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On a certain boundary value problem and its relevance in the theory of surface induction hardening.

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ON A CERTAIN BOUNDARY VALUE PROBLEM AND ITS RELEVANCE IN THE THEORY OF SURFACE INDUCTION HARDENING

by

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A Dissertation
Submitted to the Faculty of Graduate Studies through the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of
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Respectfully Dedicated to

My FATHER and MOTHER

ABSTRACT

Surface hardening accomplished by localized heating and controlled cooling permits resistance to local surface wear to be combined with good torsional and bending strength characteristics using cheaper alloys in mechanical components. Induction heating is a convenient means of applying this local heating. Although fast, convenient, and having excellent energy conversion characteristics, it is not simple to set up, nor rapid nor reliable due to the lack of easily useable mathematical solutions to the problem.

This study treats the case of induction hardening of a cylindrical metal rod of sufficient length that end effects may be ignored. The boundary value problem is given and a simplified representation developed. The physical parameters, such as conductivity and permeability, are assumed to be smooth functions of the radius of the rod, contrary to earlier studies which assume these to be constants in order to simplify the mathematics.

A representation theorem for the solution of the problem is proved using a number of lemmas which are stated and proven first. A number of methods for approximating the solution for numerical calculations are also described. Suggestions are made for some possible further studies of the problem.

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INTRODUCTION

(a) Historical Sketch

The surface hardening of metals and alloys, which are the most widely utilized basic materials for constructing all kinds of machines and mechanisms, plays an important role in modern industrial technology and engineering. Localized heating of individual regions of components is necessary in order to carry out the widely employed industrial process of surface hardening. The possibility of localized heating of surface regions of components of various mechanisms and also the duration of the process depends a great deal upon the heating method selected. Two heating methods are in common use: the application of external heat; and the generation of heat within the work piece.

The application of external heat is the traditional method used and consists of placing the work piece in a hot environment such as a furnace or crucible. This has the disadvantages characterized by a low rate of temperature increase in the work piece (usually of the order of tens or hundreds of degrees per minute), and by the low thermal efficiency of the furnace, both characteristics being unsuitable for surface hardening.

When applying external heat, not more than one-third of the thermal energy is utilized directly in heating the workpiece, the remaining heat being expended in heating the furnace itself and in other losses. The result of this is a long heating process and high speed specific expenditure of thermal energy in heating each kilogram of the material being heated. In addition to this, the localized heating of the surface regions whioh is required for some articles is practically impossible using a furnace or equivalent means [10].

The method of generating heat within the work piece usually uses the principle of induction. It is based on the fact that an eddy current is induced at the surface of an electric conductor which is subjected to a rapidly changing magnetic field. This current, flowing as it does through only the very thin surface layer of the conductor, will produce great heat because the Ohm's losses are concentrated there. The magnetic field used in surface hardening a cylindrical steel rod is produced by passing a high frequency electric current through a circular coil moving with constant speed with respect to the cylinder.

The chief advantages of this form of heating are the speed of the process, the ability to control the degree of hardening by selecting the proper speed of the moving coil, and the ability to harden only a limited portion of the metal object. Also, objects case hardened by induction combine greatly improved torsional and bending fatigue strength with good wear resistance while using cheaper alloys. The wide range of present applications to other fields where speed and convenience of heating are necessary or desirable, or where local heating is a decided asset, shows the potential for the process if the set up of the process for a given application can be made simple, rapid, and reliable.

The next advance in the application of this most useful technique of induction heating is the theoretical study of the process and development of a mathematical treatment of the process. Until the outcome of K. A. Zischka's work [14], not much work had been done to contribute to the mathematical aspect of induction heating. However, in recent years, there has been an increased interest in the construction of mathematical models for iron and steel work processing [13]. This has arisen because the possibility of performing by computer the arduous arithmetic involved in translating a model into numerical results for use in specific cases has now won universal acceptance. The development of a mathematical model depends on a knowledge of the behaviour of the system being studied, the capability of expressing these mathematically, and on the ability to solve the resulting equations. Unfortunately, the exact mathematical description in closed form of the heating of a work piece is possible only for a few simple shapes of conductors (work pieces) and configurations of the coil. Even with this limitation, the mathematical theory of the induction hardening process is far from complete, mainly because of the great difficulty encountered in solving the associated boundary value problem which usually consists of a system of partial differential equations with coupled boundary conditions. As a result, the physical parameters involved, such as electrical conductivity and permeability, are frequently assumed to be constant even though this is not the true case for induction hardening [4-6], [14-15]. In fact, these physical parameters of permeability and conductivity depend upon the spatial coordinates since they are functions of temperature, and the temperature depends upon the spatial coordinates because the induction heating process impresses a steep temperature gradient upon the work piece. It is this aspect of the problem with which we concern ourselves in this thesis.

(b) Scope of the Present Work

The present work is intended to treat a boundary value problem arising from the surface induction hardening of a cylindrical metal rod. We shall assume that the physical parameters, such as the electric conductivity and permeability of this rod, are smooth functions of the radius of the rod.

In section 1 of chapter 2, we give a description of the physical model together with its geometry. In section 2 of chapter 2, we give a brief mathematical formulation of the boundary value problem for this model. Some approximations have been made in order that the boundary value problem shall become solvable.

Chapter 3 deals with the solution of this boundary value problem. A representation theorem is proved. In section 1 of chapter 3, the representation theorem is stated with its pertinent assumptions. In section 2 of chapter 3, some fundamental lemmas which are essential in proving the theorem are stated and proved. In section 3 of chapter 3, we give a proof of the theorem in a form which has been simplified with the help of the lemmas proved in the previous section.

Chapter 4 is chiefly devoted to approximate methods. In section 1 of chapter 4, we use the layer method to approximate the whole system. In section 2 of chapter 4, we use a method of integral equations to approximate the solution of the second order differential equation which arises in the solution of the boundary value problem. In section 3 of chapter 4, we use a perturbation method to approximate the differential equation for small values of x. In section 4 of chapter 4, we give bounds for replacing the improper integral by an integral taken over a finite interval.

In the fifth and final chapter, we give a brief conclusion together with some suggestions for further investigations in the subject.

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Chapter 2

THE PHYSICAL MODEL AND ITS MATHEMATICAL FORMULATION

This chapter deals with the physical model of the surface induction hardening of a metallic cylindrical rod and the mathematical formulation of the boundary value problem associated with this model. Certain approximations and simplifying assumptions have to be made in order that the problem so described becomes solvable.

1. A Description and the Geometry of the Physical Model

In this thesis we consider the following physical problem. A long cylindrical metallic rod is surrounded by a concentric cylindrical coil of length 2L which moves with constant velocity $\vec{v}$ parallel to the axis of the cylinder. The length of the rod is assumed to be much greater than that of the coil so that we can neglect the end effects and treat the rod as having infinite length. Since the velocity $\vec{v}$ of the coil is negligible compared to that of light, we ignore the relativistic effect and use only the Newtonian transformation $u = z - vt$ in order to consider the coordinate system of the coil as a moving frame. The geometry and coordinate system being used are illustrated in Fig. 1.

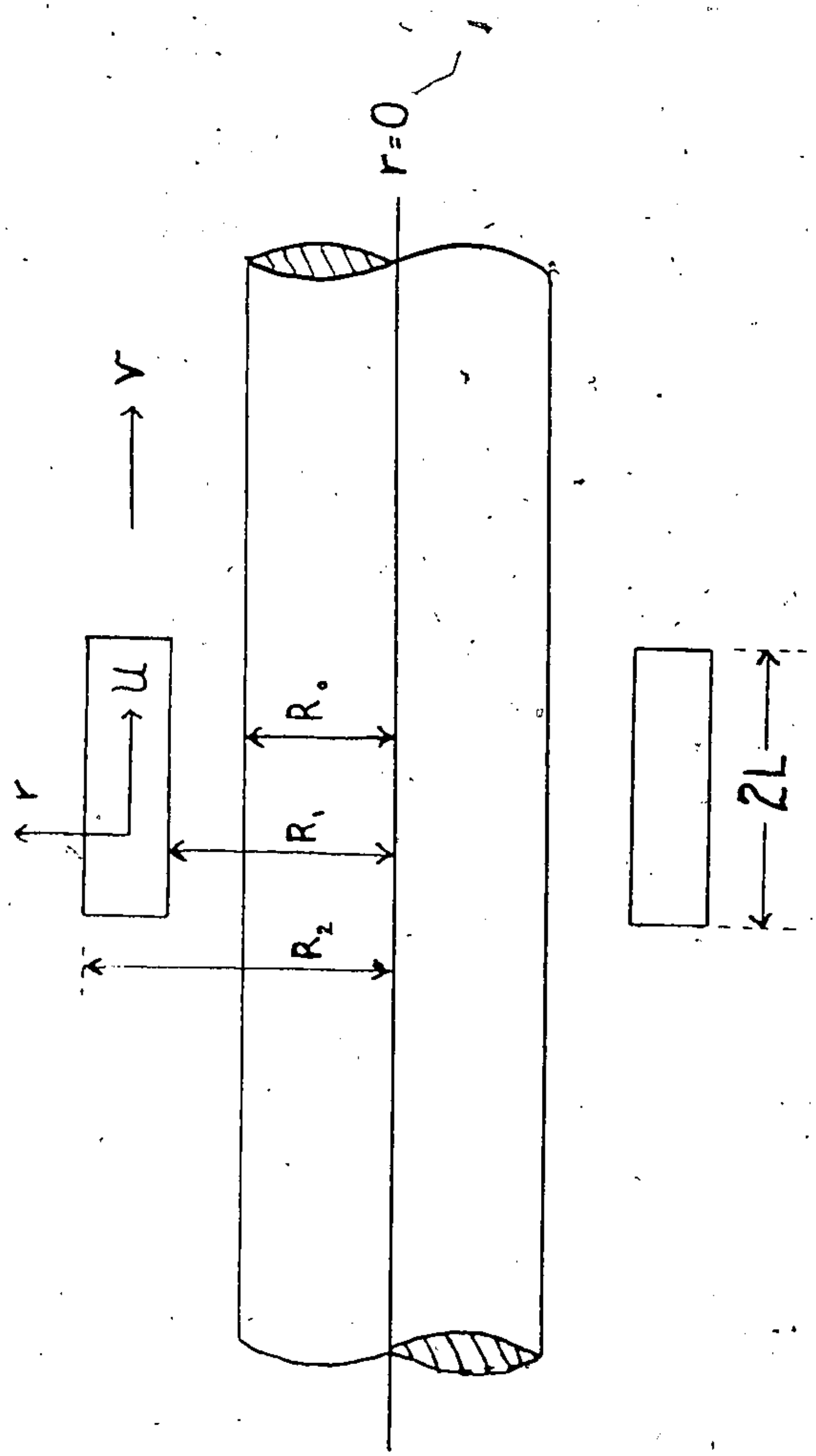

Fig 1

Fig. 1

We wish to investigate the effect on the rod when a high frequency (of order 10^6 Hertz) alternating current is passed through the coil. The current induced in the work piece by the motion of the coil is neglected as being insignificant compared to that induced by the alternating magnetic field at the frequencies being used.

2. The Mathematical Formulation of the Boundary Value Problem

The basic equations governing this problem are Maxwell's equations. In rationalized MKS units, the macroscopic Maxwell equations may be written as:

$$(2.1) \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$(2.2) \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$(2.3) \qquad \text{Div } \vec{D} = \rho \text{ , \quad and}$$

$$(2.4) \qquad \text{Div } \vec{B} = 0.$$

Here $\vec{H}$ is the magnetic intensity, $\vec{J}$ is the current density, $\vec{E}$ is the electric field, $\vec{D}$ is the displacement vector, $\vec{B}$ is the magnetic induction field, and ρ is the charge density. For the range of frequencies we are considering, the current becomes the quasistationary case [4], [14], [15], and the displacement current $\partial\vec{D}/\partial t$ is negligible compared to the true current density $\vec{J}$, and we may replace (2.1) by:

(2.5) $\quad \text{Curl } \vec{H} = \vec{J}.$

Also, from Ohm's Law we have

(2.6) $\quad \vec{J} = \sigma \vec{E},$

for nonferromagnetic material we have

(2.7) $\quad \vec{B} = \mu \vec{H}$

while for isotropic material we have

(2.8) $\quad \vec{D} = \varepsilon \vec{E};$

where σ is the conductivity, μ the permeability, and ε is the permittivity. We note that σ, μ may be different in different regions. We shall assume the following: Within the rod μ and σ may depend upon the radial coordinate r because of the high temperature gradient in the radial direction of the rod. Within the coil μ and σ are constant, $\mu = \mu_c$, $\sigma = \sigma_c$. In the air $\sigma = 0$, and $\mu = \mu_0$.

