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EXACT SOLUTIONS OF STEADY PLANE POTENTIAL COMPRESSIBLE FLOWS - A NEW APPROACH

by

IQBAL HUSAIN

A Dissertation
submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy at
the University of Windsor

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1994

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ABSTRACT

The aim of this dissertation is the integration of the governing equations of motion for steady, two-dimensional potential gas flows. Although there has been an ongoing search for the solutions of these equations for over one hundred and fifty years, only a limited number of exact solutions in closed form exist prior to this thesis. The methods or processes that were employed in the past inevitably required dealing with a non-linear partial differential equation in the potential function with unmanageable boundary conditions or pre-deciding the type of gas that flows along a flow pattern. By adopting and pursuing a new approach, exact solutions in closed form are obtained in this thesis. This approach specifies *a priori* the form of the streamline pattern or a specific geometric pattern and determines the exact solution and the permissible gas for each chosen pattern. This approach also obtains exact solutions of the non-linear partial differential equation in the potential function even though it does not deal directly with this equation.

This dissertation contains two parts. The first part treats and develops investigations when the forms for the flow patterns are considered. Following the classification of all permissible flows for the chosen forms, exact solutions for these permissible flows are determined. The second part of this thesis is concerned with specified streamline patterns defined by $\text{Re}[f(z)] = \text{constant}$ or $\text{Im}[f(z)] = \text{constant}$ or a linear combination of $\text{Re}[f(z)]$ and $\text{Im}[f(z)]$ equal to any constant when $f(z)$ is a known analytic function of z.

This new approach involves transformations of independent variables only so that systems of ordinary differential equations and linear partial differential are

dealt with. New and existing exact solutions in closed form of these equations are obtained. However, in some cases, the transformation employed yielded nonlinear ordinary differential equations for which only particular solutions were obtained. In addition, equations of state corresponding to these solutions are also determined and analyzed. The exact solutions for incompressible, inviscid and irrotational flows can also be easily obtained by this new approach.

To my wife Fotini

and my beloved parents

ACKNOWLEDGEMENT

Last but not least, thanks are also extended to my friends Mr. S. Venkatasubramaniam, Dr. B. Vellapulai, Mrs. R. Gignac and Mr. E. Oku-Ukpong for all their help and encouragement.

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NOMENCLATURE

Symbol	Meaning
c	speed of sound
e	specific internal energy
M	Mach number
M_∞	free-stream Mach number
p	pressure
q	flow speed
$\hat{q}$	Bernoulli's constant
r	radius, polar coordinates
s	specific entropy
t	time
T	temperature
u, v	cartesian components of velocity
$\underset{\sim}{V}$	velocity vector
V_∞	free-stream speed
$w = \xi + i\eta$	complex function
x, y	cartesian coordinates
$z = x + iy$	complex variable
α	flow intensity
β	angle between $\underset{\sim}{V}$ and x-axis
γ	adiabatic constant

θ	angle between $\underset{\sim}{V}$ and x−axis
ρ	density
τ	specific volume
ϕ	curvilinear coordinate
Φ	velocity potential function
ψ	streamfunction

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION.

This dissertation treats and is concerned with the subject of gas dynamics. The study of the motion of compressible fluids, namely gases, is known as gas dynamics when the effects of density and temperature variations due to pressure changes cannot be neglected. The knowledge attained in this subject over the last one hundred and fifty years has been utilized in the developments of ballistics, gas turbines, combustion, rockets, jet engines, ram jets and high speed flights. Heat transfer at high speeds and blast-wave phenomena are also investigated using gas dynamics.This field is of such significance and importance in the development of sciences that applied mathematicians, theoretical physicists, chemical engineers, mechanical engineers and aeronautical engineers have all contributed extensively to its evolution and advancement.

The subject of gas dynamics includes and encompasses both theoretical and experimental aspects of this science. Progress in this subject has relied on both branches, each complementing the other. The theoretical branch of gas dynamics rests on a foundation containing concepts, definitions and the statements of the physical laws which have been verified by experiments. All theoretical investigations of the motion of gases must begin with the statements of the four basic physical laws governing such motions which are independent of the nature of the specific gas. These laws are:

(i)   the first law of thermodynamics,

(ii)   the second law of thermodynamics,

(iii) the principle of conservation of mass,

and

(iv)   Newton's principle of conservation of linear momentum.

Together with these fundamental laws, it is necessary in the analysis of the motion of a gas to include some relationship that specifies the particular type of gas under consideration. This relationship is between the thermodynamic properties of the gas and is called the caloric equation of state or state equation for the gas.

The study of gas flows can be divided into one-, two- and three- dimensional flows. Although one-dimensional flows have been extensively investigated, these studies are inadequate in the analysis of many real problems that involve two- and three-dimensional flows. On the other hand, the most general three-dimensional or two-dimensional flows - including shocks, heat transfer, friction and a gas with a complex equation of state - present mathematical difficulties that cannot be resolved by present-day analytical methods. Therefore, it is necessary to make simplifying assumptions on two- and three-dimensional flows in order to render them to available analytical techniques. Prandtl's concept of the boundary layer considerably simplifies the analysis of many physical flows since the flow is essentially inviscid and adiabatic outside the boundary layer [1934]. According to this concept, shearing stresses and heat transfer are significant only in a thin layer adjacent to solid boundaries, usually called the boundary layer. Therefore, if the boundary of a fictitious body is formed by moving the boundary of the actual body by an amount equal to the displacement thickness of the boundary layer at each point, then the flow outside the fictitious body is essentially frictionless and adiabatic. It is also known that such a flow which is initially uniform and parallel is irrotational outside

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the boundary layer in the absence of shocks. The flow within the boundary layer can be handled separately by methods that are well-suited for dealing with features in a boundary layer. An advantage of this approach is that the irrotational flow outside the boundary layer is independent of the boundary layer flow as far as the first order effects are concerned. The assumption of irrotationality results in significant simplification in the study of gas flows. Circulation is defined as the line integral of the velocity vector field along any closed curve in the motion of a fluid. If the flow is two-dimensional in the (x, y)−plane, then circulation per unit area is $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ when the velocity field $\underset{\sim}{V}(x, y, t) = (u(x, y, t), v(x, y, t))$. A plane flow is said to be irrotational if $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$ everywhere in the flow region.

Mathematical modeling of physical problems requires idealizing assumptions. A common assumption that the flow is steady is of considerable interest since it lends itself to analytical treatment and provides valuable insight into real and more complex flows. Steady flow is an idealizing assumption and we assume that one-, two- and three-dimensional steady fluid flows exist. Steady fluid flow is defined as a fluid motion in which the dynamic variables and the thermodynamic variables do not vary in time at each point, that is, flow variables depend only on the spatial coordinates and not on time. In such a flow, all the particles passing through a certain point have the same value for the dependent variables at this point and follow the same path, the streamline through that point. The flow is, therefore, covered by streamlines which do not change in time.

For a gas in motion, we have

$$de = dq - p\,d\tau$$

$$dq = T\,ds$$

as our first and second laws of thermodynamics applied to a unit mass of gas when e, q, p, τ, T and s are respectively the specific internal energy, the specific heat gained, the pressure function, the specific volume, the temperature function and

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the specific entropy of the medium. A gas which obeys Charles' law, $p\tau = RT$, is called an ideal gas. Gases obeying any other relationship between p, T and τ are called non-ideal gases. Van der Waal non-ideal gas, Clasius non-ideal gas, and Beatty-Bridgeman non-ideal gas are some of the well known non-ideal gases. Joule and Kelvin (1843) established experimentally that internal energy was a function of temperature only for an ideal gas. This result was also analytically established in later years and is well-documented in texts (c.f. Courant and Friedrichs[1948]). An ideal gas is called a perfect or a polytropic gas if its internal energy is proportional to temperature, a gas for which internal energy is not proportional to temperature, the name semi-perfect gas or an imperfect gas or non-polytropic gas is used. The assumption that a gas is a perfect gas facilitates and simplifies the mathematical analysis of many flow problems.

Viscous flows are said to be diabatic if heat transfer cannot be neglected. If viscosity is present but heat conduction or heat transfer is absent, then the flow is termed adiabatic. Assumption of absence of viscosity and heat conduction is tantamount to assuming that specific entropy is constant everywhere and corresponding compressible fluid flows are called isentropic flows. For isentropic flows, state equation for a gas has the form

$$p = f(\rho) \qquad \text{or} \qquad p = g(\tau)$$

where f and g are single-valued known functions of ρ and τ respectively. For every compressible isentropic fluid flow, the following hold true:

(i) $\quad c^2(\rho) = \dfrac{dp}{d\rho} = \dfrac{df}{d\rho} > 0$

(ii) $\quad \dfrac{d^2p}{d\rho^2} = \dfrac{d^2f}{d\rho^2} \geq 0$

(iii) $\quad \dfrac{dp}{d\tau} = \dfrac{dg}{d\tau} = -\rho^2 c^2 < 0$

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and

$$\text{(iv)} \qquad \frac{d^2 p}{d\tau^2} = \frac{d^2 g}{d\tau^2} > 0$$

The principle of conservation of mass, Newton's second law, irrotationality assumption and the assumption that pressure is a single-valued function of density only, yield a system of five equations for steady, continuous, frictionless, irrotational, isentropic plane flow. This system is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{(Conservation of Mass)}$$

$$\left. \begin{array}{l} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial p}{\partial x} = 0 \\[2ex] \rho\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) + \dfrac{\partial p}{\partial y} = 0 \end{array} \right\} \qquad \text{(Linear Momentum)}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{(Irrotationality)}$$

$$p = f(\rho) \qquad \text{(State)}$$

where $V = (u(x,y),\, v(x,y))$ is the velocity vector field, $p(x,y)$ the pressure function and $\rho(x,y)$ the fluid density function.

Eliminating the derivatives of ρ from the conservation of mass equation by application of the linear momentum equations, we find that $\phi(x,y)$ satisfies (c.f. Courant and Friedrichs [1948]):

$$\left[c^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} - 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \left[c^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} = 0$$

where the potential function $\phi(x,y)$ is defined by the irrotationality condition such that $V = \text{grad}\phi$. Since integration of the linear momentum equations after use of the irrotationality equation yields the Bernoulli's relation for a given pressure-density relation that determines ρ as a function of $u^2 + v^2 = \left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2$, it follows that the equation satisfied by $\phi(x,y)$ is a second order non-linear partial differential equation with unknown coefficients involving first order partial derivatives of ϕ.

Analysis of different flow phenomena by formulating and solving appropriate boundary value problems for this differential equation is the direct approach. If we wish to solve a real problem of flow around a body of known shape or a channel flow along with appropriate boundary conditions then we must find a function $\phi(x,y)$ which satisfies this nonlinear partial differential equation in velocity potential and the boundary conditions. This direct approach is a difficult task to accomplish because there does not exist any process in the armoury of our presently developed mathematics that can be used to achieve this objective. Experience, intuition and the capability to search without any process are the only tools available to accomplish this mission.

In the absence of the preferred direct approach, a second approach is available. This approach requires finding functions $\phi(x,y)$ satisfying the velocity potential equation. If the streamline pattern given by a solution function $\phi(x,y)$ is such that the shape of the body or the walls of the channel coincide with this pattern, then this solution function is applicable; otherwise, it is not. However, since there does not exist any method for finding solutions for the non-linear partial differential equation we are dealing with, this process depends upon chance and is accidental. Researchers realized and understood that obtaining solutions of the velocity potential equation by direct or indirect approach was mathematically non-achievable. In addition, due to the nonlinearity of this velocity potential equation, it is not possible to use superposition principles to construct desired solutions from any known simple solutions. This appreciation of the difficulty directed researchers to search for and develop other means.

Existence of profound mathematical and physical differences was noted between subsonic and supersonic flows. A flow in a region is said to be supersonic, sonic or subsonic according as the flow speed at every point in the region is greater than,

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equal to or less than the local speed of sound everywhere in the region. The velocity potential equation is a hyperbolic partial differential equation for a supersonic flow and is an elliptic equation for a subsonic flow. This understanding of the radical change occuring in the properties of the differential equation resulted in separate treatment and analysis of the two types of flows.

Prominent methods used are:

(a)   Method of Small Perturbation

(b)   Hodograph Method

(c)   Rayleigh-Janzen Method

(d)   Prandtl-Glauert Method

(e)   Method of Characteristics.

(a) **Method of Small Perturbation.** This method is used to treat the case when the disturbance of a rectilinear flow, due to the presence of a solid body, is small. Application of this approach linearizes the differential equation for the velocity potential. Due to this fact, the method is also called the linearized theory and is applied to both supersonic and subsonic flows to determine simple approximate solutions.

By the assumption of this method, the velocity vector field is

$$\underset{\sim}{V} = (u + V_\infty, \; v) = \underset{\sim}{\nabla}\Phi = \underset{\sim}{\nabla}(\phi + V_\infty x)$$

when the perturbation velocity components are very small when compared to the free-stream speed V_∞.

Making further assumptions that

$$\left[\frac{M_\infty^2}{1 - M_\infty^2}\right]\left[\frac{u}{V_\infty}\right] << 1 \quad \text{and} \quad M_\infty^2\left[\frac{v}{V_\infty}\right] << 1$$

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where M_∞ is the free-stream Mach number, it is found that $\phi(x,y)$ satisfies

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

This linear partial diffferential equation is elliptic for subsonic flows and is very similar to the potential function equation for incompressible fluids. For the case of supersonic flow, this partial differential equation is hyperbolic and

$$\phi(x,y) = f_1 \left(x + \sqrt{M_\infty^2 - 1}\, y \right) + f_2 \left(x - \sqrt{M_\infty^2 - 1}\, y \right)$$

is its general solution where f_1, f_2 are two arbitrary functions of their arguments.

The solutions of linearized flows yield approximate solutions that predict many important features of more complex flow patterns. Approximate solutions of simple flows are useful as a first approximation for practical applications. Practical problems such as flow past thin airfoils with sharp leading and trailing edges and the flow through turbines are studied using the method of small perturbation.

There are usually stagnation points either on the surface of the body or in the flow field and the disturbance is not small. For this reason,

    (i)   the application of the linear theory is questionable in the presence of stagnation points, and

    (ii)   this theory is valid and more appreciated for subsonic flows and not for supersonic or transonic flows.

Some of the outstanding works that used this theory are that of Prandtl [1934], Gothert [1946], Glauert [1927], Laitone [1951], Taylor [1932], Liepmann [1947] and Sauer [1947].

(b) <u>Hodograph Method</u>. The hodograph method is a powerful mathematical approach. Using the magnitude of velocity q and the inclination θ of velocity to a chosen axis, polar coordinates in the hodograph plane, as independent variables,

the equation of velocity potential reduces to a linear equation

$$c^2 \frac{\partial^2 \Phi}{\partial q^2} + \left(c^2 - q^2\right) \left(\frac{1}{q} \frac{\partial \Phi}{\partial q} + \frac{1}{q^2} \frac{\partial^2 \Phi}{\partial \theta^2}\right) = 0$$

with

$$x = \sin\theta \frac{\partial \Phi}{\partial q} + \frac{\cos\theta}{q} \frac{\partial \Phi}{\partial \theta}, \qquad y = \frac{\sin\theta}{q} \frac{\partial \Phi}{\partial \theta} - \cos\theta \frac{\partial \Phi}{\partial q}$$

as the transformation equations between the physical plane and the hodograph plane. One great advantage of the obtained linear equation is that we can form and generate complex solutions by superposition of elementary solutions. This does not mean that we are able to solve exactly the flow problem for given boundary conditions in the physical plane, but we obtain exact solutions in the hodograph plane which can be transferred to the physical plane to provide certain exact flow patterns. The main complication arises from the boundary conditions of a given problem. Given a body shape in the physical plane, it is rather difficult to obtain a solution of the equation in the transformed region in the hodograph plane having transformed boundaries which are totally dissimilar and extremely complicated in most cases.

Chaplygin [1944] approximated and simplified this equation to the Laplace equation. His assumption required that the ratio of specific heats of the gas be equal to -1. Since all real gases have their ratio of specific heats between 1 and 2, the value -1 seemed without practical significance. However, Demtchenko [1932], Busemann [1933], Tsien [1939] and von Karman [1941] clarified the meaning and physical relevance of this imaginary gas. This approximation is called the tangent gas approximation and it led to the Karman-Tsien method for subsonic flows.

The hodograph method is also applied without approximations for either subsonic or mixed subsonic-supersonic flows. Some early investigators doubted whether steady mixed flows could exist stably. The investigation of simple mixed flows to illustrate general features of mixed flows is a major contribution of this method.

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Some other well known researchers who have employed this method are Bergman [1945], Bers [1945], Cherry [1959] and Lighthill [1947].

(c) **The Rayleigh-Janzen Method.** Exact solutions are required for testing the validity of approximate, experimental and numerical methods. The Rayleigh-Janzen method is called a method for nearly exact solutions. This method has provided almost exact answers to some simple problems. However, these answers are important since they are a useful guide for more complex problems.

Assuming that the potential function $\phi(x, y)$ may be expressed as

$$\phi(x, y) = \phi_0(x, y) + M_\infty^2 \phi_1(x, y) + M_\infty^4 \phi_2(x, y) + \cdots$$

where $\phi(x, y)$ is a solution of the potential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial y} \right)^2 \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x \partial y} \right]$$

where c^2 for a polytropic gas is given by the Bernoulli's equation

$$\left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] + \frac{2c^2}{\gamma - 1} = V_\infty^2 + \frac{2}{\gamma - 1} c_\infty^2,$$

one solves the following:

(a) $\dfrac{\partial^2 \phi_0}{\partial x^2} + \dfrac{\partial^2 \phi_0}{\partial y^2} = 0$

with $\dfrac{\partial \phi_0}{\partial x} = V_\infty$, $\dfrac{\partial \phi_0}{\partial y} = 0$ at $x = y = \infty$ and $\mathrm{grad}\phi_0 \cdot \underset{\sim}{n} = 0$ on the solid boundary

(b) $\dfrac{\partial^2 \phi_1}{\partial x^2} + \dfrac{\partial^2 \phi_1}{\partial y^2} = \left(\dfrac{\partial \phi_0}{\partial x} \right)^2 \dfrac{\partial^2 \phi_0}{\partial x^2} + \left(\dfrac{\partial \phi_0}{\partial y} \right)^2 \dfrac{\partial^2 \phi_0}{\partial y^2} + 2 \dfrac{\partial \phi_0}{\partial x} \dfrac{\partial \phi_0}{\partial y} \dfrac{\partial^2 \phi_0}{\partial x \partial y}$

with $\dfrac{\partial \phi_1}{\partial x} = \dfrac{\partial \phi_1}{\partial y} = 0$ at $x = y = \infty$ and $\mathrm{grad}\phi \cdot \underset{\sim}{n} = 0$ on the solid boundary and so on.

Using the assumed form in the potential equation, retaining terms up to the order of M_∞^2 and solving the problem (a) and (b), one obtains

$$\phi(x, y) = \phi_0(x, y) + M_\infty^2 \phi_1(x, y)$$

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Classical potential theory or powerful tool of analytic functions of complex variables are used to find $\phi_0(x,y)$, and $\phi_1(x,y)$ is obtained by solving the Poisson equation. This method is a method of successive approximations, applicable to subsonic flows.

This method was given by Rayleigh [1916] and the details of carrying out this method were later improved by Kaplan [1939, 1942] and Imai [1941,1942]. Works of Hasimoto [1943], Lamla [1943], and Tamatiko and Umemoto [1941] are also excellent contributions.

(d) **The Prandlt-Glauert Method**. If there is a uniform parallel flow past a thin body, this method assumes that

$$\phi = V_\infty x + \phi_1(x,y) + t^2\phi_2(x,y) + \cdots$$

where t is a characteristic parameter of the shape of the body and V_∞ is the flow speed of uniform parallel flow away from the body. Using this assumed form in

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

where c^2 for a polytropic gas is given by the Bernoulli's equation

$$\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \frac{2c^2}{\gamma-1} = V_\infty^2 + \frac{2}{\gamma-1}c_\infty^2,$$

and knowing that our obtained equation is valid for all values of t, the coefficient of each power of t is equated to zero to obtain the following equations satisfied by $\phi_1(x,y)$, $\phi_2(x,y)$ etc:

$$(1 - M_\infty^2)\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0$$

$$(1 - M_\infty^2)\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2} = \frac{M_\infty^2}{U_\infty}\left[(\gamma+1)\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial x^2} + (\gamma-1)\frac{\partial\phi_1}{\partial x}\frac{\partial^2\phi_1}{\partial y^2} + 2\frac{\partial\phi_1}{\partial y}\frac{\partial^2\phi_1}{\partial x\,\partial y}\right]$$

and equations satisfied by $\phi_3(x,y)$, $\phi_4(x,y)$ and so on.

Kaplan [1943, 1944, 1946] and others used this method to analyzed some flow problems.

(e) The Method of Characteristics. Linearized approximation theory is valuable for supersonic flow past thin profiles. However, if more accurate calculations are necessary, the method of characteristics is used. Many problems involving supersonic flow past bodies have been solved with great success using this method. The method of characteristics is developed by rigorous formal mathematical methods which helps us appreciate the concept of characteristic curves.

A flow in a region in (x, y)-plane is a simple wave if its image in the (u, v)-plane is an arc of a characteristic. These flows were discovered by Prandtl [1934] and their theory was established by Meyer [1908]. Simple waves, flow around a bend or sharp corner, flow along a bump, flow in a duct and gas jets are studied using the theory of characteristics with numerical computations.

1.2 OUTLINE OF PRESENT WORK.

The purpose of this dissertation is the integration of Euler's equations for the motion of steady potential gas flow in two dimensions. Only a small number of exact solutions in closed form existed prior to this work, even though there has been an ongoing search for these solutions for the last one hundred and fifty years. The solutions that existed prior to this work are that of radial flows, vortex flows, spiral flows and Ringleb flows. The methods or processes that were used in the past almost always required

(a) dealing with a non-linear partial differential equation in the potential function or linear hodograph equations with unmanageable boundary conditions,

and

(b) pre-deciding the type of gas that flows along a flow pattern.

By adopting and pursuing a new approach, exact solutions in closed form are obtained in this dissertation. This approach specifies *a priori* the form of the streamline pattern or a specific streamline pattern only and determines the exact solution

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and the permissible gas for each chosen pattern. This approach also determines exact solutions of the non-linear partial differential equation in the potential function even though it does not deal directly with this equation. Previously obtained four solutions are also obtained by using this new approach. These rediscovered flows are a part of the chosen forms used for this approach.

This dissertation contains two parts. The first part treats and develops investigations when the forms for flow patterns are considered. Following the classification of all permissible flows for the chosen forms, exact solutions for these permissible flows are determined. All flows having streamlines of the form $y - f(x) =$ constant are studied in Chapter III. Chapters IV to VI treat the forms $\frac{r}{g(\theta)}$, $\theta - f(r)$ and $\frac{y}{g(x)}$ respectively where f and g are continuously differentiable functions.

The second part of this dissertation is organized in Chapters VII and VIII. Chapter VII is concerned with specific streamline patterns defined by $\text{Re}[f(z)] =$ constant or $\text{Im}[f(z)] =$ constant when $f(z)$ is a known analytic function of z. Chapter VIII deals with flows when a linear combination of $\text{Re}[f(z)]$ and $\text{Im}[f(z)]$ equal to any constant defines the streamlines.

A brief outline of this dissertation is as follows:

Chapter II contains some preliminary work. The governing equations are presented in section 2.2. In section 2.3, some results from differential geometry required later are summarized. A new formulation is outlined in section 2.4. Section 2.5 discusses briefly the theory of tangent gas.

Chapter III deals with exact solutions for a class of flows whose streamlines in the physical plane can be expressed in the form $y - f(x) =$ constant, so that $y - f(x) = \Gamma(\psi)$ when $\Gamma(\psi)$ is any function of the streamfunction ψ such that $\Gamma'(\psi) \neq 0$. The flow equations are transformed into the von Mises coordinate system in section 3.2. In section 3.3, we employ this system to classify all flows

of this form. Each of these classified flows is studied in sections 3.4, 3.5, 3.6 and

3.7. All permissible streamline patterns and their exact solutions are determined

in these sections. The state equations of the gases that allow these flows are also

found. In section 3.8, we pose and solve some boundary value problems using the

exact solutions obtained. The possible flows in this chapter are:

1. flows with $y - \frac{1}{c_3}\ln|c_5\sec(c_3 x + c_4)| = $ constant as the streamline pattern

having

$$\Phi(x,y) = c_{10} - \frac{1}{c_3\sqrt{2c_2}}\text{arcsec}\left[\frac{2c_5\sqrt{c_8}\exp(-c_3 y)}{\sin(c_3 x + c_4)}\right]$$

as a solution of the potential equation for a tangent gas.

2. flows with $y - f(x) = $ constant where

$$f(x) = \left(\frac{1-\lambda}{2B_6\lambda}\right)\left(\frac{1}{1+t^2}\right) - \frac{1}{2B_6}\ln\left(1+t^2\right),$$

$$x = \left(\frac{\lambda-1}{2B_6\lambda}\right)\left(\frac{t}{1+t^2}\right) - \left(\frac{1+\lambda}{2B_6\lambda}\right)\arctan(t), \qquad t = f'(x)$$

as the streamlines having

$$\Phi(x,y) = -\frac{B_5\,|B_9|^{1-\lambda}}{\lambda B_6 B_9}\frac{A(x)\left[1+A^2(x)\right]^{\frac{1}{2}(\lambda-1)}}{\exp\left[\lambda B_6\left(y - B(x)\right)\right]}$$

as a solution of the potential equation for a gas with the equation of state given by

$$p = \begin{cases} p_0 + B_5\ln\left[\sqrt{B_5\rho}\right]; & \lambda = -1 \\[2mm] p_0 - \left[\frac{\lambda B_5^{\frac{2}{1-\lambda}}}{1+\lambda}\right]\rho^{\frac{1+\lambda}{1-\lambda}}; & \lambda \neq -1 \end{cases}$$

3. flows with $y - f(x) = $ constant as the streamline pattern where $f(x)$ is any

monotonic function, $f'(x) > 0$ or $f'(x) < 0$, and this flow corresponds to the solution

$$\Phi(x,y) = \frac{D_2}{D_1}\left[\frac{f'(x)}{|f'(x)|}y + \int\frac{1}{|f'(x)|}dx\right]$$

of the potential equation for a tangent gas.

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4. flows with $y - E_1 x - E_2 = $ constant as the streamline pattern and this flow corresponds to the solution

$$\Phi(x,y) = \frac{|E_3|}{E_3\sqrt{1+E_1^2}}X(s) + E_5$$

of the potential equation for a tangent gas.

Chapter IV is devoted to the study of all flows whose streamlines are of the form $\frac{r}{g(\theta)} =$constant in polar coordinates. Here, the $(\theta, \psi)-$coordinates are employed to obtain all permissible flow patterns of this form and their exact integrals. In section 4.2, the governing flow equations are expressed in $(\theta, \psi)-$coordinates and classified. These classified flows are studied separately in sections 4.3, 4.4, 4.5 and 4.6. The possible flows in this chapter are:

1. flows with $r = $ constant as the streamlines having

$$\Phi(r,\theta) = c_3 - \frac{(c_0 c_1)}{c_0 c_1}F(\theta)$$

or

$$\Phi(r,\theta) = d_2 - d_1\theta$$

as solutions of the potential equation for a tangent gas and a polytropic gas respectively.

2. flows with $r\cos(\theta + b_1) = $ constant as the streamline pattern with the solution

$$\Phi(r,\theta) = b_4 - b_2 H(s)$$

of the potential equation for a tangent gas.

3. flows with $re^{-m\theta} = $ constant as the streamlines having

$$\Phi(r,\theta) = a_4 - \frac{|a_1| a_3 m}{a_1\sqrt{1+m^2}}r\exp\left(\frac{\theta}{m}\right)$$

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as a solution of the potential equation for a gas with the equation of state given by

$$
p = \begin{cases} p_0 - \dfrac{a_3^{m^2 n} \left(\sqrt{1+m^2}\right)^n}{|a_1|^n (1-m)^2} \rho^{\frac{m^2-1}{m^2+1}} \; ; & m \neq 1 \\[4mm] p_0 - \dfrac{a_3}{|a_1|\sqrt{1+m^2}} \ln\left[\dfrac{\sqrt{1+m^2}}{|a_1| a_3 \rho} \right] \; ; & m = 1 \end{cases}
$$

4. flows with $re^{-\theta} = $ constant as the streamline pattern and this flow corresponds to the solution

$$
\Phi(r,\theta) = -\frac{(1+\gamma)}{2^{\frac{\gamma-1}{2(\gamma+1)}}} \left(\frac{2A\gamma}{1-\gamma} \right)^{\frac{1}{1+\gamma}} \frac{|k_3|^{\frac{2}{\gamma+1}}}{k_3} r^{\frac{1}{\gamma+1}} \exp\left(\frac{1}{\gamma+1} \right) \theta
$$

of the potential equation for a polytropic gas.

5. flows with $r\cos^{-1}(\theta + a_{13}) = $ constant as the streamlines having

$$
\Phi(r,\theta) = a_4 \sqrt{a_{12}} \int \frac{r\cos(\theta + a_{13})\, d\theta - \sin(\theta + a_{13})\, dr}{a_{12} b_0^2 r^2 + a_{14} \sin^2(\theta + a_{13})}
$$

as a solution of the potential equation for a gas with state equation given by

$$
p = p_0 - \frac{1}{a_3} \frac{1}{\rho} + \frac{a_4}{a_3^2} \ln\rho
$$

6. flows with $r\exp(-b_1\theta) = $ constant as the streamlines having

$$
\Phi(r,\theta) = \frac{a_{10} b_1}{\sqrt{1+b_1^2}\,(1-a_9 b_3)} \left[\frac{\left|b_2^{b_3}\right| \sqrt{1+b_1^2}}{b_2\,|b_4|} \right]^{a_9} r^{1-a_9 b_3} \exp\{a_9 b_1 (b_3 - 1)\theta\}
$$

as a solution of the potential equation for a polytropic gas.

In Chapter V, the streamline pattern takes the form $\theta - f(r) = $ constant in polar coordinates. The (r,ψ)−net is chosen to analyze this class of flows. In section 5.2, the flow equations are given in (r,ψ)−coordinates and all flows of the chosen form are classified. In sections 5.3, 5.4 and 5.5, exact solutions of each of these classified flows are determined and the state equations of the gases that permit these flows are obtained. The possible flows in this chapter are:

1. flows with $\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\dfrac{1}{\sqrt{2c_1}r}\right) = $ constant as the streamlines

with

$$\Phi(r,\theta) = \frac{|c_3|}{c_3\sqrt{2c_1}} H(\eta)$$

as a solution of the potential equation for a tangent gas.

2. flows with $\theta - r = $ constant as the streamline pattern having

$$\Phi(r,\theta) = \frac{c_6}{d_1}\left(\theta - \frac{1}{r}\right)$$

as a solution of the potential equation for a gas whose state equation is given by

$$p = p_0 - \frac{1}{d_1^2}\frac{1}{\rho} - \frac{2}{3d_1^2 c_6^2}\frac{1}{\rho^3}$$

3. flows with $\theta - f(r) = $ constant as the streamline pattern where $f(r)$ is any

solution of equations (5.45) and (5.51) having

$$\Phi(r,\theta) = \frac{b_6\sqrt{b_7}}{\sqrt{2}} \int \frac{dr + r^2 f'(r)d\theta}{\sqrt{b_5 b_7 \left(1 + r^2 f'^2(r)\right) + b_0 b_5 r^2 \left[\exp\left(2b_7\left[f(r) - \theta + b_9\right] - 1\right)\right]}}$$

as a solution of the potential equation for a tangent gas.

4. flows with $\theta - f(r) = $ constant as the streamline pattern where $f(r) = \lambda_1 r + \lambda_2$

or given implicitly by equation (5.71) with

$$\Phi(r,\theta) = \frac{b_{12}\,|b_{14}|^{\frac{2}{\gamma+1}}}{b_{14}} \int \frac{\left(1 + r^2 f'^2(r)\right)^{\frac{-1}{\gamma+1}}}{r^{\frac{\gamma-1}{\gamma+1}}} \exp\left(\frac{b_{13}(\gamma-1)}{\gamma+1}\left[\theta - f(r)\right]\right)\left[dr + f'(r)d\theta\right]$$

as a solution of the potential equation for a polytropic gas.

5. flows with $\theta - f(r) = $ constant as streamlines where $f(r)$ is any solution of

equations (5.87) and (5.89) having

$$\Phi(r,\theta) = |b_{22}|^{\frac{1-\gamma}{1+\gamma}}\left[\frac{\gamma-1}{2\gamma A}\right]^{\frac{1}{1+\gamma}} \int \left[\frac{\sqrt{1 + r^2 f'^2(r)}}{r}\exp\left(b_{21}\left[\theta - f(r)\right]\right)\right]^{\frac{\gamma-1}{\gamma+1}}\left[\frac{dr + r^2 f'(r)d\theta}{\sqrt{1 + r^2 f'^2(r)}}\right]$$

as a solution of the potential equation for a polytropic gas.

Chapter VI deals with exact solutions for a class of flows whose streamlines in the (x,y)–plane can be expressed in the form $\dfrac{y}{g(x)}$ =constant. Again the von Mises coordinates are employed to obtain the exact solutions of all possible flows of this form. The equations of motion are transformed into the von Mises plane and the possible flows are classified in section 6.2. In section 6.3, the only permissible non-uniform flows are investigated. The possible flows in this chapter are:

1. flows with $y(d_1 x + d_2)^{-1}$ = constant as the streamline pattern and this flow corresponds to the solution

$$\Phi(x,y) = |d_3|^{\frac{1-\gamma}{1+\gamma}} \frac{1+\gamma}{d_1} \left[d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right]^{\frac{1}{\gamma+1}}$$

of the potential equation for a polytropic gas.

2. flows with $y\tan^{-1}\left(d_4\sqrt{d_6}\,[c_1 x + c_2] \right)$ = constant as the streamlines having

$$\Phi(x,y) = d_7 \int \frac{g(x)dx + yg'(x)dy}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x)y^2}}$$

where $g(x) = d_4\sqrt{d_6}\tan^{-1}\left(d_4\sqrt{d_6}\,[c_1 x + c_2] \right)$ as a solution of the potential equation for a tangent gas.

3. flows having $y\exp\left(-\frac{m}{k}x \right)$ = constant as the streamline pattern and this flow corresponds to the solution

$$\Phi(x,y) = \frac{1}{m} \int \frac{1}{\rho y}dx + \frac{1}{m} \int \frac{g'(x)}{\rho g(x)}dy$$

of the potential equation for a tangent gas where $g(x) = \lambda_3\exp\left(\frac{m}{k}x \right)$.

Chapter VII deals with flows whose streamline pattern is given by $\xi(x,y) = Re\left[f(z)\right]$ =constant or $\eta(x,y) = Im\left[f(z)\right]$ =constant for some chosen analytic function $f(z)$ where $z = x+iy$. The (ξ,ψ)–net is employed to obtain the exact solution of these flows. In section 7.2, the flow equations are given in (ξ,ψ)–coordinates. In section 7.3, several functions $f(z)$ are chosen and the exact solutions of the corresponding flows are obtained in some examples. The equation of state is also

determined in each example. In section 7.4, the remaining possible flows for the chosen form are studied for completeness. Some of the possible flows in this chapter are:

1. flows with $\frac{y}{x^2+y^2} = $ constant as the streamline pattern having

$$\Phi(x,y) = \frac{M_0}{\sqrt{N_0 C}}\arctan\left[\frac{\sqrt{N_0}\left(x^2+y^2\right)}{\sqrt{C}x}\right]$$

as a solution of the potential equation for a gas with equation of state given by

$$p = p_0 + \frac{M_0}{C^2}\ln\left(\frac{1}{\rho}\right) - \frac{1}{C^2}\frac{1}{\rho}$$

2. flows with $\frac{x}{x^2+y^2} = $ constant as the streamlines having

$$\Phi(x,y) = -\frac{M_1}{\sqrt{N_1 C}}\arctan\left\{\frac{\sqrt{N_1}\left(x^2+y^2\right)}{\sqrt{C}y}\right\}$$

as a solution of the potential equation for a gas whose state equation is

$$p = p_0 + \frac{M_1}{C^2}\ln\left(\frac{1}{\rho}\right) - \frac{1}{C^2}\frac{1}{\rho}$$

3. flows having $\sqrt{x^2+y^2}-x = $ constant as the streamline pattern with

$$\Phi(x,y) = \frac{K}{2}y\int\frac{\sqrt{L+Cx+C\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}-x}}dx$$

$$+\frac{K}{2}\int\frac{y\sqrt{\sqrt{x^2+y^2}-x}\sqrt{L+Cx+C\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}dy$$

as a solution of the potential equation for a gas having state equation given by

$$p = p_0 + \frac{1}{2}CK\ln\left[\frac{K\sqrt{C}\left(1+K\rho\right)}{\sqrt{1-K^2\rho^2}}\right] - \frac{1}{2}\frac{K^2C\rho}{\left(1-K^2\rho^2\right)}$$

4. flows with $\sqrt{x^2+y^2}+x = $ constant as the streamline pattern having

$$\Phi(x,y) = \frac{K_1}{2}\int\frac{y\sqrt{L_1-Cx+C\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}\sqrt{\sqrt{x^2+y^2}+x}}dx$$

$$+\frac{K_1}{2}\int\frac{\sqrt{\sqrt{x^2+y^2}+x}\sqrt{L_1-Cx+C\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}dy$$

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as a solution of the potential equation for a gas whose state equation is

$$p = p_0 + \frac{1}{2}CK_1 ln \left[\frac{K_1\sqrt{C}\,(1 + K_1\rho)}{\sqrt{1 - K_1^2\rho^2}} \right] - \frac{1}{2}\frac{K_1^2 C\rho}{(1 - K_1^2\rho^2)}$$

5. flows with $xy = $ constant as the streamlines with

$$\Phi(x,y) = \int \frac{C_1 x dx - C_1 y dy}{[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4]^{\frac{1}{4}}}$$

as a solution of the potential equation for a gas having the state equation

$$p = \frac{1}{\sqrt{C}} \int \frac{1}{\rho^2\sqrt{1 - C_1\rho^4}} d\rho$$

6. flows with $x^2 - y^2 = $ constant as the streamlines with

$$\Phi(x,y) = - \int \frac{D_1 y dx + D_1 x dy}{[4D_2 - 4Cx^2y^2]^{\frac{1}{4}}}$$

as a solution of the potential equation for a gas whose state equation is

$$p = \frac{1}{\sqrt{C}} \int \frac{1}{\rho^2\sqrt{1 - D_1\rho^4}} d\rho$$

Chapter VIII deals with a class of flows when the streamline pattern is of the form $C\xi(x,y) + D\eta(x,y) =$ constant where $\xi(x,y) = Re\,[f(z)]$, $\eta(x,y) = Im\,[f(z)]$, $f(z)$ is a chosen analytic function of $z = x + iy$ and $C \neq 0$, $D \neq 0$ are arbitrary constants. The (ξ, ψ)−curvilinear coordinate net is used to analyze these flows. Flow equations are recast in (ξ, ψ)−net in section 8.2. In section 8.3, several examples of one type of possible flows of the assumed form are presented. The remaining possible flows are investigated in section 8.4. Some of the possible flows in this chapter are:

1. flows with $\frac{1}{2}Cln\,(x^2 + y^2) + Dtan^{-1}\left(\frac{y}{x}\right) = $ constant as streamlines having

$$\Phi(x,y) = \frac{A_2}{\sqrt{C^2 + D^2}} \int \alpha^{1 - AA_1} \left[\frac{(Cy + Dx)\,dx + (Dy - Cx)\,dy}{\sqrt{x^2 + y^2}} \right]$$

as a solution of the potential equation for a polytropic gas.

2. flows with $\frac{Cx+Dy}{x^2+y^2}$ = constant as the streamline pattern with

$$\Phi(x,y) = \frac{A_5}{\sqrt{C^2+D^2}} \left\{ \int \left[\frac{D\left(x^2-y^2\right)-2Cxy}{\left(x^2+y^2\right)} \right] \left[\frac{\alpha}{\left[2+A_4\sqrt{C^2+D^2}\alpha\right]} \right] dx \right.$$

$$\left. + \int \left[\frac{C\left(x^2-y^2\right)+2Dxy}{\left(x^2+y^2\right)} \right] \left[\frac{\alpha}{\left[2+A_4\sqrt{C^2+D^2}\alpha\right]} \right] dy \right\}$$

as a solution of the potential equation for a gas having the state equation

$$p = p_0 - \frac{2A_5}{A_7^2} \left[ln\left(A_5\rho\right) + \frac{2}{A_5\rho} \right]$$

CHAPTER 2

PRELIMINARIES

2.1 INTRODUCTION.

The starting point for the study of motion of a compressible medium in the absence of discontinuities is the system of differential equations that expresses the principle of conservation of mass, Newton's law of conservation of momentum and the state equation expressing the condition that the flow is isentropic. This system

$$\frac{\partial \rho}{\partial t} + \text{div}\left(\rho \underset{\sim}{V}\right) = 0$$

$$\frac{\partial \underset{\sim}{V}}{\partial t} + \left(\underset{\sim}{V} \cdot \text{grad}\right) \underset{\sim}{V} + \frac{1}{\rho} \text{grad}\, p = \underset{\sim}{0}$$

$$p = R(\rho)$$

of five differential equations in three dynamic variables $\underset{\sim}{V} = (u(x,y,z,t),\ v(x,y,z,t),\ w(x,y,z,t))$ and two thermodynamic variables $p(x,y,z,t)$ and $\rho(x,y,z,t)$ governs the flow. These equations of gas dynamics admit an important integral, called Bernoulli's equation, given by

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left|\underset{\sim}{V}\right|^2 + i = \frac{1}{2}\hat{q}^2$$

for an unsteady flow for which the circulation around every curve is zero where $\hat{q}$, which may depend upon time, is constant throughout the flow and ϕ is the velocity potential. This flow, for which $\text{curl}\underset{\sim}{V}$ vanishes identically, is called an irrotational flow. Many flows start from rest and proceed such that one of the following holds true:

(i)   entropy remains unchanged,

(ii)   energy of fluid $e = e(\tau, s)$ is separable,

(iii)   the flow is so symmetrical that τ and s depend on a single

space coordinate,

where $\tau = \frac{1}{\rho}$. Under any one of the three conditions enumerated, circulation is conserved and the flow remains irrotational once they start from rest.

The general differential equations governing a compressible fluid flow in a three-dimensional space present insurmountable mathematical challenges which are beyond the present power of analysis. Fortunately, in many problems of great interest simplifications arise when the dependent variables are dependent upon two independent variables. Steady plane or two dimensional flow is one such case. Under our assumption the flow is characterized by the two components u, v of the velocity vector field $\underset{\sim}{V}$ as functions of the rectangular coordinates x, y in the plane; similarly ρ, p are functions of x and y alone.

2.2 EQUATIONS OF MOTION.

The steady two-dimensional irrotational isentropic flow of a compressible inviscid fluid is governed by the following system of equations(c.f. von Mises [1958]):

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad \text{(continuity)} \qquad (2.1)$$

$$\left.\begin{array}{c} \rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial p}{\partial x} = 0 \\[3mm] \rho\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) + \dfrac{\partial p}{\partial y} = 0 \end{array}\right\} \qquad \text{(linear momentum)} \qquad (2.2, 2.3)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \text{(irrotationality)} \qquad (2.4)$$

$$p = R(\rho) \qquad \text{(state)} \qquad (2.5)$$

where $u(x,y)$ and $v(x,y)$ are the horizontal and vertical components of velocity respectively, $\rho(x,y)$ is the density function, $p(x,y)$ is the pressure function and

$R(\rho)$ is some function of ρ. Equations (2.1) to (2.5) form a system of five equations in four unknown functions $u(x,y)$, $v(x,y)$, $p(x,y)$ and $\rho(x,y)$ when $R(\rho)$ is a known function. However, if $R(\rho)$ is not known or is not considered, then equations (2.1) to (2.4) are a system of four equations in four unknowns.

We transform the above equations into a curvilinear coordinate system (ϕ, ψ) where the curves $\psi(x,y) =$constant are the streamlines of a flow and the curves $\phi(x,y) =$constant are left arbitrary. Before proceeding any further, we present some results from differential geometry for a general curvilinear net (ϕ, ψ) in the next section which are required for the transformation.

2.3 SOME RESULTS FROM DIFFERENTIAL GEOMETRY.

Let

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi), \tag{2.6}$$

define a system of curvilinear coordinates (ϕ, ψ) in the physical plane such that

$$0 < |J| = \left| \frac{\partial(x,y)}{\partial(\phi,\psi)} \right| < \infty$$

and the squared element of arc length along any curve is

$$ds^2 = E(\phi,\psi)d\phi^2 + 2F(\phi,\psi)d\phi d\psi + G(\phi,\psi)d\psi^2 \tag{2.7}$$

where

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2, \quad F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}, \quad G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 \tag{2.8}$$

Equation (2.6) can be solved to determine ϕ, ψ as functions of x, y so that

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y}, \qquad \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y}, \qquad \frac{\partial y}{\partial \phi} = -J\frac{\partial \psi}{\partial x}, \qquad \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x} \tag{2.9}$$

and by (2.8),

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi} = \pm\sqrt{EG - F^2} = \pm W \quad \text{(say)} \tag{2.10}$$

Let the following conditions hold

i) $P(x,y)$ is any point on a curve $\psi(x,y) = c$, where c is a constant,

ii) variable ϕ is increasing on $\psi(x,y) = c$ in the direction in which x, y are increasing,

iii) $\beta(x,y)$ or $\beta(\phi,\psi)$ denotes the angle of inclination of the tangent to the coordinate line $\psi(x,y) = c$, directed in the sense of increasing ϕ.
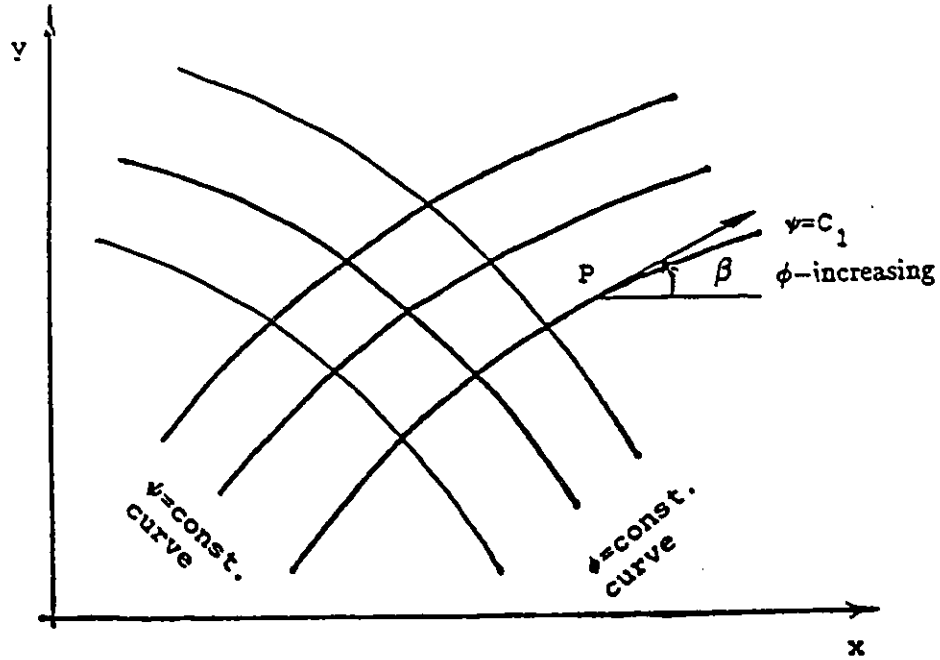
Figure 2.1: The (ϕ,ψ)—curvilinear net

The tangent vector to $\psi(x,y) = c$ at P is $\left(\dfrac{\partial x}{\partial \phi}, \dfrac{\partial y}{\partial \phi}\right)$ or $(x'(\phi), y'(\phi))$ and we have

$$\tan\beta = \frac{\dfrac{\partial y}{\partial \phi}}{\dfrac{\partial x}{\partial \phi}} \qquad \text{or} \qquad \frac{\partial x}{\partial \phi}\sin\beta = \frac{\partial y}{\partial \phi}\cos\beta, \qquad (2.11)$$

Using (2.11) in the first equation of (2.8), we get

$$\frac{\partial x}{\partial \phi} = \sqrt{E}\cos\beta, \qquad \frac{\partial y}{\partial \phi} = \sqrt{E}\sin\beta, \qquad (2.12)$$

The first two equations in (2.8) can be rewritten in the form

$$\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} = E,$$

$$\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi} = F.$$

Solving these two equations for $\dfrac{\partial x}{\partial \phi}$, we obtain

$$J\frac{\partial x}{\partial \phi} = E\frac{\partial y}{\partial \psi} - F\frac{\partial y}{\partial \phi},$$

or

$$E\frac{\partial y}{\partial \psi} = J\frac{\partial x}{\partial \phi} + F\frac{\partial y}{\partial \phi}. \tag{2.13}$$

Similarly, we find that

$$E\frac{\partial x}{\partial \psi} = F\frac{\partial x}{\partial \phi} - J\frac{\partial y}{\partial \phi}, \tag{2.14}$$

Using (2.12) in (2.13) and (2.14), we get

$$\begin{aligned}
\frac{\partial x}{\partial \psi} &= \frac{F}{\sqrt{E}}\cos\beta - \frac{J}{\sqrt{E}}\sin\beta \\
\frac{\partial y}{\partial \psi} &= \frac{J}{\sqrt{E}}\cos\beta + \frac{F}{\sqrt{E}}\sin\beta
\end{aligned} \tag{2.15}$$

Differentiating (2.10) with respect to ϕ, we have

$$\frac{\partial J}{\partial \phi} = \frac{G\dfrac{\partial E}{\partial \phi} + E\dfrac{\partial G}{\partial \phi} - 2F\dfrac{\partial F}{\partial \phi}}{2J} \tag{2.16}$$

The integrability conditions $\dfrac{\partial^2 x}{\partial \phi \partial \psi} = \dfrac{\partial^2 x}{\partial \psi \partial \phi}$, $\dfrac{\partial^2 y}{\partial \phi \partial \psi} = \dfrac{\partial^2 y}{\partial \psi \partial \phi}$ give

$$\begin{aligned}
\left[\frac{F}{\sqrt{E}}\sin\beta + \frac{J}{\sqrt{E}}\cos\beta\right]\frac{\partial \beta}{\partial \phi} - \sqrt{E}\sin\beta\frac{\partial \beta}{\partial \psi} &= \left[-\frac{1}{2\sqrt{E}}\frac{\partial E}{\partial \psi} - \frac{F}{2E\sqrt{E}}\frac{\partial E}{\partial \phi}\right. \\
\left. + \frac{1}{\sqrt{E}}\frac{\partial F}{\partial \phi}\right]\cos\beta + \left[\frac{J}{2E\sqrt{E}}\frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}}\frac{\partial J}{\partial \phi}\right]\sin\beta
\end{aligned}$$

and

$$\begin{aligned}
-\left[\frac{F}{\sqrt{E}}\cos\beta - \frac{J}{\sqrt{E}}\sin\beta\right]\frac{\partial \beta}{\partial \phi} + \sqrt{E}\cos\beta\frac{\partial \beta}{\partial \psi} &= \left[-\frac{1}{2\sqrt{E}}\frac{\partial E}{\partial \psi} - \frac{F}{2E\sqrt{E}}\frac{\partial E}{\partial \phi}\right. \\
\left. + \frac{1}{\sqrt{E}}\frac{\partial F}{\partial \phi}\right]\sin\beta - \left[\frac{J}{2E\sqrt{E}}\frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}}\frac{\partial J}{\partial \phi}\right]\cos\beta
\end{aligned}$$

Solving the above two equations for $\frac{\partial \beta}{\partial \phi}$, $\frac{\partial \beta}{\partial \psi}$ and using (2.16), we obtain

$$\frac{\partial \beta}{\partial \phi} = \frac{1}{2EJ}\left[-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\right]$$

$$\frac{\partial \beta}{\partial \psi} = \frac{1}{2EJ}\left[-F\frac{\partial E}{\partial \psi} + E\frac{\partial G}{\partial \phi}\right]$$

which can be written as

$$\frac{\partial \beta}{\partial \phi} = \frac{J}{E}\Gamma_{11}^2, \qquad \frac{\partial \beta}{\partial \psi} = \frac{J}{E}\Gamma_{12}^2 \tag{2.17}$$

where

$$\Gamma_{11}^2 = \frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\right]$$

$$\Gamma_{12}^2 = \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}\right] \tag{2.18}$$

are the Christoffel's symbols.

From (2.17), we see that the integrability condition $\frac{\partial^2 \beta}{\partial \phi \partial \psi} = \frac{\partial^2 \beta}{\partial \psi \partial \phi}$ implies that

$$\frac{\partial}{\partial \psi}\left(\frac{J}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{J}{E}\Gamma_{12}^2\right) = 0 \tag{2.19}$$

Equation (2.19) says that the Gaussian curvature

$$K = \frac{1}{W}\left[\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right)\right] = 0$$

and is referred to as the Gauss equation.

Conversely, if E, F and G are given as functions of ϕ, ψ such that the Gauss equation (2.19) is satisfied, then we show that the functions $x(\phi, \psi)$ and $y(\phi, \psi)$ can be obtained in terms of E, F and G where E, F and G satisfy (2.7).

Equation (2.19) implies the existence of $\beta = \beta(\phi, \psi)$ such that

$$\frac{\partial \beta}{\partial \phi} = \frac{J}{E}\Gamma_{11}^2, \qquad \frac{\partial \beta}{\partial \psi} = \frac{J}{E}\Gamma_{12}^2$$

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Therefore, β can be obtained from

$$\beta = \int \left(\frac{\partial \beta}{\partial \phi} d\phi + \frac{\partial \beta}{\partial \psi} d\psi \right) = \int \frac{J}{E} \left(\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi \right) \tag{2.20}$$

The functions $x(\phi, \psi)$ and $y(\phi, \psi)$ are given by

$$
\begin{aligned}
x &= \int \left\{ \left(\sqrt{E} \cos\beta \right) d\phi + \left(\frac{F}{\sqrt{E}} \cos\beta - \frac{J}{\sqrt{E}} \sin\beta \right) d\psi \right\} \\
y &= \int \left\{ \left(\sqrt{E} \sin\beta \right) d\phi + \left(\frac{F}{\sqrt{E}} \sin\beta + \frac{J}{\sqrt{E}} \cos\beta \right) d\psi \right\}
\end{aligned}
\tag{2.21}
$$

Introducing the complex variable $z = x + iy$, (2.21) can be written in a concise form as

$$z = \int \frac{1}{\sqrt{E}} e^{i\beta} \left\{ E d\phi + (F + iJ) d\psi \right\} \tag{2.22}$$

where β is given by (2.20).

Summing up, we have:

Theorem 2.1. *Three functions E, F, G of ϕ, ψ serve as coefficients in the first fundamental form*

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2$$

for a plane with a curvilinear coordinate system

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi)$$

if and only if they satisfy the Gauss equation

$$\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) = 0$$

If this condition is satisfied, then the functions $x(\phi, \psi)$ and $y(\phi, \psi)$ that define the curvilinear coordinate system are given in terms of E, F and G by (2.22).

From the relation

$$W = \sqrt{EG - F^2}$$

we find that

$$\frac{\partial}{\partial \phi}\left(\frac{E}{2W^2}\right) = \frac{1}{2W^2}\left\{\frac{\partial E}{\partial \phi} - \frac{E}{W^2}\left[E\frac{\partial G}{\partial \phi} + G\frac{\partial E}{\partial \phi} - 2F\frac{\partial F}{\partial \phi}\right]\right\}$$

$$= \frac{1}{W^2}\left[F\Gamma_{11}^2 - E\Gamma_{12}^2\right] \qquad (2.23)$$

$$\frac{\partial}{\partial \psi}\left(\frac{E}{2W^2}\right) = \frac{1}{W^2}\left[F\Gamma_{12}^2 - E\Gamma_{22}^2\right] \qquad (2.24)$$

$$\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right) = \frac{1}{W}\left(G\Gamma_{11}^2 - 2F\Gamma_{12}^2 + E\Gamma_{22}^2\right) \qquad (2.25)$$

where Γ_{22}^2 is given by

$$\Gamma_{22}^2 = \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \psi} - 2F\frac{\partial F}{\partial \psi} + F\frac{\partial G}{\partial \phi}\right] \qquad (2.26)$$

2.4 A NEW FORMULATION.

In this section, we transform the flow equations into a curvilinear coordinate system (ϕ, ψ) where the curves $\psi(x, y)$ =constant are the streamlines of the flow under consideration.

Equation of continuity implies the existence of a streamfunction $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -\rho v, \qquad \frac{\partial \psi}{\partial y} = \rho u \qquad (2.27)$$

We take two families of curves $\phi(x, y) =$ constant and $\psi(x, y) =$ constant such that a curvilinear net is defined in the physical plane when $\psi = \psi(x, y)$ is the streamfunction and $\phi = \phi(x, y)$ is an arbitrary function.

Having recorded the results from differential geometry in the previous se...ion, we transform the flow equations (2.1) to (2.4) into a new form.

Continuity equation

Employing (2.9) in (2.27), we have

$$\frac{\partial x}{\partial \phi} = J\rho u, \qquad \frac{\partial y}{\partial \phi} = J\rho v \qquad (2.28)$$

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Equations (2.28) in (ϕ, ψ) coordinates are equivalent to the continuity equation (2.1) in (x, y) coordinates.

We introduce polar coordinates q, θ in the hodograph plane by placing

$$u = q\cos\theta, \qquad v = q\sin\theta, \qquad q = \sqrt{u^2 + v^2}. \qquad (2.29)$$

Here θ is the direction of flow in the physical plane. Now equation (2.28) becomes

$$\frac{\partial x}{\partial \phi} = J\rho q \cos\theta, \qquad \frac{\partial y}{\partial \phi} = J\rho q \sin\theta \qquad (2.30)$$

When equations (2.30) are compared with (2.12) two possibilities arise, namely

$$\text{i)} \quad \theta = \beta, \qquad J\rho q = \sqrt{E}, \qquad J > 0$$

$$\text{ii)} \quad \theta = \beta + \pi, \qquad J\rho q = -\sqrt{E}, \qquad J < 0$$

In i) the fluid flows towards higher parameter values of ϕ and in ii) the fluid flows towards lower parameter values of ϕ. In either case, from (2.10), taking $J = W$ for $\theta = \beta$ and $J = -W$ for $\theta = \beta + \pi$, we have

$$\rho q = \frac{\sqrt{E}}{W} \qquad (2.31)$$

for fluid flowing in either direction.

Conversely, multiplying (2.31) by $\cos\beta$ and $\sin\beta$ and using (2.12), we have

$$\frac{\partial x}{\partial \phi} = W\rho q \cos\beta, \qquad \frac{\partial y}{\partial \phi} = W\rho q \sin\beta$$

Taking $\beta = \theta$ and $W = J$ or $\beta = \theta - \pi$ and $W = -J$ according as fluid flows along a streamline in the direction of increasing ϕ or decreasing ϕ, we get

$$\frac{1}{J}\frac{\partial x}{\partial \phi} = \rho u = \frac{\partial \psi}{\partial y}, \qquad \frac{1}{J}\frac{\partial y}{\partial \phi} = \rho v = -\frac{\partial \psi}{\partial x}$$

Employing the integrability condition $\dfrac{\partial^2 \psi}{\partial x \partial y} = \dfrac{\partial^2 \psi}{\partial y \partial x}$ yields the continuity equation. Therefore, the new equivalent form of the continuity equation (2.1) is given by

(2.31). This equivalent form was originally derived by Martin [1971] in a study of viscous incompressible flows.

Linear momentum equations

Taking ϕ, ψ as independent variables and using (2.4), equations (2.2) and (2.3) are

$$\frac{1}{2}\rho\left[\frac{\partial q^2}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial q^2}{\partial \psi}\frac{\partial \psi}{\partial x}\right] + \frac{\partial p}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial p}{\partial \psi}\frac{\partial \psi}{\partial x} = 0$$

$$\frac{1}{2}\rho\left[\frac{\partial q^2}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial q^2}{\partial \psi}\frac{\partial \psi}{\partial y}\right] + \frac{\partial p}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial p}{\partial \psi}\frac{\partial \psi}{\partial y} = 0$$

where $q^2 = u^2 + v^2$. Using the transformation equations (2.9), we obtain

$$\frac{1}{2}\rho\left[\frac{\partial q^2}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial q^2}{\partial \psi}\frac{\partial y}{\partial \phi}\right] + \frac{\partial p}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial p}{\partial \psi}\frac{\partial y}{\partial \phi} = 0$$

$$\frac{1}{2}\rho\left[-\frac{\partial q^2}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial q^2}{\partial \psi}\frac{\partial x}{\partial \phi}\right] - \frac{\partial p}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial p}{\partial \psi}\frac{\partial x}{\partial \phi} = 0$$

Multiplying these two equations by $\dfrac{\partial x}{\partial \phi}, \dfrac{\partial y}{\partial \phi}$ respectively and adding gives one equation; again, multiplying by $\dfrac{\partial x}{\partial \psi}, \dfrac{\partial y}{\partial \psi}$ respectively and adding gives the second equation of the following set of new equivalent form of the linear momentum equations:

$$\frac{1}{2}\rho\frac{\partial q^2}{\partial \phi} + \frac{\partial p}{\partial \phi} = 0$$

$$\frac{1}{2}\rho\frac{\partial q^2}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0$$

Irrotationality Condition

Employing ϕ, ψ as independent variables in (2.4) and using (2.9), we obtain

$$\frac{\partial v}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial v}{\partial \psi}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial u}{\partial \psi}\frac{\partial x}{\partial \phi} = 0$$

Using (2.29) in this equation, we get

$$\frac{\partial q}{\partial \phi}\left[\sin\theta\,\frac{\partial y}{\partial \psi} + \cos\theta\,\frac{\partial x}{\partial \psi}\right] - \frac{\partial q}{\partial \psi}\left[\sin\theta\,\frac{\partial y}{\partial \phi} + \cos\theta\,\frac{\partial x}{\partial \phi}\right]$$
$$+ q\frac{\partial \theta}{\partial \phi}\left[\cos\theta\,\frac{\partial y}{\partial \psi} - \sin\theta\,\frac{\partial x}{\partial \psi}\right] - q\frac{\partial \theta}{\partial \psi}\left[\cos\theta\,\frac{\partial y}{\partial \phi} - \sin\theta\,\frac{\partial x}{\partial \phi}\right] = 0 \tag{2.32}$$

Equation (2.32) with the use of (2.12) and (2.15) takes the form

$$\frac{\partial q}{\partial \phi}\left[\left(\frac{F}{\sqrt{E}}\sin\beta + \frac{J}{\sqrt{E}}\cos\beta\right)\sin\theta + \left(\frac{F}{\sqrt{E}}\cos\beta - \frac{J}{\sqrt{E}}\sin\beta\right)\cos\theta\right]$$

$$-\frac{\partial q}{\partial \psi}\left[\sqrt{E}\sin\beta\,\sin\theta + \sqrt{E}\cos\beta\,\cos\theta\right] + q\frac{\partial\theta}{\partial\phi}\left[\left(\frac{F}{\sqrt{E}}\sin\beta + \frac{J}{\sqrt{E}}\cos\beta\right)\cos\theta\right.$$

$$\left.-\left(\frac{F}{\sqrt{E}}\cos\beta - \frac{J}{\sqrt{E}}\sin\beta\right)\sin\theta\right] - q\frac{\partial\theta}{\partial\psi}\left\{\sqrt{E}\sin\beta\cos\theta - \sqrt{E}\cos\beta\sin\theta\right\} = 0$$

$$(2.33)$$

When fluid flows in the direction of increasing ϕ so that $\theta = \beta$ or when fluid flows in the direction of decreasing ϕ so that $\theta = \beta + \pi$, (2.33) yields

$$F\frac{\partial q}{\partial\phi} - E\frac{\partial q}{\partial\psi} + Jq\frac{\partial\beta}{\partial\phi} = 0 \qquad (2.34)$$

Using (2.17) in (2.34), we get

$$F\frac{\partial q}{\partial\phi} - E\frac{\partial q}{\partial\psi} + \frac{q}{2E}\left[2E\frac{\partial F}{\partial\phi} - F\frac{\partial E}{\partial\phi} - E\frac{\partial E}{\partial\psi}\right] = 0 \qquad (2.35)$$

Dividing (2.35) by $\sqrt{E}$ and simplifying, we get

$$\frac{\partial}{\partial\phi}\left(\frac{F}{\sqrt{E}}q\right) - \frac{\partial}{\partial\psi}\left(\sqrt{E}q\right) = 0$$

Summing up, we have

Theorem 2.2. *If the streamlines* $\psi(x,y)$ =*constant of a steady, plane, inviscid, isentropic, irrotational compressible fluid flow are taken as a set of coordinate curves in a curvilinear coordinate system* ϕ,ψ *in the physical plane, the system of flow equations (2.1) to (2.5) is replaced by the system*

$$\rho q = \frac{\sqrt{E}}{W} \qquad (2.36)$$

$$\frac{1}{2}\rho\frac{\partial}{\partial\phi}(q^2) + \frac{\partial p}{\partial\phi} = 0 \qquad (2.37)$$

$$\frac{1}{2}\rho\frac{\partial}{\partial\psi}(q^2) + \frac{\partial p}{\partial\psi} = 0 \qquad (2.38)$$

$$\frac{\partial}{\partial \phi}\left(\frac{F}{\sqrt{E}}q\right) - \frac{\partial}{\partial \psi}(\sqrt{E}q) = 0 \qquad (2.39)$$

$$\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right) = 0 \qquad (2.40)$$

$$p = R(\rho) \qquad (2.41)$$

of six equations in six unknowns E, F, G, ρ, q and p as functions of ϕ, ψ.

