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Some extensions of Cochran's theorem

Jianhua Hu
University of Windsor

SOME EXTENSIONS OF COCHRAN'S THEOREM

by

JIANHUA HU

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Abstract

The aim of this dissertation is to obtain the distributions of *matrix quadratic forms (MQFs)* in a normal random matrix and some extensions of *Cochran's theorem*. The main contribution of this dissertation consists of the following two parts.

1. Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. In this dissertation, a general covariance Σ_Y of Y means that the collection of all np elements in Y has an arbitrary $np \times np$ covariance matrix. For the symmetric matrix W, a set of general necessary and sufficient conditions is derived for the matrix quadratic form $Y'WY$ to have a noncentral Wishart distribution. Then a multivariate version of Cochran's theorem concerning the noncentral Wishartness and independence of matrix quadratic forms is obtained. Some examples and the usual versions of Cochran's theorem are presented as special cases of this result.

2. Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance matrix Σ_Y. For the symmetric matrix W, a set of general necessary and sufficient conditions is derived for a matrix quadratic form to be distributed as a *difference of independent noncentral Wishart random matrices (DINWRM)*. A multivariate version of Cochran's theorem concerning *differences of independent noncentral Wishart random matrices (DINWRMs)* is obtained. Two usual versions of Cochran's theorem concerning differences of independent noncentral Wishart random matrices are presented as special cases of our result.

In addition to the above contribution, for the first part, we use a matrix approach

to present the proven result for the zero mean $\mathbf{0}$ case. This case has been solved by Masaro and Wong (2004a). They used Jordan algebra representations to obtain a general multivariate version of Cochran's theorem concerning Wishartness and independence. Their result and proof is more mathematically involved. Further, we provide a discrete representation version of Cochran's theorem.

For the second part, we use a matrix approach to present the proven result for the mean zero case. This case has been solved by Masaro and Wong (2004b). They used Jordan algebra homomorphisms to obtain the necessary and sufficient conditions for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ to be an independent family of random matrices distributed as *differences of independent Wishart random matrices (DIWRMs)*. Their result and proof is more mathematically involved. Our presentation provides a discrete representation version of Cochran's theorem concerning DIWRMs.

I dedicate my efforts to:

my parents for their love and support;

my wife, for her love, support and encouragement

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List of Nomenclature

$\mathbb{R}$: the real number set

$\mathbb{R}^p$: the Euclidean space of dimension p consisting $p \times 1$ real vectors

$|A|$: the determinant of square matrix A

$tr(A)$: the trace of matrix A

$\mathbb{M}_{n \times p}$: the set of $n \times p$ matrices over real set $\mathbb{R}$

$<, >$: trace inner product

$< S >$: the linear span of a given set S

$\|A\|$: the trace norm of matrix A

$\mathbb{S}_p$: the set of symmetric matrices of order p

I_p : the identity matrix of order p

A^+ : the Moore-Penrose inverse of matrix A

$sr(\mathbf{s})$: the spectral radius of square matrix $\mathbf{s}$

$\mathbb{N}_p$: the set of nonnegative definite matrices of order p over real set $\mathbb{R}$

n.n.d. : nonnegative definite

$\mathcal{N}_0$: the neighborhood of $\mathbf{0}$

A^α : the αth n.n.d. root of A for $\alpha > 0$

$A^{-\alpha}$: the αth n.n.d. root of A^+ for $\alpha > 0$

$vec(\,)$: *vec* operator

$\otimes$: the Kronecker product

K_{np} : the commutation matrix of order np

E_{ij} : the symmetric matrix of order p whose ijth entry and jith entry both are 1 and all other entries 0

$\mathbb{E}_p$: the set of matrices E_{ij}, $1 \leq i \leq j \leq p$, called the basic basis of the set $\mathbb{S}_p$

$\mathbb{H}_p$: the set of matrices $HE_{ij}H'$, $1 \leq i \leq j \leq p$, for $\Sigma \in \mathbb{N}_p$, where H is an orthogonal matrix such that $H'\Sigma H$ is diagonal and $\mathbb{H}_p = \{HE_{ij}H' : E_{ij} \in \mathbb{E}_p\}$, called the similar basis (of the $\mathbb{S}_p$) associated with Σ.

$\mathbb{K}_0$ set $\{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\}$

$\mathbb{K}$ set $\{\mathbf{h} : \Sigma\mathbf{h}\Sigma = \mathbf{0},\ \mathbf{h} \in \mathbb{H}_p\}$

$\mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$: multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ

$\mathcal{N}_{n \times p}(\boldsymbol{\mu}, \Sigma_Y)$: multivariate normal distribution with mean matrix $\boldsymbol{\mu}$ and covariance matrix Σ_Y

$\mathcal{W}_p(m, \Sigma)$: Wishart distribution with m degrees of freedom and covariance matrix Σ of order p

$\mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$: noncentral Wishart distribution with m degrees of freedom, covariance matrix Σ of order p and non-centrality matrix $\boldsymbol{\lambda}$

$M_Q(\mathbf{s}), M(\mathbf{s})$: the moment generating function of matrix quadratic form Q

MQF : matrix quadratic form

$MQFs$: matrix quadratic forms

$DIWRM$: difference of independent Wishart random matrices

$DIWRMs$: differences of independent Wishart random matrices

$DINWRM$: difference of independent noncentral Wishart random matrices

$DINWRMs$: differences of independent noncentral Wishart random matrices

Contents

Chapter 1

Introduction and Literature

1.1 Cochran's Theorem

It is well-known that Cochran's theorem plays an important role in the distribution theory for (matrix) quadratic forms in normal random variables and in the application of the theory of statistics, such as the theory of least squares, variance component analysis, estimation including MINQUE theory and testing of hypothesis and time series analysis. This has attracted many scholars to research and develop the extensions of Cochran's theorem for over seventy years.

In general, for a set of symmetric matrices W_1, W_2, $\ldots$, W_l of order n, the necessary and sufficient algebraic conditions are expected to characterize the probability statement that a set of matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$, where Y is an $n \times p$ normally distributed random matrix, is an independent family of central or noncentral Wishart random matrices. This is one problem that extensions of Cochran's theorem intend to solve. For convenience of statement, it is called **the**

original problem in this dissertation.

The property of a matrix quadratic form being distributed as a (noncentral) Wishart random matrix is called the (noncentral) Wishartness of the matrix quadratic form. The property that a quadratic form in a normal random vector is distributed as a (noncentral) chi-square random variable is called the (noncentral) chi-squareness of the quadratic form.

Note that, in the original problem, the symmetric matrices $W_1, W_2, \ldots, W_l$ do not need to be nonnegative definite, then, it is quite natural for statisticians to discuss whether a matrix quadratic form is distributed as a difference of two independent (noncentral) Wishart random matrices. This is called **the extended problem** in this dissertation for the sake of differentiating from the original problem.

The extended problem is formally stated as follows: For the symmetric matrices $W_1, W_2, \ldots, W_l$ of order n, the necessary and sufficient algebraic conditions are expected to characterize the probability statement that a set of matrix quadratic forms $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$, where Y is an $n \times p$ normally distributed random matrix, is an independent family of random matrices distributed as differences of independent (noncentral) Wishart random matrices. The extended problem is also one problem that the extensions of Cochran's theorem intend to solve.

Although the extended problem is different from the original problem, a new theorem stating the necessary and sufficient algebraic conditions to characterize the original problem or the extended problem is called a new version or an extension of Cochran's theorem. So far, many versions of Cochran's theorem concerning the original problem have been obtained and some versions of Cochran's theorem concerning the extended problem have been developed.

In the next section, we shall review the literature for the original problem and the extended problem.

1.2 Literature Review

Cochran investigated the distribution of quadratic forms in a normal random vector. His well-known result of the algebraic characterization of the chi-squareness and independence of quadratic forms in a normal random vector was published in Proceedings of the Combridge Philosophical Society in 1934. Cochran (1934) proved that the sum of ranks r_1, r_2, ..., r_l of the symmetric matrices W_1, W_2, ..., W_l being their order n is a necessary and sufficient algebraic condition for the quadratic forms $\mathbf{y}'W_1\mathbf{y}$, $\mathbf{y}'W_2\mathbf{y}$, ..., $\mathbf{y}'W_l\mathbf{y}$, where $\mathbf{y}$ is an n-variate normally distributed random vector with mean vector $\mathbf{0}$ and population covariance matrix I_n, to be an independent family of chi-square random variables with r_1, r_2, ..., r_l degrees of freedom, respectively. This is the well-known Cochran's theorem in statistics.

Since 1934, Cochran's result has become a cornerstone of the theory of analysis of variance in experimental designs, regression analysis and data analysis. Many scholars have been attracted to generalize Cochran's result in the univariate normal system. Madow (1940) generalized the result for $\mathbf{y}$ with nonzero mean vector $\boldsymbol{\mu}$ while Chipman and Rao (1964) extended Cochran's theorem to $\mathbf{y}$ with positive definite population covariance matrix Σ. Various extensions of Cochran's result and their interrelationships were given by Ogasawara and Takahashi (1951), James (1952), Graybill and Marsaglia (1957), Khatri (1963, 1968), Banerjee (1964), Chipman and Rao (1964), Rayner and Livingst (1965), Loynes (1966), Shanbhag (1966, 1968), Banerjee and Nagase(1976), Good (1969), Styan (1970) and Anderson and Styan (1982).

The independence of two quadratic forms with the case $\Sigma = I_n$ were investigated by Craig (1943) and Sakamoto (1944). Parallel results were then obtained and extended to the singular case by Ogasawara and Takahashi (1951), and by Khatri (1963), for nonzero mean, and by Good (1963) and Shanbhag (1966) for zero mean only. The corresponding results for the independence of two second degree polynomial quadratic expressions were established by Laha (1956).

The interested reader can further refer to Johnson and Kotz (1970), Styan (1970), Rao and Mitra (1971), Searle (1971), Rao (1973), Khatri (1980), Driscoll and Gundberg (1986) and the references therein for various univariate versions of Cochran's theorem and their interrelationships.

In the earlier 60's, Khatri (1962, 1963) extended Cochran's theorem from the univariate case to the multivariate case. With the development and applications of statistics, matrix quadratic forms have extensive applications in multivariate analysis of dispersion and in multiple regression in time series analysis, see Anderson (1971) and Rao (1973) for several examples. It was also noted that the covariance matrix of the normal random matrix Y is the structure of a Kronecker product in those applications. For example, the asymptotic distribution of some maximum likelihood estimates in linear stochastic models is normal with dispersion matrix of the form of a Kronecker product, see Anderson (1971).

Thus, there are numerous papers developing some extensions of Cochran's theorem concerning the Wishartness and independence of matrix quadratic forms in a normal random matrix with the covariance structure of a Kronecker product. Namely, the various necessary and sufficient conditions for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of (noncentral) Wishart random matrices were established for situations where the covariance matrix of Y is the Kronecker product $A \otimes \Sigma$ of the design covariance A and the population covariance Σ. We refer the interested reader to Rao and Mitra (1971), Khatri (1980), Fang and Wu (1984), Siotani *et al.* (1985), Fan (1986), de Gunst (1987), Mathai and Provost (1992), Baksalary *et al.* (1994), Vaish and Chaganty (2004), Tian and Styan (2005) and the reference therein.

Since the application of matrix quadratic forms is more and more extensive there are also a number of important instances where the covariance matrix Σ_Y of Y cannot be represented as the form of the Kronecker product $A \otimes \Sigma$ of the design covariance A and the population covariance Σ, see Anderson *et al.* (1986), Pavur (1987), Rao and Kleffe (1988), Mathew (1989) and Wong *et al.* (1995). So scholars started to extend Cochran's theorem to the cases that the covariance of the normal random matrix Y is a general nonnegative definite matrix Σ_Y, namely, the collection of all np elements in Y has an arbitrary $np \times np$ covariance matrix.