If we assume the electric field $\vec{E}$ and the magnetic intensity $\vec{H}$ to be separable in the time variable, that is

(2.9)(a) $\quad \vec{E} = \hat{E}\, e^{i\omega t}$, and

(2.9)(b) $\quad \vec{H} = \hat{H}\, e^{i\omega t}$,

where $\hat{E}$ and $\hat{H}$ are the amplitudes of the electric field $\vec{E}$ and the magnetic intensity $\vec{H}$ respectively, and ω is the frequency of the alternating current flowing in the coil, we then have:

(2.10) $\nabla \times \hat{H} = \sigma \hat{E}$, and

(2.11) $\nabla \times \hat{E} = -i\omega\hat{B}$.

Since $\hat{B}$ has zero divergence, it may always be represented as the curl of a vector potential, that is

(2.12) $\hat{B} = \nabla \times \hat{A}$.

From the formula we then have curl $(\hat{E} + i\omega\hat{A}) = 0$. It follows that

(2.13) $\hat{E} = -i\omega\hat{A}$

with $\hat{A}$ defined through (2.12), is a particular solution of (2.11). We have within the rod

(2.14)(a) $$\nabla \times (\nabla \times \hat{A}) = k^2\hat{A} + \frac{\text{grad}\,\mu}{\mu} \times (\nabla \times \hat{A})$$

while in the air

(2.14)(b) $\nabla \times (\nabla \times \hat{A}) = 0$,

where

(2.15) $k^2 = -i\omega\mu\sigma$.

In setting up the coil we require that the current density $\hat{J}$ in the coil satisfies the following conditions:

(2.16)(a) $\nabla \times \hat{J} = 0$,

(2.16)(b) $\hat{J} = \{0, j_\varphi, 0\}$, and

(2.16)(c) $j_\varphi = j_\varphi(r)$ for $R_1 \leq r \leq R_2$.

That is to say, we consider that the coil consists of solid rings.

From (2.16) (a) - (c) we have the differential equation

$$\frac{d}{dr}(r j_\varphi) = 0.$$

Integrating this differential equation, we obtain $j_\varphi = c/r$, where c is the constant of integration.

Since the total current I flowing in the coil is kept constant, we have

$$I = \int_0^{2L}\int_{R_1}^{R_2} j_\varphi \, dr \, du = 2 \, L \, C \, \ln\left(\frac{R_2}{R_1}\right)$$

and hence

$$(2.17) \qquad j_\varphi(r) = \frac{I}{2 \, L \, \ln \, (R_2/R_1)} \, \frac{1}{r} \qquad \text{for} \quad R_1 \le r \le R_2 .$$

Furthermore, we have the following boundary conditions at the interface of the cylindrical rod and the surrounding air:

$$(2.18)\,(a) \qquad \hat{E}_t^{(r)} = \hat{E}_t^{(a)},$$

$$(b) \qquad \hat{E}_n^{(r)} = \hat{E}_t^{(a)}, \quad \text{and}$$

$$(c) \qquad B_t^{(r)} = \frac{\mu}{\mu_0} \, B_t^{(a)},$$

where the index (r) denotes the interior of the rod and the index (a) denotes the exterior region (air). The subscripts t and n signify the tangential component and the normal component respectively.

We also have the following boundary conditions at the interface of the coil and the surrounding air

$$(2.19)\,(a) \qquad \hat{E}_t^{(c)} = \hat{E}_t^{(a)},$$

$$(b) \qquad \hat{E}_n^{(c)} = \hat{E}_n^{(a)}, \quad \text{and}$$

$$(c) \qquad \frac{1}{\mu_c} \, B_t^{(c)} - \frac{1}{\mu_0} \, B_t^{(a)} = j_{s\varphi} ,$$

where the index (c) denotes the interior of the coil and $j_{s\varphi}$ is the surface current density at the interface.

We obviously must require that $\hat{E}$ and $\hat{B}$ remain bounded along the axis of the rod, and that $\hat{E}$ and $\hat{B}$ are functions of order $(1/r^2)$ for large values of r, that is, $\hat{E}$ and $\hat{B}$ vanish at infinity according to the inverse square law. Since the rod and the coil are cylindrically symmetric, the current flow is assumed to be in coaxial circles only. For this case we require:

(2.20) $\hat{A} = \{0, A_\varphi, 0\}$, with

(2.21) $A_\varphi = A_\varphi(r, u)$.

In a cylindrical coordinate system (r, φ, u) we have

$$(2.22) \quad \nabla \times (\nabla \times A) = \left\{0, -\frac{\partial^2 A_\varphi}{\partial u^2} - \frac{\partial^2 A_\varphi}{\partial r^2} - \frac{1}{r}\frac{\partial A_\varphi}{\partial r} + \frac{A_\varphi}{r^2}, 0\right\}.$$

By making use of the following definitions:

$$(2.23)(a) \quad L_1(\omega) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{\mu'}{\mu}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\mu'}{\mu} + k^2\right)\omega$$

$$(b) \quad L_2(\omega) = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega ,$$

we derive from (2.14) (a), (b) the differential equation

(2.24)(a) $L_1(A_\varphi) = 0$,

in the rod, and

(2.24)(b) $L_2(A_\varphi) = 0$

in the air.

From the boundary conditions (2.19) (a) - (c) and the relations (2.12) and (2.13), we derive the boundary conditions at the interface of the rod and the surrounding air to be

$$A_\varphi^{(r)} = A_\varphi^{(c)} \qquad \text{for } r = R_0, \quad |u| < \infty ,$$

$$\frac{\partial A_\varphi^{(r)}}{\partial u} = \frac{\partial A_\varphi^{(a)}}{\partial u} \qquad \text{for } r = R_0, \quad |u| < \infty, \quad \text{and}$$

$$\frac{1}{r} A_\varphi^{(r)} + \frac{\partial A_\varphi^{(r)}}{\partial r} = \frac{\mu}{\mu_0}\left(\frac{1}{r} A_\varphi^{(a)} + \frac{\partial A_\varphi^{(a)}}{\partial r}\right) \qquad \text{for } r = R_0, \quad |u| < \infty .$$

Also the boundary conditions between the coil and the surrounding air are

$$A_\varphi^{(c)} = A_\varphi^{(a)} \qquad \text{for } r = R_1 \text{ or } R_2, \quad |u| < \infty ,$$

$$\frac{\partial A_\varphi^{(c)}}{\partial u} = \frac{\partial A_\varphi^{(a)}}{\partial u} \qquad \text{for } r = R_1 \text{ or } R_2, \quad |u| < \infty, \quad \text{and}$$

$$\frac{1}{\mu_c}\left(\frac{1}{r} A_\varphi^{(c)} + \frac{\partial A_\varphi^{(c)}}{\partial r}\right) - \frac{1}{\mu_0}\left(\frac{1}{r} A_\varphi^{(a)} + \frac{\partial A_\varphi^{(a)}}{\partial r}\right) = j_{3\varphi}$$
$$\text{for } r = R_1 \text{ or } R_2, \quad |u| < L.$$

Since we are studying a problem of surface hardening, the exciting frequency ω is usually of the order of 10^6 Hertz (cycles per second) or higher and hence the skin depth $\delta = \sqrt{2/\omega\mu\sigma}$ [8] is quite small. If the radius R_0 of the rod to be hardened is not too small, so that the ratio R_0/δ is quite large, then for practical purposes the above boundary value problem can be approximated with sufficient accuracy by a boundary value problem in a half-space[14],[15]. Geometrically this is equivalent to allowing the radii of the cylinder and coil to become very large so that their surfaces may be approximated by their tangent planes. In the transition to the limit, these tangent planes remain separated by the distance equal to the difference of the radii, and the metallic cylinder becomes the metallic half-space and the coil becomes an infinite beam aligned parallel to the half space.

In making the transformation we discard terms containing $1/r$ and $1/r^2$ in the differential equation and in the boundary conditions and then introduce the coordinate transformation [14]:

$$
\begin{aligned}
(2.25) \qquad x &= R_0 - r, \\
h &= R_1 - R_0, \\
\hat{u} &= u.
\end{aligned}
$$

Also, we consider only a line source of length $2L$, instead of a coil, because the coil of finite thickness, $R_2 - R_1 > 0$, may be obtained from the solution for the line source by superposition, i.e. by integration. The justification for this may be found in [14].

The geometry of the original boundary value problem is shown in Fig. 2, and that of the approximated problem in Fig. 3.

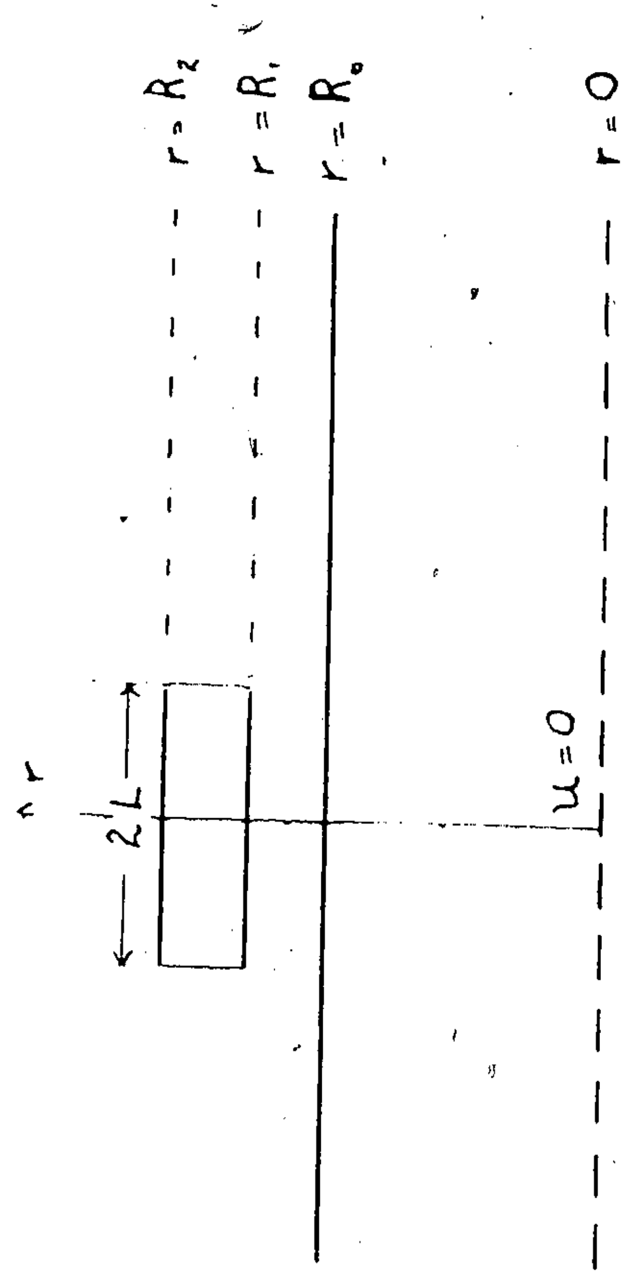

Fig. 2

Fig. 2

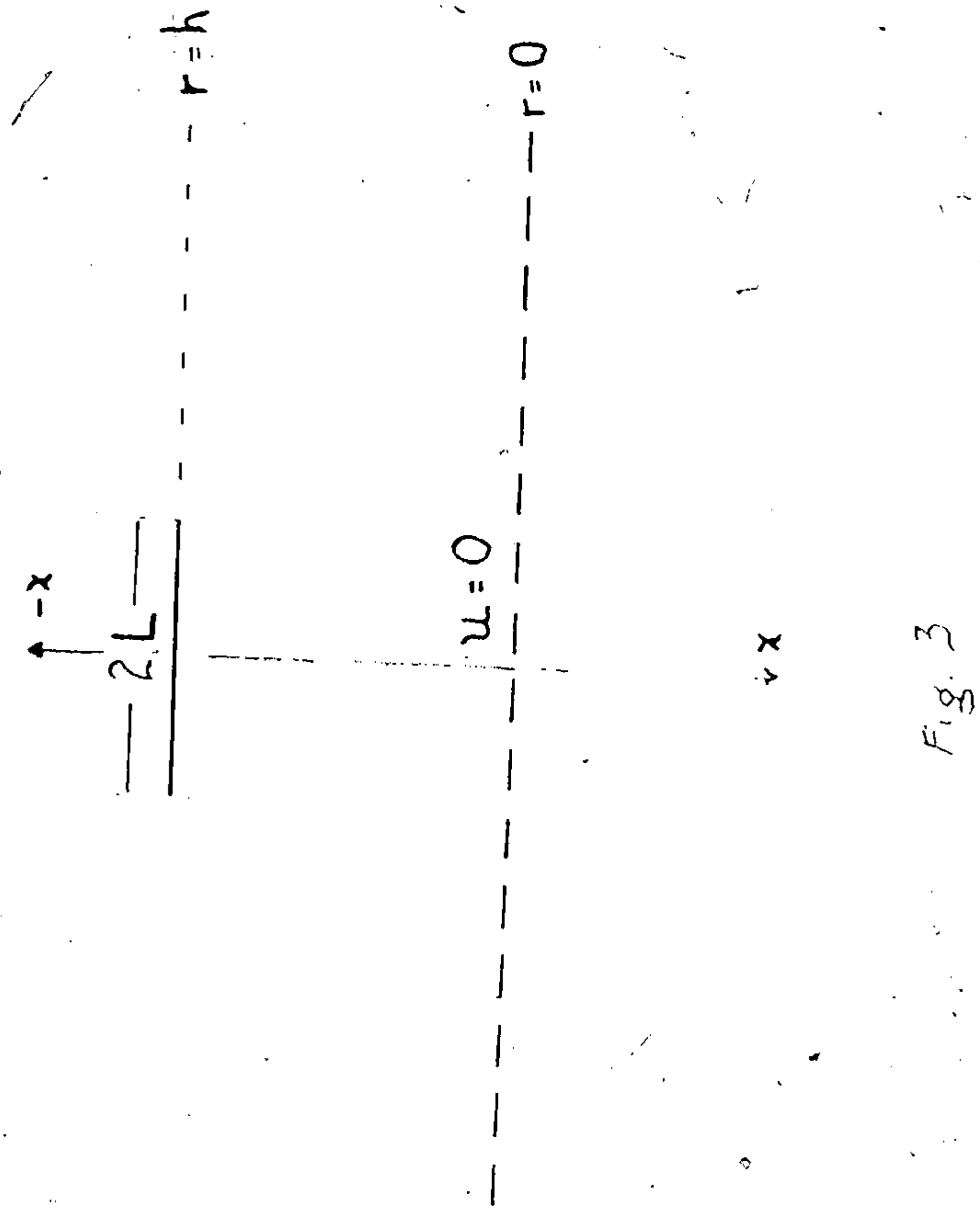

Fig. 3

The simplified problem in two dimensions may now be stated thus:

(2.26) $\psi^{(II)} = \psi^{(II)}(x, u)$ defined on $-\infty < x < -h$, $|u| < \infty$,

$\psi^{(I)} = \psi^{(I)}(x, u)$ defined on $-h < x < 0$, $|u| < \infty$,

$\psi = \psi(x, u)$ defined on $0 \leq x < \infty$, $|u| < \infty$;

(2.27) $\Delta \psi^{(II)} = 0$ defined on $-\infty < x < -h$, $|u| < \infty$,

where $\Delta \psi^{(II)} = \dfrac{\partial^2 \psi^{(II)}}{\partial x^2} + \dfrac{\partial^2 \psi^{(II)}}{\partial u^2}$,

(2.28) $\Delta \psi^{(I)} = 0$ defined on $-h < x < 0$, $|u| < \infty$,

(2.29) $\Delta \psi - \dfrac{\mu'(x)}{\mu(x)} \psi_x + k^2(x) \psi = 0$, $0 < x < \infty$, $|u| < \infty$,

with $k^2(x) = -i \omega \mu(x) \sigma(x)$.

The boundary conditions become:

(2.30) $$\psi_x^{(II)}(-h, u) - \psi_x^{(I)}(-h, u) = \begin{cases} \mu \frac{I}{2L} & \text{for } |u| < L \\ 0 & \text{for } |u| > L, \end{cases}$$

(2.31)(a) $\psi^{(II)}(-h, u) - \psi^{(I)}(-h, u) = 0$ for $|u| < \infty$,

(b) $\psi_u^{(II)}(-h, u) - \psi_u^{(I)}(-h, u) = 0$ for $|u| < \infty$,

(2.32)(a) $\psi^{(I)}(0, u) - \psi(0, u) = 0$ for $|u| < \infty$,

(b) $\psi_u^{(I)}(0, u) - \psi_u(0, u) = 0$ for $|u| < \infty$,

(c) $\mu(0) \psi_x^{(I)}(0, u) - \mu_0 \psi_x(0, u) = 0$ for $|u| < \infty$, The conditions at infinity are:

(2.33)(a) $\psi^{(II)}$, $\psi^{(I)}$, and $\psi \longrightarrow 0$ as $|u| \longrightarrow \infty$,

(b) $\psi^{(III)} = O(|x|^{-2})$, and

$\psi = o(|x|^{2})$.

We use the notation $f = O(g(x))$ to signify that $|f|/|g|$ is uniformly bounded as $x \longrightarrow \infty$; and $f = o(g(x))$ to signify that $|f|/|g| \longrightarrow 0$ as $x \longrightarrow \infty$.

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Chapter 3

THE REPRESENTATION THEOREM

In this chapter we deal with the solution of the boundary value problem formulated in Chapter 2. A representation theorem is proved for the problem.

1. The Statement of the Representation Theorem

The following assumptions are made with regard to the physical parameters $\mu(x)$, $\sigma(x)$, and ω for the purposes of the representation theorem. They follow naturally from the physical conditions found in the model.

(A.1) μ, σ, and ω are positive.

(A.2) $\mu(x),\ \sigma(x) \in C^2[0, \infty)$.

(A.3) ω is a constant.

(A.4) $\lim\limits_{x \to \infty} \mu(x) = \mu_\infty$,

$\lim\limits_{x \to \infty} \sigma(x) = \sigma_\infty$.

(A.5) $\lim\limits_{x \to \infty} \mu'(x) = 0$,

$\lim\limits_{x \to \infty} \mu''(x) = 0$,

$$\lim_{x \to \infty} \sigma'(x) = 0,$$

$$\mu'(0) = 0.$$

(A.6) $$\int_0^\infty \left|\mu(x)\,\sigma(x) - \mu_\infty \sigma_\infty\right| dx < \infty,$$

$$\int_0^\infty \left|\mu'(x)\right|^2 dx < \infty, \quad \text{and}$$

$$\int_0^\infty \left|\mu''(x)\right| dx < \infty.$$

With these assumptions in mind, the theorem is now stated.

Theorem (Representation Theorem)

Suppose that $\mu(x)$, $\sigma(x)$, and ω satisfy assumptions (A.1) to (A.6), then a solution of the boundary value problem (2.26) - (2.33) exists and can be represented by:

(3.1) $$\psi^{(I)}(x, u) = \int_0^\infty \left[B_o(\lambda)\exp\{\lambda x\} + C_o(\lambda)\exp\{-\lambda x\}\right] \cos \lambda u \, \frac{\sin \lambda L}{\lambda L} \, d\lambda,$$

(3.2) $$\psi^{(II)}(x, u) = \int_0^\infty A_o(\lambda)\exp\{\lambda x\} \cos \lambda u \, \frac{\sin \lambda L}{\lambda L} \, d\lambda,$$

(3.3) $$\psi(x, u) = \int_0^\infty A(x, \lambda) \cos \lambda u \, \frac{\sin \lambda L}{\lambda L} \, d\lambda,$$

where $A_o(\lambda)$, $B_o(\lambda)$, $C_o(\lambda)$, and $A(x, \lambda)$ satisfy the following equations:

(3.4)(a) $$-A_o(\lambda)\exp\{-\lambda h\}\lambda + B_o(\lambda)\exp\{-\lambda h\}\lambda - C_o(\lambda)\exp\{\lambda h\}\lambda = -\mu_\circ\frac{I}{\pi},$$

(b) $$A_o(\lambda)\exp\{-\lambda h\} - B_o(\lambda)\exp\{-\lambda h\} - C_o(\lambda)\exp\{\lambda h\} = 0,$$

(c) $$-B_o(\lambda) - C_o(\lambda) + A(0, \lambda) = 0,$$

(d) $$\mu(0)\, B_o(\lambda)\lambda - \mu(0)\, C_o(\lambda)\lambda - \mu_\circ A'(0, \lambda) = 0,$$

(e) $$A''(x, \lambda) - \frac{\mu'(x)}{\mu(x)} A'(x, \lambda) + \left[k^2(x) - \lambda^2\right] A(x, \lambda) = 0,$$

(f) $$A(x, \lambda) \longrightarrow 0 \quad \text{as } x \longrightarrow \infty \quad \text{for all } \lambda,$$

and ' denotes differentiation with respect to x.

2. Some Fundamental Lemmas

In order to facilitate the proof of the theorem we first state and prove the following lemmas.

Lemma 1

If $p(x) \in C\,[0, \infty) \cap L_1[0, \infty)$,

then the solution of the differential equation

(3.5) $$u'(x) + p(x)\, u(x) = q(x), \qquad 0 \le x < \infty$$

subject to the initial condition

(3.6) $$u(0) = c_1$$

when considered as a functional of q(x), is a monotone increasing linear functional of q(x).

Furthermore, the solution of the differential equation (3.5) subject to

(3.7) $$\lim_{x \to \infty} u(x) = c_2$$

is a monotone decreasing linear functional of q(x).

Proof:

If we multiply (3,5) by the integrating factor $\exp\left\{\int_0^x p(t)\,dt\right\}$, we obtain

(3.8) $$u'(x)\exp\left\{\int_0^x p(t)\,dt\right\} + \exp\left\{\int_0^x p(t)\,dt\right\} p(x)\,u(x) = \exp\left\{\int_0^x p(t)\,dt\right\} q(x).$$

Upon integrating (3.8) and making use of (3.6) we obtain

(3.9) $$u(x) = c_1 \exp\left\{-\int_0^x p(t)\,dt\right\} + \exp\left\{-\int_0^x p(t)\,dt\right\} \left[\int_0^x \exp\left\{\int_0^t p(s)\,ds\right\} q(t)\,dt\right].$$

From (3.9) it is obvious that $u(x)$ is a linear functional of $q(x)$ and further that the positivity of the exponential function permits us to assert that $u(x)$ is a monotone increasing functional of $q(x)$.

Similarly, we can show that the solution of (3.5) subject to (3.7) is

(3.10) $$u(x) = c_2 \exp\left\{\int_x^\infty p(t)\,dt\right\} - \exp\left\{\int_x^\infty p(t)\,dt\right\} \left[\int_x^\infty \exp\left\{-\int_x^\infty p(s)\,ds\right\} q(t)\,dt\right],$$

and hence, that $u(x)$ is a monotone decreasing functional of $q(x)$.

Lemma 2

For $\int_0^\infty |f(x)|\,dx < \infty$ and Real $\alpha > 0$, the integral equation

$$(3.11) \qquad z(x,\alpha) = \frac{1}{2\alpha}\int_0^x \left[\exp\{\alpha(t-x)\} - \exp\{\alpha(x-t)\}\right] \left(f(t)\left[z(t,\alpha) + \exp\{\alpha t\}\right] dt\right),$$

has a unique solution.

Also, the integral equation

$$(3.12) \qquad z(x,\alpha) = \frac{1}{2\alpha}\int_x^\infty \left[\exp\{\alpha(x-t)\} - \exp\{\alpha(t-x)\}\right] \left(f(t)\left[z(t,\alpha) + \exp\{-\alpha t\}\right] dt\right),$$

has a unique solution.

Proof:

We use the method of successive approximations to solve (3.11). We introduce a sequence of iterates $\{z_n(x,\alpha)\}$ defined by $z_0(x,\alpha) = 0$, and for $n \geq 1$

$$(3.13) \qquad z_n(x,\alpha) = \frac{1}{2\alpha}\int_0^x \left[\exp\{\alpha(t-x)\} - \exp\{\alpha(x-t)\}\right] \left(f(t)\left[z_{n-1}(t,\alpha) + \exp\{\alpha t\}\right] dt\right).$$

Since the variable of integration t is less than x in (3.13) we have

$$(3.14)\qquad \left|\exp\{\alpha(2t-x)\} - \exp\{\alpha x\}\right| \le \left|\exp\{\alpha(2t-x)\}\right| + \left|\exp\{\alpha x\}\right|$$
$$= \exp\{\text{Real } \alpha(2t-x)\} + \exp\{\text{Real } \alpha x\}$$
$$\le 2\exp\{\text{Real } \alpha x\}.$$

For $n = 1$ we have

$$z_1(x;\alpha) = \frac{1}{2\alpha}\int_0^x \left[\exp\{\alpha(t-x)\} - \exp\{\alpha(x-t)\}\right]$$
$$\left(f(t)\exp\{\alpha t\}\,dt\right)$$
$$= \frac{1}{2\alpha}\int_0^x \left(\exp\{\alpha(2t-x)\} - \exp\{\alpha x\}\right) f(t)\,dt,$$

and then by using (3.14) we obtain

$$(3.15)\qquad |z_1(x;\alpha)| \le \frac{\exp\{\text{Real}\,\alpha x\}}{\text{Real}\,\alpha}\int_0^x |f(t)|\,dt.$$

