a pure state density matrix, the expression of the fidelity is simplified

$$F(\rho,\rho') = \left(Tr\sqrt{\rho\rho'\rho}\right)^2 = Tr(\rho\rho') \tag{5.2}$$

This formula can be used to calculate the fidelity of two output states coming from an initially pure state $\rho_0 = |\Psi\rangle\langle\Psi|$. The density matrix ρ as output state could come from a unitary transformation that preserves the purity of the state and the ρ' output state could come from a more general

completely positive trace-preserving linear transformation

$$\rho = U|\Psi\rangle\langle\Psi|U^{\dagger} \tag{5.3}$$

$$\rho' = \mathcal{M}(|\Psi\rangle\langle\Psi|) \tag{5.4}$$

The average square fidelity is defined over all the possible pure initial states

$$\langle F\rangle = \langle F(\rho,\rho')\rangle_{All\,\rho_0} = \langle Tr(\rho\rho')\rangle_{All\,\rho_0} \tag{5.5}$$

Taking the case of the single qubit, one way to proceed was presented by [29], who performs an integration over the complete Bloch sphere

$$\langle F\rangle = \frac{1}{4\pi}\int Tr(\rho_1\rho_2)d\Omega, \tag{5.6}$$

where the initial pure state could be parametrized as

$$\rho_0 = \frac{1}{2}(\mathbf{1} + \sigma_x\sin(\theta)\cos(\phi) + \sigma_y\sin(\theta)\sin(\phi) + \sigma_z\cos(\theta)), \tag{5.7}$$

but the integration becomes intractable for the general n-qubit case. A possible approach is possible by defining the average over a discrete set of orthogonal density matrices as done by Evan M. Fortunato *et al.* [30]. However, the continuous average fidelity (5.5) was faithfully expressed as a discrete sum by Bagan, *et al.* [31] for the "qudit" case (N levels) by using an expansion of ρ_0 in terms of the generators of $SU(N)$. The following lines develop a simpler approach suitable to n-qubits by performing an expansion in terms of tensor product of Pauli matrices.

The density matrix of a single qubit can be written in terms of the polarization vector $\mathbf{P}$ with components along the Pauli matrices

$$\mathbf{P} = P^j\sigma_j = P^1\sigma_1 + P^2\sigma_2 + P^3\sigma_3, \tag{5.8}$$

so that a general single-qubit density matrix can be written as

$$\rho = \frac{1}{2}(1 + \mathbf{P}), \tag{5.9}$$

where, $\mathbf{P}$, for a pure states lies on the surface of a sphere

$$(P^1)^2 + (P^2)^2 + (P^3)^2 = 1 \tag{5.10}$$

and a general mixed state lies inside the sphere with zero polarization corresponding to the purely mixed state.

In this way, the output states can be written as

$$\rho = \frac{1}{2}(\mathbf{1} + U\mathbf{P}U^\dagger) \tag{5.11}$$

$$\rho' = \frac{1}{2}(\mathcal{M}(\mathbf{1}) + \mathcal{M}(\mathbf{P})). \tag{5.12}$$

The fidelity then becomes

$$F = \frac{1}{4}Tr\left[(\mathbf{1} + U\mathbf{P}U^\dagger)(\mathcal{M}(\mathbf{1}) + \mathcal{M}(\mathbf{P})\right] \tag{5.13}$$

$$= \frac{1}{4}Tr(\mathbf{1} + U\mathbf{P}U^\dagger\mathcal{M}(\mathbf{1}) + \mathcal{M}(\mathbf{P}) + U\mathbf{P}U^\dagger\mathcal{M}(\mathbf{P})) \tag{5.14}$$

$$= \frac{1}{4}Tr(\mathbf{1} + \sum_{jk} P^j P^k U\sigma_j U^\dagger \mathcal{M}(\sigma_k)), \tag{5.15}$$

from which the average fidelity is

$$\langle F\rangle = \frac{1}{4}Tr(\mathbf{1} + \sum_{jk}\langle P^j P^k\rangle U\sigma_j U^\dagger \mathcal{M}(\sigma_k)). \tag{5.16}$$

The following argument can be made from the symmetry of the geometry involved

$$\langle P^j P^k\rangle = \frac{1}{3}\delta^{jk}. \tag{5.17}$$

The P^j elements are understood as the components of a unit radius vector so that for the contribution of a given pair of different indexes jk there is another pair with opposite contribution that cancels the initial pair. On the other hand, the following average is evident

$$\langle (P^1)^2 + (P^2)^2 + (P^3)^2\rangle = 1, \tag{5.18}$$

but the contribution of each component must be same

$$\langle (P^1)^2 \rangle = \langle (P^2)^2 \rangle = \langle (P^3)^2 \rangle = \frac{1}{3}. \tag{5.19}$$

In the more general case of an N-dimensional unit vector, the result reads

$$\langle P^j P^k \rangle = \frac{1}{N} \delta^{jk}. \tag{5.20}$$

Returning to 5.16, the average fidelity is simplified to

$$\begin{aligned} \langle F \rangle &= \frac{1}{4} Tr \left(1 + \frac{1}{3} \sum_j U \sigma_j U^\dagger \mathcal{M}(\sigma_j) \right) && (5.21) \\ &= \frac{1}{4} \sum_j Tr \left(\frac{\mathbf{1}}{3} + \frac{1}{3} U \sigma_j U^\dagger \mathcal{M}(\sigma_j) \right) && (5.22) \\ &= \frac{1}{3} \sum_j Tr \left(\frac{\mathbf{1}}{4} + U \frac{\sigma_j}{2} U^\dagger \mathcal{M}(\frac{\sigma_j}{2}) \right), && (5.23) \end{aligned}$$

but the Pauli matrices can be written in terms of pure states as

$$\begin{aligned} \frac{\sigma_j}{2} &= \frac{\mathbf{1} + \sigma_j}{2} - \frac{\mathbf{1}}{2} = \rho_j - \hat{\rho} && (5.24) \\ \frac{\sigma_j}{2} &= \frac{\sigma_j - \mathbf{1}}{2} + \frac{\mathbf{1}}{2} = -\rho_{-j} + \hat{\rho}, && (5.25) \end{aligned}$$

where $\hat{\rho}$ is the purely mixed state, so that the average fidelity is

$$\langle F \rangle = \frac{1}{3} \sum_j Tr \left(\frac{\mathbf{1}}{2} + U \rho_j U^\dagger \mathcal{M}(\rho_j) - U \rho_j U^\dagger \mathcal{M}(\hat{\rho}) \right) \tag{5.26}$$

or

$$\langle F \rangle = \frac{1}{3} \sum_j Tr \left(\frac{\mathbf{1}}{2} + U \rho_{-j} U^\dagger \mathcal{M}(\rho_{-j}) - U \rho_{-j} U^\dagger \mathcal{M}(\hat{\rho}) \right). \tag{5.27}$$