Having determined a solution of this system, the (ϕ, ψ)-plane is mapped onto the physical and the hodograph plane by

$$x + iy = \int \frac{e^{i\beta}}{\sqrt{E}}[Ed\phi + (F + iW)d\psi], \qquad \beta = \int \frac{W}{E}(\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi)$$

$$u + iv = \frac{\sqrt{E}}{\rho W}\exp(i\beta)$$

to achieve the complete solution for our flow given by

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi)$$

$$u = u(\phi, \psi), \qquad v = v(\phi, \psi), \qquad \rho = \rho(\phi, \psi), \qquad p = p(\phi, \psi)$$

2.5 TANGENT GAS.

The motion of gas in this study is assumed to be such that the gas viscosity and heat conduction are neglected. This assumption means that the changes of state are isentropic at every fluid particle. The general isentropic equation is given by

$$p = f(\rho)$$

The most investigated isentropic equation has the form

$$p = A\rho^\lambda + B \qquad (p \geq 0)$$

where A, λ and B are constants and this equation for an ideal gas is

$$p = A\rho^\gamma \qquad (p \geq 0) \qquad (2.42)$$

where A and $\gamma = \dfrac{c_p}{c_v}$ are constants. γ is the ratio of the specific heat of the gas at constant pressure to that at constant volume.

The complex phenomenon of gas flow due to the non-linearity of the flow equations receives additional complexity to its theory due to the non-linearity of the state equation (2.42). This fact forced approximations of this state equation in solving most problems. The isentrope of a gas having state equation

$$p = A\tau^{-\gamma} + B, \qquad \tau = \frac{1}{\rho} \qquad (2.43)$$

is approximated and linearized. This curved isentrope in (p, τ) –plane is approximated by the tangent line at some point $\left(p_a, \ \tau_a = \frac{1}{\rho_a}\right)$ on this ideal gas curve when this point corresponds to suitably averaged thermodynamic flow condition.

It is postulated that there exists an imaginary gas which has this tangent as its isentrope. The name for such an imaginary gas is Chaplygin-Karman-Tsien tangent gas [1904,1939,1941] or just a tangent gas. The isentrope of this imaginary gas, corresponding to an ideal gas, is shown in the Figure 2.2. A detailed discussion of the properties of this gas and their applications in the theory of subsonic plane flows have been given by Woods [1961].

The tangent gas will reasonably approximate an ideal gas provided either

(a) p and $\dfrac{1}{\rho}$ do not vary markedly from p_a and $\dfrac{1}{\rho_a}$, or

(b)   the point $\left(p_a, \ \dfrac{1}{\rho_a}\right)$ lies on the ideal gas curve at a point of small curvature.

Linear perturbation theory results from (a) when variations of both pressure and density are small. When (b) applies, we have either the case of high pressure and density or the case of low pressure and density. When the tangency point is a point of relatively high density, the tangent to the ideal gas curve or isentrope becomes almost vertical and so the flow approximates incompressible flow when large variations in pressure but negligible variations in density occur. This approximation is
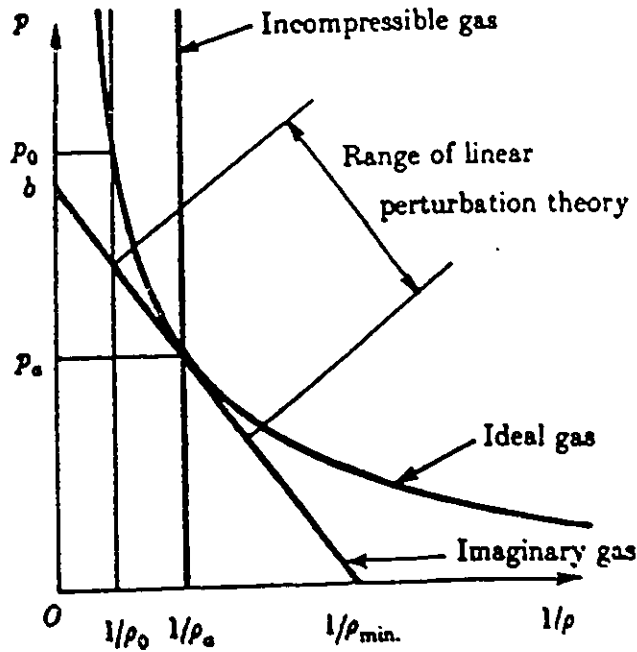
Figure 2.2   Pressure versus specific volume for ideal gas.

reasonably good provided low pressures are not attained in the flow. Therefore, the theory of tangent gas includes incompressible flow theory and linear perturbation theory.

The speed of sound c^2 at the tangency point $\left(p_a, \dfrac{1}{\rho_a} \right)$ for the equation of state (2.43) is

$$c^2 = \frac{dp}{d\rho} = -\frac{A}{\rho^2} \qquad \text{and} \qquad \frac{c^2}{c_a^2} = \frac{\rho_a^2}{\rho^2}$$

However, at the tangency point, the speed of sound for both the ideal gas and the tangent gas is the same and, therefore, from (2.42), we get

$$c_a^2 = \frac{\gamma p_a}{\rho_a} = \left(\frac{dp}{d\rho} \right)_{\text{at } \rho_a}$$

The equation of the tangent line to the ideal gas curve at the point $\left(p_a, \dfrac{1}{\rho_a} \right)$ is

$$p - p_a = \left(\frac{dp}{d\rho} \right)_a (\tau - \tau_a)$$

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Since

$$\left(\frac{dp}{d\tau}\right)_a = \left(\frac{dp}{d\rho}\frac{d\rho}{d\tau}\right)_a = c_a^2 \left(-\frac{1}{\tau_a^2}\right) = -\rho_a^2 c_a^2$$

the isentropic equation for the tangent gas is

$$p - p_a = \rho_a^2 c_a^2 \left(\tau_a - \tau\right) \tag{2.44}$$

Since

$$p = p_a + \rho_a^2 c_a^2 \left(\tau_a - \tau\right) \geq 0$$

it follows that

$$\tau \leq \frac{p_a}{\rho_a^2 c_a^2} + \tau_a = \tau_a \left[1 + \frac{p_a \tau_a}{c_a^2}\right] = \left[1 + \frac{1}{\gamma}\right] \tau_a$$

Therefore, the minimum value of ρ for a tangent gas is

$$\rho_{\min} = \frac{\gamma \rho_a}{\gamma + 1}$$

Integration of the two linear momentum equations (2.37) and (2.38) yield the Bernoulli's equation given by

$$\frac{1}{2}q^2 + \int \frac{dp}{\rho} = \text{constant}$$

For a tangent gas, this equation takes the form

$$q^2 - c^2 = q_\infty^2 - c_\infty^2$$

where q_∞ and c_∞ are the flow speed and Mach number at infinity. Some of the other researchers who have contributed to the theory of tangent gas are Lin [1946], Coburn [1944], Karpp [1984], Daripa [1986] and Sirovich [1986].

CHAPTER 3

STREAMLINE
PATTERN $y - f(x) =$ CONSTANT

3.1 INTRODUCTION.

This chapter deals with exact solutions for a class of flows whose streamlines in the (x,y)-plane can be expressed in the form $y - f(x) =$ constant, where $f(x)$ is a continuously differentiable function. For these flows, $y = f(x) + \Gamma(\psi)$ is a function of x, ψ along such streamlines, where $\Gamma(\psi)$ is some function of ψ. We choose straight lines $x =$ constant for the coordinate lines $\phi =$ constant in theorem 2.2 having flow equations in (ϕ, ψ)-coordinates. We employ von Mises coordinates (x, ψ) for our study of this class of flows.

Having made the assumption that our streamline pattern has the chosen form, we analyse the flow equations in von Mises coordinates and obtain further classification for these flows. This classification guides us to study the four different possible flows. We learn that there are three types of non-straight flows besides straight parallel flows that have the chosen flow pattern. Function $f(x)$ in $y - f(x) =$ constant is found to be

(a) a solution of $\left[1 + f'^2(x)\right] f'''(x) - 2f'(x)f''^2(x) = 0$ in the first type of non-straight flow,

(b) a solution of $\frac{d}{dx}\left[\frac{(1+\lambda f'^2(x))f''(x)}{\lambda(1+f'^2(x))^2}\right] = 0$ in the second type of non-straight flow where λ is any constant with the restriction that $\lambda \neq 0$ and $\lambda \neq 1$,

(c) any function $f(x)$ such that $f'(x) > 0$ and $f''(x) \neq 0$ in the third type of non-straight flow and

(d) a solution of $f''(x) = 0$ for straight parallel flows.

Fluid dynamics equations governing our flow patterns are studied in an *a priori* unknown system of non-orthogonal von Mises coordinates where one family of coordinate lines is assumed to coincide with the streamlines. Flow equations are completely integrated in this coordinate system. Exact solutions are determined and the state equations for the gas that permit these obtained solutions are found for each flow. Tangent gas equation introduced by Chaplygin [1904] and further developed by von Karman [1941] and Tsien [1939] show up in some of these solutions. It is found that the gas must be a tangent gas for two of the four non-straight flows and the straight parallel flows. It is also found that two permissible solutions are neither valid for any polytropic gas or its tangent approximation.

3.2 FLOW EQUATIONS IN VON MISES COORDINATES.

Steady, two-dimensional, irrotational, isentropic flow of a compressible inviscid fluid is governed by equations (2.1) to (2.5) in (x, y)- coordinates.

Transforming these flow equations into a curvilinear coordinate system (ϕ, ψ), where $\phi(x, y) = $ constant is an arbitrary family of curves and $\psi(x, y) = $constant are the streamlines, our flow is governed by a system of six equations given in theorem 2.2.

To determine continuously differentiable functions $f(x) \in C^3$ so that our compressible fluid flows along families of curves that can be expressed in the form $y - f(x) = $constant, we choose the von Mises coordinates (x, ψ) and have

$$y - f(x) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{3.1}$$

where $\Gamma(\psi)$ is an unknown function of ψ.

Taking (x,ψ)- coordinates so that the family of curves $\phi(x,y)$ =constant are the curves x = constant in theorem 2.2, as shown in Figure 3.1. we use (3.1) in equations (2.8), (2.10) and have

$$E(x,\psi) = 1 + f'^2(x), \quad G(x,\psi) = \Gamma'^2(\psi), \quad F(x,\psi) = f'(x)\Gamma'(\psi)$$

$$J(x,\psi) = \left| \frac{\partial(x,y)}{\partial(x,\psi)} \right| = \Gamma'(\psi), \quad W(x,\psi) = \sqrt{EG - F^2} = \sqrt{\Gamma'^2(\psi)} = |\Gamma'(\psi)| > 0 \tag{3.2}$$
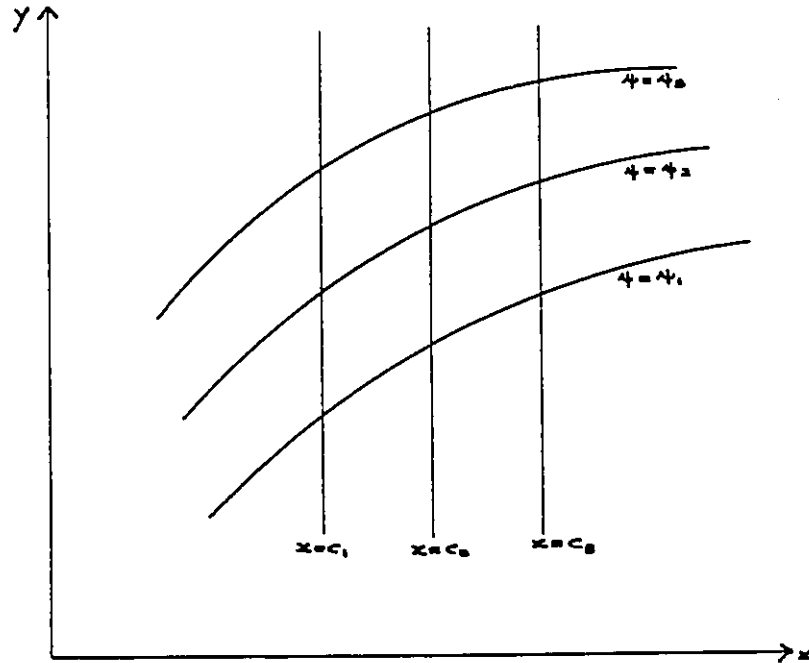
Figure 3.1.   von Mises coordinates.

From (3.2), it follows that $J > 0$ and, therefore, the fluid flows along a streamline in the direction of increasing x when $\Gamma'(\psi) > 0$. Also, $J < 0$ and, therefore, fluid flows in the direction of decreasing x along a streamline when $\Gamma'(\psi) < 0$.

From (3.2), we also have

$$F(x,\psi) = \pm\sqrt{(E-1)G}, \qquad W = \sqrt{G} \tag{3.3}$$

Writing equations of theorem 2.2 in von Mises coordinates to study flow along the curves $y - f(x) = $ constant, we use equations (3.2) and (3.3). Gauss equation is identically satisfied and the fluid flowing along a streamline is governed by the system

$$\rho q \frac{\partial q}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{3.4}$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \tag{3.5}$$

$$\frac{\partial}{\partial x} \left[\frac{f'(x)\Gamma'(\psi)}{\sqrt{1 + f'^2(x)}} q \right] - \frac{\partial}{\partial \psi} \left[\sqrt{1 + f'^2(x)} q \right] = 0 \tag{3.6}$$

$$\rho q = \frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} \tag{3.7}$$

$$p = R(\rho) \tag{2.5}$$

of five equations in five unknowns $\rho(x, \psi)$, $p(x, \psi)$, $q(x, \psi)$, $\Gamma(\psi)$ and $f(x)$.

In our search for all continuously differentiable functions $f(x)$ so that for each $f(x)$, curves $y - f(x) = $ constant define a streamline pattern for some compressible flow, we do not specify the gas we study and, therefore, do not have a state equation. Equations (3.4) to (3.7) are a system of four equations in five unknown functions and we solve this under-determined system. Given a solution of this system, state equation (2.5) is determined from $p = p(x, \psi)$, $\rho = \rho(x, \psi)$ for the gas that flows along the obtained streamline pattern.

Using equation (3.7), we eliminate ρ from equations (3.4), (3.5) and have

$$\frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} \frac{\partial q}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{3.8}$$

$$\frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \tag{3.9}$$

Employing the integrability condition $\frac{\partial^2 p}{\partial x \partial \psi} = \frac{\partial^2 p}{\partial \psi \partial x}$ to eliminate pressure p from (3.8) and (3.9), we obtain

$$\frac{\partial}{\partial \psi} \left[\frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} \right] \frac{\partial q}{\partial x} - \frac{\partial}{\partial x} \left[\frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} \right] \frac{\partial q}{\partial \psi} = 0 \tag{3.10}$$

Equations (3.6) and (3.10) are a system of two equations in three unknown functions $\Gamma(\psi)$, $f(x)$ and $q(x, \psi)$. Given a solution of this sytem, $p(x, \psi)$ is determined by the integration of equations (3.8), (3.9) and $\rho(x, \psi)$ is determined by using (3.7).

Defining

$$\rho q = \frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} = \alpha(x, \psi) \neq 0, \qquad (3.11)$$

system of equations (3.6), (3.10) is rewritten as

$$\frac{\partial}{\partial x} \left[\frac{f'(x)}{\alpha(x, \psi)} q \right] - \frac{\partial}{\partial \psi} \left[\alpha(x, \psi) \Gamma'(\psi) q \right] = 0 \qquad (3.12)$$

and

$$\left| \frac{\partial(\alpha, q)}{\partial(x, \psi)} \right| = \frac{\partial \alpha}{\partial x} \frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi} \frac{\partial q}{\partial x} = 0 \qquad (3.13)$$

where

$$\frac{\partial \alpha}{\partial x} = \frac{f'(x) f''(x)}{\alpha(x, \psi) \Gamma'^2(\psi)}, \quad \frac{\partial \alpha}{\partial \psi} = -\frac{\alpha(x, \psi) \Gamma''(\psi)}{\Gamma'(\psi)} \qquad (3.14)$$

3.3 CLASSIFICATION OF FLOWS.

Analysis of the system of equations (3.12) and (3.13) leads us to the classification of all flows with a streamline pattern of the form $y - f(x) =$ constant.

Equation (3.13) is identically satisfied only if any one of the following holds true:

(i)$q = q(\alpha)$, $\quad q'(\alpha) \neq 0$. This is the case when the curves of constant speed and the curves of constant flow intensity coincide.

(ii) $\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial \psi} = 0$. This is the case when the flow intensity is constant everywhere.

(iii) $\frac{\partial \alpha}{\partial \psi} = \frac{\partial q}{\partial \psi} = 0$. This is the case when the flow speed and the flow intensity are constant on each individual parallel straight line $x =$ constant.

(iv) $\frac{\partial q}{\partial x} = \frac{\partial q}{\partial \psi} = 0$. This is the case when the flow speed is constant everywhere.

(v) $\frac{\partial q}{\partial x} = \frac{\partial \alpha}{\partial x} = 0$. This is the case when the streamlines are the curves of constant flow speed and the curves of constant flow intensity.

We study these possibilities one by one.

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<u>Possibility (i)</u> $q = q(\alpha)$, $q'(\alpha) \neq 0$:

Taking $q = q(\alpha)$ in equation (3.12) and expanding, we obtain

$$\frac{f''(x)}{\alpha} q(\alpha) + f'(x) \left[\frac{q'(\alpha)}{\alpha} - \frac{q(\alpha)}{\alpha^2} \right] \frac{\partial \alpha}{\partial x} - \alpha q(\alpha) \Gamma''(\psi) - [q(\alpha) + \alpha q'(\alpha)] \Gamma'(\psi) \frac{\partial \alpha}{\partial \psi} = 0$$

Using (3.14), we eliminate $\frac{\partial \alpha}{\partial x}$ and $\frac{\partial \alpha}{\partial \psi}$ from this equation and obtain

$$\left[\frac{q(\alpha)}{\alpha} + \left(\frac{f'^2(x)}{\alpha^3 \Gamma'^2(\psi)} \right) \{\alpha q'(\alpha) - q(\alpha)\} \right] f''(x) + \alpha^2 q'(\alpha) \Gamma''(\psi) = 0$$

Eliminating $f'^2(x)$ from this equation by using $\alpha^2 \Gamma'^2(\psi) - 1 = f'^2(x)$ given by (3.11)

and multiplying by $\alpha^3 \Gamma'^2(\psi)$, we have

$$\left[q(\alpha) - \alpha q'(\alpha) + \alpha^3 q'(\alpha) \Gamma'^2(\psi) \right] f''(x) + \alpha^5 q'(\alpha) \Gamma'^2(\psi) \Gamma''(\psi) = 0 \qquad (3.15)$$

This equation replaces (3.12) for the case when $q = q(\alpha)$.

Since $f''(x) = 0$ in (3.15) implies that $\Gamma''(\psi) = 0$ and equations (3.14) give $\dfrac{\partial \alpha(x, \psi)}{\partial x} = \dfrac{\partial \alpha(x, \psi)}{\partial \psi} = 0$ if $f''(x) = \Gamma''(\psi) = 0$, it follows that $\alpha(x, \psi) =$ constant when $q = q(\alpha)$ and $f''(x) = 0$. Using $\alpha =$ constant in $q = q(\alpha)$ and $\rho q = \alpha$ given by equation (3.11), we note that $\rho =$ constant. Therefore, $f''(x) \neq 0$ in equation (3.15) when $q = q(\alpha)$ for our study of compressible flows and we have either

(a) non-straight flows (that is, $f''(x) \neq 0$) with

$$\Gamma''(\psi) = 0, \qquad q(\alpha) - \alpha q'(\alpha) + \alpha^3 q'(\alpha) \Gamma'^2(\psi) = 0 \qquad (3.16)$$

or

(b) non-straight flows with $\Gamma''(\psi) \neq 0$ when $q = q(\alpha)$.

We now further analyse non-straight flows with $\Gamma''(\psi) \neq 0$.

Since $q(\alpha) - \alpha q'(\alpha) = 0$ implies that q is proportional to α and we know from equation (3.11) that $\rho q = \alpha$, it follows that $q(\alpha) - \alpha q'(\alpha) \neq 0$ in (3.15) for the study of compressible flows and (3.15) may also be written as

$$\Gamma'^2(\psi) = \frac{[\alpha q'(\alpha) - q(\alpha)] f''(x)}{\alpha^3 q'(\alpha) f''(x) + \alpha^5 q'(\alpha) \Gamma''(\psi)} \qquad (3.17)$$

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when $f''(x) \neq 0$ and $\Gamma''(\psi) \neq 0$.

Differentiating (3.17) with respect to x, using (3.14), dividing by $f''(x) \neq 0$ and simplifying, we obtain

$$A_1(\alpha)f'(x)f'''^3(x) + \left[A_2(\alpha)\Gamma'^2(\psi)f'''(x) + A_3(\alpha)f'(x)f''^2(x)\right]\Gamma''(\psi) = 0 \quad (3.18)$$

where

$$A_1(\alpha) = 3q(\alpha)q'(\alpha) + \alpha q(\alpha)q''(\alpha) - 3\alpha q'^2(\alpha), \qquad (3.19)$$

$$A_2(\alpha) = \alpha^4 \left[\alpha q'^2(\alpha) - q(\alpha)q'(\alpha)\right] \qquad (3.20)$$

and

$$A_3(\alpha) = \alpha^2 \left[\alpha q(\alpha)q''(\alpha) - 5\alpha q'^2(\alpha) + 5q(\alpha)q'(\alpha)\right] \qquad (3.21)$$

Equation (3.18) requires that we classify further our non-straight flows with $\Gamma''(\psi) \neq 0$. We therefore investigate

(b') non-straight flows with $\Gamma''(\psi) \neq 0$ such that

$$A_1(\alpha) = 0 \quad \text{and} \quad A_2(\alpha)\Gamma'^2(\psi)f'''(x) + A_3(\alpha)f'(x)f''^2(x) = 0$$

and

(b'') non-straight flows with $\Gamma''(\psi) \neq 0$, such that

$$A_1(\alpha) \neq 0 \quad \text{and} \quad \Gamma''(\psi) = \frac{-A_1(\alpha)f'(x)f'''^3(x)}{A_2(\alpha)\Gamma'^2(\psi)f'''(x) + A_3(\alpha)f'(x)f''^2(x)} \qquad (3.22)$$

<u>Possibility (ii)</u> $\dfrac{\partial \alpha}{\partial x} = \dfrac{\partial \alpha}{\partial \psi} = 0$:

Using $\dfrac{\partial \alpha}{\partial x} = \dfrac{\partial \alpha}{\partial \psi} = 0$ in equations (3.14), we note that $f''(x) = \Gamma''(\psi) = 0$ and, therefore, we get

$$f(x) = D_1 x + D_2, \qquad \Gamma(\psi) = D_3\psi + D_4 \qquad (3.23)$$

where $D_1 \neq 0$, D_2, $D_3 \neq 0$ and D_4 are arbitrary constants.

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Using (3.23) in equation (3.12), $q(x, \psi)$ satisfies

$$D_1 D_3 \frac{\partial q}{\partial x} - (1 + D_1^2) \frac{\partial q}{\partial \psi} = 0 \qquad (3.24)$$

This possibility gives straight parallel flows when the flow speed is given by (3.24).

<u>Possibility (iii)</u> $\frac{\partial q}{\partial \psi} = \frac{\partial \alpha}{\partial \psi} = 0$:

Using $\frac{\partial \alpha}{\partial \psi} = 0$ in the second of the two equations (3.14), we get $\Gamma''(\psi) = 0$ and therefore equation (3.11) gives

$$\rho q = \alpha = \frac{\sqrt{1 + f'^2(x)}}{K_1}, \qquad \Gamma'(\psi) = K_1 \qquad (3.25)$$

Using (3.25) and $\frac{\partial q}{\partial \psi} = 0$ in (3.12), we integrate and have

$$q = \pm \frac{K_2 \alpha}{\sqrt{K_1^2 \alpha^2 - 1}} \qquad (3.26)$$

where $K_2 \neq 0$ is an arbitrary constant and $K_2 \gtrless 0$ according as there is $\pm$ sign on the right hand side of (3.26). Since (3.25) and (3.26) satisfy equations (3.16) in $q(\alpha)$ and $\Gamma'(\psi)$, it follows that this possibility is the first case of possibility (i) defined by (3.16).

<u>Possibility (iv)</u> $\frac{\partial q}{\partial \psi} = \frac{\partial q}{\partial x} = 0$:

Using $\frac{\partial q}{\partial \psi} = \frac{\partial q}{\partial x} = 0$ in (3.8), (3.9) , we get

$$\frac{\partial p}{\partial x} = c^2(\rho) \frac{\partial \rho}{\partial x} = 0 \qquad (3.27)$$

$$\frac{\partial p}{\partial \psi} = c^2(\rho) \frac{\partial \rho}{\partial \psi} = 0 \qquad (3.28)$$

Equations (3.27), (3.28) and our possibility lead to uniform flow of constant density which is of no interest to us.

<u>Possibility (v)</u> $\frac{\partial q}{\partial x} = \frac{\partial \alpha}{\partial x} = 0$:

Using $f'(x) =$ constant, as implied by (3.14) and $\frac{\partial \alpha}{\partial x} = 0$, in (3.11) and (3.12), we obtain $\frac{\partial q}{\partial \psi} = 0$. Therefore, $\frac{\partial q}{\partial \psi} = \frac{\partial q}{\partial x} = 0$ as in possibility (iv) and we get a uniform flow of constant density which is of no interest to us.

Summing up, we have

Theorem 3.1. *If families of curves of the form $y - f(x) = constant$ define streamline patterns for steady plane isentropic irrotational compressible fluid flow for various continuously differentiable functions $f(x)$ so that $y - f(x) = \Gamma(\psi)$ for each $f(x)$, where $\psi(x, y)$ is the flow streamfunction and $\Gamma(\psi)$ is a function of ψ with $\Gamma'(\psi) \gtrless 0$, then all possible flows are classified as*

(i) non-straight flows with $\Gamma''(\psi) \neq 0$ and $A_1(\alpha) = 0$.

(ii) non-straight flows with $\Gamma''(\psi) \neq 0$, $A_1(\alpha) \neq 0$ and

$$\Gamma''(\psi) = \frac{-A_1(\alpha)f'(x)f''^3(x)}{A_2(\alpha)\Gamma'^2(\psi)f'''(x) + A_3(\alpha)f'(x)f''^2(x)},$$

(iii) non-straight flows with $\Gamma''(\psi) = 0$ and

$$q(\alpha) - \alpha q'(\alpha) + \alpha^3 q'(\alpha)\Gamma'^2(\psi) = 0$$

(iv) straight flows with $\Gamma''(\psi) = 0$.

where $A_1(\alpha)$, $A_2(\alpha)$ and $A_3(\alpha)$ are given by equations (3.19) to (3.21).

Flows classified in this theorem are studied in the following sections.

3.4 NON-STRAIGHT FLOWS WITH $A_1(\alpha) = 0$ AND $\Gamma''(\psi) \neq 0$.

Functions $q(\alpha)$, $\Gamma(\psi)$ and $f(x)$ satisfy a system of three equations for these flows where $\alpha(x, \psi)$ is defined in (3.11). This system is comprised of equation (3.15) along with

$$A_1(\alpha) = 3q(\alpha)q'(\alpha) + \alpha q(\alpha)q''(\alpha) - 3\alpha q'^2(\alpha) = 0 \tag{3.29}$$

and

$$A_2(\alpha)\Gamma'^2(\psi)f'''(x) + A_3(\alpha)f'(x)f''^2(x) = \alpha^4 \left[\alpha q'^2(\alpha) - q(\alpha)q'(\alpha)\right]\Gamma'^2(\psi)f'''(x)$$

$$+ \alpha^2 \left[\alpha q(\alpha)q''(\alpha) + 5q(\alpha)q'(\alpha) - 5\alpha q'^2(\alpha)\right]f'(x)f''^2(x) = 0 \tag{3.30}$$

obtained from (3.18) by taking $A_1(\alpha) = 0$.

Dividing (3.29) by $\alpha q(\alpha) q'(\alpha)$ and integrating twice with respect to α, we get

$$q'(\alpha) = \frac{c_1 q^3(\alpha)}{\alpha^3}, \qquad q^2(\alpha) = \frac{\alpha^2}{c_1 - 2c_2\alpha^2} \qquad (3.31)$$

where $c_1 \neq 0$ and $c_2 \neq 0$ are arbitrary constants. These restrictions on c_1 and c_2 are the restrictions due to our study of non-uniform compressible fluid flows. Since $q^2 > 0$, it follows that $c_1 > 2c_2\alpha^2$ for every permissible value of α in the flow. Therefore, $c_2 < 0$ if $c_1 < 0$, $c_2 \lessgtr 0$ if $c_1 > 0$ and c_2 cannot be positive if c_1 is negative.

Eliminating $q'(\alpha)$ from (3.29) by using (3.31) and solving for $q''(\alpha)$, we obtain

$$q''(\alpha) = \frac{3c_1^2 q^5(\alpha)}{\alpha^6} - \frac{3c_1 q^3(\alpha)}{\alpha^4} \qquad (3.32)$$

Using (3.31) and (3.32) in (3.30) and (3.15), we eliminate $q'(\alpha)$, $q''(\alpha)$ and obtain

$$\left[c_1 q^2(\alpha) - \alpha^2 \right] \alpha^2 \Gamma'^2(\psi) f'''(x) - 2 \left[c_1 q^2(\alpha) - \alpha^2 \right] f'(x) f''^2(x) = 0$$

and

$$\left[\alpha^2 - c_1 q^2(\alpha) \right] f''(x) + c_1 \left[f''(x) + \alpha^2 \Gamma''(\psi) \right] \alpha^2 \Gamma'^2(\psi) q^2(\alpha) = 0$$

Eliminating $\alpha^2 \Gamma'^2(\psi)$ and $q^2(\alpha)$ from these two equations respectively by using (3.11), (3.31) and dividing the first equation by $c_1 q^2(\alpha) - \alpha^2 \neq 0$, we obtain the system

$$\left[1 + f'^2(x) \right] f'''(x) - 2f'(x) f''^2(x) = 0 \qquad (3.33)$$

and

$$c_1 f''(x) \Gamma'^2(\psi) + c_1 \alpha^2 \Gamma'^2(\psi) \Gamma''(\psi) - 2c_2 f''(x) = 0 \qquad (3.34)$$

of two equations in two unknown functions $f(x)$ and $\Gamma(\psi)$.

Dividing (3.33) by $\left[1 + f'^2(x) \right] f''(x) \neq 0$ and integrating the resulting equation three times with respect to x, we obtain

$$f''(x) = c_3 \left[1 + f'^2(x) \right], \qquad f'(x) = \tan(c_3 x + c_4)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.35)$

$$f(x) = \frac{1}{c_3} \ln|c_5 \sec(c_3 x + c_4)|$$

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where $c_3 \neq 0$, c_4 and $c_5 \neq 0$ are arbitrary constants.

Employing (3.11) in the first of the three equations of (3.35), we have

$$f''(x) = c_3 \left[1 + f'^2(x)\right] = c_3 \alpha^2 \Gamma'^2(\psi) \qquad (3.36)$$

Using (3.36) in (3.34) and dividing by $f''(x) = c_3 \alpha^2 \Gamma'^2(\psi) \neq 0$, we get

$$c_1 \Gamma''(\psi) + c_3 \left[c_1 \Gamma'^2(\psi) - 2c_2\right] = 0$$

Since $c_1 \neq 0$ and $\Gamma''(\psi) \neq 0$, it follows that $c_1 \Gamma'^2(\psi) - 2c_2 \neq 0$ and the above equation can be written as

$$\frac{\Gamma''(\psi)}{\Gamma'^2(\psi) - \frac{2c_2}{c_1}} + c_3 = 0 \qquad (3.37)$$

Integration of (3.37) twice with respect to ψ yields either

$$\text{(i)} \qquad \Gamma'(\psi) = k\tan(kc_6 - kc_3\psi) \qquad (3.38)$$

and

$$\Gamma(\psi) = \frac{1}{c_3}\ln|c_7\cos(kc_6 - kc_3\psi)| \qquad (3.39)$$

when $\dfrac{2c_2}{c_1} = -k^2 < 0$, c_6 and $c_7 \neq 0$ are arbitrary constants ($c_1 > 0$, $c_2 < 0$),

or

$$\text{(ii)} \qquad \Gamma'(\psi) = m + \frac{2c_8 m\exp\left(-2mc_3\psi\right)}{1 - c_8\exp\left(-2mc_3\psi\right)} \qquad (3.40)$$

and

$$\Gamma(\psi) = m\psi + \frac{1}{c_3}\ln|c_9\left\{1 - c_8 exp(-2mc_3\psi)\right\}| \qquad (3.41)$$

when $\dfrac{2c_2}{c_1} = m^2 > 0$, $c_8 \neq 0$ and $c_9 \neq 0$ are arbitrary constants (c_1, c_2 are both positive or negative).

We, therefore, have:

'A family of curves $y - \frac{1}{c_3} ln|c_5 sec(c_3 x + c_4)|$ =constant defines a streamline pattern in a steady plane irrotational isentropic compressible fluid flow for some gas such that

$$c_3 y - ln|c_5 sec(c_3 x + c_4)| = c_3 \Gamma(\psi) \tag{3.42}$$

where $\Gamma(\psi)$ is given by (3.39) or (3.41) according as $\dfrac{2c_2}{c_1} = -k^2 < 0$ when $c_1 > 0$ and $c_2 < 0$ or $\dfrac{2c_2}{c_1} = m^2 > 0$ when c_1 and c_2 are both positive or negative constants.'

The streamline pattern for this flow is shown in Figure 3.2. We now further investigate these two cases in the following:
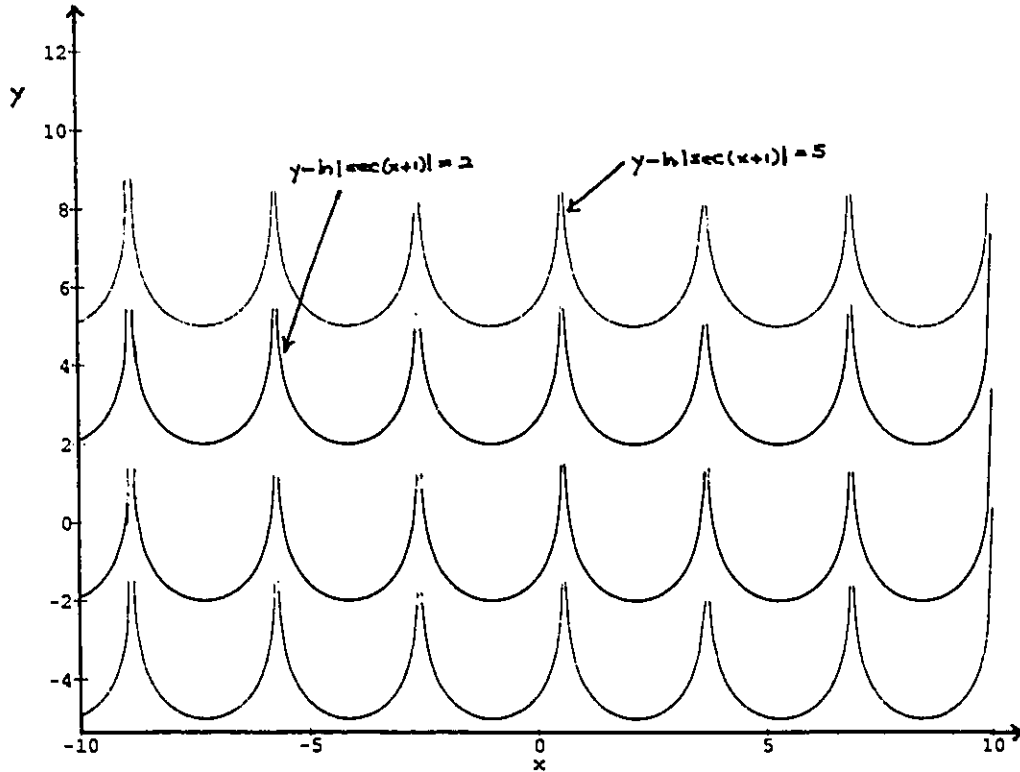
Figure 3.2.   Streamline Pattern for $y - \ln(|\sec(x + 1)|)$=constant.

<u>Case 1</u>: $\left(\dfrac{2c_2}{c_1} = -k^2 < 0 \text{ when } c_1 > 0, c_2 < 0\right)$

In this case, using (3.35) and (3.39), (3.1) reads

$$c_3 y - ln|c_5 sec(c_3 x + c_4)| = ln|c_7 cos(k c_6 - k c_3 \psi)| \tag{3.43}$$

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Solving (3.43) for $\cos(kc_6 - kc_3\psi)$ and $\psi(x,y)$, we get

$$\cos(kc_6 - kc_3\psi) = \pm\frac{1}{c_5c_7}\exp(c_3y)\cos(c_3x + c_4) \qquad (3.44)$$

and

$$\psi(x,y) = \frac{c_6}{c_3} - \frac{1}{kc_3}\cos^{-1}\left[\pm\frac{1}{c_5c_7}\exp(c_3y)\cos(c_3x + c_4)\right] \qquad (3.45)$$

where positive or negative sign is taken according as $c_5c_7\sec(c_3x + c_4)\cos(kc_6 - kc_3\psi)\gtreqless 0$ respectively.

Using (3.45) in (3.38), we obtain

$$\Gamma'(\psi) = \pm\frac{k\sqrt{c_5^2c_7^2 - \exp(2c_3y)\cos^2(c_3x + c_4)}}{\exp(c_3y)\cos(c_3x + c_4)} = \sqrt{\frac{2c_2\exp(2c_3y)\cos^2(c_3x + c_4) - 2c_2c_5^2c_7^2}{c_1\exp(2c_3y)\cos^2(c_3x + c_4)}} \qquad (3.46)$$

Employing (3.46), (3.35), (3.31), (3.36) and (3.11), we have

$$\rho^2q^2 = \alpha^2 = \frac{c_1q^2}{1 + 2c_2q^2} = \frac{1 + f'^2(x)}{\Gamma'^2(\psi)} = \frac{\exp(2c_3y)}{k^2\left[c_5^2c_7^2 - \exp(2c_3y)\cos^2(c_3x + c_4)\right]} \qquad (3.47)$$

Solving (3.47) for $q(x,\psi)$, we get

$$q = \frac{\alpha}{\sqrt{c_1}\sqrt{1 + k^2\alpha^2}} = \frac{1}{\sqrt{2c_2\cos^2(c_3x + c_4) - 2c_2c_5^2c_7^2\exp(-2c_3y) - 2c_2}} \qquad (3.48)$$

Equations (3.47) and (3.48) yield

$$\rho = \sqrt{c_1}\sqrt{1 + k^2\alpha^2} = \frac{\sqrt{c_1c_5^2c_7^2\exp(-2c_3y) - c_1\cos^2(c_3x + c_4) + c_1}}{\sqrt{c_5^2c_7^2\exp(-2c_3y) - \cos^2(c_3x + c_4)}} \qquad (3.49)$$

Differentiating (3.1) with respect to y and x respectively and using (3.35) and (3.49), we obtain

$$u(x,y) = \frac{\cos(c_3x + c_4)}{\sqrt{2c_2\cos^2(c_3x + c_4) - 2c_2c_5^2c_7^2\exp(-2c_3y) - 2c_2}} \qquad (3.50)$$

and

$$v(x,y) = \frac{\sin(c_3x + c_4)}{\sqrt{2c_2\cos^2(c_3x + c_4) - 2c_2c_5^2c_7^2\exp(-2c_3y) - 2c_2}} \qquad (3.51)$$

Employing (3.47) and (3.48) in (3.4), (3.5) and integrating, we have

$$p = \int dp = -\int \left[\rho q \frac{\partial q}{\partial x} dx + \rho q \frac{\partial q}{\partial y} dy \right] = -\int \alpha q'(\alpha) d\alpha = p_0 - \frac{\sqrt{c_1}}{2c_2 \sqrt{1 + k^2 \alpha^2}}$$

$$= p_0 - \frac{\sqrt{c_1}}{2c_2} \frac{\sqrt{c_5^2 c_7^2 \exp(-2c_3 y) - \cos^2(c_3 x + c_4)}}{\sqrt{c_5^2 c_7^2 \exp(-2c_3 y) - \cos^2(c_3 x + c_4) + 1}} \qquad (3.52)$$

where p_0 is an arbitrary constant.

From the solution obtained for the two thermodynamic variables in (3.49) and (3.52), it follows that the state equation for this flow is

$$p = p_0 - \frac{c_1}{2c_2 \rho} \qquad (3.53)$$

The speed of sound for this flow is given by

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{-\frac{1}{k^2 \rho^2}} \qquad (3.54)$$

The slope of the tangent gas represented by the straight curve given by (3.53) in $\left(p, \frac{1}{\rho} \right)$-plane is positive and the speed of sound given by (3.54) is an imaginary number. We, therefore, conclude that

'$u(x,y)$, $v(x,y)$, $p(x,y)$ and $\rho(x,y)$ given by (3.50), (3.51), (3.52) and (3.49) determine a mathematical solution set of (2.1) to (2.4) which, however, does not correspond to any physically possible state equation or physically possible gas.'

Case 2: $\left(\dfrac{2c_2}{c_1} = m^2 > 0 \text{ when } c_1, c_2 \text{ are both negative or positive} \right)$

Using (3.35) in (3.1), we have

$$y - \frac{1}{c_3} \ln|c_5 \sec(c_3 x + c_4)| = \Gamma(\psi)$$

so that

$$\pm \exp \left[c_3 y - c_3 \Gamma(\psi) \right] = c_5 \sec(c_3 x + c_4)$$

and

$$c_5^2 \exp \left[2c_3 \Gamma(\psi) \right] = \exp \left[2c_3 y \right] \cos^2(c_3 x + c_4) \qquad (3.55)$$

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Differentiating (3.40) with respect to ψ and rewriting (3.41), we get

$$\Gamma''(\psi) = \frac{-4m^2 c_3 c_8 \exp\left[-2mc_3\psi\right]}{\left(1 - c_8 \exp\left[-2mc_3\psi\right]\right)^2}$$

and

$$c_9 \left(1 - c_8 \exp\left[-2mc_3\psi\right]\right) = \pm\exp\left[c_3\Gamma(\psi)\right]\exp\left[-mc_3\psi\right]$$

Using the second equation in the first and employing (3.55), we have

$$\Gamma''(\psi) = -\frac{8c_2 c_3 c_8 c_9^2}{c_1 \exp\left[2c_3\Gamma(\psi)\right]} = -\frac{8c_2 c_3 c_5^2 c_8 c_9^2 \sec^2(c_3 x + c_4)}{c_1 \exp(2c_3 y)} \tag{3.56}$$

when $m^2 = \dfrac{2c_2}{c_1}$ is used.

Using (3.56) in (3.37), we get

$$\Gamma'^2(\psi) = \frac{2c_2}{c_1} - \frac{\Gamma''(\psi)}{c_3}$$

$$= \frac{2c_2}{c_1} + \frac{8c_2 c_8 c_9^2}{c_1 \exp\left[2c_3\Gamma(\psi)\right]} = \frac{2c_2}{c_1} + \frac{8c_2 c_5^2 c_8 c_9^2 \sec^2(c_3 x + c_4)}{c_1 \exp(2c_3 y)} \tag{3.57}$$

Using (3.57), (3.35) and $\Gamma'(\psi) > 0$ in the definition of α in (3.11), we obtain

$$\rho q = \alpha(x,\psi) = \frac{\sqrt{1 + f'^2(x)}}{|\Gamma'(\psi)|} = \sqrt{\frac{c_1}{2c_2\cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)}} \tag{3.58}$$

Employing (3.58) and (3.31), we have

$$q(x,y) = \frac{\alpha}{\sqrt{c_1 - 2c_2\alpha^2}} = \frac{1}{\sqrt{2c_2\cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)} - 2c_2} \tag{3.59}$$

and

$$\rho(x,y) = \sqrt{c_1 - 2c_2\alpha^2} = \frac{\sqrt{2c_1 c_2\cos^2(c_3 x + c_4) + 8c_1 c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)} - 2c_1 c_2}{\sqrt{2c_2\cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)}} \tag{3.60}$$

Differentiating (3.55) with respect to y and x and using (2.27), (3.57) and (3.60),

we get

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)} = \pm\frac{\cos(c_3 x + c_4)}{\sqrt{2c_2\cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)} - 2c_2} \tag{3.61}$$

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and

$$v(x,y) = \frac{\tan(c_3 x + c_4)}{\rho \Gamma'(\psi)} = \pm \frac{\sin(c_3 x + c_4)}{\sqrt{2c_2 \cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y) - 2c_2}}$$

(3.62)

Applying (3.58) and (3.59) in (3.4) and (3.5), we obtain

$$p = \int \mathrm{d}p = \int - \left[\rho q \frac{\partial q}{\partial x} \mathrm{d}x + \rho q \frac{\partial q}{\partial y} \mathrm{d}y \right] = - \int \alpha q'(\alpha) \mathrm{d}\alpha$$

and, therefore,

$$p(x,y) = p_0 - \frac{c_1}{2c_2 \sqrt{c_1 - 2c_2 \alpha^2}}$$

$$= p_0 - \frac{\sqrt{c_1}}{2c_2} \frac{\sqrt{2c_2 \cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y)}}{\sqrt{2c_2 \cos^2(c_3 x + c_4) + 8c_2 c_5^2 c_8 c_9^2 \exp(-2c_3 y) - 2c_2}}$$

(3.63)

The state equation for our flow is obtained from the solutions for the two thermo-dynamic variables given by (3.60), (3.63) and we have

$$p = p_0 - \frac{c_1}{2c_2 \rho}$$

(3.64)

Since c_1 cannot be negative for pressure to be a real valued function and $c_2 > 0$ if $c_1 > 0$ in our case, it follows that c_1 and c_2 are both positive real numbers in our solutions. Furthermore, choice of positive or negative sign in (3.58), (3.59), (3.61) and (3.62) is made to keep ρq and q as positive definite quantities.

Summing up, we have

Theorem 3.2. *The family of curves* $c_3 y - \ln|c_5 \sec(c_3 x + c_4)|$ =constant *is a permissible streamline pattern for steady plane irrotational isentropic fluid flow of a tangent gas having state equation given by (3.64) and solutions for* $u(x,y)$, $v(x,y)$, $\rho(x,y)$ *and* $p(x,y)$ *given by (3.61), (3.62), (3.60) and (3.63) respectively.*

The streamfunction for this flow is implicitly given by

$$\sqrt{c_1} c_3 y - \sqrt{c_1} \ln|c_5 \sec(c_3 x + c_4)| = \sqrt{2c_2} c_3 \psi(x,y) + \sqrt{c_1} \ln|c_9$$

$$- c_8 c_9 \exp\left(-2\sqrt{\frac{2c_2}{c_1}} c_3 \psi(x,y) \right) |$$

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The potential function $\Phi(x, y)$ and the Mach number are respectively given by

$$\Phi(x, y) = c_{10} - \frac{1}{c_3\sqrt{2c_2}} \text{arcsec} \left[\frac{2c_5\sqrt{c_8}\exp(-c_3 y)}{\sin(c_3 x + c_4)} \right]$$

and

$$M(x, y) = \frac{\sqrt{2c_2\alpha^2\rho^2}}{c_1^2 - 2c_1 c_2 \alpha^2} = \frac{1}{\sqrt{\cos^2(c_3 x + c_4) + 4c_5^2 c_8 c_9^2 \exp(-2c_3 y)}}$$

This flow is sonic along the curve

$$y = \frac{1}{2c_3}\ln|4c_5^2 c_8 c_9^2 \text{cosec}^2(c_3 x + c_4)|,$$

and is subsonic when

$$y < \frac{1}{2c_3}\ln|4c_5^2 c_8 c_9^2 \text{cosec}^2(c_3 x + c_4)|.$$

Also, for this flow, the pressure function p varies with the flow intensity $\rho q = \alpha$ according as

$$p = p_0 - \frac{c_1}{2c_2\sqrt{c_1 - 2c_2\alpha^2}}$$

The variations of pressure with density, the changes of pressure with flow intensity and the isobaric curves are respectively shown in Figures 3.3, 3.4, and 3.5.
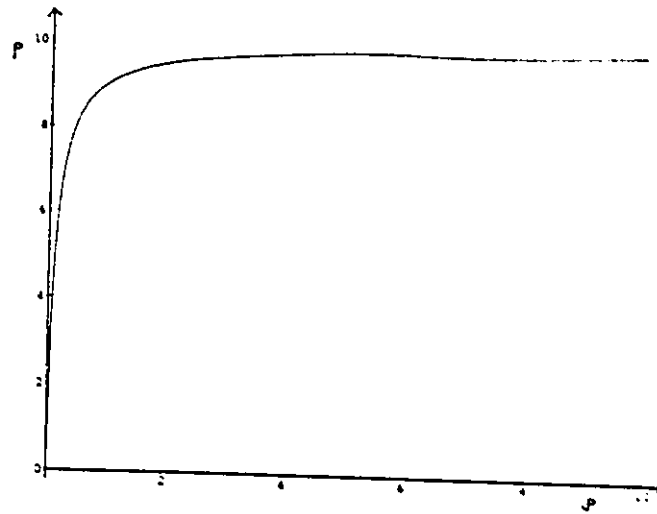
Figure 3.3.   Pressure versus density for flow in Case 2 (Equation 3.64 with $c_1 = 2$, $c_2 = 1$ and $p_0 = 10$)
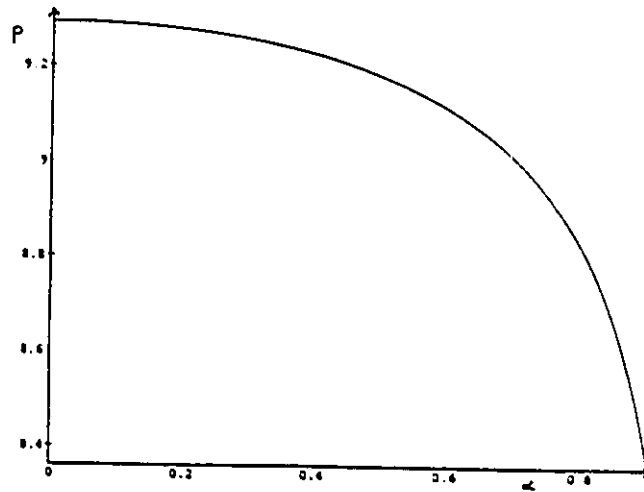
Figure 3.4.   Pressure versus flow intensity for flow in Case 2
$$\left(p = 10 - \frac{1}{\sqrt{1 - \alpha^2}} \right).$$
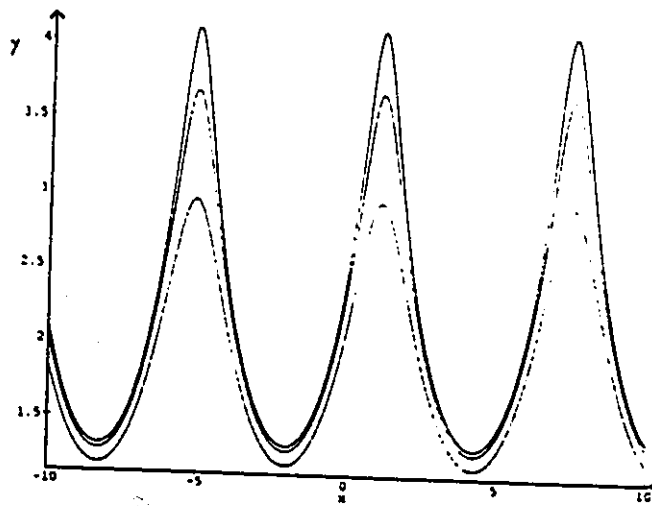
Figure 3.5.   Curves of constant pressure for flow in Case 2 (Equation 3.63 with $c_1 = 4$, $c_2 = 1$, $c_3 = 0.3$, $c_4 = 1$, $c_5 = 1$, $c_6 = 1$, $c_9 = 1$ and $p_0 = 10$)

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3.5 NON-STRAIGHT FLOWS WITH $\Gamma''(\psi) \neq 0$ AND $A_1(\alpha) \neq 0$.

In this case, we eliminate $\Gamma''(\psi)$ and $\Gamma'^2(\psi)$ from (3.15) by using (3.22) and (3.11) respectively and have

$$\left\{\alpha^3 q'^2(\alpha)\left[\alpha q'(\alpha) - q(\alpha)\right]\left[1 + f'^2(x)\right] - \alpha^2 q'(\alpha)\left[\alpha q'(\alpha) - \varsigma(\alpha)\right]^2\right\}\left\{\left[1 + f'^2(x)\right]f''(x)f'''(x)\right\}$$

$$+ \left\{\alpha^2\left[\alpha q'(\alpha) - q(\alpha)\right]\left[5\alpha q'^2(\alpha) - \alpha q(\alpha)q''(\alpha) - 5q(\alpha)q'(\alpha)\right]\right\}\left\{f'(x)f''^3(x)\right\}$$

$$- \left\{2\alpha^3 q'^2(\alpha)\left[\alpha q'(\alpha) - q(\alpha)\right]\right\}\left\{\left[1 + f'^2(x)\right]f'(x)f''^3(x)\right\} = 0$$

Dividing this equation by $\alpha^2\left[\alpha q'(\alpha) - q(\alpha)\right]f''(x) \neq 0$ and writing in a suitable form, we obtain

$$\left\{\alpha q'^2(\alpha)f'^2(x) + q(\alpha)q'(\alpha)\right\}\left\{\left[1 + f'^2(x)\right]f'''(x)\right\}$$

$$+ \left\{5\alpha q'^2(\alpha) - \alpha q(\alpha)q''(\alpha) - 5q(\alpha)q'(\alpha)\right\}\left\{f'(x)f''^2(x)\right\}$$

$$- \left\{2\alpha q'^2(\alpha)\right\}\left\{\left[1 + f'^2(x)\right]f'(x)f''^2(x)\right\} = 0$$

Dividing by $q(\alpha)q'(\alpha)f'(x)f''^2(x)$, we have

$$\left\{\frac{\left[1 + f'^2(x)\right]f'''(x)}{f'(x)f''^2(x)} + \frac{\alpha q'(\alpha)}{q(\alpha)}\left[1 + f'^2(x)\right]\frac{f'(x)f'''(x)}{f''^2(x)}\right\}$$

$$+ \left\{\frac{5\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)} - 5\right\} - \left\{\frac{2\alpha q'(\alpha)}{q(\alpha)}\right\}\left[1 + f'^2(x)\right] = 0$$

Rewriting in a suitable form, this equation is

$$\frac{\left[1 + f'^2(x)\right]f'''(x)}{f'(x)f''^2(x)} + \left[5\frac{\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)} - 5\right] + \left[1 + f'^2(x)\right]\left[\frac{f'(x)f'''(x)}{f''^2(x)} - 2\right]\left(\frac{\alpha q'(\alpha)}{q(\alpha)}\right) = 0$$

$$\tag{3.65}$$

Differentiating (3.65) with respect to ψ and using $\frac{\partial \alpha}{\partial \psi} \neq 0$, we get

$$\frac{d}{d\alpha}\left[5\frac{\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)}\right] + \left[1 + f'^2(x)\right]\left[\frac{f'(x)f'''(x)}{f''^2(x)} - 2\right]\frac{d}{d\alpha}\left(\frac{\alpha q'(\alpha)}{q(\alpha)}\right) = 0 \tag{3.66}$$

For this equation to hold true we study the following possible subcases:

a) $\quad \dfrac{d}{d\alpha}\left(\dfrac{\alpha q'(\alpha)}{q(\alpha)}\right) \neq 0 \quad$ and $\quad \dfrac{d}{d\alpha}\left[5\dfrac{\alpha q'(\alpha)}{q(\alpha)} - \dfrac{\alpha q''(\alpha)}{q'(\alpha)}\right] \neq 0$

b) $\quad \dfrac{f'(x)f'''(x)}{f''^2(x)} - 2 = 0$

c) $\quad \dfrac{d}{d\alpha}\left(\dfrac{\alpha q'(\alpha)}{q(\alpha)}\right) = 0$

We now consider these subcases separately.

Subcase (a):

In this case dividing (3.66) by $\dfrac{d}{d\alpha}\left(\dfrac{\alpha q'(\alpha)}{q(\alpha)}\right) \neq 0$ and separating the variables, we get

$$\frac{d}{d\alpha}\left[5\frac{\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)}\right] + B_1\frac{d}{d\alpha}\left(\frac{\alpha q'(\alpha)}{q(\alpha)}\right) = 0 \qquad (3.67)$$

and

$$(1 + f'^2(x))\left[\frac{f'(x)f'''(x)}{f''^2(x)} - 2\right] = B_1 \qquad (3.68)$$

where $B_1 \neq 0$ is an arbitrary constant.

Integrating (3.67) twice, we obtain

$$q'(\alpha) = B_3\left(\frac{q^{5+B_1}(\alpha)}{\alpha^{B_2}}\right), \quad q''(\alpha) = \frac{(5+B_1)\,B_3^2 q^{9+2B_1}(\alpha)}{\alpha^{2B_2}} - \frac{B_2 B_3 q^{5+B_1}(\alpha)}{\alpha^{B_2+1}}$$

$$(3.69)$$

where B_2 and $B_3 \neq 0$ are arbitrary constants.

Eliminating $q'(\alpha)$ and $q''(\alpha)$ from (3.65) by using (3.69) and simplifying, we get

$$\left(\frac{B_3 q^{4+B_1}(\alpha)}{\alpha^{B_2-1}}\right)\left\{-B_1 + (1 + f'^2(x))\left[\frac{f'(x)f'''(x)}{f''^2(x)} - 2\right]\right\} = (5-B_2) - \frac{\left[1 + f'^2(x)\right]f'''(x)}{f'(x)f''^2(x)}$$

$$(3.70)$$

Employing (3.68) in (3.70), we find that the function $f(x)$ satisfies (3.68) and

$$(5 - B_2) - \frac{\left[1 + f'^2(x)\right]f'''(x)}{f'(x)f''^2(x)} = 0 \qquad (3.71)$$

Multiplying (3.68) by $\dfrac{1}{f'^2(x)}$ and adding to (3.71), we get

$$(3 - B_2)\,f'^2(x) - (B_1 + 2) = 0$$

This equation is satisfied if either $f'(x) = \sqrt{\dfrac{B_1 + 2}{3 - B_2}}$ =constant and, therefore, $f''(x) = 0$ or if $B_1 = -2$ and $B_2 = 3$. Since $f''(x) = 0$ is contrary to the assumption of this section, it follows that we only need to consider $B_1 = -2$, $B_2 = 3$ and have

$$q'(\alpha) = B_3\frac{q^3(\alpha)}{\alpha^3}$$

and, therefore,

$$q''(\alpha) = \frac{3B_3^2 q^5(\alpha)}{\alpha^6} - \frac{3B_3 q^3(\alpha)}{\alpha^4}$$

as given by (3.69).

Employing these in (3.19), we obtain

$$A_1(\alpha) = 0$$

for this case and this is also contrary to the assumption of this section. Therefore, there does not exist any solution for this subcase.

Subcase (b):

For this case, (3.66) yields after one integration

$$\frac{5\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)} = B_4$$

Employing this equation and our assumption $f'(x)f'''(x) = 2f''^2(x)$ in (3.65), we get $f'(x)$ is a constant and, therefore, $f''(x) = 0$ which is contrary to the assumption that $f''(x) \neq 0$.

Subcase (c):

In this case, (3.66) yields

$$\frac{d}{d\alpha}\left(\frac{\alpha q'(\alpha)}{q(\alpha)}\right) = 0 \quad \text{and} \quad \frac{d}{d\alpha}\left[5\frac{\alpha q'(\alpha)}{q(\alpha)} - \frac{\alpha q''(\alpha)}{q'(\alpha)}\right] = 0$$

The solution for these two equations is

$$q(\alpha) = B_5 \alpha^\lambda \tag{3.72}$$

where $B_5 \neq 0$ and $\lambda \neq 0$ or 1 are arbitrary constants. $\lambda \neq 1$ or $\lambda \neq 0$ so that q is not proportional to α or a constant.

Using (3.72) in (3.65) and using (3.72) and $\alpha^2 = [1 + f'^2(x)]/\Gamma'^2(\psi)$ in (3.15), we respectively get

$$\frac{\left[1 + f'^2(x)\right]f'''(x)}{f'(x)f''^2(x)} + 4(\lambda - 1) + \lambda\left[1 + f'^2(x)\right]\left[\frac{f'(x)f'''(x)}{f''^2(x)} - 2\right] = 0$$

$$\left[1 + \lambda f'^2(x)\right] f''(x) + \lambda \left[1 + f'^2(x)\right]^2 \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0$$

Separating variables for the second equation, we have

$$\left[1 + \lambda f'^2(x)\right] f''(x) + B_6 \lambda \left[1 + f'^2(x)\right]^2 = 0 \qquad (3.73)$$

which identically satisfies the first equation and

$$\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = B_6 \qquad (3.74)$$

where $B_6 \neq 0$ is the separation constant.

Equation (3.73) is a non-linear ordinary differential equation with λ as a parameter. We integrate it in two ways to get (a) $f(x)$ as a function of $t = f'(x)$ and (b) x as a function of $f'(x) = t$.