In a similar fashion we obtain

$$z_2(x,\alpha) - z_1(x,\alpha) = \frac{1}{2\alpha}\int_0^x \left(\exp\{\alpha(t-x)\} - \exp\{\alpha(x-t)\}\right)$$
$$\left(f(t)\, z_1(t,\alpha)\,dt\right)$$
$$= \frac{1}{2\alpha}\int_0^x \left(\exp\{\alpha(2t-x)\} - \exp\{\alpha x\}\right)$$
$$\left(f(t)\exp\{-\alpha t\}\, z_1(t,\alpha)\,dt\right),$$

$$|z_2(x,\alpha) - z_1(x,\alpha)| \le \frac{\exp\{\text{Real}\,\alpha x\}}{\text{Real}\,\alpha}\,.$$
$$\int_0^x |f(t)|\exp\{-\text{Real}\,\alpha t\}\,|z_1(t,\alpha)|\,dt$$

$$\leq \frac{\exp\{\text{Real}\,\alpha x\}}{\text{Real}\,\alpha}\int_0^x |f(t)| \frac{\int_0^t |f(s)|\,ds}{\text{Real}\,\alpha}\,dt$$

$$= \frac{\exp\{\text{Real}\,\alpha x\}}{(\text{Real}\,\alpha)^2}\int_0^x \frac{d}{dt}\,\frac{\left[\int_0^t |f(s)|\,ds\right]^2}{2}\,dt$$

$$= \exp\{\text{Real}\,\alpha x\}\,\frac{\left[\dfrac{\int_0^t |f(t)|\,dt}{\text{Real}}\right]^2}{2}$$

Similarly we may also show by induction that for $n \geq 0$.

$$(3.16) \qquad |z_n(x,\alpha) - z_{n-1}(x,\alpha)| \leq \frac{\exp\{\text{Real}\,\alpha x\}}{n!}\left[\frac{\int_0^x |f(t)|\,dt}{\text{Real}\,\alpha}\right]^n.$$

Since $z_{n+1}(x,\alpha)$ may be written as a series in the form

$$(3.17) \qquad z_{n+1}(x,\alpha) = \sum_{i=1}^{n}\left[z_{i+1}(x,\alpha) - z_i(x,\alpha)\right],$$

the convergence of the sequence $\{z_n(x,\alpha)\}$ to a limit function $z(x,\alpha)$ as $n \longrightarrow \infty$ is equivalent to the convergence of the series (3.17). However, it is obvious that the series (3.17) is also absolutely convergent because by (3.16) each term in (3.17) is less in absolute value than $\exp\{\text{Real}\,\alpha x\}$ times a term in the convergent series for $\exp\left\{\dfrac{\int_0^x |f(t)|\,dt}{\text{Real}\,\alpha}\right\}$. Thus the sequence of iterates $\{z_n(x,\alpha)\}$ converges to a limit function $z(x,\alpha)$ uniformly in every finite subinterval of $[0, \infty)$. Thus we now have the solution of the integral equation (3.11) by the method of successive approximations. It remains to be shown that the solution for (3.11) is unique.

To show uniqueness we suppose instead that there exist two solutions $z^{(1)}(x,\alpha)$ and $z^{(2)}(x,\alpha)$ satisfying (3.11). We then have

$$z^{(2)}(x,\alpha) - z^{(1)}(x,\alpha) = \frac{1}{2\alpha}\int_0^x \left[\exp\{\alpha(t-x)\} - \exp\{\alpha(x-t)\}\right] \left(f(t)\left[z^{(2)}(x,\alpha) - z^{(1)}(x,\alpha)\right] dt\right)$$

$$= \frac{1}{2\alpha}\int_0^x \left[\exp\{\alpha(2t-x)\} - \exp\{\alpha x\}\right] \left(f(t) \exp\{-\alpha t\}\right) \cdot \left[z^{(2)}(t,\alpha) - z^{(1)}(t,\alpha)\right] dt,$$

$$\left|z^{(2)}(x,\alpha) - z^{(1)}(x,\alpha)\right| \leq \frac{\exp\{\text{Real}\,\alpha x\}}{\text{Real}\,\alpha}\int_0^x |f(t)| \exp\{-\text{Real}\,\alpha t\} \left|z^{(2)}(t,\alpha) - z^{(1)}(t,\alpha)\right| dt.$$

(3.18) $$\exp\{-\text{Real}\,\alpha x\} \left|z^{(2)}(x,\alpha) - z^{(1)}(x,\alpha)\right| \leq \int_0^x \frac{|f(t)|}{\text{Real}\,\alpha} \exp\{-\text{Real}\,\alpha t\} \left|z^{(2)}(t,\alpha) - z^{(1)}(t,\alpha)\right| dt.$$

If we put

(3.19) $$|g(x,\alpha)| = \exp\{-\text{Real}\,\alpha x\} \left|z^{(2)}(x,\alpha) - z^{(1)}(x,\alpha)\right|$$

we have from (3.18)

$$(3.20) \qquad \left|g(x,\alpha)\right| \leq \int_0^x \frac{|f(t)|}{\text{Real}\,\alpha} \left|g(t,\alpha)\right| dt.$$

If we can show that the solution of this integral inequality is just the trivial solution, then by (3.19) we have $z^{(2)}(x,\alpha) \equiv z^{(1)}(x,\alpha)$. In order to show this we let

$$(3.21) \qquad h_1(x,\alpha) = \int_0^x \frac{|f(t)|}{\text{Real}\,\alpha} \left|g(t,\alpha)\right| dt.$$

From (3.20) we have

$$(3.22) \qquad \left|g(x,\alpha)\right| \leq h_1(x,\alpha).$$

Differentiating (3.21) with respect to x we obtain

$$h_1'(x,\alpha) = \frac{|f(x)|}{\text{Real}\,\alpha} \left|g(x,\alpha)\right|$$

$$\leq \frac{|f(x)|}{\text{Real}\,\alpha} h_1(x,\alpha)$$

and thus deduce that

$$(3.23)(a) \qquad h_1'(x,\alpha) - \frac{|f(x)|}{\text{Real}\,\alpha} h_1(x,\alpha) \leq 0$$

while from (3.21) we obtain

$$(3,23)(b) \qquad h_1(0,\alpha) = 0.$$

Now, if we compare (3.23)(a) and (b) with the following differential equation

$$(3.24)(a) \qquad h_2'(x,\alpha) - \frac{|f(x)|}{\text{Real}\,\alpha} h_2(x,\alpha) = 0$$

subject to

$$(3.24)(b) \qquad h_2(0,\alpha) = 0,$$

and apply Lemma 1 and [1] p.134, we conclude that

(3.25) $\qquad h_1(x, \alpha) \leq h_2(x, \alpha).$

However, the solution $h_2(x, \alpha)$ of (3.24)(a) and (b) is just the trivial solution. Hence by (3.22) and (3.25) we have $|g(x, \alpha)| = 0$.

Similarly we can employ the method of successive approximations to prove that (3.12) has a unique solution.

Lemma 3

For $\int_0^\infty |f(x)| dx < \infty$ and Real $\alpha > 0$, the differential equation

(3.26) $\qquad Y''(x, \alpha) - \left[\alpha^2 - f(x)\right] Y(x, \alpha) = 0$

has two linearly independent solutions for all $0 \leq x < \infty$, and for all α with Real $\alpha > 0$, and these are given by

(3.27)(a) $\qquad Y_1(x, \alpha) = \exp\{\alpha x\}\left[1 + b_1(x, \alpha)\right]$, and

(b) $\qquad Y_2(x, \alpha) = \exp\{-\alpha x\}\left[1 + b_2(x, \alpha)\right]$, with

(3.28)(a) $\qquad |b_1(x, \alpha)| \leq \exp\left\{\dfrac{\int_0^x |f(t)|\, dt}{\text{Real } \alpha}\right\} - 1$, and

(b) $\qquad |b_2(x, \alpha)| \leq \exp\left\{\dfrac{\int_x^\infty |f(t)|\, dt}{\text{Real } \alpha}\right\} - 1.$

Proof:

Let

(3.29)(a) $\qquad z(x, \alpha) = \exp\{\alpha x\}\, b_1(x, \alpha)$

so that

(3.29)(b) $\quad Y_1(x,\alpha) = \exp\{\alpha x\} + z(x,\alpha).$

Substituting (3.29)(b) into (3.26) we obtain

(3.30) $\quad z''(x,\alpha) - \alpha^2 z(x,\alpha) = -f(x)\left[z(x,\alpha) + \exp\{\alpha x\}\right].$

Now (3.30) may be transformed into the integral equation (3.11). By Lemma 2 the solution to this integral equation exists and hence $Y_1(x,\alpha)$ exists.

It remains to show that $b_1(x,\alpha)$ satisfies the estimate (3.28)(a).

From (3.16) and (3.17) we obtain

(3.31)
$$|z(x,\alpha)| \leq \sum_{j=0}^{\infty} \left|z_{j+1}(x,\alpha) - z_j(x,\alpha)\right|$$
$$\leq \exp\{\text{Real}\,\alpha x\}\left[\exp\left\{\frac{\int_0^x |f(t)|\,dt}{\text{Real}\,\alpha}\right\} - 1\right].$$

Using (3.29)(a) and (3.31) we obtain

$$|b_1(x,\alpha)| \leq \exp\{-\text{Real}\,\alpha x\}\,|z(x,\alpha)|$$
$$\leq \exp\left\{\frac{\int_0^x |f(t)|\,dt}{\text{Real}\,\alpha}\right\} - 1\,.$$

Similarly we can show that $Y_2(x,\alpha)$ exists and that $b_2(x,\alpha)$ satisfies the estimate (3.28)(b).

Lemma 4

There exists a unique solution for the differential equation (3.4)(e) subject to (3.4)(f) and the following condition (3.32):

(3.32) $\quad \mu(0)\lambda A(0,\lambda) - \mu_0 A'(0,\lambda) = \dfrac{\mu_0\mu(0)}{\pi}\, I \exp\{-\lambda h\}.$

Proof:

We first note that (3.32) is obtained by eliminating $A_o(\lambda)$, $B_o(\lambda)$, and $C_o(\lambda)$ from equations (3.4)(a) -.(d). If we introduce the one to one transformation:

$$(3.33) \qquad a(x,\lambda) = \sqrt{\frac{\mu(0)}{\mu(x)}}\; A(x;\lambda)$$

then equation (3.4)(e) is reduced to

$$(3.34) \qquad a''(x,\lambda) - \left[\lambda^2 - K^2(x)\right] a(x,\lambda) = 0$$

where

$$(3.35) \qquad K^2(x) = k^2(x) - \frac{3}{4}\left[\frac{\mu'(x)}{\mu(x)}\right]^2 + \frac{1}{2}\,\frac{\mu''(x)}{\mu(x)}.$$

By (A.5) and (3.33) we also have:

$$(3.36)(a) \qquad a(0,\lambda) = A(0,\lambda),$$

$$(b) \qquad a'(0,\lambda) = A'(0,\lambda).$$

Substitution of (3.36)(a) and (b) into (3.32) and substitution of (3.33) into (3.4)(f) yields:

$$(3.37)(a) \qquad \mu(0)\lambda\, a(0,\lambda) - \mu_o a'(0,\lambda) = \mu_o \frac{\mu(0)}{\pi} I \exp\{-\lambda h\}, \text{ and}$$

$$(b) \qquad a(x,\lambda) \to 0 \text{ as } x \to \infty, \quad \text{for all } \lambda.$$

Hence if we can prove that a solution exists for (3.34) subject to conditions (3.37)(a) and (b) and that this solution is unique, then this is also true for $A(x,\lambda)$.

Let us then define

$$(3.38)(a) \qquad K_\infty^2 = \lim_{x\to\infty} K^2(x)$$

then by (A.5) we have

(3.38)(b) $K_\infty^2 = k_\infty^2 = -i\,\omega\mu_\infty\sigma_\infty$,

and hence

(3.39) $$\sqrt{\lambda^2 - K_\infty^2} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\lambda^4 + \omega^2\mu_\infty^2\sigma_\infty^2} + \lambda^2} + i\sqrt{\sqrt{\lambda^4 + \omega^2\mu_\infty^2\sigma_\infty^2} - \lambda^2},$$

for all $\infty > \lambda \geq 0$.

Also from (3.35), (3.38) and (A.1) - (A.6) we obtain

(3.40) $$\int_0^\infty \left| K^2(x) - K_\infty^2 \right| dx < \infty.$$

Now equation (3.34) may be rewritten as

$$a''(x,\lambda) - \left\{ (\lambda^2 - K_\infty^2) - \left[K^2(x) - K_\infty^2 \right] \right\} a(x,\lambda) = 0,$$

hence the assumptions of Lemma 3 are satisfied by (3.34). Then by Lemma 3 there are two linearly independent solutions $a_1(x,\lambda)$ and $a_2(x,\lambda)$ to (3.34) such that for all $0 \leq x < \infty$ and $\infty > \lambda \geq 0$ it holds that:

(3.41)(a) $$a_1(x,\lambda) = \exp\left\{\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\left[1 + d_1(x,\lambda)\right]$$

(b) $$a_2(x,\lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\left[1 + d_2(x,\lambda)\right]$$

where

(3.42)(a) $$\left| d_1(x,\lambda) \right| \leq \exp\left\{ \frac{\int_0^x \left| K^2(t) - K_\infty^2 \right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right\} - 1,$$

(b) $$\left| d_2(x,\lambda) \right| \leq \exp\left\{ \frac{\int_x^\infty \left| K^2(t) - K_\infty^2 \right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right\} - 1.$$

Now from (3.41)(a) and (b) and from (3.42)(a) and (b) we have for large values of x and for all values $0 \leq \lambda < \infty$

(3.43)(a) $\quad a_1(x, \lambda) = \exp\left\{\sqrt{\lambda^2 - K_\infty^2}\ x\right\}\left[1 + o(1)\right]$

(b) $\quad a_2(x, \lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\ x\right\}\left[1 + o(1)\right].$

Since (3.34)' is linear we have

$$a(x, \lambda) = c_1(\lambda)\, a_1(x, \lambda) + c_2(\lambda)\, a_2(x, \lambda)$$

for all $0 \leq x < \infty$ and for all $0 \leq \lambda < \infty$.