The average of the last two expressions gives finally

$$\begin{aligned}
\langle F \rangle &= \frac{1}{6}\sum_j Tr(1 + \\
&\quad U\rho_j U^\dagger \mathcal{M}(\rho_j) - U\rho_j U^\dagger \mathcal{M}(\hat{\rho}) + \\
&\quad U\rho_{-j} U^\dagger \mathcal{M}(\rho_{-j}) - U\rho_{-j} U^\dagger \mathcal{M}(\hat{\rho})) \\
&= \frac{1}{6}\sum_j Tr(1 + \\
&\quad U\rho_j U^\dagger \mathcal{M}(\rho_j) + U\rho_{-j} U^\dagger \mathcal{M}(\rho_{-j}) + \\
&\quad -U(\rho_j + \rho_{-j}) U^\dagger \mathcal{M}(\hat{\rho})) \\
&= \frac{1}{6}\sum_j Tr(U\rho_j U^\dagger \mathcal{M}(\rho_j) + \\
&\quad U\rho_{-j} U^\dagger \mathcal{M}(\rho_{-j}))
\end{aligned}$$

or in a more compact form

$$\langle F \rangle = \frac{1}{6}\sum_{j=\pm 1,\pm 2,\pm 3} Tr\left(U\rho_j U^\dagger \mathcal{M}(\rho_j)\right). \tag{5.28}$$

This result can be extended to n-qubits by defining the proper basis in which to expand the density matrix. For two qubits the density matrix can be expanded in terms of a linear combination of a basis set $\mathbf{f}_j$ as

$$\rho = \frac{1}{4}(1 + \sum_j^{15} w^j \mathbf{f}_j), \tag{5.29}$$

where the basis set is defined in terms of the Kronecker product of the Pauli matrices

$$\mathbf{f}_j = \sigma_\mu \otimes \sigma_\nu, \quad \mu, \nu = 0, 1, 2, 3 \tag{5.30}$$

where σ_0 is the 2×2 identity matrix and the pair $\sigma_0 \otimes \sigma_0$ is not included. The elements of this basis obey the following orthogonality condition

$$\mathbf{f}_j \mathbf{f}_j = \mathbf{1} \tag{5.31}$$

$$\frac{1}{4} Tr(\mathbf{f}_i \mathbf{f}_j) = \delta_{ij} \tag{5.32}$$

so that in general

$$\begin{aligned} Tr\left(\rho_1\rho_2\right) &= \frac{1}{16}Tr\left[\left(\mathbf{1}+\sum_{j=1}^{15}w_1^j\mathbf{f}_j\right)\left(\mathbf{1}+\sum_{k=1}^{15}w_2^k\mathbf{f}_k\right)\right] \\ &= \frac{1}{4}\left(1+\sum_{j=1}^{15}w_1^jw_2^j\right), \end{aligned}$$

but for a pure state such as ρ_0 this is

$$Tr\left(\rho_0^2\right) = \mathrm{tr}\left(\rho_0\right) = 1 = \frac{1}{4}\left(1+\sum_{j=1}^{15}\left(w_0^j\right)^2\right) \tag{5.33}$$

so that the following equation holds

$$\sum_{j=1}^{15}\left(w_0^j\right)^2 = 3. \tag{5.34}$$

More generally, for a state that may be mixed,

$$\sum_{j=1}^{15}\left(w^j\right)^2 \leq 3 \tag{5.35}$$

with the equality holding only when the state is pure. The average of (5.34) can be reduced to the average of any term in the sum arguing the evident symmetry

$$3 = \langle\sum_{j=1}^{15}\left(w_0^j\right)^2\rangle \tag{5.36}$$

$$3 = 15\langle(w_o^j)^2\rangle \tag{5.37}$$

so that the average square of any given component is the same

$$\langle(w_o^j)^2\rangle = \frac{1}{5}. \tag{5.38}$$

This result is important for calculating the average including the cross terms as

$$\langle w_0^i w_0^j\rangle = \delta^{ij}\langle(w_o^j)^2\rangle = \frac{1}{5}\delta^{ij}, \tag{5.39}$$

where the cross terms are zero because they are not correlated with each other at all, once the average over all the states is taken.

For the general n-qubit case the basis $\mathbf{f}_j$ contains $N^2 - 1$ elements such that

$$Tr(\mathbf{f}_i \mathbf{f}_j) = N\delta_{ij} \tag{5.40}$$

with $N = 2^n$. For some applications it is useful to notice that any given basis element $\mathbf{f}_j$ anticommutes with $\frac{N^2}{2}$ elements of the basis and commutes with the rest $\frac{N^2}{2} - 1$ elements, including itself.

The general density matrix for n-qubits can be written as

$$\rho = \frac{1}{N}(1 + \sum_{j=1}^{N^2-1} w^j \mathbf{f}_j) \tag{5.41}$$

The trace square of a pure density matrix is one $Tr(\rho_0^2) = 1$ so that

$$\begin{aligned} 1 &= \frac{1}{N^2} Tr \left[\left(1 + \sum_{j=1}^{N^2-1} w^j \mathbf{f}_j \right) \left(1 + \sum_{k=1}^{N^2-1} w^k \mathbf{f}_k \right) \right] \\ &= \frac{1}{N} + \frac{1}{N} \sum_{jk} w^j w^k \delta_{jk} = \frac{1}{N} \left(1 + \sum_{j=1}^{N^2-1} \left(w^j \right)^2 \right), \end{aligned}$$

which gives

$$\sum_{j=1}^{N^2-1} \left(w^j \right)^2 = N - 1 \tag{5.42}$$

Taking the average of this result gives

$$\langle (w^j)^2 \rangle = \frac{N-1}{N^2-1} = \frac{1}{1+N}, \tag{5.43}$$

which finally allows to write the general average including the cross terms as

$$\langle w^i w^j \rangle = \frac{1}{1+N} \delta^{ij} \tag{5.44}$$

The expression 5.16 is then generalized and calculated as

$$\langle F\rangle = \frac{1}{N^2}Tr\left[\mathbf{1}+\sum_{jk}\langle w^j w^k\rangle U\mathbf{f}_j U^\dagger \mathcal{M}(\mathbf{f}_k)\right] \tag{5.45}$$

$$= \frac{1}{N^2}Tr\left[\mathbf{1}+\frac{1}{1+N}\sum_{j}U\mathbf{f}_j U^\dagger \mathcal{M}(\mathbf{f}_j)\right] \tag{5.46}$$

$$= \frac{1}{N}+\frac{1}{(N+1)N^2}\sum_{j}^{N^2-1}Tr\left[U\mathbf{f}_j U^\dagger \mathcal{M}(\mathbf{f}_j)\right] \tag{5.47}$$

The remaining step is to express the basis elements $\mathbf{f}_j$ in terms of a set of pure density matrices. In the 2-qubit case, this is possible by using the following identities

$$\sigma_j\otimes\sigma_k = (\rho_{jk}\rho_{kj}-\bar{\rho}_{jk}\rho_{kj}+\rho_{jk}\bar{\rho}_{kj}-\bar{\rho}_{jk}\bar{\rho}_{kj}) \tag{5.48}$$

$$\sigma_j\otimes\sigma_0 = -(\rho_{j0}\rho_{0j}-\bar{\rho}_{j0}\rho_{0j}+\rho_{j0}\bar{\rho}_{0j}-\bar{\rho}_{j0}\bar{\rho}_{0j}) \tag{5.49}$$

$$\sigma_0\otimes\sigma_k = -(\rho_{0k}\rho_{k0}-\bar{\rho}_{0k}\rho_{k0}+\rho_{0k}\bar{\rho}_{k0}-\bar{\rho}_{0k}\bar{\rho}_{k0}), \tag{5.50}$$

where j and k are different from zero and

$$\rho_{jk} = \frac{1}{2}(\mathbf{1}+\sigma_j\otimes\sigma_k) \tag{5.51}$$

$$\bar{\rho}_{jk} = \frac{1}{2}(\mathbf{1}-\sigma_j\otimes\sigma_k) \tag{5.52}$$

A similar procedure is possible for the general n-qubit case.