Pavur (1987) obtained the distribution of matrix quadratic forms on condition that the underlying matrix W is nonnegative definite, the population covariance Σ is nonsingular and the covariance structure of Y does not need to be the form of a Kronecker product. Wong *et al.* (1991) obtained a set of necessary and sufficient conditions for the general case except for placing restrictions on the column space of Σ_Y, extending Pavur's results. A verifiable version of Cochran's theorem was also obtained by Wong and Wang (1993) for the case where the underlying matrices are nonnegative definite. Later, refinements and simpler proofs of the main result in Wong and Wang (1993) were obtained by Mathew and Nordstrom (1997) and Wong *et al.* (1999). Wang *et al.* (1996) obtained a version of Cochran's theorem for multivariate components of variance models, where the covariance Σ_Y is the sum of a series Kronecker products and underlying matrices $W_1, W_2, \ldots, W_l$ are nonnegative definite. They extended the result of Wong and Wang (1993) to the case of matrix quadratic expressions. Wang (1997) still obtained versions of Cochran's theorem for matrix quadratic express. Wong (2000) collects these necessary and sufficient algebraic conditions developed in the 90's. Also see Dumais and Styan (1998) for an extensive bibliography on Cochran's theorem prior to 1998.

The underlying matrices $W_1, W_2, \ldots, W_l$ associated with the matrix quadratic forms $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are symmetric rather than nonnegative definite. This was a condition assumed in Cochran's univariate version in 1934. So it is our motivation to obtain a fully general multivariate version of Cochran's theorem. Under the condition that the underlying matrices W_i's associated with matrix quadratic forms $Y'W_iY$'s are symmetric, the original problem becomes much harder. because the matrices W_i's can no longer be factorized into square roots as can be done in the case of nonnegative definite matrices.

Masaro and Wong (2003) obtained a set of verifiable, but cumbersome, necessary and sufficient conditions about the Wishartness of the matrix quadratic form $Y'WY$

with the symmetric matrix W in a normal random matrix Y with mean $\mathbf{0}$ and general covariance Σ_Y.

Recently, Masaro and Wong (2004a) used Jordan algebra homomorphisms to further obtained a quite generalized extension of Cochran's theorem. They extended the problem to the case of the $\mathbb{H}_p^d$-valued matrix quadratic forms in a real, complex or quaternionic normal random matrix Y with zero mean and general covariance, where $\mathbb{H}_p^d$ denotes the family of $n \times n$ Hermitian matrices over $\mathbb{A}_d$. Here $\mathbb{A}_d$ is $\mathbb{R}$ (the set of real numbers), $\mathbb{C}$ (the set of complex numbers) or $\mathbb{H}$ (the set of the division ring of quarternions) according to $d = 1, 2$ or 4. However, their generalization is far away from the topic with which some statisticians are concerned.

In addition, Cochran's theorem has been extended and generalized in other directions. It was noted that Cochran's theorem on the distributions of quadratic forms in normal random variables can be equivalently formulated as a rank additivity result for symmetric idempotent matrices. The various generalizations of Cochran's theorem regarding the underlying matrices W_1, W_2, ..., W_l with certain properties were obtained by Anderson and Styan (1982), Styan and Takemura (1983), Baksalary and Hauke (1990), Šemrl (1996), Behboodian (2001), Waterhouse (2001), Lešnjak (2004) and Tian and Styan (2005). Anderson and Fang (1987) extended Cochran's theorem from normal distributions to elliptical contoured distributions including the case of tripotent matrices and left-special distributions. An interested reader can find the further extensions of Cochran's theorem in elliptically contoured distributions in Zhang (1989), Anderson and Fang (1990), Fang and Zhang (1990), Wang and Wong (1995), Wong and Cheng (1998, 1999).

Another important extension of Cochran's theorem is one called the extended problem in Section 1.1. Luther (1965) established the equivalence between a quadratic form distributed as the unique difference of two independent chi-square random variables and the tripotency of the underlying matrix of this quadratic form. The other discussion and corresponding results concerning a difference of two independent chi-square random variables and the tripotency of the underlying matrix can be found in Graybill (1969), Rao and Mitra (1971) and Anderson and Styan (1982). More generally, Baldessari (1967) obtained the necessary and sufficient conditions for a quadratic form, in normal random variables, to be distributed as a given linear combination of independent chi-square random variables, generalizing the results of Graybill and Marsaglia (1957) and Luther (1965). Later, Tan (1977) extended Baldessari's result to singular normal random variables. Khatri (1977) further extended the result of Baldessari (1967) to a singular covariance matrix, to a quadratic form family and to a quadratic expression family.

Tan (1975) extended the extended problem from univariate case to multivariate case. He gave a set of necessary and sufficient conditions for matrix quadratic expressions, in a normal random matrix with a Kronecker product $A \otimes \Sigma$ covariance matrix, to be independent family of random matrices distributed as differences of independent noncentral Wishart random matrices. Mathai (1993) introduced the noncentral generalized Laplacian distribution and obtained some univariate versions of Cochran's theorem. The distribution considered in Mathai (1993) is, with a change of scale, one considered in Graybill (1969). Wong and Wang (1995) extended Tan's results to the case of a general covariance matrix. They gave a set of necessary and sufficient conditions for matrix quadratic expressions to be an independent family of random matrices distributed as differences of independent noncentral Wishart random matrices.

Recently, Masaro and Wong (2004b) obtained a set of verifiable, but cumbersome, necessary and sufficient conditions for a matrix quadratic form, in a normal random matrix Y with zero mean and general covariance, to be distributed as a difference of two independent Wishart random matrices with a diagonal common covariance Λ. Further, they used certain Jordan algebra homomorphisms to derive a set of general necessary and sufficient conditions for matrix quadratic forms, in a normal random matrix Y with zero mean and general covariance, to be an independent family of random matrices distributed as differences of independent Wishart random matrices.

1.3 Our Motivation and Research Results

The underlying matrices associated with matrix quadratic forms are symmetric rather than nonnegative definite. This was a condition assumed in Cochran's univariate version in 1934. So, differentiating from the existing research results, it has been our motivation and goal to **use a matrix approach to establish a fully general multivariate version of Cochran's theorem** concerning the (noncentral) Wishartness and independence of matrix quadratic forms under the quit general conditions. For instance, the underlying matrices of matrix quadratic forms are symmetric, not necessarily nonnegative definite, and the random matrix Y has a normal distribution with general covariance structure, not necessarily Kronecker product nor positive definite.

Moreover, our motivation and goal also include **the development of a fully general multivariate version of Cochran's theorem** concerning differences of independent (noncentral) Wishart random matrices under the same conditions via a matrix approach.

This dissertation will focus its attention on our goals: 1) to develop the multivariate versions of Cochran's theorem concerning the central or noncentral Wishartness and independence of matrix quadratic forms; 2) to develop the multivariate versions of Cochran's theorem concerning the independence and the differences of independent central or noncentral Wishart distributions.

Our research results consist of the following two parts.

1. Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. In the dissertation, the general covariance Σ_Y of Y means that the collection of all np elements in Y has an arbitrary $np \times np$ covariance matrix. For the symmetric matrix W, a set of general necessary and sufficient conditions is derived for the matrix quadratic form $Y'WY$ to have a noncentral Wishart distribution. Then a multivariate version of Cochran's theorem concerning the noncentral Wishartness and independence of matrix quadratic forms is obtained. Some examples and the usual versions of Cochran's theorem are presented as special cases of this result.

2. Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance matrix Σ_Y. For the symmetric matrix W, a set of general necessary and sufficient conditions is derived for a matrix quadratic form to be distributed as a *difference of independent noncentral Wishart random matrices (DINWRM)*. A multi-

variate version of Cochran's theorem concerning *differences of independent noncentral Wishart random matrices (DINWRMs)* is obtained. Two usual versions of Cochran's theorem concerning differences of independent noncentral Wishart random matrices are presented as special cases of our result.

In addition, for the first part, we use a matrix approach to present the proven result for the zero mean case. This case has been solved by Masaro and Wong (2004a). They used Jordan algebra representations to obtain a general multivariate version of Cochran's theorem concerning Wishartness and independence. Their result and proof is more mathematically involved. Our presentation provides a discrete representation version of Cochran's theorem.

For the second part, we use a matrix approach to present the proven result for the mean zero case. This case has also been solved by Masaro and Wong (2004b). They used Jordan algebra homomorphisms to obtain the necessary and sufficient conditions for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ to be an independent family of random matrices distributed as *differences of independent Wishart random matrices (DIWRMs)*. Their result and proof is also more mathematically involved. Our presentation provides a discrete representation version of Cochran's theorem concerning DIWRMs.

1.4 The Organization of this Dissertation

This dissertation falls into five chapters. Chapter 1 introduces Cochran's theorem, follows the track of its development and reviews the literature on various versions of

Cochran's theorem during several decades. Chapter one also states our motivation and research results which are some extensions of Cochran's theorem.

Chapter 2 deals with the notations and preliminaries which are useful to the subsequent chapters. It includes matrix algebra, e.g. Kronecker products, the Moore-Penrose inverse. It introduces Wishart distributions, noncentral Wishart distributions, matrix quadratic forms and the moment generating functions. It also states some useful lemmas which are used in the derivation of our main results.

Chapter 3 is entirely devoted to the development of the multivariate version of Cochran's theorem concerning the central or noncentral Wishartness and independence of matrix quadratic forms in normal random matrix Y with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. For the symmetric matrix W, a set of general necessary and sufficient conditions (Theorem 3.3.1) is derived for the matrix quadratic form $Y'WY$ to be distributed as a noncentral Wishart random matrix. For the symmetric matrices $W_1, W_2, \ldots, W_l$, a set of general necessary and sufficient conditions (Theorem 3.4.2) is obtained for the matrix quadratic forms $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ to be an independent family of Wishart random matrices. Some examples and applications are presented. The usual versions of Cochran's theorem are presented as special cases of these results (from Corollary 3.3.4 to Corollary 3.4.5). As the intermediate result, we use a matrix approach to obtain a discrete representation version of Masaro and Wong's recent result (2004a). Namely, a set of succinct and verifiable necessary and sufficient conditions is established for the matrix quadratic form $Y'WY$ with the symmetric matrix W to be distributed as a Wishart random matrix (Theorem 3.1.3 and Theorem 3.1.1 for a special case). Then a set of succinct and verifiable necessary and sufficient conditions is developed for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ with the symmetric matrices W_1, W_2, ..., W_l to be an independent family of Wishart random matrices (Theorem 3.2.4 and Theorem 3.2.3 for a special case). Some examples and applications are presented. Also, we use the matrix approach to present the main result (Theorem 3.2.9) obtained by Masaro and Wong (2004a).

Chapter 4 is devoted to the multivariate version of Cochran's theorem concerning differences of independent central or noncentral Wishart random matrices. Let Y be normal random matrix with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. For the symmetric matrix W, a set of general necessary and sufficient conditions (Theorem 4.3.1) is derived for the matrix quadratic form $Y'WY$ to be distributed as a difference of two independent noncentral Wishart random matrices. For the symmetric matrices W_1, W_2, ..., W_l of order n, a set of general necessary and sufficient conditions (Theorem 4.4.1) is obtained for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of random matrices distributed as differences of independent noncentral Wishart random matrices. Some special cases are presented (from Corollary 4.4.2 to Corollary 4.4.4). As an intermediate result, we use a matrix approach to obtain a refined and improved version of Masaro and Wong's recent result (2004b). Namely, a set of succinct and verifiable necessary and sufficient conditions is established for the matrix quadratic form $Y'WY$ with the symmetric matrices W to be distributed as a difference of two independent Wishart random matrices (Theorem 4.1.2 and Theorem 4.1.1 for a special case). Then a set of succinct and verifiable necessary and sufficient conditions is developed for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ with the symmetric matrices W_1, W_2, ..., W_l to be an independent family of random matrices distributed as differences of independent Wishart random matrices (Theorem 4.2.2 and Theorem 4.2.1 for a special case). Some special cases are presented (from Corollary 4.2.3 to Corollary 4.2.6). Also, we use the matrix approach to present the main result (Theorem 4.1.8) obtained by Masaro and Wong (2004b).

Chapter 5 outlines some considerable topics and ideas on the problems discussed in above chapters for the future research.

The appendix attaches the proof of a set of necessary and sufficient conditions, obtained by Masaro and Wong (2004b), for a matrix quadratic form, in a normal random matrix with zero mean $\mathbf{0}$ and general covariance Σ_Y, to be distributed as a difference of two independent Wishart random matrices with a diagonal common covariance Λ.