First approach:

Equation (3.73) can be written as

$$\frac{f''(x)}{\left[1 + f'^2(x)\right]^2} + \frac{\lambda f'^2(x) f''(x)}{\left[1 + f'^2(x)\right]^2} + \lambda B_6 = 0$$

Multiplying by $2f'(x)$, we have

$$\frac{2f'(x)f''(x)}{\left[1 + f'^2(x)\right]^2} + \frac{2\lambda f'(x)f''(x)\left[1 + f'^2(x) - 1\right]}{\left[1 + f'^2(x)\right]^2} + 2B_6\lambda f'(x) = 0$$

Rewriting this equation, we get

$$(\lambda - 1)\left[\frac{-2f'(x)f''(x)}{\left[1 + f'^2(x)\right]^2}\right] + \lambda \left[\frac{2f'(x)f''(x)}{\left[1 + f'^2(x)\right]}\right] + 2B_6\lambda f'(x) = 0$$

Integrating the above equation, we get

$$\frac{(\lambda - 1)}{\left[1 + f'^2(x)\right]} + \lambda \ln \left[1 + f'^2(x)\right] + 2B_6\lambda f(x) = B_7 \qquad (3.75)$$

<u>Second approach:</u>

Taking $f'(x) = t$, equation (3.73) can be written as

$$\frac{1 + \lambda t^2}{(1 + t^2)^2} \frac{dt}{dx} + B_6 \lambda = 0$$

Rewriting this equation as

$$\left[\frac{\lambda}{1 + t^2} + \frac{1 - \lambda}{[1 + t^2]^2} \right] \frac{dt}{dx} + B_6 \lambda = \left[\frac{\lambda}{1 + t^2} + \left(\frac{1 - \lambda}{2} \right) \left\{ \frac{1}{1 + t^2} + \frac{1 - t^2}{(1 + t^2)^2} \right\} \right] \frac{dt}{dx}$$
$$+ B_6 \lambda = 0$$

Integrating with respect to x, we get

$$\lambda \arctan(t) + \left(\frac{1 - \lambda}{2} \right) \left(\arctan(t) + \frac{t}{(1 + t^2)} \right) + B_6 \lambda x = B_8 \tag{3.76}$$

Taking $B_7 = 0$ and $B_8 = 0$ without any loss of generality, equations (3.75) and (3.76) yield

$$f(x) = \left(\frac{1 - \lambda}{2B_6 \lambda} \right) \left(\frac{1}{1 + t^2} \right) - \frac{1}{2B_6} \ln \left(1 + t^2 \right)$$
$$x = \left(\frac{\lambda - 1}{2B_6 \lambda} \right) \left(\frac{t}{1 + t^2} \right) - \left(\frac{1 + \lambda}{2B_6 \lambda} \right) \arctan(t) \tag{3.77}$$
$$t = f'(x)$$

Since $\dfrac{dx}{dt} = - \left[\dfrac{1 + \lambda t^2}{B_6 \lambda (1 + t^2)^2} \right]$ from the second equation of (3.77), it follows that

$\frac{dx}{dt} \neq 0$ for any t when $\lambda > 0$ and $\frac{dx}{dt} = 0$ only if $t = \pm \sqrt{-\frac{1}{\lambda}}$ when $\lambda < 0$. Therefore,

the second equation in (3.77) defines x as a monotonic function of t and also defines

t as a unique function of x when λ is positive. However, this equation defines x

as a monotonic function of t and defines t as a unique function of x only when

$t = \pm \sqrt{-\frac{1}{\lambda}}$ does not belong to the domain of our function for negative values of λ.

Having solved the second equation of (3.77) so that $t = t(x) = f'(x)$ is deter-

mined, we use this solution in the first equation and determine $f(x)$ giving us the

streamline pattern $y - f(x) =$ constant. For this flow pattern, we have

$$y - f(x) = \Gamma(\psi)$$

such that

$$\Gamma'(\psi) = B_9 \exp\left[B_6 \Gamma(\psi)\right] = B_9 \exp\left[B_6\left(y - f(x)\right)\right] \qquad (3.78)$$

given by (3.74) wherein $B_9 \neq 0$ is an arbitrary constant. Also $B_9 \gtrless 0$ according as fluid flows along the streamline in the direction of increasing or decreasing x.

The streamline pattern for this flow is shown in Figure 3.6.
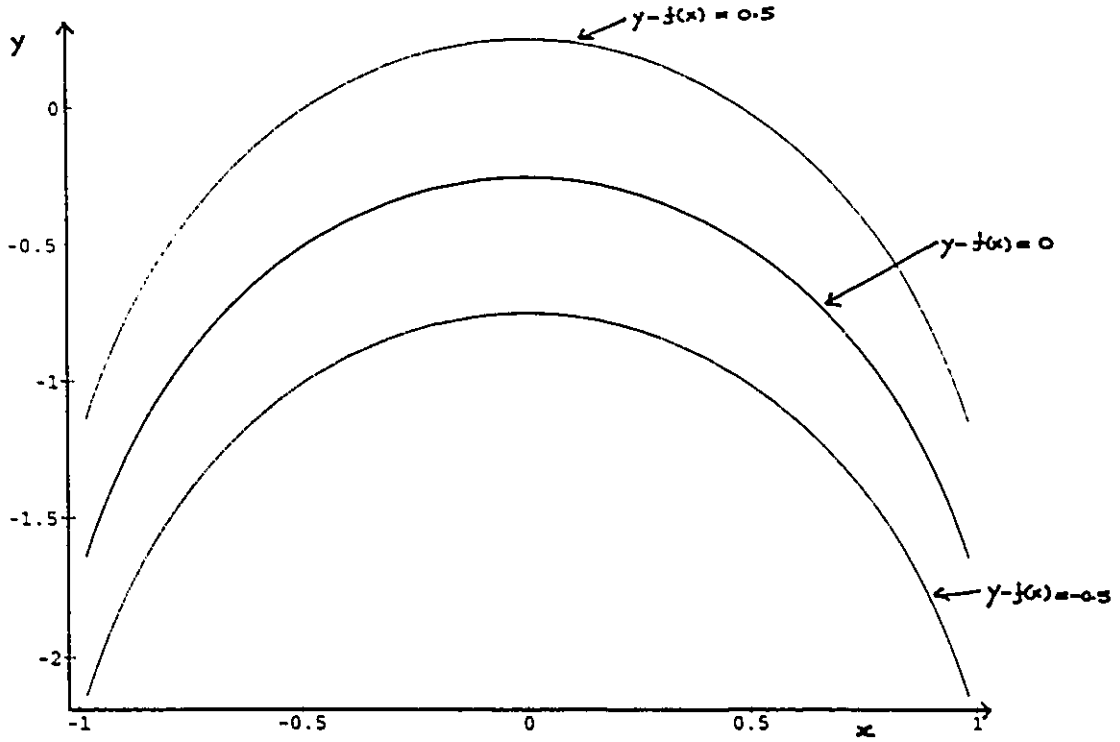
Figure 3.6.   Streamline pattern as given by equation (3.77) with $\lambda = 2$ and $B_6 = 1$.

Having solved (3.77) for $t = f'(x)$ and $f(x)$ given by

$$t = f'(x) = A(x) \qquad \text{(say)}$$

and

$$f(x) = \left(\frac{1-\lambda}{2B_6\lambda}\right)\left[\frac{1}{1 + A^2(x)}\right] - \frac{1}{2B_6}\ln\left[1 + A^2(x)\right] = B(x) \qquad \text{(say)},$$

and using (3.78), we obtain

$$\alpha(x, y) = \frac{\sqrt{1 + A^2(x)}}{|B_9| \exp\left[B_6(y - B(x))\right]},$$

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$$q(x,y) = B_5 \alpha^\lambda = B_5 \left[\frac{\sqrt{1 + A^2(x)}}{|B_9| \exp[B_6(y - B(x))]} \right]^\lambda ,$$

$$p(x,y) = \frac{\alpha}{q(x,y)} = \frac{1}{B_5} \left[\frac{\sqrt{1 + A^2(x)}}{|B_9| \exp[B_6(y - B(x))]} \right]^{1-\lambda} , \qquad (3.79)$$

$$u(x,y) = \frac{1}{\rho \Gamma'(\psi)} = \left[\frac{B_5 |B_9|^{1-\lambda}}{B_9} \right] \frac{[1 + A^2(x)]^{\frac{1}{2}(\lambda-1)}}{[\exp[\lambda B_6(y - B(x))]]} , \qquad (3.80)$$

$$v(x,y) = \frac{f'(x)}{\rho \Gamma'(\psi)} = \left[\frac{B_5 |B_9|^{1-\lambda}}{B_9} \right] \frac{A(x)[1 + A^2(x)]^{\frac{1}{2}(\lambda-1)}}{[\exp[\lambda B_6(y - B(x))]]} , \qquad (3.81)$$

and

$$p(x,y) = - \int \alpha q'(\alpha) d\alpha = \begin{cases} p_0 + B_5 \ln \left[\dfrac{\sqrt{1 + A^2(x)}}{|B_9| \exp[B_6(y - B(x))]} \right] ; & \lambda = -1 \\[4mm] p_0 - \dfrac{B_5 \lambda}{1 + \lambda} \left[\dfrac{\sqrt{1 + A^2(x)}}{|B_9| \exp[B_6(y - B(x))]} \right]^{\lambda+1} ; & \lambda \neq -1 \end{cases}$$

$$(3.82)$$

where p_0 is an arbitrary constant and for $\lambda = -1$, ρ, u and v are given by (3.79), (3.80) and (3.81) respectively where $A(x)$ and $B(x)$ are given by

$$A(x) = \frac{1 \pm \sqrt{1 - 4(B_6 x)^2}}{2(B_6 x)}$$

and

$$B(x) = \frac{1}{2B_6 x} \ln(B_6 x) \pm \frac{1}{2B_6} \left[\sqrt{1 - 4(B_6 x)^2} - \ln \left(\frac{1 + \sqrt{1 - 4(B_6 x)^2}}{2(B_6 x)} \right) \right]$$

obtained by integrating (3.73) twice.

The state equations for these flows are given by

$$p = \begin{cases} p_0 + B_5 \ln \left[\sqrt{B_5 \rho} \right] ; & \lambda = -1 \\[4mm] p_0 - \left[\dfrac{\lambda B_5^{\frac{2}{1-\lambda}}}{1 + \lambda} \right] \rho^{\frac{1+\lambda}{1-\lambda}} ; & \lambda \neq -1 \end{cases} \qquad (3.83)$$

Summing up, we have

Theorem 3.3. *A family of curves* $y - f(x) =$ *constant where* $f(x)$ *is given by* (3.77) *is a permissible streamline pattern in a steady plane isentropic compressible fluid flow of a gas with the solutions* ρ, u, v *and* p *given by* (3.79), (3.80), (3.81) *and* (3.82) *and the state equations given by* (3.83).

The streamfunction, the potential function, pressure as a function of flow intensity and the Mach number for this flow are respectively found to be

$$\psi(x,y) = -\frac{1}{B_6 B_9} e^{-B_6(y - B(x))},$$

$$\Phi(x,y) = -\frac{B_5 |B_9|^{1-\lambda}}{\lambda B_6 B_9} \frac{A(x) \left[1 + A^2(x)\right]^{\frac{1}{2}(\lambda - 1)}}{[\exp\{\lambda B_6(y - B(x))\}]}, \tag{3.84}$$

$$p = \begin{cases} p_0 + B_5 \ln\alpha, & \lambda = -1 \\ p_0 - \dfrac{\lambda B_5}{1 + \lambda} \alpha^{1+\lambda}, & \lambda \neq -1 \end{cases}$$

and

$$M = \sqrt{1 - \frac{1}{\lambda}}$$

Since Mach number is real when the flow is supersonic with $\lambda < 0$ or when the flow is subsonic with $\lambda > 1$, it follows that λ cannot be a real number such that $0 \leq \lambda \leq 1$. For any value of the adiabatic constant γ for monoatomic, diatomic or polyatomic gas, $\frac{1+\lambda}{1-\lambda} = \gamma$ yields $0 < \lambda = \frac{\gamma-1}{\gamma+1} < 1$. Also, there does not exist any value for γ such that $\frac{\gamma+1}{1-\gamma} = -1$. Therefore, the non-parallel flows studied in this section are neither valid for the adiabatic relation of a polytropic gas or a tangent gas associated with such a gas.

State equations given by (3.83) yield

$$\frac{dp}{d\rho} = \begin{cases} \frac{B_5}{\rho}; & \lambda = -1 \\ \frac{\lambda}{\lambda - 1} B_5^{\frac{2}{1-\lambda}} \rho^{\frac{2\lambda}{1-\lambda}}; & \lambda \neq -1 \end{cases}$$

Since $\frac{dp}{d\rho} = c^2 > 0$, it follows that $B_5 > 0$ when $\lambda = -1$. However, $\frac{dp}{d\rho} = c^2 > 0$ for every permissible choice of $\lambda \neq -1$. Since $\frac{d^2p}{d\rho^2} > 0$ for all compressible real media

and we have

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} = \begin{cases} -\frac{B_5}{2\rho^2}; & \lambda=-1 \\ -\frac{2\lambda^2}{(\lambda-1)^2} B_5^{\frac{2}{1-\lambda}} \rho^{\frac{3\lambda-1}{1-\lambda}}; & \lambda\neq-1 \end{cases}$$

which is negative for $\lambda = -1$ or $\lambda \neq -1$, it follows that our solutions are valid for an imaginary gas having (3.83) as the state equations giving p as a simple-valued function of ρ for every permissible choice of B_5 and λ.

3.6 NON-STRAIGHT FLOWS WITH $\Gamma''(\psi) = 0$.

For these flows, (3.15) requires that $q(\alpha)$ must satisfy

$$q(\alpha) - \alpha q'(\alpha) + D_1^2 \alpha^3 q'(\alpha) = 0, \qquad \Gamma'(\psi) = D_1 \tag{3.85}$$

where $D_1 \gtrless 0$ according as fluid flows in the direction of increasing x or decreasing x along a streamline and is an arbitrary constant as shown in Figure 3.7.

Rewriting (3.85) as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\ln q(\alpha) - \ln\alpha + \frac{1}{2}\ln\left(D_1^2\alpha^2 - 1\right)\right] = 0 \tag{3.86}$$

wherein

$$\alpha = \frac{\sqrt{1 + f'^2(x)}}{|D_1|} > 0 \tag{3.87}$$

we integrate and get

$$q(x,y) = q(\alpha) = \frac{D_2\alpha}{\sqrt{D_1^2\alpha^2 - 1}} = \frac{D_2\sqrt{1 + f'^2(x)}}{|D_1||f'(x)|} \tag{3.88}$$

where $D_2 > 0$ is an arbitrary constant and $|f'(x)| = \sqrt{D_1^2\alpha^2 - 1} > 0$,

Employing (3.87) and (3.88) in (3.11), the density function is

$$\rho(x,y) = \frac{\sqrt{D_1^2\alpha^2 - 1}}{D_2} = \frac{|f'(x)|}{D_2}, \tag{3.89}$$

Differentiating (3.1) with respect to x and y, and using (3.85) and (3.89), we obtain velocity components given by

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)} = \frac{D_2}{D_1|f'(x)|}, \qquad v(x,y) = \frac{f'(x)}{\rho\Gamma'(\psi)} = \frac{D_2}{D_1}\frac{f'(x)}{|f'(x)|} \tag{3.90}$$

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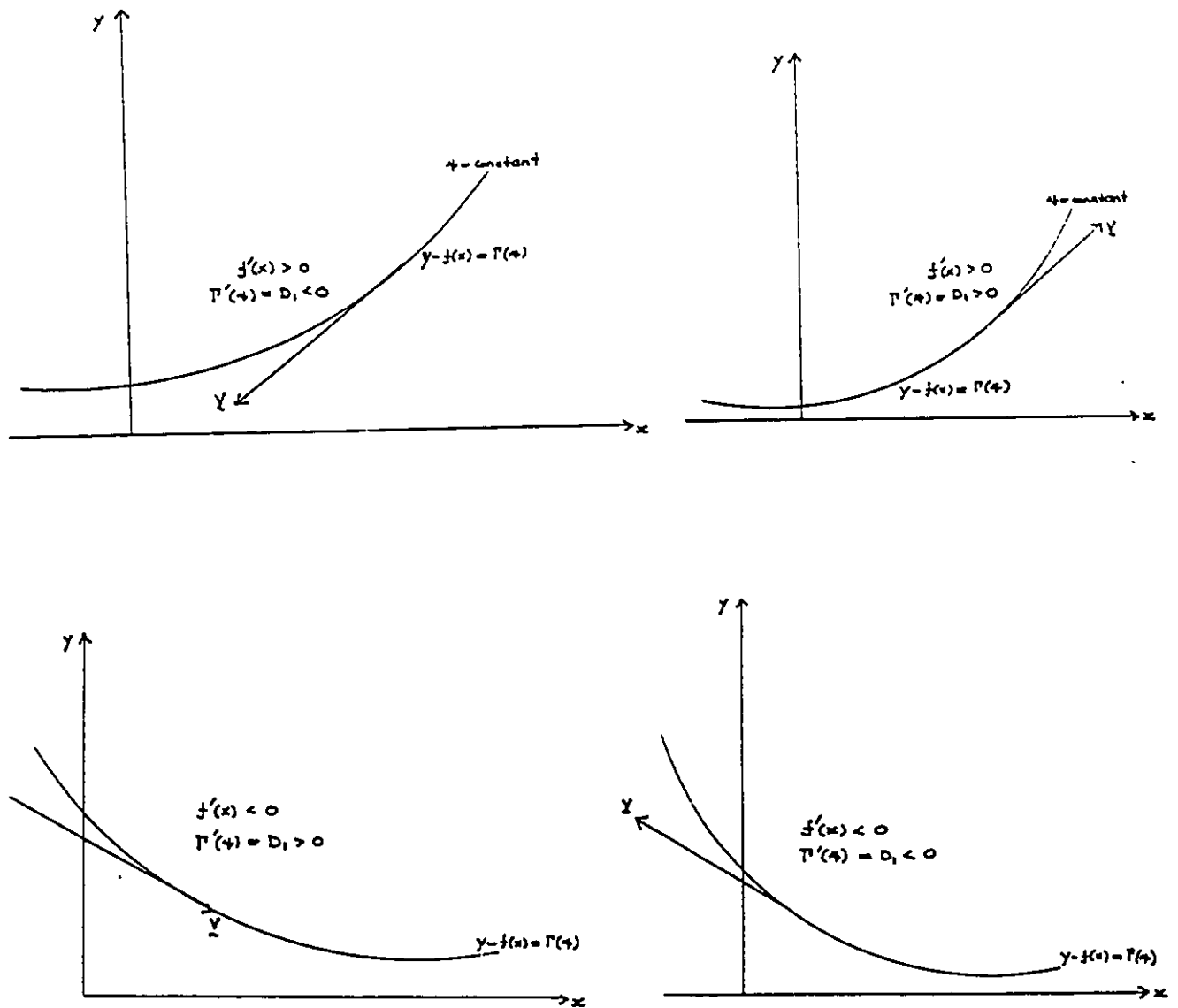
Figure 3.7.   Direction of flow along a streamline.

Integrating (3.4), (3.5) after using (3.87) and (3.88), we have

$$p(x,y) = -\int \alpha q'(\alpha)d\alpha = \int \frac{D_2 \alpha}{(D_1^2 \alpha^2 - 1)^{\frac{3}{2}}}d\alpha = p_0 - \frac{D_2}{D_1^2}\frac{1}{|f'(x)|} \qquad (3.91)$$

The state equation, obtained from solutions for pressure and density functions, is

given by

$$p = p_0 - \frac{1}{D_1^2 \rho} \tag{3.92}$$

Summing up, we have

Theorem 3.4. *Every family of monotonic curves of the form $y - f(x) =$ constant, so that $f'(x) > 0$ or $f'(x) < 0$, is a permissible streamline pattern in a steady plane irrotational isentropic compressible fluid flow of a tangent gas having the state equation given by (3.92) and the solutions u, v, ρ and p given by (3.90), (3.89) and (3.91).*

The streamfunction, the potential function, pressure as a function of flow intensity and the Mach number for this flow are respectively given by

$$\psi(x,y) = \frac{1}{D_1}[y - f(x)] - \frac{D_3}{D_1}$$

$$\Phi(x,y) = \frac{D_2}{D_1} \left(\frac{f'(x)}{|f'(x)|} y + \int \frac{1}{|f'(x)|} dx \right)$$

$$p = p_0 - \frac{D_2}{D_1^2} \frac{1}{\sqrt{D_1^2 \alpha^2 - 1}}$$

and

$$M = \sqrt{1 + f'^2(x)}$$

where D_3 is arbitrary constants.

Since obtained potential function must satisfy

$$\left(\Phi_y^2 - \hat{q}^2 \right) \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + \left(\Phi_x^2 - \hat{q}^2 \right) \Phi_{yy} = 0$$

for a supersonic flow of a tangent gas, it follows that constants D_1 and D_2 in our solutions are related by $D_2^2 = D_1^2 \hat{q}^2$. Here $\hat{q}$ is the Bernoulli's constant in Bernoulli's equation $q^2 - c^2 = \hat{q}^2$.

3.7 STRAIGHT FLOWS WITH $\Gamma''(\psi) = 0$.

For these flows, we have

$$f(x) = E_1 x + E_2, \qquad \Gamma(\psi) = E_3 \psi + E_4 \qquad (3.93)$$

where $E_3 \neq 0$, E_1, E_2 and E_4 are arbitrary constants.

The streamline pattern for this flow is shown in Figure 3.8.
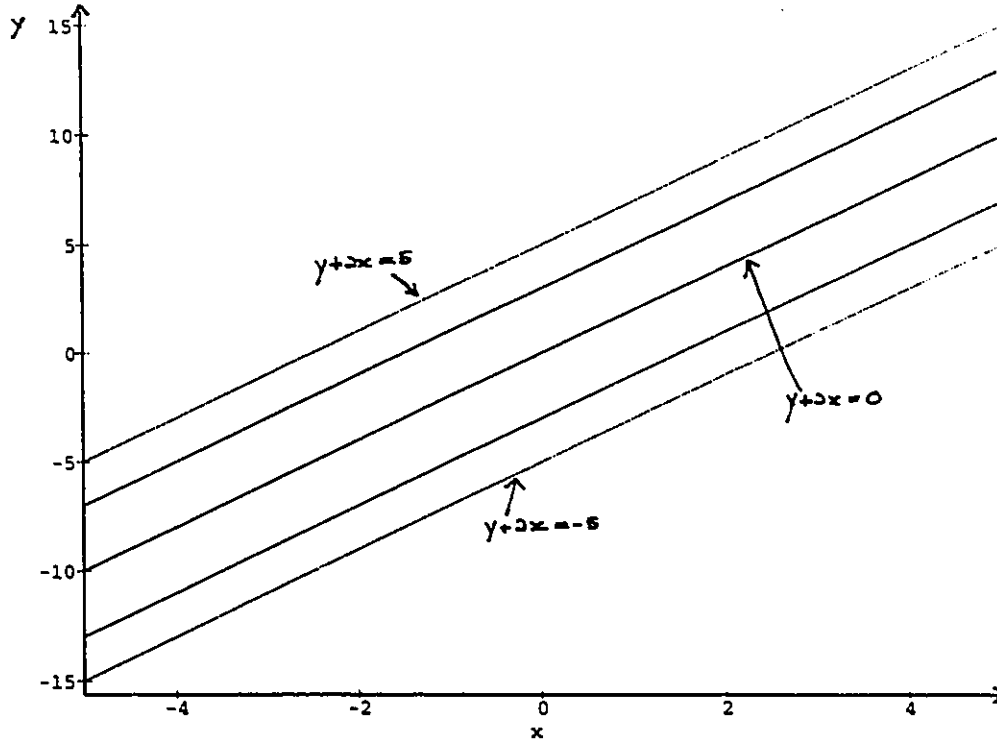
Figure 3.8.   Streamline pattern for $y - E_1 x - E_2 =$ constant $(E_1 = -2, E_2 = 0)$.

Employing (3.93), equations (3.4) to (3.7) take the form

$$\frac{\sqrt{1 + E_1^2}}{|E_3|} \frac{\partial q}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\sqrt{1 + E_1^2}}{|E_3|} \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0$$

$$E_1 E_3 \frac{\partial q}{\partial x} - (1 + E_1^2) \frac{\partial q}{\partial \psi} = 0$$

$$\rho q = \frac{\sqrt{1 + E_1^2}}{|E_3|}$$

Integrating this system, we have

$$q(x, y) = g(s), \qquad \rho(x, y) = \frac{\sqrt{1 + E_1^2}}{|E_3| g(s)}, \qquad (3.94)$$

$$u(x, y) = \frac{|E_3| g(s)}{E_3 \sqrt{1 + E_1^2}}, \qquad v(x, y) = \frac{E_1 |E_3| g(s)}{E_3 \sqrt{1 + E_1^2}}, \qquad (3.95)$$

and

$$p(x, y) = p_0 - \frac{\sqrt{1 + E_1^2}}{|E_3|} g(s) \qquad (3.96)$$

where $g(s)$ is an arbitrary positive valued function of $s = x + E_1 y - E_1 E_2 - E_1 E_4$.

The state equation for this flow is

$$p = p_0 - \left(\frac{1 + E_1^2}{E_3^2} \right) \frac{1}{\rho} \qquad (3.97)$$

where p_0 is an arbitrary constant.

Summing up, we have

Theorem 3.5. *The family of curves $y - E_1 x - E_2 =$ constant is a permissible streamline pattern for steady plane irrotational isentropic fluid flow of a tangent gas having the equation of state given by (3.97) and the solutions $\rho(x, y)$, $u(x, y)$, $v(x, y)$ and $p(x, y)$ given by (3.94), (3.95) and (3.96) respectively.*

The variation of pressure with flow intensity is given by

$$p = p_0 - g(s)\alpha$$

The streamfunction and the potential function for this flow are respectively given by

$$\psi(x, y) = \frac{1}{E_3}(y - f(x)) - \frac{E_4}{E_3} = \frac{1}{E_3}[y - E_1 x - E_2 - E_4]$$

and

$$\Phi(x,y) = \frac{|E_3|}{E_3\sqrt{1 + E_1^2}}X(s) + E_5$$

where $X'(s) = g(s)$ and E_5 is an arbitrary constant. The Mach number for this flow is given by

$$M = \frac{q}{c} = \frac{g(s)}{g(s)} = 1$$

Therefore, this flow is sonic througout the flow domain.

3.8 BOUNDARY VALUE PROBLEMS.

There are two definitions of exact integrals in fluid dynamics. According to the first definition, a set of known functions for dynamic and thermodynamic variables is said to constitute an exact integral if the flow equations are identically satisfied by these functions in the flow domain with no boundaries. In the second definition, a set of 'known' functions for dynamic and thermodynamic variables constitutes an exact integral if the set also gives the solution to a problem with boundaries arising from a real physical problem or a physical problem that could be realistically imposed.

Knowledge of exact integrals or solutions determined according to the first definition mostly leads to some real problem that may correspond to the solution. An exact solution which does not seem to correspond to any real problem now may one day be applied to a real problem.

Sections 3.4 to 3.7 were devoted to exact integrals for the flow patterns of this chapter according to the first definition. This section deals with simple boundary-value problems for three of the four sets of exact integrals found in those sections and are exact integrals according to the second definition.

FLOW BETWEEN PARALLEL PLATES

Steady potential isentropic compressible flow between two plates $2x - y - 5 = 0$ and $2x - y + 5 = 0$ can be studied as a plane flow in the x, y plane when the
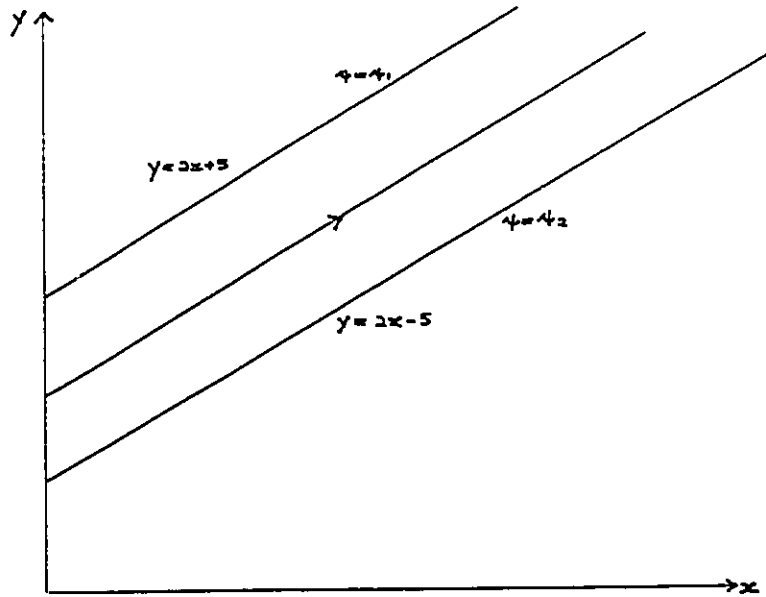
Figure 3.9.   Streamline pattern for flow between parallel plates.

flow is identical in all z =constant planes that is, the flow variables are functions of x, y only. Considering the flow bounded by straight lines $2x - y - 5 = 0$ and $2x - y + 5 = 0$ as shown in Figure 3.9, with pressure function prescribed on these bounding lines, we study the flow problem with flow streamlines

$$2x - y = c(\text{constant}); \qquad -5 \leq c \leq 5$$

We take the case when fluid flows along any streamline in the direction of increasing x. It follows that the pressure function decreases in the direction of increasing x and we prescribe one such pressure distribution on the wall $2x - y - 5 = 0$ given by

$$p(x, 2x - 5) = P_0 - 2e^x; \qquad -L < x < L \qquad (3.98)$$

Letting the bounding streamlines be $\psi(x,y) = \psi_1$, $\psi(x,y) = \psi_2$ and using $f(x) = E_1 x + E_2$, $\Gamma(\psi) = E_3 \psi + E_4$ in flow pattern $y - f(x) = \Gamma(\psi)$, we get

$$y - 2x - 5 = y - E_1 x - E_2 - E_3 \psi_1 - E_4$$

and

$$y - 2x + 5 = y - E_1 x - E_2 - E_3 \psi_2 - E_4$$

Rewriting, we get

$$(E_1 - 2)x + (E_3 \psi_1 + E_2 + E_4 - 5) = 0$$

and

$$(E_1 - 2)x + (E_3 \psi_2 + E_2 + E_4 + 5) = 0$$

For these equations to hold true for every x, we get

$$E_1 = 2, \qquad E_3 \psi_1 + (E_2 + E_4) - 5 = 0 \qquad \text{and} \qquad E_3 \psi_2 + (E_2 + E_4) + 5 = 0$$

Therefore, we have

$$E_1 = 2, \qquad E_2 + E_4 = \frac{5(\psi_1 + \psi_2)}{\psi_2 - \psi_1} \qquad \text{and} \qquad E_3 = \frac{10}{\psi_1 - \psi_2}$$

Using these constants in (3.94) to (3.97), we get

$$u(x,y) = \frac{1}{\sqrt{5}} g(s), \qquad v(x,y) = \frac{2}{\sqrt{5}} g(s), \qquad \rho(x,y) = \frac{\psi_1 - \psi_2}{2\sqrt{5}} \frac{1}{g(s)} \qquad (3.99)$$

and

$$p(x,y) = p_0 - \frac{(\psi_1 - \psi_2)}{2\sqrt{5}} g(s) \qquad (3.100)$$

where $g(s)$ is an arbitrary function of $s = x + 2y - 10\left(\frac{\psi_1 + \psi_2}{\psi_2 - \psi_1}\right)$. Using the boundary condition (3.98), we obtain

$$P_0 - 2e^x = p_0 - \frac{(\psi_1 - \psi_2)}{2\sqrt{5}} g\left(5x - 10\left[1 + \frac{\psi_1 + \psi_2}{\psi_2 - \psi_1}\right]\right) \qquad \text{on} \quad y = 2x - 5$$

This equation gives

$$p_0 = P_0 \qquad \text{and} \qquad g(t) = \frac{4\sqrt{5}}{\psi_1 - \psi_2} \exp\left(\frac{t}{5} + \frac{4\psi_2}{\psi_2 - \psi_1}\right) \qquad (3.101)$$

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Employing (3.101) in (3.99) and (3.100), the exact integral for our boundary value problem is

$$u(x,y) = \frac{4}{(\psi_1 - \psi_2)} \exp\left(\left(\frac{x+2y}{5}\right) + 2\right), \quad v(x,y) = \frac{8}{(\psi_1 - \psi_2)} \exp\left(\left(\frac{x+2y}{5}\right) + 2\right)$$

$$\rho(x,y) = \frac{(\psi_1 - \psi_2)^2}{40} \exp\left(-\left(\frac{x+2y}{5}\right) - 2\right) \quad \text{and} \quad p(x,y) = p_\infty - 2\exp\left(\left(\frac{x+2y}{5}\right) + 2\right)$$

The state equation for this flow is

$$p = P_0 - \frac{(\psi_1 - \psi_2)^2}{20} \frac{1}{\rho}$$

Comparing this state equation with the equation of the tangent gas, for an isentropic pressure-volume curve $p = k\rho^\gamma$, at $\left(\frac{1}{\rho_a}, p_a\right)$, we get

$$P_0 = p_a + k\gamma\rho_a^k, \qquad k\gamma\rho_a^{\gamma+1} = \frac{(\psi_1 - \psi_2)^2}{20}$$

Therefore, our flow between two plates with the prescribed pressure distribution is for the tangent gas approximated to the isentropic pressure-volume curve $p = k\rho^\gamma$ at the point $\left(\left[\frac{20k\gamma}{(\psi_1 - \psi_2)^2}\right]^{\frac{1}{\gamma+1}}, P_0 - k\gamma \left[\frac{20k\gamma}{(\psi_1 - \psi_2)^2}\right]^{-\frac{\gamma}{\gamma+1}}\right)$.

AN EXPONENTIAL CHANNEL FLOW WITH GIVEN MASS FLUX

The family of curves $y - e^x = C$, $\psi_1 \leq C \leq \psi_1 + e - \frac{1}{e}$, define a streamline pattern for a steady plane compressible potential flow in an exponential channel whose walls are given by

$$y - e^x = \psi_1$$

and

$$y - e^x = e - \frac{1}{e} + \psi_1$$

where $-\infty < x < \infty$.

Points A $(0, 1 + \psi_1)$ and D $(0, 1 + e - e^{-1} + \psi_1)$ are the respective points of intersection of the walls of the channel and the y-axis as shown in Figure 3.10.

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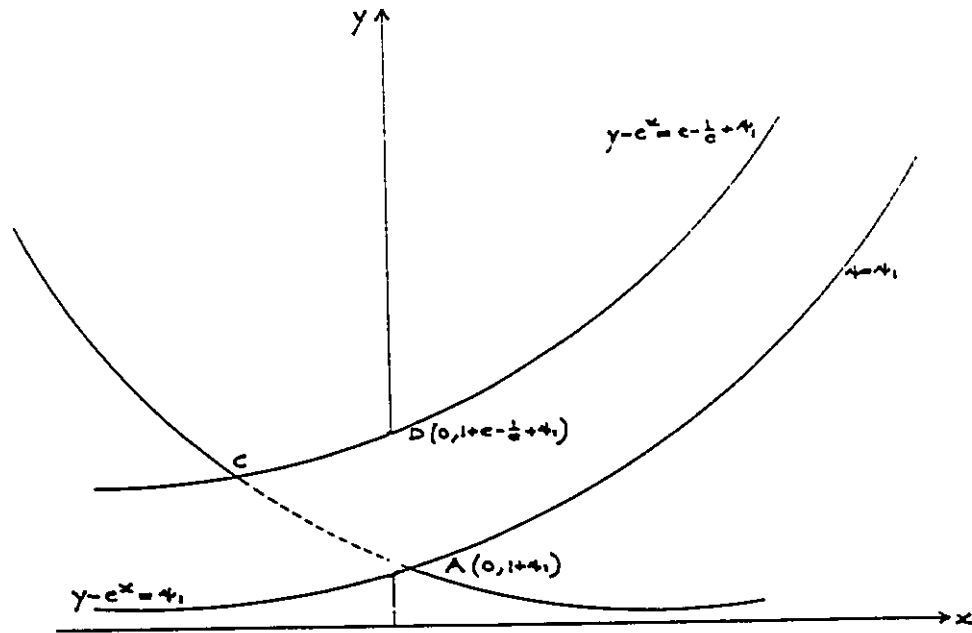
Figure 3.10.   Exponential channel flow.

The curve $y - e^{-x} = \psi_1$ is orthogonal to the flow streamlines passing through the point A $(0,\, 1 + \psi_1)$ on wall $y - e^x = \psi_1$ and the point C $(-1,\, e + \psi_1)$ on wall $y - e^x = e - \frac{1}{e} + \psi_1$.

Vector equation

$$\underset{\sim}{r} = \underset{\sim}{r}(t) = (t,\, e^{-t} + \psi_1); \qquad -1 \leq t \leq 0 \tag{3.102}$$

gives us the parametric representation of arc AC of the orthogonal trajectory.

Using (3.102), the unit normal vector field $\underset{\sim}{n}$ to AC and the differential element of arc length along AC are given by

$$\underset{\sim}{n} = \left(\frac{e^{-t}}{\sqrt{1 + e^{-2t}}},\, \frac{1}{\sqrt{1 + e^{-2t}}} \right) \tag{3.103}$$

and

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + e^{-2t}}\, dt \tag{3.104}$$

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Assuming that mass influx across the section AC of our channel is known to be m, then

$$\int_{AC} \rho \vec{V} \cdot \vec{n}\, ds = m \qquad (3.105)$$

Since streamlines are a family of monotonically increasing curves in our channel flow, it follows that the results of section 3.6 obtained for the monotonic streamline pattern $y - f(x) = \Gamma(\psi) = D_1\psi + D_1^*$ are valid for our flow. The walls of our channel have $\Gamma(\psi) = \psi_1$ and $\Gamma(\psi) = e - e^{-1} + \psi_1$ respectively. Monotonic function $f(x)$ is the monotonically increasing function e^x in our channel flow.

Using $f(x) = e^x$ in (3.89) to (3.91), solutions for our flow are given by

$$\rho = \frac{e^x}{D_2} \qquad , u = \frac{D_2}{D_1 e^x}, \qquad v = \frac{D_2}{D_1} \qquad (3.106)$$

and

$$p = p_0 - \frac{D_2}{D_1^2 e^x}$$

such that (3.105) is satisfied. Substituting (3.106) in (3.105) and using (3.103) and (3.104), we have

$$\int_{-1}^{0} \frac{1}{D_1}\left(e^{-t} + e^t\right) dt = \frac{1}{D_1}\left(e - \frac{1}{e}\right) = m \qquad (3.107)$$

Using $D_1 = \frac{e^2-1}{me}$ and $D_2 = |D_1|\hat{q} = \frac{e^2-1}{me}\hat{q}$ as given by (3.107) and $D_2^2 = D_1^2\hat{q}^2$ in (3.106), solution for our flow problem is determined.

Given mass influx m, ρ_a is given by $\rho_a^2 c_a^2 = \left[\frac{e^2-1}{me}\right]^2$. Also, given ρ_a, we can find mass influx $m = \frac{e-e^{-1}}{\rho_a c_a}$.

CHAPTER 4

STREAMLINE
PATTERN $\frac{r}{g(\theta)}$ = CONSTANT

4.1 INTRODUCTION.

In this chapter, we determine exact solutions for steady plane irrotational isentropic fluid flows whose streamlines are of the form $\frac{r}{g(\theta)}$ = constant in the physical plane where r and θ are the polar coordinates and $g(\theta)$ is a continuously differentiable function. Along these streamlines, $r = g(\theta)\Gamma(\psi)$ is a function of r and ψ, where $\Gamma(\psi)$ is some function of ψ. We choose the curves ϕ = constant to be θ = constant in the flow equations in (ϕ,ψ)-coordinates given in theorem 2.2. We obtained all permissible streamline patterns of this form and their exact solutions by employing the (θ,ψ)-coordinate system.

The governing equations are analyzed in this coordinate net and classified for this assumed form. This classification leads us to investigate six different types of possible flows, the first two of which are circular and straight flows. It is found that the function $g(\theta)$ is

(a) a solution of $g''(\theta) = 0$ for circular or vortex flows,

(b) a solution of $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = 0$ giving straight parallel flows,

(c) any function such that $g'(\theta) \neq 0$ and $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$,

(d) any solution of equations (4.45) and (4.46),

(e) a solution of $g(\theta)g''(\theta) + 2\left(\frac{1}{a_1} - 1\right)g'^2(\theta) + \left(\frac{2}{a_1} - 1\right)g^2(\theta) = 0$ where $a_1 \neq 0$

is an arbitrary constant and

(f) a solution of $g'^2(\theta) - b_1^2 g^2(\theta) = 0$ where $b_1 \neq 0$ is an arbitrary constant.

For each of the flows above, the flow equations are completely integrated in the (θ, ψ)-coordinate system giving us the exact solutions for all permissible flow patterns of the chosen form. The state equations for the gases that allow these flows are also determined. It is found that the gas that permits vortex flow can be either a polytropic gas or a tangent gas and the state equation for the straight flow is that of a tangent gas. It is also found that two solutions of permissible flows are not valid for a polytropic gas or a tangent gas.

4.2 FLOW EQUATIONS AND CLASSIFICATION OF FLOWS.

To determine all continuously differentiable functions $g(\theta) \in C^3$ so that steady plane irrotational isentropic fluid flows along a family of curves $\dfrac{r}{g(\theta)}$ =constant and to find the exact solutions of the resulting permissible flow patterns, we choose the (θ, ψ)-coordinates and initiate our study with

$$\frac{r}{g(\theta)} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{4.1}$$

where $\Gamma(\psi)$ is an arbitrary function of ψ and $\psi(r, \theta)$ is the streamfunction for our flow such that

$$\mathrm{d}\psi(r, \theta) = -\rho V_2 \mathrm{d}r + r\rho V_1 \mathrm{d}\theta, \qquad \underset{\sim}{V} = V_1(r, \theta)\underset{\sim}{e}_r + V_2(r, \theta)\underset{\sim}{e}_\theta \tag{4.2}$$

in polar coordinates.

This choice of coordinates that we have made requires us to choose the curves ϕ =constant to be θ =constant curves in the nonlinear system of governing flow equations in (ϕ, ψ)-coordinates given in theorem 2.2.

We use (4.1) in (2.8), (2.10) and have

$$E(\theta,\psi) = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 = \left[g^2(\theta) + g'^2(\theta)\right]\Gamma^2(\psi),$$

$$F(\theta,\psi) = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \psi} = g(\theta)g'(\theta)\Gamma(\psi)\Gamma'(\psi),$$

$$G(\theta,\psi) = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 = g^2(\theta)\Gamma'^2(\psi), \qquad (4.3)$$

$$J(\theta,\psi) = \left|\frac{\partial(x,y)}{\partial(\theta,\psi)}\right| = -g^2(\theta)\Gamma(\psi)\Gamma'(\psi),$$

and

$$W(\theta,\psi) = \sqrt{EG - F^2} = g^2(\theta)|\Gamma(\psi)||\Gamma'(\psi)|$$

where $x = g(\theta)\cos(\theta)\Gamma(\psi)$ and $y = g(\theta)\sin(\theta)\Gamma(\psi)$.

$J(\theta,\psi) = -W(\theta,\psi) = -g^2(\theta)\Gamma(\psi)\Gamma'(\psi) < 0$ whenever $\Gamma(\psi)$ and $\Gamma'(\psi)$ are either both positive or both negative. In this case, fluid flows along the streamlines $\frac{r}{g(\theta)}$ =constant in the direction of decreasing θ. However, $J(\theta,\psi) = W(\theta,\psi) = -g^2(\theta)\Gamma(\psi)\Gamma'(\psi) > 0$ whenever one of $\Gamma(\psi)$ and $\Gamma'(\psi)$ is positive and the other is negative. In this case, fluid flows along the streamlines in the direction of increasing θ.

Writing equations of theorem 2.2 in (θ,ψ)−coordinates, we use (4.3) to find that the Gauss equation is identically satisfied and the fluid flowing along the streamlines, given by (4.1) with ψ =constant, is governed by the system

$$\rho q \frac{\partial q}{\partial \theta} + \frac{\partial p}{\partial \theta} = 0 \qquad (4.4)$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \qquad (4.5)$$

$$\frac{\partial}{\partial \theta}\left[\frac{g(\theta)g'(\theta)\Gamma'(\psi)}{\sqrt{g^2(\theta) + g'^2(\theta)}}q\right] - \frac{\partial}{\partial \psi}\left[\sqrt{g^2(\theta) + g'^2(\theta)}\,\Gamma(\psi)q\right] = 0 \qquad (4.6)$$

$$\rho q = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|} \qquad (4.7)$$

$$p = R(\rho) \qquad (2.5)$$

of five equations in five unknowns $\rho(\theta,\psi)$, $p(\theta,\psi)$, $q(\theta,\psi)$, $\Gamma(\psi)$ and $g(\theta)$. Here $R(\rho)$ is some function of ρ to be determined.

In our search for permissible streamline patterns of the chosen form $\dfrac{r}{g(\theta)}$ =constant and in determination of the exact solutions for the resulting isentropic flows, we do not make any choice of the gas and, therefore, do not have a state equation for the study. Equations (4.4) to (4.7) are a system of four equations in five unknowns and we solve this underdetermined system.

Given a solution of the system (4.4) to (4.7), the state equation is determined from the solutions $p = p(\theta,\psi)$, $\rho = \rho(\theta,\psi)$ giving us the gas that flows along the obtained streamline pattern.

Using (4.7) with $\Gamma'(\psi) > 0$ and eliminating ρ from (4.4) and (4.5), we get

$$\frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|}\frac{\partial q}{\partial \theta} + \frac{\partial p}{\partial \theta} = 0 \tag{4.8}$$

$$\frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|}\frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \tag{4.9}$$

Employing the integrability condition $\dfrac{\partial^2 p}{\partial\theta\partial\psi} = \dfrac{\partial^2 p}{\partial\psi\partial\theta}$, we eliminate pressure p from (4.8) and (4.9) and have

$$\frac{\partial}{\partial \psi}\left[\frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|}\right]\frac{\partial q}{\partial \theta} - \frac{\partial}{\partial \theta}\left[\frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|}\right]\frac{\partial q}{\partial \psi} = 0 \tag{4.10}$$

Equations (4.6) and (4.10) are a system of two equations in three unknown functions $\Gamma(\psi)$, $g(\theta)$ and $q(\theta,\psi)$. Given a solution of this system, $p(\theta,\psi)$ is determined by the integration of (4.8), (4.9) and $\rho(\theta,\psi)$ is determined by using (4.7).

Defining

$$pq = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|} = \alpha(\theta,\psi) > 0 \tag{4.11}$$

the two equations (4.6) and (4.10) are rewritten as

$$\frac{\partial}{\partial \theta}\left[\frac{g'(\theta)}{g(\theta)\alpha(\theta,\psi)}q\right] - \frac{\partial}{\partial \psi}\left[g^2(\theta)\Gamma(\psi)\Gamma'(\psi)\alpha(\theta,\psi)q\right] = 0 \tag{4.12}$$

and

$$\left|\frac{\partial(\alpha,q)}{\partial(\theta,\psi)}\right| = \frac{\partial\alpha}{\partial\theta}\frac{\partial q}{\partial\psi} - \frac{\partial\alpha}{\partial\psi}\frac{\partial q}{\partial\theta} = 0 \qquad (4.13)$$

where

$$\frac{\partial\alpha}{\partial\theta} = \frac{g'(\theta)A(\theta)}{g^5(\theta)\Gamma'^2(\psi)\alpha(\theta,\psi)} = \frac{g'(\theta)A(\theta)\alpha(\theta,\psi)}{g^3(\theta) + g(\theta)g'^2(\theta)}, \quad \frac{\partial\alpha}{\partial\psi} = \frac{-\Gamma''(\psi)}{\Gamma'(\psi)}\alpha(\theta,\psi) \quad (4.14)$$

with

$$A(\theta) = g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)$$

In the study of non-uniform flows, (4.13) is identically satisfied if any one of the following holds true:

1. $\dfrac{\partial\alpha}{\partial\theta} = \dfrac{\partial q}{\partial\theta} = 0$, i.e., both flow intensity and speed are constant along each individual streamline.

2. $\dfrac{\partial\alpha}{\partial\theta} = \dfrac{\partial\alpha}{\partial\psi} = 0$, i.e., flow intensity is constant in the flow region.

3. $\dfrac{\partial q}{\partial\psi} = \dfrac{\partial\alpha}{\partial\psi} = 0$, i.e., flow intensity and speed are constant on each individual radial line.

4. $q = q(\alpha), q'(\alpha) \neq 0$, i.e., curves of constant flow intensity and speed coincide in the flow region.

Now, $\dfrac{\partial\alpha}{\partial\theta} = \dfrac{g'(\theta)\left[g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right]}{g^5(\theta)\Gamma'^2(\theta)\alpha(\theta,\psi)} = 0$ if either

(a) $g(\theta)$ =constant so that the streamlines $\dfrac{r}{g(\theta)}$ =constant are concentric circles r =constant,

or (b) $g(\theta) = b_2|\sec(\theta+b_1)|$, obtained by solving $g(\theta)g''(\theta)-2g'^2(\theta)-g^2(\theta) = 0$, so that the streamlines $\dfrac{r}{g(\theta)}$ =constant are a family of parallel straight lines.

Therefore, we classify our flows as follows:

Theorem 4.1. *If families of curves of the form* $\dfrac{r}{g(\theta)}$ *=constant define streamline patterns in a steady plane isentropic irrotational compressible fluid flow for some continuously differentiable function $g(\theta)$ so that* $\dfrac{r}{g(\theta)} = \Gamma(\psi)$, *where $\psi(r,\theta)$ is the*

flow streamfunction and $\Gamma(\psi)$ is some function of ψ with $\Gamma'(\psi) \gtrless 0$, then all possible

flows may be classified as

(i) circular flows with $g(\theta) =$ constant

(ii) flows with $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = A(\theta) = 0$

(iii) flows with $q = q(\theta)$ and any $g(\theta)$ such that $g'(\theta) \neq 0$ and $A(\theta) \neq 0$

(iv) flows with $q = q(\alpha)$ such that $q'(\alpha) \neq 0$, $\dfrac{\partial \alpha}{\partial \theta} \neq 0$ and $\dfrac{\partial \alpha}{\partial \psi} \neq 0$.

Flows classified in this theorem are studied in the following sections.

4.3 CIRCULAR FLOWS WITH $g(\theta) =$ CONSTANT.

Taking $g(\theta) =$ constant in (4.14) and using the result in (4.13) and (4.14), we find

that $\dfrac{\partial \alpha}{\partial \theta} = 0$ and, therefore, we have the following two cases:

1. $g'(\theta) = 0$ and $\Gamma''(\psi) = 0$

2. $g'(\theta) = 0$ and $\dfrac{\partial q}{\partial \theta} = 0$

We now study these two cases.

<u>Case 1</u> : In this case,

$$g(\theta) = c_0, \qquad \Gamma(\psi) = c_1\psi + c_2 = \frac{r}{c_0} \qquad (4.15)$$

where $c_0 \neq 0$, $c_1 \neq 0$ and c_2 are arbitrary constants and (4.1) is used. The streamline

pattern for this flow is shown in Figure 4.1.

Using (4.15) and (4.14) in (4.11) to (4.13), equation (4.13) is identically satisfied

and we have

$$pq = \alpha(\theta, \psi) = \frac{1}{|c_0 c_1|}, \qquad \frac{\partial}{\partial \psi}[rq] = 0 \qquad (4.16)$$

Solutions of (4.16) and integration of (4.4) and (4.5) give

$$q(r, \theta) = \frac{f(\theta)}{r} \qquad (4.17)$$

$$\rho(r, \theta) = \frac{r}{|c_0 c_1| f(\theta)} \qquad (4.18)$$
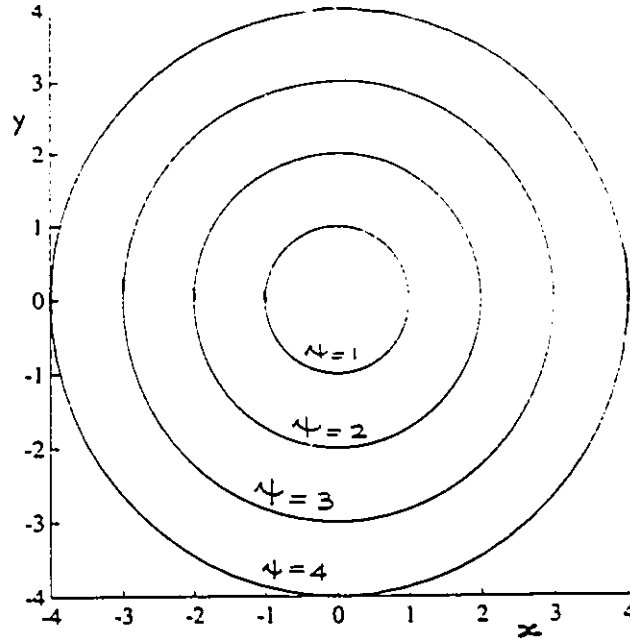
Figure 4.1.   Streamline pattern for vortex flow r =constant.

and

$$p(r,\theta) = p_0 - \frac{f(\theta)}{|c_0 c_1| r} \qquad (4.19)$$

where $f(\theta)$ is a single-valued and positive valued function.

Using (4.15) in (4.2), we obtain

$$\underset{\sim}{V} = \frac{1}{r\rho} \frac{\partial \psi}{\partial \theta} \underset{\sim}{e}_r - \frac{1}{\rho} \frac{\partial \psi}{\partial r} \underset{\sim}{e}_\theta = -\frac{|c_0 c_1|}{c_0 c_1} \frac{f(\theta)}{r} \underset{\sim}{e}_\theta \qquad (4.20)$$

Finally, the state equation for this flow is given by

$$p = p_0 - \frac{1}{c_0^2 c_1^2 \rho} \qquad (4.21)$$

The streamfunction, the potential function and the Mach number for this flow are given by

$$\psi(r) = \frac{1}{c_1} \left(\frac{r}{c_0} - c_2 \right),$$

$$\Phi(\theta) = c_3 - \frac{(c_0 c_1)}{c_0 c_1} F(\theta)$$

and

$$M = \frac{q}{c} = 1$$

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respectively where $F'(\theta) = f(\theta)$. Therefore, this flow is sonic throughout the flow domain.

Case 2: In this case,

$$g(\theta) = d_0, \qquad d_0 \Gamma(\psi) = r, \qquad q = q(\psi) = q(r) \qquad (4.22)$$

where $d_0 \neq 0$ is an arbitrary constant, $\Gamma(\psi)$ is an arbitrary function and (4.1) is used.

Using (4.22) in (4.11) to (4.13), we find that (4.13) is identically satisfied and we get

$$\alpha(\theta, \psi) = \rho q = \frac{1}{|d_0||\Gamma'(\psi)|}, \qquad \frac{d}{d\psi}\left[\frac{|d_0|\Gamma(\psi)\Gamma'(\psi)}{|\Gamma'(\psi)|}q\right] = 0 \qquad (4.23)$$

Solving (4.23), we find ρ, q and use the results in (4.4), (4.5) to obtain

$$q = \frac{d_1|\Gamma'(\psi)|}{|d_0|\Gamma(\psi)\Gamma'(\psi)}, \qquad \rho = \frac{\Gamma(\psi)}{d_1\Gamma'(\psi)}, \qquad p = p_0 + \frac{d_1}{d_0^2}\int\frac{1}{\Gamma^2(\psi)}d\psi \qquad (4.24)$$

where $d_1 \neq 0$ is an arbitrary constant.

Using (4.22) in (4.2), we get

$$\underset{\sim}{V} = V_2(r,\theta)\underset{\sim}{e_\theta} = -\frac{1}{\rho}\frac{\partial\psi}{\partial r}\underset{\sim}{e_\theta} = -\frac{d_1}{r}\underset{\sim}{e_\theta} \qquad (4.25)$$

Solutions given by (4.24) and (4.25) are valid for every choice of $\Gamma(\psi)$ with the only requirement that $\Gamma'(\psi) \neq 0$.

An Example: We study vortex flow of a polytropic gas as an example. The choice of $\Gamma(\psi) = \psi^{\frac{1-\gamma}{2}}$ is required for this flow and we use this choice in (4.22) to obtain

$$\Gamma(\psi) = \psi^{\frac{1-\gamma}{2}} = \frac{r}{d_0} \qquad (4.26)$$

Using (4.26), we get

$$\psi = \left(\frac{r}{d_0}\right)^{\frac{2}{1-\gamma}}, \qquad \Gamma'(\psi) = \left(\frac{1-\gamma}{2}\right)\psi^{-\left(\frac{\gamma+1}{2}\right)} = \left(\frac{1-\gamma}{2}\right)\left(\frac{r}{d_0}\right)^{\frac{\gamma+1}{\gamma-1}} \qquad (4.27)$$

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This equation gives $\Gamma'(\psi) < 0$ for $\gamma > 1$. Requiring $d_1 < 0$ in (4.25), we study counter-clockwise vortex flow and using (4.27), (4.24) with $d_0 > 0$, we obtain

$$q = -\frac{d_1 d_0}{|d_0|r} = -\frac{d_1}{r}, \qquad \rho = -\frac{2d_0^{\frac{2}{\gamma-1}}}{(\gamma-1)d_1}\left(\frac{1}{r}\right)^{\frac{2}{\gamma-1}}$$

and

$$p = p_0 + \frac{d_1 d_0^{\frac{2}{\gamma-1}}}{\gamma}\left(\frac{1}{r}\right)^{\frac{2\gamma}{\gamma-1}} \tag{4.28}$$

Taking $p_0 = 0$ and eliminating r between expressions for ρ and p, we obtain the state equation given by

$$p = A\rho^\gamma, \qquad A = \frac{d_1^{\gamma+1}(1-\gamma)^\gamma}{2^\gamma \gamma d_0^2}$$

The streamfunction for this flow is given by (4.27). The potential function and the Mach number are respectively given by

$$\Phi(\theta) = -d_1\theta + d_2 \qquad \text{and} \qquad M = \sqrt{\frac{d_1^2}{c_s^2 r^2 - \left(\frac{\gamma-1}{2}\right)d_1^2}}$$

where c_s is the sound speed at a stagnation point. Vortex flow studied in Case 2 has been studied by hodograph method in the literature and is well documented in texts on the subject [c.f. von Mises, 1958].

Theorem 4.2. *A family of curves* $\dfrac{r}{g(\theta)}$ *=constant, when* $g(\theta)$ *=constant in (4.1), is a permissible streamline pattern for steady plane potential isentropic compressible fluid flow either of a tangent gas in case* $\Gamma''(\psi) = 0$ *in (4.1) or of a polytropic gas in case* $\Gamma(\psi) = \psi^{\frac{1-\gamma}{2}}$. *The solutions for the two cases are given in the study of these two cases.*

4.4 STRAIGHT FLOWS WITH $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = 0$.

Using $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = 0$ in (4.14), we find that $\dfrac{\partial\alpha}{\partial\theta} = 0$ and, therefore, equations (4.13) and (4.14) give the following two cases:

1. $\quad g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = 0 \qquad$ and $\qquad \dfrac{\partial q}{\partial \theta} = 0$

2. $\quad g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) = 0 \qquad$ and $\qquad \Gamma''(\psi) = 0$

Since $g(\theta)g''(\theta) - g'^2(\theta) = g'^2(\theta) + g^2(\theta)$ in both cases, it follows that

$$\frac{\left[\dfrac{g'(\theta)}{g(\theta)}\right]'}{1 + \left[\dfrac{g'(\theta)}{g(\theta)}\right]^2} = 1$$

and, therefore, we get

$$\frac{g'(\theta)}{g(\theta)} = \tan(\theta + b_1), \qquad g(\theta) = b_2|\sec(\theta + b_1)| \qquad (4.29)$$

where b_1 and $b_2 \neq 0$ are arbitrary constants.

Using (4.29) in (4.1), the streamline pattern for this section is given by

$$r\cos(\theta + b_1) = \pm b_2\Gamma(\psi) \qquad (4.30)$$

<u>Case 1:</u> $\left(g(\theta) = b_2|\sec(\theta + b_1)| \qquad$ and $\qquad \dfrac{\partial q}{\partial \theta} = 0\right)$

In this case, equation (4.11) yields

$$\alpha(\theta, \psi) = \rho q = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|} = \frac{1}{|b_2||\Gamma'(\psi)|} \qquad (4.31)$$

Using (4.29), (4.31) and $\dfrac{\partial q}{\partial \theta} = 0$ in (4.12), we obtain $\dfrac{\partial q}{\partial \psi} = 0$ as well. This case is of uniform flow and is, therefore, removed from further consideration.

<u>Case 2:</u> $(g(\theta) = b_2|\sec(\theta + b_1)| \qquad$ and $\qquad \Gamma''(\psi) = 0)$

In this case, equation (4.11) yields

$$\alpha(\theta, \psi) = \rho q = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{g^2(\theta)|\Gamma'(\psi)|} = \frac{1}{|b_2||b_3|} \qquad (4.32)$$

where $\Gamma'(\psi) = b_3 \neq 0$ is an arbitrary constant.

Using (4.29) and (4.31) in (4.12), $q(\theta, \psi)$ satisfies

$$b_3 \tan(\theta + b_1) \frac{\partial q}{\partial \theta} - (b_3 \psi + b_4) \sec^2(\theta + b_1) \frac{\partial q}{\partial \psi} = 0$$

where b_4 is an arbitrary constant in $\Gamma(\psi) = b_3 \psi + b_4$.

Solution of this linear first order partial differential equation is

$$q(r, \theta) = q(\theta, \psi) = h(s), \quad s = (b_3 \psi + b_4) \tan(\theta + b_1) = \pm \frac{1}{b_2} r \sin(\theta + b_1) \quad (4.33)$$

where (4.30) has been used and $h(s)$ is an arbitrary function of s.

Using (4.33) in (4.32), (4.4) and (4.5), we obtain

$$\rho(r, \theta) = \frac{1}{|b_2 b_3| h(s)}, \qquad p(r, \theta) = p_0 - \frac{1}{|b_2 b_3|} h(s) \qquad (4.34)$$

where s is given by (4.33).

Also, the polar components of the velocity vector field and the state equation are given by

$$V = \mp \sin(\theta + b_1) h(s) \underset{\sim}{e}_r \mp \cos(\theta + b_1) h(s) \underset{\sim}{e}_\theta,$$

$$p = p_0 - \frac{1}{b_2^2 b_3^2 \rho} \qquad (4.35)$$

Taking the flow to be an isentropic flow of a polytropic gas $p = A\rho^\gamma$, our state equation in (4.35) is that for a tangent gas. Since tangent gas at $\left(p_i, \frac{1}{\rho_i} \right)$ for the polytropic gas is

$$p = (p_i + A\gamma \rho_i^\gamma) - A\gamma \rho_i^{\gamma+1} \left(\frac{1}{\rho} \right)$$

it follows that p_0 and $b_2 b_3$ are given by

$$p_0 = p_i + A\gamma \rho_i^\gamma, \qquad \frac{1}{|b_2 b_3|} = \sqrt{A\gamma \rho_i^{\gamma+1}}$$

Theorem 4.3. *A family of curves $r\cos(\theta + b_1)$ =constant, when $g(\theta) = b_2|\sec(\theta + b_1)|$ in (4.1), is a permissible streamline pattern for steady plane compressible isentropic potential flow with solutions given by (4.34) and (4.35).*

The potential function, the streamfunction and the Mach number for this flow are respectively given by

$$\Phi(r,\theta) = -b_2 H(s) + b_4,$$

$$\psi(r,\theta) = \frac{1}{b_3}\left\{\pm\frac{1}{b_2}r\cos(\theta + b_1) - b_5\right\}$$

and

$$M = \frac{q}{c} = 1$$

where $H'(s) = h(s)$. This flow is sonic throughout the flow domain.