If the application of general linear map $\mathcal{M}$ is restricted to the following form

$$\mathcal{M}(\rho) = M\rho M^\dagger, \tag{5.53}$$

the fidelity can be formated as

$$F(\rho,\rho') = Tr(U\rho_0 U^\dagger M\rho_0 M^\dagger) = Tr(\rho_0 U^\dagger M\rho_0 M^\dagger U) \tag{5.54}$$

or by expressing the pure state as $\rho_0 = |\Psi_0\rangle\langle\Psi_0|$, the fidelity becomes

$$F(\rho, \rho') = \langle\Psi_0|U^\dagger M|\Psi_0\rangle\langle\Psi_0|M^\dagger U|\Psi_0\rangle = |\langle\Psi_0|U^\dagger M|\Psi_0\rangle|^2, \tag{5.55}$$

so that the average fidelity is calculated by integrating over the pure states lying on a unit hypersphere of dimension $2N-1$.

$$\langle F\rangle = \int_{S^{2N-1}} dS|\langle\Psi_0|U^\dagger M|\Psi_0\rangle|^2, \tag{5.56}$$

where dS is the normalized measure on the hypersphere. A closed expression of the average fidelity from this formula was given by Pedersen *et al.* [32], as a quadratic invariant form constructed from $U^\dagger M$, equivalent to the formula given by Emerson *et al.* [33]. According to them [32], the average fidelity could be expressed in following form

$$\langle F\rangle = a\,Tr(U^\dagger M M^\dagger U) + b\,|Tr(U^\dagger M)|^2, \tag{5.57}$$

where a and b are constants to be fit and depend on N.

Taking the cases $U^\dagger M = \mathbf{1}$ and $U^\dagger M = \sigma_1 \otimes \sigma_1$, and applying the formula (5.47), the invariant form gives the following respective equations

$$U^\dagger M = \mathbf{1} \quad \rightarrow \quad 1 = aN + bN^2 \tag{5.58}$$

$$U^\dagger M = \sigma_1 \otimes \sigma_1 \quad \rightarrow \quad \frac{1}{N+1} = aN, \tag{5.59}$$

which allows us to fit the values of a and b as

$$a = b = \frac{1}{N(N+1)}, \tag{5.60}$$

in complete agreement with [32], so that the average fidelity is expressed as

$$\langle F\rangle = \frac{1}{N(N+1)}(Tr(U^\dagger M M^\dagger U) + |Tr(U^\dagger M)|^2). \tag{5.61}$$

Chapter 6

N-Fermion Algebras

It has been shown that the Lie algebras of groups locally isomorphic to special orthogonal groups $SO(2N)$ can be written in terms of all the possible biproducts of the creation and annihilation operators of N different fermions [34, 35, 36](bilinear space of N fermions), where the $su(N)$ Lie algebra appears as a subgroup of $so(2N)$ when the number of particles is conserved. In this chapter we show how to formulate complete Clifford algebras in terms of the fermion creation-annihilation operators [5, 37].

6.1 Single Fermion Algebra

The properties that define the fermion algebra of creation and annihilation operators are

$$\left\{a, a^\dagger\right\} \equiv aa^\dagger + a^\dagger a = 1 \tag{6.1}$$

$$a\ a\ = 0 \tag{6.2}$$

The fermion algebra with the complex numbers is enough to define the $\mathcal{C}\ell_3$ algebra. This can be done by constructing two hermitian linear combi-

nations of the creation-annihilation operators and identifying the complex number with the volume element as

$$\mathbf{e}_1 = \quad a^\dagger + a \tag{6.3}$$

$$\mathbf{e}_2 = -i(a^\dagger - a) \tag{6.4}$$

$$\mathbf{e}_{123} \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i. \tag{6.5}$$

All basis elements of the linear space of the $C\ell_3$ algebra can be generated from these. For example, the bivectors can be found directly as

$$\mathbf{e}_1\mathbf{e}_{123} = \mathbf{e}_{23} \quad = \quad i(a^\dagger + a) \tag{6.6}$$

$$\mathbf{e}_2\mathbf{e}_{123} = \mathbf{e}_{31} \quad = \quad (a^\dagger - a) \tag{6.7}$$

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12} \quad = \quad ia^\dagger a - iaa^\dagger = i[a^\dagger, a], \tag{6.8}$$

and the remaining vector element $\mathbf{e}_3$ is calculated as

$$\mathbf{e}_{12}\mathbf{e}_{123} = -\mathbf{e}_3 = -[a^\dagger, a], \tag{6.9}$$

so that the complete basis can be written as

$$1 = 1 \tag{6.10}$$

$$\mathbf{e}_1 = a^\dagger + a \tag{6.11}$$

$$\mathbf{e}_2 = i(a - a^\dagger) \tag{6.12}$$

$$\mathbf{e}_3 = [a^\dagger, a] \tag{6.13}$$

$$\mathbf{e}_{12} = i[a^\dagger, a] \tag{6.14}$$

$$\mathbf{e}_{13} = (a - a^\dagger) \tag{6.15}$$

$$\mathbf{e}_{23} = i(a^\dagger + a) \tag{6.16}$$

$$\mathbf{e}_{123} = i, \tag{6.17}$$

satisfying the usual Clifford-algebra relations

$$\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j = 2\delta_{jk} \ .$$

The fact that the $C\ell_3$ can be generated from the single particle fermion algebra is highly suggestive, and it is natural to ask whether this is simply a mathematical curiosity or there is a more fundamental truth lying behind it.

From a computational perspective, these relations could be used to perform calculations in $C\ell_3$ with a computer program based on fermion algebras.

Conversely, the single-fermion algebra can be obtained from the $C\ell_3$ algebra with the annihilation and creation operators represented by null flags

$$a^\dagger = \frac{1}{2}(1 + \mathbf{e}_3)\,\mathbf{e}_1 \tag{6.18}$$

$$a = \frac{1}{2}\mathbf{e}_1\,(1 + \mathbf{e}_3)\,. \tag{6.19}$$

Alternatively, the equivalent relations

$$a^\dagger = \frac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2) \tag{6.20}$$

$$a = \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2) \tag{6.21}$$

can be understood as a representation within the complexified Clifford algebra $C\ell_2$. These approaches could be useful if we want to perform calculations involving the fermion algebra within a computer program based on Clifford algebras.

6.2 Bi-Fermion Algebra

The properties of a multiple fermion algebra are defined as

$$\left\{a_j, a_k^\dagger\right\} = \delta_{jk} \tag{6.22}$$

$$\{a_j\,, a_k\} = 0. \tag{6.23}$$

The bi-fermion algebra is enough to define the $C\ell_4$ algebra. The orthonormal vector basis can be expressed in terms of the bi-fermion algebra as

$$\mathbf{e}_1 = a_1^\dagger + a_1 \tag{6.24}$$

$$\mathbf{e}_2 = i(a_1 - a_1^\dagger) \tag{6.25}$$

$$\mathbf{e}_3 = a_2^\dagger + a_2 \tag{6.26}$$

$$\mathbf{e}_4 = i(a_2 - a_2^\dagger) \tag{6.27}$$

The complete the $C\ell_4$ algebra is obtained taking the respective multiple products.

There are three Lie subalgebras of interest:

- The anti-Hermitian $C\ell_4$ algebra made with both bi and triproducts of the creation-annihilation operators as shown in table 6.1. This algebra is isomorphic to $sp(4)$.
- The bivector algebra that corresponds to the $spin(4)$ Lie algebra can be written only using the biproducts of the creation-annihilation operators. Within the bivector algebra, it is possible to construct a $su(2)$ Lie algebra considering the terms that maintain the number of particles invariant.