Finally, the index of symbols lists the common symbols used in this dissertation.

Chapter 2

Notations and Preliminaries

Chapter 2 will deal with the notations and preliminaries which are useful to the subsequent chapters. It includes some concepts of matrix algebra, e.g. the trace inner product, the Cartesian product, Kronecker Products, the Moore-Penrose inverse, the idempotency, the tripotency and the commutation matrix. It defines Wishart distributions and noncentral Wishart distributions, and matrix quadratic forms and their moment generating functions. It also states some useful lemmas which are used in the derivation of our main results in the subsequent chapters.

2.1 Matrix Algebra

In this dissertation, we shall use $\mathbb{M}_{n\times p}$ to denote the set of $n \times p$ matrices over the real set $\mathbb{R}$. The trace inner product $<,>$ equipped in $\mathbb{M}_{n\times p}$ is defined as

$$< A, B >= tr(AB') \text{ for all } A,\ B \in \mathbb{M}_{n\times p}, \tag{2.1}$$

where B' is the transpose of B. We shall use $\|.\|$ to denote the trace norm in the matrices set $\mathbb{M}_{n\times p}$, defined as $\|A\|^2 =< A, A >$. We shall use $\mathbb{S}_p$ to denote the set of symmetric matrices of order p over the real set $\mathbb{R}$ and use $\prod_{i=1}^{l} \mathbb{S}_p$ to denote the Cartesian product of the symmetric matrices set $\mathbb{S}_p$ equipped with the trace inner product $<, >$ defined as

$$< (\mathbf{s_i}), (\tilde{\mathbf{s}}_\mathbf{i}) >= \sum_{i=1}^{l} < \mathbf{s_i}, \tilde{\mathbf{s}}_\mathbf{i} > \text{ for all } \mathbf{s_i}, \tilde{\mathbf{s}}_\mathbf{i} \in \mathbb{S}_p,\ i = 1, 2, \ldots, l. \tag{2.2}$$

We shall use $r(A)$ to denote the rank of matrix A and use A^+ to denote the Moore-Penrose inverse of matrix A if, for the matrix A, there exists a matrix A^+ such that $A^+AA^+ = A^+$, $AA^+A = A$, $(AA^+)' = AA^+$ and $(A^+A)' = A^+A$. When A is nonnegative definite (n.n.d.) and $\alpha > 0$, A^α will denote the αth n.n.d. root of A, $A^{-\alpha}$ will denote the αth n.n.d root of A^+, and A^0 will denote A^+A; thus $A^0 = A^\alpha A^{-\alpha} = A^{-\alpha} A^\alpha$.

We shall use bold-face lower case symbols or light-face upper case symbols to denote matrices or vectors. We shall use $\mathbb{N}_p$ to denote the set of nonnegative definite matrices of order p over the real set $\mathbb{R}$.

We shall use $\mathbf{e_{ij}}$ to denote the matrix whose ijth entry is 1 and all other entries 0 and E_{ij} to denote the symmetric matrix of order p whose ijth entry and jith entry both are 1 and all other entries 0. Write

$$\mathbb{E}_p = \{E_{ij} :\ 1 \le i \le j \le p\}.$$

We shall call $\mathbb{E}_p$ **the basic basis** of the set $\mathbb{S}_p$.

We shall use $sr(A)$ to denote the spectral radius of the square matrix A, i.e. $sr(A) = max\{|\lambda| : \lambda \text{ is an eigenvalue of square matrix } A\}$.

The square matrix A is said to be idempotent if $A^2 = A$, to be tripotent if $A^3 = A$. The matrices A and B are said to be orthogonal if $A'B = \mathbf{0}$, and A is said to be primitive in a family if A is non-zero and can not be written as the sum of two nonzero orthogonal idempotent elements in the family.

For the nonnegative definite matrix Σ of order p, there exists an orthogonal matrix H, i.e. $H'H = I_p$ where I_p denotes the identity matrix of order p, such that $H'\Sigma H = diag[\sigma_1, \sigma_2, \ldots, \sigma_p]$. Write

$$\mathbb{H}_p = \{H_{ij} \equiv HE_{ij}H' : 1 \le i \le j \le p,\ E_{ij} \in \mathbb{E}_p\}.$$

We shall call $\mathbb{H}_p$ **the similar basis** (of the set $\mathbb{S}_p$) associated with Σ. The "similar" is due to the similarity between the matrix H_{ij} in $\mathbb{H}_p$ and the matrix E_{ij} in $\mathbb{E}_p$.

Lemma 2.1.1. *If A is a nonnegative definite matrix of order np with rank $r(A) = q$, then there exists a $q \times np$ matrix L of rank q such that*

$$A = L'L,\ L \equiv [L_1, L_2, \ldots, L_p]. \tag{2.3}$$

with $L_i \in \mathbb{M}_{q \times n}$.

For the $n \times p$ matrix Y, we shall write Y into $Y = [\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_n}]'$, $\mathbf{y_i} \in \mathbb{R}^p$, where $\mathbb{R}^p$ is the p dimensional real space, and use $vec(Y)$ to denote the np dimensional vector $[\mathbf{y_1}', \mathbf{y_2}', \ldots, \mathbf{y_n}']'$. Here the *vec* operator transforms a matrix into a vector by stacking the rows of the matrix one underneath the other. For A in $\mathbb{M}_{n\times q}$ and B in $\mathbb{M}_{p\times r}$, we shall define the Kronecker product of matrices A and B, denoted by $A \otimes B$, as $A \otimes B = [a_{ij}B]$. The Kronecker product is also often called the direct product or the tensor product. The connection between the Kronecker product and the *vec* of matrices is often used in the calculations of our results.

Lemma 2.1.2. *If $A \in \mathbb{M}_{n\times q}$, $B \in \mathbb{M}_{p\times r}$ and $C \in \mathbb{M}_{q\times r}$ then*

$$(A \otimes B)vec(C) = vec(ACB'). \tag{2.4}$$

Moreover, the Kronecker product $\otimes$ also has the following properties, see Rao and Mitra (1971), Chapter 1, Kruskal (1975) and Muirhead (1982), Chapter 2.

$$(A \otimes C)(B \otimes D) = (AB) \otimes (CD), \;\; (A \otimes B)' = A' \otimes B', \tag{2.5}$$

$$tr(A \otimes B) = tr(A)tr(B), \;\; (A \otimes B)^+ = A^+ \otimes B^+. \tag{2.6}$$

The following lemma will be often used in the proofs of our results in the subsequent chapters.

Lemma 2.1.3. *For A, B and C, $AB'B = CB'B$ is equivalent to $AB' = CB'$, and $B'BA = B'BC$ is equivalent to $BA = BC$.*

Proof. For A, B and C, multiplying both sides of the equation $AB' = CB'$ on the right by B yields equation $AB'B = CB'B$.

Conversely, multiplying both sides of the equation $AB'B = CB'B$ on the right, respectively, by A' and C' yields equations $AB'BA' = CB'BA'$ and $AB'BC' = CB'BC'$. It follows that

$$||AB' - CB'||^2 = < AB' - CB', AB' - CB' > = tr((AB' - CB')(AB' - CB')')$$
$$= tr(AB'BA' - AB'BC' - CB'BA' + CB'BC') = 0,$$

i.e. $AB' = CB'$.

Similarly, it is easy to prove the equivalence between $BA = BC$ and $B'BA = B'BC$. □

The following lemma is due to Masaro and Wong (2004a).

Lemma 2.1.4. *Suppose A and B are symmetric matrices of order p with $A^2 = A$ and $AB + BA = 2B$. Then $AB = BA$.*

The vectors $vec(Y)$ and $vec(Y')$ clearly contain the same np components, but in a different order. We shall define the commutation matrix K_{np} of order np as follows

$$K_{np}vec(Y') = vec(Y), \ Y \in \mathbb{M}_{n\times p}.$$

Note the fact that the commutation matrix K_{np} is the unique $np \times np$ permutation matrix which transforms $vec(Y')$ into $vec(Y)$. The commutation matrix $K_{n\times p}$ has the following properties, see Magnus and Neudecker (1975) or Magnus and Neudecker (1991), Chapter 2.

$$K'_{np} = K_{pn} \text{ and } K_{np}K_{pn} = I_{np}. \tag{2.7}$$

The key property of the commutation matrix $K_{n\times p}$ enables us to interchange the two matrices of a Kronecker product.

Lemma 2.1.5. *Let A be a $p \times q$ matrix and B an $n \times r$ matrix. Then*

$$K_{np}(A \otimes B)K_{qr} = B \otimes A, \ \ K_{pn}(B \otimes A)K_{rq} = A \otimes B. \tag{2.8}$$

With the commutation matrix $K_{n\times p}$, the relation of the covariance matrix Σ_Y of Y and the covariance matrix $\Sigma_{Y'}$ of Y' can be easily expressed as

$$\Sigma_Y = \Sigma_{vec(Y)} = \Sigma_{K_{np}vec(Y')} = K_{np}\Sigma_{Y'}K'_{np} \ \text{ or } \ \Sigma_{Y'} = K'_{np}\Sigma_Y K_{np}. \tag{2.9}$$

(2.9) implies that the covariance matrix $\Sigma_{Y'}$ and the covariance matrix Σ_Y are similar.

Lemma 2.1.6. *Let Σ_Y and Σ be nonnegative definite matrices of order np and p, respectively, W be a symmetric matrix of order n and $\mathbf{s}$, $\tilde{\mathbf{s}}$ be symmetric matrices of order p. Let*

$$F(\mathbf{s}, \tilde{\mathbf{s}}, W, \Sigma_Y) = \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \tilde{\mathbf{s}})\Sigma_Y$$

and

$$\Theta(\mathbf{s}, \tilde{\mathbf{s}}, W, L) = L(\mathbf{s} \otimes W)L'L(\tilde{\mathbf{s}} \otimes W)L'.$$

Then

$$\Sigma_Y[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})]\Sigma_Y = F(\mathbf{s}, \tilde{\mathbf{s}}, W, \Sigma_Y) + F(\tilde{\mathbf{s}}, \mathbf{s}, W, \Sigma_Y) \tag{2.10}$$

is equivalent to

$$L[(\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s}) \otimes W]L' = \Theta(\mathbf{s}, \tilde{\mathbf{s}}, W, L) + \Theta(\tilde{\mathbf{s}}, \mathbf{s}, W, L) \tag{2.11}$$

where $\Sigma_{Y'} = L'L$, $L = [L_1, L_2, \ldots, L_p]$, $q = rank(\Sigma_Y)$, $L_i \in \mathbb{M}_{q\times n}$, $i = 1, 2, \ldots, p$.

Proof. There is a $q \times np$ matrix L with $q = rank(\Sigma_{Y'})$ such that $\Sigma_{Y'} = L'L$ from Lemma 2.1.1. Since by (2.7)-(2.9)

$$\begin{aligned}\Sigma_{Y'}(\mathbf{s}\Sigma\tilde{\mathbf{s}} \otimes W)\Sigma_{Y'} =& K'_{np}\Sigma_Y K_{np}K'_{np}(W \otimes \mathbf{s}\Sigma\tilde{\mathbf{s}})K_{np}K'_{np}\Sigma_Y K_{np} \\ =& K'_{np}\Sigma_Y(W \otimes \mathbf{s}\Sigma\tilde{\mathbf{s}})\Sigma_Y K_{np},\end{aligned}$$

and

$$\begin{aligned}&\Sigma_{Y'}(\mathbf{s} \otimes W)\Sigma_{Y'}(\tilde{\mathbf{s}} \otimes W)\Sigma_{Y'} \\ &= K'_{np}\Sigma_Y K_{np}K'_{np}(W \otimes \mathbf{s})K_{np}K'_{np}\Sigma_Y \times K_{np}K'_{np}(W \otimes \tilde{\mathbf{s}})K_{np}K'_{np}\Sigma_Y K_{np} \\ &= K'_{np}\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \tilde{\mathbf{s}})\Sigma_Y K_{np}.\end{aligned}$$

(2.10) is equivalent to

$$\Sigma_{Y'}[(\mathrm{s}\Sigma\tilde{\mathrm{s}}+\tilde{\mathrm{s}}\Sigma\mathrm{s})\otimes W]\Sigma_{Y'}=\Theta(\mathrm{s},\tilde{\mathrm{s}},W,\Sigma_{Y'})+\Theta(\tilde{\mathrm{s}},\mathrm{s},W,\Sigma_{Y'}). \tag{2.12}$$

Using $L'L$ to replace $\Sigma_{Y'}$ in Eq. (2.12) above equation and then by Lemma 2.1.3, we have completed the proof of the desired result. □

We shall repeatedly use (2.11) to replace (2.10) in the proofs of the results in the subsequent chapters.