4.5 FLOWS SATISFYING $q = q(\theta)$ WITH $q'(\theta) \neq 0$.

Taking $q = q(\theta)$ such that $q'(\theta) \neq 0$, we ignore uniform flow and note that (4.13) is identically satisfied when α =constant or $\alpha = \alpha(\theta)$ such that $\alpha'(\theta) \neq 0$. In the first case, we have $\dfrac{\partial q}{\partial \psi} = 0$, $\Gamma(\psi) = m_0\psi + m_1$ and $g(\theta) = m_2$ or $\dfrac{\partial q}{\partial \psi} = 0$, $\Gamma(\psi) = m_0\psi + m_1$ and $g(\theta) = m_3|\sec(\theta + m_4)|$ in (4.6), equation (4.6) is satisfied only if $m_0 = 0$ and or $m_3 = 0$ contrary to the restriction that $m_0 \neq 0$, $m_3 \neq 0$.

Therefore, $q = q(\theta)$, $\alpha = \alpha(\theta)$ such that $\alpha'(\theta) \neq 0$ and $g(\theta)$ is any function such that $g'(\theta) \neq 0$ and $A(\theta) = g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$. Equations (4.1), (4.11), (4.12) and (4.14), using $\Gamma(\psi) = a_1\psi + a_2$, yield

$$\rho q = \alpha(\theta) = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{|a_1|g^2(\theta)}, \qquad \frac{r}{g(\theta)} = \Gamma(\psi) = a_1\psi + a_2$$

and

$$\frac{1}{q}\frac{dq}{d\theta} + \left[\frac{g(\theta)\left\{g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right\}}{g'(\theta)\left\{g^2(\theta) + g'^2(\theta)\right\}}\right] = 0 \qquad (4.36)$$

where $a_1 \neq 0$, a_2 are arbitrary constants and $\alpha = \alpha(\theta)$ such that $\alpha'(\theta) \neq 0$.

Equations (4.36) and (4.2) give

$$q(\theta) = a_3 \exp\left[\int \frac{-g(\theta)\left\{g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right\}}{g'(\theta)\left\{g^2(\theta) + g'^2(\theta)\right\}}\,d\theta\right], \qquad (4.37)$$

$$\rho(\theta) = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{|a_1|a_3 g^2(\theta)} \exp\left[\int \frac{g(\theta)\left\{g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right\}}{g'(\theta)\left\{g^2(\theta) + g'^2(\theta)\right\}}\,d\theta\right],$$

$$V_1(\theta) = \frac{1}{r\rho}\frac{\partial\psi}{\partial\theta} = \frac{-|a_1|a_3 g'(\theta)}{a_1\sqrt{g^2(\theta) + g'^2(\theta)}} \exp\left[\int \frac{-g(\theta)\left\{g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right\}}{g'(\theta)\left\{g^2(\theta) + g'^2(\theta)\right\}}\,d\theta\right],$$

and

$$V_2(\theta) = -\frac{1}{\rho}\frac{\partial\psi}{\partial r} = \frac{-|a_1|a_3 g(\theta)}{a_1\sqrt{g^2(\theta) + g'^2(\theta)}} \exp\left[\int \frac{-g(\theta)\left\{g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)\right\}}{g'(\theta)\left\{g^2(\theta) + g'^2(\theta)\right\}}\,d\theta\right]$$

where $a_3 > 0$ is an arbitrary constant and $g(\theta)$ is any function of θ that must satisfy $g'(\theta) \neq 0$ and $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$ in the flow region.

Using (4.37) and (4.36) in (4.4), (4.5) and integrating, we obtain

$$p(\theta) = p_0 + \int \left\{ \frac{[g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta)]}{|a_1|g(\theta)g'(\theta)\sqrt{g^2(\theta) + g'^2(\theta)}} \right\} [q(\theta)]\,d\theta \qquad (4.38)$$

Theorem 4.4. *If $g'(\theta) \neq 0$, $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$ and q is constant on each individual radial line then every family of curves $\dfrac{r}{g(\theta)} =$constant is a permissible streamline pattern for a steady, plane, potential isentropic flow of some gas with solutions given by (4.37) and (4.38).*

An Example

Taking $g(\theta) = e^{m\theta}$, $m \neq 0$ is any real number, we note that $g'(\theta) \neq 0$ and $g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$. Using our choice of $g(\theta)$ in (4.37) and (4.38), solutions for the streamline pattern $re^{-m\theta} =$constant are given by

$$q = a_3 \exp\left(\frac{\theta}{m}\right), \qquad \rho = \frac{\sqrt{1 + m^2}}{|a_1|a_3} \exp\left[-\left(\frac{1 + m^2}{m}\right)\theta\right]$$

$$V_1 = \frac{-|a_1|a_3 m}{a_1\sqrt{1+m^2}}\exp\left(\frac{\theta}{m}\right), \qquad V_2 = \frac{-|a_1|a_3}{a_1\sqrt{1+m^2}}\exp\left(\frac{\theta}{m}\right)$$

and

$$p = \begin{cases} p_0 - \dfrac{a_3\sqrt{1+m^2}}{|a_1|(1-m^2)}\exp\left[\left(\dfrac{1}{m}-m\right)\theta\right]; & m \neq \pm 1 \\[3mm] p_0 - \dfrac{a_3\sqrt{1+m^2}}{|a_1|m}\theta; & m = \pm 1 \end{cases}$$

The state equation for this flow is given by

$$p = \begin{cases} p_0 - \dfrac{a_3^{m^2 n}\left(\sqrt{1+m^2}\right)^n}{|a_1|^n(1-m^2)}\rho^{\frac{m^2-1}{m^2+1}}; & m \neq \pm 1 \\[3mm] p_0 - \dfrac{a_3}{|a_1|\sqrt{1+m^2}}\ln\left[\dfrac{\sqrt{1+m^2}}{|a_1|a_3\rho}\right]; & m = \pm 1 \end{cases} \tag{4.39}$$

where $n = \frac{2}{1+m^2}$.

The flow pattern in this example is shown in Figure 4.2.
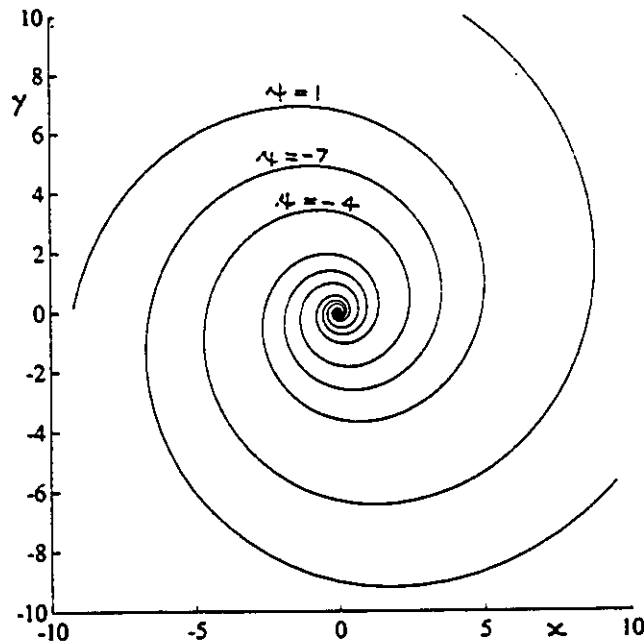
Figure 4.2.   Streamline pattern for $re^{-m\theta}$ =constant.

The potential function, the streamfunction and the Mach number for this flow are given by

$$\Phi(r,\theta) = -\frac{|a_1|a_3 m}{a_1\sqrt{1+m^2}}r\exp\left(\frac{\theta}{m}\right) + a_4,$$

$$\psi(r,\theta) = \frac{1}{a_1}\left(re^{-m\theta} - a_2\right)$$

and

$$M = \begin{cases} \sqrt{1 + m^2}\; ; & m \neq \pm 1 \\[2ex] \sqrt{\dfrac{1 + m^2}{a_3}\exp\left(-m\theta\right)}\; ; & m = \pm 1 \end{cases}$$

respectively where a_4 is an arbitrary constant.

Since for $m \neq \pm 1$, the Mach number corresponding to this flow is constant, this flow is of no physical interest and, therefore, removed from further consideration. For $m = \pm 1$, state equation (4.39) yields

$$\frac{dp}{d\rho} = \frac{a_3^2}{1 + m^2}\exp\left(\frac{m^2 + 1}{m}\right)\theta$$

and

$$\frac{d^2p}{d\rho^2} = \frac{a_3^2}{m}\exp\left(\frac{m^2 + 1}{m}\right)\theta$$

Since $\frac{dp}{d\rho} = c^2 > 0$ and $\frac{d^2p}{d\rho^2} > 0$ for all real compressible media, it follows that the solutions obtained in this example for $m = \pm 1$ corresponds to some real gas having p as a simple function of ρ, equation (4.39), as the state equation provided $m = 1$.

4.6 FLOWS SATISFYING $q = q(\alpha)$ SUCH THAT $q'(\alpha) \neq 0$, $\frac{\partial \alpha}{\partial \theta} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$.

Using the assumption $q = q(\alpha)$, the expression for $\alpha(\theta, \psi)$ from (4.11) and the expression for $\frac{\partial \alpha}{\partial \theta}$ and $\frac{\partial \alpha}{\partial \psi}$ from (4.14), equation (4.13) is identically satisfied while equation (4.12) takes the form

$$\left[\frac{g^2(\theta)A(\theta)}{g^2(\theta) + g'^2(\theta)}\right]q(\alpha)\Gamma'(\psi) + \left[\frac{g'^2(\theta)A(\theta)}{\alpha g^4(\theta)\Gamma'(\psi)} + \alpha^3 g^4(\theta)\Gamma(\psi)\Gamma'(\psi)\Gamma''(\psi)\right]q'(\alpha) = 0$$

where $A(\theta) = g(\theta)g''(\theta) - 2g'^2(\theta) - g^2(\theta) \neq 0$.

Dividing this equation by $\Gamma'(\psi)$, using (4.11) to eliminate $\Gamma'^2(\psi)$ and simplifying, we get

$$\left[\frac{g^2(\theta)A(\theta)}{g^2(\theta) + g'^2(\theta)}\right]q(\alpha) + \left[\frac{\alpha g'^2(\theta)A(\theta)}{g^2(\theta) + g'^2(\theta)} + \alpha^3 g^4(\theta)\Gamma(\psi)\Gamma''(\psi)\right]q'(\alpha) = 0$$

Dividing this equation by $\alpha^3 g^4(\theta)q'(\alpha)$, equation (4.12) for this case when $q = q(\alpha)$ is

$$\frac{A(\theta)}{g^2(\theta)\,[g^2(\theta)+g'^2(\theta)]}\left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right] + \frac{g'^2(\theta)A(\theta)}{g^4(\theta)\,[g^2(\theta)+g'^2(\theta)]}\left[\frac{1}{\alpha^2}\right] + \Gamma(\psi)\Gamma''(\psi) = 0 \tag{4.40}$$

Differentiating (4.40) with respect to θ, we have

$$\left(\frac{A(\theta)}{g^2(\theta)\,[g^2(\theta)+g'^2(\theta)]}\right)'\left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right] + \left(\frac{g'^2(\theta)A(\theta)}{g^4(\theta)\,[g^2(\theta)+g'^2(\theta)]}\right)'\left[\frac{1}{\alpha^2}\right]$$
$$+ \left\{\left(\frac{A(\theta)}{g^2(\theta)\,[g^2(\theta)+g'^2(\theta)]}\right)\left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]' - \left(\frac{g'^2(\theta)A(\theta)}{g^4(\theta)\,[g^2(\theta)+g'^2(\theta)]}\right)\left[\frac{-2}{\alpha^3}\right]\right\}\frac{\partial\alpha}{\partial\theta} = 0$$

Multiplying this equation by α^2 and using (4.11) to eliminate $\dfrac{\partial\alpha}{\partial\theta}$, we have

$$\left[\frac{A(\theta)}{g^2(\theta)\,\{g^2(\theta)+g'^2(\theta)\}}\right]'\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right] + \left[\frac{g'(\theta)A^2(\theta)}{g^3(\theta)\,\{g^2(\theta)+g'^2(\theta)\}^2}\right]\left[\alpha^3\left\{\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right\}'\right]$$
$$+ \left[\frac{g'^2(\theta)A(\theta)}{g^4(\theta)\,\{g^2(\theta)+g'^2(\theta)\}}\right]' - \left[\frac{2g'^3(\theta)A^2(\theta)}{g^5(\theta)\,\{g^2(\theta)+g'^2(\theta)\}^2}\right] = 0 \tag{4.41}$$

Differentiating (4.41) with respect to ψ and dividing the result by $\dfrac{\partial\alpha}{\partial\psi} \neq 0$, we get

$$\left(\frac{A(\theta)}{g^2(\theta)\,\{g^2(\theta)+g'^2(\theta)\}}\right)'\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' + \left(\frac{g'(\theta)A^2(\theta)}{g^3(\theta)\,\{g^2(\theta)+g'^2(\theta)\}^2}\right)\left[\alpha^3\left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0 \tag{4.42}$$

Equations (4.12) and (4.13) forming a system of two equations in three unknowns $q(\theta,\psi)$, $\Gamma(\psi)$ and $g(\theta)$ reduced to one equation (4.40) for flows satisfying $q = q(\alpha)$. Differentiation of this equation with respect to θ followed by differentiation with respect to ψ yielded (4.41) and (4.42). We shall use separation of variables technique on equation (4.42) to analyze our problem after two cases are studied where separation of variables cannot be used. In equation (4.42), variables θ and α are independent variables since $\left|\dfrac{\partial(\alpha,\theta)}{\partial(\theta,\psi)}\right| = -\dfrac{\partial\alpha}{\partial\psi} \neq 0$ in the flow domain. Knowing that $g'(\theta)A^2(\theta) \neq 0$, equation (4.42) requires us to study the following three cases:

Case 1: $\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0$ and $\left[\alpha^3 \left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$

Case 2: $\left(\dfrac{A(\theta)}{g^2(\theta)\,(g^2(\theta) + g'^2(\theta))}\right)' = 0$ and $\left[\alpha^3 \left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$

Case 3: $\left[\alpha^3 \left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' \neq 0$

We consider each of these cases separately.

Case 1: $\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0$ and $\left[\alpha^3 \left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$

In this case, we integrate the above differential equations and obtain the solution given by

$$q(\alpha) = k_1 \alpha^{k_0}, \qquad (4.43)$$

where $k_0 \neq 0$, $k_0 \neq 1$ and $k_1 > 0$ are arbitrary constants.

Using (4.43) and (4.11) in equations (4.40), (4.41) and (4.42), equation (4.42) is identically satisfied and we obtain

$$\frac{g^2(\theta) A(\theta)}{k_0 \left[g^2(\theta) + g'^2(\theta)\right]^2} + \frac{g'^2(\theta) A(\theta)}{\left[g^2(\theta) + g'^2(\theta)\right]^2} + \frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0 \qquad (4.44)$$

and

$$\frac{1}{k_0} \left(\frac{A(\theta)}{g^2(\theta)\left[g^2(\theta) + g'^2(\theta)\right]}\right)' - \frac{2}{k_0} \left(\frac{g'(\theta) A^2(\theta)}{g^3(\theta)\left[g^2(\theta) + g'^2(\theta)\right]^2}\right)$$

$$+ \left(\frac{g'^2(\theta) A(\theta)}{g^4(\theta)\left[g^2(\theta) + g'^2(\theta)\right]}\right)' - 2\left[\frac{g'^2(\theta)}{g^2(\theta)}\right]\left(\frac{g'(\theta) A^2(\theta)}{g^3(\theta)\left[g^2(\theta) + g'^2(\theta)\right]^2}\right) = 0 \qquad (4.45)$$

Equations (4.44) and (4.45) are a system of two equations in two unknowns $g(\theta)$ and $\Gamma(\psi)$. The function $g(\theta)$ is a solution of equation (4.45) and

$$\frac{\left[g^2(\theta) + k_0 g'^2(\theta)\right] A(\theta)}{k_0 \left[g^2(\theta) + g'^2(\theta)\right]^2} + k_2 = 0 \qquad (4.46)$$

obtained by using separation of variables on (4.44). Also, $\Gamma(\psi)$ is a solution of

$$\frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)} = k_2 \qquad (4.47)$$

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where $k_2 \neq 0$ is the separation constant in (4.46) and (4.47).

Integrating (4.47) and using (4.1), we have

$$\Gamma'(\psi) = k_3 \left[\Gamma(\psi)\right]^{k_2} = k_3 \left[\frac{r}{g(\theta)}\right]^{k_2} \tag{4.48}$$

where $k_3 \neq 0$ is any constant and $g(\theta)$ is any solution of equations (4.45) and (4.46).

Using (4.48) in (4.11), we obtain

$$\rho q = \alpha(r,\theta) = \frac{\sqrt{g^2(\theta) + g'^2(\theta)}}{\left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|} \tag{4.49}$$

Employing (4.49), (4.43), (4.1), (4.2) and (4.48), we have

$$q(r,\theta) = k_1 \alpha^{k_0} = \frac{k_1 \left[g^2(\theta) + g'^2(\theta)\right]^{\frac{1}{2}k_0}}{\left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|^{k_0}} \tag{4.50}$$

$$\rho(r,\theta) = \frac{\alpha}{q(\alpha)} = \frac{1}{k_1} \frac{\left[g^2(\theta) + g'^2(\theta)\right]^{\frac{1}{2}(1-k_0)}}{\left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|^{1-k_0}} \tag{4.51}$$

$$V_1(r,\theta) = \frac{1}{r\rho} \frac{\partial \psi}{\partial \theta} = -\frac{g'(\theta)}{\rho \Gamma'(\psi) g^2(\theta)} = \frac{-k_1 g'(\theta) \left[g^2(\theta) + g'^2(\theta)\right]^{\frac{1}{2}(k_0-1)}}{\left(k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right) \left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|^{k_0-1}} \tag{4.52}$$

$$V_2(r,\theta) = -\frac{1}{\rho} \frac{\partial \psi}{\partial r} = \frac{-1}{\rho g(\theta) \Gamma'(\psi)} = -\frac{k_1 g(\theta) \left[g^2(\theta) + g'^2(\theta)\right]^{\frac{1}{2}(k_0-1)}}{\left(k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right) \left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|^{k_0-1}} \tag{4.53}$$

Using (4.49) and (4.43) in (4.4), (4.5) and integrating, we obtain

$$p(r,\theta) = -\int \alpha(r,\theta) dq = -\int \left(\frac{q}{k_1}\right)^{\frac{1}{k_0}} dq = p_0 - \frac{k_0 k_1}{(1+k_0)} \left\{\frac{\left(g^2(\theta) + g'^2(\theta)\right)^{\frac{1}{2}(1+k_0)}}{\left\{\left|k_3 r^{k_2} \left[g(\theta)\right]^{2-k_2}\right|\right\}^{1+k_0}}\right\} \tag{4.54}$$

where p_0 is an arbitrary constant.

The equation of state for every solution function $g(\theta)$, is given by solutions for $\rho(r,\theta)$, $p(r,\theta)$ and is

$$p = p_0 - \frac{k_0 k_1^{\lambda_1}}{1+k_0} \rho^{\lambda_2} \tag{4.55}$$

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where $\lambda_1 = \dfrac{2}{1 - k_0}$ and $\lambda_2 = \dfrac{1 + k_0}{1 - k_0}$.

Summing up, we have

Theorem 4.5. *Every family of curves $\frac{r}{g(\theta)} = constant$ where $g(\theta)$ is a solution of (4.45) and (4.46) is a permissible streamline pattern for steady plane potential isentropic compressible fluid flow with solutions given by (4.51), (4.52), (4.53) and (4.54).*

Flow Examples

For every solution $g(\theta)$ of two non-linear ordinary differential equations (4.45) and (4.46) in $g(\theta)$, we have a streamline pattern $\dfrac{r}{g(\theta)} =$ constant and the exact integral for the flow is given by (4.50) to (4.54).

Taking $g(\theta) = e^{m\theta}$, where $m \neq 0$ is any real constant in (4.45) and (4.46), (4.45) is identically satisfied and (4.46) requires m to be a solution of

$$k_0(k_2 - 1)m^2 + (k_0 k_2 - 1) = 0, \qquad k_2 \neq 1 \tag{4.56}$$

if chosen $g(\theta)$ is to be a solution of (4.46).

For any choice of k_0, k_2 such that $k_0 \neq 0$, $k_0 \neq 1$, $k_2 \neq 1$ and $\dfrac{k_0 k_2 - 1}{k_0 k_2 - k_0} < 0$, we get

$$g(\theta) = \exp\left[\left(\sqrt{\frac{k_0 k_2 - 1}{k_0 - k_0 k_2}}\right)\theta\right]$$

or

$$g(\theta) = \exp\left[-\left(\sqrt{\frac{k_0 k_2 - 1}{k_0 - k_0 k_2}}\right)\theta\right] \tag{4.57}$$

as a solution of the system of two equations (4.45) and (4.46).

For every permissible choice of k_0 and k_2, a family of flow pattern is defined and the exact integral corresponding to the flow is given by (4.50) to (4.54) with $g(\theta)$ given by (4.57).

Polytropic Gas If the gas is a polytropic gas having the ratio of specific heats to be γ, then the equation of state in (4.55) requires k_0 to be $\dfrac{\gamma - 1}{\gamma + 1}$. Flow pattern is given by $\dfrac{r}{g(\theta)} =$ constant where

$$g(\theta) = \exp\left[\left(\sqrt{\frac{1}{\gamma - 1}\left\{\frac{k_2 + 1}{k_2 - 1} - \gamma\right\}}\right)\theta\right]$$

or

$$g(\theta) = \exp\left[-\left(\sqrt{\frac{1}{\gamma - 1}\left\{\frac{k_2 + 1}{k_2 - 1} - \gamma\right\}}\right)\theta\right]$$

(4.58)

Here $k_2 \neq 1$ and $k_2 < \dfrac{\gamma + 1}{\gamma - 1}$.

Spiral Flow Along $re^{-\theta} = $ constant for Polytropic Gas $p = A\rho^{\gamma}$

This flow is defined by $g(\theta) = e^{\theta}$. The streamline pattern for this flow is shown in Figure 4.3.
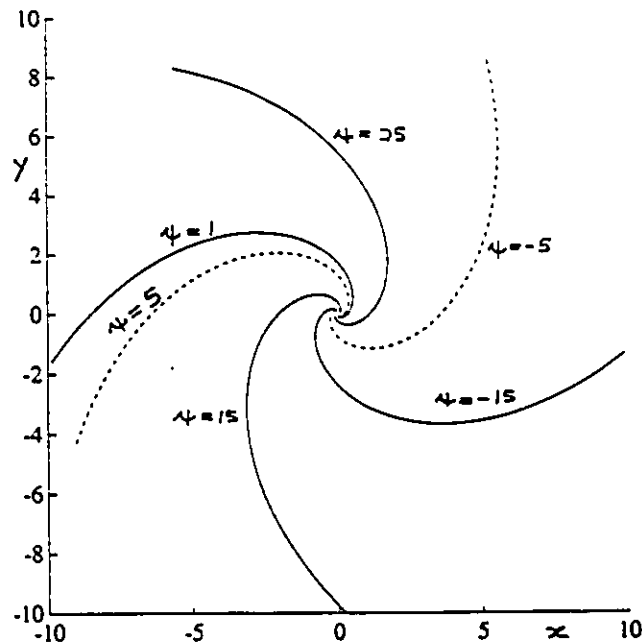
Figure 4.3.   Streamline pattern for $re^{-\theta} = $ constant.

Equation (4.56) takes the form $k_0(k_2 - 1) + k_0 k_2 - 1 = 0$ and we have

$$k_2 = \frac{k_0 + 1}{2k_0}$$

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Also, since the state equation (4.55) is $p = A\rho^\gamma$, it follows that

$$p_0 = 0, \qquad A = -\left(\frac{k_0}{1+k_0}\right) k_1^{\frac{2}{1-k_0}}, \qquad \frac{1+k_0}{1-k_0} = \gamma \qquad (4.59)$$

Solving these equations, we get

$$k_0 = \frac{\gamma-1}{\gamma+1}, \qquad k_1 = \left[-\frac{A(1+k_0)}{k_0}\right]^{\frac{1-k_0}{2}} = \left[\frac{-2A\gamma}{\gamma-1}\right]^{\frac{1}{\gamma+1}}, \qquad k_2 = \frac{\gamma}{\gamma-1} \qquad (4.60)$$

Using $g(\theta) = e^\theta$, (4.59) and (4.60) in (4.50) to (4.54), we get

$$\rho(r,\theta) = \left(\frac{2A\gamma}{1-\gamma}\right)^{-\frac{1}{\gamma+1}} \frac{2^{\frac{1}{\gamma+1}}}{|k_3|^{2/\gamma+1}r^{2\gamma/(\gamma^2-1)}} \exp\left(\frac{2}{(\gamma+1)(\gamma-1)}\right)\theta$$

$$V_1(r,\theta) = -\left(\frac{2A\gamma}{1-\gamma}\right)^{\frac{1}{\gamma+1}} \frac{|k_3|^{\frac{2}{\gamma+1}}}{2^{(\gamma-1)/2(\gamma+1)}k_3 r^{\gamma/\gamma+1}} \exp\left(\frac{1}{\gamma+1}\right)\theta$$

$$\qquad (4.61)$$

$$V_2(r,\theta) = -\left(\frac{2A\gamma}{1-\gamma}\right)^{\frac{1}{\gamma+1}} \frac{|k_3|^{\frac{2}{\gamma+1}}}{2^{(\gamma-1)/2(\gamma+1)}k_3 r^{\gamma/\gamma+1}} \exp\left(\frac{1}{\gamma+1}\right)\theta$$

$$p(r,\theta) = \frac{1-\gamma}{2\gamma}\left(\frac{2A\gamma}{1-\gamma}\right)^{\frac{1}{\gamma+1}} \frac{2^{\frac{\gamma}{\gamma+1}}}{|k_3|^{2\gamma/\gamma+1}r^{2\gamma^2/(\gamma^2-1)}} \exp\left(\frac{2\gamma}{(\gamma+1)(\gamma-1)}\right)\theta$$

The potential function, the streamfunction and the Mach number for this flow are respectively given by

$$\Phi(r,\theta) = -\frac{(\gamma+1)}{2^{\frac{\gamma-1}{2(\gamma+1)}}}\left(\frac{2A\gamma}{1-\gamma}\right)^{\frac{1}{\gamma+1}} \frac{|k_3|^{\frac{2}{\gamma+1}}}{k_3} r^{\frac{1}{\gamma+1}} \exp\left(\frac{1}{\gamma+1}\right)\theta,$$

$$\psi(r,\theta) = \frac{1}{k_3}\left[(1-\gamma)r^{\frac{1}{1-\gamma}}\exp\left(\frac{1}{\gamma-1}\right)\theta - l_0\right]$$

and

$$M = \sqrt{\frac{2l_1^2 \exp\left(\frac{2}{\gamma+1}\right)\theta}{c_s^2 r^{\frac{2\gamma}{\gamma+1}} - l_1^2(\gamma-1)\exp\left(\frac{2}{\gamma+1}\right)\theta}}$$

where l_0 is an arbitrary constant, c_s is the sound speed at a stagnation point and

$$l_1^2 = \left(\frac{2A\gamma}{1-\gamma}\right)^{2/1+\gamma} \frac{|k_3|^{4/\gamma+1}}{2^{\gamma-1/\gamma+1}k_3^2} \cdot$$

<u>Case 2:</u> $\left(\dfrac{A(\theta)}{g^2(\theta)\,(g^2(\theta)+g'^2(\theta))}\right)' = 0$ and $\left[\alpha^3\left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$

In this case, we have

$$\frac{A(\theta)}{g^2(\theta)\,(g^2(\theta)+g'^2(\theta))} = c_0 \tag{4.62}$$

where $c_0 \neq 0$ is an arbitrary constant and

$$\frac{q(\alpha)}{\alpha^3 q'(\alpha)} = c_2 - \frac{c_1}{2\alpha^2} \tag{4.63}$$

where c_1 and c_2 are arbitrary constants.

Employing (4.62) and (4.63) in (4.41), we get

$$c_0\left[c_1 - 2\left(\frac{g'^2(\theta)}{g^2(\theta)}\right)\right]g(\theta)g'(\theta) + \left(\frac{g'^2(\theta)}{g^2(\theta)}\right)' = 0 \tag{4.64}$$

Using (4.11), (4.62) and (4.63) in (4.40) and separating variables, we obtain

$$c_0\frac{g^2(\theta)g'^2(\theta)}{g^2(\theta)+g'^2(\theta)} - \frac{1}{2}c_0 c_1\frac{g^4(\theta)}{g^2(\theta)+g'^2(\theta)} = -\left[\frac{c_0 c_2}{\Gamma'^2(\psi)} + \frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)}\right] = c_4 \tag{4.65}$$

where $c_4 \neq 0$ is the separation constant.

Equation (4.65) yields

$$c_0 g^2(\theta)g'^2(\theta) - \frac{1}{2}c_0 c_1 g^4(\theta) - c_4\left(g^2(\theta)+g'^2(\theta)\right) = 0 \tag{4.66}$$

and

$$\frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)} + c_0 c_2\frac{1}{\Gamma'^2(\psi)} + c_4 = 0 \tag{4.67}$$

Since $g'(\theta) \neq 0$, (4.64) is satisfied if

$$c_1 - 2\left(\frac{g'^2(\theta)}{g^2(\theta)}\right) = 0 \tag{4.68}$$

Employing (4.68) in (4.62) and simplifying yields $g'(\theta) = 0$ which is contrary to the assumption of this case and, therefore, from (4.64), we have

$$c_0 g(\theta)g'(\theta) + \left(\frac{g'^2(\theta)}{g^2(\theta)}\right)'\left[c_1 - 2\left(\frac{g'^2(\theta)}{g^2(\theta)}\right)\right]^{-1} = 0$$

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Integrating this equation once, we obtain

$$g^2(\theta)\left[c_1 - \exp\left(c_0 g^2(\theta) - c_3\right)\right] = 2g'^2(\theta) \tag{4.69}$$

where c_3 is an arbitrary constant.

Using (4.69) in (4.62) and simplifying, we have

$$c_0\left(1 + \frac{c_1}{2}\right)g^2(\theta) - \frac{1}{2}\exp\left(c_0 g^2(\theta) - c_3\right) + \left(1 + \frac{c_1}{2}\right) = 0 \tag{4.70}$$

Employing (4.69) in (4.66), we get

$$\frac{1}{2}\left(c_4 - c_0 g^2(\theta)\right)\exp\left(c_0 g^2(\theta) - c_3\right) - c_4\left(1 + \frac{c_1}{2}\right) = 0 \tag{4.71}$$

Using (4.71), equation (4.70) yields

$$c_0^2 g^2(\theta) - c_0\left(c_4 - 1\right) = 0$$

since $g(\theta) \neq 0$ and $1 + \frac{1}{2}c_1 \neq 0$ from (4.70).

This equation gives $g'(\theta) = 0$ which is contrary to the assumption of this case and, therefore, this case does not yield any solution.

Case 3: $\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]' \neq 0$ and $\left[\alpha^3\left(\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' \neq 0$

In this case from (4.42), we have

$$\frac{\left\{\alpha^3\left[\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]'\right\}'}{\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]'} = -\frac{\left\{\dfrac{A(\theta)}{g^2(\theta)\left(g^2(\theta) + g'^2(\theta)\right)}\right\}'}{\left\{\dfrac{g'(\theta)A^2(\theta)}{g^3(\theta)\left(g^2(\theta) + g'^2(\theta)\right)^2}\right\}} = \text{constant} = -a_1 \tag{4.72}$$

Equation (4.72) yields

$$\left\{\alpha^3\left[\dfrac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]'\right\}' + a_1\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0 \tag{4.73}$$

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and

$$\left\{ \frac{A(\theta)}{g^2(\theta)\left(g^2(\theta) + g'^2(\theta)\right)} \right\}' - a_1 \left\{ \frac{g'(\theta)A^2(\theta)}{g^3(\theta)\left(g^2(\theta) + g'^2(\theta)\right)^2} \right\} = 0 \qquad (4.74)$$

Integrating (4.73), we get

$$\alpha^3 \left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)} \right)' + a_1 \left(\frac{q(\alpha)}{\alpha q'(\alpha)} \right) = a_2, \qquad (4.75)$$

$$q(\alpha) = \begin{cases} a_4 \left\{ a_2 \alpha^{a_1 - 2} + (a_1 - 2)a_3 \right\}^{\frac{1}{a_2}}; & a_1 \neq 2, \ a_2 \neq 0 \\[2mm] a_6 \exp\left(\frac{a_5}{a_1 - 2} \alpha^{a_1 - 2} \right); & a_1 \neq 2, \ a_2 = 0 \\[2mm] a_2^{\frac{1}{a_2}} \left[\ln\left(a_8 \alpha^{a_7} \right) \right]^{\frac{1}{a_2}}; & a_1 = 2, \ a_2 \neq 0 \\[2mm] a_{10} \alpha^{a_9}; & a_1 = 2, \ a_2 = 0 \end{cases} \qquad (4.76)$$

and

$$\frac{q(\alpha)}{\alpha q'(\alpha)} = \begin{cases} \dfrac{a_2}{a_1 - 2} + \dfrac{a_3}{\alpha^{a_1 - 2}}; & a_1 \neq 2, \ a_2 \neq 0 \\[2mm] \dfrac{1}{a_5 \alpha^{a_1 - 2}}; & a_1 \neq 2, \ a_2 = 0 \\[2mm] \dfrac{a_2}{a_7} \ln\left(a_8 \alpha^{a_7} \right); & a_1 = 2, \ a_2 \neq 0 \\[2mm] \dfrac{1}{a_9}; & a_1 = 2, \ a_2 = 0 \end{cases} \qquad (4.77)$$

where $a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, $a_5 \neq 0$, $a_6 \neq 0$, $a_7 \neq 0$, $a_8 \neq 0$, $a_9 \neq 0$ or 1 and $a_{10} \neq 0$ are arbitrary constants.

Various values of the constants a_1 and a_2 yield different flows. However, the cases when $a_1 \neq 2$, $a_2 = 0$ and $a_1 = 2$, $a_2 \neq 0$ lead to contradictions and so are not possible. Therefore, we consider the remaining two cases separately.

$\underline{a_1 \neq 2, \ a_2 \neq 0}$

Integrating (4.74) once with respect to θ, we obtain

$$\frac{A(\theta)}{g^2(\theta) + g'^2(\theta)} = \frac{2g^2(\theta)}{2a_{11} - a_1 g^2(\theta)} \qquad (4.78)$$

where a_{11} is an arbitrary constant.

Employing (4.74), (4.75) and (4.78) in (4.41) and simplifying, we obtain

$$(a_2 + 2 - a_1) g^4(\theta) + 2a_{11} \left[g^2(\theta) + g'^2(\theta) \right] = 0 \qquad (4.79)$$

If $a_{11} \neq 0$, then differentiating (4.79) with respect to θ, we get

$$g''(\theta) = \frac{(a_1 - a_2 - 2)}{a_{11}} g^3(\theta) - g(\theta) \qquad (4.80)$$

Using (4.80) in the expression for $A(\theta)$ gives

$$A(\theta) = 0$$

which implies that

$$\frac{\partial \alpha}{\partial \theta} = 0$$

contrary to the assumption of this case. Therefore, we must have from (4.79)

$$a_{11} = 0 \qquad \text{and} \qquad a_2 = a_1 - 2 \qquad (4.81)$$

Using (4.81) in (4.78) and integrating the resulting equation, we get

$$g(\theta) = \left[\frac{1}{\sqrt{a_{12}}} \cos \left\{ \frac{a_2}{a_1} \theta + \frac{a_2}{a_1} a_{13} \right\} \right]^{-\frac{a_1}{a_2}} \qquad (4.82)$$

where $a_{12} \neq 0$ and a_{13} are arbitrary constants.

Employing (4.11), (4.77), (4.78) and (4.81) in (4.40) and separating variables, we have

$$a_3 \frac{g^{2a_2+2}(\theta)}{(g^2(\theta) + g'^2(\theta))^{\frac{1}{2}a_2+1}} = \frac{1}{[\Gamma'(\psi)]^{a_2}} \left[\left(\frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)} \right) - \frac{2}{a_1} \right] = \text{constant} = a_{14}$$

This equation gives

$$a_3 \left[g(\theta) \right]^{2a_2+2} - a_{14} \left[g^2(\theta) + g'^2(\theta) \right]^{\frac{1}{2}a_2+1} = 0 \qquad (4.83)$$

and

$$\frac{a_1}{2} \Gamma(\psi)\Gamma''(\psi) - \left[\Gamma'^2(\psi) + a_{14}\Gamma'^{a_1}(\psi) \right] = 0 \qquad (4.84)$$

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Using (4.81) and (4.82) in (4.83) and simplifying, we obtain

$$\left[a_3 \left(\sqrt{a_{12}}\right)^{a_1} - a_{14}\right] \sec^{a_1} \lambda = 0 \tag{4.85}$$

where $\lambda = \dfrac{a_2}{a_1}\theta + \dfrac{a_2}{a_1}a_{13}$.

Equation (4.84) yields $\Gamma'(\psi)$ for various values of a_1. Choosing $a_1 = 1$ and solving (4.84), we get

$$\Gamma'(\psi) = b_0^2 \Gamma^2(\psi) - a_{14}$$

where $b_0 \neq 0$ is an arbitrary constant. Using (4.1), (4.82) and (4.81) with $a_1 = 1$ in the above equation, we have

$$\Gamma'(\psi) = \frac{a_{12} b_0^2 r^2 - a_{14}\cos^2\lambda}{\cos^2\lambda} \tag{4.86}$$

Employing $a_1 = 1$ in (4.85), we get

$$\left[a_3 \sqrt{a_{12}} - a_{14}\right]\sec\lambda = 0$$

Since $\sec\lambda \neq 0$, this equation yields

$$a_3 \sqrt{a_{12}} - a_{14} = 0 \tag{4.87}$$

Using $a_1 = 1$ and (4.87) in (4.82) and (4.11), we get

$$g(\theta) = \frac{1}{\sqrt{a_{12}}}\cos\lambda \tag{4.88}$$

and

$$\alpha(r,\theta) = \frac{\sqrt{a_{12}}}{a_{12} b_0^2 r^2 - a_{14}\cos^2\lambda} \tag{4.89}$$

respectively where $\lambda = -\theta - a_{13}$.

The streamlines for this flow are shown in Figure 4.4.
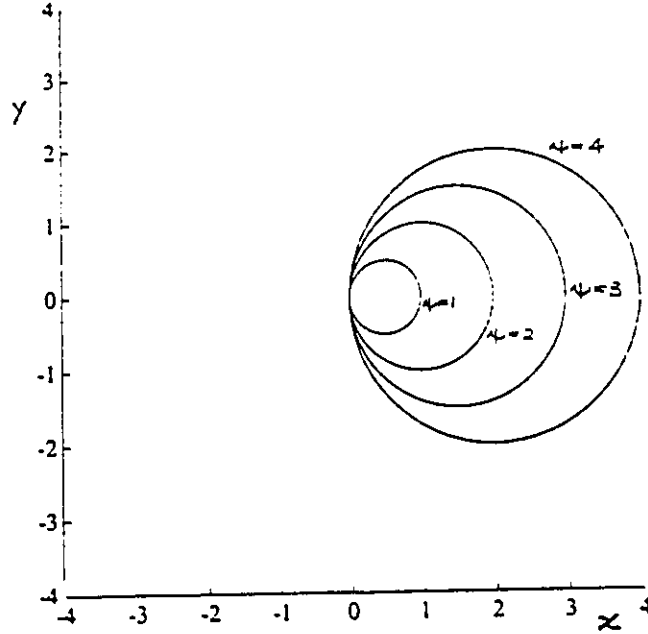
Figure 4.4. Streamline pattern $\dfrac{r}{\cos(\theta + a_{13})}$ =constant. $(a_{13} = 0)$.

Employing $a_1 = 1$, (4.81), (4.87) and (4.89) in (4.76), we get

$$q(r,\theta) = \frac{-a_4\sqrt{a_{12}}}{a_{12}b_0^2 r^2 + a_{14}\sin^2\lambda} \tag{4.90}$$

Employing (4.87), (4.89) and (4.90) in (4.11), the density function is given by

$$\rho(r,\theta) = \frac{a_{12}b_0^2 r^2 + a_{14}\sin^2\lambda}{a_4(a_{14}\cos^2\lambda - a_{12}b_0^2 r^2)} \tag{4.91}$$

Using (4.86), (4.87), (4.88) and (4.91), the polar components of velocity are given by

$$V_1(r,\theta) = \frac{a_4\sqrt{a_{12}}\sin\lambda}{a_{12}b_0^2 r^2 + a_{14}\sin^2\lambda} \tag{4.92}$$

and

$$V_2(r,\theta) = \frac{a_4\sqrt{a_{12}}\cos\lambda}{a_{14}\sin^2\lambda + a_{12}b_0^2 r^2} \tag{4.93}$$

The pressure function is given by

$$p(r,\theta) = p_0 + \frac{1}{a_3}q(r,\theta) - \frac{a_4}{a_3^2}\ln\left[a_4 + a_3 q(r,\theta)\right] \tag{4.94}$$

where $q(r,\theta)$ is given by (4.90) and p_0 is an arbitrary constant.

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The state equation for this flow is

$$p = p_0 - \frac{1}{a_3}\frac{1}{\rho} + \frac{a_4}{a_3^2}\ln\rho \qquad (4.95)$$

Summing up, we have

Theorem 4.6. *A family of curves* $r\cos^{-1}(\theta + a_{13}) = $ *constant when* $g(\theta) = \cos(\theta + a_{13})$ *in (4.1) is a permissible streamline pattern for steady plane compressible isentropic potential flow with solutions given by (4.91), (4.92), (4.93) and (4.94).*

The potential function and the streamfunction for the above flow are given by

$$\Phi(r,\theta) = a_4\sqrt{a_{12}}\int \frac{\sin\lambda dr + r\cos\lambda d\theta}{a_{12}b_0^2 r^2 + a_{14}\sin^2\lambda}$$

and

$$\psi(r,\theta) = \frac{1}{2b_0\sqrt{a_{14}}}\left[\ln\left(\frac{b_0\sqrt{a_{12}}r - \sqrt{a_{14}}\cos\lambda}{b_0\sqrt{a_{12}}r + \sqrt{a_{14}}\cos\lambda}\right) - l_1\right]$$

where l_1 is an arbitrary constant.

We have from (4.95)

$$\frac{dp}{d\rho} = \frac{1}{a_3}\frac{1}{\rho^2} + \frac{a_4}{a_3^2}\frac{1}{\rho}$$

$$\frac{d^2p}{d\rho^2} = -\frac{2}{a_3}\frac{1}{\rho^3} - \frac{a_4}{a_3^2}\frac{1}{\rho^2}$$

Since $\dfrac{dp}{d\rho} > 0$ and $\dfrac{d^2p}{d\rho^2} > 0$ for all real compressible media, we obtain from the two equations above the restrictions

$$\rho < -\frac{2a_3}{a_4} \qquad \text{and} \qquad \rho > -\frac{a_3}{a_4}$$

Since the density function is positive definite, we conclude that the ratio $\frac{a_3}{a_4}$ must be a negative but arbitrary constant. Therefore, (4.95) is the state equation that corresponds to some real gas giving p as a simple-valued function of ρ for every choice of permissible choice of a_3 and a_4.
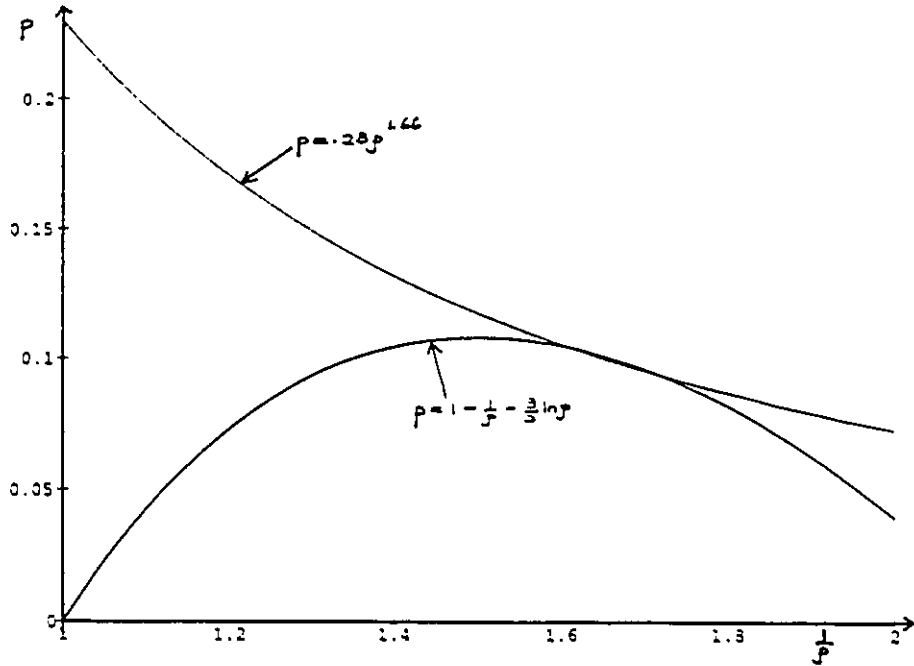
Figure 4.5. Pressure versus $1/\rho$ as given by equation (4.95) with $a_3 = 1$, $a_4 = -\frac{3}{2}$.

Figure 4.5 shows the isentrope of state equation (4.95) with $a_3 = 1, a_4 = -\frac{3}{2}$ against that of a polytropic gas $p = .23\rho^{1.66}$. We observed that for a small range of values of ρ, the two isentropes are in good agreement. We may, therefore, consider that the solutions obtained above describe reasonably the behaviour of an ideal gas within this range.

$\underline{a_1 = 2, \ a_2 = 0}$

Integrating (4.74) once with respect to θ, we obtain

$$\frac{A(\theta)}{g^2(\theta) + g'^2(\theta)} = \frac{2g^2(\theta)}{2\lambda_0 - 2g^2(\theta)} \tag{4.96}$$

where λ_0 is an arbitrary constant.

Employing (4.74), (4.75) and (4.96) in (4.41) and simplifying, we get

$$\left(\frac{g'^2(\theta)}{g^2(\theta)}\right)' = 0 \tag{4.97}$$

Solving (4.97), we have

$$g(\theta) = b_2 \exp(b_1\theta) \tag{4.98}$$

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where $b_1 \neq 0$ and $b_2 \neq 0$ are arbitrary constants. The flow pattern in this case is identical to Figure 4.2.

Using (4.98) in (4.96), equation (4.96) yields $\lambda_0 = 0$. Therefore, employing (4.11), (4.77) and (4.96) with $\lambda_0 = 0$ and (4.98) in (4.40) and separating variables, we have

$$\frac{1 + a_9 b_1^2}{a_9 (1 + b_1^2)} = \frac{\Gamma(\psi)\Gamma''(\psi)}{\Gamma'^2(\psi)} = \text{constant} = b_3 \neq 0$$

This equation gives

$$a_9 = \frac{1}{b_3 + b_1^2(b_3 - 1)}$$

and

$$\Gamma'(\psi[r, \theta]) = \frac{b_4}{b_2^{b_3}} r^{b_3} \exp\left(-b_1 b_3 \theta\right) \tag{4.99}$$

Employing (4.98) and (4.99) in (4.11), we get

$$\alpha(r, \theta) = \frac{|b_2^{b_3}|\sqrt{1 + b_1^2}}{b_2 |b_4| r^{b_3} \exp\left(b_1 [1 - b_3] \theta\right)} \tag{4.100}$$

The speed, density, components of velocity and the pressure for this flow are given by

$$q(r, \theta) = a_{10} \left\{ \frac{|b_2^{b_3}|\sqrt{1 + b_1^2}}{b_2 |b_4| r^{b_3} \exp\left(b_1 [1 - b_3] \theta\right)} \right\}^{a_9}, \tag{4.101}$$

$$\rho(r, \theta) = \frac{1}{a_{10}} \left\{ \frac{|b_2^{b_3}|\sqrt{1 + b_1^2}}{b_2 |b_4| r^{b_3} \exp\left(b_1 [1 - b_3] \theta\right)} \right\}^{1 - a_9}, \tag{4.102}$$

$$V_1(r, \theta) = -\frac{a_{10} b_1}{\sqrt{1 + b_1^2}} \left\{ \frac{|b_2^{b_3}|\sqrt{1 + b_1^2}}{b_2 |b_4| r^{b_3} \exp\left(b_1 [1 - b_3] \theta\right)} \right\}^{a_9}, \tag{4.103}$$

$$V_2(r, \theta) = -\frac{a_{10}}{\sqrt{1 + b_1^2}} \left\{ \frac{|b_2^{b_3}|\sqrt{1 + b_1^2}}{b_2 |b_4| r^{b_3} \exp\left(b_1 [1 - b_3] \theta\right)} \right\}^{a_9}, \tag{4.104}$$

and

$$p(r, \theta) = p_0 - \frac{a_9}{1 + a_9} \left(\frac{1}{a_{10}}\right)^{\frac{1}{a_9}} q^{\frac{1 + a_9}{a_9}}(\alpha) \tag{4.105}$$

respectively where $q(r, \theta)$ is given by (4.101).

The state equation for this flow is given by

$$p = p_0 - \frac{a_9}{1 + a_9} a_{10}^{\frac{2}{1 - a_9}} \rho^{\frac{1 + a_9}{1 - a_9}} \tag{4.106}$$

Summing up, we have

Theorem 4.7. *A family of curves* $r\exp(-b_1\theta) = constant$ *is a permissible flow pattern for a steady plane potential isentropic flow of a polytropic gas with solutions given by (4.102), (4.103), (4.104) and (4.105).*

The potential function, the streamfunction and the Mach number for the above flow are given by

$$\Phi(r,\theta) = \frac{a_{10}b_1}{\sqrt{1+b_1^2}\,(1-a_9b_3)} \left\{ \frac{|b_2^{b_3}|\sqrt{1+b_1^2}}{b_2|b_4|} \right\}^{a_9} r^{1-a_9b_3} \exp\left[a_9 b_1 \left(b_3 - 1\right)\theta\right],$$

$$\psi(r,\theta) = \frac{1}{b_4}\left[\frac{1}{(1-b_3)}\left(\frac{r}{b_2\exp\left(b_1\theta\right)}\right)^{1-b_3} - l_2\right]$$

and

$$M = \sqrt{\frac{q^2}{c_s^2 - \left(\frac{\gamma-1}{2}\right)q^2}}$$

where l_2 is an arbitrary constant, c_s is the sound speed at a stagnation point and $q(r,\theta)$ is given by (4.101).

As in the example above, we may consider the spiral flow $re^{-b_1\theta} = $ constant of a polytropic gas and obtain the relationships between the constants in the solutions (4.102) to (4.105) and the adiabatic constant γ.

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CHAPTER 5

STREAMLINE
PATTERN $\theta - f(r) =$ CONSTANT

5.1 INTRODUCTION.

This chapter deals with a class of flows for which the streamlines in the physical plane take the form $\theta - f(r) =$ constant where r and θ are the polar coordinates and $f(r)$ is a continuously differentiable function. For these flows, $\theta = f(r) + \Gamma(\psi)$ is a function of r and ψ along such streamlines. The coordinate lines $\phi =$ constant is chosen to be $r =$ constant in the flow equations in (ϕ,ψ)-coordinates of theorem 2.2. We determined the permissible streamline patterns of this form and their exact solutions by employing the (r,ψ) coordinate system.

Having assumed that streamlines can be expressed as $\theta - f(r) =$ constant, the flow equations are analyzed and classified for this chosen form. This classification results in the investigation of six different possible flows. It is found that the function $f(r)$ is

    (a) a solution of $r^3 f'(r) f''(r) - 1 = 0$,

    (b) any $f(r)$ such that $f'(r) \neq 0$,

    (c) any solution of equations (5.44) and (5.51),

    (d) the solution of equation (5.66).

    (e) any solution of (5.87) and (5.89),

    (f) a solution of $r f'(r) =$ constant.

The governing equations are completely integrated in the (r,ψ)-coordinate sys-

tem. Exact solutions are determined and the state equation for the gas that allows these obtained solutions are found for each flow. Two of the permissible flows are valid for a polytropic gas and two more are valid for a tangent gas. It is also found that the remaining two flows are neither valid for a polytropic gas nor its tangent approximation.

5.2 FLOW EQUATIONS AND CLASSIFICATION OF FLOWS.

We consider flow streamlines of the form $\theta - f(r) =$ constant where $f(r) \in C^3$ is any continuously differentiable function. Since $\psi(r, \theta) =$ constant are also the streamlines for steady plane flows when

$$d\psi(r, \theta) = -\rho V_2 dr + r\rho V_1 d\theta \quad \text{and} \quad \vec{V} = V_1(r, \theta)\vec{e}_r + V_2(r, \theta)\vec{e}_\theta \quad (5.1)$$

in polar coordinates, it follows that there exists some function $\Gamma(\psi)$ such that

$$\theta = f(r) + \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \qquad (5.2)$$

Choosing (r, ψ)-coordinates and using (5.2) in equations (2.8) and (2.10), we have

$$E(r, \psi) = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 = 1 + r^2 f'^2(r),$$

$$F(r, \psi) = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \psi} = r^2 f'(r)\Gamma'(\psi),$$

$$G(r, \psi) = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 = r^2\Gamma'^2(\psi), \qquad J(r, \psi) = r\Gamma'(\psi), \qquad W(r, \psi) = r\Gamma'(\psi)$$

$$(5.3)$$

where $x = r\cos[f(r) + \Gamma(\psi)]$ and $y = r\sin[f(r) + \Gamma(\psi)]$ are used. $J(r, \psi) \gtrless 0$ according as $\Gamma'(\psi) \gtrless 0$. Fluid flows along a streamline in the direction of increasing or decreasing r according as $\Gamma'(\psi) \gtrless 0$.

Employing (5.3) in equations of theorem 2.2 written in (r, ψ)-coordinates, we find that the Gauss equation is identically satisfied and the fluid flowing along our streamlines, given by (5.2) with $\psi =$ constant, is governed by the system

$$\rho q \frac{\partial q}{\partial r} + \frac{\partial p}{\partial r} = 0, \qquad (5.4)$$

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$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0. \tag{5.5}$$

$$\frac{\partial}{\partial r}\left(\frac{r^2 f'(r)\Gamma'(\psi)}{\sqrt{1 + r^2 f'^2(r)}}q\right) - \frac{\partial}{\partial \psi}\left(\sqrt{1 + r^2 f'^2(r)}\,q\right) = 0, \tag{5.6}$$

$$\rho q = \frac{\sqrt{1 + r^2 f'^2(r)}}{r|\Gamma'(\psi)|} = \pm\frac{\sqrt{1 + r^2 f'^2(r)}}{r\Gamma'(\psi)}, \tag{5.7}$$

$$p = R(\rho) \tag{2.5}$$

where $\pm$ sign is taken in (5.7) according as $\Gamma'(\psi) \gtrless 0$, that is, according as fluid flows along a streamline in the direction of increasing or decreasing r.

Equations (5.4) to (5.7) and (2.5) are a system of five equations in five unknowns $\rho(r, \psi)$, $p(r, \psi)$, $q(r, \psi)$, $\Gamma(\psi)$ and $f(r)$. $R(\rho)$ is some function to be determined.

In our search for exact solutions, when streamlines are a family of curves of the form $\theta - f(r) =$ constant, we do not make any choice of the gas and, therefore, do not have a state equation (2.5) to use. Equations (5.4) to (5.7) are a system of four equations in five unknown functions and we solve this underdetermined system.

Once a solution for a determined flow is known, the state equation is obtained from solutions for $p(r, \psi)$ and $\rho(r, \psi)$. We use (5.7) in (5.4), (5.5) to eliminate ρ and employ $\frac{\partial^2 p}{\partial r \partial \psi} = \frac{\partial^2 p}{\partial \psi \partial r}$ to get

$$\frac{\partial}{\partial \psi}\left[\frac{\sqrt{1 + r^2 f'^2(r)}}{r\Gamma'(\psi)}\right]\frac{\partial q}{\partial r} - \frac{\partial}{\partial r}\left[\frac{\sqrt{1 + r^2 f'^2(r)}}{r\Gamma'(\psi)}\right]\frac{\partial q}{\partial \psi} = 0 \tag{5.8}$$

Equations (5.6) and (5.8) are a system of two equations in three unknown functions $\Gamma(\psi)$, $f(r)$ and $q(r, \psi)$. Having found a solution of this system, we find $\rho(r, \psi)$ from (5.7) and $p(r, \psi)$ by integrating (5.4) and (5.5).

Defining

$$\rho q = \alpha(r, \psi) = \pm\frac{\sqrt{1 + r^2 f'^2(r)}}{r\Gamma'(\psi)} > 0, \tag{5.9}$$

system of equations (5.6) and (5.8) can be written as

$$\frac{\partial}{\partial r}\left[\frac{rf'(r)}{\alpha}q\right] - \frac{\partial}{\partial \psi}\left[r\Gamma'(\psi)\alpha q\right] = 0 \tag{5.10}$$

and

$$\frac{\partial \alpha}{\partial r}\frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial r} = 0 \tag{5.11}$$

where

$$\frac{\partial \alpha}{\partial r} = \frac{r^3 f'(r)f''(r) - 1}{r^3 \Gamma'^2(\psi)\alpha(r,\psi)}, \qquad \frac{\partial \alpha}{\partial \psi} = -\frac{\alpha \Gamma''(\psi)}{\Gamma'(\psi)} \tag{5.12}$$

Equation (5.11) is identically satisfied if any of the following holds true:

(1) $\frac{\partial \alpha}{\partial r} = 0,$ $\frac{\partial \alpha}{\partial \psi} = 0$, i.e., flow intensity is constant throughout the flow domain.

(2) $\frac{\partial \alpha}{\partial r} = 0,$ $\frac{\partial q}{\partial r} = 0$, i.e., both flow intensity and flow speed are constant along each individual streamline.

(3) $\frac{\partial \alpha}{\partial \psi} = 0,$ $\frac{\partial q}{\partial \psi} = 0$, i.e., flow intensity and speed are constant on each individual circle $r = $ constant.

(4) $\frac{\partial q}{\partial r} = 0,$ $\frac{\partial q}{\partial \psi} = 0$, i.e., flow speed is constant throughout the flow region and, therefore, of no interest and removed from further consideration.

(5) $q = q(\alpha),$ $q'(\alpha) \neq 0$, i.e., curves of constant flow intensity and speed coincide in the flow domain.

The first equation of (5.12) and $\frac{\partial \alpha}{\partial r} = 0$ yield

$$r^3 f'(r)f''(r) - 1 = 0$$

which has the general solution given by

$$f(r) = \sqrt{2c_1 r^2 - 1} - \arccos\left(\frac{1}{\sqrt{2c_1}r}\right) + c_2$$

so that the family of streamlines for flows (1) and (2) above is given by $\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\sqrt{2c_1}r\right)^{-1}$ =constant.

Therefore, we classify the flows above as follows:

Theorem 5.1. *If families of curves of the form $\theta - f(r)$ =constant define streamline patterns in a steady plane isentropic irrotational compressible fluid flow for various*

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continuously differentiable function $f(r)$ so that $\theta - f(r) = \Gamma(\psi)$ for each $f(r)$, where $\psi(r, \theta)$ is the flow streamfunction and $\Gamma(\psi)$ is a function of ψ with $\Gamma'(\psi) \gtrless 0$, then all possible flows are classified as

    (i) flows with $r^3 f'(r) f''(r) - 1 = 0$,

    (ii) flows with $q = q(r)$ and any $f(r)$ provided $f'(r) \neq 0$,

    (iii) flows with $q = q(\alpha)$ such that $q'(\alpha) \neq 0$, $\dfrac{\partial \alpha}{\partial r} \neq 0$ and $\dfrac{\partial \alpha}{\partial \psi} \neq 0$.

Flows classified in this theorem are studied in the following sections.

5.3 FLOWS SATISFYING $r^3 f'(r) f''(r) - 1 = 0$.

Employing $r^3 f'(r) f''(r) - 1 = 0$ in (5.12), we find that $\dfrac{\partial \alpha}{\partial r} = 0$ and, therefore, equations (5.11) and (5.12) give the following two cases:

1. $r^3 f'(r) f''(r) - 1 = 0$ and $\Gamma''(\psi) = 0$

2. $r^3 f'(r) f''(r) - 1 = 0$ and $\frac{\partial q}{\partial r} = 0$

Since $r^3 f'(r) f''(r) - 1 = 0$ in both cases, we can rewrite this equation as

$$f'(r) f''(r) - \frac{1}{r^3} = 0$$

Integrating the above equation twice, we get

$$f'(r) = \frac{\sqrt{2c_1 r^2 - 1}}{r} \tag{5.13}$$

and

$$f(r) = \sqrt{2c_1 r^2 - 1} - \arccos\left(\frac{1}{\sqrt{2c_1}\, r}\right) + c_2 \tag{5.14}$$

where $c_1 \neq 0$ and c_2 are arbitrary constants.

Employing (5.14) in (5.2), the streamline pattern for this section is

$$\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\frac{1}{\sqrt{2c_1}\, r}\right) = \Gamma(\psi) \tag{5.15}$$

The streamlines for this flow are shown in Figure 5.1.
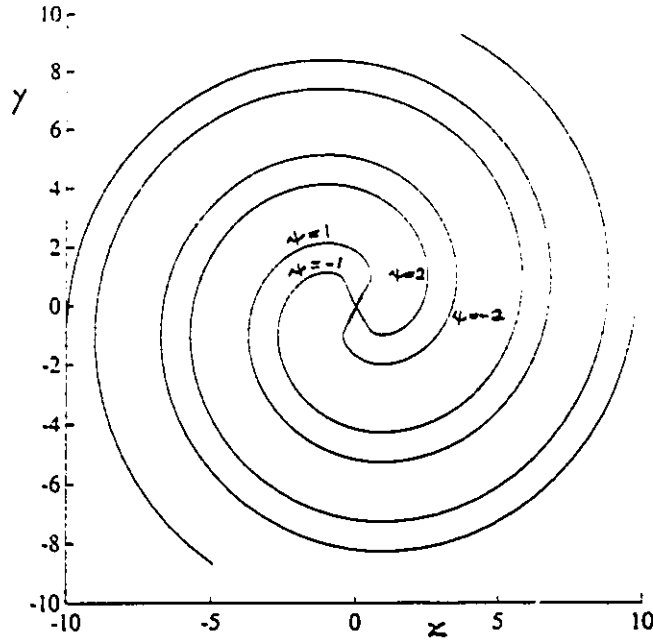
Figure 5.1. Streamline pattern $\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\dfrac{1}{\sqrt{2c_1}\,r}\right) = $ constant $(c_1 = 1)$.

Case 1:

In this case, using (5.13), equation (5.9) gives

$$\rho q = \alpha(r, \psi) = \frac{\sqrt{2c_1}}{|c_3|} \tag{5.16}$$

where $\Gamma'(\psi) = c_3 \neq 0$ is an arbitrary constant.

Employing (5.13), (5.16) and $\Gamma'(\psi) = c_3$ in (5.10), we get

$$c_3\left(2c_1 r^2 - 1\right)\frac{\partial q}{\partial r} - 2c_1 r\sqrt{2c_1 r^2 - 1}\frac{\partial q}{\partial \psi} + 2c_1 c_3 r q = 0 \tag{5.17}$$

The general solution of (5.17) is given by

$$q(r, \psi) = q(r, \theta) = h(\eta)\left[\frac{1}{\sqrt{2c_1 r^2 - 1}}\right] \tag{5.18}$$

where $\eta = \sqrt{2c_1 r^2 - 1} + |c_3|\psi = \theta + \arccos\left(\sqrt{2c_1}\,r\right)^{-1}$.