In particular for large values of x we have

$$a(x, \lambda) = c_1(\lambda) \exp\left\{\sqrt{\lambda^2 - K_\infty^2}\ x\right\}\left[1 + o(1)\right]$$

$$+ c_2(\lambda) \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\ x\right\}\left[1 + o(1)\right].$$

From (3.37)(b) we see that in order for $a(x, \lambda) \to 0$ as $x \to \infty$ for all $0 \leq \lambda < \infty$, we must have $c_1(\lambda) = 0$ for all $0 \leq \lambda < \infty$. Therefore we have

(3.44) $\quad a(x, \lambda) = c_2(\lambda)\, a_2(x, \lambda).$

If we substitute (3.44) into (3.34)(c) we obtain

$$c_2(\lambda) = \frac{\mu_0(\mu(0)/\pi)\ I \exp\{-\lambda h\}}{\mu(0)\lambda\, a_2(0, \lambda) - \mu_0 a_2'(0, \lambda)}$$

for all $0 \leq \lambda < \infty$. Hence we have:

(3.45) $\quad a(x, \lambda) = \dfrac{\mu_0(\mu(0)/\pi)\ I \exp\{-\lambda h\}}{\mu(0)\lambda\, a_2(0, \lambda) - \mu_0 a_2'(0, \lambda)}\, a_2(x, \lambda).$

From (3.33) we deduce that

(3.46) $\quad A(x, \lambda) = \sqrt{\dfrac{\mu(x)}{\mu(0)}}\ a(x, \lambda)$

$$= \sqrt{\frac{\mu(x)}{\mu(0)}}\left[\frac{\mu_0(\mu(0)/\pi)\ I \exp\{-\lambda h\}}{\mu(0)\lambda\, a_2(0, \lambda) - \mu_0 a_2'(0, \lambda)}\right] a\ (x, \lambda)$$

Thus we have shown the existence and uniqueness of the solution to (3.4)(c) subject to (3.4)(f) and (3.32).

As a corollary to Lemma 4 we obtain the behaviour of $A(x, \lambda)$ for large values of x and for all values of $0 \leq \lambda < \infty$.

Corollary to Lemma 4

For large values of x, $A(x, \lambda)$ behaves as $\exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\, x\right\}$.

Proof:

From (3.43)(b) we have for all values of $0 \leq \lambda < \infty$ and for large x, that $a_2(x, \lambda)$ behaves as $\exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\, x\right\}$. Hence, from (3.46), we conclude that $A(x, \lambda)$ has the same behaviour.

Lemma 5

For large values of λ and for all values of $0 \leq x < \infty$ then there exist solutions to the Ricatti equations

(3.47)(a) $\quad \phi'(x, \lambda) + \phi^2(x, \lambda) = \lambda^2 - K^2(x)$

subject to

(3.47)(b) $\quad \phi(x, \lambda) \longrightarrow -\sqrt{\lambda^2 - K_\infty^2} \quad$ as $x \rightarrow \infty$.

Proof:

If we let $w(x, \lambda) = \phi(x, \lambda) + \sqrt{\lambda^2 - K_\infty^2}$ and substitute this into equation (3.47)(a) and (b) we obtain:

$$(3.48)(a) \quad w'(x,\lambda) + w^2(x,\lambda) - 2\sqrt{\lambda^2 - K_\infty^2}\, w(x,\lambda) = \left[K_\infty^2 - K^2(x)\right],$$

$$(b) \quad w(x,\lambda) \longrightarrow 0 \quad \text{as } x \to \infty.$$

From (3.38)(a) we deduce that $\lim_{x\to\infty} \left|K^2(x) - K_\infty^2\right| = 0$ and from assumption (A.2) we conclude that $\left|K^2(x) - K_\infty^2\right| \in C\,[0, \infty)$, and hence that $\left|K^2(x) - K_\infty^2\right|$ is uniformly bounded. Now let M be such a bound, then we have

$$(3.49) \quad \left|K^2(x) - K_\infty^2\right| < M \quad \text{for all } 0 \leq x < \infty.$$

From (3.39) we observe that Real$\sqrt{\lambda^2 - K_\infty^2}$ is a positive monotone increasing function of λ, hence there exists a λ_0 such that for all $\lambda \geqslant \lambda_0$:

$$(3.50) \quad \mathrm{Re}\sqrt{\lambda^2 - K_\infty^2} > 1 + M.$$

Now equation (3.48)(a) may be reduced to the following integral equation:

$$(3.51) \quad w(x,\lambda) = c(\lambda)\exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\, x\right\} + \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \int_0^x \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\, s\right\} \left[K_\infty^2 - K^2(s) - w^2(s,\lambda)\right] ds,$$

where $c(\lambda)$ is an arbitrary constant which depends on the parameter λ. The presence of $\exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\, x\right\}$ can lead to difficulty, so we rewrite the equation as

$$(3.52)\quad w(x,\lambda) = c(\lambda)\exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\} + \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\int_0^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[K_\infty^2 - K^2(s) - w^2(s,\lambda)\right] ds$$
$$- \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\int_x^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[K_\infty^2 - K^2(s) - w^2(x,\lambda)\right] ds.$$

In order that $w(x, \lambda)$ shall satisfy (3.48)(b) the $c(\lambda)$ must satisfy

$$c(\lambda) = -\int_0^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[K_\infty^2 - K^2(s) - w^2(s,\lambda)\right] ds.$$

Hence (3.52) reduces to the following:

$$(3.53)\quad w(x,\lambda) = \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\int_x^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[w^2(s,\lambda) - K^2(s) - K_\infty^2\right] ds.$$

We now apply the method of successive approximations ([2] pp. 29-30) with

$$(3.54)(a)\quad w_0(x,\lambda) = \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\int_x^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[K^2(s) - K_\infty^2\right] ds$$

$$(b)\quad w_{n+1}(x,\lambda) = \exp\left\{2\sqrt{\lambda^2 - K_\infty^2}\,x\right\}\int_x^\infty \exp\left\{-2\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left[w_n^2(s,\lambda) + K^2(s) - K_\infty^2\right] ds$$

for $n \geqslant 0$. We next wish to show that this iteration converges for all $0 \leq x < \infty$ and for all $\lambda \geq \lambda_0$.

We first show by induction that for all $0 \leq x < \infty$ and for all $\lambda_0 \leq \lambda < \infty$:

$$(3.55) \qquad |w_n(x, \lambda)| < \frac{M}{1 + M} < 1 \qquad \text{for all } n \geq 0.$$

From (3.54)(a) we observe that

$$\begin{aligned}
|w_0(x, \lambda)| &\leq \exp\left\{2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \int_x^\infty \exp\left\{-2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, s\right\} \\
&\qquad |K^2(s) - K_\infty^2|\, ds \\
&\leq M \exp\left\{2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \\
&\qquad \int_x^\infty \exp\left\{-2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, s\right\} ds \\
&= \frac{M}{2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}} < \frac{M}{2(1+M)} < \frac{M}{1+M} < 1 .
\end{aligned}$$

If we assume that the relation holds for n, it then follows from (3.54)(b) that

$$\begin{aligned}
|w_{n+1}(x, \lambda)| &\leq \exp\left\{2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \\
&\qquad \int_x^\infty \exp\left\{-2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, s\right\} \\
&\qquad |w_n^2(s, \lambda)|\, ds + |w_0(x, \lambda)| \\
&\leq \left(\frac{M}{1 + M}\right)^2 \exp\left\{2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \\
&\qquad \int_x^\infty \exp\left\{-2 \operatorname{Re}\sqrt{\lambda^2 - K_\infty^2}\, s\right\} ds + |w_0(x, \lambda)|
\end{aligned}$$

$$\leq \left(\frac{M}{1+M}\right)^2 \frac{1}{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}} + \frac{M}{2(1+M)}$$

$$\leq \left(\frac{M}{1+M}\right)^2 \frac{1}{2(1+M)} + \frac{M}{2(1+M)}$$

$$\leq \left(\frac{M}{1+M}\right)^2 \frac{1}{2} + \frac{M}{2(1+M)}$$

$$= \left(\frac{M}{1+M}\right)\left(\frac{1}{2}\,\frac{M}{1+M} + \frac{1}{2}\right) < \frac{M}{1+M}.$$

Since $M/(1+M) < 1$ we have that:

$$(3.56)' \quad \left|w_{n+1}(x,\lambda) - w_n(x,\lambda)\right| \leq \exp\left\{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,x\right\}$$

$$\int_x^\infty \exp\left\{-2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left|w_n(s,\lambda) - w_{n-1}(s,\lambda)\right|$$

$$\left|w_n(s,\lambda) + w_{n-1}(s,\lambda)\right| ds$$

$$\leq 2\,\frac{M}{1+M}\,\exp\left\{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,x\right\}$$

$$\int_x^\infty \exp\left\{-2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,s\right\}$$

$$\left|w_n(s,\lambda) - w_{n-1}(s,\lambda)\right| ds.$$

Furthermore, for $n = 1$ we have

$$\left|w_1(x,\lambda) - w_0(x,\lambda)\right| \leq \exp\left\{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,x\right\}$$

$$\int_x^\infty \exp\left\{-2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,s\right\}\left|w_0^2(s,\lambda)\right| ds$$

$$\leq \left(\frac{M}{1+M}\right)^2 \exp\left\{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,x\right\}$$

$$\int_x^\infty \exp\left\{-2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}\,s\right\} ds$$

$$= \left(\frac{M}{1+M}\right)^2 \frac{1}{2\,\mathrm{Re}\sqrt{\lambda^2 - K_\infty^2}}$$

$$\leq \left(\frac{M}{1+M}\right)^2 \frac{1}{2(1+M)} < \frac{1}{2}\left(\frac{M}{1+M}\right)^2 .$$

Iterating this relation in (3.52) we obtain

(3.57) $$\left| w_{n+1}(x, \lambda) - w_n(x, \lambda) \right| \quad \frac{1}{2}\left(\frac{M}{1+M}\right)^{n+2} .$$

Hence the series

$$\sum_{n=0}^{\infty} \left| w_{n+1}(x, \lambda) - w_n(x, \lambda) \right| \leq \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{M}{1+M}\right)^{2+n} = \frac{1}{2}\,\frac{M}{1+M}^2$$

converges uniformly for all $0 \leq x < \infty$ and for all $\lambda \geqslant \lambda_0$ to a function $w(x, \lambda)$ which satisfies the integral equation (3.53) and hence satisfies the Ricatti equations.

Lemma 6

For large values of λ, $A(x, \lambda)$ behaves as $A(0, \lambda) \exp\{-\lambda x\}$.

Proof:

We introduce the transformation:

(3.58) $$A(x, \lambda) = A(0, \lambda)\sqrt{\frac{\mu(x)}{\mu(0)}} \exp\left\{\int_0^x \phi(x, \lambda)\, dx\right\}$$

which we substitute into (3.4)(e) to obtain (3.47)(a). From the corollary to Lemma 4, we have that $A(x, \lambda)$ behaves as $\exp\{-\sqrt{\lambda^2 - K_\infty^2}\, x\}$ for large values of x, and hence we obtain conditions (3.47)(b).

Further, if we introduce the transformation:

(3.59) $\quad \Phi(x, \lambda) = (1/\lambda)\, \phi(x, \lambda)$

then (3.47)(a) and (b) are transformed into:

(3.60)(a) $\quad \lambda \Phi'(x, \lambda) + \lambda^2 \Phi^2(x, \lambda) - \lambda^2 + K^2(x) = 0,$

(b) $\quad \Phi(x, \lambda) \to -(1/\lambda)\sqrt{\lambda^2 - K_\infty^2}$ as $x \to \infty$.

It has been shown in [7] p. 191, in [3], and in [9] that for large values of λ, we have :

(3.61) $\quad \Phi(x, \lambda) = [-1 + O(1/\lambda^2)]$ for all $0 \leq x < \infty$.

From (2.47) we deduce that

(3.62) $\quad \Phi(x, \lambda) = \lambda[-1 + O(1/\lambda^2)]$.