The following property is useful when we focus our attention on the set $\mathbb{S}_p$.

Lemma 2.1.7. *The following statements (a) and (b) are equivalent.*

(a) For any $\mathrm{s}\in\mathbb{S}_p$*,*

$$\Sigma_Y(W\otimes\mathrm{s}\Sigma\mathrm{s})\Sigma_Y=\Sigma_Y(W\otimes\mathrm{s})\Sigma_Y(W\otimes\mathrm{s})\Sigma_Y; \tag{2.13}$$

(b) For any $\mathrm{s},\tilde{\mathrm{s}}\in\mathbb{S}_p$*,*

$$\Sigma_Y[W\otimes(\mathrm{s}\Sigma\tilde{\mathrm{s}}+\tilde{\mathrm{s}}\Sigma\mathrm{s})]\Sigma_Y=F(\mathrm{s},\tilde{\mathrm{s}},W,\Sigma_Y)+F(\tilde{\mathrm{s}},\mathrm{s},W,\Sigma_Y). \tag{2.14}$$

Proof. Since

$$\Sigma_Y(W\otimes\mathrm{s}\Sigma\mathrm{s})\Sigma_Y=\frac{1}{2}[\Sigma_Y(W\otimes\mathrm{s}\Sigma\mathrm{s})\Sigma_Y+\Sigma_Y(W\otimes\mathrm{s}\Sigma\mathrm{s})\Sigma_Y],$$

it suffices to show that (a) $\Longrightarrow$ (b). Note that for $\mathrm{s},\tilde{\mathrm{s}}\in\mathbb{S}_p$,

$$\Sigma_Y[W\otimes(\mathrm{s}\Sigma\tilde{\mathrm{s}}+\tilde{\mathrm{s}}\Sigma\mathrm{s})]\Sigma_Y=\frac{1}{2}\Sigma_Y\left(W\otimes[(\mathrm{s}+\tilde{\mathrm{s}})\Sigma(\mathrm{s}+\tilde{\mathrm{s}})-(\mathrm{s}-\tilde{\mathrm{s}})\Sigma(\mathrm{s}-\tilde{\mathrm{s}})]\right)\Sigma_Y.$$

Since $\mathrm{s}+\tilde{\mathrm{s}},\mathrm{s}-\tilde{\mathrm{s}}\in\mathbb{S}_p$, with (a) and simple operations, we obtain (2.14) and that completes the proof. □

We shall use $< S >$ to denote the linear span of a given set S. The following lemma is due to Wong and Wang (1995).

Lemma 2.1.8. *Let $\Sigma \in \mathbb{N}_p$, then the following conditions are equivalent.*

(a) $\Sigma \neq \mathbf{0}$;

(b) $< \{(\mathbf{s}\Sigma)^n \mathbf{s} : \mathbf{s} \in \mathbb{S}_p\} >= \mathbb{S}_p$ *for positive integer* n.

Lemma 2.1.9. *The following conditions are equivalent.*

(a) $< \{\mathbf{s}E_{ii}\mathbf{s} : \mathbf{s} \in \mathbb{S}_p\} >= \mathbb{S}_p$ *for any* $E_{ii} \in \mathbb{E}_p$, $i = 1, 2, \ldots, p$;

(b) $< \{\mathbf{s}E_{ii}\mathbf{s}E_{jj}\mathbf{s} : \mathbf{s} \in \mathbb{S}_p\} >= \mathbb{S}_p$ *for any* $i, j \in \{1, 2, \ldots, p\}$.

2.2 Central and Noncentral Wishart Distributions

Let $\mathbf{y}$ be a $p \times 1$ real random vector with mean vector μ and covariance matrix Σ. Namely,

$$\boldsymbol{\mu} = E(\mathbf{y});\ \Sigma = Cov(\mathbf{y}) = E(\mathbf{y} - E(\mathbf{y}))(\mathbf{y} - E(\mathbf{y}))',$$

where E denotes the expected value. If $\mathbf{y}$ is a multivariate normal distribution then we write $\mathbf{y} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ where $\sim$ means distributed as. In the case of Σ being nonsingular the density function of $\mathbf{y}$ is given by

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} exp\{-\frac{1}{2} < \mathbf{y} - \boldsymbol{\mu}, \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}) >\}. \tag{2.15}$$

In the case of Σ being singular, there exists a $p \times q$ matrix L of rank q $(< p)$ such that

$$\mathbf{y} = \boldsymbol{\mu} + L\mathbf{z}.$$

Then

$$E(\mathbf{y}) = \boldsymbol{\mu} + LE(\mathbf{z}), \quad \Sigma = Cov(\mathbf{y}) = LCov(\mathbf{z})L',$$

implying that there exists a $q \times 1$ vector $\mathbf{z}$ such that $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, I_q)$ and $\mathbf{z}$ has its density function. See Mathai *et al.* (1995), Chapter 1, and Muirhead (1982), Chapter 1, for more details about the multivariate normal distribution.

The $n \times p$ random matrix Y taking the real values in set $\mathbb{M}_{n\times p}$ is said to have a real multivariate normal distribution with mean (matrix) $\boldsymbol{\mu}_Y \in \mathbb{M}_{n\times p}$ and covariance (matrix) $\Sigma_Y \in \mathbb{N}_{np}$ if the vector $vec(Y)$ has a multivariate normal distribution $\mathcal{N}_{np}(vec(\boldsymbol{\mu}_Y), \Sigma_Y)$. In this case we write $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$. $\mathcal{N}_{p\times 1}(\boldsymbol{\mu}, \Sigma)$ is nothing else but $\mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$.

Definition 2.2.1. *If $A = X'X$, where X is an $m \times p$ random matrix normally distributed as $\mathcal{N}_{m\times p}(\mathbf{0}, I_m \otimes \Sigma)$ with $\Sigma \in \mathbb{N}_p$, then A is said to have the* **(central) Wishart distribution** *with m degrees of freedom and covariance matrix Σ.*

We shall use $\mathcal{W}_p(m, \Sigma)$ to denote the Wishart distribution with m degrees of freedom and covariance Σ of order p, and write $A \sim \mathcal{W}_p(m, \Sigma)$ if A has this distribution, where the subscript on $\mathcal{W}$ denotes the size of the matrix A. A is also said to be a Wishart random matrix.

When $m \geq p$, $A = X'X$ is nonsingular and then the Wishart distribution $\mathcal{W}_p(m, \Sigma)$ has a density function. When $m < p$, $A = X'X$ is singular and the Wishart distribution $\mathcal{W}_p(m, \Sigma)$ does not have a density function, see the references Muirhead (1982), Chapter 3 and Chapter 10, Eaton (1983), Chapter 8, and Srivastava (2003) for more details.

The Wishart distribution generalizes the chi-square distribution.

We shall use $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ to denote that $Y'WY$ has the distribution of the difference of two independent Wishart random matrices with distribution $\mathcal{W}_p(m_1, \Sigma)$ and $\mathcal{W}_p(m_2, \Sigma)$. In this case, $Y'WY$ is said to be distributed as a **difference of independent Wishart random matrices (DIWRM)**.

Properties of Wishart distributions are given in the following two lemma.

Lemma 2.2.1. *If $Y'WY \sim \mathcal{W}_p(m, \Sigma)$ and H is a $p \times k$ matrix of rank k, then $(YH)'W(YH) \sim \mathcal{W}_k(m, H'\Sigma H)$.*

Lemma 2.2.2. *If $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ and H is a $p \times k$ matrix of rank k, then $(YH)'W(YH) \sim \mathcal{W}_p(m_1, H'\Sigma H) - \mathcal{W}_p(m_2, H'\Sigma H)$.*

Definition 2.2.2. *If $A = X'X$, where X is an $m \times p$ random matrix normally distributed as $\mathcal{N}_{m \times p}(\boldsymbol{\mu}, I_m \otimes \Sigma)$ with $\Sigma \in \mathbb{N}_p$, then A is said to have the* **noncentral Wishart distribution** *with m degrees of freedom, covariance matrix Σ and noncentrality matrix $\boldsymbol{\lambda} = \boldsymbol{\mu}'\boldsymbol{\mu}$.*

We shall write that A is $\mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$ or $A \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$. A is also said to be a noncentral Wishart random matrix

The noncentral Wishart distribution generalizes the noncentral chi-square distribution in the same way that the Wishart distribution generalizes the chi-square distribution. The Wishart distribution is one special noncentral Wishart distribution with $\boldsymbol{\mu} = \mathbf{0}$ and then $\boldsymbol{\lambda} = \mathbf{0}$.

We shall use $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ to denote that $Y'WY$ has the distribution of the difference of two independent noncentral Wishart random matrices with $\mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$ and $\mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$. In this case, $Y'WY$ is said to be distributed as a **difference of independent noncentral Wishart random matrices (DINWRM)**.

The properties of noncentral Wishart distributions are given in the following two lemma.

Lemma 2.2.3. *If* $Y'WY \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$ *and* H *is a* $p \times k$ *matrix of rank* k*, then* $(YH)'W(YH) \sim \mathcal{W}_k(m, H'\Sigma H, H'\boldsymbol{\lambda}H)$.

Lemma 2.2.4. *If* $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ *and* H *is a* $p \times k$ *matrix of rank* k*, then* $(YH)'W(YH) \sim \mathcal{W}_p(m_1, H'\Sigma H, H'\boldsymbol{\lambda}_1 H) - \mathcal{W}_p(m_2, H'\Sigma H, H'\boldsymbol{\lambda}_2 H)$.

Above lemmas will be used repeatedly in the subsequent chapters.

2.3 Matrix Quadratic Forms (MQFs) and Moment Generating Functions

Let $\mathbf{y}$ be a $p \times 1$ real normal random vector. Then for the symmetric matrix W of order p, $\mathbf{y}'W\mathbf{y}$ is called a quadratic form in a normal random vector $\mathbf{y}$. Theoretical results on $\mathbf{y}'W\mathbf{y}$ as well as applications and generalizations are available from Mathai and Provost (1992). To distinguish it from the quadratic form, we shall call $\mathbf{y}'W\mathbf{y} + \mathbf{a}'\mathbf{y} + d$ a quadratic expression in $\mathbf{y}$ where $\mathbf{a}$ is a $p \times 1$ vector and d is a real number. Regarding quadratic forms and quadratic expressions we refer the interested reader to Mathai *et al.* (1995), Chapter 2, for more details.

For the symmetric matrix W of order n, we shall call $Y'WY$ a **matrix quadratic**

form (MQF) in a normal random matrix Y and call $Y'WY + B'Y + Y'B + C$ a **matrix quadratic expression** in a normal random matrix Y where B is a $p \times n$ matrix and C is a symmetric matrix of order p. In this dissertation, we shall focus our attention on the matrix quadratic form $Y'WY$ as well as a family of matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$. We shall use Q to denote $Y'WY$ as well as $Y'WY + B'Y + Y'B + C$ without distinction.

We are interested in whether the matrix quadratic form $Y'WY$ is distributed as a Wishart random matrix in the case $\boldsymbol{\mu} = \mathbf{0}$ or a noncentral Wishart random matrix in the case $\boldsymbol{\mu} \neq \mathbf{0}$. Necessary and sufficient conditions for the matrix quadratic form $Y'WY$ to have a Wishart distribution $\mathcal{W}_p(m, \Sigma)$ or a noncentral Wishart distribution $\mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$ will be investigated. The property of the matrix quadratic form $Y'WY$ distributed as a Wishart random matrix is called its **Wishartness**. In the similar way, the property of the matrix quadratic form $Y'WY$ distributed as a noncentral Wishart random matrix is called its **noncentral Wishartness**. For a set of symmetric matrices W_1, W_2, ..., W_l of order n, we are interested in the independence as well as Wishartness or noncentral Wishartness of the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$. We are also interested whether a set of matrix quadratic forms is an independent family of random matrices distributed as differences of independent (noncentral) Wishart random matrices.