Using (5.18) in (5.16), we obtain

$$\rho(r, \theta) = \frac{\sqrt{2c_1}}{|c_3| q(r, \theta)} \tag{5.19}$$

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Differentiating (5.2) with respect to r and θ respectively, we have

$$\frac{\partial \psi}{\partial r} = -\frac{f'(r)}{\Gamma'(\psi)}, \qquad \frac{\partial \psi}{\partial \theta} = \frac{1}{\Gamma'(\psi)} \tag{5.20}$$

The radial and axial components of velocity by (5.1) are given by

$$V_1(r,\theta) = \frac{1}{r\rho}\frac{\partial \psi}{\partial \theta}, \qquad V_2(r,\theta) = -\frac{1}{\rho}\frac{\partial \psi}{\partial r} \tag{5.21}$$

Employing (5.13), (5.19) and (5.20) in (5.21), we get

$$V_1(r,\theta) = \frac{|c_3|}{c_3\sqrt{2c_1}r}q(r,\theta), \qquad V_2(r,\theta) = \frac{|c_3|}{c_3\sqrt{2c_1}}\frac{\sqrt{2c_1r^2-1}}{r}q(r,\theta) \tag{5.22}$$

Using (5.16) and (5.18) in (5.4) and (5.5) and integrating, the pressure function is given by

$$p(r,\theta) = -\int \alpha dq = p_0 - \frac{\sqrt{2c_1}}{|c_3|}q(r,\theta) \tag{5.23}$$

where p_0 is an arbitrary constant. Using (5.19) in (5.23), the (p,ρ)−relation for this flow is

$$p = p_0 - \frac{2c_1}{c_3^2}\frac{1}{\rho} \tag{5.24}$$

The potential function, the streamfunction and the Mach number for this flow are respectively given by

$$\Phi(r,\theta) = \frac{|c_3|}{c_3\sqrt{2c_1}}H(\eta),$$

$$\psi(r,\theta) = \frac{1}{c_3}\left[\theta - \sqrt{2c_1r^2-1} + \arccos\left(\frac{1}{\sqrt{2c_1}r}\right)\right] - \frac{c_4}{c_3}$$

and

$$M = \frac{q}{c} = 1$$

where $H'(\eta) = h(\eta)$ and c_4 is an arbitrary constant.

<u>Case 2</u>:

In this case, using (5.2),

$$\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\frac{1}{\sqrt{2c_1}\,r}\right) = \Gamma(\psi) \qquad \text{and} \qquad q = q(\psi) \qquad (5.25)$$

where $\Gamma(\psi)$ is an arbitrary function.

Using (5.13) in (5.9), we get

$$\rho q = \alpha(r, \psi) = \frac{2c_1}{\Gamma'(\psi)} \qquad (5.26)$$

Employing (5.13), (5.25) and (5.26) in (5.10) and separating variables, we obtain

$$\frac{1}{\sqrt{2c_1 r^2 - 1}} = \frac{1}{\Gamma'(\psi)}\frac{q'(\psi)}{q(\psi)} = \text{constant} = c_5 \neq 0 \qquad (5.27)$$

since r and ψ are independent variables.

Equation (5.27) yields two equations, the first of which can be written as

$$2c_1 c_5^2 r^2 - \left(1 + c_5^2 +\right) = 0$$

For this equation to hold true for all values of r, we must have

$$c_1 = 0 \qquad \text{and} \qquad 1 + c_5^2 = 0$$

This gives a non-real $f(r)$ and therefore this case is removed from further consideration.

Summing up, we have

Theorem 5.2. *A family of curves* $\theta - \sqrt{2c_1 r^2 - 1} + \arccos\left(\sqrt{2c_1}\,r\right)^{-1} = constant$ *when* $f(r) = \sqrt{2c_1 r^2 - 1} - \arccos\left(\sqrt{2c_1}\,r\right)^{-1} + c_2$ *in (5.2) is a permissible streamline pattern for steady plane compressible isentropic potential flow with solutions given by (5.19), (5.22) and (5.23).*

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5.4 FLOWS SATISFYING $q = q(r)$ WITH $q'(r) \neq 0$.

Having considered flows with flow intensity $\alpha = $ constant througout the flow domain and ignoring flows with constant speed, we observed that equation (5.11) is identically satisfied when $q = q(r)$ such that $q'(r) \neq 0$ and $\alpha = \alpha(r)$ such that $\alpha'(r) \neq 0$.

Therefore, employing $q = q(r)$, $\alpha = \alpha(r)$ with $\alpha'(r) \neq 0$ which yields $\Gamma(\psi) = d_1\psi + d_2$ from (5.12) and any function $f(r)$ such that $f'(r) \neq 0$ in equations (5.2), (5.9) and (5.10), we get

$$\rho q = \alpha(r) = \frac{\sqrt{1 + r^2 f'^2(r)}}{|d_1| r}, \qquad \theta - f(r) = \Gamma(\psi) = d_1\psi + d_2$$

and

$$\frac{d}{dr}\left[\frac{rf'(r)}{\alpha} q(r)\right] = 0$$

(5.28)

where $d_1 \neq 0$ and d_2 are arbitrary constants.

Equations (5.28), (5.9), and (5.21) give

$$q(r) = \frac{c_6 \sqrt{1 + r^2 f'^2(r)}}{|d_1| r^2 f'(r)}$$

$$\rho(r) = \frac{1}{c_6} r f'(r)$$

(5.29)

and

$$V_1(r, \theta) = \frac{c_6}{d_1 r^2 f'(r)}, \qquad V_2(r, \theta) = \frac{c_6}{d_1 r}$$

where $c_6 \neq 0$ is an arbitrary constant and $f(r)$ is an arbitrary function of its argument.

Employing (5.29) in (5.4) and (5.5) and integrating, we obtain

$$p(r) = p_0 - \frac{c_6 \left(1 + r^2 f'^2(r)\right)}{d_1^2 r^3 f'(r)} + \frac{c_6}{d_1^2} \int \frac{\left(r^3 f'(r) f''(r) - 1\right)}{r^4 f'(r)} dr$$

(5.30)

where p_0 is an arbitrary constant.

Theorem 5.3. *If q is constant on each individual circle $r = $ constant then every family of curves $\theta - f(r) = $ constant such that $f'(r) \neq 0$ is a permissible streamline pattern for a steady, plane, potential isentropic flow of some gas with solutions given by (5.29) and (5.30).*

An Example

Choosing $f(r) = r$, the streamline pattern for this flow is given by $\theta - r = $ constant. Employing this chosen function in the solutions (5.29) and (5.30), we have

$$q(r) = \frac{c_6 \sqrt{1 + r^2}}{|d_1| r^2}, \qquad \rho(r) = \frac{1}{c_6} r,$$

$$V_1(r) = \frac{c_6}{d_1 r^2}, \qquad V_2(r) = \frac{c_6}{d_1 r}$$

and

$$p(r) = p_0 - \frac{c_6}{d_1^2} \left[\frac{1}{r} + \frac{2}{3} \frac{1}{r^3} \right]$$

The state equation for this flow is given by

$$p = p_0 - \frac{1}{d_1^2} \frac{1}{\rho} - \frac{2}{3 d_1^2 c_6^2} \frac{1}{\rho^3}$$

The streamline pattern for this flow is shown in Figure 5.2.

The potential function and the streamfunction for this flow are respectively given by

$$\Phi(r, \theta) = \frac{c_6}{d_1} \left(\theta - \frac{1}{r} \right)$$

and

$$\psi(r, \theta) = \frac{1}{d_1} \left(\theta - r - d_2 \right)$$

The state equation above gives

$$\frac{dp}{d\rho} = \frac{c_6^2 \rho^2 + 2}{d_1^2 c_6^2 \rho^4}$$
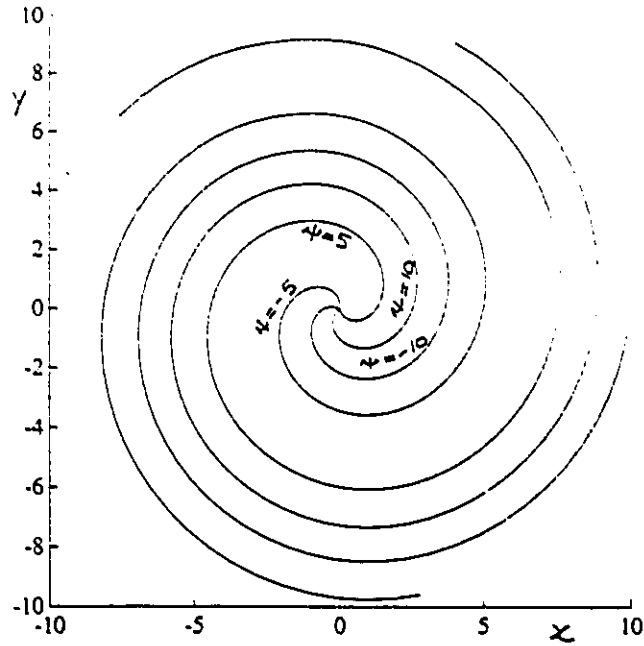
Figure 5.2.   Streamline pattern $\theta - r = $ constant.

and

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} = -\frac{\left(2c_6^2\rho^2 + 8\right)}{d_1^2 c_6^2 \rho^5}$$

Since $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$ and $\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} > 0$ for all real gases, it follows that ρ is required to satisfy

$$c_6^2 \rho^2 + 2 > 0 \qquad \text{and} \qquad c_6^2 \rho^2 + 4 < 0$$

Since density is a positive-valued function, the above inequalities cannot be satisfied for any value of c_6. Therefore, we may conclude that compressible flow along the streamlines $\theta - r = $ constant is only possible for an imaginary gas having the above state equation.

Figure 5.3 shows the graph of pressure versus density for this state equation with $d_1 = \frac{1}{\sqrt{3}}$, $c_6 = \sqrt{\frac{4}{5}}$ and $p_0 = 5$ together with the state equation for a polytropic gas given by $p = 1.3\rho^{1.44}$. It can be seen that the isentrope for this state equation is in close approximation to the ideal gas for a range of values of ρ. Therefore, we

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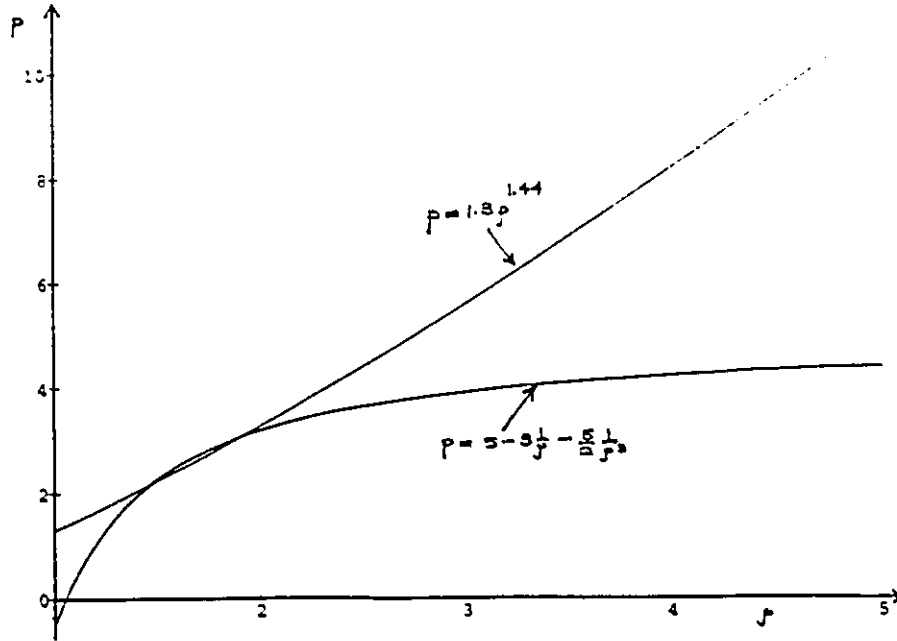
Figure 5.3. Pressure versus density for the flow $\theta - r$ =constant
(Equation (5.30) with $d_1 = 1/\sqrt{3}$, $c_4 = \sqrt{4/5}$, $p_0 = 5$).

may consider the solutions obtained above to be applicable for an ideal gas for this range of values of ρ.

5.5 FLOWS SATISFYING $q = q(\alpha)$, $q'(\alpha) \neq 0$.

Taking $q = q(\alpha)$, equation (5.11) is identically satisfied. Employing this assumption, the expression for $\alpha(r, \psi)$ given by (5.9) and (5.12) in (5.10), we obtain

$$\left[\frac{2f'(r) + r^2 f'^3(r) + r f''(r)}{\alpha \left(1 + r^2 f'^2(r)\right)} \right] q(\alpha) + \left[\frac{r^3 f'^2(r) f''(r) - f'(r)}{1 + r^2 f'^2(r)} + r\alpha^2 \Gamma''(\psi) \right] q'(\alpha) = 0$$

Dividing this equation by $r\alpha^2 q'(\alpha)$, equation (5.10) for the case when $q = q(\alpha)$ is

$$A(r) \left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)} \right] + \frac{r^3 f'^2(r) f''(r) - f'(r)}{r \left(1 + r^2 f'^2(r)\right)} \left[\frac{1}{\alpha^2} \right] + \Gamma''(\psi) = 0 \qquad (5.31)$$

where

$$A(r) = \frac{2f'(r) + r^2 f'^3(r) + r f''(r)}{r \left(1 + r^2 f'^2(r)\right)} \qquad (5.32)$$

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Differentiating (5.31) with respect to r, we have

$$A'(r)\left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right] + \left(\frac{r^3 f'^2(r) f''(r) - f'(r)}{r(1 + r^2 f'^2(r))}\right)'\left[\frac{1}{\alpha^2}\right]$$

$$\left\{A(r)\left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]' + \left(\frac{r^3 f'^2(r) f''(r) - f'(r)}{r(1 + r^2 f'^2(r))}\right)\left[-\frac{2}{\alpha^3}\right]\right\}\frac{\partial \alpha}{\partial r} = 0$$

Multiplying this equation by α^2 and using (5.12) to remove $\frac{\partial \alpha}{\partial r}$, we obtain

$$A'(r)\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right] + B(r)\left[\alpha^3\left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right] + \left[\frac{f'(r)\left(r^3 f'(r) f''(r) - 1\right)}{r(1 + r^2 f'^2(r))}\right]'$$

$$- 2\left[\frac{f'(r)\left(r^3 f'(r) f''(r) - 1\right)^2}{r^2(1 + r^2 f'^2(r))^2}\right] = 0$$

(5.33)

where

$$B(r) = A(r)\frac{(r^3 f'(r) f''(r) - 1)}{r(1 + r^2 f'^2(r))}$$

(5.34)

Differentiating (5.33) with respect to ψ, we have

$$A'(r)\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' + B(r)\left[\alpha^3\left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$$

(5.35)

since $\frac{\partial \alpha}{\partial \psi} \neq 0$.

The system of two equations (5.10) and (5.11) in three unknowns $q(r, \psi)$, $f(r)$ and $\Gamma(\psi)$ reduced to one equation (5.31) for flows satisfying $q = q(\alpha)$. Differentiation of this equation with respect to r followed by differentiation with respect to ψ produced (5.33) and (5.35). The separation of variables technique is employed in (5.35) to determine all flows satisfying $q = q(\alpha)$. The variables r and α in (5.35) are independent variables since $|\frac{\partial(\alpha, r)}{\partial(r, \psi)}| = -\frac{\partial \alpha}{\partial \psi} \neq 0$ in the flow domain. This equation is satisfied for the following five cases:

Case 1:

$$A'(r) = 0 \quad \text{and} \quad B(r) = 0$$

(5.36)

Case 2:

$$\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0 \quad \text{and} \quad B(r) = 0$$

(5.37)

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Case 3:

$$A'(r) = 0 \quad \text{and} \quad \left[\alpha^3 \left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)} \right)' \right]' = 0 \tag{5.38}$$

Case 4:

$$\left[\frac{q(\alpha)}{\alpha q'(\alpha)} \right]' = 0 \quad \text{and} \quad \left[\alpha^3 \left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)} \right)' \right]' = 0 \tag{5.39}$$

Case 5:

$$A'(r) \neq 0 \quad \text{and} \quad \left[\frac{q(\alpha)}{\alpha q'(\alpha)} \right]' \neq 0 \tag{5.40}$$

We consider each of these cases separately.

Case 1: $A'(r) = 0$ and $B(r) = 0$.

In this case, equations (5.32) and (5.34) yield

$$2f'(r) + r^2 f'^3(r) + r f''(r) = 0 \tag{5.41}$$

since $r^3 f'(r) f''(r) - 1 \neq 0$.

Equation (5.33) then reduces to

$$\left[\frac{f'(r) \left(r^3 f'(r) f''(r) - 1 \right)}{r \left(1 + r^2 f'^2(r) \right)} \right]' - 2 \left[\frac{f'(r) \left(r^3 f'(r) f''(r) - 1 \right)^2}{r^2 \left(1 + r^2 f'^2(r) \right)^2} \right] = 0 \tag{5.42}$$

Employing (5.41) in (5.42), we obtain

$$f'(r) \left(1 + r^2 f'^2(r) \right)^2 = 0$$

which implies that

$$f'(r) = 0 \tag{5.43}$$

since $1 + r^2 f'^2(r) \neq 0$.

Using (5.43) in (5.31), we get

$$\Gamma''(\psi) = 0$$

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Employing this equation in the second equation of (5.12) gives

$$\frac{\partial \alpha}{\partial \psi} = 0$$

which is contrary to the assumption that $q = q(\alpha)$, $\frac{\partial \alpha}{\partial \psi} \neq 0$.

<u>Case 2:</u> $\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0$ and $B(r) = 0$. Using $B(r) = 0$ in (5.34) yields $A(r) = 0$. Therefore, equation (5.32) gives (5.41). As in Case 1, equation (5.33) reduces to (5.42). Employing (5.41) in (5.42) results in $\frac{\partial \alpha}{\partial \psi} = 0$ and, therefore, this subcase does not yield any solution when $q = q(\alpha)$.

<u>Case 3:</u> $A'(r) = 0$ and $\left[\alpha^3 \left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0$. In this case, we integrate the first differential equation twice and obtain

$$f(r) + \arctan(rf'(r)) = \frac{1}{2}b_0 r^2 + b_1 \tag{5.44}$$

where $b_0 \neq 0$ and b_1 are arbitrary constants.

Integrating the second differential equation, we get

$$\alpha^3 \left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)' = b_2 \tag{5.45}$$

and

$$q(\alpha) = \begin{cases} b_4 \exp[-\frac{b_3}{\alpha^2}]; & b_2 = 0 \\ b_6 (2b_5 - \frac{b_3}{\alpha^2})^{\frac{1}{b_2}}; & b_2 \neq 0 \end{cases} \tag{5.46}$$

where b_2, $b_3 \neq 0$, $b_4 \neq 0$, b_5 and $b_6 \neq 0$ are arbitrary constants.

Employing (5.44) and (5.45) in (5.33), we obtain

$$b_0 b_2 \left[\frac{(r^3 f'(r) f''(r) - 1)}{r(1 + r^2 f'^2(r))}\right] + \left[\frac{f'(r)(r^3 f'(r) f''(r) - 1)}{r(1 + r^2 f'^2(r))}\right]'$$
$$- 2 \left[\frac{f'(r)(r^3 f'(r) f''(r) - 1)^2}{r^2(1 + r^2 f'^2(r))^2}\right] = 0 \tag{5.47}$$

Using (5.9), (5.44) and (5.45) in (5.31) and separating variables, we have

$$\frac{r f'(r) \left(r^3 f'(r) f''(r) - 1 \right)}{\left(1 + r^2 f'^2(r) \right)^2} - \frac{1}{2} \frac{b_5 b_6 r^2}{\left(1 + r^2 f'^2(r) \right)} = -\frac{\left(\Gamma''(\psi) + b_5 b_8 \right)}{\Gamma'^2(\psi)} = \text{constant} = b_7$$

(5.48)

since r, ψ are independent variables.

Equation (5.48) gives

$$r f'(r) \left(r^3 f'(r) f''(r) - 1 \right) - b_7 \left(1 + r^2 f'^2(r) \right)^2 - \frac{1}{2} b_0 b_2 r^2 \left(1 + r^2 f'^2(r) \right) = 0 \quad (5.49)$$

and

$$\Gamma''(\psi) + b_7 \Gamma'^2(\psi) + b_0 b_5 = 0 \tag{5.50}$$

Employing (5.49) in (5.47), we find that (5.47) is identically satisfied.

Using (5.44) in (5.49) and simplifying, we obtain

$$r^3 f'^3(r) - r^2 \left(b_0 r^2 - b_7 \right) f'^2(r) + r f'(r) + \frac{1}{2} b_0 b_2 r^2 + b_7 = 0 \tag{5.51}$$

Therefore, the function $f(r)$ must satisfy equations (5.44) and (5.51).

Integrating (5.50) once yields

$$\Gamma'(\psi) = \frac{\sqrt{b_0 b_5}}{\sqrt{b_7}} \left| \tan \left(\frac{\sqrt{b_0 b_5}}{\sqrt{b_7}} \left[b_8 - b_7 \psi \right] \right) \right| \tag{5.52}$$

where b_8 is an arbitrary constant.

Integrating (5.52) once more gives

$$\Gamma(\psi) = -\frac{1}{b_7} \ln \left| \sec \left(\frac{\sqrt{b_0 b_5}}{\sqrt{b_7}} \left[b_8 - b_7 \psi \right] \right) \right| + b_9 \tag{5.53}$$

where b_9 is an arbitrary constant.

Using (5.53) in (5.2), we have

$$\sec \left(\frac{\sqrt{b_0 b_5}}{\sqrt{b_7}} \left[b_8 - b_7 \psi \right] \right) = \pm \exp \left[b_7 \left(f(r) - \theta + b_9 \right) \right]$$

Employing the above equation in (5.52). we get

$$\Gamma'(\psi[r,\theta]) = \frac{\sqrt{b_0 b_5}}{\sqrt{b_7}} \sqrt{\exp\left(2b_7\left[b_9 - \theta + f(r)\right]\right) - 1} \tag{5.54}$$

Using (5.54) in (5.9). we have

$$\alpha(r,\theta) = \frac{\sqrt{1 + r^2 f'^2(r)}}{r|\Gamma'(\psi)|} = \frac{\sqrt{b_7}}{\sqrt{b_0 b_5}} \frac{\sqrt{1 + r^2 f'^2(r)}}{r\sqrt{\exp\left(2b_7\left[b_9 - \theta + f(r)\right]\right) - 1}} \tag{5.55}$$

where $f(r)$ is given by (5.44) and (5.51).

Employing (5.46) in (5.9). the density function is given by

$$\rho(r,\theta) = \frac{1}{b_6}\left(\frac{\alpha^{2+b_2}}{2b_5\alpha^2 - b_2}\right)^{\frac{1}{b_2}} \tag{5.56}$$

where $\alpha(r,\theta)$ is given by (5.55).

The velocity components in polar coordinates using (5.20) and (5.21) are

$$V_1(r,\theta) = \frac{1}{\sqrt{1 + r^2 f'^2(r)}} q(\alpha), \qquad V_2(r,\theta) = \frac{rf'(r)}{\sqrt{1 + r^2 f'^2(r)}} q(\alpha) \tag{5.57}$$

Using (5.9) and (5.46) in (5.4) and (5.5) and integrating, we obtain

$$p(r,\theta) = -\int \alpha q'(\alpha) d\alpha = p_0 - 2b_6 \int \alpha^{-\frac{2}{b_6}}\left(2b_5\alpha^2 - b_2\right)^{\frac{1}{b_2}-1} d\alpha \tag{5.58}$$

For $b_2 = 0$, the arbitrary function $f(r)$ must also satisfy equations (5.44) and (5.51) with $b_2 = 0$. Equation (5.50) is identical with $b_0 b_5$ replaced by $\frac{b_0}{2b_3}$. Therefore, the solution set for $b_2 = 0$ is given by

$$\alpha(r,\theta) = \frac{\sqrt{2b_3 b_7}}{\sqrt{b_0}} \frac{\sqrt{1 + r^2 f'^2(r)}}{r\sqrt{\exp\left(2b_7\left[b_{10} - \theta + f(r)\right]\right) - 1}}, \tag{5.59}$$

$$\rho(r,\theta) = \frac{1}{b_4}\alpha\exp\left(\frac{b_3}{\alpha^2}\right), \tag{5.60}$$

$$V_1(r,\theta) = \frac{1}{\sqrt{1 + r^2 f'^2(r)}} q(\alpha), \qquad V_2(r,\theta) = \frac{rf'(r)}{\sqrt{1 + r^2 f'^2(r)}} q(\alpha) \tag{5.61}$$

and

$$p(r,\theta) = p_0 - 2b_3 b_4 \int \alpha^{-2}\exp\left(-\frac{b_3}{\alpha^2}\right) d\alpha \tag{5.62}$$

where b_{10} is an arbitrary constant.

Theorem 5.4. *A family of curves $\theta - f(r) = $ constant such that $f(r)$ is a solution of both (5.44) and (5.51) is a permissible streamline pattern for a steady, plane, potential isentropic flow of some gas with solutions given by (5.56), (5.57) and (5.58) or (5.59), (5.60), (5.61) and (5.62).*

An Example

For $b_2 \neq 0$, equation (5.58) yields various state equations corresponding to the flow of different gases for various choices of b_2. For example, choosing $b_2 = -2$ in (5.58) and integrating, we get

$$p(r, \theta) = p_0 + \frac{b_6}{b_5} \frac{1}{\sqrt{2b_5\alpha^2 + 2}}$$

Employing $b_2 = -2$ in (5.56) gives

$$\rho(r, \theta) = \frac{1}{b_6}\sqrt{2b_5\alpha^2 + 2}$$

Therefore, the state equation for this flow is

$$p = p_0 + \frac{1}{b_5}\frac{1}{\rho}$$

For every solution $f(r)$ of the two nonlinear ordinary differential equations (5.45) and (5.51), we have a streamline pattern $\theta - f(r) = $ constant and the exact integral for the flow of a tangent gas, obtained by choosing $b_2 = -2$, having this streamline pattern is given by

$$\rho(r, \theta) = \frac{\sqrt{2b_7}}{b_6\sqrt{b_0}} \frac{\sqrt{1 - r^2 + r^2 f'^2(r) + r^2\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right)}}{\sqrt{r^2\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right) - r^2}}$$

$$V_1(r, \theta) = \frac{\sqrt{b_7}b_6}{\sqrt{2}}\left\{b_5b_7\left(1 + r^2 f'^2(r)\right) + b_0b_5r^2[\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right) - 1]\right\}^{\frac{1}{2}}$$

$$V_2(r, \theta) = \frac{\sqrt{b_7}b_6}{\sqrt{2}}rf'(r)\left\{b_5b_7\left(1 + r^2 f'^2(r)\right) + b_0b_5r^2[\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right) - 1]\right\}^{\frac{1}{2}}$$

$$p(r, \theta) = p_0 + \frac{\sqrt{b_0}b_6}{\sqrt{2b_5}\sqrt{b_7}} \frac{\sqrt{r^2\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right) - r^2}}{\sqrt{1 - r^2 + r^2 f'^2(r) + r^2\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right)}}$$

The potential function and the streamfunction for this flow are

$$\Phi(r,\theta) = \frac{b_6\sqrt{b_7}}{\sqrt{2}} \int \frac{dr + r^2 f'(r)d\theta}{\sqrt{b_5 b_7 \left(1 + r^2 f'^2(r)\right) + b_0 b_5 r^2 \left[\exp\left(2b_7\left[f(r) - \theta + b_9\right]\right) - 1\right]}},$$

$$\psi(r,\theta) = \frac{1}{b_7}\left\{b_8 - \frac{\sqrt{b_7}}{\sqrt{b_0 b_5}}\arccos\left[\exp\left(b_7\left\{\theta - f(r)\right\} - b_7 b_9\right)\right]\right\}$$

respectively.

<u>Case 4:</u> $\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0$ and $\left[\alpha^3\left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right]' = 0.$

In this case, we have

$$\frac{q(\alpha)}{\alpha q'(\alpha)} = \text{constant} = \frac{1}{\lambda}(\text{say})$$

This equation gives

$$q(\alpha) = b_{12}\alpha^\lambda, \qquad \lambda \neq 0, 1 \tag{5.63}$$

where $b_{12} \neq 0$ is an arbitrary constant.

Equation (5.63) identically satisfies the second assumption of this case.

Using (5.63) in (5.33), we get

$$\begin{aligned}
&\frac{1}{\lambda}\left[A'(r) - 2B(r)\right] + \left[\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)}{r\left(1 + r^2 f'^2(r)\right)}\right]' \\
&- 2\left[\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)^2}{r^2\left(1 + r^2 f'^2(r)\right)^2}\right] = 0
\end{aligned} \tag{5.64}$$

Also employing (5.9) and (5.63) in (5.31), we have

$$\frac{r}{\lambda}\frac{\left(2f'(r) + r^2 f'^3(r) + rf''(r)\right)}{\left(1 + r^2 f'^2(r)\right)^2} + \frac{rf'(r)\left(r^3 f'(r)f''(r) - 1\right)}{\left(1 + r^2 f'^2(r)\right)^2} = -\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = b_{13} \tag{5.65}$$

where $b_{13} \neq 0$ is the separation constant.

Equation (5.65) gives

$$2rf'(r) + r^3 f'^3(r) + r^2 f''(r) + \lambda rf'(r)\left(r^3 f'(r)f''(r) - 1\right) - \lambda b_{13}\left(1 + r^2 f'^2(r)\right) = 0 \tag{5.66}$$

and

$$\Gamma''(\psi) + b_{13}\Gamma'^2(\psi) = 0 \qquad (5.67)$$

Employing (5.66) in (5.64), we find that (5.64) is identically satisfied. Therefore, $f(r)$ is given by (5.66).

Letting $u(r) = rf'(r)$, then $\frac{du(r)}{dr} = \frac{u(r)}{r} + rf''(r)$. Replacing $f'(r)$ and $f''(r)$ by $u(r)$ and $\frac{du(r)}{dr}$ in (5.66), we obtain

$$r\left(1 + \lambda u^2(r)\right)\frac{du(r)}{dr} + (1 + u(r))\left[(1 - \lambda)u(r) - \lambda b_{13}(1 + u^2(r))\right] = 0 \qquad (5.68)$$

This equation is identically satisfied if $u(r)$ =constant = λ_1(say) such that

$$(1 - \lambda)\lambda_1 - \lambda b_{13}(1 + \lambda_1^2) = 0 \qquad (5.69)$$

Since $u(r) = rf'(r)$, we obtain

$$f(r) = \lambda_1 \ln r + \lambda_2 \qquad (5.70)$$

where λ_2 is an arbitrary constant and λ_1 satisfies (5.69). The streamline pattern for (5.70) with $\lambda_1 = 1$ and $\lambda_2 = 0$ is shown in Figure 5.4.

If $\left[(1 - \lambda)u(r) - \lambda b_{13}(1 + u^2(r))\right] \neq 0$, rewriting (5.68), we obtain

$$\frac{1 + \lambda u^2(r)}{(1 + u^2(r))\left[(1 - \lambda)u(r) - \lambda b_{13}(1 + u^2(r))\right]}du = -\frac{1}{r}dr$$

Integrating the above equation, we get

$$r = \begin{cases} h_1(u); & m^2 < 1 \\ h_2(u); & m^2 = 1 \\ h_3(u); & m^2 > 1 \end{cases} \qquad (5.71)$$

where $u(r) = rf'(r)$, $m = \frac{1-\lambda}{2\lambda b_{13}}$ and

$$h_1(u) = d_1 \frac{\sqrt{1 + u^2}}{\sqrt{u^2 - 2mu + 1}}\exp\left[\frac{1}{\sqrt{1 - m^2}}\arctan\left(\frac{u - m}{\sqrt{1 - m^2}}\right)\right]$$

$$h_2(u) = d_2\left(1 + u^2\right)^{\pm\frac{1-\lambda}{4}}(u \mp 1)^{\mp\frac{1-\lambda}{2}}\exp\left[-\frac{(1 + \lambda)}{2}\frac{1}{(u \mp 1)}\right] \qquad (5.72)$$

$$h_3(u) = d_3 \frac{\sqrt{1 + u^2}}{(u - m_1)^{\lambda_3}(u - m_2)^{\lambda_4}}$$

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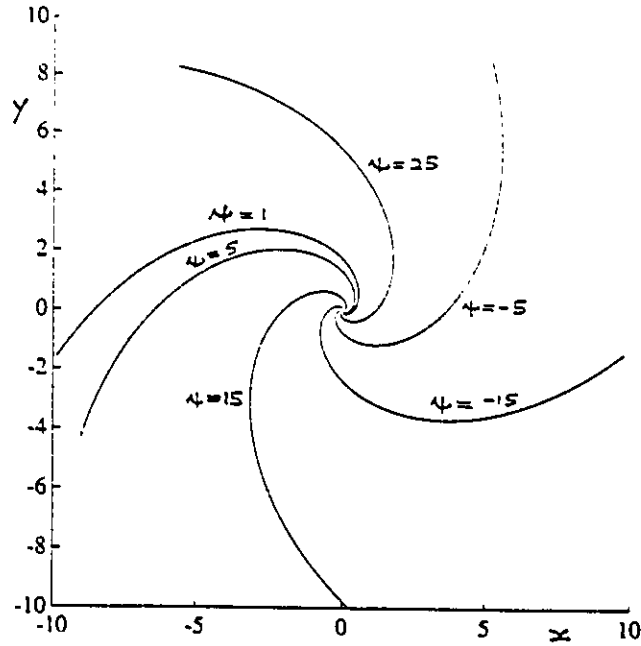
Figure 5.4. Streamline pattern $\theta - \ln r = $ constant.

where $m_1 = m + \sqrt{m^2 - 1}$, $m_2 = m - \sqrt{m^2 - 1}$, $\lambda_3 = \dfrac{(am_1 - 1)}{\sqrt{(1-\lambda)^2 - 4\lambda^2 b_{13}^2}}$, $\lambda_4 = \dfrac{(1 - am_2)}{\sqrt{(1-\lambda)^2 - 4\lambda^2 b_{13}^2}}$, $a = \dfrac{1-\lambda}{2m}$ and d_1, d_2 and d_3 are arbitrary constants.

Integrating (5.67) once and using (5.2) in the resulting equation, we get

$$\Gamma'(\psi[r, \theta]) = b_{14}\exp\left(-b_{13}\left[\theta - f(r)\right]\right) \tag{5.73}$$

where $b_{14} \neq 0$ is an arbitrary constant.

Employing (5.73) in (5.9), we obtain

$$\alpha(r, \theta) = \frac{1}{|b_{14}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r}\exp\left(b_{13}\left[\theta - f(r)\right]\right) \tag{5.74}$$

where $f(r)$ is given by (5.70) or (5.71).

The density function, the velocity components and the pressure function are given by

$$\rho(r, \theta) = \frac{1}{b_{12}|b_{14}|^{1-\lambda}} \frac{\left(1 + r^2 f'^2(r)\right)^{\frac{1}{2}(1-\lambda)}}{r^{1-\lambda}}\exp\left\{b_{13}(1 - \lambda)\left[\theta - f(r)\right]\right\} \tag{5.75}$$

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$$V_1(r, \theta) = \frac{b_{12}|b_{14}|^{1-\lambda}}{b_{14}} \frac{\left(1 + r^2 f'^2(r)\right)^{\frac{1}{2}(\lambda-1)}}{r^\lambda} \exp\left(\lambda b_{13}\left[\theta - f(r)\right]\right) \qquad (5.76)$$

$$V_2(r, \theta) = \frac{b_{12}|b_{14}|^{1-\lambda}}{b_{14}} \frac{f'(r)\left(1 + r^2 f'^2(r)\right)^{\frac{1}{2}(\lambda-1)}}{r^{1-\lambda}} \exp\left(\lambda b_{13}\left[\theta - f(r)\right]\right) \qquad (5.77)$$

$$p(r, \theta) = p_0 - \frac{\lambda}{1 + \lambda} \frac{1}{b_{12}^{\frac{1}{\lambda}}} q^{\frac{1+\lambda}{\lambda}}(\alpha) \qquad (5.78)$$

where $q(r, \theta)$ is given by (5.63) and (5.74).

Employing (5.63) and (5.75) in (5.78), the state equation for this flow is given by

$$p = p_0 - \frac{\lambda}{1 + \lambda} b_{12}^{\frac{2}{1-\lambda}} \rho^{\frac{1+\lambda}{1-\lambda}} \qquad (5.79)$$

Theorem 5.5. *The streamline pattern defined by a family of curves of the form* $\theta - f(r) = constant$ *where* $f(r)$ *is given by (5.70) or (5.71) is permissible in a steady, plane, isentropic, irrotational compressible fluid flow with solutions given by (5.75), (5.76), (5.77) and (5.78).*

<u>An Example of a flow for a Polytropic Gas $p = A\rho^\gamma$</u>

If we consider a polytropic gas having the state equation $p = A\rho^\gamma$ where γ is the ratio of specific heats, the equation of state (5.79) above requires

$$p_0 = 0, \qquad A = -\left(\frac{\lambda}{1 + \lambda}\right) b_{12}^{\frac{2}{1-\lambda}} \qquad \text{and} \qquad \frac{1 + \lambda}{1 - \lambda} = \gamma$$

Solving these equations, we get

$$\lambda = \frac{\gamma - 1}{\gamma + 1} \qquad \text{and} \qquad A = \frac{(1 - \gamma)}{2\gamma} b_{12}^{1+\gamma}$$

Therefore, for the flow pattern given by $\theta - f(r) = $ constant where $f(r)$ is given by (5.70) or (5.71), the solution set of a polytropic gas having this flow pattern is

given by

$$\rho(r,\theta) = \frac{1}{b_{12}} \left\{ \frac{1}{|b_{14}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{13}\left[\theta - f(r)\right]\right) \right\}^{\frac{2}{\gamma+1}}$$

$$V_1(r,\theta) = \frac{b_{12}|b_{14}|^{\frac{2}{\gamma+1}}}{b_{14}} \frac{\left(1 + r^2 f'^2(r)\right)^{-\frac{1}{\gamma+1}}}{r^{\frac{\gamma-1}{\gamma+1}}} \exp\left(b_{13}\frac{\gamma-1}{\gamma+1}\left[\theta - f(r)\right]\right)$$

$$V_2(r,\theta) = \frac{b_{12}|b_{14}|^{\frac{2}{\gamma+1}}}{b_{14}} \frac{f'(r)\left(1 + r^2 f'^2(r)\right)^{-\frac{1}{\gamma+1}}}{r^{\frac{2}{\gamma+1}}} \exp\left(b_{13}\frac{\gamma-1}{\gamma+1}\left[\theta - f(r)\right]\right)$$

$$p(r,\theta) = \frac{b_{12}(1-\gamma)}{2\gamma} \left\{ \frac{1}{|b_{14}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{13}\left[\theta - f(r)\right]\right) \right\}^{\frac{2\gamma}{\gamma+1}}$$

The potential function and the streamfunction for this flow are given by

$$\Phi(r,\theta) = \frac{b_{12}|b_{14}|^{\frac{2}{\gamma+1}}}{b_{14}} \int \left[dr + f'(r)d\theta\right] \frac{\left(1 + r^2 f'^2(r)\right)^{-\frac{1}{\gamma+1}}}{r^{\frac{\gamma-1}{\gamma+1}}} \exp\left(b_{13}\frac{\gamma-1}{\gamma+1}\left[\theta - f(r)\right]\right)$$

and

$$\psi(r,\theta) = \frac{1}{b_{14}} \left\{ \frac{1}{b_{13}} \exp\left[b_{13}\left(\theta - f(r)\right)\right] - \psi_1 \right\}$$

respectively where ψ_1 is an arbitrary constant.

<u>Case 5</u>: $A'(r) \neq 0$ and $\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' \neq 0$.

In this case, we have

$$\frac{\left\{ \alpha^3 \left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]' \right\}'}{\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'} = -\frac{A'(r)}{B(r)} = \text{constant} = b_{15} \neq 0 \qquad (5.80)$$

Equation (5.80) yields

$$\left\{ \alpha^3 \left[\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right]' \right\}' - b_{15} \left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]' = 0 \qquad (5.81)$$

and

$$A'(r) + b_{15}B(r) = 0 \qquad (5.82)$$

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Solving (5.81), we get

$$\frac{q(\alpha)}{\alpha^3 q'(\alpha)} = \begin{cases} \dfrac{1}{b_{17}} \alpha^{b_{15}} q^{b_{16}}(\alpha); & b_{16} \neq 0 \\[3mm] b_{17}\alpha^{b_{15}}; & b_{16} = 0 \end{cases} \tag{5.83}$$

and

$$q(\alpha) = \begin{cases} \left(b_{16} b_{18} - \dfrac{b_{16} b_{17}}{b_{15}+2} \dfrac{1}{\alpha^{b_{15}+2}} \right)^{\frac{1}{b_{16}}}; & b_{15} \neq -2, b_{16} \neq 0 \\[3mm] (b_{16}b_{17}\ln\alpha + b_{16}b_{18})^{\frac{1}{b_{16}}}; & b_{15} = -2, b_{16} \neq 0 \\[3mm] b_{19}\alpha^{\frac{1}{b_{17}}}; & b_{15} = -2, b_{16} = 0 \\[3mm] b_{20}\exp\left(-\dfrac{1}{b_{17}(b_{15}+2)\alpha^{b_{15}+2}} \right); & b_{15} \neq -2, b_{16} = 0 \end{cases} \tag{5.84}$$

where $b_{16} \neq 0$, $b_{17} \neq 0$, b_{18}, $b_{19} \neq 0$ and $b_{20} \neq 0$ are arbitrary constants.

We consider the most general case when $b_{15} \neq -2$ and $b_{16} \neq 0$ and obtain the exact solution for this case. A similar analysis can be carried out for the remaining three possibilities.

Employing (5.81) and (5.82) in (5.33), we obtain

$$b_{16}B(r) + \left[\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)}{r\left(1 + r^2 f'^2(r)\right)} \right]' - 2\left[\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)^2}{r^2\left(1 + r^2 f'^2(r)\right)^2} \right] = 0 \tag{5.85}$$

Using (5.9) (5.83) and (5.84) in (5.31), we get

$$\frac{\left(2f'(r) + r^2 f'^3(r) + rf''(r)\right)}{r\left(1 + r^2 f'^2(r)\right)}\left[\frac{b_{16}b_{18}}{b_{17}} \frac{\left(1 + r^2 f'^2(r)\right)^{\frac{1}{2}(b_{15}+2)}}{(r\Gamma'(\psi))^{b_{15}+2}} - \frac{1}{b_{15}+2} \right]$$

$$+ \frac{r^3 f'^2(r)f''(r) - f'(r)}{r\left(1 + r^2 f'^2(r)\right)} + \frac{\left(1 + r^2 f'^2(r)\right)}{r^2}\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0 \tag{5.86}$$

In order to separate variables, we assume $b_{18} = 0$ in (5.86) and therefore, have

$$\frac{rf'(r)\left(r^3 f'(r)f''(r) - 1\right)}{\left(1 + r^2 f'^2(r)\right)^2} - \frac{b_{16}}{b_{15}+2}\frac{r\left(2f'(r) + r^2 f'^3(r) + rf''(r)\right)}{\left(1 + r^2 f'^2(r)\right)^2} = -\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = b_{21}$$

where $b_{21} \neq 0$ is an arbitrary constant. This equation gives

$$\frac{b_{16}}{b_{15}+2}r\left(2f'(r) + r^2 f'^3(r) + rf''(r)\right) - rf'(r)\left(r^3 f'(r)f''(r) - 1\right)$$

$$+ b_{21}\left(1 + r^2 f'^2(r)\right)^2 = 0 \tag{5.87}$$

and

$$\Gamma''(\psi) + b_{21}\Gamma'^2(\psi) = 0 \tag{5.88}$$

Employing (5.87) in (5.85) and simplifying, we obtain

$$b_{15}\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)^2}{r^2\left(1 + r^2 f'^2(r)\right)^2} - b_{21}\left(b_{15} + 2\right)\frac{r^3 f'(r)f''(r) - 1}{r^3}$$

$$+ \left[\frac{f'(r)\left(r^3 f'(r)f''(r) - 1\right)}{r\left(1 + r^2 f'^2(r)\right)}\right]' = 0 \tag{5.89}$$

Using (5.89) in (5.82), we find that (5.82) is identically satisfied. Therefore, $f(r)$ must satisfy (5.87) and (5.89).

Integrating (5.88) once yields

$$\Gamma'(\psi[r,\theta]) = b_{22}\exp\left(-b_{21}\left[\theta - f(r)\right]\right) \tag{5.90}$$

where $b_{22} \neq 0$ is an arbitrary constant and $f(r)$ is given by (5.87) and (5.89).

The solution set for this flow is given by

$$\alpha(r,\theta) = \frac{1}{|b_{22}|}\frac{\sqrt{1 + r^2 f'^2(r)}}{r}\exp\left(b_{21}\left[\theta - f(r)\right]\right)$$

$$q(r,\theta) = c_3\frac{1}{\alpha^{c_4}}$$

$$\rho(r,\theta) = \frac{1}{c_3}\alpha^{1+c_4} \tag{5.91}$$

$$V_1(r,\theta) = \frac{c_3}{\sqrt{1 + r^2 f'^2(r)}}\frac{1}{\alpha^{c_4}} \tag{5.92}$$

$$V_2(r,\theta) = \frac{c_3 r f'(r)}{\sqrt{1 + r^2 f'^2(r)}}\frac{1}{\alpha^{c_4}} \tag{5.93}$$

and

$$p(r,\theta) = \begin{cases} p_0 - \dfrac{c_3 c_4}{(1 - c_4)}\alpha^{1-c_4}; & c_4 \neq 1 \\[2mm] p_0 + c_3 c_4\ln\alpha; & c_4 = 1 \end{cases} \tag{5.94}$$

where

$$c_3 = \left(-\frac{b_{16}b_{17}}{b_{15} + 2}\right)^{\frac{1}{b_{16}}} \quad \text{and} \quad c_4 = \frac{b_{15} + 2}{b_{16}}$$

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and $f(\tau)$ is a solution of both (5.87) and (5.89).

The state equation for this flow is

$$p = \begin{cases} p_0 - \dfrac{c_3^{\frac{2}{1+c_4}} c_4}{(1 - c_4)} \rho^{\frac{1-c_4}{1+c_4}}; & c_4 \neq 1 \\[4mm] p_0 + \dfrac{c_3 c_4}{1 + c_4} \ln\rho; & c_4 = 1 \end{cases} \qquad (5.95)$$

Therefore, we have

Theorem 5.6. *A family of curves $\theta - f(\tau) = constant$ such that $f(\tau)$ is a solution of both (5.87) and (5.89) is a permissible streamline pattern for a steady, plane, potential isentropic flow of some gas with solutions given by (5.91), (5.92), (5.93) and (5.94).*

Flow along $\theta - f(\tau)$ =constant for a Polytropic Gas $p = A\rho^\gamma$

Consider the flow of a polytropic gas $p = A\rho^\gamma$ where γ is the ratio of the specific heats along the streamlines $\theta - f(\tau) = $ constant where $f(\tau)$ is given by (5.87) and (5.89). The equation of state (5.95) for $c_4 \neq 1$ requires

$$p_0 = 0, \qquad A = \frac{c_3^{\frac{2}{1+c_4}} c_4}{c_4 - 1} \qquad \text{and} \qquad \frac{1 - c_4}{1 + c_4} = \gamma$$

Solving these equations, we obtain

$$c_4 = \frac{1 - \gamma}{1 + \gamma} \qquad \text{and} \qquad c_3 = \left(\frac{2\gamma A}{(\gamma - 1)} \right)^{\frac{1}{1+\gamma}}$$

Therefore, the solution set for the flow of a polytropic gas having the streamline

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pattern $\theta - f(r) = $ constant for every $f(r)$ satisfying (5.87) and (5.89) is given by

$$\rho(r,\theta) = \left(\frac{2\gamma A}{(\gamma - 1)}\right)^{-\frac{1}{1+\gamma}} \left(\frac{1}{|b_{22}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{21}\left[\theta - f(r)\right]\right)\right)^{\frac{2}{1+\gamma}},$$

$$V_1(r,\theta) = \left(\frac{(\gamma - 1)}{2\gamma A}\right)^{\frac{1}{1+\gamma}} \frac{1}{\sqrt{1 + r^2 f'^2(r)}} \left(\frac{1}{|b_{22}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{21}\left[\theta - f(r)\right]\right)\right)^{\frac{\gamma-1}{\gamma+1}}$$

$$V_2(r,\theta) = \left(\frac{(\gamma - 1)}{2\gamma A}\right)^{\frac{1}{1+\gamma}} \frac{r f'(r)}{\sqrt{1 + r^2 f'^2(r)}} \left(\frac{1}{|b_{22}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{21}\left[\theta - f(r)\right]\right)\right)^{\frac{\gamma-1}{\gamma+1}}$$

$$p(r,\theta) = A^{-\frac{1}{1+\gamma}} \left(\frac{(\gamma - 1)}{2\gamma}\right)^{\frac{2+\gamma}{1+\gamma}} \left(\frac{1}{|b_{22}|} \frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{21}\left[\theta - f(r)\right]\right)\right)^{\frac{2\gamma}{\gamma+1}}$$

The potential function and the streamfunction for this flow are given by

$$\Phi(r,\theta) = |b_{22}|^\delta \left(\frac{(\gamma - 1)}{2\gamma A}\right)^{\frac{1}{1+\gamma}} \int \frac{dr + r^2 f'(r) d\theta}{\sqrt{1 + r^2 f'^2(r)}} \left(\frac{\sqrt{1 + r^2 f'^2(r)}}{r} \exp\left(b_{21}\left[\theta - f(r)\right]\right)\right)^{\frac{\gamma-1}{\gamma+1}}$$

and

$$\psi(r,\theta) = \frac{1}{b_{22}} \left\{\frac{1}{b_{21}} \exp\left[b_{21}\left\{\theta - f(r)\right\}\right] - \psi_2\right\}$$

respectively where $\delta = \frac{1-\gamma}{1+\gamma}$ and ψ_2 is an arbitrary constant.

CHAPTER 6

STREAMLINE
PATTERN $\frac{y}{g(x)}$ = CONSTANT

6.1. INTRODUCTION.

This chapter deals with determining the permissible streamline patterns and their exact solutions for a class of flows whose streamlines in the (x, y)-plane can be expressed in the form $\dfrac{y}{g(x)}$ = constant, where $g(x)$ is a continuously differentiable function. Along these streamlines, $y = g(x)\Gamma(\psi)$ is a function of x and ψ where $\Gamma(\psi)$ is some function of ψ. We choose the curves ϕ = constant to be x = constant in the flow equations in (ϕ, ψ)-coordinates given in theorem 2.2. We employ von Mises coordinates (x, ψ) for the investigation of this class of flows.

Given this assumed form for the streamline pattern, the governing equations are analyzed with von Mises coordinates and classified. This classification results in the study of three different possible flows. It is found that the first of these flows are straight parallel flows while the remaining two are non-straight flows. The function $g(x)$ in the chosen form is shown to be a

(a) a solution of $g''(x) = 0$ yielding straight flows,

(b) a solution of $\left[1 + d_4^2 d_6 g^2(x)\right] g''(x) - 2d_4^2 d_6 g(x) g'^2(x) = 0$ in the first type of non-straight flows where $d_4 \neq 0$ and $d_6 \neq 0$ are arbitrary constants,

(c) a solution of $\lambda_1 g''(x) + m^2 g(x) = 0$ in the second type of non-straight flows with the additional assumption that $\Gamma(\psi) = e^{m\psi}$, where $\lambda_1 \neq 0$ and $m \neq 0$ are any

constants.

The flow equations are integrated in the von Mises coordinate system and the resulting exact solutions are expressed in the physical plane. The state equation for the gas that permits each of the above flows is also determined. It is found that the gas is a polytropic gas for straight parallel flows and a tangent gas for both non-straight flows.

6.2. EQUATIONS OF MOTION AND CLASSIFICATION OF FLOWS.

We investigate compressible fluid flows along families of curves that can be expressed in the form $\frac{y}{g(x)}$ =constant where $g(x) \in C^3$ is any continuously differentiable function. Since $\psi(x,y)$ =constant are also the streamlines in a steady plane flow, it follows that there exists some function $\Gamma(\psi)$ such that

$$\frac{y}{g(x)} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{6.1}$$

We choose the von Mises coordinates (x, ψ) to study these flows and employing this coordinate system and (6.1) in (2.8) and (2.10), we obtain

$$E(x,\psi) = 1 + g'^2(x)\Gamma^2(\psi), \qquad G(x,\psi) = g^2(x)\Gamma'^2(\psi)$$

$$F(x,\psi) = g(x)g'(x)\Gamma(\psi)\Gamma'(\psi), \qquad J(x,\psi) = \left|\frac{\partial(x,y)}{\partial(x,\psi)}\right| = g(x)\Gamma'(\psi) \tag{6.2}$$

$$W(x,\psi) = \sqrt{EG - F^2} = |g(x)\Gamma'(\psi)| > 0$$

Whenever $g(x)$ and $\Gamma'(\psi)$ are both positive or negative, $g(x)\Gamma'(\psi) > 0$, it follows that $J > 0$ from (6.2) and, therefore, fluid flows along a streamline in the direction of increasing x. Also, $g(x)\Gamma'(\psi) < 0$ whenever one of $g(x)$ or $\Gamma'(\psi)$ is positive and the other is negative which yields $J < 0$ and, therefore, fluid flows in the direction of decreasing x along a streamline.

To study flow along the curves $\frac{y}{g(x)}$ =constant, we employ (6.2) in the equations of theorem 2.2 and write these equations in von Mises coordinates. Gauss equation

is identically satisfied and the fluid flowing along a streamline is governed by the system

$$\rho q \frac{\partial q}{\partial x} + \frac{\partial p}{\partial x} = 0 \tag{6.3}$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \tag{6.4}$$

$$\frac{\partial}{\partial x}\left[\frac{g(x)g'(x)\Gamma(\psi)\Gamma'(\psi)}{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}q\right] - \frac{\partial}{\partial \psi}\left[\sqrt{1 + g'^2(x)\Gamma^2(\psi)}\,q\right] = 0 \tag{6.5}$$

$$\rho q = \frac{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}{g(x)\,|\Gamma'(\psi)|} = \pm\frac{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}{g(x)\Gamma'(\psi)} \tag{6.6}$$

$$p = R(\rho) \tag{2.5}$$

of five equations in five unknowns $g(x)$, $\Gamma(\psi)$, $\rho(x,\psi)$, $q(x,\psi)$ and $p(x,\psi)$. In (6.6), the $\pm$ sign is chosen according as $\Gamma'(\psi) \gtrless 0$, that is, according as fluid flows along a streamline in the direction of increasing or decreasing x.

In seeking exact solutions of compressible flows whose streamlines are families of curves of the form $\frac{y}{g(x)}$ =constant, we do not choose *a priori* a particular gas, and, therefore, do not have a specific state equation (2.5) to employ. Equations (6.3) to (6.6) form a system of four equations in five unknown functions and we solve this underdetermined system. Having found a solution of this system, the state equation is determined from solutions for $p(x,\psi)$ and $\rho(x,\psi)$ giving us the gas that flows along the obtained streamline pattern.

Employing (6.6) in (6.3) and (6.4) to eliminate ρ and using the integrability condition $\frac{\partial^2 p}{\partial x \partial \psi} = \frac{\partial^2 p}{\partial \psi \partial x}$ to eliminate p from the resulting equations , we get

$$\frac{\partial}{\partial x}\left[\frac{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}{g(x)\,|\Gamma'(\psi)|}\right]\frac{\partial q}{\partial \psi} - \frac{\partial}{\partial \psi}\left[\frac{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}{g(x)\,|\Gamma'(\psi)|}\right]\frac{\partial q}{\partial x} = 0 \tag{6.7}$$

Equations (6.5) and (6.7) form a system of two equations in three unknown functions $g(x)$, $\Gamma(\psi)$ and $q(x,\psi)$. Once a solution of this system is found, $\rho(x,\psi)$ is determined from (6.6) and $p(x,\psi)$ is determined by integrating (6.3) and (6.4).

Defining

$$pq = \alpha(x, \psi) = \frac{\sqrt{1 + g'^2(x)\Gamma^2(\psi)}}{g(x)\,|\Gamma'(\psi)|} > 0 \qquad (6.8)$$

the system of equations (6.5) and (6.7) can be written as

$$\frac{\partial}{\partial x}\left[\frac{g'(x)\Gamma(\psi)}{\alpha(x,\psi)}\right] - \frac{\partial}{\partial \psi}\left[g(x)\Gamma'(\psi)\,\alpha(x,\psi)\,q(x,\psi)\right] = 0 \qquad (6.9)$$

and

$$\frac{\partial \alpha}{\partial x}\frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial x} = 0 \qquad (6.10)$$

where

$$\frac{\partial \alpha}{\partial x} = \frac{g'(x)g''(x)\Gamma^2(\psi)}{g^2(x)\alpha(x,\psi)\Gamma'^2(\psi)} - \alpha\frac{g'(x)}{g(x)}, \qquad \frac{\partial \alpha}{\partial \psi} = \frac{g'^2(x)\,\Gamma(\psi)}{g^2(x)\,\alpha(x,\psi)\,\Gamma'(\psi)} - \frac{\alpha(x,\psi)\,\Gamma''(\psi)}{\Gamma'(\psi)}$$
$$(6.11)$$

Equation (6.10) is identically satisfied if any of the following holds true:

(1) $q = q(\alpha)$, $q'(\alpha) \neq 0$. This is the case when the curves of constant speed

and the curves of constant flow intensity coincide in the flow domain.

(2) $\dfrac{\partial \alpha}{\partial x} = \dfrac{\partial \alpha}{\partial \psi} = 0$. This is the case when the flow intensity is constant

everywhere.

(3) $\dfrac{\partial \alpha}{\partial x} = \dfrac{\partial q}{\partial x} = 0$. This is the case when the streamlines are the curves of

constant flow speed and the curves of constant flow intensity.

(4) $\dfrac{\partial \alpha}{\partial \psi} = \dfrac{\partial q}{\partial \psi} = 0$. This is the case when the flow speed and the flow intensity

are constant on each individual parallel straight line $x =$ constant.

(5) $\dfrac{\partial q}{\partial x} = \dfrac{\partial q}{\partial \psi} = 0$. This is the case when the flow is of constant speed

everywhere and so is of no interest.

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Flows resulting from cases (2), (3) and (4) are not possible since the assumptions $\frac{\partial \alpha}{\partial x} = 0$ and $\frac{\partial \alpha}{\partial \psi} = 0$ give $\Gamma'(\psi) = 0$. This can be seen easily by assuming first $\frac{\partial \alpha}{\partial x} = 0$ and employing (6.8) in the first equation of (6.11) which gives

$$\frac{g''(x)\Gamma^2(\psi)}{g(x)\Gamma'^2(\psi)} - \frac{1 + g'^2(x)\Gamma^2(\psi)}{g^2(x)\Gamma'^2(\psi)} = 0$$

Since x and ψ are independent variables, separating variables in the above equation, we get

$$g(x)g''(x) - g'^2(x) = \frac{1}{\Gamma^2(\psi)} = \text{constant}$$

which implies that $\Gamma'(\psi) = 0$ contrary to the assumption (6.1).

Similarly, assuming $\frac{\partial \alpha}{\partial \psi} = 0$ and employing (6.8) in the second equation of (6.11), we obtain

$$\Gamma(\psi)\Gamma'^2(\psi) - \Gamma^2(\psi)\Gamma''(\psi) = \frac{1}{g'^2(x)} = \text{constant}$$

Using these equations in (6.5), we obtain $\Gamma'(\psi) = 0$.

Summing up, we have

Theorem 6.1. *If families of curves of the form $\frac{y}{g(x)} = \text{constant}$ define streamline patterns in a steady plane isentropic irrotational compressible fluid flow for some continuously differentiable function $g(x)$ so that $\frac{y}{g(x)} = \Gamma(\psi)$, where $\psi(x,y)$ is the streamfunction and $\Gamma(\psi)$ is some function of ψ with $\Gamma'(\psi) \gtrless 0$, then all possible flows are flows with $q = q(\alpha)$ such that $q'(\alpha) \neq 0$, $\frac{\partial \alpha}{\partial x} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$.*

Therefore, we study flows when $q = q(\alpha)$ in the next section.

6.3 FLOWS SATISFYING $q = q(\alpha)$, $q'(\alpha) \neq 0$.

Taking $q = q(\alpha)$, equation (6.10) is identically satisfied. Employing this assumption in equation (6.9) and expanding, we obtain

$$\frac{g''(x)\,\Gamma(\psi)}{\alpha}q + \frac{g'(x)\Gamma(\psi)}{\alpha}\left[q'(\alpha) - \frac{1}{\alpha}q(\alpha)\right]\frac{\partial \alpha}{\partial x} - g(x)\Gamma'(\psi)\left[q(\alpha) + \alpha q(\alpha)\right]\frac{\partial \alpha}{\partial \psi}$$
$$- g(x)\alpha q(\alpha)\Gamma''(\psi) = 0$$

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Using (6.11) to eliminate $\dfrac{\partial \alpha}{\partial x}$ and $\dfrac{\partial \alpha}{\partial \psi}$ from this equation, we obtain

$$g'^2(x)g''(x)\left[q(\alpha) - \alpha q'(\alpha)\right]\Gamma^3(\psi) - g^2(x)g''(x)\alpha^2 q(\alpha)\Gamma(\psi)\Gamma'^2(\psi)$$
$$+ 2g(x)g'^2(x)\alpha^3 q'(\alpha)\Gamma(\psi)\Gamma'^2(\psi) - g^3(x)\alpha^5 q'(\alpha)\Gamma''(\psi)\Gamma'^2(\psi) = 0$$

(6.12)

This equation replaces (6.9) for this case when $q = q(\alpha)$. Since $q(\alpha) - aq'(\alpha) = 0$ implies that q is proportional to α and from equation (6.8), $pq = \alpha$, it follows that $q(\alpha) - \alpha q'(\alpha) \neq 0$ in (6.12) for the compressible flows and, therefore, equation (6.12) is satisfied if one of the following holds true:

(1) straight flows, that is, $g''(x) = 0$ with

$$2g'^2(x)\Gamma(\psi) - g^2(x)\alpha^2\Gamma''(\psi) = 0$$

since $g(x) \neq 0$, $\alpha \neq 0$, $\Gamma'^2(\psi) \neq 0$ and $q'(\alpha) \neq 0$.

(2) non-straight flows with $\Gamma''(\psi) = 0$ and

$$g^2(x)g''(x)\alpha^2 q(\alpha) - 2g(x)g'^2(x)\alpha^3 q'(\alpha) - \frac{\Gamma^2(\psi)}{\Gamma'^2(\psi)}\left[g'^2(x)g''(x)\left\{q(\alpha) - \alpha q'(\alpha)\right\}\right] = 0$$

(3) non-straight flows with $g''(x) \neq 0$ and $\Gamma''(\psi) \neq 0$.

We consider each of these flows separately:

Case 1: Straight Flows with $g''(x) = 0$.

In this case, we have

$$g(x) = d_1 x + d_2$$

(6.13)

and

$$2g'^2(x)\Gamma(\psi) - g^2(x)\alpha^2\Gamma''(\psi) = 0$$

(6.14)

where $d_1 \neq 0$ and d_2 are arbitrary constants.

Employing (6.8) and (6.13) in (6.14), we obtain

$$2d_1^2\Gamma(\psi) - \left[1 + d_1^2\Gamma^2(\psi)\right]\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0 \qquad (6.15)$$

Integrating (6.15) once, we get

$$\Gamma'(\psi) = d_3\left[1 + d_1^2\Gamma^2(\psi)\right] \qquad (6.16)$$

where $d_3 \neq 0$ is an arbitrary constant.