If we can use (3.62) in (3.58) we conclude that $A(x, \lambda)$ behaves as $A(0, \lambda) \exp\{-\lambda x\}$, which proves the lemma.

Lemma 7

Let $f(x) \in L_1[0, \infty)$, $F(x)$ be defined and uniformly bounded on $(-\infty, \infty)$, and for large values of $|t|$,

$$\int_0^t F(x)\, dx = o(t),$$

then $\lim\limits_{|t| \to \infty} \int_0^\infty f(x)\, F(tx)\, dx = 0.$

Proof:

Suppose that

$$f(x) = \begin{cases} c & x \in [\alpha, \beta] \subset [0, \infty) \\ 0 & x \in [0, \infty) \setminus [\alpha, \beta], \end{cases}$$

then we have:

$$(3.63) \quad \int_0^{\infty} f(x)\, F(tx)\, dx = c \int_{\alpha}^{\beta} F(tx)\, dx$$

$$= \frac{c}{t} \int_0^{\beta t} F(s)\, ds - \frac{c}{t} \int_0^{\alpha t} F(s)\, ds,$$

so we conclude that for this particular choice of $f(x)$

$$\lim_{t \to \infty} \int_0^{\infty} f(x)\, F(tx)\, dx = 0.$$

Now $f(x) \in L_1[0, \infty)$, so for any $\varepsilon > 0$ there exists a simple function $g_m(x) = \sum_{i=1}^{m} f_i(x)$, such that

$$\int_0^{\infty} |f(x) - g_m(x)|\, dx < \varepsilon,$$

where the $f_i(x)$ are of the form

$$f_i(x) = \begin{cases} c_i & x \in [\alpha_i, \beta_i] \subset [0, \infty) \\ 0 & x \in [0, \infty) \setminus [\alpha_i, \beta_i], \end{cases}$$

and the $[\alpha_i, \beta_i]$ are pairwise disjoint [12] p. 67. Then the following inequality becomes obvious:

$$\left| \int_0^{\infty} f(x)\, F(tx)\, dx \right| \le \left| \int_0^{\infty} \{f(x) - g_m(x)\}\, F(tx)\, dx \right|$$

$$+ \left| \int_0^{\infty} F(tx)\, g_m(x)\, dx \right|$$

$$= I_1 + I_2.$$

Now $F(x)$ is bounded, $|F(x)| \le M$, and

$$I_1 \le M \int_0^{\infty} |f(x) - g_m(x)|\, dx < \varepsilon M.$$

From the definition of $g_m(x)$ we deduce that

$$\int_0^\infty g_m(x)\ F(tx)\ dx = \sum_{i=1}^{m} \int_0^\infty f_i(x)\ F(tx)\ dx$$

and since

$$\lim_{t \to \infty} \int_0^\infty f_i(x)\ F(tx)\ dx = 0 \quad \text{for } i = 1, \ldots, m,$$

we conclude that

$$\lim_{t \to \infty} \int_0^\infty g_m(x)\ F(tx)\ dx = 0.$$

Now there exists a $T > 0$ such that $\forall\ t > T,\ I_2 < \varepsilon$.
Hence for $t > T$,

$$\left| \int_0^\infty f(x)\ F(tx)\ dx \right| < \varepsilon(M+1), \qquad \text{i.e.}$$

$$\lim_{t \to \infty} \int_0^\infty f(x)\ F(tx)\ dx = 0.$$

It can be shown similarly that

$$\lim_{t \to -\infty} \int_0^\infty f(x)\ F(tx)\ dx = 0.$$

Hence

$$\lim_{|t| \to \infty} \int_0^\infty f(x)\ F(tx)\ dx = 0.$$

As a result of this Lemma we conclude that

$$\lim_{|t| \to \infty} \int_0^\infty f(x)\ \cos tx\ dx = 0, \quad \text{and}$$

$$\lim_{t \to \infty} \int_0^\infty f(x)\ \sin tx\ dx = 0,$$

for all $f \in L_1[0, \infty)$.

Lemma 8

For large values of λ we have

$$A(0,\lambda) \sim \exp\{-\lambda h\}/\lambda,$$

$$A_o(\lambda) \sim \exp\{+\lambda h\}/\lambda,$$

$$B_o(\lambda) \sim \exp\{-\lambda h\}/\lambda,$$

$$C_o(\lambda) \sim \exp\{-\lambda h\}/\lambda.$$

Proof:

We have shown in Lemma 5 that, for large values of (3.47)(a) and (b) has a solution for all x, in particular $\phi(0,\lambda)$ exists. Also from (3.62) we have:

$$\phi(0,\lambda) = \lambda\left[-1 + o(1/\lambda^2)\right]. \tag{3.64}$$

From (3.58) we obtain by differentiation

$$A'(0,\lambda) - \phi(0,\lambda)\, A(0,\lambda) = 0, \quad \text{for all } \lambda \geqslant \lambda_o. \tag{3.65}$$

From (3.4)(a) - (d) and (3.65) we have for all values of $\lambda \geqslant \lambda_o$:

(3.66)(a) $$A(0,\lambda) - \frac{\mu_o(I/\pi)\exp\{-\lambda h\}}{\lambda - (\mu_o/\mu(0))\,\phi(0,\lambda)},$$

(b) $$A_o(\lambda) = A(0,\lambda) + \mu_o \frac{I}{\pi}\,\frac{\exp\{-\lambda h\}}{2\lambda}\left[\exp\{2\lambda h\} - 1\right]$$

(c) $$B_o(\lambda) = A(0,\lambda) - \mu_o \frac{I}{\pi}\,\frac{\exp\{-\lambda h\}}{2\lambda}$$

(d) $$C_o(\lambda) = \mu_o \frac{I}{\pi}\,\frac{\exp\{-\lambda h\}}{2\lambda}.$$

If we substitute (3.64) into (3.66)(a) we obtain for large values of λ:

$$A(0,\lambda) = \frac{\mu_o(I/\pi)\exp\{-\lambda h\}}{\lambda\left[1 + \frac{\mu_o}{\mu(0)} + \frac{\mu_o}{\mu(0)}\, o\left(\frac{1}{\lambda^2}\right)\right]}.$$

Thus we have demonstrated the asymptotic behaviour of $A(0,\lambda)$, $A_o(\lambda)$, $B_o(\lambda)$, and $C_o(\lambda)$ proposed in the Lemma.

Lemma 9

$A(0,\lambda)$, $A_o(\lambda)$, $B_o(\lambda)$, $C_o(\lambda)$, and $A(x,\lambda)$ may be determined from (3.4)(a) - (f) for all $0 \leq \lambda < \infty$, and for all $0 < x < \infty$.

Proof:

From (3.46) and (3.41)(b) we determine $A(x,\lambda)$ and $A(0,\lambda)$ in terms of $a_2(x,\lambda)$, $a_2(0,\lambda)$, and $a_2'(0,\lambda)$. Since $a_2(x,\lambda)$, and hence $a_2(0,\lambda)$ and $a_2'(0,\lambda)$, are assumed to be known, we have

(3.67)(a) $$A(0,\lambda) = \frac{\mu_o(\mu(0)/\pi)\, I \exp\{-\lambda h\}}{\mu(0)\lambda a_2(0,\lambda) - \mu_o a_2'(0,\lambda)}\, a_2(0,\lambda)$$

(b) $$A'(0,\lambda) = \frac{\mu_o(\mu(0)/\pi)\, I \exp\{-\lambda h\}}{\mu(0)\lambda a_2(0,\lambda) - \mu_o a_2'(0,\lambda)}\, a_2'(0,\lambda)$$

(c) $$A_o(\lambda) = B_o(\lambda) + C_o(\lambda)\exp\{2\lambda h\}$$

(d) $$B_o(\lambda) = A(0,\lambda) - C_o(\lambda)$$

(e) $$C_o(\lambda) = \frac{I}{2\pi}\,\frac{\mu_o \exp\{-\lambda h\}}{\lambda}.$$

3. The Proof of the Theorem

Proof of the Theorem:

With the aid of the nine Lemmas we may now proceed to prove the theorem.

Since we have the asymptotic behaviour in λ for $A_o(\lambda)$, $B_o(\lambda)$, $C_o(\lambda)$, $A(0, \lambda)$, and $A(x, \lambda)$, and since all three integrands in (3.1) - (3.3) are continuous, all three integrals converge. Here we have defined the values of the integrands at $\lambda = 0$ as the value of $\lim_{\lambda \to 0}$.

Similarly all the integrals ψ, $\psi^{(I)}$, $\psi^{(II)}$, and their derivatives up to the second order converge. The integrands for $\psi^{(I)}$, $\psi^{(II)}$, and ψ also satisfy the condition at infinity. The latter follows from Lemmas 4 and 7.

If we substitute $\psi^{(I)}$, $\psi^{(II)}$, and ψ into the differential equations and the boundary conditions and make use of the Dirichlet Integral

$$\int_0^{\infty} \cos \lambda u \, \frac{\sin \lambda L}{\lambda L} \, d\lambda = \begin{cases} \dfrac{\pi}{2L} & |u| < L \\ 0 & |u| > L \end{cases}$$

we see that they satisfy the differential equations and the boundary conditions. Hence the representation theorem has been proved.

- o -

Chapter 4

APPROXIMATE METHODS

The representation theorem of the last chapter shows that one of the integrands is composed of the solution of a second order linear differential equation. Since it is well known that the exact solution of a second order linear differential equation with non-constant coefficients cannot be obtained using a finite number of algebraic, differential, or integral operations [11] p. 26, it becomes imperative to provide some methods for approximating the solution. Because the engineer and physicist are also interested in the quantitative aspects of the solution, only those approximate solutions are discussed which can be evaluated numerically on a computer. For computational reasons, we are interested only in the values of $\psi(x, u)$ in the range of x which is of the order of the skin depth being heated. The frequency used in induction hardening is so high that the skin depth is very small compared with the radius of the cylinder, hence it suffices to have approximate methods which are good for small values of x only. We note also that in applications the permeability and conductivity are functions which vary relatively slowly, that is, deviate from a constant value by a function whose order of magnitude is very small compared to that of the constant. We shall make use of this property in constructing the approximate solution.

1. The Layer Method

The Layer Method consists essentially of replacing the model previously described -- where the permeability $\mu(x)$ and the conductivity $\sigma(x)$ are assumed to depend continuously upon the spatial coordinate x -- by a model which may be characterized by postulating that the permeability μ_ν and the conductivity σ_ν in the metallic medium which occupies the half space $R = \{(x, u) \mid x \geqslant 0, |u| < \infty\}$ are constants in the layers $L_\nu = \{(x, u) \mid (\nu - 1)\Delta x \leq x \leq \nu\Delta x, |u| < \infty\}$ $\nu = 1, \ldots, N$, of uniform width Δx.

If we adopt this layer model, we will obtain the following system of differential equations and boundary conditions:

$$\Delta \psi^{(I)} = 0,$$

$$\Delta \psi^{(III)} = 0,$$

$$\Delta \psi^{(\nu)} + k_\nu^2 \psi^{(\nu)} = 0 \quad \text{in } R,$$

where $k_\nu^2 = \begin{cases} -i\,\omega\mu_\nu\sigma_\nu & \nu = 1, \ldots, N, \\ 0 & \nu = 0, \end{cases}$

subject to:

$$\frac{\partial \psi^{(II)}}{\partial x} - \frac{\partial \psi^{(I)}}{\partial x} = \begin{cases} \mu_1 \frac{I}{2L}, & x = -h, \quad |u| < L \\ 0, & x = -h, \quad |u| > L \end{cases}$$

$$\frac{\partial \psi^{(II)}}{\partial u} - \frac{\partial \psi^{(I)}}{\partial u} = 0, \qquad x = -h, \quad |u| < \infty$$

$$\psi^{(II)} - \psi^{(I)} = 0, \qquad x = -h, \quad |u| < \infty$$

$$\frac{\partial \psi^{(I)}}{\partial u} - \frac{\partial \psi^{(1)}}{\partial u} = 0, \qquad x = 0, \quad |u| < \infty$$

$$\mu_1 \frac{\partial \psi^{(I)}}{\partial x} - \mu_0 \frac{\partial \psi^{(1)}}{\partial x} = 0, \qquad x = 0, \quad |u| < \infty$$

$$\psi^{(I)} - \psi^{(1)} = 0, \qquad x = 0, \quad |u| < \infty$$

$$\frac{\partial \psi^{(\nu)}}{\partial u} - \frac{\partial \psi^{(\nu+1)}}{\partial u} = 0, \qquad x = \nu \Delta x, \ |u| < \infty$$

$$\mu_{\nu+1} \frac{\partial \psi^{(\nu)}}{\partial x} - \mu_\nu \frac{\partial \psi^{(\nu+1)}}{\partial x} = 0, \qquad x = \nu \Delta x, \ |u| < \infty \quad 1, \ldots, N$$