- Finally, it is possible to construct an independent $su(2)$ Lie algebra with the rest of the generators that do not maintain the number of particles invariant.

Anti-Hermitian $C\ell_4$	Bi-fermion algebra
$\mathbf{e}_{12}$	$i[a_1^\dagger, a_1]$
$\mathbf{e}_{13}$	$(a_1^\dagger + a_1)(a_2^\dagger + a_2)$
$\mathbf{e}_{14}$	$i(a_1^\dagger + a_1)(a_2^\dagger - a_2)$
$\mathbf{e}_{23}$	$i(a_1 - a_1^\dagger)(a_2^\dagger + a_2)$
$\mathbf{e}_{24}$	$-(a_1^\dagger - a_1)(a_2^\dagger - a_2)$
$\mathbf{e}_{34}$	$i[a_2^\dagger, a_2]$
$\mathbf{e}_{123}$	$i[a_1^\dagger, a_1](a_2^\dagger + a_2)$
$\mathbf{e}_{124}$	$-[a_1^\dagger, a_1](a_2 - a_2^\dagger)$
$\mathbf{e}_{134}$	$i(a_1^\dagger + a_1)[a_2^\dagger, a_2]$
$\mathbf{e}_{234}$	$-(a_1 - a_1^\dagger)[a_2^\dagger, a_2]$

Table 6.1. The anti-Hermitian $C\ell_4$ algebra in terms of the bi-fermion algebra. This set of generators forms a closed Lie algebra isomorphic to $sp(4)$

The number operator that counts the number of particles is

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 = 1 - \frac{i}{2}\mathbf{e}_{12} - \frac{i}{2}\mathbf{e}_{34} \tag{6.28}$$

and the biproducts that commute with this operator can be written in terms of the bivectors as

$$T_0 = a_1^\dagger a_1 - a_2^\dagger a_2 = \frac{i}{2}(\mathbf{e}_{34} - \mathbf{e}_{12}) \tag{6.29}$$

$$T_+ = a_1^\dagger a_2 = \frac{1}{4}(\mathbf{e}_{13} + \mathbf{e}_{24} + i\mathbf{e}_{23} - i\mathbf{e}_{14}) \tag{6.30}$$

$$T_- = a_2^\dagger a_1 = \frac{1}{4}(-\mathbf{e}_{13} - \mathbf{e}_{24} + i\mathbf{e}_{23} - i\mathbf{e}_{14}) \tag{6.31}$$

so that they form the canonical representation for the $su(2)$ Lie algebra. The anti-Hermitian generators of the $SU(2)$ group are

$$iT_0 = \frac{1}{2}(\mathbf{e}_{12} - \mathbf{e}_{34}) \tag{6.32}$$

$$i(T_+ + T_-) = \frac{1}{2}(\mathbf{e}_{14} - \mathbf{e}_{23}) \tag{6.33}$$

$$(T_+ - T_-) = \frac{1}{2}(\mathbf{e}_{13} - \mathbf{e}_{24}). \tag{6.34}$$

A more symmetric form can be obtained by relabeling some of the basis vectors as

$$\mathbf{f}_1 = \mathbf{e}_3 \tag{6.35}$$

$$\mathbf{f}_2 = \mathbf{e}_4 \tag{6.36}$$

to obtain the following form of the anti-Hermitian generators

$$iT_0 = \frac{1}{2}(\mathbf{e}_{12} - \mathbf{f}_{12}) \tag{6.37}$$

$$i(T_+ + T_-) = \frac{1}{2}(\mathbf{e}_1\mathbf{f}_2 - \mathbf{e}_2\mathbf{f}_1) \tag{6.38}$$

$$(T_+ - T_-) = \frac{1}{2}(\mathbf{e}_1\mathbf{f}_1 + \mathbf{e}_2\mathbf{f}_2). \tag{6.39}$$

There is another set of generators that commutes with the generators that conserve the number of particles. One such generator is N, but it is convenient to define an offset operator as $S_0 = N - 1$, in order to be part of this set. This new set of generators does not maintain the number of particles invariant, but remarkably it forms another independent $su(2)$ Lie algebra. The canonical representation of this Lie algebra is

$$S_0 = \frac{1}{2}(a_1^\dagger a_1 - a_2 a_2^\dagger) = -\frac{i}{2}(\mathbf{e}_{34} + \mathbf{e}_{12}) \tag{6.40}$$

$$S_+ = a_1^\dagger a_2^\dagger = \frac{1}{4}(\mathbf{e}_{13} - \mathbf{e}_{24} + i\mathbf{e}_{23} + i\mathbf{e}_{14}) \tag{6.41}$$

$$S_- = a_2 a_1 = \frac{1}{4}(-\mathbf{e}_{13} + \mathbf{e}_{24} + i\mathbf{e}_{23} + i\mathbf{e}_{14}). \tag{6.42}$$

The corresponding anti-Hermitian generators in terms of the relabeled basis vectors are

$$iS_0 = \frac{1}{2}(\mathbf{e}_{12} + \mathbf{f}_{12}) \tag{6.43}$$

$$i(S_+ + S_-) = -\frac{1}{2}(\mathbf{e}_2\mathbf{f}_1 + \mathbf{e}_1\mathbf{f}_2) \tag{6.44}$$

$$(S_+ - S_-) = \frac{1}{2}(\mathbf{e}_1\mathbf{f}_1 - \mathbf{e}_2\mathbf{f}_2). \tag{6.45}$$

Thus, the $spin(4)$ made with all the possible biproducts of the creation annihilation operators, which are six, can be divided into two independent $su(2)$ groups. This allows us to write the familiar isomorphism

$$spin(4) \simeq su(2) \otimes su(2). \tag{6.46}$$

6.3 Tri-Fermion Algebra

The tri-fermion algebra is richer because it can be used to define the $\mathcal{C}\ell_7$ algebra and therefore all the lower dimensional Clifford algebras as well. The connection of the tri-fermion algebra with the $\mathcal{C}\ell_7$ algebra is given by the following definition of six orthonormal vectors and the volume element in terms of the fermion algebra with complex numbers

$$\mathbf{e}_1 = a_1^\dagger + a_1 \tag{6.47}$$

$$\mathbf{e}_2 = i(a_1 - a_1^\dagger) \tag{6.48}$$

$$\mathbf{e}_3 = a_2^\dagger + a_2 \tag{6.49}$$

$$\mathbf{e}_4 = i(a_2 - a_2^\dagger) \tag{6.50}$$

$$\mathbf{e}_5 = a_3^\dagger + a_3 \tag{6.51}$$

$$\mathbf{e}_6 = i(a_3 - a_3^\dagger) \tag{6.52}$$

$$\mathbf{e}_{1234567} = i \tag{6.53}$$

These unit basis vectors generate $\mathcal{C}\ell_6$ with a volume element $\mathbf{e}_{123456}$ that squares to -1. If we identify $\mathbf{e}_7 = -i\mathbf{e}_{123456}$, we get $\mathcal{C}\ell_7$ with a volume element identified with i. The $\mathcal{C}\ell_7$ algebra allows in this way to express the $spin(7)$ Lie algebra along with its subgroups in terms of the three particle fermion algebra. This list includes for example the $spin(6)$ and $su(3)$ Lie algebras that are studied in the following lines.

The Lie algebra made with all the possible biproducts of the tri-fermion algebra forms the $spin(6)$ Lie algebra. The operators that conserve the number of particles form a representation of the $su(3)$ Lie algebra.