Definition 2.3.1. *If Q is a matrix quadratic form, the moment generating function, denoted by $M_Q(\mathbf{s})$, of Q is defined as*

$$M_Q(\mathbf{s}) = E(e^{<\mathbf{s}, Q>}),\ \mathbf{s} \in \mathbb{S}_p. \tag{2.16}$$

We often write $M(\mathbf{s})$ instead of $M_Q(\mathbf{s})$ for short.

For the symmetric matrices $W_1, W_2, \ldots, W_l$ of order n, we have a set of matrix quadratic expressions $\{Q_i\}_{i=1}^l$, where $Q_i = Y'W_iY + B'Y + Y'B + C$, $B_i \in \mathbb{M}_{n\times p}$, $C_i \in \mathbb{S}_p$, $i = 1, 2, \ldots, l$. The following lemma, due to Wong *et al.* (1991), gives the joint moment generating function $M(\mathbf{s})$ of a set of matrix quadratic expressions $\{Q_i\}_{i=1}^l$.

Lemma 2.3.1. *Let Y be an $n\times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ and $W_1, W_2, \ldots, W_l$ be symmetric matrices of order n. Then the joint moment generating function $M(\mathbf{s})$ of matrix quadratic expressions $Q_1, Q_2, \ldots, Q_l$ is given by*

$$M(\mathbf{s}) = |I_{np} - 2\Sigma^*|^{-1/2} exp\{< \mathbf{s}, \boldsymbol{\lambda} > +2 < \boldsymbol{\mu}^*, \Sigma_Y^{1/2}(I_{np} - 2\Sigma^*)^{-1}\Sigma_Y^{1/2}\boldsymbol{\mu}^* >\} \quad (2.17)$$

where $\mathbb{S} = \mathbb{S}_p \times \mathbb{S}_p \times \cdots \times \mathbb{S}_p$ (l times), $\mathbf{s} = (\mathbf{s}_i) \in \mathbb{S}$, $\Sigma^* = \Sigma_Y^{1/2}[\sum_{i=1}^l (W_i \otimes \mathbf{s}_i)]\Sigma_Y^{1/2}$, $\boldsymbol{\mu}^* = \sum_{i=1}^l vec(W_i\boldsymbol{\mu}\mathbf{s}_i + B_i\mathbf{s}_i)$, $\boldsymbol{\lambda}_i = \boldsymbol{\mu}'W_i\boldsymbol{\mu} + B_i'\boldsymbol{\mu} + \boldsymbol{\mu}'B_i + C_i \in \mathbb{S}_p$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_i) \in \mathbb{S}$ and $sr(\Sigma^*) < \frac{1}{2}$.

Proof. Let $Q = (Q_i)$, then for $\mathbf{s} \in \mathbb{S}_p$,

$$\begin{aligned} < \mathbf{s}, Q > = < (\mathbf{s}_i), (Q_i) > &= \sum_{i=1}^l < \mathbf{s}_i, Q_i > \\ &= \sum_{i=1}^l \{< \mathbf{s}_i, Y'W_iY > + < \mathbf{s}_i, B_i'Y > + < \mathbf{s}_i, Y'B_i > + < \mathbf{s}_i, C_i >\} \\ &= \sum_{i=1}^l \{< Y, W_iY\mathbf{s}_i > +2 < Y, B_i\mathbf{s}_i > + < \mathbf{s}_i, C_i >\} \\ &= \sum_{i=1}^l \{< vec(Y), (W_i \otimes \mathbf{s}_i)vec(Y) > +2 < vec(Y), vec(B_i\mathbf{s}_i) > + < \mathbf{s}_i, C_i >\}. \end{aligned} \quad (2.18)$$

Let $vec(Z) \sim \mathcal{N}_{np}(\mathbf{0}, I)$, then $vec(Y) = vec(\boldsymbol{\mu}) + \Sigma_Y vec(Z) \sim \mathcal{N}_{np}(vec(\boldsymbol{\mu}), \Sigma_Y)$ and (2.18) becomes

$$< \mathbf{s}, Q > = < vec(Z), \Sigma^* vec(Z) > + 2 < vec(Z), \Sigma_Y^{1/2} \boldsymbol{\mu}^* > + < \mathbf{s}, \boldsymbol{\lambda} > .$$

Thus by (2.16)

$$\begin{aligned} M(\mathbf{s}) =& E(exp\{< \mathbf{s}, Q >\} \\ =& \frac{1}{(2\pi)^{np/2}} \int_{\Re^{np}} exp\{-\frac{1}{2} < vec(\mathbf{z}), vec(\mathbf{z}) > + < vec(\mathbf{z}), \Sigma^* vec(\mathbf{z}) > \\ & + 2 < vec(\mathbf{z}), \Sigma_Y^{1/2} \boldsymbol{\mu}^* > + < \mathbf{s}, \boldsymbol{\lambda} >\} d\mathbf{z} \\ =& \frac{1}{(2\pi)^{np/2}} \int_{\Re^{np}} exp\{-\frac{1}{2} < vec(\mathbf{z}) - \boldsymbol{\alpha}, (I_{np} - 2\Sigma^*)(vec(\mathbf{z}) - \boldsymbol{\alpha}) > \\ & + < \mathbf{s}, \boldsymbol{\lambda} > + \frac{1}{2} < (I_{np} - 2\Sigma^*)\boldsymbol{\alpha}, \boldsymbol{\alpha} >\} d\mathbf{z}. \end{aligned}$$

where $\boldsymbol{\alpha} = 2(I_{np} - 2\Sigma^*)^{-1} \Sigma_Y^{1/2} \boldsymbol{\mu}^*$. Therefore, for $sr(\Sigma^*) < 1/2$,

$$\begin{aligned} M(\mathbf{s}) =& |I_{np} - 2\Sigma^*|^{-1/2} exp\{< \mathbf{s}, \boldsymbol{\lambda} > + \frac{1}{2} < (I_{np} - 2\Sigma^*)\boldsymbol{\alpha}, \boldsymbol{\alpha} >\} \\ =& |I_{np} - 2\Sigma^*|^{-1/2} exp\{< \mathbf{s}, \boldsymbol{\lambda} > + 2 < \boldsymbol{\mu}^*, \Sigma_Y^{1/2} |I_{np} - 2\Sigma^*|^{-1} \Sigma_Y^{1/2} \boldsymbol{\mu}^* >\}. \end{aligned}$$

□

Let us discuss the moment generating functions of some special and useful matrix quadratic forms. The following two corollaries are the immediate consequences from Lemma 2.3.1.

Corollary 2.3.2. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\boldsymbol{\mu}, I_n \otimes \Sigma)$, then, the moment generating function $M(\mathbf{s})$ of $Y'Y$ is given by*

$$M(\mathbf{s}) = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-n/2} exp\{< \mathbf{s}, \boldsymbol{\lambda} > + 2 < \boldsymbol{\lambda}, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >\} \tag{2.19}$$

for all $\mathbf{s} \in \mathbb{S}_p$ *such that* $sr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) < 1/2$ *with* $\boldsymbol{\lambda} = \boldsymbol{\mu}'\boldsymbol{\mu}$.

Proof. In Lemma 2.3.1, taking $\Sigma_Y = I_n \otimes \Sigma$, $l = 1$, $W_1 = I_n$ and $\mathbf{s} = \mathbf{s}_1 \in \mathbb{S}_p$, for all $\mathbf{s} \in \mathbb{S}_p$ such that $sr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) < 1/2$, we obtain from (2.5)

$$\Sigma^* = \Sigma_Y^{1/2}(W_1 \otimes \mathbf{s}_1)\Sigma_Y^{1/2} = (I_n \otimes \Sigma^{1/2})(I_n \otimes \mathbf{s})(I_n \otimes \Sigma^{1/2}) = I_n \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}.$$

So

$$\begin{aligned}|I_{np} - 2\Sigma^*|^{-1/2} =&|I_n \otimes I_p - 2I_n \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-1/2}\\ =&(|I_n|^p|I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^n)^{-1/2} = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-n/2},\end{aligned}$$

and

$$\begin{aligned}\Sigma_Y^{1/2}(I_{np} - 2\Sigma^*)^{-1}\Sigma_Y^{1/2} =&(I_n \otimes \Sigma^{1/2})(I_{np} - 2I_p \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}(I_n \otimes \Sigma^{1/2})\\ =&I_n \otimes [\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}].\end{aligned}$$

Thus

$$\begin{aligned}< \boldsymbol{\mu}^*, \Sigma_Y^{1/2}(I_{np} - 2\Sigma^*)^{-1}\Sigma_Y^{1/2}\boldsymbol{\mu}^* >=& < vec(\boldsymbol{\mu}\mathbf{s}),\ vec(\boldsymbol{\mu}\mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}) >\\ =& < \boldsymbol{\mu}'\boldsymbol{\mu},\ \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >,\end{aligned}$$

implying that (2.19) holds. □

By Definition 2.2.2, (2.19) is the moment generating function of a random matrix distributed as the noncentral Wishart distribution $\mathcal{W}_p(n, \Sigma, \boldsymbol{\lambda})$ with $\boldsymbol{\lambda} = \boldsymbol{\mu}'\boldsymbol{\mu}$. For convenience, we can use (2.19) to extend $\mathcal{W}_p(n, \Sigma, \boldsymbol{\lambda})$ so that the case $n = 0$ or $\Sigma = \mathbf{0}$ is included.

Corollary 2.3.3. *Let* $\mathbf{y}$ *be a* $p \times 1$ *random vector normally distributed as* $\mathcal{N}_p(\mathbf{0}, \Sigma)$. *Then the moment generating function* $M(\mathbf{s})$ *of* $\mathbf{y}\mathbf{y}'$ *is given by*

$$M(\mathbf{s}) = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-1/2}$$

for all $\mathbf{s} \in \mathbb{S}_p$ *such that* $sr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) < 1/2$.

The following corollary follows from Lemma 2.3.1, Corollary 2.3.2 and independence.

Corollary 2.3.4. *Let Q_1 and Q_2 be independent symmetric matrices of order p distributed, respectively, as $\mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$ and $\mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$. Then the moment generating function $M(\mathbf{s})$ of $Q = Q_1 - Q_2$ is given by*

$$M(\mathbf{s}) = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_1/2}|I_p + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_2/2} exp\{< \mathbf{s}, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 > +2\Phi_1 + 2\Phi_2\} \tag{2.20}$$

for all $\mathbf{s} \in \mathbb{S}_p$ such that $sr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) < 1/2$, where $\Phi_1 =< \boldsymbol{\lambda}_1, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}$ $\Sigma^{1/2}\mathbf{s} >$ and $\Phi_2 =< \boldsymbol{\lambda}_2, \mathbf{s}\Sigma^{1/2}(I_p + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >$.

In fact, (2.20) is the moment generating function of a random matrix distributed as a difference of two independent noncentral Wishart random matrices with $\mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$ and $\mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$. For convenience, we can use (2.20) to extend $\mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$-$\mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ so that the case $m_1 = 0$ or m_2 or $\Sigma = \mathbf{0}$ is included.

The following lemma is useful in studying a difference of two independent Wishart random matrices. It can be obtained by imitating the proof of Theorem 2.3 in Wong *et al.* (1991).

Lemma 2.3.5. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and $\Sigma \in \mathbb{N}_p$. Then the following statements are equivalent propositions.*

(a) $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$;

(b) For any $\mathbf{s} \in \mathbb{S}_p$,

$$|I_{np} - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_1}|I_p + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_2};$$

(c) The matrix $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ *and the diagonal matrix* $diag[I_{m_1} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, -I_{m_2} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, \mathbf{0}] \in \mathbb{S}_{np}$ *have the same characteristic polynomial for all* $\mathbf{s} \in \mathbb{S}_p$*; and*

(d) For any positive integer k *and any* $\mathbf{s} \in \mathbb{S}_p$*,*

$$tr(\Sigma_Y(W \otimes \mathbf{s}))^k = [m_1 + (-1)^k m_2] tr(\Sigma \mathbf{s})^k.$$

Proof. By Corollary 2.3.4 and analytic continuation, (a) and (b) are equivalent. Note that (b) amounts to

(b') $|I_{np} - \Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |I_p - \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_1} |I_p + \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_2}$.