$$\psi^{(\nu)} - \psi^{(\nu+1)} = 0, \qquad x = \nu \Delta x, \ |u| < \infty$$

$$\psi^{(N+1)} = \psi_x^{(N+1)} = \psi_u^{(N+1)} = 0.$$

We then use the following representations to approximate the solution of the above boundary value problem:

$$(4.1) \quad \tilde{\psi}^{(I)}(x, u) = \int_0^\infty \left[B_0(\lambda) \exp\{\lambda x\} + C_0(\lambda) \exp\{-\lambda x\} \right] \cos \lambda u \frac{\sin \lambda L}{\lambda L} \, d\lambda,$$

$$(4.2)\qquad \hat{\psi}^{(\mathrm{II})}(x,\ u) = \int_0^\infty A_0(\lambda)\ \exp\{\lambda x\}\ \cos \lambda u\ \frac{\sin \lambda L}{\lambda L}\ d\lambda,$$

$$(4.3)\qquad \psi^{(\nu)}(x,\ u) = \int_0^\infty A_\nu(\lambda)\ \exp\left\{-\sqrt{\lambda^2 - k_\nu^2}\ x\right\}$$

$$\cos \lambda u\ \frac{\sin \lambda L}{\lambda L}\ d\lambda,$$

which use the following relations:

$$(4.4)(a)\qquad -A_0(\lambda)\ \exp\{-\lambda h\}\lambda + B_0(\lambda)\ \exp\{-\lambda h\}\lambda$$

$$- C_0(\lambda)\ \exp\{\lambda h\}\lambda = -\mu_0 \frac{I}{\pi},$$

$$(b)\qquad A_0(\lambda)\ \exp\{-\lambda h\} - B_0(\lambda)\ \exp\{-\lambda h\}$$

$$- C_0(\lambda)\ \exp\{\lambda h\} = 0,$$

$$(c)\qquad B_0(\lambda)\mu_1\lambda - C_0(\lambda)\mu_1\lambda + A_1(\lambda)\sqrt{\lambda^2 - k_1^2}\,\mu_0 = 0,$$

$$(d)\qquad -B_0(\lambda) - C_0(\lambda) + A_1(\lambda) = 0,$$

$$(e)\qquad A_{\nu+1}(\lambda)\sqrt{\lambda^2 - k_{\nu+1}^2}\,\mu_\nu\ \exp\left\{-\sqrt{\lambda^2 - k_{\nu+1}^2}\ \nu\Delta x\right\}$$

$$- A_\nu(\lambda)\sqrt{\lambda^2 - k_{\nu+1}^2}\,\mu_{\nu+1}\ \exp\left\{\sqrt{\lambda^2 - k_\nu^2}\ \nu\Delta x\right\} = 0,$$

$$(f)\qquad A_{\nu+1}(\lambda)\ \exp\left\{-\sqrt{\lambda^2 - k_{\nu+1}^2}\ \nu\ \Delta x\right\}$$

$$- A_\nu(\lambda)\ \exp\left\{-\sqrt{\lambda^2 - k_\nu^2}\ \nu\Delta x\right\} = 0.$$

In order that (4.4)(e) and (4.4)(f) may have a nontrivial solution we require that:

$$(4.5)\qquad \frac{\mu_{\nu+1}}{\mu_\nu} - \frac{\sqrt{\lambda^2 - k_{\nu+1}^2}}{\sqrt{\lambda^2 - k_\nu^2}} = 0.$$

This can be satisfied approximately when the following hold:

$$(4.6)(a) \quad \frac{\sigma_{\nu+1}}{\sigma_{\nu}} \approx 1,$$

$$(b) \quad \frac{\mu_{\nu+1}}{\mu_{\nu}} \approx 1.$$

It is clear that (4.6)(a) and (b) can hold only if $\mu(x)$ and $\sigma(x)$ are slowly varying functions of x. In general (4.5) is not always satisfied, the forms (4.1) - (4.4) representing the approximate solution only.

From (4.4)(a) we derive

$$(4.7) \quad A_1(\nu) = \frac{\mu_0}{\pi} I \frac{\exp\{-\lambda h\}}{\lambda + (\mu_0/\mu_1)\sqrt{\lambda^2 - k_1^2}} .$$

Similarly from (4.5)(a) and (b) we derive

$$(4.8) \quad A_\nu(\lambda) = \frac{\mu_0}{\pi} I \frac{\exp\{-\lambda h\}}{\lambda + (\mu_0/\mu_1)\sqrt{\lambda^2 - k_1^2}} \exp\left\{-\sum_{j=0}^{\nu-1}\sqrt{\lambda^2 - k_j^2}\,\Delta x + \sqrt{\lambda^2 - k_\nu^2}\,(\nu-1)\Delta x\right\}.$$

Substitution of (4.8) into (4.3) yields:

$$(4.9) \quad \tilde{\psi}^{(\nu)}(x, u) = \frac{\mu_0}{\pi} I \int_0^\infty \frac{\exp\{-\lambda h\}}{\lambda + (\mu_0/\mu_1)\sqrt{\lambda^2 - k_1^2}} \exp\left\{-\sum_{j=1}^{\nu-1}\sqrt{\lambda^2 - k_j^2}\,\Delta x - \sqrt{\lambda^2 - k_\nu^2}\,\xi\right\} \cos\lambda u \,\frac{\sin\lambda L}{\lambda L}\, d\lambda,$$

with $x = (\nu - 1)\Delta x - \xi, \quad \Delta x \geqslant \xi \geqslant 0.$

For $\Delta x \longrightarrow 0$, (4.9) reduces to:

$$(4.10)\quad \widetilde{\psi}(x, u) = \frac{\mu_0}{\pi} I \int_0^{\infty} \exp\left\{-\int_0^x \sqrt{\lambda^2 - k^2(s)}\, ds\right\} \frac{\exp\{-\lambda h\}}{\lambda + (\mu_0/\mu(0))\sqrt{\lambda^2 - k^2(0)}} \cos \lambda u \frac{\sin \lambda L}{\lambda L}\, d\lambda,$$

and (4.6)(a) and (b) reduce to $\sigma' \approx 0$, $\mu' \approx 0$. Even though (4.10) is an integral over an infinite interval, we may approximate it arbitrarily closely by an integral over a finite interval. To see this we can calculate the following:

$$\sqrt{\lambda^2 - k^2(x)} = \frac{1}{\sqrt{2}}\left\{\frac{\omega\mu\sigma(x) + i\left(\sqrt{\lambda^4 + \omega^2\mu^2(x)\sigma^2(x)} - \lambda^2\right)}{\sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(x)\sigma^2(x)} - \lambda^2}}\right\},$$

$$\exp\left\{-\int_0^x \sqrt{\lambda^2 - k^2(s)}\, ds\right\} = \exp\left\{-\frac{1}{\sqrt{2}}\int_0^x \frac{\omega\,\mu(s)\sigma(s)\, ds}{\sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(s)\sigma^2(s)} - \lambda^2}}\right\}$$

$$\cos \frac{1}{\sqrt{2}} \int_0^x \sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(s)\sigma^2(s)} - \lambda^2}\, ds$$

$$- i \exp\left\{-\frac{1}{\sqrt{2}}\int_0^x \frac{\omega\,\mu(s)\sigma(s)}{\sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(s)\sigma^2(s)} - \lambda^2}}\, ds\right\}$$

$$\sin \frac{1}{\sqrt{2}} \int_0^x \sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(s)\sigma^2(s)} - \lambda^2}\, ds,$$

and

$$\lambda + \frac{\mu_0}{\mu(0)}\sqrt{\lambda^2 - k^2(0)} = \lambda + \frac{\mu_0}{\mu(0)}\frac{1}{\sqrt{2}}\frac{\omega\,\mu(0)\sigma(0)}{\sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(0)\sigma^2(0)} - \lambda^2}}$$

$$+ i\frac{\mu_0}{\mu(0)}\frac{1}{\sqrt{2}}\sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(0)\sigma^2(0)} - \lambda^2}.$$

We now set $K_1(x, \lambda) = \sqrt{\sqrt{\lambda^4 + \omega^2\mu^2(x)\sigma^2(x)} - \lambda^2}$,

$$\eta(x, \lambda) = \frac{\omega\mu(x)\sigma(x)}{K_1(x, \lambda)},$$

$$\alpha(0, \lambda) = \lambda + \frac{\mu_0}{\mu(0)}\frac{1}{\sqrt{2}}\,\eta(0, \lambda),$$

$$\beta(0, \lambda) = \frac{\mu_0}{\mu(0)}\frac{1}{\sqrt{2}}\,K_1(0, \lambda)$$

$$J_0(x, \lambda) = \int_0^x K_1(t, \lambda)\,dt,$$

$$\gamma(x, \lambda) = \cos\frac{1}{\sqrt{2}}\,J_0(x, \lambda),$$

$$\delta(x, \lambda) = -\sin\frac{1}{\sqrt{2}}\,J_0(x, \lambda), \quad \text{and}$$

$$J_1(x, \lambda) = \int_0^x \eta(t, \lambda)\,dt;$$

whereupon (4.10) becomes:

$$(4.11)\qquad \tilde{\psi}(x, u) = \frac{\mu_0}{\pi}\frac{I}{L}\int_0^\infty \frac{\sin\lambda L}{\lambda}\cos\lambda u$$

$$\exp\{-\lambda h\}\exp\left\{-\frac{1}{\sqrt{2}}\,J_1(x, \lambda)\right\}$$

$$\left[\frac{\alpha(0, \lambda)\,\gamma(x, \lambda) + \beta(0, \lambda)\,\delta(x, \lambda)}{\alpha^2(0, \lambda) + \beta^2(0, \lambda)}\right] d\lambda$$

$$+ \; i \frac{\mu_o}{\pi} \frac{I}{L} \int_0^\infty \frac{\sin \lambda L}{\lambda} \cos \lambda u$$

$$\exp\left\{-\lambda h\right\} \exp\left\{-\frac{1}{\sqrt{2}} \quad J_1(x, \lambda)\right\}$$

$$\left[\frac{\alpha(0, \lambda)\,\delta(x, \lambda) - \beta(0, \lambda)\,\gamma(x, \lambda)}{\alpha^2(0, \lambda) + \beta^2(0, \lambda)}\right] d\lambda .$$

If we assume that $\sigma(0) \geqslant (8\mu(0)/(\omega\mu_o^2))$ and denote the first and second integrands of (4.11) by $f_1(x, u; \lambda)$ and $f_2(x, u; \lambda)$, then we find that $f_1(x, u; \lambda)$ and $f_2(x, u; \lambda)$ are both dominated by $\exp\{-\lambda h\}$. We define:

(4.12) $$\widetilde{\psi}_s(x, u) = \frac{\mu_o}{\pi} \frac{I}{L} \int_0^s \left[f_1(x, u; \lambda) + i\, f_2(x, u; \lambda)\right] d\lambda$$

Then for all $\epsilon > 0$, we can approximate (4.11) by (4.12) with

$$s_\epsilon = \frac{1}{h} \ln \left\{\frac{\mu_o}{\pi} \frac{I}{L} \frac{\sqrt{2}}{\epsilon h}\right\}, \quad \text{i.e.}$$

$$\left|\widetilde{\psi}_{s_\epsilon}(x, u) - \widetilde{\psi}(x, u)\right| < \epsilon .$$

2. The Method of Integral Equations

From Lemma 4 of section 2 of Chapter 3, we have the existence of a solution $a_2(x, \lambda)$. We now wish to construct an approximation to this $a_2(x, \lambda)$ using the method of integral equations.