The operator that counts the number of particles is

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 + a_1^\dagger a_2 = -\frac{1}{2}(i\mathbf{e}_{12} + i\mathbf{e}_{34} + i\mathbf{e}_{56} - 3) \tag{6.54}$$

The operators that commute with N are

$$T_1 = a_1^\dagger a_2 \tag{6.55}$$

$$T_2 = a_2^\dagger a_1 \tag{6.56}$$

$$T_3 = a_1^\dagger a_3 \tag{6.57}$$

$$T_4 = a_3^\dagger a_1 \tag{6.58}$$

$$T_5 = a_1^\dagger a_3 \tag{6.59}$$

$$T_6 = a_3^\dagger a_2 \tag{6.60}$$

$$T_7 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \tag{6.61}$$

$$T_8 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3), \tag{6.62}$$

so, the anti-Hermitian generators that can be constructed in terms of the

bivectors of $C\ell_6$ are

$$i(T_1 + T_2) = \frac{1}{2}(\mathbf{e}_{14} - \mathbf{e}_{23}) \tag{6.63}$$

$$(T_1 - T_2) = \frac{1}{2}(\mathbf{e}_{13} + \mathbf{e}_{24}) \tag{6.64}$$

$$i(T_3 + T_4) = \frac{1}{2}(\mathbf{e}_{16} - \mathbf{e}_{25}) \tag{6.65}$$

$$(T_3 - T_4) = \frac{1}{2}(\mathbf{e}_{15} + \mathbf{e}_{26}) \tag{6.66}$$

$$i(T_5 + T_6) = \frac{1}{2}(\mathbf{e}_{36} - \mathbf{e}_{45}) \tag{6.67}$$

$$(T_5 - T_6) = \frac{1}{2}(\mathbf{e}_{35} + \mathbf{e}_{46}) \tag{6.68}$$

$$iT_7 = \frac{1}{4}(\mathbf{e}_{12} - \mathbf{e}_{34}) \tag{6.69}$$

$$iT_8 = \frac{1}{4}(\mathbf{e}_{12} + \mathbf{e}_{34} - 2\mathbf{e}_{56}) \tag{6.70}$$

As was done in the bi-fermion algebra, it is convenient to relabel the unit vectors as

$$\mathbf{f}_1 = \mathbf{e}_3 \tag{6.71}$$

$$\mathbf{f}_2 = \mathbf{e}_4 \tag{6.72}$$

$$\mathbf{g}_1 = \mathbf{e}_5 \tag{6.73}$$

$$\mathbf{g}_2 = \mathbf{e}_6. \tag{6.74}$$

The anti-Hermitian generators are written as

$$
\begin{aligned}
i(T_1 + T_2) &= \frac{1}{2}(\mathbf{e}_1\mathbf{f}_2 - \mathbf{e}_2\mathbf{f}_1) && (6.75)\\
(T_1 - T_2) &= \frac{1}{2}(\mathbf{e}_1\mathbf{f}_1 + \mathbf{e}_2\mathbf{f}_2) && (6.76)\\
i(T_3 + T_4) &= \frac{1}{2}(\mathbf{e}_1\mathbf{g}_2 - \mathbf{e}_2\mathbf{g}_1) && (6.77)\\
(T_3 - T_4) &= \frac{1}{2}(\mathbf{e}_1\mathbf{g}_1 + \mathbf{e}_2\mathbf{g}_2) && (6.78)\\
i(T_5 + T_6) &= \frac{1}{2}(\mathbf{f}_1\mathbf{g}_2 - \mathbf{f}_2\mathbf{g}_1) && (6.79)\\
(T_5 - T_6) &= \frac{1}{2}(\mathbf{f}_1\mathbf{g}_1 + \mathbf{f}_2\mathbf{g}_2) && (6.80)\\
iT_7 &= \frac{1}{4}(\mathbf{f}_{12} - \mathbf{e}_{12}) && (6.81)\\
iT_8 &= -\frac{1}{4}(\mathbf{e}_{12} + \mathbf{f}_{12} - 2\mathbf{g}_{12}). && (6.82)
\end{aligned}
$$

In general it is always possible to express higher-dimensional Clifford algebras by using a certain number of fermions. Given a number of m Fermions it is always possible to construct a $C\ell_{2m}$ Clifford algebra. Another possibility appears if the number of fermions is odd, $m = 2k + 1$ ($k \geq 0$). In this case it is possible to construct a $C\ell_{4k+3}$ algebra by representing the volume element with the complex number i. For example, the $C\ell_8$ algebra could be obtained by using the 4-Fermion algebra and the $C\ell_{11}$ algebra could be obtained by using the 5-Fermion algebra.

The equivalence between the Clifford and fermion algebras has the potential to identity new algebraic properties that may be hidden when expressed in only one of them. For example, we were able to construct a Lie algebra isomorphic to $sp(4)$ by using the bilinear plus trilinear elements of the bi-fermion algebra. There are general methods to extract Lie algebras within Clifford algebras as done by Doran *et al.* [5] (using the non-Euclidean Clifford algebras $C\ell_{n,n}$ as well). The association between the fermion and

Clifford algebras can be used ultimately to identify more Lie groups of multilinear fermion creation-annihilation operators.

Chapter 7

Conclusions and Further Work

This thesis has shown how appropriate Clifford algebras can be applied in order to represent n-qubit systems.

The Lie algebras of interest, $su(4)$, $su(8)$, $sp(4)$, $sp(8)$ appeared naturally in the context of some Clifford algebras and they were used in chapters 3 and 4 to control and access the Hilbert space of two and three-qubit systems.

The fifth chapter treated the problem of calculating the average fidelity of n-qubits in terms of a finite sum, which is easier to evaluate than the original integral form. This implies that only a finite number of measurements is necessary in order to establish the fidelity of a quantum operation.

The sixth chapter showed the deep connection between fermion and Clifford algebras and how they are equivalent to each other. This reveals hidden algebraic properties of the fermion algebra that may have potential applications in particle models or second-quantized schemes.

A natural extension of this work could be done in non-coherent control and decoherence in general. The measurement of the entanglement of mixed systems is still an open problem and more work could be done, perhaps by doubling the system and finding its equivalent pure state. This complete

thesis could be generalized by introducing the spatial dependence of the wave function and relativistic effects.

Appendix A

Clifford Conjugation

Clifford conjugation [12], also known as spatial reversal conjugation or informally as *bar conjugation*, is an anti-automorphism that changes the sign of vectors. This means that if $\mathbf{v}$ is a vector, its Clifford conjugation is

$$\overline{\mathbf{v}} = -\mathbf{v} \tag{A.1}$$

Moreover, as an anti-automorphism, the Clifford conjugation is distributed on a product as

$$\overline{AB} = \overline{B}\,\overline{A} \tag{A.2}$$

The application of the spatial reversal on the elements of the $C\ell_3$ basis is given in table (A)

Basis element	Clifford conjugation
1	1
$\mathbf{e}_1$	$-\mathbf{e}_1$
$\mathbf{e}_2$	$-\mathbf{e}_2$
$\mathbf{e}_3$	$-\mathbf{e}_3$
$\mathbf{e}_{12}$	$-\mathbf{e}_{12}$
$\mathbf{e}_{23}$	$-\mathbf{e}_{23}$
$\mathbf{e}_{13}$	$-\mathbf{e}_{13}$
$\mathbf{e}_{123}$	$\mathbf{e}_{123}$

Table A.1. Clifford conjugation of the elements of the the $C\ell_3$ basis

Appendix B

Classical Semisimple Lie Algebras

The classical Lie algebras are the fundamental semisimple Lie algebras [36, 38]. The list of the classical Lie algebras with their Cartan matrices follows below, with their Dynkin diagrams in figure (B.1)

- $A_n = su(n-1)$ generators of the $SU(n-1)$ group. (Upper case is used for groups and lower case for algebras).

$$C_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & . & & . \\ 0 & -1 & . & & & . \\ & & & & -1 & 0 \\ & & & -1 & 2 & -1 \\ . & . & . & 0 & -1 & 2 \end{pmatrix}. \tag{B.1}$$

- $C_n = sp(2n)$ generators of the $Sp(2n)$ group.

$$C_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & . & & . \\ 0 & -1 & . & & & . \\ & & & & -1 & 0 \\ & & & -1 & 2 & -1 \\ . & . & . & 0 & -2 & 2 \end{pmatrix}. \tag{B.2}$$

- $D_n = so(2n)$ (isomorphic to $spin(2n)$) generators of the $SO(2n)$ group.