Using (6.13) and (6.16) in (6.8), we have

$$\alpha(x,\psi) = \frac{1}{|d_3(d_1x + d_2)|\sqrt{1 + d_1^2\Gamma^2(\psi)}}$$

Replacing $\Gamma(\psi)$ in the above equation by (6.1), we get

$$\alpha(x,y) = \frac{1}{|d_3|\sqrt{d_1^2(x^2 + y^2) + 2d_1d_2x + d_2^2}} \qquad (6.17)$$

The speed function is given by

$$q(x,y) = q(\alpha) = q\left(\frac{1}{|d_3|\sqrt{d_1^2(x^2 + y^2) + 2d_1d_2x + d_2^2}}\right) \qquad (6.18)$$

where q is an arbitrary function of its argument. From (6.8), the density function is given by

$$\rho(x,y) = \frac{\alpha(x,y)}{q(x,y)} \qquad (6.19)$$

Differentiating (6.1) with respect to x and y respectively and using (6.1), (6.13) and (6.14) in the resulting equations, we have

$$\frac{\partial\psi}{\partial x} = -\frac{d_1y}{d_3\left(d_1^2[x^2 + y^2] + 2d_1d_2x + d_2^2\right)}, \qquad \frac{\partial\psi}{\partial y} = \frac{d_1x + d_2}{d_3\left(d_1^2[x^2 + y^2] + 2d_1d_2x + d_2^2\right)}$$

Employing the above equations and (6.19) in (2.27), we get

$$u(x,y) = \frac{(d_1x + d_2)}{\sqrt{(d_1^2[x^2 + y^2] + 2d_1d_2x + d_2^2)}}q(x,y) \qquad (6.20)$$

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and

$$v(x,y) = \frac{d_1 y}{\sqrt{(d_1^2[x^2 + y^2] + 2d_1 d_2 x + d_2^2)}} q(x,y) \tag{6.21}$$

Using (6.5) and (6.18) in (6.3) and (6.4) and integrating, we obtain

$$p(x,y) = -\int \alpha \, q'(\alpha) \, d\alpha = p_0 + \frac{1}{d_3^2} \int \frac{\left[(d_1^2 x + d_1 d_2) \, dx + d_1^2 y dy\right] q'(\alpha)}{[d_1^2 (x^2 + y^2) + 2d_1 d_2 x + d_2^2]^2} \tag{6.22}$$

where p_0 is an arbitrary constant. The state equation for this flow is obtained by expressing (6.22) in terms of ρ after choosing the speed function $q(x,y)$.

Therefore, we have

Theorem 6.2. *A family of curves* $\dfrac{y}{d_1 x + d_2}$ =*constant where* $g(x) = d_1 x + d_2$ *in (6.1) is a permissible streamline pattern for steady plane compressible isentropic potential flow with solutions given by (6.19), (6.20), (6.21) and (6.22)*

An Example of a Flow for a Polytropic Gas $p = A\rho^\gamma$

Choosing $q(\alpha) = \alpha^{\frac{1}{m}}$ where $m \neq 0$ or $m \neq \pm 1$, the pressure function for the flow above is obtained by employing this chosen $q(\alpha)$ in (6.22) and is given by

$$p(\alpha) = -\int \alpha \, dq = -\int q^m(\alpha) dq = p_0 - \frac{1}{m+1} q^{m+1}(\alpha)$$

Using $q(\alpha) = \alpha^{\frac{1}{m}}$ in (6.19), we get

$$\rho = \frac{\alpha}{q} = \alpha^{1 - \frac{1}{m}}$$

Therefore, the state equation for our flow above is given by

$$p = p_0 - \frac{1}{m+1} \rho^{\frac{m+1}{m-1}}$$

Consider a polytropic gas having the state equation $p = A\rho^\gamma$ where γ is the ratio of specific heats, the state equation for our flow above requires

$$p_0 = 0, \qquad A = \frac{1}{m+1} \qquad \text{and} \qquad \frac{m+1}{m-1} = \gamma$$

Solving these equations, we get

$$m = \frac{\gamma + 1}{\gamma - 1} \qquad \text{and} \qquad A = \frac{\gamma - 1}{2\gamma}$$

Therefore, for the flow pattern given by $\dfrac{y}{d_1 x + d_2}$ =constant, the solution set of a polytropic gas having this flow pattern is given by

$$\rho(x,y) = |d_3|^{-\frac{2}{\gamma+1}} \left\{ d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right\}^{-\frac{1}{\gamma+1}}$$

$$u(x,y) = |d_3|^{\frac{1-\gamma}{1+\gamma}} \left(d_1 x + d_2 \right) \left\{ d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right\}^{-\frac{\gamma}{\gamma+1}}$$

$$v(x,y) = |d_3|^{\frac{1-\gamma}{1+\gamma}} d_1 y \left\{ d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right\}^{-\frac{\gamma}{\gamma+1}}$$

and

$$p(x,y) = \frac{\gamma - 1}{2\gamma} |d_3|^{-\frac{2\gamma}{\gamma+1}} \left\{ d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right\} - \frac{\gamma}{\gamma + 1}$$

The potential function, the streamfunction and the Mach number for this flow are respectively given by

$$\Phi(x,y) = |d_3|^{\frac{1-\gamma}{1+\gamma}} \frac{\gamma + 1}{d_1} \left\{ d_1^2 \left(x^2 + y^2 \right) + 2d_1 d_2 x + d_2^2 \right\}^{\frac{1}{\gamma+1}} + \Phi_0,$$

$$\psi(x,y) = \frac{d_1}{d_3} \arctan \left(\frac{d_1 y}{d_1 x + d_2} \right) - \psi_0$$

and

$$M = \sqrt{ \frac{\alpha^{\frac{2}{m}}}{c_s^2 - \left(\frac{\gamma - 1}{2} \right) \alpha^{\frac{2}{m}}} }$$

where Φ_0 and ψ_0 are arbitrary constants, c_s is the sound speed at a stagnation point and $\alpha(x,y)$ is given by (6.17).

Case 2: Non-straight Flows with $\Gamma''(\psi) = 0$.

In this case, we have

$$\Gamma(\psi) = d_4 \psi + d_5 \tag{6.23}$$

and

$$g^2(x)g''(x)\alpha^2 q(\alpha) - 2g(x)g'^2(x)\alpha^3 q'(\alpha) - g'^2(x)g''(x) \left[q(\alpha) - aq'(\alpha) \right] \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0 \tag{6.24}$$

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where $d_4 \neq 0$ and d_5 are arbitrary constants.

Since $g''(x) \neq 0$ and $q(\alpha) - \alpha q'(\alpha) \neq 0$, equation (6.24) gives

$$\frac{g(x)\alpha^2 \left[g(x)g''(x)q(\alpha) - 2g'^2(x)\alpha q'(\alpha) \right]}{g'^2(x)g''(x)\left[q(\alpha) - \alpha q'(\alpha)\right]} = \frac{\Gamma^2(\psi)}{\Gamma'^2(\psi)} \tag{6.25}$$

Differentiating (6.25) with respect to ψ and using (6.11), we get

$$\left[\frac{\alpha^2 q(\alpha)}{q(\alpha) - \alpha q'(\alpha)} \right]' - \frac{2g'^2(x)}{g(x)g''(x)} \left[\frac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)} \right]' = 2\alpha \tag{6.26}$$

Equation (6.26) yields the following two subcases:

Subcase (a): $\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)} \right]' = 0$

Subcase (b): $\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)} \right]' \neq 0$

We consider each of these subcases separately:

Subcase (a):

In this case, we have

$$\left[\frac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)} \right] = \text{constant} = d_6 \neq 0 \tag{6.27}$$

Integrating (6.27), we get

$$q(\alpha) = \frac{d_7 \alpha}{\sqrt{d_6 + \alpha^2}} \tag{6.28}$$

where $d_7 \neq 0$ is an arbitrary constant.

Using (6.28), equation (6.26) is identically satisfied. Using (6.23) and (6.28) in (6.25) and simplifying, we obtain

$$\left[1 + d_4^2 d_6 g^2(x) \right] g''(x) - 2d_4^2 d_6 g(x)g'^2(x) = 0 \tag{6.29}$$

Solving (6.29), we have

$$g(x) = \frac{1}{d_4\sqrt{d_6}} \tan \left(d_4 \sqrt{d_6} \left[c_1 x + c_2 \right] \right) \tag{6.30}$$

where $c_1 \neq 0$ and c_2 are arbitrary constants.

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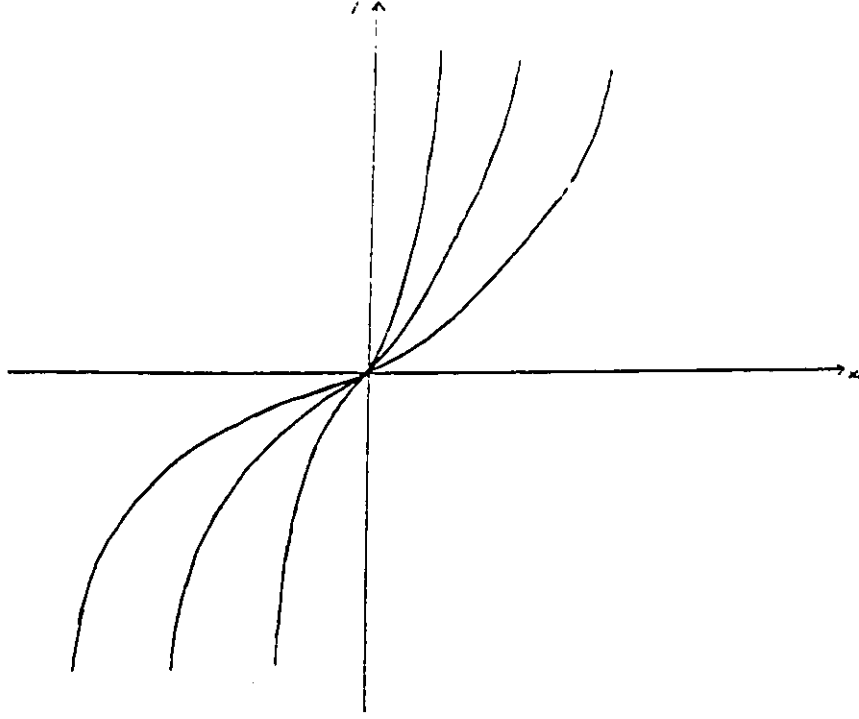
Figure 6.1.   Streamline pattern $\dfrac{y}{\tan\left(d_4\sqrt{d_n}\;[c_1 x + c_3]\right)}$ =constant
$(c_1 = 1,\quad c_2 = 0,\quad d_4 = d_6 = 1)$.

The streamline pattern for this flow is shown in Figure 6.1.

Substitution of (6.1) and (6.23) in (6.8) yields

$$\alpha(x,y) = \frac{\sqrt{g^2(x) + g'^2(x)y^2}}{d_4 g^2(x)} \tag{6.31}$$

where $g(x)$ is given by (6.30).

Using (6.31) in (6.28), we get

$$q(x,y) = \frac{d_7\sqrt{g^2(x) + g'^2(x)y^2}}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x)y^2}} \tag{6.32}$$

Employing (6.31) and (6.32) in (6.8), we obtain

$$\rho(x,y) = \frac{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x)y^2}}{d_4 d_7 g^2(x)} \tag{6.33}$$

Differentiating (6.1) with respect to x and y respectively and using the resulting equations, (6.23) and (6.33) in (2.27) gives

$$u(x,y) = \frac{d_7 g(x)}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x)y^2}} \tag{6.34}$$

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and

$$v(x,y) = \frac{d_7 y g'(x)}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x) y^2}} \tag{6.35}$$

Integrating (6.3) and (6.4), the pressure function is

$$p(x,y) = p_0 + \frac{d_4 d_6 d_7 g^2(x)}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x) y^2}} \tag{6.36}$$

where p_0 is an arbitrary constant.

The state equation for this flow is given by

$$p = p_0 + \frac{d_6}{\rho} \tag{6.37}$$

Therefore, we have

Theorem 6.3. *A family of curves* $\dfrac{d_4 \sqrt{d_6}\, y}{\tan\left(d_4 \sqrt{d_6}\, [c_1 x + c_2]\right)} = $ *constant is a permissible streamline pattern for steady plane compressible isentropic potential flow with solutions given by (6.33), (6.34), (6.35) and (6.36).*

The potential function, the streamfunction and the Mach number of this flow are

$$\Phi(x,y) = d_7 \int \frac{g(x)dx + y g'(x)dy}{\sqrt{g^2(x) + d_4^2 d_6 g^4(x) + g'^2(x) y^2}},$$

$$\psi(x,y) = \sqrt{d_6}\, y \tan^{-1}\left(d_4 \sqrt{d_6}\, [c_1 x + c_2]\right) - \frac{d_5}{d_4}$$

and

$$M = \frac{\sqrt{d_4^2 d_6 d_7}\, g^2(x)}{\sqrt{d_4^2 d_6 \left(d_7 - 1\right) g^4(x) - g^2(x) - g'^2(x) y^2}}$$

respectively.

Subcase (b): Dividing equation (6.26) by $\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]' \neq 0$, we have

$$\frac{\left[\dfrac{\alpha^2 q(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'}{\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'} - \frac{2\alpha}{\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'} = \frac{2g'^2(x)}{g(x)g''(x)} \tag{6.38}$$

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Since $\dfrac{\partial(\alpha, x)}{\partial(x, \psi)} = -\dfrac{\partial \alpha}{\partial \psi} \neq 0$, equation (6.38) yields

$$\frac{\left[\dfrac{\alpha^2 q(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'}{\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'} - \frac{2\alpha}{\left[\dfrac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]'} = \frac{2g'^2(x)}{g(x)g''(x)} = \text{constant} = d_{11} \neq 0$$

This equation gives

$$\left[\frac{\alpha^2 q(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]' - 2\alpha - d_{11}\left[\frac{\alpha^3 q'(\alpha)}{q(\alpha) - \alpha q'(\alpha)}\right]' = 0 \qquad (6.39)$$

and

$$\frac{2g'^2(x)}{g(x)g''(x)} = d_{11} \qquad (6.40)$$

Integration of (6.39) with respect to α yields

$$d_{12} q(\alpha) - \left[d_{12}\alpha + \alpha^3 - d_{11}\alpha^3\right] q'(\alpha) = 0 \qquad (6.41)$$

where d_{12} is an arbitrary constant. Integrating (6.41) once again with respect to α, we have

$$q(\alpha) = \frac{d_{13}\alpha}{\sqrt{d_{12} + (1 - d_{11})\alpha^2}} \qquad (6.42)$$

provided $d_{11} \neq 1$ where $d_{13} \neq 0$ is an arbitrary constant. Integrating equation (6.40) once with respect to x, we get

$$g'(x)g^{-\frac{2}{d_{11}}}(x) = d_{14} \qquad (6.43)$$

where $d_{14} \neq 0$ is an arbitrary constant. Employing (6.43) and (6.42) in (6.25) and simplifying the resulting equation, we get

$$g'(x) = 0$$

which contradicts our assumption that $q = q(\alpha)$, $\dfrac{\partial \alpha}{\partial x} \neq 0$.

Suppose now that $d_{11} = 1$, then (6.41) gives

$$d_{12}\left[q(\alpha) - \alpha q'(\alpha)\right] = 0$$

which implies that

$$d_{12} = 0$$

since $q(\alpha) - \alpha q'(\alpha) \neq 0$. Using (6.40) with $d_{11} = 1$ and (6.23) in (6.25) and simplifying the resulting equation, we get

$$\frac{1}{g'^2(x)\Gamma'^2(\psi)} = 0$$

which is again a contradiction.

Case 3: Non-straight Flows with $g''(x) \neq 0$ and $\Gamma''(\psi) \neq 0$.

In this case, equation (6.12) can be rewritten as

$$\begin{aligned}
&\frac{g'^2(x)g''(x)}{g^3(x)}\left(\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^5 q'(\alpha)}\right)\left(\frac{\Gamma^2(\psi)}{\Gamma'^2(\psi)}\right) - \frac{g''(x)}{g(x)}\left(\frac{q(\alpha)}{\alpha^3 q'(\alpha)}\right) \\
&+ \frac{2g'^2(x)}{g^2(x)}\left(\frac{1}{\alpha^2}\right) - \frac{\Gamma''(\psi)}{\Gamma(\psi)} = 0
\end{aligned} \tag{6.44}$$

Equation (6.44) is a non-linear equation involving three unknown functions $g(x)$, $q(\alpha)$ and $\Gamma(\psi)$. This equation does not allow the separation of these three unknowns and, therefore, an assumption on one of these unknowns has to be made in order to solve for the other two. We choose to solve (6.44) for $g(x)$ and $q(\alpha)$ and so, we assume that

$$\Gamma(\psi) = e^{m\psi} \tag{6.45}$$

where $m \neq 0$ is an arbitrary constant. Using (6.45) in (6.44) and multiplying the resulting equation by α^2, we obtain

$$\frac{1}{m^2}\frac{g'^2(x)g''(x)}{g^3(x)}\left(\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^3 q'(\alpha)}\right) - \frac{g''(x)}{g(x)}\left(\frac{q(\alpha)}{\alpha q'(\alpha)}\right) + \frac{2g'^2(x)}{g^2(x)} - m^2\alpha^2 = 0 \tag{6.46}$$

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Differentiating equation (6.46) with respect to ψ, we get

$$\frac{1}{m^2} \frac{g'^2(x)g''(x)}{g^3(x)} \left(\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^3 q'(\alpha)}\right)' - \frac{g''(x)}{g(x)} \left(\frac{q(\alpha)}{\alpha q'(\alpha)}\right)' - 2m^2\alpha = 0 \qquad (6.47)$$

since $\dfrac{\partial \alpha}{\partial \psi} \neq 0$. Dividing equation (6.47) by α and differentiating the resulting equation with respect to ψ, we have

$$\frac{g'^2(x)}{g^2(x)} \left\{\frac{1}{\alpha} \left(\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^3 q'(\alpha)}\right)'\right\}' - m^2 \left\{\frac{1}{\alpha} \left(\frac{q(\alpha)}{\alpha q'(\alpha)}\right)'\right\}' = 0 \qquad (6.48)$$

since $g''(x) \neq 0$.

Equation (6.48) leads us to the following two subcases:

$$(i) \quad \left\{\frac{1}{\alpha} \left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' = 0 \qquad \text{and} \qquad (ii) \quad \left\{\frac{1}{\alpha} \left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' \neq 0$$

We study these two subcases separately in the following:

Subcase (i): $\left\{\dfrac{1}{\alpha} \left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' = 0.$

Integrating this equation twice with respect to α, we get

$$\frac{q(\alpha)}{\alpha q'(\alpha)} = \lambda_1 \alpha^2 + \lambda_2 \qquad (6.49)$$

where λ_1 and λ_2 are arbitrary constants. Using (6.49), equation (6.48) yields

$$\lambda_2 = 1$$

since $g'(x) \neq 0$.

Employing (6.49) with $\lambda_2 = 1$ in equations (6.46) and (6.47), we obtain

$$\frac{\lambda_1}{m^2} \frac{g'^2(x)g''(x)}{g^3(x)} - \frac{g''(x)}{g(x)} + \frac{2g'^2(x)}{g^2(x)} - \left(\lambda_1 \frac{g''(x)}{g(x)} + m^2\right)\alpha^2 = 0$$

and

$$\lambda_1 g''(x) + m^2 g(x) = 0$$

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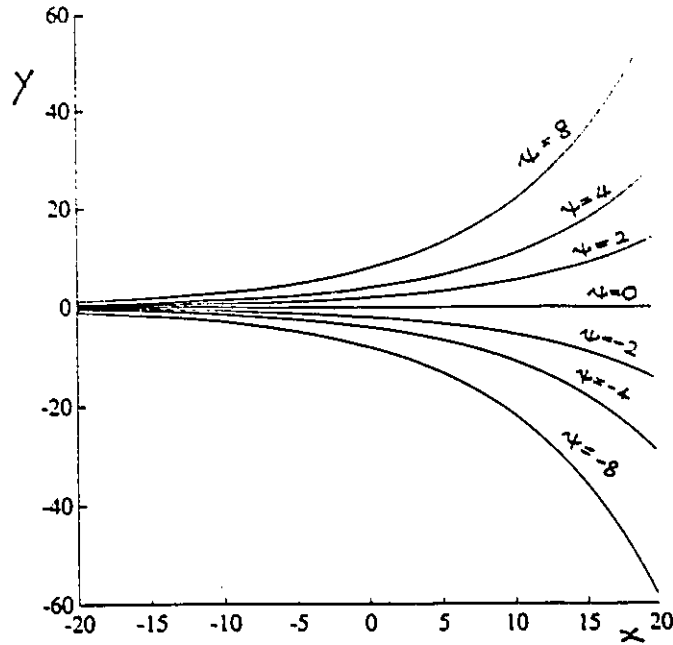
Figure 6.2.　　Streamline pattern $y \exp\left(-\dfrac{m}{k}x\right)$ =constant $(m = k = 1)$.

respectively. The solution of the above two equations is given by

$$g(x) = \lambda_3 \exp\left(\frac{m}{k}x\right) \tag{6.50}$$

where λ_3 is an arbitrary constant and $\lambda_1 = -k^2 < 0$.

Figure 6.2 shows the flow pattern for this flow.

Using $\lambda_1 = -k^2$ and $\lambda_2 = 1$ in (6.49) and integrating the resulting equation, we have

$$q(\alpha) = \lambda_4 \frac{\alpha}{\sqrt{1 - k^2\alpha^2}} \tag{6.51}$$

where λ_4 is an arbitrary constant.

Employing $\Gamma(\psi) = e^{m\psi}$ and (6.1) in (6.8), we get

$$\alpha(x,y) = \frac{\sqrt{g^2(x) + g'^2(x)y^2}}{myg^2(x)} \tag{6.52}$$

where $g(x)$ is given by equation (6.50). Substitution of (6.51) in $pq = \alpha$, yields

$$\rho(x,y) = \frac{1}{\lambda_4}\sqrt{1 - k^2\alpha^2} \tag{6.53}$$

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where α is given by (6.52).

Proceeding as in the previous cases, we find that the velocity components and the pressure function are given by

$$u(x,y) = \frac{1}{m\rho y}$$
$$v(x,y) = \frac{g'(x)}{m\rho g(x)} \tag{6.54}$$
$$p(x,y) = p_0 - \frac{\lambda_4}{k^2\sqrt{1 - k^2\alpha^2}}$$

where p_0 is an arbitrary constant. $g(x)$ and $\rho(x.y)$ are given by equation (6.50) and (6.53) respectively.

The state equation for this flow is given by

$$p = p_0 - \frac{1}{k^2\rho}$$

Therefore, we have

Theorem 6.4. *A family of curves* $\frac{y}{\lambda_3}\exp\left(-\frac{m}{k}x\right)$ =constant *is a permissible streamline pattern for steady plane compressible isentropic potential flow with the solutions given by (6.53) and (6.54).*

The potential function, the streamfunction and the Mach number of this flow are respectively given by

$$\Phi(x,y) = \frac{1}{m}\int \frac{1}{\rho y}dx + \frac{1}{m}\int \frac{g'(x)}{\rho g(x)}dy,$$

$$\psi(x,y) = \frac{1}{m}\ln y - \frac{1}{k}x - \psi_1,$$

and

$$M = k\frac{\sqrt{g^2(x) + g'^2(x)y^2}}{myg^2(x)}$$

Subcase (ii): $\left\{\frac{1}{\alpha}\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' \neq 0.$

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Dividing equation (6.48) by $\dfrac{g'^2(x)}{g^2(x)}\left\{\dfrac{1}{\alpha}\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' \neq 0$, we have

$$\frac{\left\{\dfrac{1}{\alpha}\left[\dfrac{q(\alpha)-\alpha q'(\alpha)}{\alpha^3 q'(\alpha)}\right]'\right\}'}{\left\{\dfrac{1}{\alpha}\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}'} - m^2\frac{g^2(x)}{g'^2(x)} = 0$$

Since $\dfrac{\partial(\alpha,x)}{\partial(x,\psi)} = -\dfrac{\partial\alpha}{\partial\psi} \neq 0$, this equation implies that

$$m^2\frac{g^2(x)}{g'^2(x)} = a_1^2 \tag{6.55}$$

and

$$\left\{\frac{1}{\alpha}\left[\frac{q(\alpha)-\alpha q'(\alpha)}{\alpha^3 q'(\alpha)}\right]'\right\}' = a_1^2\left\{\frac{1}{\alpha}\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' \tag{6.56}$$

where $a_1^2 \neq 0$ is an arbitrary constant. Integrating equation (6.55) with respect to x, we get

$$g(x) = a_2\exp(\frac{m}{a_1}x) \tag{6.57}$$

where $a_2 \neq 0$ is an arbitrary constant. Integrating equation (6.56) three times with respect to α, we obtain

$$\frac{q(\alpha)-\alpha q'(\alpha)}{\alpha^3 q'(\alpha)} = a_1^2\frac{q(\alpha)}{\alpha q'(\alpha)} + a_3\alpha^2 + a_4 \tag{6.58}$$

where a_3 and a_4 are arbitrary constants of integration.

Substitution of (6.57) and (6.58) in equations (6.46) and (6.47) yield

$$a_3 = a_1^4, \qquad a_4 = -2a_1^2 \tag{6.59}$$

Using (6.59) in (6.58) and simplifying the resulting equation, we get

$$\frac{q(\alpha)}{\alpha q'(\alpha)} = 1 - a_1^2\alpha^2$$

which implies that

$$\left\{\frac{1}{\alpha}\left[\frac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' = 0$$

contrary to our assumption that $\left\{\dfrac{1}{\alpha}\left[\dfrac{q(\alpha)}{\alpha q'(\alpha)}\right]'\right\}' \neq 0$.

CHAPTER 7

STREAMLINE PATTERN $\eta(x,y)$=CONSTANT

7.1 INTRODUCTION.

Re$[f(z)]$ =constant and Im$[f(z)]$ =constant define a streamline pattern for some steady plane irrotational inviscid incompressible fluid motion when $f(z)$ is some analytic function of z. This chapter deals with flows when streamlines are Re$[f(z)]$ =constant or Im$[f(z)]$ =constant for potential steady plane compressible flows for some chosen analytic functions $f(z)$.

Prim [1949] studied steady plane adiabatic rotational flows and established that only four flow patterns are possible for gases with product equation of state when Re$[f(z)]$ =constant or Im$[f(z)]$ =constant define a streamline pattern. No such restriction exists for isentropic potential flows we investigated in this chapter.

7.2 EQUATIONS OF MOTION.

Curvilinear coordinate systems in the (x,y)-plane or z-plane are transformations of the rectangular map in the (ξ,η)-plane or w-plane.

Taking an arbitrary function

$$z = f(w) \tag{7.1}$$

where $z = x + iy$ and $w = \xi + i\eta$, the Cauchy-Riemann equations

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta}, \qquad \frac{\partial x}{\partial \eta} = -\frac{\partial y}{\partial \xi} \qquad (7.2)$$

apply and the angles are preserved by this transformation.

Separation of (7.1) into real and imaginary parts gives

$$x = x(\xi, \eta), \qquad y = y(\xi, \eta) \qquad (7.3)$$

Plotting the two families of curves $\xi =$constant and $\eta =$constant, we have an orthogonal curvilinear map in the $z-$plane for every choice of function f in (7.1).

The squared infinitesimal differential element of arc length along any curve in the z-plane is

$$
\begin{aligned}
ds^2 = dx^2 + dy^2 &= \left(\frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \right)^2 + \left(\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \right)^2 \\
&= \bar{h}^2(\xi, \eta) \left[d\xi^2 + d\eta^2 \right]
\end{aligned}
\qquad (7.4)
$$

where

$$\bar{h}^2(\xi, \eta) = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2, \qquad \bar{h}(\xi, \eta) > 0 \qquad (7.5)$$

when (7.2) are employed.

A practical application dictates the proper choice of the function $f(w)$ in (7.1) and the resulting coordinate system.

Since

$$J = \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \bar{h}^2(\xi, \eta) \neq 0$$

it follows that (7.1) and (7.3) can be solved to get

$$\xi + i\eta = w = f^{-1}(z) = g(z) - (say)$$

so that

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y)$$

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These functions are harmonic functions and are harmonic conjugates of each other. Two families of curves defined by $\xi(x,y) = \text{Re}[g(z)] = $ constant and $\eta(x,y) = \text{Im}[g(z)] = $ constant are the streamlines and their orthogonal trajectories forming an isometric curvilinear coordinate net of a steady plane incompressible irrotational motion when $g(z)$ is some analytic function of z.

Steady plane compressible irrotational isentropic flows when the streamline pattern is defined by a family of curves $\eta(x,y) =$ constant, we have

$$\eta(x,y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{7.6}$$

where $\psi(x,y)$ is the streamfunction.

Employing (7.6) in (7.4), we get

$$ds^2 = h^2(\xi,\psi) \left[d\xi^2 + \Gamma'^2(\psi) d\psi^2 \right] \tag{7.7}$$

where $h^2(\xi,\psi) = \bar{h}^2(\xi, \Gamma(\psi))$.

Comparing (7.7) with (2.7) and (2.8), we get

$$E(\xi,\psi) = h^2(\xi,\psi), \qquad F(\xi,\psi) = 0, \qquad G(\xi,\psi) = h^2(\xi,\psi)\Gamma'^2(\psi)$$

$$J(\xi,\psi) = \left| \frac{\partial(x,y)}{\partial(\xi,\psi)} \right| = \left| \frac{\partial(x,y)}{\partial(\xi,\eta)} \frac{\partial(\xi,\eta)}{\partial(\xi,\psi)} \right| = h^2(\xi,\psi)\Gamma'(\psi), \qquad W(\xi,\psi) = h^2(\xi,\psi)\Gamma'(\psi) \tag{7.8}$$

Choosing $\Gamma'(\psi) > 0$, we have $J = W > 0$ and, therefore, fluid flows along a streamline in the direction of increasing ξ.

Taking the arbitrary family of curves $\phi(x,y) =$ constant to be $\xi(x,y) =$ constant and using (7.8), flow equations of Theorem (2.2) in (ξ,ψ)-coordinates are

$$\rho q \frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} = 0 \tag{7.9}$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \tag{7.10}$$

$$\rho q = \frac{1}{h(\xi,\psi)\Gamma'(\psi)} \tag{7.11}$$

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$$\frac{\partial}{\partial \psi}[h(\xi,\psi)q] = 0 \tag{7.12}$$

$$\frac{\partial}{\partial \psi}\left[\frac{1}{h(\xi,\psi)\Gamma'(\psi)}\frac{\partial h}{\partial \psi}\right] + \frac{\partial}{\partial \xi}\left[\frac{\Gamma'(\psi)}{h(\xi,\psi)}\frac{\partial h}{\partial \xi}\right] = 0 \tag{7.13}$$

$$p = R(\rho) \tag{7.14}$$

which form a system of six equations in five unknowns $q(\xi,\psi)$, $\rho(\xi,\psi)$, $p(\xi,\psi)$, $h(\xi,\psi)$ and $\Gamma'(\psi)$.

Using (7.11) in (7.9) and (7.10), we eliminate ρ and employ the integrability condition $\dfrac{\partial^2 p}{\partial \xi \partial \psi} = \dfrac{\partial^2 p}{\partial \psi \partial \xi}$ to obtain

$$\frac{\partial}{\partial \psi}\left[\frac{1}{h(\xi,\psi)\Gamma'(\psi)}\right]\frac{\partial q}{\partial \xi} - \frac{\partial}{\partial \xi}\left[\frac{1}{h(\xi,\psi)\Gamma'(\psi)}\right]\frac{\partial q}{\partial \psi} = 0 \tag{7.15}$$

For a chosen orthogonal curvilinear coordinate net (ξ,η) define by an analytic function $w = \xi + i\eta = f(z)$ and for the chosen streamline pattern $\eta(x,y)$ =constant, the function $h(\xi,\psi)$ is known, Gauss equation (7.13) is identically satisfied and we are left to deal with a system of two equations (7.12) and (7.15) in two unknown functions $q(\xi,\psi)$ and $\Gamma(\psi)$.

In the following section, we choose various orthogonal curvilinear coordinate systems (ξ,η) and, therefore, various streamline patterns and obtain the exact solution of the flows these streamlines represent.

7.3 STREAMLINE PATTERNS $\eta(x,y)$ =CONSTANT.

We investigate various potential compressible flows by choosing specifically several orthogonal curvilinear net (ξ,η) and determine the exact solution of these flows in the following examples:

Example 1 : <u>Vertical Doublet</u>

Equations

$$x = x(\xi,\eta) = \frac{\xi}{\xi^2 + \eta^2}, \qquad -\infty < \xi < \infty$$
$$y = y(\xi,\eta) = \frac{\eta}{\xi^2 + \eta^2}, \qquad -\infty < \eta < \infty \tag{7.16}$$

define an orthogonal curvilinear coordinate net in the (x, y)-plane. This coordinate system is given by the analytic function

$$w = \frac{1}{\bar{z}}$$

so that

$$\xi(x, y) = \text{Re}(w) = \frac{x}{x^2 + y^2}$$
$$\eta(x, y) = \text{Im}(w) = \frac{y}{x^2 + y^2} \tag{7.17}$$

In this example, we investigate flows when the streamlines are $\eta(x, y) =$ constant and, therefore, we have

$$\eta(x, y) = \frac{y}{x^2 + y^2} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{7.18}$$

The streamline pattern for this flow is shown in Figure 7.1.
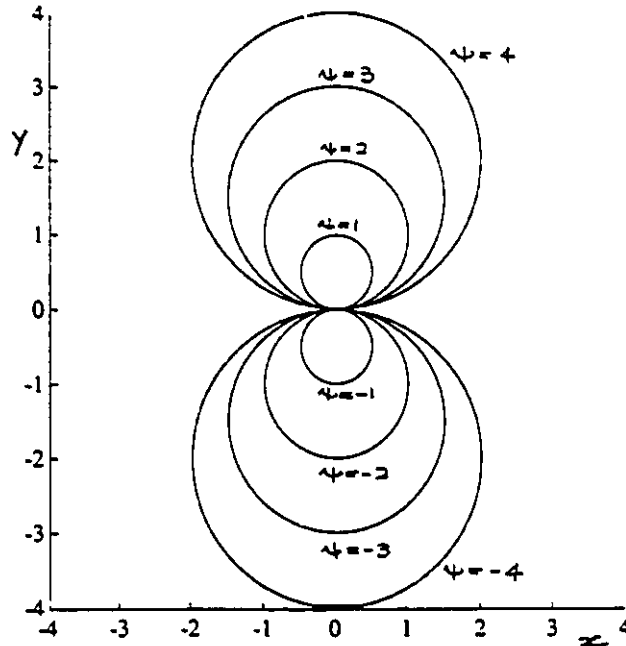
Figure 7.1.   Streamline pattern $\dfrac{y}{x^2 + y^2} =$ constant.

We employ the (ξ, η)–net and the squared differential element of arc length for this net, using (7.18), is given by (7.7) where $h(\xi, \eta) = h(\xi, \Gamma(\psi))$ is

$$h(\xi, \psi) = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} = \frac{1}{\xi^2 + \eta^2} = \frac{1}{\xi^2 + \Gamma^2(\psi)} \tag{7.19}$$

Using (7.19) in the Gauss equation (7.13), we find that (7.13) is identically satisfied. Employing (7.19) in (7.11), we get

$$\rho q = \alpha(\xi, \eta) = \frac{\xi^2 + \Gamma^2(\psi)}{\Gamma'(\psi)} \tag{7.20}$$

Substituting (7.19) and (7.20) in (7.15), we obtain

$$\frac{\partial \alpha}{\partial \xi} \frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi} \frac{\partial q}{\partial \xi} = 0 \tag{7.21}$$

Since $\frac{\partial \alpha}{\partial \xi} = \frac{2\xi}{\Gamma'(\psi)} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} = \frac{2\Gamma(\psi)\Gamma'^2(\psi) - (\xi^2 + \Gamma^2(\psi))\Gamma''(\psi)}{\Gamma'^2(\psi)} \neq 0$ and we do not consider uniform flow, (7.21) is satisfied only if

$$q = q(\alpha) \tag{7.22}$$

Employing (7.19), (7.20) and (7.22) in (7.12) and simplifying, we get

$$\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^2 q'(\alpha)} + \frac{\Gamma''(\psi)}{2\Gamma(\psi)\Gamma'(\psi)} = 0 \tag{7.23}$$

Since

$$\left| \frac{\partial(\alpha, \psi)}{\partial(\xi, \psi)} \right| = \frac{\partial \alpha}{\partial \xi} = \frac{2\xi}{\Gamma'(\psi)} \neq 0$$

provided $\xi \neq 0$, (7.23) yields

$$\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^2 q'(\alpha)} = -\frac{\Gamma''(\psi)}{2\Gamma(\psi)\Gamma'(\psi)} = \text{constant} = C \neq 0 \tag{7.24}$$

Equation (7.24) gives two equations in $q(\alpha)$ and $\Gamma(\psi)$. Integrating these equations once, we obtain

$$q(\alpha) = \frac{M_0 \alpha}{1 + C\alpha} \tag{7.25}$$

and

$$\Gamma'(\psi) = N_0 - C\Gamma^2(\psi) \tag{7.26}$$

where $M_0 \neq 0$ and $N_0 \neq 0$ are arbitrary constants.

Substituting (7.17), (7.18) and (7.26) in (7.20), we get

$$\alpha(x,y) = \frac{x^2 + y^2}{N_0(x^2 + y^2)^2 - Cy^2} \tag{7.27}$$

Employing (7.27) in (7.25) and (7.20), we obtain

$$q(x,y) = \frac{M_0(x^2 + y^2)}{N_0(x^2 + y^2)^2 + Cx^2}$$
$$\rho(x,y) = \frac{N_0(x^2 + y^2)^2 + Cx^2}{M_0[N_0(x^2 + y^2)^2 - Cy^2]} \tag{7.28}$$

Differentiating (7.18) with respect to x and y respectively and using (2.27), we have

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)}\frac{\partial\eta}{\partial y}, \qquad v(x,y) = -\frac{1}{\rho\Gamma'(\psi)}\frac{\partial\eta}{\partial x} \tag{7.29}$$

Using (7.18), (7.26) and (7.28) in (7.29) gives

$$u(x,y) = \frac{M_0[x^2 - y^2]}{N_0(x^2 + y^2)^2 + Cx^2}, \qquad v(x,y) = \frac{2M_0xy}{N_0(x^2 + y^2)^2 + Cx^2} \tag{7.30}$$

Using (7.20), (7.25) and (7.28) in (7.9), (7.10) and integrating, the pressure is given by

$$p(x,y) = p_0 + \frac{M_0}{C^2}\ln\left[\frac{M_0[N_0(x^2 + y^2)^2 - Cy^2]}{N_0(x^2 + y^2)^2 + Cx^2}\right] - \frac{1}{C^2}\left[\frac{M_0[N_0(x^2 + y^2)^2 - Cy^2]}{N_0(x^2 + y^2)^2 + Cx^2}\right] \tag{7.31}$$

The state equation for this flow is given by

$$p = p_0 + \frac{M_0}{C^2}\ln\left(\frac{1}{\rho}\right) - \frac{1}{C^2}\frac{1}{\rho} \tag{7.32}$$

The potential function and the streamfunction for this flow are respectively given by

$$\Phi(x,y) = \frac{M_0}{\sqrt{N_0}\sqrt{C}}\arctan\left[\frac{\sqrt{N_0}\,(x^2 + y^2)}{\sqrt{C}x}\right]$$

and

$$\psi(x,y) = \frac{1}{2\sqrt{CN_0}}\ln\left[\frac{\sqrt{C}y + \sqrt{N_0}(x^2 + y^2)}{\sqrt{C}y - \sqrt{N_0}(x^2 + y^2)}\right] - \psi_0$$
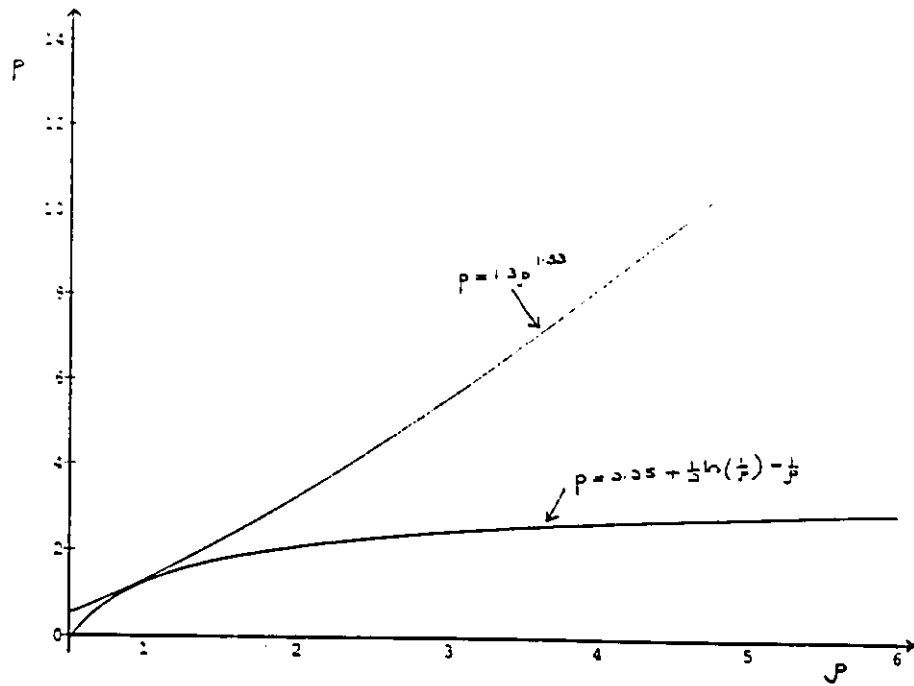
Figure 7.2.   Pressure versus density for the state equation (7.32)
with $p_0 = 2.25$, $M = 0.5$, $C = 1$.

where ψ_0 is an arbitrary constant.

The state equation (7.32) yields

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{1 - M_0\rho}{C^2\rho^2}$$

and

$$\frac{\mathrm{d}^2p}{\mathrm{d}\rho^2} = \frac{M_0\rho - 2}{C^2\rho^3}$$

Since $\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2 > 0$ and $\frac{\mathrm{d}^2p}{\mathrm{d}\rho^2} > 0$ for all compressible real media, it follows that

$$\frac{2}{M} < \rho < \frac{1}{M_0}$$

which cannot be satisfied for all $M_0 > 0$. Therefore, we may consider the solutions

obtained in this example as being valid for an imaginary gas with (7.32) as its state

equation. However, Figure 7.2 shows pressure versus density for the state equation

(7.32) for certain values of p_0, M_0 and C, in particular, $p_0 = 2.25$, $M_0 = 0.5$ and

$C = 1.0$, as compared to the curve of a polytropic gas $p = A\rho^\gamma$ where $A = 1.3$ and

$\gamma = 1.33$. We observed that the state equation (7.32) is a reasonable approximation

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of the ideal gas for a small range of values of ρ. Therefore, we can consider the solutions obtained above to be valid for an ideal gas provided the variations in density are confined to this range of values for ρ.

Example 2 : <u>Horizontal Doublet</u>
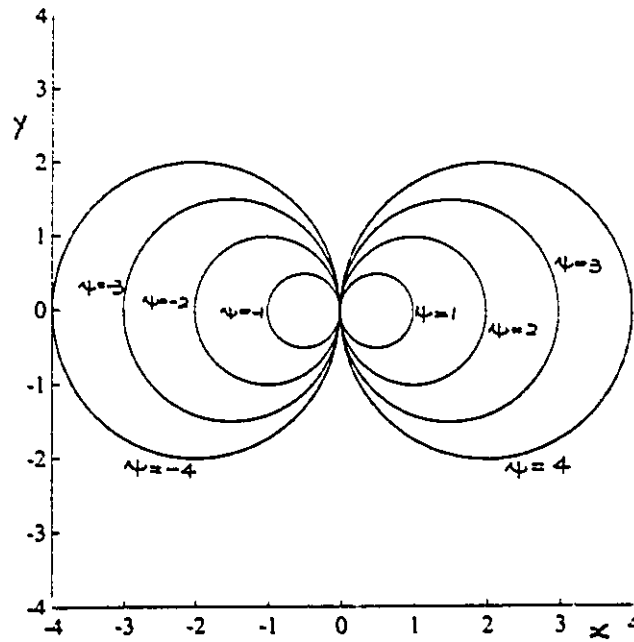
Figure 7.3.   Streamline pattern $\dfrac{x}{x^2 + y^2}$ =constant.

The orthogonal curvilinear coordinate system defined by the analytic function

$$w = \frac{i}{z}$$

is

$$
\begin{aligned}
x = x(\xi, \eta) = \frac{\eta}{\xi^2 + \eta^2}, && -\infty < \xi < \infty \\
y = y(\xi, \eta) = \frac{\xi}{\xi^2 + \eta^2}, && -\infty < \eta < \infty
\end{aligned}
$$

(7.33)

Solving (7.33) for ξ and η, we get

$$
\begin{aligned}
\xi(x, y) = \mathrm{Re}(w) = \frac{y}{x^2 + y^2} \\
\eta(x, y) = \mathrm{Im}(w) = \frac{x}{x^2 + y^2}
\end{aligned}
$$

(7.34)

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The streamlines for this flow are given by the family of curves $\eta(x,y) = $constant. Therefore, we have

$$\eta(x,y) = \frac{x}{x^2 + y^2} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{7.35}$$

The streamlines for this flow are shown in Figure 7.3.

To study this flow, we use the (ξ,ψ)–net and the squared differential element of arc length for this net, using (7.35), is given by (7.7) where

$$h(\xi,\psi) = \frac{1}{\xi^2 + \Gamma^2(\psi)} \tag{7.36}$$

Equation (7.36) identically satisfies (7.13) and employing (7.36) in (7.11), we get

$$\rho q = \alpha(\xi,\psi) = \frac{\xi^2 + \Gamma^2(\psi)}{\Gamma'(\psi)} \tag{7.37}$$

Differentiating (7.37) with respect to ξ and ψ, we obtain

$$\frac{\partial \alpha}{\partial \xi} = \frac{2\xi}{\Gamma'(\psi)}, \qquad \frac{\partial \alpha}{\partial \psi} = \frac{2\Gamma(\psi)\Gamma'^2(\psi) - (\xi^2 + \Gamma^2(\psi))\Gamma''(\psi)}{\Gamma'^2(\psi)} \tag{7.38}$$

Employing (7.37) in (7.15) yields

$$\frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} = 0 \tag{7.39}$$

Since $\frac{\partial \alpha}{\partial \xi} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} = 0$ requires $\Gamma'(\psi) = 0$ and ignoring flows with constant speed, (7.39) is satisfied only if

$$q = q(\alpha) \tag{7.40}$$

Using (7.36), (7.38) and (7.40) in (7.12), we obtain

$$\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^2 q'(\alpha)} = -\frac{\Gamma''(\psi)}{2\Gamma(\psi)\Gamma'(\psi)} = \text{constant} = C \neq 0$$

since α and ψ are independent variables provided $\xi \neq 0$. This equation yields two equations in $q(\alpha)$ and $\Gamma(\psi)$ and solving these equations, we get

$$q(\alpha) = \frac{M_1 \alpha}{1 + C\alpha} \tag{7.41}$$

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and

$$\Gamma'(\psi) = N_1 - C\Gamma^2(\psi) \tag{7.42}$$

where $M_1 \neq 0$ and $N_1 \neq 0$ are arbitrary constants.

Substituting (7.34), (7.35) and (7.42) in (7.37), we get

$$\alpha(x,y) = \frac{x^2 + y^2}{N_1(x^2 + y^2)^2 - Cx^2} \tag{7.43}$$

Using (7.43) in (7.41) and (7.11), we obtain

$$q(x,y) = \frac{M_1(x^2 + y^2)}{N_1(x^2 + y^2)^2 + Cy^2}$$

$$\rho(x,y) = \frac{N_1(x^2 + y^2)^2 + Cy^2}{M_1[N(x^2 + y^2)^2 - Cx^2]} \tag{7.44}$$

Employing (7.35), (7.42) and (7.44) in (7.29) gives

$$u(x,y) = -\frac{2M_1 xy}{N_1(x^2 + y^2)^2 + Cy^2}, \qquad v(x,y) = \frac{M_1[x^2 - y^2]}{N_1(x^2 + y^2)^2 + Cy^2} \tag{7.45}$$

Using (7.41), (7.44) in (7.9) and (7.10) and integrating, the pressure is given by

$$p(x,y) = p_0 - \frac{1}{C^2}\left[\frac{M_1[N_1(x^2 + y^2)^2 - Cx^2]}{N_1(x^2 + y^2)^2 + Cy^2}\right] + \frac{M_1}{C^2}\ln\left[\frac{M_1[N_1(x^2 + y^2)^2 - Cx^2]}{N_1(x^2 + y^2)^2 + Cy^2}\right] \tag{7.46}$$

The state equation for this flow is given by

$$p = p_0 + \frac{M_1}{C^2}\ln\left(\frac{1}{\rho}\right) - \frac{1}{C^2}\left(\frac{1}{\rho}\right) \tag{7.47}$$

The potential function and the streamfunction for this flow are given by

$$\Phi(x,y) = -\frac{M_1}{\sqrt{N_1}\sqrt{C}}\arctan\left[\frac{\sqrt{N_1}\,(x^2 + y^2)}{\sqrt{C}y}\right]$$

and

$$\psi(x,y) = \frac{1}{2\sqrt{CN_1}}\ln\left[\frac{\sqrt{C}x + \sqrt{N_1}(x^2 + y^2)}{\sqrt{C}x - \sqrt{N_1}(x^2 + y^2)}\right] - \psi_1$$

respectively where ψ_1 is an arbitrary constant.
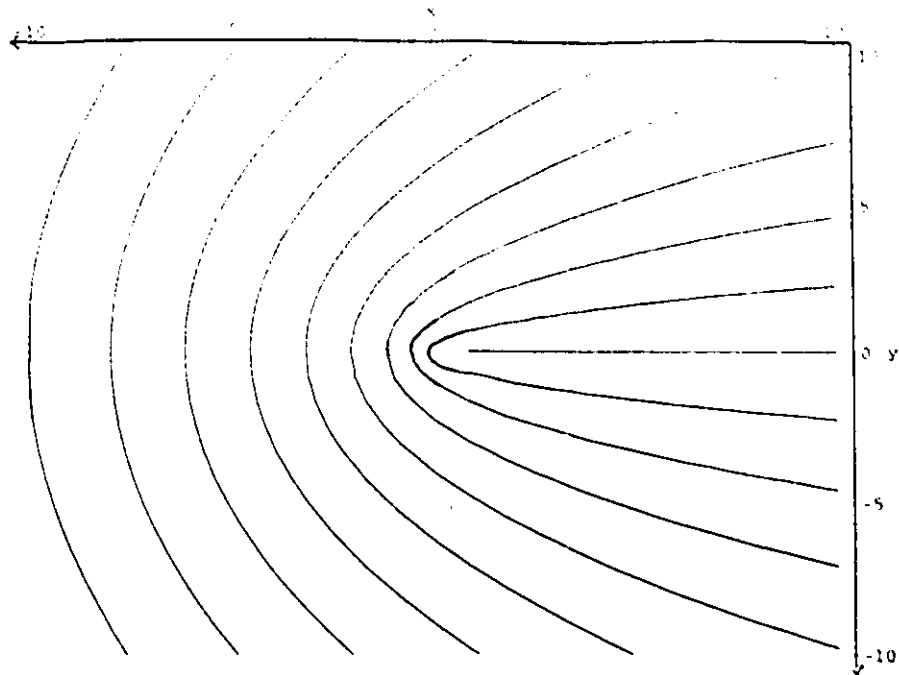
Figure 7.4. Streamline pattern $\sqrt{\sqrt{x^2 + y^2} - x}$ =constant .

The state equation for this flow (7.47) is identical to (7.32) with the exception of a constant. Therefore, the same analysis applies as in the previous example.

Example 3 : Parabolic flow

Equations

$$x(\xi,\eta) = \frac{1}{2}\left(\xi^2 - \eta^2\right), \qquad 0 \leq \xi < \infty$$

$$y(\xi,\eta) = \xi\eta, \qquad\qquad 0 \leq \eta < \infty$$

(7.48)

defined by

$$w = \sqrt{2z}$$

define an orthogonal curvilinear coordinate system (ξ,η) in the (x,y)-plane. From (7.48), we obtain

$$\xi(x,y) = \text{Re}(w) = \sqrt{\sqrt{x^2 + y^2} + x}$$

$$\eta(x,y) = \text{Im}(w) = \sqrt{\sqrt{x^2 + y^2} - x}$$

(7.49)

In this example, we study flows when the streamlines are given by the family of curves $\eta(x,y)$ =constant. Therefore, we have

$$\eta(x,y) = \sqrt{\sqrt{x^2 + y^2} - x} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0$$

(7.50)

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The flow pattern for this example is shown in Figure 7.4.

For this study, we employ the (ξ, ψ)–net and the squared differential element of arc length for this net, using (7.50), is given by (7.7) where

$$h(\xi, \psi) = \sqrt{\xi^2 + \eta^2} = \sqrt{\xi^2 + \Gamma^2(\psi)} \tag{7.51}$$

Substituting (7.51) in (7.13) results in this equation being identically satisfied.

Using (7.51) in (7.11), we obtain

$$\rho q = \alpha(\xi, \psi) = \frac{1}{\Gamma'(\psi)\sqrt{\xi^2 + \Gamma^2(\psi)}} \tag{7.52}$$

Differentiating (7.52) with respect to ξ and ψ, we get

$$\frac{\partial \alpha}{\partial \xi} = -\frac{\xi}{\Gamma'(\psi)\left[\xi^2 + \Gamma^2(\psi)\right]^{\frac{3}{2}}}, \qquad \frac{\partial \alpha}{\partial \psi} = -\frac{\left(\Gamma(\psi)\Gamma'^2(\psi) + \left[\xi^2 + \Gamma^2(\psi)\right]\Gamma''(\psi)\right)}{\Gamma'^2(\psi)\left[\xi^2 + \Gamma^2\right]^{\frac{3}{2}}} \tag{7.53}$$

Using (7.52) in (7.15), we have

$$\frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} = 0 \tag{7.54}$$

Since $\frac{\partial \alpha}{\partial \xi} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$ and we do not consider uniform flow, (7.54) gives

$$q = q(\alpha) \tag{7.55}$$

Employing (7.52), (7.53) and (7.55) in (7.12) and separating variables, we obtain

$$\frac{\alpha\left[\gamma(\alpha) - \alpha q'(\alpha)\right]}{q'(\alpha)} = \frac{\Gamma''(\psi)}{\Gamma(\psi)\Gamma'^4(\psi)} = \text{constant} = C \neq 0$$

since $\frac{\partial(\alpha, \psi)}{\partial(\xi, \psi)} = \frac{\partial \alpha}{\partial \xi} \neq 0$ when $\xi \neq 0$. Solving the two equations in $q(\alpha)$ and $\Gamma(\psi)$, given by the above equation, we get

$$q(\alpha) = K\sqrt{\alpha^2 + C} \tag{7.56}$$

and

$$\Gamma'(\psi) = \frac{1}{\sqrt{L - C\Gamma^2(\psi)}} \tag{7.57}$$

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where $K \neq 0$ and $L \neq 0$ are arbitrary constants. Employing (7.50) and (7.57) in (7.52) yields

$$\alpha(x, y) = \frac{\sqrt{L - C \left[\sqrt{x^2 + y^2} - x\right]}}{\sqrt{2\sqrt{x^2 + y^2}}} \qquad (7.58)$$

Substituting (7.58) in (7.56) and (7.11), we get

$$q(x, y) = \frac{K\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}{\sqrt{2\sqrt{x^2 + y^2}}}$$

$$\qquad (7.59)$$

$$p(x, y) = \frac{\sqrt{L - C\left[\sqrt{x^2 + y^2} - x\right]}}{K\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}$$

Employing (7.50), (7.57) and (7.59) in (7.29) gives

$$u(x, y) = \frac{Ky\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}{2\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} - x}}$$

$$v(x, y) = \frac{K\left(\sqrt{x^2 + y^2} - x\right)\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}{2\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} - x}}$$

Integrating (7.9) and (7.10) using (7.59), the pressure function is given by

$$p(x, y) = p_0 + \frac{1}{2}CK\ln\left[K\sqrt{\alpha^2 + C} + K\alpha\right] - \frac{1}{2}K\alpha\sqrt{\alpha^2 + C} \qquad (7.60)$$

where $\alpha(x, y)$ is given by (7.58).

The state equation for this flow is given by

$$p = p_0 + \frac{1}{2}CK\ln\left[\frac{K\sqrt{C}}{\sqrt{1 - K^2\rho^2}} + \frac{K^2\sqrt{C}\rho}{\sqrt{1 - K^2\rho^2}}\right] - \frac{1}{2}\frac{K^2C\rho}{1 - K^2\rho^2} \qquad (7.61)$$

The changes of pressure with flow intensity α are given by (7.60).

The potential function for this flow is

$$\Phi(x, y) = \frac{K}{2}y\int \frac{\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} - x}}dx$$

$$+ \frac{K}{2}\int \frac{y\sqrt{\sqrt{x^2 + y^2} - x}\sqrt{L + Cx + C\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}}dy$$

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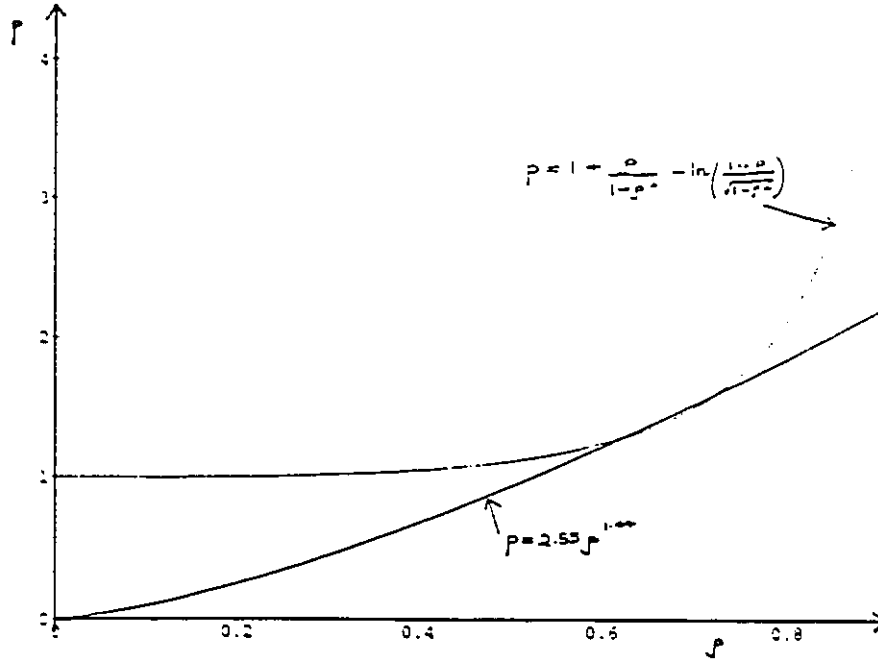
Figure 7.3. Pressure versus density for the state equation (7.61) with $p_0 = 1$, $C = -2$, $K = 1$.

The streamfunction for this flow is given by

$$\psi(x,y) = \frac{1}{2}\left[\left(\sqrt{x^2+y^2}-x\right)\left\{L-\sqrt{x^2+y^2}+x\right\}\right]^{\frac{1}{2}}$$
$$+ \frac{1}{2C}\arcsin\left(\frac{\sqrt{C}}{\sqrt{L}}\sqrt{\sqrt{x^2+y^2}-x}\right) - \psi_2$$

where ψ_2 is an arbitrary constant.

From the state equation (7.61), we have

$$\frac{dp}{d\rho} = -CK^4\frac{\rho^2}{\left(1-K^2\rho^2\right)^2}$$

and

$$\frac{d^2p}{d\rho^2} = -CK^4\frac{2\rho\left(1+2K^2\rho^2\right)}{\left(1-K^2\rho^2\right)^2}$$

Since $\frac{dp}{d\rho} = c^2 > 0$ and $\frac{d^2p}{d\rho^2} > 0$ for all real gases, it follows that

$$C < 0$$

Therefore, the solutions obtained above are valid for some real gas having (7.61) as the equation of state giving p as a function of ρ for every permissible choice of C and K.

Figure 7.5 shows the plot of state equation (7.61) for $p_0 = 1$, $C = -2$ and $K = 1$ and the curve of a polytropic gas $p = A\rho^\gamma$ where $A = 2.55$ and $\gamma = 1.44$. It can be seen that the state equation (7.61) approximates the ideal gas reasonably well for a small range of values of ρ. Therefore, we may restrict the application of the above solutions to this range of values of ρ in order to study the behaviour of an ideal gas in such a flow.

Example 4 : Reverse Parabolic flow
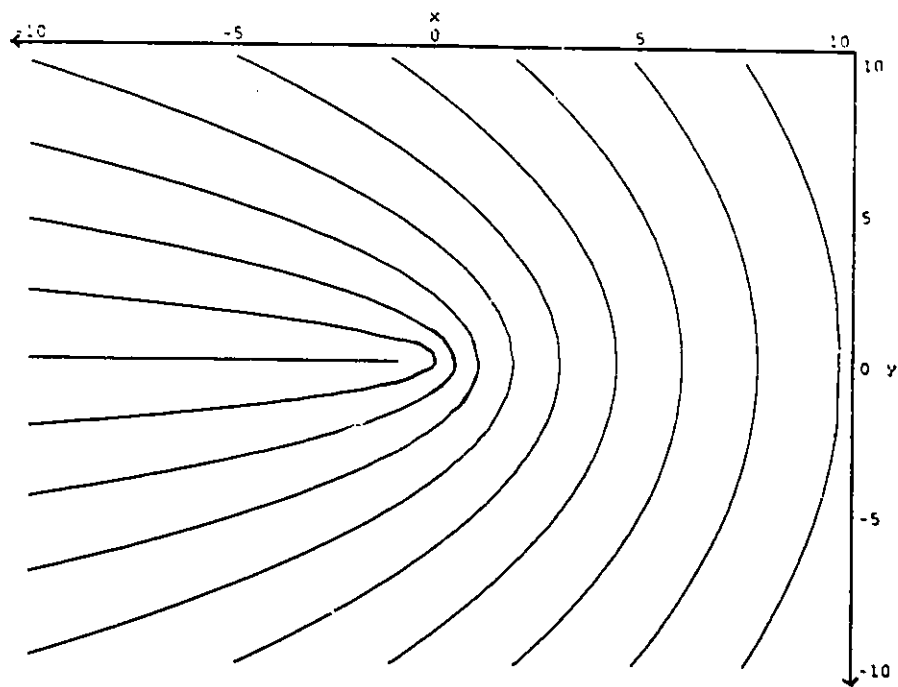
Figure 7.6.   Streamline pattern $\sqrt{\sqrt{x^2 + y^2} + x}$ =constant.

Equations

$$x = x(\xi, \eta) = \frac{1}{2}\left(\eta^2 - \xi^2\right), \qquad 0 \le \xi < \infty$$

$$y = y(\xi, \eta) = -\xi\eta, \qquad\qquad 0 \le \eta < \infty$$

(7.62)

define an orthogonal curvilinear coordinate net in the (x, y)-plane. This coordinate system is given by the analytic function

$$w = \sqrt{-2z}$$

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so that

$$\xi(x,y) = \text{Re}(w) = \sqrt{\sqrt{x^2 + y^2} - x}$$

$$\eta(x,y) = \text{Im}(w) = \sqrt{\sqrt{x^2 + y^2} + x} \qquad (7.63)$$

The streamline for this flow are given by the curves $\eta(x,y) =$ constant and from (7.6) we, therefore, have

$$\sqrt{\sqrt{x^2 + y^2} + x} = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \qquad (7.64)$$

The streamlines for this flow are shown in Figure 7.6.

We employ the (ξ, ψ)−net and the squared differential element of arc length for this net using (7.64) is given by (7.7) where

$$h(\xi, \psi) = \sqrt{\xi^2 + \Gamma^2(\psi)} \qquad (7.65)$$

Using (7.65) in the Gauss equation (7.13), we find that (7.13) is identically satisfied.