From (3.41)(b) we have:

$$(4.13) \qquad a_2(x, \lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; x\right\} + \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; x\right\} d_2(x, \lambda).$$

If we let $U(x, \lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; x\right\} d_2(x, \lambda)$, we obtain

$$(4.14) \qquad a_2(x, \lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; x\right\} + U(x, \lambda),$$

where $U(x, \lambda)$ satisfies the integral equation

$$(4.15) \qquad U(x, \lambda) = \frac{1}{2\sqrt{\lambda^2 - K_\infty^2}} \int_x^\infty \left[\exp\left\{\sqrt{\lambda^2 - K_\infty^2}\;(x-t)\right\} - \exp\left\{\sqrt{\lambda^2 - K_\infty^2}\;(t-x)\right\}\right]\left[K^2(t) - K_\infty^2\right] \left[U(t, \lambda) + \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; t\right\}\right] dt.$$

We again employ successive approximations to solve (4.15) by defining:

$$U_0(x, \lambda) = 0,$$

$$(4.16) \qquad U_n(x, \lambda) = \frac{1}{2\sqrt{\lambda^2 - K_\infty^2}} \int_x^\infty \left[\exp\left\{\sqrt{\lambda^2 - K_\infty^2}\;(x-t)\right\} - \exp\left\{\sqrt{\lambda^2 - K_\infty^2}\;(t-x)\right\}\right]\left(K^2(t) - K_\infty^2\right) \left[U_{n-1}(t, \lambda) + \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\; t\right\}\right] dt.$$

The same arguments as were used in the proof of Lemma 2 permit us to show that:

$$(4.17) \qquad \left| U_n(x, \lambda) - U_{n-1}(x, \lambda) \right| \leq \exp\left\{-\mathrm{Real}\sqrt{\lambda^2 - K_\infty^2}\; x\right\} \left[\frac{\int_0^\infty \left|K^2(t) - K_\infty^2\right| dt}{\mathrm{Real}\sqrt{\lambda^2 - K_\infty^2}}\right]^n \Big/ n!$$

and

$$(4.18) \qquad U(x, \lambda) = \sum_{j=0}^{\infty} \left\{ U_{j+1}(x, \lambda) - U_j(x, \lambda) \right\},$$

that is, the sequence $\{U_n(x, \lambda)\}$ of iterates converges uniformly to a limit function $U(x, \lambda)$. From (4.17) and (4.18) we can derive the following estimate:

$$(4.19) \qquad \left| U(x, \lambda) - U_n(x, \lambda) \right| \leq \exp\left\{-\text{Real}\sqrt{\lambda^2 - K_\infty^2}\, x\right\} \sum_{j=n}^{\infty} \left[\frac{\int_x^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right]^j \Big/ j!$$

$$\leq \sum_{j=n}^{\infty} \left[\frac{\int_x^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right]^j \Big/ j!$$

$$\leq M \exp\left\{ \frac{\int_\xi^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right\} \left[\frac{\int_x^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right]^{n+1} \Big/ (n+1)!$$

$$\leq M \exp\left\{ \frac{\int_0^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right\} \left[\frac{\int_0^\infty \left|K^2(t) - K_\infty^2\right| dt}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \right]^{n+1} \Big/ n+1\ !$$

where $x < \xi < \infty$.

From (3.39) we have that

$$\frac{1}{\text{Real}\sqrt{\lambda^2 - K_\infty^2}} \leq \sqrt{\frac{2}{\omega\mu_\infty\sigma_\infty}} \qquad \text{for all } \lambda \geqslant 0.$$

Hence we have from (4.19)

$$(4.20) \qquad \left|U(x,\lambda) - U_n(x,\lambda)\right| \leq M\exp\left\{\sqrt{\frac{2}{\omega\mu_\infty\sigma_\infty}} \int_0^\infty \left|K^2(t) - K_\infty^2\right| dt\right\} \left[\sqrt{\frac{2}{\omega\mu_\infty\sigma_\infty}} \int_0^\infty \left|K^2(t) - K_\infty^2\right| dt\right]^{n+1} \Big/ (n+1)!$$

We now define $a^{(n)}(x,\lambda) = \exp\left\{-\sqrt{\lambda^2 - K_\infty^2}\, x\right\} + U_n(x,\lambda)$ so that from (4.14) we have:

$$(4.21) \qquad \left|a_2(x,\lambda) - a^{(n)}(x,\lambda)\right| = \left|U(x,\lambda) - U_n(x,\lambda)\right| \leq M\exp\left\{\sqrt{\frac{2}{\omega\mu_\infty\sigma_\infty}} \int_0^\infty \left|K^2(t) - K_\infty^2\right| dt\right\} \left[\sqrt{\frac{2}{\omega\mu_\infty\sigma_\infty}} \int_0^\infty \left|K^2(t) - K_\infty^2\right| dt\right]^{n+1} \Big/ (n+1)!$$

We note that the right side of (4.21) may be made arbitrarily small independently of x and λ if we choose n sufficiently large.

3. The Perturbation Method

Using this method we approach the problem of constructing approximations to (3.34) subject to:

(4.22)(a) $$a(0, \lambda) = \frac{\mu_0 (I/\pi) \exp\{-\lambda h\}}{\lambda + (\mu_0/\mu(0))\sqrt{\lambda^2 - K^2(0)}},$$

and

(4.22)(b) $$a'(0, \lambda) = -\sqrt{\lambda^2 - K^2(0)}\ a(0, \lambda),$$

by assuming that $a(x, \lambda)$ can be represented as:

(4.23)(a) $$a(x, \lambda) = a_0(x, \lambda) + \eta(x, \lambda)$$

subject to

(4.23)(b) $$\varepsilon(x)\,\eta(x, \lambda) \simeq 0;$$

where $\varepsilon(x)$ is defined by $K^2(x) = K^2(0) + \varepsilon(x)$ and $a_0(x, \lambda)$ is a solution of

(4.24) $$a_0''(x, \lambda) + [K^2(0) - \lambda^2]\, a_0(x, \lambda) = 0$$

subject to the initial conditions (4.22)(a) and (b).

Substitution of (4.23)(a) into (3.34) and (4.22)(a) yields:

(4.25)(a) $$\eta''(x, \lambda) + a_0''(x, \lambda) + [K^2(0) - \lambda^2 + \varepsilon(x)][a_0(x, \lambda) + \eta(x, \lambda)] = 0$$

(b) $$\eta(0, \lambda) = 0, \qquad \eta'(0, \lambda) = 0.$$

Using the assumptions (4.23)(b) we obtain

$$\eta''(x, \lambda) + [K^2(0) - \lambda^2]\, \eta(x, \lambda) \cong -\varepsilon(x)\, a_0(x, \lambda).$$

This suggests use of the approximations $\eta(x, \lambda) \cong \tilde{\eta}(x, \lambda)$ where $\tilde{\eta}(x, \lambda)$ is defined by

(4.26) $\quad \tilde{\eta}''(x, \lambda) + [K^2(0) - \lambda^2]\tilde{\eta}(x, \lambda) = -\mathcal{E}(x)\, a_o(x, \lambda)$

subject to (4.25)(b). The solution is found to be

$$(4.27) \quad \tilde{\eta}(x, \lambda) = a_o(0, \lambda) \frac{\int_o^x - \mathcal{E}(s)\, ds}{2\sqrt{\lambda^2 - K^2(s)}} \exp\left\{-\sqrt{\lambda^2 - K^2(0)}\; x\right\}$$

$$-a_o(0, \lambda) \frac{\int_o^x - \mathcal{E}(s) \exp\left\{-2\sqrt{\lambda^2 - K^2(0)}\, s\right\} ds}{2\sqrt{\lambda^2 - K^2(0)}} \exp\left\{\sqrt{\lambda^2 - K^2(0)}\; x\right\}.$$

Replacing $\eta(x, \lambda)$ by $\tilde{\eta}(x, \lambda)$ in (4.23)(b) yields the proposed approximation to the equation, which is (3.34) subject to (4.22)(a) and (b).

4. Some Error Estimates.

From (3.3) we note that the solution $\psi(x, u)$ is represented by an improper integral with the integrand being the solution of a second order ordinary differential equation subject to two-point boundary conditions. In practice, we can not obtain the exact solution of the differential equation using a finite number of steps, while for computational purposes we usually replace the improper integral by an integral taken over a finite interval. In this section we give some error bounds based upon the above considerations.

(a) The Error due to Changing the Limits of Integration

We have seen that:

(3.3) $$\psi(x, u) = \int_0^\infty A(x, \lambda) \cos \lambda u \frac{\sin \lambda L}{\lambda L} d\lambda.$$

If we now define:

(4.28) $$\psi_\theta(x, u) = \int_0^\theta A(x, \lambda) \cos \lambda u \frac{\sin \lambda L}{\lambda L} d\lambda,$$

we obtain

(4.29) $$\left|\psi(x, u) - \psi_\theta(x, u)\right| \leq \int_\theta^\infty \left|A(x, \lambda) \cos \lambda u \frac{\sin \lambda L}{\lambda L}\right| d\lambda.$$

We have already shown in Lemma 6 and in Lemma 8 that $|A(x, \lambda)|$ is dominated by $\exp\{-\lambda h\}$ for large values of λ. We may assume without loss of generality that θ is sufficiently large that this domination holds and we then have

(4.30) $$\left|\psi(x, u) - \psi_\theta(x, u)\right| \leq \int_\theta^\infty \exp\{-\lambda h\} d\lambda = \frac{\exp\{-h\theta\}}{h} \leq \varepsilon_1,$$

providing that

(4.31) $$\theta = \frac{1}{h} \ln \frac{1}{h\varepsilon_1}.$$

(b) The Error due to Approximating the Integrand

If we replace $A(x, \lambda)$ by an approximating function $\bar{A}(x, \lambda)$, where $\bar{A}(x, \lambda)$ satisfies:

(4.32) $$\left| A(x, \lambda) - \bar{A}(x, \lambda) \right| < \epsilon_2$$

and if we define

(4.33) $$\tilde{\psi}_\theta(x, u) = \int_0^\theta \bar{A}(x, \lambda) \cos \lambda u \frac{\sin \lambda L}{\lambda L} \, d\lambda,$$

we obtain

(4.34) $$\begin{aligned} &\left| \psi_\theta(x, u) - \tilde{\psi}_\theta(x, u) \right| \\ &= \left| \int_0^\theta A(x, \lambda) - \bar{A}(x, \lambda) \cos \lambda u \frac{\sin \lambda L}{\lambda L} \, d\lambda \right| . \\ &\leq \int_0^1 \left| A(x, \lambda) - \bar{A}(x, \lambda) \right| \left| \cos \lambda u \frac{\sin \lambda L}{\lambda L} \right| d\lambda \\ &\quad + \int_1^\theta \left| A(x, \lambda) - \bar{A}(x, \lambda) \right| \left| \cos \lambda u \frac{\sin \lambda L}{\lambda L} \right| d\lambda \\ &\leq \int_0^1 \left| A(x, \lambda) - \bar{A}(x, \lambda) \right| d\lambda \\ &\quad + \int_1^\theta \left| A(x, \lambda) - \bar{A}(x, \lambda) \right| \frac{1}{L} \frac{d\lambda}{\lambda} \\ &\leq \mathcal{E}_2 \left[1 + \frac{1}{L} \ln \theta \right] . \end{aligned}$$

By combining (4.30) and (4.34) we finally obtain

$$(4.35) \qquad \left|\psi(x,\ u) - \widetilde{\psi}_{\theta}(x,\ u)\right| \le \left|\psi(x,\ u) - \psi_{\theta}(x,\ u)\right| + \left|\psi_{\theta}(x,\ u) - \widetilde{\psi}_{\theta}(x,\ u)\right| \le \varepsilon_1 + \varepsilon_2\left[1 + \frac{1}{L}\ln\theta\right].$$

This error estimate is composed of two parts; the first due to the use of a finite interval of integration and which may be made arbitrarily small; the second due to approximating the integrand. The latter obviously depends on the goodness of the approximation.

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CHAPTER 5

CONCLUSION

This study has been based upon a physical problem of considerable interest to industry and technology -- that of surface hardening of metallic components to improve their performance characteristics in various mechanisms. The particular application chosen is the surface hardening of a long circular rod by the use of induction heating, for which the mathematical formulation of the electromagnetic part of the problem has been chosen for study.

The case of the cylindrical rod has been formulated mathematically as a system of differential equations with coupled boundary conditions. A simplification in this problem has been made by approximating it by a problem in the half plane. The representation theorem for the solution of the latter problem is then stated and proved.

Being an applied problem, some methods for evaluating the solution have been provided. These are the layer method, the method of integral equations, and the perturbation method, together with some error estimates. It is hoped that this presentation will be useful not only for the present problem, but also as a method for attacking other problems in engineering and physics having similar formulations.

Few investigators are so blessed as to work on a problem to which they can supply all the solutions, and this problem likewise leaves questions unanswered. Some further topics which require investigation in order to further facilitate the more general adoption of the method of surface hardening by induction heating will be mentioned in closing.

The first physical problem that comes to mind is that of the end conditions, that is to say, the effects on the rod on entering and leaving the coil. The consideration of this problem will involve much more complicated statements of boundary conditions, but will be of considerable interest in applications.

Another physical problem concerns the hardening of workpieces of different shapes. At the outset it is obvious that the simplification to a plane problem is not likely possible for even simple shapes to be used. The addition of another dimension as well as the new boundary conditions will be the principle difficulties met in this kind of extension.

From the fact that induction hardening is reproducible in practice, we infer that the solution is unique. Another mathematical problem for investigation is that of showing that the solution for any of the problems formulated has a solution which is unique.

The final problem, and that of the greatest importance for engineering applications, is the inverse problem. That is to say, the given workpiece has a specified shape and is to be hardened in a desired pattern, which need not be uniform or even continuous. Many parameters are required to be defined for the set up of the process, some of which may permit alternative combinations which may involve optimizing trade offs in application. The shape of the coil to conform to the shape of the workpiece is the most obvious parameter which may be varied. Another involves the end effects, or the orientation of small workpieces for presentation to the process. Velocity of passage, and the frequency and intensity of the current in the coil are other parameters that come readily to mind as being available for variation.

While the inverse problem is probably beyond the possibility of mathematical analysis in the near future, its approximation empirically or heuristically is needed for the general adoption of the method in manufacturing technology.

It is hoped that these thoughts may encourage others to consider the further ramifications of the problem.

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