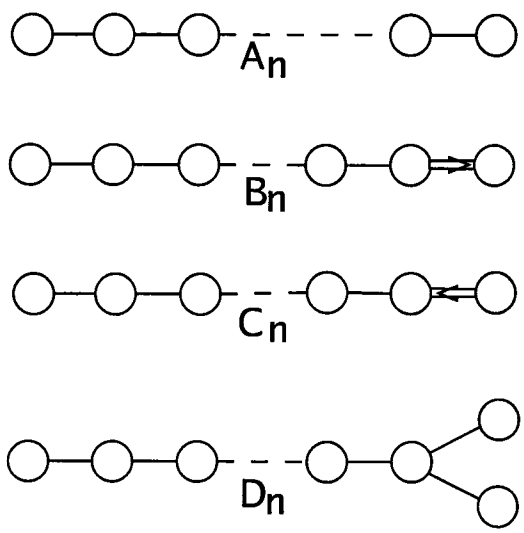

Figure B.1. The Dynkin diagrams of the classical Lie algebras.

$$C_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & & & & . \\ -1 & 2 & -1 & . & & & . \\ 0 & -1 & . & & & & . \\ & & & & -1 & 0 & 0 \\ & & & -1 & 2 & -1 & -1 \\ & & & 0 & -1 & 2 & 0 \\ & & & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{B.3}$$

- $B_n = so(2n+1)$ (isomorphic to $spin(2n+1)$) generators of the $SO(2n+1)$ group.

$$C_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & . & & . \\ 0 & -1 & . & & & . \\ & & & & -1 & 0 \\ & & & -1 & 2 & -2 \\ . & . & . & 0 & -1 & 2 \end{pmatrix}. \tag{B.4}$$

Appendix C

Anti-Hermitian $\mathcal{C}\ell_4$ Cartan Matrix

The purpose of this appendix is to characterize the Lie algebra of the anti-Hermitian space of the $\mathcal{C}\ell_4$ Clifford algebra by calculating the roots and the Cartan matrix [38]. It is convenient to relabel the generators as shown in Table C.1 that correspond to the matrices given in Table 3.4. This basis is

$F_1 = \mathbf{e}_{12}$	$F_6 = \mathbf{e}_{34}$
$F_2 = \mathbf{e}_{13}$	$F_7 = \mathbf{e}_{123}$
$F_3 = \mathbf{e}_{14}$	$F_8 = \mathbf{e}_{124}$
$F_4 = \mathbf{e}_{23}$	$F_9 = \mathbf{e}_{134}$
$F_5 = \mathbf{e}_{24}$	$F_{10} = \mathbf{e}_{234}$

Table C.1. Relabeling of the $\mathcal{C}\ell_4$ Lie algebra

orthogonal according to the trace when the matrix representation is used

$$Tr(F_j F_k) = -4\delta_{jk}. \tag{C.1}$$

The basis of the Cartan subalgebra can be taken as

$$S = \{F_1, F_7\}, \tag{C.2}$$

where each generator of the Cartan subalgebra is usually called a weight generator whereas the remaining eight generators are called root generators.

Their matrix adjoint representations are

$$Ad_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{C.3}$$

$$Ad_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.4}$$

The matrix that diagonalizes both matrices is

$$U = U_1 U_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & i \\ 0 & -i & i & 0 & -i & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ i & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & -i & 0 & i & i & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.5}$$

where U_1 diagonalizes Ad_1 only and U_7 diagonalizes $U_1^{-1} Ad_7 U_1$. The generators of the Cartan subalgebra are diagonalized as

$$U^{-1} Ad_1 U = diag(\ 0,\ \ 2, -2,\ 0,\ \ 2, -2,\ 0,\ 0,\ \ 2, -2) \tag{C.6}$$

$$U^{-1} Ad_7 U = diag(-2, -2, -2,\ 2,\ 2,\ \ 2,\ 0,\ 0,\ 0,\ \ 0). \tag{C.7}$$

The positive roots are

$$\begin{aligned} \boldsymbol{\alpha}_1 &= \{2,2\} && \text{(C.8)} \\ \boldsymbol{\alpha}_2 &= \{2,0\} && \text{(C.9)} \\ \boldsymbol{\alpha}_3 &= \{2,-2\} && \text{(C.10)} \\ \boldsymbol{\alpha}_4 &= \{0,2\}, && \text{(C.11)} \end{aligned}$$

where the simple roots are α_3 and α_4.

The next step is to calculate the canonical representation of the root generators $\{G_j\}$. This is done by solving for four eigen-equations, where the eigenvalues are given by the components of the four positive root vectors

$$\{\,[iF_1, G]\,,\,[iF_7, G]\,\} = \boldsymbol{\alpha}_k G, \quad k = 1,2,3,4, \tag{C.12}$$

and G is given by a linear superposition over the root generators

$$G = a^j F_j \quad j = 2,3,4,5,6,8,9,10. \tag{C.13}$$

The the solutions of G, associated with the positive roots are

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \end{pmatrix} \tag{C.14}$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \tag{C.15}$$

$$G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{C.16}$$

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}. \tag{C.17}$$

The solutions associated with the negative roots are calculated by applying Hermitian conjugation to the solutions associated with the respective

positive roots

$$G_5 = G_1^\dagger \tag{C.18}$$
$$G_6 = G_2^\dagger \tag{C.19}$$
$$G_7 = G_3^\dagger \tag{C.20}$$
$$G_8 = G_4^\dagger. \tag{C.21}$$

It is convenient to redefine the weight generators as

$$H_1 = iF_1 \tag{C.22}$$
$$H_2 = iF_7. \tag{C.23}$$

The dual roots h^j are defined as members of the Cartan subalgebra

$$h^j = b^{jk} H_k \tag{C.24}$$

such that the following expression is satisfied

$$\{\, Tr(h^j H_1)\,,\, Tr(h^j H_2)\,\} = \alpha_j. \tag{C.25}$$

The dual roots associate with the positive roots are explicitly shown below

$$h^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1-i & 0 \\ 0 & 0 & 0 & 1+i \\ -1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \end{pmatrix} \tag{C.26}$$

$$h^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{C.27}$$

$$h^3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1+i & 0 \\ 0 & 0 & 0 & 1-i \\ -1-i & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \end{pmatrix} \tag{C.28}$$

$$h^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{C.29}$$

The simple roots are associated with the generators $\{G_3, G_4\}$. These generators have lengths defined in terms of their traces as

$$Tr(h^3 h^3) = 2 \tag{C.30}$$
$$Tr(h^4 h^4) = 1. \tag{C.31}$$

Finally, the Cartan mat$(C^{jk}) = \left(\frac{2Tr(h^j h^k)}{Tr(h^j h^j)}\right) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$

$$(C^{jk}) = \left(\frac{2Tr(h^j h^k)}{Tr(h^j h^j)}\right) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \tag{C.32}$$

which corresponds either to the $sp(4)$ or $spin(5)$ Lie algebra.