Replacing $\mathbf{s}$ with $\mathbf{s}/\lambda$ ($\lambda \in \mathbb{R}$) in (b'), we have

$$|\lambda I_{np} - \Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |\lambda I_p - \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_1} |\lambda I_p + \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_2} |\lambda I_{(n-m_1-m_2)p} - \mathbf{0}|,$$

implying that (c) holds and vice versa. (c) means that $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ and $diag[I_{m_1} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, -I_{m_2} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, \mathbf{0}]$ in $\mathbb{S}_{np}$ have the same spectrum $\{\lambda_j\}_{j=1}^{np}$, equivalently, for any positive integer k and any $\mathbf{s} \in \mathbb{S}_p$,

$$tr\left(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right)^k = tr\left(diag[I_{m_1} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, -I_{m_2} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, \mathbf{0}]\right)^k$$

namely,

$$\begin{aligned} tr\left(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right)^k &= tr\left(diag[I_{m_1} \otimes (\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^k, -I_{m_2} \otimes (\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^k, \mathbf{0}]\right) \\ &= [tr(I_{m_1}) + (-1)^k tr(I_{m_2})] tr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^k, \end{aligned}$$

which proves the equivalence between (c) and (d). □

Chapter 3

A Multivariate Version of Cochran's Theorem on Noncentral Wishartness and Independence

Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. The expression "general covariance" Σ_Y of Y implies that the collection of all np elements in Y has an arbitrary $np \times np$ covariance matrix. For a set of nonzero symmetric matrices W_1, W_2, $\ldots$, W_l of order n, we shall discuss necessary and sufficient conditions for matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ to be an independent family of noncentral Wishart random matrices with some integers m_1, m_2, $\ldots$, m_l and non-centrality matrices $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_1$, $\ldots$, $\boldsymbol{\lambda}_l$.

For the symmetric matrix W of order n, a set of necessary and sufficient conditions is derived for the noncentral Wishartness of matrix quadratic form $Y'WY$ (Theorem 3.3.1) in Section 3.3. For a set of symmetric matrices W_1, W_2, $\ldots$, W_l of order n, a set of necessary and sufficient conditions is obtained for matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of noncentral Wishart random matrices (Theorem 3.4.2) in Section 3.4. An example and the usual versions of Cochran's theorem are presented as the special cases of our result.

In addition, as the intermediate result, we use a matrix approach to present the proven result for the mean $\mathbf{0}$ case. This case has been solved by Masaro and Wong (2004a). They used Jordan algebra representations to obtain a general multivariate version of Cochran's theorem concerning Wishartness and independence. Their result and proof is more mathematically involved. In our presentation, we provide a discrete representation version of Cochran's theorem in Section 3.1-3.2. For details, in Section 3.1, we shall establish a set of succinct necessary and sufficient conditions, in terms of verifiable matrix equations, for the matrix quadratic form $Y'WY$ with the symmetric matrix W to have a Wishart distribution. In Section 3.2, we shall develop a set of succinct necessary and sufficient conditions for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ with the symmetric matrices W_1, W_2, ..., W_l to be an independent family of Wishart random matrices. In addition some examples and applications or corollaries are discussed.

3.1 Wishartness of a Matrix Quadratic Form (MQF)

First let us consider the simple case where the covariance Σ of the Wishart distribution is a diagonal matrix. In this dissertation, without a special claim, we shall use Λ to denote a diagonal matrix of order p in the following form.

$$\Lambda \equiv diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0], \; r = r(\Lambda), \; \sigma_i > 0 \; (i = 1, 2, \ldots, r).$$

The following theorem provides us with the necessary and sufficient conditions for a matrix quadratic form to have a Wishart distribution $\mathcal{W}_p(m, \Lambda)$.

Theorem 3.1.1. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n \times p}(\mathbf{0}, \Sigma_Y)$ with general covariance Σ_Y and W be a symmetric matrix of order n. Then the matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Lambda)$ for a nonnegative integer m if and only if there exists a Λ in $\mathbb{N}_p$ such that for any elements $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathbb{E}_p$,*

$$\Sigma_Y[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})]\Sigma_Y = F(\mathbf{t}, \tilde{\mathbf{t}}, W, \Sigma_Y) + F(\tilde{\mathbf{t}}, \mathbf{t}, W, \Sigma_Y) \tag{3.1}$$

where $F(\mathbf{t}, \tilde{\mathbf{t}}, W, \Sigma_Y) = \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y(W \otimes \tilde{\mathbf{t}})\Sigma_Y$ with

$$\{\mathbf{t} : \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \mathbf{0}, \; \mathbf{t} \in \mathbb{E}_p\} = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}, \; \mathbf{t} \in \mathbb{E}_p\} \tag{3.2}$$

and

$$m = tr(\Sigma_Y(W \otimes \Lambda^+))/r(\Lambda). \tag{3.3}$$

Proof. By Lemma 2.1.1, decompose $\Sigma_{Y'}$ as

$$\Sigma_{Y'} = L'L, \quad L = [L_1, L_2, \ldots, L_p]$$

with $L_i \in \mathbb{M}_{q \times n}$ $(i = 1, 2, \ldots, p)$ and $r(\Sigma_{Y'}) \leq q \leq np$.

Let

$$B_{ij} = (L_i W L_j' + L_j W L_i')/2\sqrt{\sigma_i \sigma_j}, \; i, j \leq r.$$

Then Theorem 2.1 of Masaro and Wong (2003) tells us that (3.1)-(3.3) are equivalent to the following conditions:

(A1) $r(L_i W L_i') = m > 0$ $(i = 1, 2, \ldots, r)$;

(A2) $tr(L_i W L_i') = m\sigma_i$ $(i = 1, 2, \ldots, r)$;

(A3) $L_i W L_j' + L_j W L_i' = \mathbf{0}$ for i or $j > r$;

(B1) $B_{ii}^2 = B_{ii}$;

(B2) $4B_{ij}^2 = B_{ii} + B_{jj}$ $i \neq j$;

(B3) $B_{ii}B_{jj} = \mathbf{0}$ $i \neq j$;

(B4) $B_{ii}B_{ij} + B_{ij}B_{ii} = B_{ij}$ $i \neq j$; and

(B5) $B_{ij} = 2(B_{ik}B_{jk} + B_{jk}B_{ik})$ for distinct i, j, k.

Note that from (B1)-(B5), we also obtain

$$B_{ij} = B_{ii}B_{ij}B_{jj} + B_{jj}B_{ij}B_{ii}, \quad i \neq j \tag{3.4}$$

and then

$$B_{ii}B_{jk} = \mathbf{0}, \qquad B_{ij}B_{kl} = \mathbf{0} \quad \text{for distinct } i, j, k, l. \tag{3.5}$$

First of all, suppose conditions (A1)-(A3) and (B1)-(B5) hold. We shall show that (3.1)-(3.3) hold.

For convenience, we shall use the 4-dimensional subscript to represent a pair of elements in the basic base $\mathbb{E}_p$. For example, if $\mathbf{t} = E_{ii}$ and $\tilde{\mathbf{t}} = E_{ij}$, $1 \leq i < j \leq r$, we use (ii, ij) to represent $(\mathbf{t}, \tilde{\mathbf{t}})$. By the structure of Λ and (A3), we only need to consider these elements E_{ij}, $1 \leq i \leq j \leq r$, in the basic base $\mathbb{E}_p$. Then we divided all 4-dimensional subscripts from these elements E_{ij}, $1 \leq i \leq j \leq r$ into the following seven classes. Let

$$C_1 = \{(ii, ii) : 1 \le i \le r\},$$

$$C_2 = \{(ij, ij) : 1 \le i < j \le r\},$$

$$C_3 = \{(ii, jj) : 1 \le i, j \le r; i \ne j\},$$

$$C_4 = \{(ii, ij) \cup (ij, ii) : 1 \le i < j \le r\},$$

$$C_5 = \{(ik, jk) : 1 \le i, j < k \le r; i, j \text{ distinct}\},$$

$$C_6 = \{(ii, i'j') \cup (i'j', ii) : 1 \le i, i' < j' \le r; i, i', j' \text{ distinct}\}; \text{ and}$$

$$C_7 = \{(ij, i'j') : 1 \le i < j \le r, 1 \le i' < j' \le r; i, j, i', j' \text{ distinct}\}.$$

Then

$$\bigcup_{i=1}^{7} C_i = \{(ij, i'j') : 1 \le i \le j \le r, 1 \le i' \le j' \le r\}.$$

Write $\Omega = \{E_{ij} : 1 \le i \le j \le r\}$. Then $\Omega = \cup_{i=1}^{7} C_i$. So any 4-dimensional subscript $(ij, i'j')$ must be the element of one and only one set of C_1, C_2, ..., C_7.

From Lemma 2.1.6, to prove (3.1), it is equivalent to show that for any pair of elements in the basic base $\mathbb{E}_p$, we have

$$L[(\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t}) \otimes W]L' = \Theta(\mathbf{t}, \tilde{\mathbf{t}}, W, L) + \Theta(\tilde{\mathbf{t}}, \mathbf{t}, W, L). \tag{3.6}$$

Eq. (3.6) follows from (B1)-(B5) and (3.4) with simple matrix calculations.

Exactly as in the proof of Lemma 2.1.6, (3.2) is equivalent to Eq. (3.7). So, proving (3.2) is equivalent to showing that for any element $\mathbf{t}$ in the basic base $\mathbb{E}_p$,

$$\{\mathbf{t} : L(\mathbf{t} \otimes W)L' = \mathbf{0}\} = \mathbb{K}_0 \tag{3.7}$$

where $\mathbb{K}_0 = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\}$.

Since

$$\mathbb{K}_0 = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\} = \{E_{ij} : \ i \text{ or } j > r\}$$

and from (A3)

$$\{\mathbf{t} : \ L(\mathbf{t} \otimes W)L' = \mathbf{0}\} = \{E_{ij} : \ i \text{ or } j > r\},$$

This means that Eq. (3.7) holds.

By (2.8), (2.9), (A1) and (A2),

$$tr(\Sigma_Y(W \otimes \Lambda^+)) = tr(L(\Lambda^+ \otimes W)L') = tr(Ldiag[\frac{1}{\sigma_1} \otimes W, \frac{1}{\sigma_2} \otimes W, \ldots, \frac{1}{\sigma_l} \otimes W]L')$$
$$= \sum_{i=1}^{r} tr(L_i W L_i')/\sigma_i = rm,$$

which proves (3.3).

Next, suppose (3.1), (3.2) and (3.3) hold, we shall show that (A1)-(A3) and (B1)-(B5) hold.

Taking $(ij, i'j') \in C_1$, the left side value of (3.6) is

$$2L((E_{ii}\Lambda E_{ii}) \otimes W)L' = 2\sigma_i^2 B_{ii},$$

while the right side value of (3.6) is $2\sigma_i^2 B_{ii} B_{ii}$. Eq. (3.6) implies that (B1)holds.

Taking $(ij, i'j') \in C_2$, the left side value of of (3.6) is

$$2L((E_{ij}\Lambda E_{ij}) \otimes W)L' = 2\sigma_i\sigma_j(B_{ii} + B_{jj}),$$

while the right side value of (3.6) is $8\sigma_i\sigma_j B_{ij} B_{ij}$. Eq. (3.6) means that (B2) holds.

Taking $(ij, i'j') \in C_3$, the left side of (3.6) is

$$L((E_{ii}\Lambda E_{jj} + E_{jj}\Lambda E_{ii}) \otimes W)L' = \mathbf{0},$$

while the right side of (3.6) is

$$\sigma_i\sigma_j(B_{ii}B_{jj} + B_{jj}B_{ii}).$$

Eq. (3.6) implies that $B_{ii}B_{jj} = -B_{ii}B_{jj}$, or $B_{ii}B_{jj}$ is skew-symmetric. Then

$$\|B_{ii}B_{jj}\| =< B_{ii}B_{jj}, B_{ii}B_{jj} >= tr(B_{ii}B_{jj}(B_{ii}B_{jj})') = tr(B_{ii}B_{jj}) = 0,$$

so $B_{ii}B_{jj} = \mathbf{0}$, thus (B3) holds.