Employing (7.65) in (7.11), we obtain

$$\alpha(\xi, \psi) = \frac{1}{\Gamma'(\psi)\sqrt{\xi^2 + \Gamma^2(\psi)}} \qquad (7.66)$$

Substitution of (7.66) in (7.15) yields (7.54) which is only satisfied if

$$q = q(\alpha) \qquad (7.67)$$

since $\frac{\partial \alpha}{\partial \xi} \neq 0$, $\frac{\partial \alpha}{\partial \psi} \neq 0$ and we do not consider uniform flow.

Since α and ψ are independent variables, using (7.65), (7.66) and (7.67) in (7.12), separating variables and solving the resulting equations, we get

$$q(\alpha) = K_1 \sqrt{\alpha^2 + C} \qquad (7.68)$$

and

$$\Gamma'(\psi) = \frac{1}{\sqrt{L_1 - C\Gamma^2(\psi)}} \qquad (7.69)$$

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where $C \neq 0$, $K_1 \neq 0$ and $L_1 \neq 0$ are arbitrary constants. Using (7.64) and (7.69) in (7.66) yields

$$\alpha(x,y) = \frac{\sqrt{L_1 - C\left[\sqrt{x^2 + y^2} + x\right]}}{\sqrt{2\sqrt{x^2 + y^2}}} \tag{7.70}$$

Substituting (7.70) in (7.68) and (7.11), we get

$$q(x,y) = \frac{K_1\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}}{\sqrt{2\sqrt{x^2 + y^2}}}$$

$$\rho(x,y) = \frac{\sqrt{L_1 - C\left[\sqrt{x^2 + y^2} + x\right]}}{K_1\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}} \tag{7.71}$$

Employing (7.64), (7.69) and (7.71) in (7.29) gives

$$u(x,y) = \frac{K_1 y\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}}{2\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} + x}}$$

$$v(x,y) = -\frac{K_1\left(\sqrt{x^2 + y^2} + x\right)\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}}{2\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} + x}}$$

Integrating (7.9) and (7.10) using (7.71), the pressure function is given by

$$p(x,y) = p_0 + \frac{1}{2}CK_1\ln\left[K_1\sqrt{\alpha^2 + C} + K_1\alpha\right] - \frac{1}{2}K_1\alpha\sqrt{\alpha^2 + C} \tag{7.72}$$

where $\alpha(x,y)$ is given by (7.70).

The state equation for this flow is given by

$$p = p_0 + \frac{1}{2}CK_1\ln\left[\frac{K_1\sqrt{C}}{\sqrt{1 - K_1^2\rho^2}} + \frac{K_1^2\sqrt{C}\rho}{\sqrt{1 - K_1^2\rho^2}}\right] - \frac{1}{2}\frac{K_1^2 C\rho}{1 - K_1^2\rho^2} \tag{7.73}$$

The potential function for this flow is

$$\Phi(x,y) = \frac{K_1}{2}\left[\int \frac{y\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}\sqrt{\sqrt{x^2 + y^2} + x}}dx\right.$$

$$\left. + \int \frac{y\sqrt{\sqrt{x^2 + y^2} + x}\sqrt{L_1 - Cx + C\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}}dy\right]$$

The streamfunction for this flow is

$$\psi(x,y) = \frac{1}{2}\left[\left(\sqrt{x^2+y^2}+x\right)\left\{L-\sqrt{x^2+y^2}-x\right\}\right]^{\frac{1}{2}}$$

$$+ \frac{1}{2C}\arcsin\left(\frac{\sqrt{C}}{\sqrt{L}}\sqrt{\sqrt{x^2+y^2}+x}\right) - \psi_3$$

where ψ_3 is an arbitrary constant.

The state equation (7.73) is similar to (7.61). Therefore, we can conclude that the solutions obtained above are valid for some real gas having (7.73) as its state equation.

Example 5 : <u>Stagnation-point flow</u>
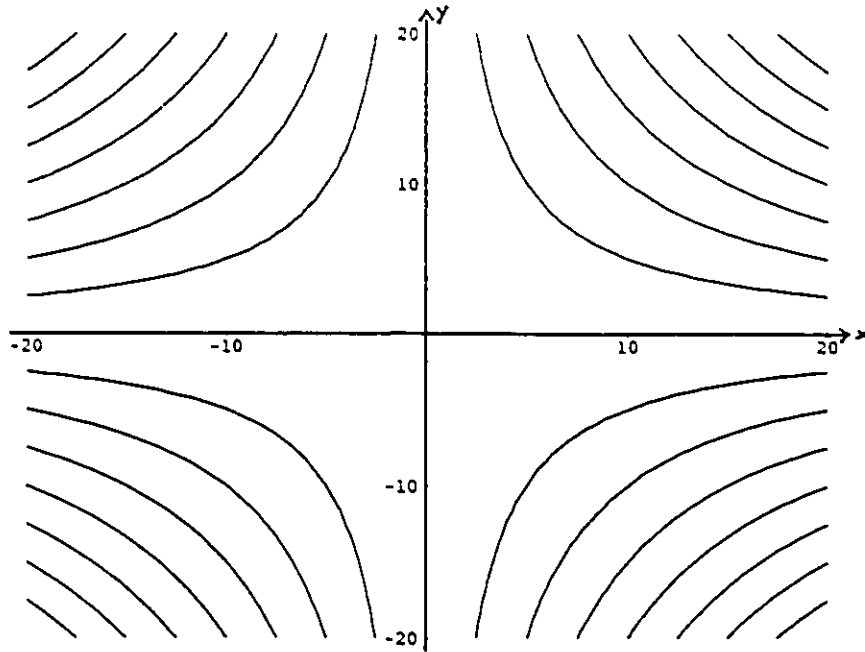
Figure 7.7.   Streamline pattern xy =constant.

The orthogonal curvilinear coordinate system defined by the analytic function

$$w = \frac{1}{2}z^2$$

is

$$\begin{aligned}
x &= x(\xi,\eta) = \sqrt{\sqrt{\xi^2+\eta^2}+\xi}, & -\infty < \xi < \infty \\
y &= y(\xi,\eta) = \sqrt{\sqrt{\xi^2+\eta^2}-\xi}, & -\infty < \eta < \infty
\end{aligned} \qquad (7.74)$$

or

$$\xi(x,y) = \text{Re}(w) = \frac{1}{2}(x^2 - y^2)$$

(7.75)

$$\eta(x,y) = \text{Im}(w) = xy$$

In this example, we investigate flows when the streamlines are given by the curves $\eta(x,y) = $constant. Therefore, we have

$$\eta(x,y) = xy = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0$$

(7.76)

The streamline pattern for this flow is shown in Figure 7.7.

For this investigation, we employ the (ξ,ψ)−net and the squared differential element of arc length for this net using (7.76) is given by (7.7) where

$$h(\xi,\psi) = \frac{1}{\sqrt{2}} \frac{1}{[\xi^2 + \Gamma^2(\psi)]^{\frac{1}{4}}}$$

(7.77)

Equation (7.77) identically satisfies (7.13). Employing (7.77) in (7.11), we get

$$\alpha(\xi,\psi) = \frac{\sqrt{2}\,[\xi^2 + \Gamma^2(\psi)]^{\frac{1}{4}}}{\Gamma'(\psi)}$$

(7.78)

Differentiating (7.78) with respect to ξ and ψ, we obtain

$$\frac{\partial \alpha}{\partial \xi} = \frac{\xi}{\sqrt{2}\Gamma'(\psi)\,[\xi^2 + \Gamma^2(\psi)]^{\frac{3}{4}}}, \frac{\partial \alpha}{\partial \psi} = \frac{\Gamma(\psi)\Gamma'^2(\psi) - 2(\xi^2 + \Gamma^2(\psi))\Gamma''(\psi)}{\sqrt{2}\Gamma'^2(\psi)\,[\xi^2 + \Gamma^2(\psi)]^{\frac{3}{4}}}$$

(7.79)

Using (7.78) in (7.15) yields

$$q = q(\alpha)$$

(7.80)

since $\frac{\partial \alpha}{\partial \xi} \neq 0$, $\frac{\partial \alpha}{\partial \psi} \neq 0$ and we do not consider flows with constant speed.

Employing (7.78), (7.79) and (7.80) in (7.12) and separating variables, we obtain

$$\frac{[\alpha q'(\alpha) - q(\alpha)]}{\alpha^5 q'(\alpha)} = \frac{\Gamma'^2(\psi)\Gamma''(\psi)}{2\Gamma(\psi)} = \text{constant} = C \neq 0$$

since $\frac{\partial(\alpha,\psi)}{\partial(\xi,\psi)} = \frac{\partial \alpha}{\partial \xi} \neq 0$. This equation yields two equations in $q(\alpha)$ and $\Gamma(\psi)$. Integrating these two equations once, we have

$$q(\alpha) = \frac{C_1 \alpha}{[1 - C\alpha^4]^{\frac{1}{4}}}$$

(7.81)

and

$$\Gamma'(\psi) = \left[4C\Gamma^2(\psi) + 4C_2\right]^{\frac{1}{4}} \tag{7.82}$$

where $C_1 \neq 0$ and C_2 are arbitrary constants. Using (7.76) and (7.82) in (7.78), we get

$$\alpha(x,y) = \frac{\sqrt{x^2 + y^2}}{\left[4Cx^2y^2 + 4C_2\right]^{\frac{1}{4}}} \tag{7.83}$$

Employing (7.83) in (7.81), we obtain

$$q(x,y) = \frac{C_1\sqrt{x^2 + y^2}}{\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{1}{4}}} \tag{7.84}$$

The density function ρ from (7.11) and (7.84) is given by

$$\rho(x,y) = \frac{\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{1}{4}}}{C_1\left[4Cx^2y^2 + 4C_2\right]^{\frac{1}{4}}} \tag{7.85}$$

Using (7.76), (7.82) and (7.85) in (7.29) yields

$$u(x,y) = \frac{C_1 x}{\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{1}{4}}} \tag{7.86}$$

and

$$v(x,y) = -\frac{C_1 y}{\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{1}{4}}} \tag{7.87}$$

Employing (7.11), (7.81) and (7.83) in (7.9) and (7.10) and integrating, we get

$$\begin{aligned} p(x,y) = 2C_1 \int \frac{x\left[2C_2 + Cx^2y^2 - Cy^4\right]}{\left[4Cx^2y^2 + 4C_2\right]^{\frac{1}{4}}\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{5}{4}}}dx \\ 2C_1 \int \frac{y\left[2C_2 + Cx^2y^2 - Cx^4\right]}{\left[4Cx^2y^2 + 4C_2\right]^{\frac{1}{4}}\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{5}{4}}}dy \end{aligned} \tag{7.88}$$

The state equation for this flow is given by

$$p = \frac{1}{\sqrt{C}} \int \frac{1}{\rho^2\sqrt{1 - C_1\rho^4}}d\rho \tag{7.89}$$

The potential function and the streamfunction for this flow are given by

$$\Phi(x,y) = \int \frac{C_1 x dx - C_1 y dy}{\left[4C_2 + 2Cx^2y^2 - Cx^4 - Cy^4\right]^{\frac{1}{4}}}$$

$$\psi(x,y) = \int \frac{d(xy)}{[4Cx^2y^2 + 4C_2]^{\frac{1}{4}}} - \psi_4$$

respectively where ψ_4 is an arbitrary constant.

The state equation (7.89) yields

$$\frac{dp}{d\rho} = \frac{1}{\sqrt{C}} \frac{1}{\rho^2 \sqrt{1 - C_1^4 \rho^4}}$$

and

$$\frac{d^2p}{d\rho^2} = -\frac{1}{\sqrt{C}} \frac{2\rho \left(1 - 2C_1^4 \rho^4\right)}{\rho^4 \sqrt{1 - C_1^4 \rho^4} \left(1 - C_1^4 \rho^4\right)}$$

In order for this state equation to satisfy the requirements of a real gas, that is, $\frac{dp}{d\rho} > 0$ and $\frac{d^2p}{d\rho^2} > 0$, we must have

$$C > 0, \qquad C_1 > 0 \qquad \text{and} \qquad \frac{1}{2^{\frac{1}{4}}C_1} < \rho < \frac{1}{C_1}$$

Therefore, for every permissible value of C and C_1, the solution for the stagnation flow above is valid for some real gas having (7.89) as the state equation for the range of values of ρ given above.

Example 6: <u>Double hyperbolic flow</u>

Equations

$$\begin{aligned}
x &= x(\xi,\eta) = \sqrt{\sqrt{\xi^2 + \eta^2} + \eta}, & -\infty < \xi < \infty \\
y &= y(\xi,\eta) = \sqrt{\sqrt{\xi^2 + \eta^2} - \eta}, & -\infty < \eta < \infty
\end{aligned} \qquad (7.90)$$

define an orthogonal curvilinear coordinate net in the (x,y)-plane. This coordinate system is given by the analytic function

$$w = \frac{i}{2}z^2$$

so that

$$\begin{aligned}
\xi(x,y) &= \text{Re}(w) = xy \\
\eta(x,y) &= \text{Im}(w) = \frac{1}{2}(x^2 - y^2)
\end{aligned} \qquad (7.91)$$

The streamlines for this flow are given by the curves $\eta(x,y)$ =constant. Therefore, we have

$$\eta(x,y) = \frac{1}{2}(x^2 - y^2) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \qquad (7.92)$$

The flow pattern for this example is shown in Figure 7.8.
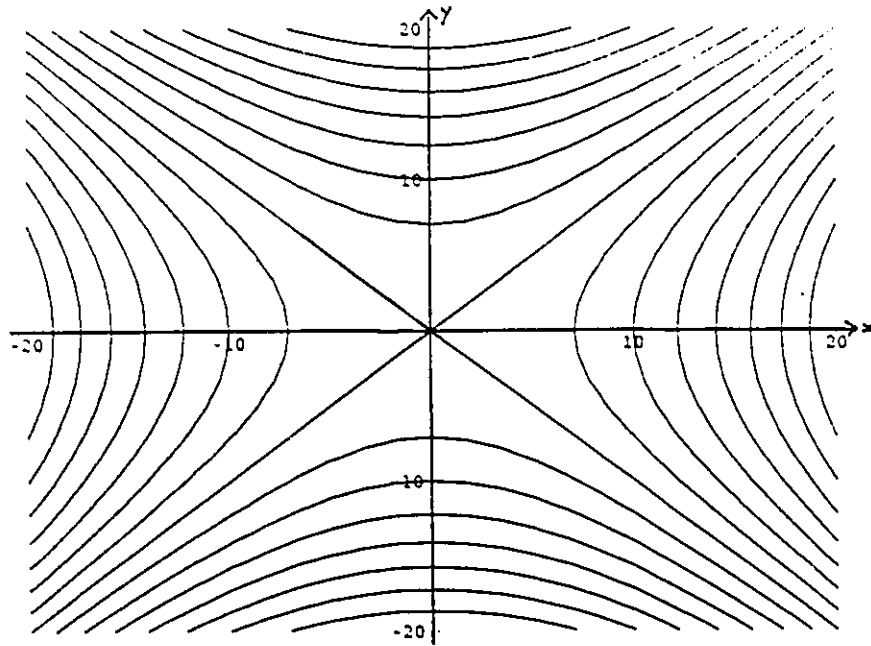
Figure 7.8.   Streamline pattern $x^2 - y^2$ =constant.

To study this flow, we employ the (ξ,ψ)—net and have (7.7) where

$$h(\xi,\psi) = \frac{1}{\sqrt{2}} \frac{1}{[\xi^2 + \Gamma^2(\psi)]^{\frac{1}{4}}} \qquad (7.93)$$

Equation (7.93) identically satisfies the Gauss equation (7.13). Using (7.93) in (7.11), we get

$$\alpha(\xi,\psi) = \frac{\sqrt{2}\,[\xi^2 + \Gamma^2(\psi)]^{\frac{1}{4}}}{\Gamma'(\psi)} \qquad (7.94)$$

Employing (7.94) in (7.15) yields

$$q = q(\alpha) \qquad (7.95)$$

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since $\frac{\partial \alpha}{\partial \xi} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$ as given in (7.79). Employing (7.94), (7.95) and (7.79) in (7.12) and separating variables, we obtain

$$\frac{[\alpha q'(\alpha) - q(\alpha)]}{\alpha^5 q'(\alpha)} = \frac{\Gamma'^2(\psi)\Gamma''(\psi)}{2\Gamma(\psi)} = \text{constant} = C \neq 0$$

since α and ψ are independent variables. Solving the two equations in $q(\alpha)$ and $\Gamma(\psi)$ obtained from the above equation, we have

$$q(\alpha) = \frac{D_1 \alpha}{[1 - C\alpha^4]^{\frac{1}{4}}} \tag{7.96}$$

and

$$\Gamma'(\psi) = \left[4C\Gamma^2(\psi) + 4D_2\right]^{\frac{1}{4}} \tag{7.97}$$

where $D_1 \neq 0$ and D_2 are arbitrary constants. Using (7.92) and (7.97) in (7.94), we get

$$\alpha(x, y) = \frac{\sqrt{x^2 + y^2}}{[C(x^2 - y^2)^2 + 4D_2]^{\frac{1}{4}}} \tag{7.98}$$

Employing (7.98) in (7.96), we obtain

$$q(x, y) = \frac{D_1 \sqrt{x^2 + y^2}}{[4D_2 - 4Cx^2 y^2]^{\frac{1}{4}}} \tag{7.99}$$

The density function ρ, using (7.98) and (7.99) in (7.11) is

$$\rho(x, y) = \frac{[4D_2 - 4Cx^2 y^2]^{\frac{1}{4}}}{D_1 [C(x^2 - y^2)^2 + 4D_2]^{\frac{1}{4}}} \tag{7.100}$$

Using (7.92), (7.97) and (7.100) in (7.29) gives

$$u(x, y) = -\frac{D_1 y}{[4D_2 - 4Cx^2 y^2]^{\frac{1}{4}}} \tag{7.101}$$

and

$$v(x, y) = -\frac{D_1 x}{[4D_2 - 4Cx^2 y^2]^{\frac{1}{4}}} \tag{7.102}$$

Employing (7.11), (7.96) and (7.98) in (7.9) and (7.10) and integrating, we get

$$p(x,y) = D_1 \int \frac{x \left[4D_2 - 2Cx^2y^2 + 2Cy^4\right]}{[C(x^2-y^2)^2 + 4D_2]^{\frac{1}{4}} [4D_2 - 4Cx^2y^2]^{\frac{5}{4}}} dx$$
$$D_1 \int \frac{y \left[4D_2 - 2Cx^2y^2 + 2Cx^4\right]}{[C(x^2-y^2)^2 + 4D_2]^{\frac{1}{4}} [4D_2 - 4Cx^2y^2]^{\frac{5}{4}}} dy \qquad (7.103)$$

The state equation for this flow is given by

$$p = \frac{1}{\sqrt{C}} \int \frac{1}{\rho^2 \sqrt{1 - D_1 \rho^4}} d\rho \qquad (7.104)$$

The potential function and the streamfunction for this flow are respectively given . by

$$\Phi(x,y) = -\int \frac{D_1 y \, dx + D_1 x \, dy}{[4D_2 - 4Cx^2y^2]^{\frac{1}{4}}}$$

$$\psi(x,y) = \int \frac{d(x^2 - y^2)}{[C(x^2-y^2)^2 + 4D_2]^{\frac{1}{4}}} - \psi_5$$

where ψ_5 is an arbitrary constant.

The state equation (7.104) is identical to (7.89) with the exception of a constant. Therefore, the solutions obtained above are valid for some real gas having (7.104) as the state equation provided

$$C > 0, \qquad D_1 > 0 \qquad \text{and} \qquad \frac{1}{2^{\frac{1}{4}} D_1} < \rho < \frac{1}{D_1}$$

Example 7 : <u>Hyperbolic flow</u>

The orthogonal curvilinear coordinate system defined by the analytic function

$$w = \cosh^{-1}\left(\frac{1}{a} z\right)$$

is

$$x = x(\xi,\eta) = a \cosh\xi \cos\eta, \qquad -\infty < \xi < \infty$$
$$y = y(\xi,\eta) = a \sinh\xi \sin\eta, \qquad 0 \le \eta \le 2\pi \qquad (7.105)$$

where $a \neq 0$ is an arbitrary constant. Equations (7.105) can be solved for ξ and η to give

$$\xi = \xi(x, y)$$
$$\eta = \eta(x, y)$$

(7.106)

The streamline pattern for this flow is given by the curves $\eta(x, y) =$ constant so that

$$\eta(x, y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0$$

(7.107)

To investigate this flow, we employ the (ξ, ψ)–net and the squared differential element of arc length for this net with (7.107) is given by (7.7) where

$$h(\xi, \psi) = a\sqrt{cosh^2\xi - cos^2\Gamma(\psi)}$$

(7.108)

The Gauss equation (7.13) is identically satisfied when (7.108) is used.

Employing (7.108) in (7.11), we get

$$\alpha(\xi, \psi) = \frac{1}{a\Gamma'(\psi)\sqrt{cosh^2\xi - cos^2\Gamma(\psi)}}$$

(7.109)

Differentiating (7.109) with respect to ξ and ψ, we obtain

$$\frac{\partial \alpha}{\partial \xi} = -\frac{cosh\xi sinh\xi}{a\Gamma'(\psi)\left[cosh^2\xi - cos^2\Gamma(\psi)\right]^{\frac{3}{2}}},$$
$$\frac{\partial \alpha}{\partial \psi} = -\frac{cos\Gamma(\psi)sin\Gamma(\psi)\Gamma'(\psi)}{a\Gamma'(\psi)\left[cosh^2\xi - cos^2\Gamma(\psi)\right]^{\frac{3}{2}}} - \frac{\Gamma''(\psi)}{a\Gamma'^2(\psi)\sqrt{cosh^2\xi - cos^2\Gamma(\psi)}}$$

(7.110)

Using (7.109) in (7.15), we obtain

$$\frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} = 0$$

(7.111)

Since $\frac{\partial \alpha}{\partial \xi} = 0$ only if $\xi = 0$ and $\frac{\partial \alpha}{\partial \psi} = 0$ requires $\Gamma'(\psi) = 0$ and we omit flows where $q =$ constant, (7.111) is satisfied only if

$$q = q(\alpha)$$

(7.112)

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Employing (7.109), (7.110) and (7.112) in (7.12) and separating variables, we get

$$\frac{\alpha\left[q(\alpha) - \alpha q'(\alpha)\right]}{q'(\alpha)} = \frac{\Gamma''(\psi)}{a^2 \cos\Gamma(\psi)\sin\Gamma(\psi)\Gamma'^4(\psi)} = \text{constant} = C \neq 0$$

since $\frac{\partial(\alpha,\psi)}{\partial(\xi,\psi)} = \frac{\partial\alpha}{\partial\xi} \neq 0$. This equation gives two equations in $q(\alpha)$ and $\Gamma(\psi)$ and solving these two equations, we have

$$q(\alpha) = E_1\sqrt{\alpha^2 + C} \tag{7.113}$$

and

$$\Gamma'(\psi) = \frac{1}{\sqrt{a^2 C\cos^2\Gamma(\psi) + 2E_2}} \tag{7.114}$$

where $E_1 \neq 0$ and E_2 are arbitrary constants. Using (7.107) and (7.114) in (7.109), we get

$$\alpha(x,y) = \frac{\sqrt{a^2 C\cos^2\eta(x,y) + 2E_2}}{a\sqrt{\cosh^2\xi(x,y) - \cos^2\eta(x,y)}} \tag{7.115}$$

Employing (7.115) in (7.113), we obtain

$$q(x,y) = \frac{E_1\sqrt{a^2 C\cosh^2\xi(x,y) + 2E_2}}{a\sqrt{\cosh^2\xi(x,y) - \cos^2\eta(x,y)}} \tag{7.116}$$

The density function ρ from (7.11) and (7.116) is given by

$$\rho(x,y) = \frac{\sqrt{a^2 C\cos^2\eta(x,y) + 2E_2}}{E_1\sqrt{a^2 C\cosh^2\xi(x,y) + 2E_2}} \tag{7.117}$$

Using (7.107), (7.114) and (7.117) in (7.29) yields

$$u(x,y) = E_1\frac{\partial\eta}{\partial y}\sqrt{a^2 C\cosh^2\xi(x,y) + 2E_2} \tag{7.118}$$

and

$$v(x,y) = -E_1\frac{\partial\eta}{\partial x}\sqrt{a^2 C\cosh^2\xi(x,y) + 2E_2} \tag{7.119}$$

Employing (7.113) and (7.115) in (7.9) and (7.10) and integrating, we get

$$p(x,y) = p_0 + \frac{E_1 C}{2}\ln\left[\frac{E_1\sqrt{a^2 C\cos^2\eta + 2E_2} + E_1\sqrt{a^2 C\cosh^2\xi + 2E_2}}{a\sqrt{\cosh^2\xi - \cos^2\eta}}\right] -$$

$$\frac{1}{2} \frac{E_1^2 \sqrt{a^4 C^2 cos^2 \eta cosh^2 \xi + 2E_2 a^2 C cos^2 \eta + 2E_2 a^2 C cosh^2 \xi + 4E_2^2}}{a^2 \left[cosh^2 \xi - cos^2 \eta \right]} \tag{7.120}$$

The state equation for this flow is given by

$$p = p_0 + \frac{E_1 C}{2} ln \left[\frac{E_1 \sqrt{C}(1 + E_1 \rho)}{\sqrt{1 - E_1^2 \rho^2}} \right] - \frac{1}{2} \frac{E_1^3 C \rho}{(1 - E_1^2 \rho^2)} \tag{7.121}$$

The potential function and the streamfunction for this flow are

$$\Phi(x, y) = E_1 \int \sqrt{a^2 C cosh^2 \xi(x, y) + 2E_2} \left(\frac{\partial \eta}{\partial y} dx - \frac{\partial \eta}{\partial x} dy \right)$$

$$\psi(x, y) = \int \sqrt{a^2 C cos^2 \eta(x, y) + 2E_2} d\eta - \psi_6$$

respectively where ψ_6 is an arbitrary constant.
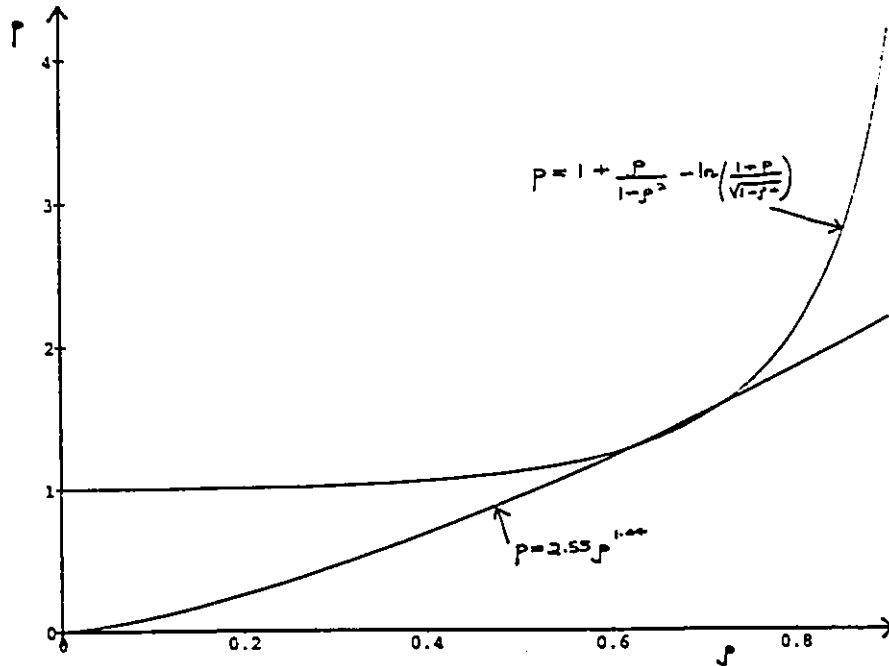
Figure 7.9.   Pressure versus density for the state equation (7.121)
with $p_0 = 1$, $C = -2$, $E = 1$.

The state equation for this flow is similar to (7.61) and so the above solutions for

hyperbolic flow are valid for some real gas possessing (7.121) as its state equation

with $C < 0$. Figure 7.9 shows pressure against density for state equation (7.121)

with $E_1 = 1$, $C = -2$ and $p_0 = 1$ and an ideal gas $p = 2.5 \rho^{1.44}$.

Example 8 : <u>Elliptic flow</u>

Equations

$$x = x(\xi, \eta) = a sinh\xi sin\eta. \qquad -\infty < \xi < \infty$$

$$y = y(\xi, \eta) = a cosh\xi cos\eta. \qquad 0 \leq \eta \leq \infty$$

(7.122)

where $a \neq 0$ is an arbitrary constant defined by

$$z = iacosh(\bar{w})$$

define an orthogonal curvilinear coordinate system (ξ, η) in the (x, y)-plane Equation (7.122) can be solved for ξ and η yielding

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

(7.123)

In this example, we investigate flows when the streamline pattern for this flow is given by the curves $\eta(x, y) =$ constant so that

$$\eta(x, y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0$$

(7.124)

For this investigation, we employ the (ξ, ψ)-net and we have (7.7) where

$$h(\xi, \psi) = a\sqrt{cosh^2\Gamma(\psi) - cos^2\xi}$$

(7.125)

Using (7.125) in the Gauss equation (7.13), we find that this equation is satisfied.

Using (7.125) in (7.11), we get

$$\alpha(\xi, \psi) = \frac{1}{a\Gamma'(\psi)\sqrt{cosh^2\Gamma(\psi) - cos^2\xi}}$$

(7.126)

Employing (7.126) in (7.15) yields

$$q = q(\alpha)$$

(7.127)

since $\frac{\partial \alpha}{\partial \xi} \neq 0$, $\frac{\partial \alpha}{\partial \psi} \neq 0$ and we do not consider uniform flow. Using (7.126) and (7.127) in (7.12) and simplifying, we obtain

$$\frac{\alpha[q(\alpha) - \alpha q'(\alpha)]}{q'(\alpha)} = \frac{\Gamma''(\psi)}{a^2 cosh\Gamma(\psi)sinh\Gamma(\psi)\Gamma'^4(\psi)} = \text{constant} = C \neq 0$$

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This equation yields two equations in $q(\alpha)$ and $\Gamma(\psi)$. Integrating these equations once, we have

$$q(\alpha) = F_1 \sqrt{\alpha^2 + C} \tag{7.128}$$

and

$$\Gamma'(\psi) = \frac{1}{\sqrt{2F_2 - a^2 C \cosh^2 \Gamma(\psi)}} \tag{7.129}$$

Using (7.124) and (7.129) in (7.126), we get

$$\alpha(x,y) = \frac{\sqrt{2F_2 - a^2 C \cosh^2 \eta(x,y)}}{a\sqrt{\cosh^2 \eta - \cos^2 \xi}} \tag{7.130}$$

Employing (7.130) in (7.128), we obtain

$$q(x,y) = \frac{F_1 \sqrt{2F_2 - a^2 C \cos^2 \xi(x,y)}}{a\sqrt{\cosh^2 \eta(x,y) - \cos^2 \xi(x,y)}} \tag{7.131}$$

The density function ρ from (7.11) and (7.131) is given by

$$\rho(x,y) = \frac{\sqrt{2F_2 - a^2 C \cosh^2 \eta(x,y)}}{F_1 \sqrt{2F_2 - a^2 C \cos^2 \xi(x,y)}} \tag{7.132}$$

Using (7.124), (7.129) and (7.132) in (7.29) gives

$$u(x,y) = F_1 \frac{\partial \eta}{\partial y} \sqrt{2F_2 - a^2 C \cos^2 \xi(x,y)} \tag{7.133}$$

and

$$v(x,y) = -F_1 \frac{\partial \eta}{\partial x} \sqrt{2F_2 - a^2 C \cos^2 \xi(x,y)} \tag{7.134}$$

Employing (7.128) and (7.130) in (7.9) and (7.10) and integrating, we get

$$p(x,y) = p_0 + \frac{F_1 C}{2} \ln \left[\frac{F_1 \sqrt{2F_2 - a^2 C \cos^2 \xi} + F_1 \sqrt{2F_2 - a^2 C \cosh^2 \eta}}{a\sqrt{\cosh^2 \eta - \cos^2 \xi}} \right] - $$
$$\frac{1}{2} \frac{F_1^2 \sqrt{4F_2^2 + a^4 C^2 \cos^2 \xi \cosh^2 \eta - 2F_2 a^2 C \cosh^2 \eta - 2F_2 a^2 C \cos^2 \xi}}{a^2 [\cosh^2 \xi - \cos^2 \eta]} \tag{7.135}$$

The state equation for this flow is given by

$$p = p_0 + \frac{F_1 C}{2} \ln \left[\frac{F_1 \sqrt{C}(1 + F_1 \rho)}{\sqrt{1 - F_1^2 \rho^2}} \right] - \frac{1}{2} \frac{F_1^3 C \rho}{(1 - F_1^2 \rho^2)} \tag{7.136}$$

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The potential function and the streamfunction for this flow are respectively given by

$$\Phi(x,y) = F_1 \int \sqrt{2F_2 - a^2 C \cos^2 \xi(x,y)} \left(\frac{\partial \eta}{\partial y} dx - \frac{\partial \eta}{\partial x} dy \right)$$

$$\psi(x,y) = \int \sqrt{2F_2 - a^2 C \cosh^2 \eta(x,y)} d\eta - \psi_6$$

where ψ_6 is an arbitrary constant.

The state equation for this flow (7.136) is identical to (7.121) and therefore the same conclusion applies.

Example 9: <u>Radial flow</u>

The orthogonal curvilinear coordinate system defined by the analytic function

$$w = \ln z$$

is

$$x = x(\xi,\eta) = e^\xi \cos\eta, \qquad 0 < \xi < \infty$$

$$y = y(\xi,\eta) = e^\xi \sin\eta, \qquad 0 \leq \eta < 2\pi$$

(7.137)

Solving for ξ and η from (7.137) yields

$$\xi(x,y) = \text{Re}(w) = \frac{1}{2}\ln(x^2 + y^2)$$

$$\eta(x,y) = \text{Im}(w) = \tan^{-1}\left(\frac{y}{x}\right)$$

(7.138)

In this example, we investigate flows when the streamlines are the family of curves $\eta(x,y) =$ constant. Therefore, we have

$$\eta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0$$

(7.139)

The streamline pattern for this flow is shown in Figure 7.10.

For this investigation, we employ the (ξ,ψ)−net and the squared differential element of arc length for this net, using (7.139), is given by (7.7) where

$$h(\xi,\psi) = e^\xi$$

(7.140)
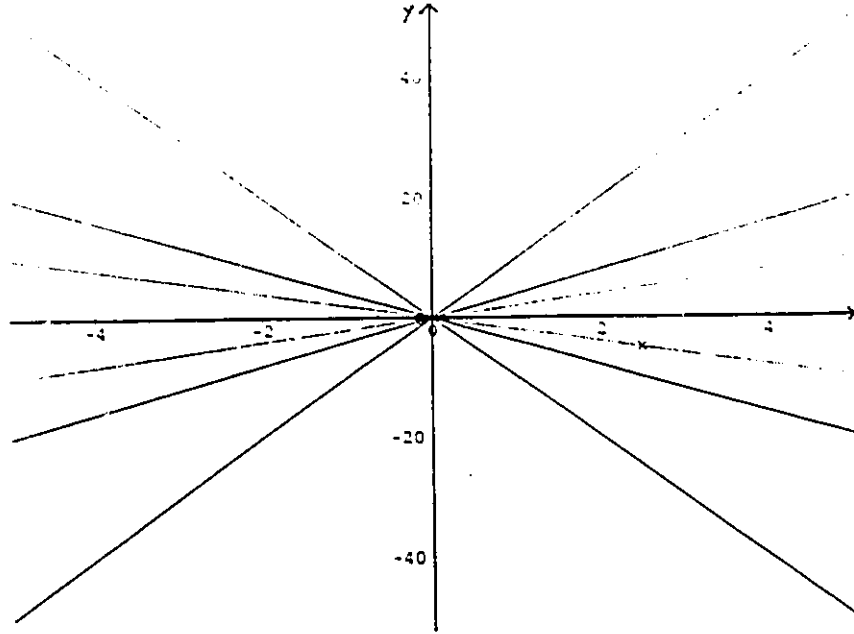
Figure 7.10. Streamline pattern $\tan\left(\frac{y}{x}\right) = $ constant.

Using (7.140) in (7.13), we find that (7.13) is identically satisfied.

Employing (7.140) in (7.11), we get

$$\alpha = \frac{1}{e^{\xi}\Gamma'(\psi)} \tag{7.141}$$

Using (7.141) in (7.15) and simplifying, we obtain

$$\frac{1}{e^{\xi}\Gamma'(\psi)}\frac{\partial q}{\partial \psi} - \frac{\Gamma''(\psi)}{e^{\xi}\Gamma'^{2}(\psi)}\frac{\partial q}{\partial \xi} = 0 \tag{7.142}$$

This equation is satisfied if one of the following holds true:

(i) $\Gamma''(\psi) = 0$ and $\frac{\partial q}{\partial \psi} = 0$

(ii) $q = q(\alpha)$, $q'(\alpha) \neq 0$

In the first possibility, we have

$$\Gamma'(\psi) = A \qquad \text{and} \qquad q = q(\xi) \tag{7.143}$$

where $A \neq 0$ is an arbitrary constant. Using (7.140) and $q = q(\xi)$ in (7.12), we find that (7.12) is identically satisfied.

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In the second possibility, employing (7.140), (7.141) and $q = q(\alpha)$ in (7.12) gives

$$\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} q'(\alpha) = 0$$

Since $q'(\alpha) \neq 0$, this equation implies that $\Gamma''(\psi) = 0$ and so $\Gamma'(\psi) = A \neq 0$, $\alpha = \alpha(\xi) = \frac{1}{A} e^{-\xi}$ and, therefore, $q = q[\alpha(\xi)] = q(\xi)$. Possibilities (i) and (ii) yield the same solution and we need to consider only one of these two possibilities.

Using the first equations of both (7.143) and (7.138) in (7.141), we obtain

$$\alpha(x, y) = \frac{1}{A\sqrt{x^2 + y^2}} \tag{7.144}$$

Since $q = q(\alpha)$, we have

$$q(x, y) = q\left(\frac{1}{A\sqrt{x^2 + y^2}}\right) \tag{7.145}$$

where q is an arbitrary function of its argument.

Using (7.144) and (7.145) in (7.11), the density function ρ is given by

$$\rho(x, y) = \frac{1}{Aq(x, y)\sqrt{x^2 + y^2}} \tag{7.146}$$

Employing (7.138), (7.146) and $\Gamma'(\psi) = A$ in (7.29), we get

$$u(x, y) = \frac{xq}{\sqrt{x^2 + y^2}}, \qquad v(x, y) = \frac{yq}{\sqrt{x^2 + y^2}} \tag{7.147}$$

Using (7.141), (7.143) and (7.145) in (7.9) and (7.10), the pressure function is given by

$$p(x, y) = \int q(\alpha) d\alpha - \alpha q(\alpha) \tag{7.148}$$

where $\alpha(x, y)$ is given above.

Letting $q(\alpha) = \alpha^m$ where $m \neq 0, \pm 1$ in (7.11) and (7.148), we obtained

$$\alpha = \rho^{\frac{1}{1-m}} \qquad \text{and} \qquad p(x, y) = p_o - \frac{m}{m+1} q^{\frac{1+m}{m}}(\alpha)$$

The state equation for this flow is

$$p = p_0 - \frac{m}{m+1}\rho^{\frac{1 \pm m}{1-m}}$$

The equi potential curves for this flow are concentric circles given by $\xi(x,y) =$ constant. Employing (7.143) in (7.139), the streamfunction for this flow is

$$\psi(x,y) = \frac{1}{A}\tan^{-1}\left(\frac{y}{x}\right) - \frac{A_1}{A} \tag{7.149}$$

where A_1 is an arbitrary constant.

Example 10: <u>Circular flow</u>

Equations

$$x = x(\xi,\eta) = e^{\eta}\cos\xi, \qquad 0 < \eta < \infty$$
$$y = y(\xi,\eta) = e^{\eta}\sin\xi, \qquad 0 \le \xi < 2\pi \tag{7.150}$$

define an orthogonal curvilinear coordinate net in the (x,y)-plane. This coordinate system is given by the analytic function

$$w = i\ln\bar{z}$$

so that

$$\xi(x,y) = \text{Re}(w) = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\eta(x,y) = \text{Im}(w) = \frac{1}{2}\ln(x^2 + y^2) \tag{7.151}$$

The streamlines representing this flow are the family of curves $\eta(x,y) =$ constant and therefore, we have

$$\eta(x,y) = \frac{1}{2}\ln(x^2 + y^2) = \Gamma(\psi); \qquad \Gamma'(\psi) \ne 0 \tag{7.152}$$

The streamline pattern for this flow is shown in Figure 7.11.
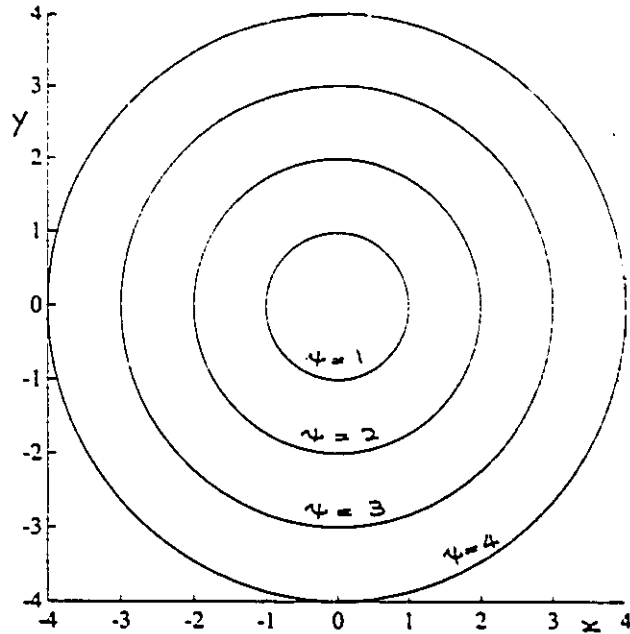
Figure 7.11.   Streamline pattern $x^2 + y^2$ =constant.

We employ the (ξ, ψ)—net to study this flow and the squared element of arc length for this net, using (7.152), is given by (7.7) where we have

$$h(\xi, \psi) = e^{\Gamma(\psi)} \qquad (7.153)$$

Equation (7.153) identically satisfies the Gauss equation (7.13).

Using (7.153) in (7.11), we obtain

$$\alpha(\xi, \psi) = \frac{1}{e^{\Gamma(\psi)}\Gamma'(\psi)} \qquad (7.154)$$

Using (7.154) in (7.15), we get

$$\left(\frac{\Gamma'^2(\psi) + \Gamma''(\psi)}{e^{\Gamma(\psi)}\Gamma'^2(\psi)} \right) \frac{\partial q}{\partial \xi} = 0 \qquad (7.155)$$

Equation (7.155) is satisfied if one of the following holds true:

(i) $\frac{\partial q}{\partial \xi} = 0$

(ii) $\Gamma'^2(\psi) + \Gamma''(\psi) = 0$

or

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(iii) $\quad \frac{\partial q}{\partial \xi} = 0 \quad$ and $\quad \Gamma'^2(\psi) + \Gamma''(\psi) = 0$

Possibilities (i) and (ii) are considered in the cases $\frac{\partial \alpha}{\partial \xi} = 0$, $\frac{\partial q}{\partial \xi} = 0$[Case (2)] and $\frac{\partial \alpha}{\partial \xi} = 0$, $\frac{\partial \alpha}{\partial \psi} = 0$[Case (1)] respectively below. The solutions presented in these cases hold true for any orthogonal curvilinear coordinate system satisfying the restrictions on $h(\xi, \psi)$. Using (7.152) and (7.153) in the solutions obtained for Case (1) and Case (2), we have the solutions for possibilities (ii) and (i) respectively. Therefore, we need to consider possibility (iii) only. This possibility yields

$$q = q(\psi) \tag{7.156}$$

and

$$\Gamma'(\psi) = B e^{-\Gamma(\psi)} \tag{7.157}$$

where $B \neq 0$ is an arbitrary constant.

Employing (7.156) and (7.157) in (7.12), we get

$$q(\psi) = \frac{D}{e^{\Gamma(\psi)}} \tag{7.158}$$

where $D \neq 0$ is an arbitrary constant.

Using (7.152) in (7.158) yields

$$q(x, y) = \frac{D}{\sqrt{x^2 + y^2}} \tag{7.159}$$

Using (7.152), (7.154), (7.157) and (7.159) in (7.11), we get

$$\rho(x, y) = \frac{1}{BD} \sqrt{x^2 + y^2} \tag{7.160}$$

Employing (7.152), (7.157) and (7.160) in (7.29), we obtain

$$u(x, y) = \frac{Dy}{x^2 + y^2}, \qquad v(x, y) = -\frac{Dx}{x^2 + y^2} \tag{7.161}$$

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Integrating (7.9) and (7.10) using (7.159) and (7.160), the pressure function is given by

$$p(x,y) = p_0 - \frac{D}{B} \frac{1}{\sqrt{x^2 + y^2}} \qquad (7.162)$$

The state equation for this flow takes the form

$$p = p_0 - \frac{1}{B^2 \rho} \qquad (7.163)$$

The potential function of this flow is given by $\xi(x,y)$ and the streamfunction is

$$\psi(x,y) = \frac{B_1}{B} \div \frac{1}{B} \sqrt{x^2 + y^2} \qquad (7.164)$$

where B_1 is an arbitrary constant.

In the following section, we consider flows not satisfying $q = q(\alpha)$ but identically satisfying equation (7.15).

7.4 FLOWS SATISFYING $q \neq q(\alpha)$.

Equation (7.15) may be written as

$$\frac{\partial \alpha}{\partial \xi} \frac{\partial q}{\partial \psi} - \frac{\partial \alpha}{\partial \psi} \frac{\partial q}{\partial \xi} = 0 \qquad (7.165)$$

where $\alpha = \rho q$ is given by (7.11) and

$$\frac{\partial \alpha}{\partial \xi} = -\frac{1}{h^2(\xi,\psi)\Gamma'(\psi)} \frac{\partial h}{\partial \xi}, \qquad \frac{\partial \alpha}{\partial \psi} = -\frac{1}{h^2(\xi,\psi)\Gamma'(\psi)} \frac{\partial h}{\partial \psi} - \frac{\Gamma''(\psi)}{h(\xi,\psi)\Gamma'^2(\psi)} \qquad (7.166)$$

Equation (7.165) is identically satisfied if $q = q(\alpha)$ as in Examples 1 to 9 above. However, this equation is also satisfied if one of the following holds true:

(1) $\quad \frac{\partial \alpha}{\partial \xi} = 0 \quad$ and $\quad \frac{\partial \alpha}{\partial \psi} = 0$

(2) $\quad \frac{\partial \alpha}{\partial \xi} = 0 \quad$ and $\quad \frac{\partial q}{\partial \xi} = 0$

(3) $\quad \frac{\partial \alpha}{\partial \psi} = 0 \quad$ and $\quad \frac{\partial q}{\partial \psi} = 0$

(4) $\quad \frac{\partial q}{\partial \xi} = 0 \quad$ and $\quad \frac{\partial q}{\partial \psi} = 0$

For completeness, we consider Cases (1) to (3) and obtain the solutions of the system of equations (7.9) to (7.14) for each of these cases. Case (4) yields uniform flow and is, therefore, removed from further consideration.

Case (1): In this case we have

$$\frac{\partial \alpha}{\partial \xi} = 0 \tag{7.167}$$

and

$$\frac{\partial \alpha}{\partial \psi} = 0 \tag{7.168}$$

Employing (7.167) and (7.168) in (7.166), we get

$$\frac{\partial h}{\partial \xi} = 0 \tag{7.169}$$

and

$$\Gamma'(\psi)\frac{\partial h}{\partial \psi} + h(\xi, \psi)\Gamma''(\psi) = 0 \tag{7.170}$$

Equation (7.169) implies that $h = h(\psi)$ and (7.170) yields

$$\Gamma'(\psi) = \frac{a_0}{h(\psi)} \tag{7.171}$$

where $a_0 \neq 0$ is an arbitrary constant. Using (7.171) in (7.11), we have

$$\alpha = \frac{1}{a_0} \tag{7.172}$$

Equation (7.12) gives

$$q(\xi, \psi) = \frac{a_1(\xi)}{h(\psi)} \tag{7.173}$$

where $a_1(\xi)$ is an arbitrary function of ξ.

Employing (7.172) and (7.173) in (7.11), we get

$$\rho(\xi, \psi) = \frac{1}{a_0}\frac{h(\psi)}{a_1(\xi)} \tag{7.174}$$

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Differentiating (7.6) with respect to x and y respectively and using (7.171), (7.174) and (2.27), we obtain

$$u(x,y) = \frac{\partial \eta}{\partial y} a_1\left(\xi(x,y)\right), \qquad v(x,y) = -\frac{\partial \eta}{\partial x} a_1\left(\xi(x,y)\right) \qquad (7.175)$$

Employing (7.172) and (7.173) in (7.9) and (7.10) and integrating, we get

$$p(x,y) = p_0 - \frac{1}{a_0} \frac{a_1\left(\xi(x,y)\right)}{h\left(\psi(x,y)\right)} \qquad (7.176)$$

where p_0 is an arbitrary constant.

The state equation for this flow from (7.174) and (7.176) is

$$p = p_0 - \frac{1}{a_0^2 \rho} \qquad (7.177)$$

Case (2): In this case, we have

$$\frac{\partial \alpha}{\partial \xi} = 0 \qquad (7.178)$$

and

$$\frac{\partial q}{\partial \xi} = 0 \qquad (7.179)$$

Equations (7.178) and (7.179) give

$$h = h(\psi) \qquad \text{and} \qquad q = q(\psi) \qquad (7.180)$$

Using (7.180) in (7.11), we get

$$\alpha(\psi) = \frac{1}{h(\psi)\Gamma'(\psi)} \qquad (7.181)$$

where $\Gamma'(\psi)$ is an arbitrary function of its argument.

Employing (7.180) in (7.12), we have

$$q(\psi) = \frac{a_2}{h(\psi)} \qquad (7.182)$$

where $a_2 \neq 0$ is an arbitrary constant. Using (7.181) and (7.182) in (7.11), we obtain

$$\rho(\psi) = \frac{1}{a_2 \Gamma'(\psi)} \qquad (7.183)$$

Employing (7.181) and (7.182) in (7.9) and (7.10) and integrating, we get the pressure function given by

$$p(\psi) = p_0 - a_2 \int \frac{h'(\psi)}{h^3(\psi)\Gamma'(\psi)} d\psi \qquad (7.184)$$

Differentiating (7.6) with respect to x and y respectively and using (7.183) and (2.27), we get

$$u(x,y) = a_2 \frac{\partial \eta}{\partial y}, \qquad v(x,y) = -a_2 \frac{\partial \eta}{\partial x} \qquad (7.185)$$

The (p, ρ)-relation for this flow can be obtained from (7.183) and (7.184) for a given $h(\psi)$ and $\Gamma'(\psi)$.

Case (3): In this case, we have

$$\frac{\partial \alpha}{\partial \psi} = 0 \qquad (7.186)$$

and

$$\frac{\partial q}{\partial \psi} = 0 \qquad (7.187)$$

Equation (7.187) implies that $q = q(\xi)$ and using (7.186) in the second equation of (7.166) yields

$$\Gamma'(\psi) = \frac{a_3(\xi)}{h(\xi, \psi)} \qquad (7.188)$$

where $a_3(\xi)$ is an arbitrary function of its argument. Employing (7.187) in (7.12), we get

$$\frac{\partial h}{\partial \psi} = 0$$

This equation implies that $h = h(\xi)$ and therefore, from (7.188), we have

$$\Gamma'(\psi) = \frac{a_3(\xi)}{h(\xi)} = constant = a_4 \neq 0(say) \qquad (7.189)$$

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Using (7.189) in (7.11). we get

$$\alpha(\xi) = \frac{1}{a_4 h(\xi)} \tag{7.190}$$

Employing (7.190) in (7.11). we obtain

$$\rho(\xi) = \frac{1}{a_4 h(\xi) q(\xi)} \tag{7.191}$$

where $q(\xi)$ is an arbitrary function of its argument.

Using (7.187) and (7.191) in (7.9) and (7.10). the pressure is given by

$$p(\xi) = p_0 - \frac{1}{a_4} \int \frac{1}{h(\xi)} \frac{\partial q}{\partial \xi} d\xi \tag{7.192}$$

where p_0 is an arbitrary constant.

Differentiating (7.6) with respect to x and y respectively and using (7.192) and (2.27), we obtain

$$u(x,y) = \frac{1}{a_4^2 h(\xi) q(\xi)} \frac{\partial \eta}{\partial y}, \qquad v(x,y) = -\frac{1}{a_4^2 h(\xi) q(\xi)} \frac{\partial \eta}{\partial x} \tag{7.193}$$

The state equation for this flow can be obtained from (7.192) and (7.193) for a given $h(\xi)$ and $q(\xi)$.

CHAPTER 8

STREAMLINE
PATTERN $C\xi(x,y) + D\eta(x,y) =$ CONSTANT

8.1 INTRODUCTION.

This chapter deals with a class of flows when the streamline pattern is of the
form

$$C\xi(x,y) + D\eta(x,y) = \text{constant}$$

where $\xi(x,y) = \text{Re}[f(z)]$, $\eta(x,y) = \text{Im}[f(z)]$, $f(z)$ is an analytic function of z and
$C \neq 0$ and $D \neq 0$ are arbitrary constants. Having considered whether the family
of curves $\text{Re}[f(z)] = \text{constant}$ or $\text{Im}[f(z)] = \text{constant}$ is a permissible streamline
pattern for steady, plane, potential compressible flow, we now investigate whether a
linear combination of these two curves will allow inviscid compressible fluid to flow
along it.

We adopt the same approach as in the previous chapter, employing the (ξ, ψ)-
coordinate system to determine some possible flows and obtain the exact solutions
of these flows.

8.2. EQUATIONS OF MOTION.

Curvilinear coordinate systems in the (x,y)–plane or z–plane are transforma-
tions of the rectangular map in the (ξ, η)–plane or w–plane.

Taking an arbitrary function

$$z = f(w) \tag{8.1}$$

where $z = x + iy$ and $w = \xi + i\eta$, the Cauchy-Riemann equations

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta}, \qquad \frac{\partial x}{\partial \eta} = -\frac{\partial y}{\partial \xi} \tag{8.2}$$

apply and the angles are preserved by the transformation. Separation of equation (8.1) into real and imaginary parts gives

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \tag{8.3}$$

Plotting the two families of curves $\xi =$ constant and $\eta =$ constant, we have an orthogonal curvilinear map in the $z-$plane for every choice of function f in (8.1).

The squared infinitesimal differential element of arc length along any curve in $z-$plane is

$$ds^2 = dx^2 + dy^2 = \left(\frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta \right)^2 + \left(\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \right)^2 = \overline{h}^2(\xi, \eta) \left[d\xi^2 + d\eta^2 \right] \tag{8.4}$$

where

$$\overline{h}^2(\xi, \eta) = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2, \quad \overline{h}(\xi, \eta) > 0 \tag{8.5}$$

where equations (8.2) are employed.

A practical application dictates the proper choice of the function $f(w)$ in (8.1) and the resulting coordinate system.

Since

$$J = \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 = \overline{h}^2(\xi, \eta) \neq 0, \tag{8.6}$$

it follows that (8.1) and (8.3) can be solved to get

$$\xi + i\eta = w = f^{-1}(z) = g(z) \text{ (say)}$$

so that

$$\xi = \xi(x,y), \qquad \eta = \eta(x,y)$$

These functions are harmonic functions and are harmonic conjugates of each other. Two families of curves defined by $\xi(x,y) = Re[g(z)] = $ constant and $\eta(x,y) = Im[g(z)] = $ constant are the streamlines and their orthogonal trajectories forming an isometric curvilinear coordinate net of a steady plane incompressible irrotational motion when $g(z)$ is an analytic function of z.

We investigate steady plane compressible irrotational isentropic flows when streamline patterns are of the form

$$C\xi(x,y) + D\eta(x,y) = \text{constant}$$

where C and $D \neq 0$ are arbitrary constants. Since $\psi(x,y) =$ constant also defines the streamline pattern when function $\psi(x,y)$ is the streamfunction for a flow, it follows that

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \quad \Gamma'(\psi) \neq 0 \tag{8.7}$$

Since

$$\left| \frac{\partial(\xi,\eta)}{\partial(x,y)} \right| = \left\{ \frac{\partial(x,y)}{\partial(\xi,\eta)} \right\}^{-1} = \left| \frac{\partial(\xi,\eta)}{\partial(\xi,\psi)} \frac{\partial(\xi,\psi)}{\partial(x,y)} \right|,$$

we use (8.6), (8.7) and have

$$\frac{1}{\overline{h}^2(\xi,\eta)} = \frac{1}{D}\Gamma'(\psi)\left| \frac{\partial(\xi,\psi)}{\partial(x,y)} \right|, \quad \left| \frac{\partial(x,y)}{\partial(\xi,\psi)} \right| = \frac{\Gamma'(\psi)}{D}h^2(\xi,\psi) \tag{8.8}$$

where $\xi(x,y) =$ constant, $C\xi(x,y) + D\eta(x,y) =$ constant generate a coordinate net in (x,y)–plane and $h^2(\xi,\psi) = \overline{h}^2\left(\xi, \frac{\Gamma(\psi)}{D} - \frac{C}{D}\xi\right)$. Using (8.7) in (8.4), the squared element of infinitesimal differential element of arc lenth along any curve in (ξ,ψ)–coordinates is

$$ds^2 = \frac{h^2(\xi,\psi)}{D^2}\left[(C^2 + D^2)\,d\xi^2 - 2C\Gamma'(\psi)\,d\xi\,d\psi + \Gamma'^2(\psi)\,d\psi^2 \right] \tag{8.9}$$

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By the choice of our coordinates (ξ, ψ), the family of curves $\phi(x, y) = $ constant in theorem 2.2 are the curves $\xi(x, y) = $ constant and we have

$$E(\xi, \psi) = \left(1 + \frac{C^2}{D^2}\right) h^2(\xi, \psi), \qquad F(\xi, \psi) = -\frac{C}{D^2}\Gamma'(\psi)h^2(\xi, \psi),$$

$$G(\xi, \psi) = \frac{\Gamma'^2(\psi)h^2(\xi, \psi)}{D^2}, \qquad J(\xi, \psi) = \frac{\partial(x, y)}{\partial(\xi, \psi)} = \frac{\Gamma'(\psi)}{D}h^2(\xi, \psi) \qquad (8.10)$$

and

$$W(\xi, \psi) = \sqrt{EG - F^2} = \frac{\Gamma'(\psi)h^2(\xi, \psi)}{D}$$

Taking $D > 0$ and $\Gamma'(\psi) > 0$, we have $J = W > 0$ and so fluid flows along a streamline in the direction of increasing ξ.

Using equations (8.10) in theorem 2.2 and taking (ξ, ψ) coordinates, the flow is governed by

$$\rho q \frac{\partial q}{\partial \xi} + \frac{\partial p}{\partial \xi} = 0 \qquad (8.11)$$

$$\rho q \frac{\partial q}{\partial \psi} + \frac{\partial p}{\partial \psi} = 0 \qquad (8.12)$$

$$\rho q = \frac{\sqrt{C^2 + D^2}}{h(\xi, \psi)\Gamma'(\psi)} \qquad (8.13)$$

$$\frac{\partial}{\partial \xi}\left[\frac{C\Gamma'(\psi)}{\sqrt{C^2 + D^2}}h(\xi, \psi)q(\xi, \psi)\right] + \frac{\partial}{\partial \psi}\left[\sqrt{C^2 + D^2}\, h(\xi, \psi)\, q(\xi, \psi)\right] = 0 \qquad (8.14)$$

$$\frac{\partial}{\partial \xi}\left[\frac{C}{h(\xi, \psi)}\frac{\partial h}{\partial \psi} + \frac{\Gamma'(\psi)}{h(\xi, \psi)}\frac{\partial h}{\partial \xi}\right] + \frac{\partial}{\partial \psi}\left[\frac{C}{h(\xi, \psi)}\frac{\partial h}{\partial \xi} + \frac{C^2 + D^2}{h(\xi, \psi)\Gamma'(\psi)}\frac{\partial h}{\partial \psi}\right] = 0 \qquad (8.15)$$

$$p = R(\rho) \qquad (2.5)$$

of six equations in five unknown functions $\rho(\xi, \psi)$, $p(\xi, \psi)$, $q(\xi, \psi)$, $h(\xi, \psi)$ and $\Gamma(\psi)$.

Using (8.13) in (8.11) and (8.12) to eliminate ρ and employing the integrability condition $\frac{\partial^2 p}{\partial \xi \partial \psi} = \frac{\partial^2 p}{\partial \psi \partial \xi}$, we obtain

$$\frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} = 0 \qquad (8.16)$$

where

$$\alpha(\xi, \psi) = \frac{\sqrt{C^2 + D^2}}{h(\xi, \psi)\Gamma'(\psi)} \qquad (8.17)$$

For a chosen curvilinear coordinate net (ξ, η) define by an analytic function $w = \xi + i\eta = f(z)$ and for the chosen streamline pattern $C\xi(x,y) + D\eta(x,y) = $ constant so that $C\xi(x,y) + D\eta(x,y) = \Gamma(\psi)$, function $h(\xi, \psi)$ is known and the Gauss equation (8.15) is identically satisfied. The system of equations (8.11) to (8.15) reduces to a system of two equations (8.14) and (8.16) in two unknowns $q(\xi, \psi)$ and $\Gamma(\psi)$.

In the following section, we study examples for which $q = q(\alpha)$ such that $\frac{\partial \alpha}{\partial \xi} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$.

8.3 FLOWS SATISFYING $q = q(\alpha)$, $q'(\alpha) \neq 0$.

Example 1: Spiral Flows
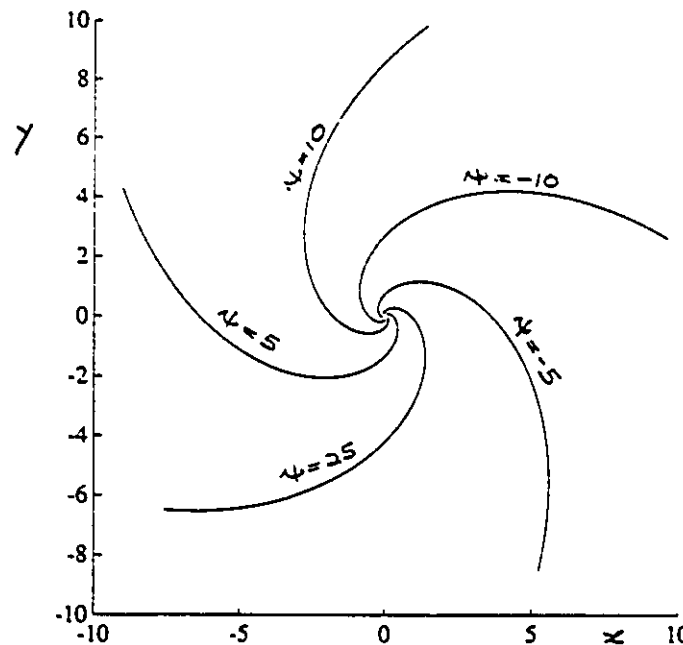
Figure 8.1.   Streamline pattern $\frac{1}{2}C\ln\left(x^2 + y^2\right) + D\arctan\left(\frac{y}{x}\right) = $constant.
$(C = D = 1)$.