Appendix D

Paravectors

A *paravector* is a Clifford number that extends the space of real vectors to include real scalars as well [12]. For example, the following expression is a general paravector in a given $C\ell_3$ basis

$$p = p^0 + p^1\mathbf{e}_1 + p^2\mathbf{e}_2 + p^3\mathbf{e}_3, \tag{D.1}$$

where $p^j \in \mathbb{R}$. The square length of a paravectors is defined in terms of the product by its Clifford conjugation

$$p\overline{p} \tag{D.2}$$

The projector P_3, defined as

$$P_3 = \frac{1}{2}(1 + \mathbf{e}_3), \tag{D.3}$$

is an example of null paravector, because its square length is zero

$$P_3\overline{P}_3 = 0. \tag{D.4}$$

In $C\ell_3$, the following elements form a null basis of paravectors

$$\{P_3, \mathbf{e}_1P_3, \bar{P}_3, \mathbf{e}_1\overline{P}_3\} \tag{D.5}$$

so that any Clifford number Ψ in $C\ell_3$ can be expressed as

$$\Psi = \psi_1\overline{P}_3 + \psi_2\mathbf{e}_1\overline{P}_3 + \psi_3P_3 + \psi_4\mathbf{e}_1\overline{P}, \tag{D.6}$$

where ψ_j belongs to the combined real-scalar and pseudo-scalar space, which is isomorphic to the standard complex algebra.

Appendix E

Pacwoman Property

The application of a projector on a given Clifford algebra $C\ell_n$ produces an ideal with a number of independent elements. For example, the following *pacwoman* identity can be established by direct calculation [12]

$$\mathbf{e}_3 P_3 = P_3 = P_3 \mathbf{e}_3 \tag{E.1}$$

In general, the eight different basis elements of $C\ell_3$ are reduced to four, namely

$$\{P_3, \mathbf{e}_1 P_3, \mathbf{e}_2 P_3, \mathbf{e}_{123} P_3\}, \tag{E.2}$$

as seen explicitly in the following identities:

$$\begin{aligned}
1P_3 &= P_3 && \text{(E.3)}\\
\mathbf{e}_1 P_3 &= \mathbf{e}_1 P_3 && \text{(E.4)}\\
\mathbf{e}_2 P_3 &= \mathbf{e}_2 P_3 && \text{(E.5)}\\
\mathbf{e}_3 P_3 &= P_3 && \text{(E.6)}\\
\mathbf{e}_{12} P_3 &= \mathbf{e}_{123} P_3 && \text{(E.7)}\\
\mathbf{e}_{23} P_3 &= \mathbf{e}_2 P_3 && \text{(E.8)}\\
\mathbf{e}_{13} P_3 &= \mathbf{e}_1 P_3 && \text{(E.9)}\\
\mathbf{e}_{123} P_3 &= \mathbf{e}_{123} P_3. && \text{(E.10)}
\end{aligned}$$

The pacwoman property is also applied in higher dimensions, where there are multiple commuting projectors. The primitive projector is defined as the projector made from the product of all the possible commuting independent projectors for a given dimension. In $C\ell_4$, there is only one independent projector, so the primitive projector can be defined as

$$P = \frac{1}{4}(1 + \mathbf{e}_3). \tag{E.11}$$

The complete $C\ell_4$ basis is reduced to eight different elements as shown in Table (E.1)

$\mathbf{e}_{12}P = \mathbf{e}_{12}P$	$\mathbf{e}_{123}P = \mathbf{e}_{12}P$
$\mathbf{e}_{13}P = \mathbf{e}_{1}P$	$\mathbf{e}_{1}P = \mathbf{e}_{1}P$
$\mathbf{e}_{14}P = \mathbf{e}_{14}P$	$\mathbf{e}_{134}P = -\mathbf{e}_{14}P$
$\mathbf{e}_{23}P = \mathbf{e}_{2}P$	$\mathbf{e}_{2}P = \mathbf{e}_{2}P$
$\mathbf{e}_{24}P = \mathbf{e}_{24}P$	$\mathbf{e}_{234}P = -\mathbf{e}_{24}P$
$\mathbf{e}_{34}P = -\mathbf{e}_{4}P$	$\mathbf{e}_{4}P = \mathbf{e}_{4}P$
$\mathbf{e}_{124}P = \mathbf{e}_{124}P$	$\mathbf{e}_{1234}P = -\mathbf{e}_{124}P$
$\mathbf{e}_{3}P = P$	1 P = P

Table E.1. The $C\ell_4$ algebra projected into the eight independent elements.

In $C\ell_4 \times \mathbb{C}$, there are two independent commuting projectors and a primitive projector can be defined as

$$P = \frac{1}{4}(1 - i\mathbf{e}_{12})(1 - i\mathbf{e}_{123}). \tag{E.12}$$

In this case, the complete $C\ell_4 \times \mathbb{C}$ algebra with 32 basis multivectors is reduced to eight.

As the product among vectors is well defined, it is possible to define functions of vectors with the help of their series expansions, whenever that is possible. Let $f(\mathbf{n})$ to be an analytic function at 0 and $\mathbf{n}$ a unit vector in a space of any dimension. Expanding this function in a Taylor series we have the following expression

$$f(\mathbf{n}) = f(0) + \mathbf{n}f^{(1)}(0) + \mathbf{n}^2\frac{1}{2!}f^{(2)}(0) + \mathbf{n}^3\frac{1}{3!}f^{(3)}(0) + ... \tag{E.13}$$

Evaluating the powers of the unit vector $\mathbf{n}$, the series becomes

$$f(\mathbf{n}) = f(0) + \mathbf{n}f^{(1)}(0) + \frac{1}{2!}f^{(2)}(0) + \mathbf{n}\frac{1}{3!}f^{(3)}(0) + ... \tag{E.14}$$

By using the identity resolution we have

$$f(\mathbf{n}) = f(0)(P_\mathbf{n} + \bar{P}_\mathbf{n}) + (P_\mathbf{n} - \bar{P}_\mathbf{n})f^{(1)}(0) + \frac{1}{2!}f^{(2)}(0)(P_\mathbf{n} + \bar{P}_\mathbf{n}) + ... \tag{E.15}$$

and collecting terms we get

$$f(\mathbf{n}) = \left(f(0) + f^{(1)}(0) + ...\right)P_\mathbf{n} + \left(f(0) - f^{(1)}(0) + \frac{1}{2!}f^{(2)}(0) - ...\right)\bar{P}_\mathbf{n}. \tag{E.16}$$

Finally, identifying the Taylor series, the following compact result is found

$$f(\mathbf{n}) = f(1)P_{\mathbf{n}} + f(-1)\bar{P}_{\mathbf{n}} \tag{E.17}$$

By using the complementarity of the projectors

$$P_{\mathbf{n}}\overline{P}_{\mathbf{n}} = 0, \tag{E.18}$$

the following identities are also found

$$f(\mathbf{n})P_{\mathbf{n}} = f(1)P_{\mathbf{n}} \tag{E.19}$$

$$f(\mathbf{n})\bar{P}_{\mathbf{n}} = f(-1)\bar{P}_{\mathbf{n}} \tag{E.20}$$

Appendix F

Scalar Part

There are certain Clifford algebras where the volume element squares to -1 and commutes with every other Clifford number in the given Clifford algebra. These Clifford algebras are $\mathcal{C}\ell_3$, $\mathcal{C}\ell_7$, $\mathcal{C}\ell_{11}$ and so on as long as the dimension n obeys $(n+1) \mod 4 = 0$. In these algebras it is possible to combine the real scalars and the pseudoscalars (scalars times the volume element) to obtain an algebra isomorphic to the standard complex algebra. In this thesis the term scalar is applied to the scalar plus pseudo-scalar space.