Taking $(ij, i'j') \in C_4$, the left side of Eq. (3.6) is

$$L(E_{ii}\Lambda E_{ij} \otimes W)L' + L(E_{ij}\Lambda E_{ii} \otimes W)L' = 2\sqrt{\sigma_i\sigma_j}\sigma_i B_{ij},$$

while the right side of Eq. (3.6) is

$$2\sqrt{\sigma_i\sigma_j}\sigma_i(B_{ii}B_{ij} + B_{ij}B_{ii}).$$

(B4) follows from Eq. (3.6) and equivalently from (3.1).

Taking $(ij, i'j') \in C_5$, the left side of Eq. (3.6) is

$$L(E_{ik}\Lambda E_{jk} \otimes W)L' + L(E_{jk}\Lambda E_{ik} \otimes W)L' = 2\sqrt{\sigma_i\sigma_j}\sigma_k B_{ij},$$

while the right side of Eq. (3.6) is

$$4\sqrt{\sigma_i\sigma_j}\sigma_k(B_{ik}B_{jk} + B_{jk}B_{ik}).$$

Eq. (3.6) implies that (B5) holds.

As above discussed, (B1)-(B5) follow from (3.1).

By eq. (3.7),

$$\{\mathbf{t} : L(\mathbf{t} \otimes W)L' = \mathbf{0}\} \cap \mathbb{E}_p = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\} \cap \mathbb{E}_p = \{E_{ij} : \ i \text{ or } j > r\},$$

that implies

$$L(E_{ij} \otimes W)L' = \mathbf{0} \text{ for } i \text{ or } j > r, \text{ i.e. } L_i W L_j' + L_j W L_i' = \mathbf{0} \text{ for } i \text{ or } j > r.$$

So (A3) follows.

Let

$$\mathbb{S}_a = \{S \equiv L(\mathbf{s}^* \otimes W)L' : \mathbf{s}^* = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathbf{a} \in \mathbb{S}_r\},$$

then we define the operation $\circ$ on $\mathbb{S}_a$ as

$$S_1 \circ S_2 = \frac{1}{2}(S_1 S_2 + S_2 S_1) \text{ for any } S_1,\ S_2 \in \mathbb{S}_a.$$

Note that $B_{ij} = L(\frac{1}{\sqrt{\sigma_i \sigma_j}} E_{ij} \otimes W)L' \in \mathbb{S}_a$ $(1 \leq i < j \leq r)$. By (3.1), the set $\mathbb{S}_a$ is closed under the operation $\circ$. From the above proofs of (B1)-(B5), we have obtained these facts that under the operation $\circ$, $\{B_{ij} : 1 \leq i < j \leq r\}$ is a basis of the set $\mathbb{S}_a$, only if $B_{11}, B_{22}, \ldots, B_{rr}$ and $B_{i_1 i_1} + B_{i_2 i_2} + \cdots + B_{i_k i_k}$ $(\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, r\})$ are idempotent elements of $\mathbb{S}_a$. Moreover, B_{11}, B_{22}, $\ldots$, B_{rr} are nonzero, orthogonal. They can not be written as the sum of two non-zero orthogonal idempotent elements of $\mathbb{S}_a$. So each of B_{11}, B_{22}, $\ldots$, B_{rr} is a primitive idempotent of $\mathbb{S}_a$ and therefore each of them has same rank, say $(c > 0)$, i.e. $r(B_{ii}) = r(L_i W L_i') = c$, $i = 1, 2, \ldots, r$, (see for example, Jacobson (1968)). Moreover, by (3.3),

$$m = tr(\Sigma_Y (W \otimes \Lambda^+))/r(\Lambda) = tr(L(\Lambda^+ \otimes W)L')/r = \sum_{i=1}^{r} tr(B_{ii})/r = \sum_{i=1}^{r} r(B_{ii})/r = c,$$

which proves condition (A1).

And since

$$tr(L_i W L_i') = \sigma_i tr(B_{ii}) = \sigma_i r(B_{ii}) = \sigma_i r(L_i W L_i') = \sigma_i m,$$

(A2) holds and, therefore, the proof is complete. □

Note that (3.1) and (3.2) determine the Wishartness of the matrix quadratic forms $Y'WY$ while (3.3) determines its m degrees of freedom if $Y'WY$ has a Wishart distribution.

We provide an example to illustrate the application of Theorem 3.1.1. This example was discussed in Masaro and Wong (2003).

Examples 3.1.2. *Let* $Y = (Y_{ij})_{3\times 2} \sim \mathcal{N}_{3\times 2}(\mathbf{0}, \Sigma_Y)$ *with*

$$\Sigma_Y = \begin{bmatrix} I_2 & \mathbf{0}_{2\times 2} & A \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ A' & \mathbf{0}_{2\times 2} & I_2 \end{bmatrix} \quad \textit{where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$W = \begin{bmatrix} 1 & a & 0 \\ a & b & c \\ 0 & c & 0 \end{bmatrix}, \quad a, b, c \in \mathbb{R}.$$

Then, we discuss the Wishartness of the matrix quadratic form $Y'WY$ *and determine its degrees of freedom if* $Y'WY$ *has a Wishart distribution.*

Proof. The basic base is

$$\mathbb{E}_2 = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

For the diagonal matrix $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, to determine the Wishartness of the matrix quadratic form $Y'WY$, by Theorem 3.1.1, it suffices to verify (3.1)-(3.2).

Taking $(\mathbf{t}, \tilde{\mathbf{t}}) = (E_{11}, E_{12})$, we have

$$\Sigma_Y[W \otimes (E_{11}\Lambda E_{12} + E_{12}\Lambda E_{11})]\Sigma_Y = \begin{bmatrix} E_{12} & \mathbf{0} & \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$F(E_{11}, E_{12}, W, \Sigma_Y) + F(E_{12}, E_{11}, W, \Sigma_Y) = \begin{bmatrix} E_{12} & \mathbf{0} & \mathbf{0} & B \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

So (3.1) holds for $(\mathbf{t}, \tilde{\mathbf{t}}) = (E_{11}, E_{12})$. Similarly, when $(\mathbf{t}, \tilde{\mathbf{t}}) = (E_{11}, E_{11})$, (E_{12}, E_{12}), (E_{11}, E_{22}), (E_{12}, E_{22}), (E_{22}, E_{22}), respectively, (3.1) also holds. Here we can use a Matlab programming to do these computations.

Obviously, $\mathbb{K}_0 = \emptyset$. Let $\mathbf{t} = \begin{bmatrix} t_1 & t_3 \\ t_3 & t_2 \end{bmatrix} \in \mathbb{H}_p$. Then $\Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \mathbf{0}$, or

$$\begin{bmatrix} t_1 & t_3 & at_1 + ct_3 & at_3 + ct_2 & 0 & 0 \\ t_3 & t_2 & at_3 & at_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ct_1 & ct_3 & 0 & 0 \\ t_1 & t_3 & at_1 + ct_3 & at_3 + ct_2 & 0 & 0 \end{bmatrix} = \mathbf{0},$$

i.e. $t_1 = t_2 = t_3 = 0$. We obtain

$$\{\mathbf{t} : \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\} = \emptyset,$$

which implies that (3.2) holds.

So (3.1) and (3.2) imply that the matrix quadratic form $Y'WY$ has a Wishart distribution.

Finally, its degrees of freedom are given by (3.3). We have

$$m = \frac{1}{2}tr\left(\begin{bmatrix} I_2 & \mathbf{0} & A \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A' & \mathbf{0} & I_2 \end{bmatrix}\begin{bmatrix} I_2 & aI_2 & \mathbf{0} \\ aI_2 & bI_2 & cI_2 \\ \mathbf{0} & cI_2 & \mathbf{0} \end{bmatrix}\right)$$

$$= \frac{1}{2}tr\left(\begin{bmatrix} I_2 & aI_2 + cA & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A' & aA' + cI_2 & \mathbf{0} \end{bmatrix}\right) = \frac{1}{2}tr(I_2) = 1.$$

Hence, it follows from Theorem 3.1.1 that $Y'WY \sim \mathcal{W}_2(m, I_2)$ with $m = 1$. □

Now we shall discuss the general case, where the covariance Σ of the Wishart distribution $W_p(m, \Sigma)$ is a general nonnegative definite matrix of order p.

Theorem 3.1.3. *Let Y be an $n\times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W be a symmetric matrix of order n. Then the matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$ for a nonnegative integer m if and only if there exists a matrix Σ in $\mathbb{N}_p$ such that for any elements $\mathbf{h}, \tilde{\mathbf{h}}$ in the similar base $\mathbb{H}_p$ associated with Σ,*

$$\Sigma_Y[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})]\Sigma_Y = F(\mathbf{h}, \tilde{\mathbf{h}}, W, \Sigma_Y) + F(\tilde{\mathbf{h}}, \mathbf{h}, W, \Sigma_Y) \quad (3.8)$$

with

$$\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \mathbb{K} \quad (3.9)$$

where $\mathbb{K} \equiv \{\mathbf{h} :\ \Sigma\mathbf{h}\Sigma = \mathbf{0},\ \mathbf{h} \in \mathbb{H}_p\}$ *and*

$$m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma). \tag{3.10}$$

Proof. Since $\Sigma \in \mathbb{N}_p$, by lemma 2.1.1, there is an orthogonal matrix H of order p such that $H'H = I_p$ and

$$H'\Sigma H = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0] \equiv \Lambda,\ r = r(\Sigma),\ \sigma_i > 0,\ i = 1, 2, \ldots, r.$$

And $YH \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_{YH})$, where $\Sigma_{YH} = (I \otimes H')\Sigma_Y(I \otimes H)$, follows from Lemma 2.1.2 and $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$.

Let

$$\mathbf{t} = H'\mathbf{h}H \text{ for any } \mathbf{h} \in \mathbb{H}_p.$$

The function $\mathbf{t} = H'\mathbf{h}H$ is a one to one map from the similar base $\mathbb{H}_p$ associated with Σ onto the basic base $\mathbb{E}_p$. By replacing $\mathbf{h}$, $\tilde{\mathbf{h}}$, Σ and Σ_Y, respectively, with $H\mathbf{t}H'$, $H\tilde{\mathbf{t}}H'$, $H\Lambda H'$ and $(I\otimes H)\Sigma_{YH}(I\otimes H')$ in (3.8)-(3.10), we obtain that for any elements $\mathbf{t}$ and $\tilde{\mathbf{t}}$ in the basic base $\mathbb{E}_p$,

$$\Sigma_{YH}[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})]\Sigma_{YH} = F(\mathbf{t}, \tilde{\mathbf{t}}, W, \Sigma_{YH}) + F(\tilde{\mathbf{t}}, \mathbf{t}, W, \Sigma_{YH}), \tag{3.11}$$

$$\{\mathbf{t} :\ \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \mathbf{0}\} = \mathbb{K}_0 \text{ and} \tag{3.12}$$

$$m = tr(\Sigma_{YH}(W \otimes \Lambda^+))/r(\Lambda). \tag{3.13}$$

By Theorem 3.1.1, (3.11)-(3.13) are the necessary and sufficient conditions for $H'Y'WYH$ to have a Wishart distribution $W_p(m, \Lambda)$. So $Y'WY \sim W_p(m, \Sigma)$ follows from Lemma 2.2.1. The equivalence between (3.11)-(3.13) and (3.8)-(3.10) tells us that the converse holds as well. □

Remark 3.1.4. *Note that, given the covariance matrix Σ, (3.8) and (3.9) determine the Wishartness of the matrix quadratic form $Y'WY$ while (3.10) determines its degrees of freedom if $Y'WY$ has a Wishart distribution.*

Theorem 3.1.3 is an important result of this chapter. Now let us discuss an example and some corollaries as special cases of Theorem 3.1.3.