We know that $w = \xi(x,y) + i\eta(x,y) = \ln z$ is an analytic function of z and the two families of curves $\xi(x,y) = $constant, $\eta(x,y) = $constant generate an orthogonal curvilinear coordinate net. This net and its squared element of arc length are given

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by

$$x = e^{\xi}\cos\eta, \qquad y = e^{\xi}\sin\eta \qquad (8.18)$$

and

$$ds^2 = e^{2\xi}\left[d\xi^2 + d\eta^2\right]$$

Solving equations (8.18) for ξ and η, we have

$$\xi(x,y) = \frac{1}{2}\ln\left(x^2 + y^2\right), \qquad \eta(x,y) = \tan^{-1}\left(\frac{y}{x}\right) \qquad (8.19)$$

Here, we study a spiral flow when the streamlines are defined by

$$C\xi(x,y) + D\eta(x,y) = \text{constant}$$

so that

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \quad \Gamma'(\psi) \neq 0 \qquad (8.20)$$

where $\Gamma(\psi)$ is some function of ψ and C, $D \neq 0$ are arbitrary constants. The streamline pattern for this flow is shown in Figure 8.1.

To study these flows, we employ the $(\xi,\psi)-$coordinates and the squared differential element of arc length for this net is given by (8.9) where

$$h(\xi,\psi) = e^{\xi} \qquad (8.21)$$

Using (8.21), Gauss equation (8.15) is identically satisfied and (8.14) and (8.16) become a system of two equations

$$C\Gamma'(\psi)\frac{\partial q}{\partial \xi} + \left(C^2 + D^2\right)\frac{\partial q}{\partial \psi} + C\Gamma'(\psi)q = 0 \qquad (8.22)$$

and

$$\frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} = 0 \qquad (8.23)$$

in two unknown functions $\Gamma(\psi)$ and $q(\xi, \psi)$ where

$$\alpha(\xi, \psi) = \frac{\sqrt{C^2 + D^2}}{e^\xi \Gamma'(\psi)} \tag{8.24}$$

Since $\dfrac{\partial \alpha}{\partial \xi} \neq 0$ and our flow is not a uniform flow, it follows that (8.23) is identically satisfied if either (i) $\dfrac{\partial q}{\partial \psi} = 0$, $\Gamma''(\psi) = 0$ or (ii) $q = q(\alpha)$, $q'(\alpha) \neq 0$.

Taking the first case, when $\dfrac{\partial q}{\partial \psi} = \Gamma''(\psi) = 0$, equations (8.22) and (8.13) gives

$$\left.\begin{array}{c} q'(\xi) + q(\xi) = 0 \\[2mm] \rho q = \dfrac{C^2 + D^2}{\Gamma_0 e^\xi} \end{array}\right\} \tag{8.25}$$

where $\Gamma'(\psi) = \Gamma_0 \neq 0$ is some constant. Equations (8.25) hold true only if $\rho =$ constant and, therefore, we cannot have $\dfrac{\partial q}{\partial \psi} = \Gamma''(\psi) = 0$ for our flow.

Taking the second case, we get

$$\left[C + (C^2 + D^2) \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} \right] q'(\alpha) - \frac{C}{\alpha} q(\alpha) = 0$$

Separating the variables, we get

$$\left(\frac{C^2 + D^2}{C} \right) \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = \frac{q(\alpha)}{\alpha q'(\alpha)} - 1 = A \tag{8.26}$$

where $A \neq 0$ is an arbitrary constant. Solving equations (8.26), we get

$$q(\alpha) = A_2 \alpha^{A_1}, \qquad A_1 = \frac{1}{1 + A} \tag{8.27}$$

and

$$\Gamma'(\psi) = A_3 \exp \left[\frac{AC}{C^2 + D^2} \Gamma(\psi) \right] \tag{8.28}$$

where $A_2 \neq 0$ and $A_3 \neq 0$ are arbitrary constants. Using (8.19) and (8.28) in (8.24), we obtain

$$\alpha(x, y) = \frac{\sqrt{C^2 + D^2}}{A_3} \left(\sqrt{x^2 + y^2} \right)^\delta \exp \left[-\frac{ACD}{C^2 + D^2} \tan \left(\frac{y}{x} \right) \right] \tag{8.29}$$

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where $\delta = -\dfrac{AC^2 + C^2 + D^2}{C^2 + D^2}$.

Using (8.27) in (8.13), the density function is given by

$$\rho(x,y) = \frac{1}{A_2}\alpha^{AA_1} \tag{8.30}$$

where $\alpha(x,y)$ is given by (8.29). Differentiating (8.20) with respect to x and y respectively and using (2.27), we have

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)}\left(C\frac{\partial\xi}{\partial y} + D\frac{\partial\eta}{\partial y}\right), \qquad v(x,y) = -\frac{1}{\rho\Gamma'(\psi)}\left(C\frac{\partial\xi}{\partial x} + D\frac{\partial\eta}{\partial x}\right) \tag{8.31}$$

Using (8.19),(8.28) and (8.30) in (8.31), we get

$$u(x,y) = \frac{A_2(Cy + Dx)}{\sqrt{C^2 + D^2}\sqrt{x^2 + y^2}}\alpha^{1-AA_1} \tag{8.32}$$

and

$$v(x,y) = \frac{A_2(Dy - Cx)}{\sqrt{C^2 + D^2}\sqrt{x^2 + y^2}}\alpha^{1-AA_1} \tag{8.33}$$

Integrating (8.11) and (8.12), the pressure function is given by

$$p = -\int \alpha dq \tag{8.34}$$

Employing (8.27) in (8.34), we get

$$p(x,y) = p_0 - \frac{A_2}{(2 + A)}\alpha^{(2+A)A_1} \tag{8.35}$$

where $\alpha(x,y)$ is given by (8.29). The state equation for this flow is

$$p = p_0 - \frac{A_2^{\frac{2+2A}{A}}}{(2 + A)}\rho^{\frac{2+A}{A}}$$

If we consider the spiral flow of a polytropic gas $p = A_0\rho^\gamma$ where A_0 is a known constant and γ is the ratio of specific heats, the above state equation requires

$$A = \frac{2}{\gamma - 1}, \qquad A_2 = -A_0\left(\frac{2\gamma}{\gamma - 1}\right)^{\frac{1}{\gamma+1}} \qquad \text{and} \qquad p_0 = 0$$

The streamfunction and the potential function for this flow are respectively given
by

$$\psi(x,y) = \psi_0 - \frac{C^2 + D^2}{AA_3C} \left(\sqrt{x^2 + y^2}\right)^{\delta_1} \exp\left[-\frac{ACD}{C^2 + D^2} \tan\left(\frac{y}{x}\right)\right]$$

and

$$\Phi(x,y) = \frac{A_2}{\sqrt{C^2 + D^2}} \int \alpha^{1-AA_1} \left[\frac{(Cy + Dx)\mathrm{d}x + (Dy - Cx)\mathrm{d}y}{\sqrt{x^2 + y^2}}\right]$$

where ψ_0 is an arbitrary constant and $\delta_1 = -\frac{AC^2}{C^2 + D^2}$.

Example 2: Doublet

The function $w = \xi(x,y) + i\eta(x,y) = \frac{1}{z}$ is an analytic function of z and the
two families of curves $\xi(x,y) =$constant, $\eta(x,y) =$constant generate an orthogonal
curvilinear coordinate net. This net and its squared element of arc length are given
by

$$x = \frac{\xi}{\xi^2 + \eta^2}, \qquad y = \frac{\eta}{\xi^2 + \eta^2} \tag{8.36}$$

and

$$ds^2 = \frac{1}{(\xi^2 + \eta^2)^2} \left[d\xi^2 + d\eta^2\right] \tag{8.37}$$

Equations (8.36) can be solved for ξ and η and we have

$$\xi(x,y) = \frac{x}{x^2 + y^2}, \qquad \eta(x,y) = \frac{y}{x^2 + y^2} \tag{8.38}$$

We assume that the streamlines are of the form

$$C\xi(x,y) + D\eta(x,y) = \text{constant}$$

so that

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{8.39}$$

where $\Gamma(\psi)$ is some arbitrary function of ψ and $D \neq 0$ is an arbitrary constant.
The flow pattern in this example is given in Figure 8.2.
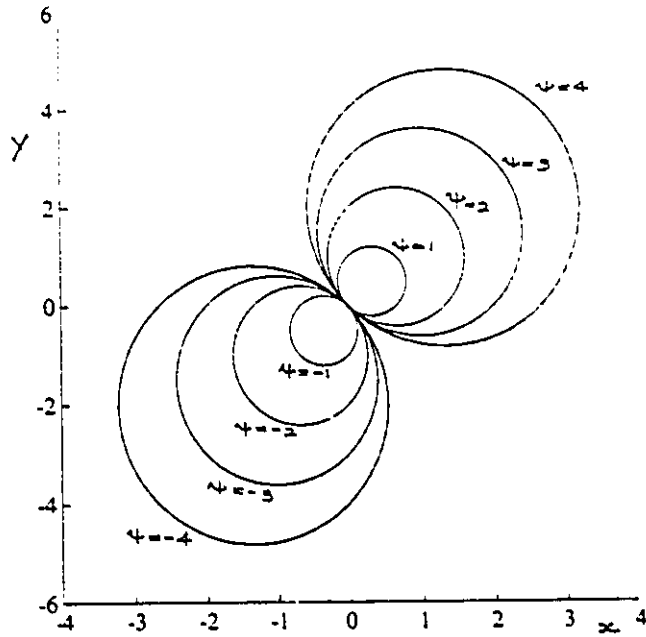
Figure 8.2.   Streamline pattern $\dfrac{Cx + Dy}{x^2 + y^2}$ =constant $(C = D = 1)$.

To study these flows, we use the (ξ, ψ)−net. The squared differential element of arc length for this net is given by (8.9) where

$$h(\xi, \psi) = \frac{D^2}{[(C^2 + D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)]} \tag{8.40}$$

Employing (8.40), Gauss equation (8.15) is identically satisfied and (8.14), (8.16) become a system of two equations

$$C\frac{\partial}{\partial \xi}\left[\frac{q}{\alpha}\right] + (C^2 + D^2)\frac{\partial}{\partial \psi}\left[\frac{q}{\alpha\Gamma'(\psi)}\right] = 0 \tag{8.41}$$

and

$$\frac{\partial \alpha}{\partial \psi}\frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi}\frac{\partial q}{\partial \psi} = 0 \tag{8.42}$$

in two unknown functions $\Gamma(\psi)$ and $q(\xi, \psi)$ where $\alpha(\xi, \psi)$ is given by

$$\alpha(\xi, \psi) = \frac{\sqrt{C^2 + D^2}}{D^2}\frac{[(C^2 + D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)]}{\Gamma'(\psi)} \tag{8.43}$$

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Since we do not consider uniform flow, it follows that equation (8.42) is identically satisfied if one of the following holds true

$$(i) \ \frac{\partial \alpha}{\partial \xi} = \frac{\partial q}{\partial \xi} = 0$$

$$(ii) \ \frac{\partial q}{\partial \psi} = \frac{\partial \alpha}{\partial \psi} = 0$$

$$(iii) \ \frac{\partial \alpha}{\partial \xi} = \frac{\partial \alpha}{\partial \psi} = 0$$

$$(iv) \ q = q(\alpha), \quad q'(\alpha) \neq 0$$

Cases (i) to (iii) yield $D = 0$ which is contrary to our assumption that $D \neq 0$. Taking $q = q(\alpha)$ and using (8.43), equation (8.41) yields

$$2 \left[\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^2} \right] \Gamma(\psi) + \sqrt{C^2 + D^2} q'(\alpha) \frac{\Gamma''(\psi)}{\Gamma'(\psi)} = 0 \qquad (8.44)$$

Separating the variables, we get

$$\frac{2 \left[q(\alpha) - \alpha q'(\alpha) \right]}{\sqrt{C^2 + D^2} \alpha^2 q'(\alpha)} = -\frac{\Gamma''(\psi)}{\Gamma(\psi) \Gamma'(\psi)} = \text{constant} = A_4 \neq 0 \ (\text{say}) \qquad (8.45)$$

Solving equations (8.45), we have

$$q(\alpha) = \frac{A_5 \alpha}{\left[2 + A_4 \sqrt{C^2 + D^2} \alpha \right]} \qquad (8.46)$$

and

$$\Gamma'(\psi) = A_6 - \frac{1}{2} A_4 \Gamma^2(\psi) \qquad (8.47)$$

where $A_5 \neq 0$ and $A_6 \neq 0$ are arbitrary constants. Employing (8.38), (8.39) and (8.47) in (8.43), we obtain

$$\alpha(x, y) = \frac{2\sqrt{C^2 + D^2}(x^2 + y^2)}{2A_6(x^2 + y^2)^2 - A_4(Cx + Dy)^2} \qquad (8.48)$$

Using (8.46) in (8.13), the density function is given by

$$\rho(\alpha) = \frac{1}{A_5} \left[2 + A_4 \sqrt{C^2 + D^2} \alpha \right] \qquad (8.49)$$

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where $\alpha(x, y)$ is given by (8.48).

Differentiating (8.20) with respect to x and y respectively and using (2.27), we have

$$u(x,y) = \frac{1}{\rho \Gamma'(\psi)} \left(C \frac{\partial \xi}{\partial y} + D \frac{\partial \eta}{\partial y} \right), \qquad v(x,y) = -\frac{1}{\rho \Gamma'(\psi)} \left(C \frac{\partial \xi}{\partial x} + D \frac{\partial \eta}{\partial x} \right) \quad (8.50)$$

Employing (8.38), (8.47) and (8.49) in (8.50), we get

$$u(x,y) = \frac{A_5}{\sqrt{C^2 + D^2}} \left[\frac{D(x^2 - y^2) - 2Cxy}{(x^2 + y^2)} \right] \left[\frac{\alpha}{\left[2 + A_4 \sqrt{C^2 + D^2} \alpha \right]} \right]$$

and

$$v(x,y) = \frac{A_5}{\sqrt{C^2 + D^2}} \left[\frac{C(x^2 - y^2) + 2Dxy}{(x^2 + y^2)} \right] \left[\frac{\alpha}{\left[2 + A_4 \sqrt{C^2 + D^2} \alpha \right]} \right]$$

The pressure function from (8.34) is given by

$$p(x,y) = p_0 - \frac{2A_5}{A_7^2} \left[\ln\left(2 + A_7 \alpha\right) + \frac{2}{2 + A_7 \alpha} \right]$$

where $A_7 = A_4 \sqrt{C^2 + D^2}$.

The state equation for this flow is

$$p = p_0 - \frac{2A_5}{A_7^2} \left[\ln\left(A_5 \rho\right) + \frac{2}{A_5 \rho} \right] \quad (8.51)$$

The state equation (8.51) yields

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{1}{A_7^2} \left[\frac{4 - 2A_5 \rho}{\rho^2} \right]$$

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} = \frac{1}{A_7^2} \left[\frac{2A_5 \rho - 8}{\rho^3} \right]$$

Since $\frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2 > 0$ and $\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} > 0$ for all real compressible media, it follows that $A_5 > 0$ and

$$\rho < \frac{2}{A_5} \qquad \text{and} \qquad \rho > \frac{4}{A_5}$$

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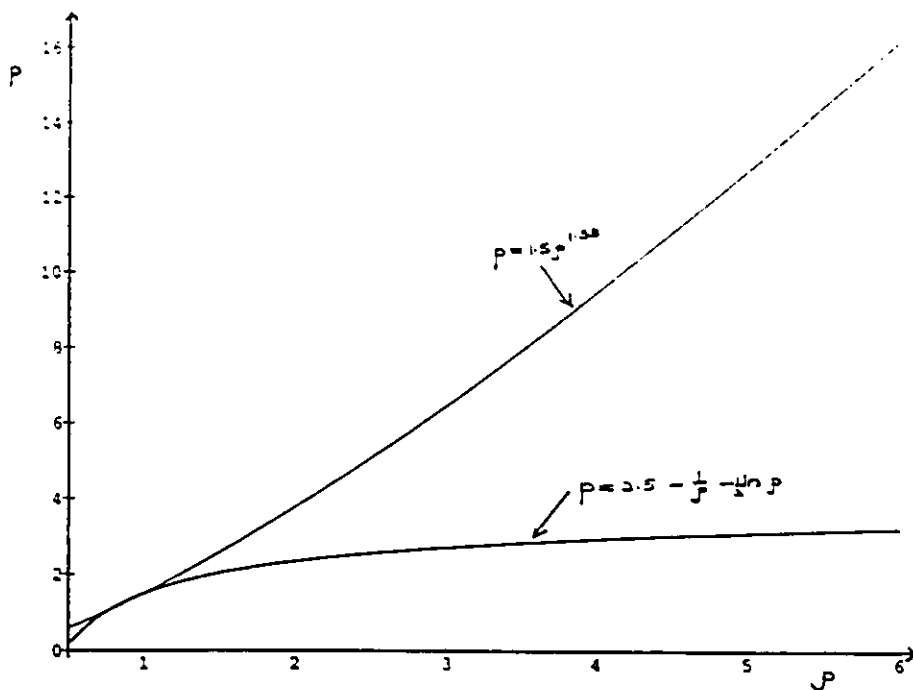
Figure 8.3.   Pressure versus density for the state equation (8.51)
with $A_5 = 1$, $A_7 = 2$, $p_0 = 2.5$.

However, the above restrictions on ρ cannot be satisfied simultaneously and, therefore, we may conclude that the solutions obtained above are valid for some imaginary gas having the state equation (8.51) giving p as a simple-valued function of ρ.

However, Figure 8.3 shows the plot of state equation (8.51) with $p_0 = 2.5$, $A_5 = 1$ and $A_7^2 = 4$ as compared with the state equation of an ideal gas $p = 1.5\rho^{1.33}$. We observed that (8.51) is a reasonable approximation of the ideal gas for a small range of value of ρ. Therefore, we may consider the solutions above as being valid for an ideal gas for this range of values of ρ.

The streamfunction and the potential function for the above flow are

$$\psi(x,y) = \frac{1}{\sqrt{2A_4 A_6}} \ln \left[\frac{\sqrt{A_4}\,(Cx + Dy) + \sqrt{2A_6}\,(x^2 + y^2)}{\sqrt{A_4}\,(Cx + Dy) - \sqrt{2A_6}\,(x^2 + y^2)} \right] - \psi_1,$$

$$\Phi(x,y) = \frac{A_5}{\sqrt{C^2 + D^2}} \int \left[\frac{D(x^2 - y^2) - 2Cxy}{(x^2 + y^2)} \right] \left[\frac{\alpha}{\left[2 + A_4 \sqrt{C^2 + D^2}\alpha \right]} \right] dx$$

$$+ \frac{A_5}{\sqrt{C^2 + D^2}} \int \left[\frac{C(x^2 - y^2) + 2Dxy}{(x^2 + y^2)} \right] \left[\frac{\alpha}{\left[2 + A_4 \sqrt{C^2 + D^2}\alpha \right]} \right] dy$$

respectively where ψ_1 is an arbitrary constant.

Example 3:

We let $w = \sqrt{2z}$. Then w is an analytic function of $z = x + iy$ and the two families of curves $\xi(x,y) =$ constant and $\eta(x,y) =$ constant form an orthogonal coordinate system. This system and the squared element of arc length are given by

$$x = \frac{1}{2}\left(\xi^2 + \eta^2 \right), \qquad y = \xi\eta \tag{8.52}$$

and

$$ds^2 = \left(\xi^2 + \eta^2 \right) \left[d\xi^2 + d\eta^2 \right] \tag{8.53}$$

Solving equations (8.52) for $\xi(x,y)$ and $\eta(x,y)$, we get

$$\xi(x,y) = \sqrt{\sqrt{x^2 + y^2} + x}, \qquad \eta(x,y) = \sqrt{\sqrt{x^2 + y^2} - x} \tag{8.54}$$

We investigate whether inviscid compressible fluid can flow along the family of curves $C\xi(x,y) + D\eta(x,y) =$ constant. Therefore, we have

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \tag{8.55}$$

where $\Gamma(\psi)$ is some arbitrary function of ψ. C and $D \neq 0$ are arbitrary constants. The streamlines for this flow are shown in Figure 8.4.

To proceed with our investigation, we employ the (ξ, ψ)−net whose squared differential element of arc length is given by (8.9) with

$$h(\xi, \psi) = \frac{1}{D}\sqrt{(C^2 + D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)} \tag{8.56}$$
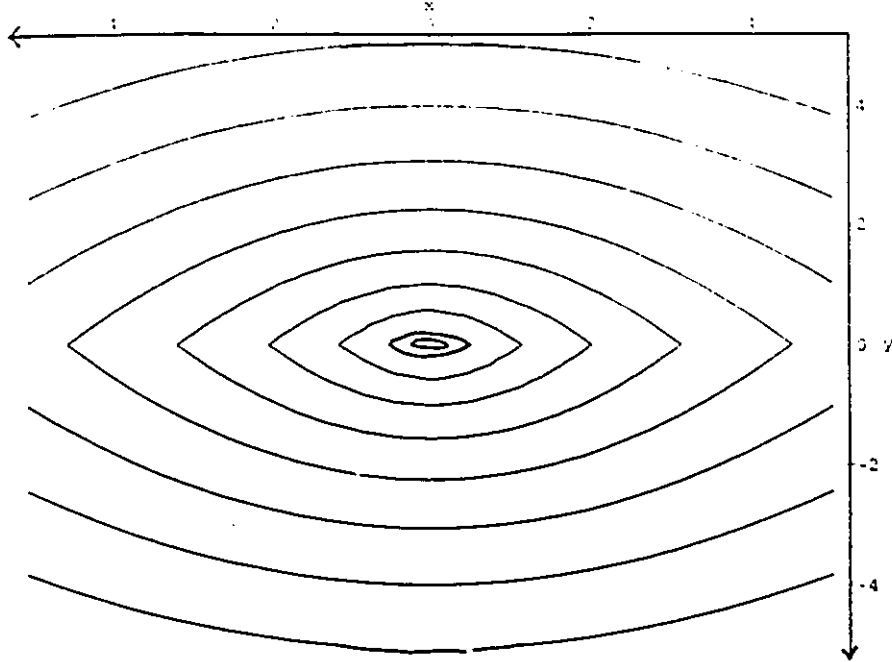
Figure 8.4. Streamline pattern $C\sqrt{\sqrt{x^2+y^2}+z} + D\sqrt{\sqrt{x^2+y^2}-z} = $ constant.
$(C = D = 1)$

Upon substitution of (8.56), Gauss equation (8.15) is identically satisfied and equations (8.14), (8.16) become a system of two equations

$$C\frac{\partial}{\partial\xi}\left[\frac{q}{\alpha}\right] + (C^2+D^2)\frac{\partial}{\partial\psi}\left[\frac{q}{\alpha\Gamma'(\psi)}\right] = 0 \qquad (8.57)$$

and

$$\frac{\partial\alpha}{\partial\psi}\frac{\partial q}{\partial\xi} - \frac{\partial\alpha}{\partial\xi}\frac{\partial q}{\partial\psi} = 0 \qquad (8.58)$$

in two unknown functions $\Gamma(\psi)$ and $q(\xi,\psi)$ where $\alpha(\xi,\psi)$ is given by

$$\alpha(\xi,\psi) = \frac{D\sqrt{C^2+D^2}}{\Gamma'(\psi)\sqrt{(C^2+D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)}} \qquad (8.59)$$

Equation (8.58) is identically satisfied if $q = q(\alpha)$, $q'(\alpha) \neq 0$ since all other cases lead us to a contradiction as we saw in the previous example. Using $q = q(\alpha)$ and equation (8.59), equation (8.57) becomes, after some simplification,

$$\left[\alpha q(\alpha) - \alpha^2 q'(\alpha)\right]\Gamma(\psi)\Gamma'^2(\psi) - \left(C^2+D^2\right)^2 q'(\alpha)\frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0$$

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Separation of variables yields

$$\frac{\alpha q(\alpha) - \alpha^2 q'(\alpha)}{(C^2 + D^2)^2 q'(\alpha)} = \frac{\Gamma''(\psi)}{\Gamma(\psi)\Gamma'^4(\psi)} = \text{constant} = A_8 \neq 0 \text{ (say)} \qquad (8.60)$$

Solving equations (8.60), we get

$$q(\alpha) = A_9 \left[\alpha^2 + A_8 \left(C^2 + D^2\right)^2\right]^{\frac{1}{2}} \qquad (8.61)$$

and

$$\Gamma'(\psi) = \left[2A_{10} - A_8\Gamma^2(\psi)\right]^{-\frac{1}{2}} \qquad (8.62)$$

where $A_9 \neq 0$ and A_{10} are arbitrary constants and $\alpha(\xi,\psi)$ is given by equation (8.59). Using (8.61) in (8.13), the density function takes the form

$$\rho(\alpha) = \frac{\alpha}{A_9 \left[\alpha^2 + A_8 \left(C^2 + D^2\right)^2\right]^{\frac{1}{2}}} \qquad (8.63)$$

Employing (8.62), (8.54) and (8.55) in (8.59) and simplifying, we get

$$\alpha(x,y) = \frac{\sqrt{C^2 + D^2}}{\sqrt{2}\,(x^2 + y^2)^{\frac{1}{4}}} M(x,y) \qquad (8.64)$$

where $M(x,y) = \sqrt{2A_{10} - A_8(C^2 - D^2)x - A_8(C^2 + D^2)\sqrt{x^2 + y^2} - 2A_8 CDy}$.

Differentiating (8.20) with respect to x and y respectively and using (2.27), we have

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)}\left(C\frac{\partial \xi}{\partial y} + D\frac{\partial \eta}{\partial y}\right), \qquad v(x,y) = -\frac{1}{\rho\Gamma'(\psi)}\left(C\frac{\partial \xi}{\partial x} + D\frac{\partial \eta}{\partial x}\right)$$

Upon substitution of equations (8.54), (8.62), (8.63) and (8.64), above equations yield

$$u(x,y) = \frac{A_9\sqrt{\alpha^2 + A_{11}}}{\sqrt{2}\sqrt{C^2 + D^2}\,(x^2 + y^2)^{\frac{1}{4}}}\left[C\sqrt{\sqrt{x^2 + y^2} - x} + D\sqrt{\sqrt{x^2 + y^2} + x}\right]$$

and

$$v(x,y) = -\frac{A_9\sqrt{\alpha^2 + A_{11}}}{\sqrt{2}\sqrt{C^2 + D^2}\,(x^2 + y^2)^{\frac{1}{4}}}\left[C\sqrt{x + \sqrt{x^2 + y^2}} - D\sqrt{\sqrt{x^2 + y^2} - x}\right]$$

where $A_{11} = A_8 \left(C^2 + D^2\right)^2$. Integrating (8.11) and (8.12), the pressure function is given by

$$p = p_0 - \frac{1}{2} A_9 \alpha \sqrt{A_{11} + \alpha^2} + \frac{1}{2} A_9 A_{11} \ln \left(\frac{\alpha + \sqrt{A_{11} + \alpha^2}}{\sqrt{A_{11}}} \right) \qquad (8.65)$$

where $\alpha(x, y)$ is given by (8.64).

Employing (8.61) and (8.63) in (8.65), the (p, ρ)-relation for this flow is given by

$$p = p_0 - \frac{A_9^2 A_{11} \rho}{2 \left(1 - A_9^2 \rho^2\right)} + \frac{1}{2} A_9 A_{11} \ln \left(\frac{\rho A_9 + 1}{\sqrt{1 - A_9^2 \rho^2}} \right)$$
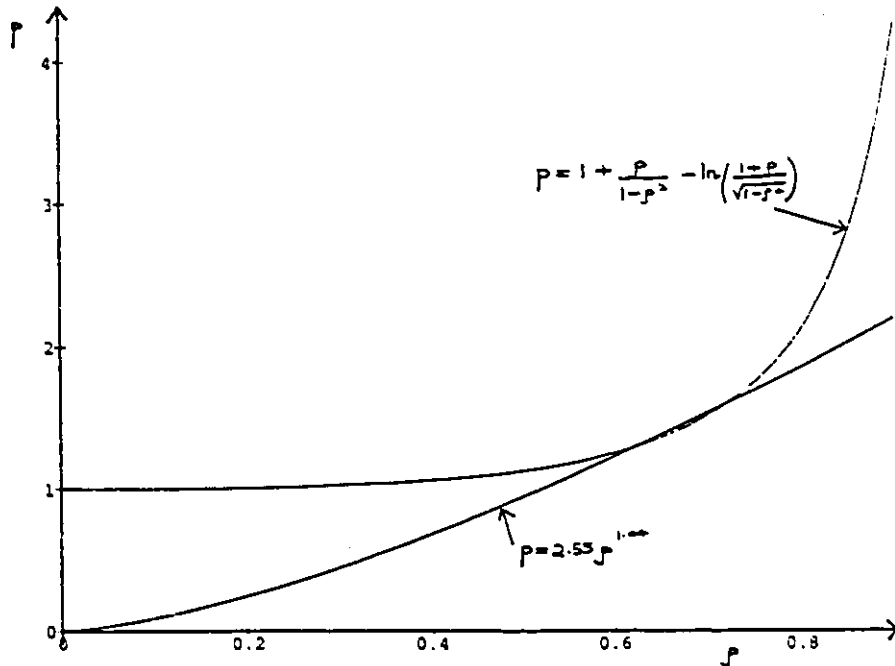

Figure 8.5.   Pressure versus density for the equation (8.65) with $A_9 = p_0 = 1$, $A_{11} = -2$.

The state equation above gives

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = -A_9^4 A_{11} \frac{\rho^2}{\left(1 - A_9^2 \rho^2\right)^2}$$

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2} = -A_9^4 A_{11} \frac{2\rho \left(1 + 2A_9^2 \rho^2\right)}{\left(1 - A_9^2 \rho^2\right)^2}$$

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Since for all real gases, $\frac{dp}{d\rho} > 0$ and $\frac{d^2p}{d\rho^2} > 0$, we have from the two equations above that $A_{11} < 0$. Therefore, the solutions obtained in this example are valid for some real gas having the state equation above for all permissible values of A_9 and A_{11}.

Figure 8.5 shows the graph of pressure versus density for the state equation above taking $A_9 = 1$, $A_{11} = -2$ and $p_0 = 1$ and the state equation of a polytropic gas $p = 2.55\rho^{1.44}$. It can be seen that this state equation is in good agreement with the ideal gas for a small range of values of ρ. Therefore, we may conclude that the solutions obtained above can be employed to study the behaviour of an ideal gas for this flow provided the variations of ρ are confined to this range.

The streamfunction and the potential function for this flow are respectively given by

$$\psi(x,y) = \frac{1}{2}\Gamma(x,y)\left[2A_{10} - \Gamma^2(x,y)\right]^{\frac{1}{2}} + \frac{A_{10}}{A_8}\arcsin\left(\frac{\sqrt{A_8}}{\sqrt{2A_{10}}}\Gamma(x,y)\right) - \psi_2,$$

$$\Phi(x,y) = \frac{A_9}{\sqrt{2}\sqrt{C^2 + D^2}}\int \frac{\sqrt{\alpha^2 + A_{11}}}{(x^2 + y^2)^{\frac{1}{4}}}\left[C\sqrt{\sqrt{x^2 + y^2} - x} + D\sqrt{\sqrt{x^2 + y^2} + x}\right]dx$$

$$-\frac{A_9}{\sqrt{2}\sqrt{C^2 + D^2}}\int \frac{\sqrt{\alpha^2 + A_{11}}}{(x^2 + y^2)^{\frac{1}{4}}}\left[C\sqrt{x + \sqrt{x^2 + y^2}} - D\sqrt{\sqrt{x^2 + y^2} - x}\right]dy$$

where $\Gamma(x,y)$ is given by (8.55) and ψ_2 is an arbitrary constant.

Example 4: Oblique Stagnation Point Flow

We let $w = \xi(x,y) + i\eta(x,y) = \frac{1}{2}z^2$. Then w is an analytic function of $z = x + iy$ and the two families of curves $\xi(x,y) =$constant and $\eta(x,y) =$constant form an orthogonal curvilinear coordinate system. This system and its squared element of arc length are given by

$$x = \sqrt{\xi + \sqrt{\xi^2 + \eta^2}}, \qquad y = \sqrt{-\xi + \sqrt{\xi^2 + \eta^2}} \tag{8.66}$$

and

$$ds^2 = \frac{1}{2\sqrt{\xi^2 + \eta^2}}\left[d\xi^2 + d\eta^2\right] \tag{8.67}$$

Solving equations (8.66) for $\xi(x,y)$ and $\eta(x,y)$, we have

$$\xi(x,y) = \frac{1}{2}\left(x^2 - y^2\right), \qquad \eta(x,y) = xy \qquad (8.68)$$

We assume that the streamlines are of the form

$$C\xi(x,y) + D\eta(x,y) = \text{constant}$$

so that

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \qquad (8.69)$$

The streamline pattern for this flow is shown in Figure 8.6.
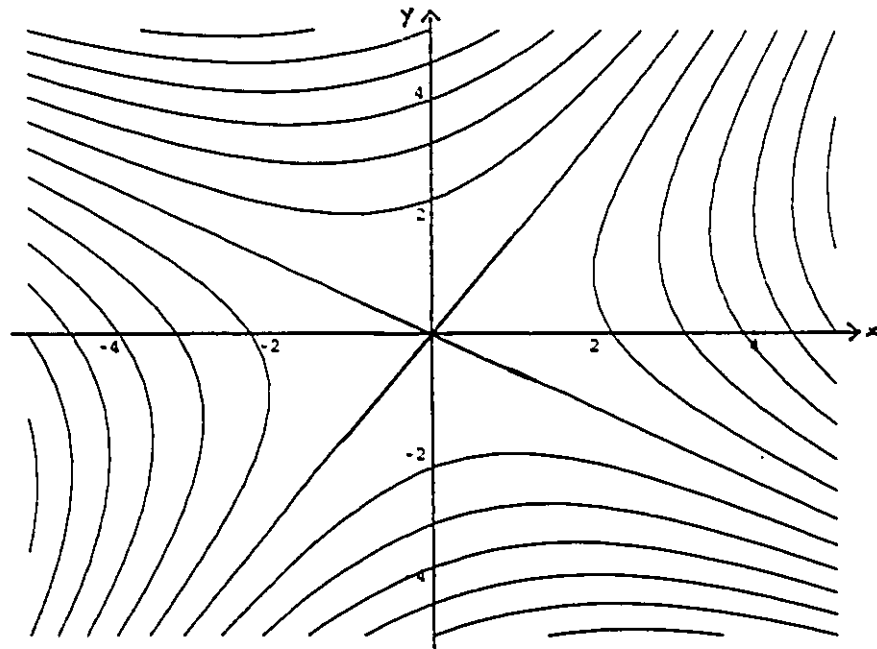

Figure 8.6.   Streamline pattern $\frac{1}{2}C\left(x^2 - y^2\right) + Dxy = \text{constant}$ $(C = D = 1)$.

To study these flows, we employ the (ξ, ψ)−net whose squared differential element of arc length is given by

$$h(\xi, \psi) = \frac{1}{\sqrt{2}} \frac{\sqrt{D}}{\left[(C^2 + D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)\right]^{\frac{1}{4}}} \qquad (8.70)$$

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Gauss equation (8.15) is identically satisfied when (8.70) is used and equations (8.14), (8.16) become a system of two equations

$$C \frac{\partial}{\partial \xi} \left[\frac{q}{\alpha} \right] + (C^2 + D^2) \frac{\partial}{\partial \psi} \left[\frac{q}{\alpha \Gamma'(\psi)} \right] = 0 \tag{8.71}$$

and

$$\frac{\partial \alpha}{\partial \psi} \frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi} \frac{\partial q}{\partial \psi} = 0 \tag{8.72}$$

in two unknowns $\Gamma(\psi)$ and $q(\xi, \psi)$ where $\alpha(\xi, \psi)$ is given by

$$\alpha(\xi, \psi) = \frac{\sqrt{2}\sqrt{C^2 + D^2}}{\sqrt{D}} \frac{\left[(C^2 + D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi) \right]^{\frac{1}{4}}}{\Gamma'(\psi)} \tag{8.73}$$

Since our flow is not a uniform flow , equation (8.72) is identically satisfied if one of the following cases hold true (i) $\dfrac{\partial \alpha}{\partial \psi} = \dfrac{\partial q}{\partial \psi} = 0$ or (ii) $\dfrac{\partial \alpha}{\partial \xi} = \dfrac{\partial q}{\partial \xi} = 0$ or (iii) $\dfrac{\partial \alpha}{\partial \xi} = \dfrac{\partial \alpha}{\partial \psi} = 0$ or (iv) $q = q(\alpha)$, $q'(\alpha) \neq 0$. Cases (i) to (iii) lead us to $D = 0$ which contradicts our assumption that $D \neq 0$. Therefore, we take $q = q(\alpha)$ in (8.71) and have

$$2\left(C^2 + D^2\right) \left[q(\alpha) - \alpha q'(\alpha)\right] \Gamma(\psi) + \alpha^5 q'(\alpha)\Gamma''(\psi)\Gamma'^2(\psi) = 0$$

Separating the variables, we obtain

$$\left[\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^5 q'(\alpha)} \right] = -\frac{1}{2(C^2 + D^2)} \frac{\Gamma''(\psi)\Gamma'^2(\psi)}{\Gamma(\psi)} = \text{constant} = A_{12} \neq 0 \text{ (say) } \tag{8.74}$$

Solving equations (8.74), we get

$$q(\alpha) = A_{13} \frac{\alpha}{(1 + A_{12}\alpha^4)^{\frac{1}{4}}} \tag{8.75}$$

and

$$\Gamma'(\psi) = \left[4A_{14} - 4A_{12}(C^2 + D^2)\Gamma^2(\psi) \right]^{\frac{1}{4}} \tag{8.76}$$

where $A_{13} \neq 0$ and $A_{14} \neq 0$ are arbitrary constants. Employing (8.69) and (8.76) in (8.73), we obtain

$$\alpha(x, y) = \frac{\sqrt{C^2 + D^2}\sqrt{x^2 + y^2}}{\left\{ 4A_{14} - 4A_{12}(C^2 + D^2) \left[\frac{1}{2}C\left(x^2 - y^2\right) + Dxy \right]^2 \right\}^{\frac{1}{4}}} \tag{8.77}$$

where $\xi(x,y)$ and $\eta(x,y)$ are given by (8.68).

Using (8.75) in (8.13), the density function is given by

$$\rho(\alpha) = \frac{(1 + A_{12}\alpha^4)^{\frac{1}{4}}}{A_{13}} \tag{8.78}$$

where $\alpha(x,y)$ is given by (8.77). Differentiating (8.20) with respect to x and y respectively and using (2.27), we have

$$u(x,y) = \frac{1}{\rho\Gamma'(\psi)} \left(C\frac{\partial \xi}{\partial y} + D\frac{\partial \eta}{\partial y} \right), \qquad v(x,y) = -\frac{1}{\rho\Gamma'(\psi)} \left(C\frac{\partial \xi}{\partial x} + D\frac{\partial \eta}{\partial x} \right)$$

Employing (8.68), (8.69), (8.76) and (8.78) in these equations, we get

$$u(x,y) = \frac{A_{13}\left[Dx - Cy\right]}{(1 + A_{12}\alpha^4)^{\frac{1}{4}} \left\{ 4A_{14} - 4A_{12}(C^2 + D^2)\left[\frac{1}{2}C\left(x^2 - y^2\right) + Dxy\right]^2 \right\}^{\frac{1}{4}}}$$

and

$$v(x,y) = -\frac{A_{13}\left[Cx + Dy\right]}{(1 + A_{12}\alpha^4)^{\frac{1}{4}} \left\{ 4A_{14} - 4A_{12}(C^2 + D^2)\left[\frac{1}{2}C\left(x^2 - y^2\right) + Dxy\right]^2 \right\}^{\frac{1}{4}}}$$

Finally, the pressure function is given by

$$p(x,y) = p_0 - A_{13} \int \frac{\alpha}{[A_{12}\alpha^4 + 1]^{\frac{1}{4}}} d\alpha$$

Example 5:

The function $w = \xi + i\eta = \frac{1}{\sqrt{2z}}$ is an analytic function of z and the curves $\xi(x,y) =$constant and $\eta(x,y) =$constant generate an orthogonal curvilinear coordinate system. This system and its squared element of arc length are given by

$$x = \frac{\xi^2 - \eta^2}{2\left(\xi^2 + \eta^2\right)^2}, \qquad y = \frac{\xi\eta}{\left(\xi^2 + \eta^2\right)^2} \tag{8.79}$$

and

$$ds^2 = \frac{1}{\left(\xi^2 + \eta^2\right)^3} \left[d\xi^2 + d\eta^2\right] \tag{8.80}$$

Solving equations (8.79) for $\xi(x,y)$ and $\eta(x,y)$, we have

$$\xi(x,y) = \frac{1}{2} \frac{\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}}, \qquad \eta(x,y) = \frac{1}{2} \frac{\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}} \qquad (8.81)$$

We investigate whether the family of curves given by $C\xi(x,y)+D\eta(x,y) = $ constant will allow inviscid compressible fluid to flow along it. Thus, we have

$$C\xi(x,y) + D\eta(x,y) = \Gamma(\psi), \qquad \Gamma'(\psi) \neq 0 \qquad (8.82)$$

The flow pattern for this example is shown in Figure 8.7.
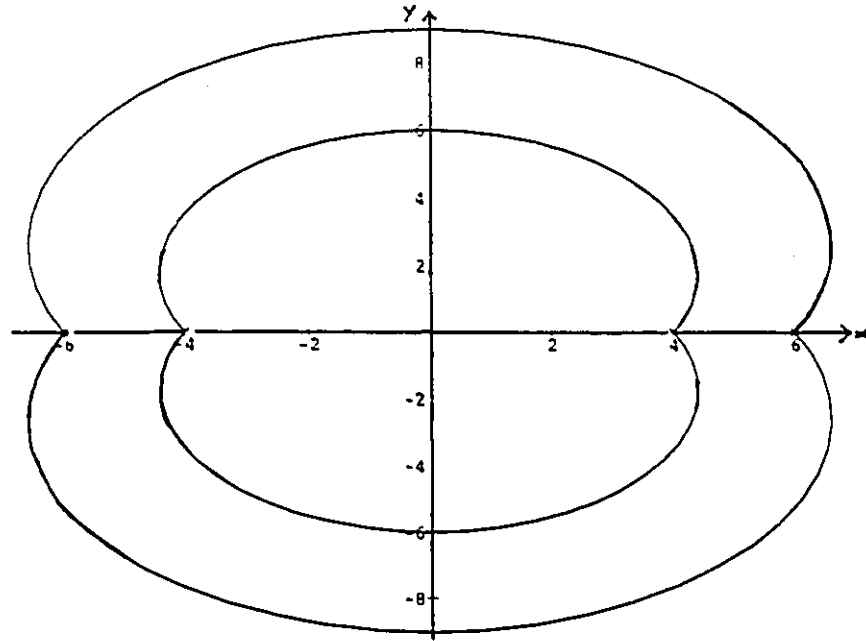
Figure 8.7. Streamline pattern $\dfrac{C\sqrt{\sqrt{x^2+y^2}+x}}{\sqrt{x^2+y^2}} + \dfrac{D\sqrt{\sqrt{x^2+y^2}-x}}{\sqrt{x^2+y^2}} = $ constant $(C = D = 1)$.

We employ the (ξ,ψ)-net with the squared differential element of arc length given by

$$h(\xi,\psi) = \frac{D^3}{[(C^2+D^2)\xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi)]^{\frac{3}{2}}} \qquad (8.83)$$

Using (8.83), Gauss equation (8.15) is identically satisfied and equations (8.14) and (8.16) are a system of two equations

$$C\frac{\partial}{\partial \xi}\left[\frac{q}{\alpha}\right] + (C^2+D^2)\frac{\partial}{\partial \psi}\left[\frac{q}{\alpha\Gamma'(\psi)}\right] = 0 \qquad (8.84)$$

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$$\frac{\partial \alpha}{\partial \psi} \frac{\partial q}{\partial \xi} - \frac{\partial \alpha}{\partial \xi} \frac{\partial q}{\partial \psi} = 0 \tag{8.85}$$

in two unknown functions $q(\xi, \psi)$ and $\Gamma(\psi)$ where $\alpha(\xi, \psi)$ is given by

$$\alpha(\xi, \psi) = \frac{\sqrt{C^2 + D^2}}{D^3 \Gamma'(\psi)} \left[\left(C^2 + D^2 \right) \xi^2 - 2C\xi\Gamma(\psi) + \Gamma^2(\psi) \right]^{\frac{3}{2}} \tag{8.86}$$

Equation (8.85) is identically satisfied only if $q = q(\alpha)$, $q'(\alpha) \neq 0$ since all other cases lead us to a contradiction as we mentioned in the previous examples. Therefore, we take $q = q(\alpha)$ in (8.84) and have

$$3 \left[\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^{\frac{5}{3}}} \right] \Gamma(\psi) + \left(C^2 + D^2 \right)^{\frac{2}{3}} q'(\alpha) \frac{\Gamma''(\psi)}{\Gamma'^{\frac{4}{3}}(\psi)} = 0$$

Separating variables, we have

$$\frac{q(\alpha) - \alpha q'(\alpha)}{\alpha^{\frac{5}{3}} q'(\alpha)} = -\frac{1}{3} \left(C^2 + D^2 \right)^{\frac{2}{3}} \frac{\Gamma''(\psi)}{\Gamma(\psi)\Gamma'^{\frac{4}{3}}(\psi)} = \text{constant} = A_{15} \neq 0 \text{ (say)} \tag{8.87}$$

Solving equations (8.87), we obtain

$$q(\alpha) = \frac{A_{16}\alpha\sqrt{1 - A_{15}\alpha^{\frac{2}{3}} + A_{15}^2\alpha^{\frac{4}{3}}}}{\left(1 + A_{15}\alpha^{\frac{2}{3}} \right)\sqrt{1 + A_{15}^3\alpha^2}} \tag{8.88}$$

and

$$\Gamma'(\psi) = \left[-A_{15} \left(C^2 + D^2 \right)^{-\frac{2}{3}} \Gamma^2(\psi) + A_{17} \right]^{\frac{3}{2}} \tag{8.89}$$

where A_{16} and A_{17} are arbitrary constants.

Proceeding as in the previous examples, we find that the density ρ and the velocity components are given by

$$\rho(\alpha) = \frac{\alpha}{q(\alpha)},$$

$$u(x, y) = \frac{\sqrt{2}\alpha}{2\sqrt{C^2 + D^2}\rho(\alpha)(x^2 + y^2)^{\frac{3}{4}}} \left\{ -(DM_1 + CN_1)\sqrt{x^2 + y^2} + 2x(DM_1 - CN_1) \right\},$$

$$v(x, y) = \frac{\sqrt{2}\alpha}{2\sqrt{C^2 + D^2}\rho(\alpha)(x^2 + y^2)^{\frac{3}{4}}} \left\{ (CM_1 - DN_1)\sqrt{x^2 + y^2} - 2x(CM_1 + DN_1) \right\}$$

where M_1 and N_1 are given by

$$M_1 = \sqrt{\sqrt{x^2 + y^2} + x} \qquad \text{and} \qquad N_1 = \sqrt{\sqrt{x^2 + y^2} - x}$$

Finally, the pressure function is given by

$$p = p_0 - \int \alpha\, dq$$

where $\alpha(x, y)$ is given by (8.86) and (8.82).

This section dealt with examples of flows satisfying $q = q(\alpha)$ such that $\frac{\partial \alpha}{\partial \xi} \neq 0$ and $\frac{\partial \alpha}{\partial \psi} \neq 0$. However, equation (8.16) is also satisfied when $q \neq q(\alpha)$ and this is considered in the following section for completeness.

8.4 FLOWS SATISFYING $q \neq q(\alpha)$.

Differentiating (8.17) with respect to ξ and ψ respectively, we obtain

$$\frac{\partial \alpha}{\partial \xi} = -\frac{\sqrt{C^2 + D^2}}{h^2(\xi, \psi)\Gamma'(\psi)}\frac{\partial h}{\partial \xi}, \qquad \frac{\partial \alpha}{\partial \psi} = -\frac{\sqrt{C^2 + D^2}}{h(\xi, \psi)\Gamma'(\psi)}\left[\frac{1}{h(\xi, \psi)}\frac{\partial h}{\partial \psi} + \frac{\Gamma''(\psi)}{\Gamma'(\psi)}\right]$$

$$(8.90)$$

Equation (8.16) yields the following five cases:

(i) $\dfrac{\partial \alpha}{\partial \xi} = 0, \qquad \dfrac{\partial \alpha}{\partial \psi} = 0$

(ii) $\dfrac{\partial \alpha}{\partial \xi} = 0, \qquad \dfrac{\partial q}{\partial \xi} = 0$

(iii) $\dfrac{\partial \alpha}{\partial \psi} = 0, \qquad \dfrac{\partial q}{\partial \psi} = 0$

(iv) $\dfrac{\partial q}{\partial \xi} = 0, \qquad \dfrac{\partial q}{\partial \psi} = 0$

(v) $q = q(\alpha)$

Case (iv) gives flow with constant speed and is of no interest and therefore removed from further consideration and case (v) was considered in the previous section. We study cases (i) to (iii) in the following:

Case (i): $\dfrac{\partial \alpha}{\partial \xi} = 0$, $\qquad \dfrac{\partial \alpha}{\partial \psi} = 0$

In this case from (8.90), we get

$$\frac{\partial h}{\partial \xi} = 0 \tag{8.91}$$

and

$$\frac{1}{h(\xi, \psi)} \frac{\partial h}{\partial \psi} + \frac{\Gamma''(\psi)}{\Gamma'(\psi)} = 0 \tag{8.92}$$

Equations (8.91) and (8.92) yield respectively

$$h = h(\psi)$$

and

$$\Gamma'(\psi) = \frac{B_1}{h(\psi)} \tag{8.93}$$

where $B_1 \neq 0$ is an arbitrary constant. Employing $h = h(\psi)$ and (8.92) in Gauss equation (8.15), we get

$$h'(\psi) = 0$$

which upon integration yields

$$h(\psi) = B_2 \psi + B_3 \tag{8.94}$$

where $B_2 \neq 0$ and B_3 are arbitrary constants. Employing (8.91) and (8.93) in (8.17), we obtain

$$\alpha = \frac{\sqrt{C^2 + D^2}}{B_1} \tag{8.95}$$

Using (8.91) and (8.93) in (8.14) and simplifying, we obtain

$$B_1 C \frac{\partial q}{\partial \xi} + (C^2 + D^2)(B_2 \psi + B_3) \frac{\partial q}{\partial \psi} + (C^2 + D^2) B_2 q = 0 \tag{8.96}$$

Solving (8.96) for $q(\xi, \psi)$, we get

$$q(\xi, \psi) = F \left[\ln (B_2 \psi + B_3) - B_2 (C^2 + D^2) \xi \right] exp \left\{ \frac{-(C^2 + D^2) B_2 \xi}{B_1 C} \right\} \tag{8.97}$$

where F is an arbitrary function of its argument.

Employing (8.94) and (8.97) in (8.13), we get

$$\rho(\xi, \psi) = \frac{\sqrt{C^2 + D^2} \, exp \left[\frac{1}{B_1 C} \left(C^2 + D^2 \right) B_2 \xi \right]}{B_1 F \left[\ln \left(B_2 \psi + B_3 \right) - B - 2 \left(C^2 + D^2 \right) \xi \right]} \tag{8.98}$$

Differentiating (8.7) with respect to x and y respectively and using (2.27), we have

$$u(x, y) = \frac{1}{\rho \Gamma'(\psi)} \left(C \frac{\partial \xi}{\partial y} + D \frac{\partial \eta}{\partial y} \right), \qquad v(x, y) = -\frac{1}{\rho \Gamma'(\psi)} \left(C \frac{\partial \xi}{\partial x} + D \frac{\partial \eta}{\partial x} \right) \tag{8.99}$$

Substituition of (8.93) and (8.98) in (8.99) yields

$$u(x, y) = \frac{B_2 \psi + B_3}{\sqrt{C^2 + D^2}} F \left[\chi \right] exp \left\{ \frac{- \left(C^2 + D^2 \right) B_2 \xi}{B_1 C} \right\} \left(C \frac{\partial \xi}{\partial y} + D \frac{\partial \eta}{\partial y} \right)$$

and

$$v(x, y) = -\frac{B_2 \psi + B_3}{\sqrt{C^2 + D^2}} F \left[\chi \right] exp \left\{ \frac{- \left(C^2 + D^2 \right) B_2 \xi}{B_1 C} \right\} \left(C \frac{\partial \xi}{\partial x} + D \frac{\partial \eta}{\partial x} \right)$$

where $\chi = \ln \left(B_2 \psi + B_3 \right) - B_2 \left(C^2 + D^2 \right) \xi$.

Employing (8.97) and (8.98) in (8.11) and (8.12) and integrating, the pressure function is given by

$$p(x, y) = p_0 - \frac{\sqrt{C^2 + D^2}}{B_1} F \left[\chi \right] exp \left\{ \frac{- \left(C^2 + D^2 \right) B_2 \xi}{B_1 C} \right\}$$

where p_0 is an arbitrary constant.

The state equation for this flow is given by

$$p = p_0 - \frac{\left(C^2 + D^2 \right)}{B_1^2} \frac{1}{\rho}$$

Case (ii): $\dfrac{\partial \alpha}{\partial \xi} = 0, \quad \dfrac{\partial q}{\partial \xi} = 0.$

Using $\dfrac{\partial \alpha}{\partial \xi} = 0$ in the first equation of (8.90), we get

$$h = h(\psi) \tag{8.100}$$

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Since $\dfrac{\partial q}{\partial \xi} = 0$, then

$$q = q(\psi) \qquad (8.101)$$

Using (8.100) in Gauss equation (8.15) and integrating the resulting equation twice with respect to ψ, we get

$$h(\psi) = B_5 \exp[B_4 \Gamma(\psi)] \qquad (8.102)$$

where $B_4 \neq 0$ and $B_5 \neq 0$ are arbitrary constants and $\Gamma(\psi)$ is an arbitrary function of ψ. Employing (8.100) and (8.101) in (8.14), we get

$$[h(\psi)q]' = 0 \qquad (8.103)$$

where prime denotes differentiation with respect to ψ.

Equation (8.103) yields, after one integration,

$$q(\psi) = \frac{B_6}{h(\psi)} \qquad (8.104)$$

where $B_6 \neq 0$ is an arbitrary constant and $h(\psi)$ is given by equation (8.102).

Using (8.104) in (8.13), the density function is given by

$$\rho(\psi) = \frac{\sqrt{C^2 + D^2}}{B_6 \Gamma'(\psi)} \qquad (8.105)$$

Employing (8.105) in (8.99), we get

$$u(x,y) = \frac{B_6}{\sqrt{C^2 + D^2}} \left(C \frac{\partial \xi}{\partial y} + D \frac{\partial \eta}{\partial y} \right)$$

and

$$v(x,y) = -\frac{B_6}{\sqrt{C^2 + D^2}} \left(C \frac{\partial \xi}{\partial x} + D \frac{\partial \eta}{\partial x} \right)$$

Using (8.104) and (8.105) in (8.11) and (8.12) and integrating, we obtain

$$p(\psi) = B_4 B_6 \sqrt{C^2 + D^2} \int \frac{1}{h^2(\psi)} d\psi \qquad (8.106)$$

The state equation for this flow can be obtained from (8.106) for a chosen $h(\psi)$.

Case (iii): $\dfrac{\partial \alpha}{\partial \psi} = 0$, $\dfrac{\partial q}{\partial \psi} = 0$

In this case, we have

$$q = q(\xi) \tag{8.107}$$

and from the second equation of (8.90), we get

$$h(\xi, \psi) = \frac{M_2(\xi)}{\Gamma'(\psi)} \tag{8.108}$$

where $M_2(\xi)$ is an arbitrary function. Using (8.108) in (8.17), we obtain

$$\alpha(\xi) = \frac{\sqrt{C^2 + D^2}}{M_2(\xi)} \tag{8.109}$$

Employing (8.107) and (8.108) in (8.14) and simplifying, we have

$$\frac{q'(\xi)}{q(\xi)} + \frac{M_2'(\xi)}{M_2(\xi)} - \frac{(C^2 + D^2)}{C} \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = 0$$

This equation implies that

$$\frac{q'(\xi)}{q(\xi)} + \frac{M_2'(\xi)}{M_2(\xi)} = \frac{(C^2 + D^2)}{C} \frac{\Gamma''(\psi)}{\Gamma'^2(\psi)} = \text{constant} = B_7 \neq 0(say) \tag{8.110}$$

Solving equations (8.110), we get

$$q(\xi) = \frac{B_8}{M_2(\xi)} \exp[B_7 \xi] \tag{8.111}$$

and

$$\Gamma'(\psi) = \frac{C^2 + D^2}{B_9 (C^2 + D^2) - CB_7 \psi} \tag{8.112}$$

where $B_8 \neq 0$ and B_9 are arbitrary constants. Integrating (8.112) once, we get

$$\Gamma(\psi) = B_{10} - \frac{(C^2 + D^2)}{B_7 C} \ln \left[B_9 (C^2 + D^2) - B_7 C \psi\right] \tag{8.113}$$

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where B_{10} is an arbitrary constant. Using (8.113) in (8.112), we have

$$\Gamma'(\psi) = \frac{C^2 + D^2}{exp\left[\frac{B_7 C}{C^2+D^2}\left\{B_{10} - \Gamma(\psi)\right\}\right]} \tag{8.114}$$

Using (8.108) and (8.113) in (8.15), we have

$$\left[\frac{M_2'(\xi)}{M_2(\xi)}\right]' = 0$$

which upon integration yields

$$M_2(\xi) = B_{12}\exp(B_{11}\xi) \tag{8.115}$$

where B_{11} and $B_{12} \neq 0$ are arbitrary constants. Employing (8.108) and (8.111) in (8.13), we obtain

$$\rho(\xi) = \frac{\sqrt{C^2 + D^2}}{B_8\exp(B_7\xi)} \tag{8.116}$$

Using (8.114) and (8.116) in (8.99), we get

$$u(x,y) = \frac{B_8\exp(B_7\xi)\exp\left[\frac{B_7 C}{C^2+D^2}\left\{B_{10} - \Gamma(\psi)\right\}\right]}{(C^2 + D^2)^{\frac{3}{2}}}\left(C\frac{\partial\xi}{\partial y} + D\frac{\partial\eta}{\partial y}\right) \tag{8.117}$$

and

$$v(x,y) = -\frac{B_8\exp(B_7\xi)\exp\left[\frac{B_7 C}{C^2+D^2}\left\{B_{10} - \Gamma(\psi)\right\}\right]}{(C^2 + D^2)^{\frac{3}{2}}}\left(C\frac{\partial\xi}{\partial x} + D\frac{\partial\eta}{\partial x}\right) \tag{8.118}$$

Employing (8.111) and (8.116) in (8.11) and (8.12) and integrating, we obtain

$$p(\xi) = -\frac{B_8\sqrt{C^2 + D^2}}{2M_2^2(\xi)}\exp(B_7\xi) - \frac{B_7 B_8\sqrt{C^2 + D^2}}{2}\int\frac{1}{M_2^2(\xi)}\exp(B_7\xi)\,d\xi \tag{8.119}$$

The (p,ρ) relation for this flow can be obtained from (8.116) and (8.119) for $M_2(\xi)$ given by equation (8.115).

CHAPTER 9

CONCLUSIONS

In the absence of a general theory for solving a system of nonlinear partial differential equations, transformations are the most powerful analytic tool available to obtain solutions of these systems of equations. The use of transformations in fluid dynamics is almost as old as the subject itself. These transformations result in simplifications and are used to achieve one of the following:

(a)   linearize the system, for example, hodograph and Kirchhoff transformations,

(b)   reduce the system to a system of ordinary differential equations, for example, similarity transformations

(c)   transform the system to another system which has already been solved.

In general, these transformations are classified into three groups: Class I includes those which are transformations only of the dependent variables; Class II includes transformations only of the independent variable(s); Class III consists of transformations of both dependent and independent variables. Ames [1965] has given an excellent treatment of the various transformations employed in fluid dynamics.

This dissertation involves transformations of the independent variables so that systems of ordinary differential equations and linear partial differential equations

are dealt with. New and existing exact solutions in closed form of these equations are obtained. However, in four cases, specifically in Chapters 4 and 5, the transformations employed yielded nonlinear ordinary differential equations for which only particular solutions were obtained. Other solutions of these equations which may give more new permissible flow patterns are yet to be found.

Some of the new solutions obtained for steady, plane, potential compressible flows do not correspond to a real gas. In such cases, an attempt has been made to determine a range of values of ρ where these gases reasonably approximate an ideal gas, for example, the solutions corresponding to a tangent gas are valid provided p and ρ do not vary significantly from some point $\left(p_a, \frac{1}{\rho_a}\right)$ on the ideal gas curve when this point corresponds to suitably average thermodynamic flow conditions. The Mach numbers for all flows having equation of state of a tangent gas have been obtained using this equation of state. Since the tangent gas is an approximation of a polytropic gas, it would be more appropriate to determine the Mach numbers for such flows using the equation of state of a polytropic gas.

It is interesting to observe that three forms of the equation of state given by

$$p = p_0 - \frac{A}{\rho}$$

$$p = p_0 - \frac{A}{B^2}\ln\left(\frac{1}{\rho}\right) - \frac{1}{B^2}\frac{1}{\rho}$$

and

$$p = p_0 + \frac{1}{2}AB\ln\left[\frac{\sqrt{A}\,B\,(1 + B\rho)}{\sqrt{1 - B^2\rho^2}}\right] - \frac{1}{2}\frac{AB^2\rho}{(1 - B^2\rho^2)}$$

have appeared frequently in many of the new solutions obtained suggesting a detail analysis of these equations may shed more light into the nature and usefulness of the gases represented by these state equations as well as the solutions obtained for them.

The exact solutions for incompressible, inviscid and irrotational flows can be easily obtained by the new approach in this dissertation either directly from the

solutions already obtained for compressible flows by taking $\rho = \text{constant}$ in them or by removing restrictions in the analysis of the governing equations that led to incompressible flows. These equations are then solved and the solutions for incompressible fluid flow for the forms chosen in Chapters 3, 4 and 5 are as follows:

1. flows with $y - c_1 x - c_2 = \text{constant}$ as the streamline pattern

    and the velocity components are given by

    $$u = \frac{1}{D_1}, \qquad v = \frac{c_1}{D_1}$$

2. flows with $y - \frac{1}{k_1}\ln|\sec(k_1 x + k_2)| = \text{constant}$ as the streamlines

    with

    $$u = \exp\left[k_1\left(y - f(x) - k_3\right)\right], \qquad v = f'(x)\exp\left[k_1\left(y - f(x) - k_3\right)\right]$$

    where $f(x) = \frac{1}{k_1}\ln|\sec(k_1 x + k_2)|$.

3. flows with $r = \text{constant}$ as the streamline pattern with velocity

    components given by

    $$V_1 = 0, \qquad V_2 = -\frac{d_1}{r}$$

4. flows with $\frac{r}{g(\theta)} = \text{constant}$ as the streamlines with

    $$V_1 = -\frac{1}{d_5}r^{d_1}\frac{g'(\theta)}{g^{2+d_1}(\theta)}, \qquad V_2 = -\frac{1}{d_5}r^{d_1}g^{-(1+d_1)}(\theta)$$

    where $g(\theta) = \left[\frac{1}{\sqrt{d_3}}\sec\left(\{d_1 + 1\}\theta + d_4\right)\right]^{\frac{1}{d_1+1}}$.

5. flows with $\theta - C\ln r = \text{constant}$ as the streamline pattern with velocity

    components given by

    $$V_1 = \frac{1}{d_1}\frac{1}{r}, \qquad V_2 = \frac{C}{d_1}\frac{1}{r}$$

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6.   flows with $\theta - f(r) = $ constant as the streamlines with

$$V_1 = \frac{1}{r}\exp\left(-c_1\left[\theta - f(r)\right]\right), \qquad V_2 = f'(r)\exp\left(-c_1\left[\theta - f(r)\right]\right)$$

where $f(r) = \frac{1}{c_1}\ln\left|\sec\left(c_1\ln r + c_2\right)\right| + c_3.$

Most of the work done in this dissertation is in obtaining closed form solutions in unbounded domains. Application of obtained solutions for solving boundary value problems in bounded domains is still to be carried out.

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