In $\mathcal{C}\ell_3$ the scalar part [12] can be extracted from a general Clifford number simply by applying the Clifford conjugation as

$$\langle x \rangle_S = \frac{1}{2}(x + \bar{x}), \tag{F.1}$$

which is evidently invariant under the reversion conjugation. In $\mathcal{C}\ell_3$, the complementary space is called the vector space, which contains both vectors and bivectors. As complementary space to the scalar space, the vector part can be extracted as

$$\langle x \rangle_V = \frac{1}{2}(x - \bar{x}) \tag{F.2}$$

The scalar part is invariant under cyclic permutations such as

$$\langle abc \rangle_S = \langle bca \rangle_S, \tag{F.3}$$

which implies an invariance under transformations of the form

$$\langle RxR^{-1} \rangle_S = \langle x \rangle_S \tag{F.4}$$

Appendix G

Partial Scalar Part

The scalar part is obtained as an operation that can be expressed in term of the trace when the matrix representation is used. In the case of $C\ell_3$ and a representation in terms of $2 \otimes 2$ matrices, the scalar part is equivalent to

$$\langle A \rangle_S = \frac{1}{2} Tr[A]. \tag{G.1}$$

A Clifford number and its matrix representation are usually denoted with the same variable, and to avoid cumbersome notation they are identified by the context.

The scalar part has a special significance in Clifford algebras where the volume element is anti-Hermitian and commutes with all the algebra. This happens when the grade of the volume element is 3,7,11,... or 1 mod 4. In these cases the scalar part is a complex number.

For an expression made with the direct product of two Clifford numbers a and b, the partial trace on the left slot is defined as

$$\langle a \otimes b \rangle_{S \otimes 1} = \langle a \rangle_S b \tag{G.2}$$

On the other hand, the partial trace on the right slot is defined as

$$\langle a \otimes b \rangle_{1 \otimes S} = a \langle b \rangle_S \tag{G.3}$$

The partial scalar part is equivalent to the partial trace so that if the the slots are filled with $C\ell_3$ numbers, the reduced trace is respectively

$$\langle a \otimes b \rangle_{S \otimes 1} = [\frac{1}{2} Tr \otimes 1](a \otimes b) \tag{G.4}$$

$$\langle a \otimes b \rangle_{1 \otimes S} = [1 \otimes \frac{1}{2} Tr](a \otimes b) \tag{G.5}$$

The partial scalar part is easily generalized to multiple direct products.

Appendix H

Single-Qubit Entropy

The density matrix of a single qubit in algebraic form is

$$\rho = \frac{1}{2}(1 + \alpha \hat{\mathbf{p}}), \tag{H.1}$$

with $\alpha = 1$ for pure states and $0 \leq \alpha < 1$ for mixed states, where the 0 value corresponds to the purely mixed state.

The Von Neumann entropy for a single qubit in terms of the $C\ell_3$ algebra is

$$S = -2\langle \rho \ln(\rho) \rangle_S \tag{H.2}$$

The first step to calculate the entropy is to take the unit vector $\hat{\mathbf{p}}$ along an arbitrary direction, say $\mathbf{e}_3$, and then use the resolution of the identity to get

$$\rho = \frac{1}{2}(1 + \alpha \mathbf{e}_3)(P_3 + \overline{P}_3), \tag{H.3}$$

and by using the pacwoman property we obtain

$$\rho = \frac{1}{2}(1 + \alpha)P_3 + \frac{1}{2}(1 - \alpha)\overline{P}_3. \tag{H.4}$$

Assuming that $\alpha < 1$, the logarithm is easily calculated taking into account that projectors are idempotent

$$\ln(\rho) = \ln \frac{1+\alpha}{2} P_3 + \ln \frac{1-\alpha}{2} \overline{P}_3 \tag{H.5}$$

By using this expression, the Von Neumann entropy is finally calculated as

$$\begin{aligned} S(\alpha) &= -\left\langle \frac{1+\alpha}{2} \ln \frac{1+\alpha}{2} P_3 + \frac{1-\alpha}{2} \ln \frac{1-\alpha}{2} \overline{P}_3 \right\rangle_S \\ &= -\left(\frac{1+\alpha}{2} \ln(\frac{1+\alpha}{2}) + \frac{1-\alpha}{2} \ln(\frac{1-\alpha}{2}) \right). \end{aligned}$$

Appendix I

Computer Programs

The following are some useful *Mathematica* programs

Entry Elements of $Sp(N)$:

```
SymmetricInitialLieBasis::usage =
    "SymmetricInitialLieBasis[n] gives n elements of the
sp(n) Lie. The remaining elements could be calculated with
all the possible commutators";

SymmetricInitialLieBasis[n_] := Join[
      Table[
        Table[

     KroneckerDelta[i, k]KroneckerDelta[j, k + 1] -
     KroneckerDelta[i - 1, k]KroneckerDelta[j, k]+
     KroneckerDelta[i, n - k]KroneckerDelta[j, n - k + 1]-
     KroneckerDelta[i - 1, n - k]KroneckerDelta[j, n - k]
          , {i, 1, n}, {j, 1, n}]
        , {k, 1, n/2}]
      ,
      Table[
        Table[

        I*KroneckerDelta[i, k]KroneckerDelta[j, k + 1] +
        I*KroneckerDelta[i - 1, k]KroneckerDelta[j, k] +
     I*KroneckerDelta[i, n - k]KroneckerDelta[j, n - k + 1]+
     I*KroneckerDelta[i - 1, n - k]KroneckerDelta[j, n - k]
          , {i, 1, n}, {j, 1, n}]
```

```
  , {k, 1, n/2}]
] /. {2 -> 1, -2 -> -1, 2*I -> I}
```

Calculating the Complete Lie Algebra:
This routine requires the following standard package

```
<<<<DiscreteMath'Combinatorica'

LinearlyIndependentMatrixQ::usage =
    "LinearlyDependentQ[u_List] establishes the status of
linear dependence of matrices u. Every number involved must
be in exact representation (No float decimal point).";

LinearlyIndependentMatrixQ[u_List] := Module[{v},
      v = Flatten /@ u;
      If[MatrixRank[v] == Length[v], True, False]
      ];

AppendNewIndependentGenerator[u_List, w_] := Module[{newu},
      newu = Append[u, w];
      If[LinearlyIndependentMatrixQ[newu], newu, u]
      ];

NewIndependentCommutator::usage =
    "NewIndependentCommutator[w_List], From the list of
matrices w, this function calculates the commutators made
from all the possible pairs and appends the Independent
generators found in this way. Many recursive applications
may be needed to find the complete set of all the possible
commutators";

NewIndependentGenerators[w_List] :=
  Module[{pairs, NewComm},
    pairs = KSubsets[Range[Length@w], 2];
    NewComm =
        pairs /. {i_, j_} :> MCommutator[w[[i]], w[[j]]];
    Fold[AppendNewIndependentGenerator, w, NewComm]]
```

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