Examples 3.1.5. *Let $Y = (Y_{ij})_{3\times 2} \sim \mathcal{N}_{3\times 2}(0, \Sigma_Y)$ with*

$$\Sigma_Y = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix} \quad \text{where } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and } B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and

$$W = \begin{bmatrix} 4 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \in \mathbb{S}_3,$$

then, we discuss the Wishartness of the matrix quadratic form $Y'WY$ and determine its degrees of freedom if $Y'WY$ has a Wishart distribution.

Proof. Consider the covariance $\Sigma = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix}$. There exists an orthogonal matrix $H = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ such that $H'\Sigma H = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. Then the similar base $\mathbb{H}_2$ associated with Σ is given by

$$\mathbb{H}_2 = \left\{ H_{11} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, H_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, H_{22} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}.$$

To determine the Wishartness of the matrix quadratic form $Y'WY$, by Theorem 3.1.3, it suffices to verify (3.8) and (3.9).

Taking $(\mathbf{h}, \tilde{\mathbf{h}}) = (H_{11}, H_{12})$, we have

$$\Sigma_Y[W \otimes (H_{11}\Lambda H_{12} + H_{12}\Lambda H_{11})]\Sigma_Y = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 2AH_{12}B \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2BH_{12}A & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$F(H_{11}, H_{12}, W, \Sigma_Y) + F(H_{12}, H_{11}, W, \Sigma_Y) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & 2AH_{12}H_{11}B \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2BH_{11}BH_{12}A & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Since

$$2AH_{12}B = 2AH_{12}H_{11}B \text{ and } 2BH_{12}A = 2BH_{11}BH_{12}A,$$

(3.8) holds for $(\mathbf{h}, \tilde{\mathbf{h}}) = (H_{11}, H_{12})$. Similarly, when

$$(\mathbf{h}, \tilde{\mathbf{h}}) = (H_{11}, H_{11}), (H_{11}, H_{22}), (H_{12}, H_{12}), (H_{12}, H_{22}), (H_{22}, H_{22}),$$

respectively, (3.8) holds for them. Here we also use a Matlab programming for this algebraic computation.

Obviously, $\mathbb{K} = \emptyset$. Let $\mathbf{h} = \begin{bmatrix} h_1 & h_3 \\ h_3 & h_2 \end{bmatrix} \in \mathbb{H}_2$. Then $\Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}$ or

$$\begin{bmatrix} 4A\mathbf{h}A & \mathbf{0} & 2A\mathbf{h}B \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2B\mathbf{h}A & \mathbf{0} & B\mathbf{h}B \end{bmatrix} = \mathbf{0},$$

i.e. $h_1 + h_2 + 2h_3 = 0,\ h_1 = h_2,\ h_1 + h_2 - 2h_3 = 0 \Rightarrow h_1 = h_2 = h_3 = 0$. It means that $\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0},\ \mathbf{h} \in \mathbb{H}_2\} = \emptyset$, which (3.9) holds.

So (3.8) and (3.9) imply that matrix quadratic form $Y'WY$ has a Wishart distribution.

Finally, its degrees of freedom are given by (3.10). We have

$$\begin{aligned} m &= tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma) \\ &= \frac{1}{2}tr\left(\begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix}\begin{bmatrix} 4\Sigma^+ & 2\Sigma^+ & 2\Sigma^+ \\ 2\Sigma^+ & -\Sigma^+ & \mathbf{0} \\ 2\Sigma^+ & \mathbf{0} & \Sigma^+ \end{bmatrix}\right) \\ &= \frac{1}{2}tr\left(\begin{bmatrix} 4A\Sigma^+ & 2A\Sigma^+ & 2A\Sigma^+ \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2B\Sigma^+ & \mathbf{0} & B\Sigma^+ \end{bmatrix}\right) = \frac{1}{2}[tr(4A\Sigma^+) + tr(B\Sigma^+)] = 1. \end{aligned}$$

Hence, it follows from Theorem 3.1.1 that the matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_2(1, \Sigma)$. □

In Theorem 3.1.3, if the covariance Σ_Y of Y is replaced with the Kronecker product $A \otimes \Sigma$ where A is a nonnegative definite matrix of order n, Theorem 3.1.3 is reduced to the following corollary which was proved by Khatri (1963) and de Gunst (1987).

Corollary 3.1.6. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, A \otimes \Sigma)$ for some $A \in \mathbb{N}_n$. Then, for $W \in \mathbb{S}_n$, the matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$ for some $m \in \{0, 1, 2, \dots\}$ if and only if*

$$AWAWA = AWA; \tag{3.14}$$

$$AWA \neq \mathbf{0}; \ \textit{and} \tag{3.15}$$

$$m = r(AW). \tag{3.16}$$

Proof. Using $A \otimes \Sigma$ to replace Σ_Y in (3.8) and (3.9), by (2.5), we obtain (3.14) and (3.15). By (2.5), (2.6) and (3.10),

$$\begin{aligned} m =& tr((A \otimes \Sigma)(W \otimes \Sigma^+))/r(\Sigma) = tr(AW \otimes \Sigma\Sigma^+)/r(\Sigma) \\ =& tr(AW)tr(\Sigma^0)/r(\Sigma) = tr(AW), \end{aligned}$$

so (3.16) holds and the desired result has been obtained. □

In Theorem 3.1.3, if we replace the covariance Σ_Y of Y with the sum of special Kronecker products, we have the following corollary which was also discussed in Masaro and Wong (2004a).

Corollary 3.1.7. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$. Suppose $\Sigma_Y = \sum_{i=1}^{r} A_i \otimes E_{ii}$, $r \leq p$, $A_i \in \mathbb{N}_n$, $i = 1, 2, \ldots, r$, and $W \in \mathbb{S}_n$. Then the matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$, where $\Sigma = \sum_{i=1}^{r} \sigma_i E_{ii}$, for some $m \in \{0, 1, 2, \ldots\}$ if and only if there exist real numbers $\sigma_k > 0$, $k = 1, 2, \ldots, r$, such that for all $i, j, k \leq r$,*

$$A_iWA_kWA_j = \sigma_k A_iWA_j; \tag{3.17}$$

$$A_iWA_j \neq \mathbf{0}; \ \textit{and} \tag{3.18}$$

$$m = \frac{1}{r}\sum_{i=1}^{r} \frac{1}{\sigma_i} tr(A_iW). \tag{3.19}$$

Proof. From (3.1), (3.2) and (3.3), replacing Σ_Y and Σ with $\sum_{i=1}^{r} A_i \otimes E_{ii}$ and $\sum_{i=1}^{r} \sigma_i E_{ii}$, respectively, we obtain that for any $\mathbf{t} = (t_{ij}), \tilde{\mathbf{t}} = (\tilde{t}_{ij}) \in \mathbb{E}_p$,

$$\sum_{i,j,k=1}^{r} (\sigma_k A_i W A_j - A_i W A_k W A_j) \otimes E_{ii}(\mathbf{t} E_{kk} \tilde{\mathbf{t}} + \tilde{\mathbf{t}} E_{kk} \mathbf{t}) E_{jj} = \mathbf{0}$$

or

$$\sum_{i,j,k=1}^{r} (\sigma_k A_i W A_j - A_i W A_k W A_j) \otimes (t_{ik}\tilde{t}_{kj} + \tilde{t}_{ik} t_{kj}) \mathbf{e_{ij}} = \mathbf{0}, \tag{3.20}$$

$$\left\{\mathbf{t} : \sum_{i,j=1}^{r} A_i W A_j \otimes E_{ii} \mathbf{t} E_{jj} = \mathbf{0}\right\} = \left\{\mathbf{t} : \sum_{i,j=1}^{r} E_{ii} \mathbf{t} E_{jj} = \mathbf{0}\right\}.$$

Namely,

$$\left\{\mathbf{t} : \sum_{i,j=1}^{r} A_i W A_j \otimes E_{ii} \mathbf{t} E_{jj} = \mathbf{0}\right\} = \left\{\mathbf{t} : \ \mathbf{t} = \begin{bmatrix} \mathbf{0}_{r \times r} & * \\ * & * \end{bmatrix} \in \mathbb{E}_p\right\} \tag{3.21}$$

and

$$m = tr\left(\sum_{i=1}^{r} A_i W \otimes E_{ii} \Sigma^+\right) / r(\Sigma) = \frac{1}{r} \sum_{i=1}^{r} tr(A_i W) tr(E_{ii} \Sigma^+). \tag{3.22}$$

Note that $t_{ik}\tilde{t}_{kj} + \tilde{t}_{ik} t_{kj}$ can take the value 0 or 1 or 2 for any $i, k, j \leq r$. So (3.20) is equivalent to (3.17) and (3.21) is equivalent to (3.18). Since

$$\Sigma^+ = \sum_{i=1}^{r} \frac{1}{\sigma_i} E_{ii},$$

(3.22) becomes

$$m = \frac{1}{r} \sum_{i=1}^{r} tr(A_i W) tr\left(E_{ii} \sum_{i=1}^{r} \frac{1}{\sigma_k} E_{kk}\right) = \frac{1}{r} \sum_{i=1}^{r} \frac{1}{\sigma_i} tr(A_i W),$$

therefore, we have completed the proof of the desired result. □

In Theorem 3.1.3, if $\mathbf{y}$ is an $n \times 1$ normal random vector with mean vector $\mathbf{0}$ and covariance C, (3.8)-(3.10) are reduced to the familiar conditions which were shown by many scholars in the sixties.

Corollary 3.1.8. *Let* $\mathbf{y}$ *be an* $n \times 1$ *random vector normally distributed as* $\mathcal{N}_n(\mathbf{0}, C)$ *and* W *be a symmetric matrix of order* n*. Then the quadratic form* $\mathbf{y}'W\mathbf{y}$ *has a Wishart distribution* $\mathcal{W}_1(m, 1)$*, that is, a chi-square distribution with* m *degrees of freedom for a nonnegative integer* m *if and only if*

$$CWCWC = CWC; \text{ and} \tag{3.23}$$

$$m = r(CW). \tag{3.24}$$

Proof. In the univariate case $p = 1$, $\Sigma_Y = C$, $\Sigma = 1$ (if $Q \sim \mathcal{W}_1(m, \sigma)$, then $Q/\sigma \sim \chi^2(m)$) and $\mathbf{h} = 1$. (3.8) is reduced to (3.23), (3.9) is reduced to an identity and (3.10) is reduced to (3.24). □

In fact, (3.23) and (3.24) imply that $m = r(CWC)$, see Styan (1970).

In Theorem 3.1.3, if the covariance Σ_Y of Y is nonsingular, Theorem 3.1.3 reduces to the following corollary.

Corollary 3.1.9. *In Theorem 3.1.3, suppose* Σ_Y *is nonsingular. Then the matrix quadratic* $Y'WY$ *follows a Wishart distribution* $\mathcal{W}_p(tr(W), \Sigma)$ *if and only if there exists a matrix* $\Sigma \in \mathbb{N}_p$ *such that*

$$W \otimes \Sigma = (W \otimes I)\Sigma_Y(W \otimes I).$$

Proof. Since Σ_Y is nonsingular, Σ must be nonsingular from (3.9). The desired condition follows from (3.8). Since

$$\begin{aligned}
&\Sigma_Y^{1/2}(W \otimes \Sigma^{-1})\Sigma_Y^{1/2}\Sigma_Y^{1/2}(W \otimes \Sigma^{-1})\Sigma_Y^{1/2} \\
&= \Sigma_Y^{1/2}(I \otimes \Sigma^{-1})(W \otimes I)\Sigma_Y(W \otimes I)(I \otimes \Sigma^{-1})\Sigma_Y^{1/2} \\
&= \Sigma_Y^{1/2}(I \otimes \Sigma^{-1})(W \otimes \Sigma)(I \otimes \Sigma^{-1})\Sigma_Y^{1/2} \\
&= \Sigma_Y^{1/2}(W \otimes \Sigma^{-1})\Sigma_Y^{1/2},
\end{aligned}$$

$\Sigma_Y^{1/2}(W \otimes \Sigma^{-1})\Sigma_Y^{1/2}$ is idempotent. So by (3.10),

$$\begin{aligned}
m &= tr(\Sigma_Y(W \otimes \Sigma^{+}))/r(\Sigma) = r(\Sigma_Y^{1/2}(W \otimes \Sigma^{-1})\Sigma_Y^{1/2})/r(\Sigma) \\
&= r(W \otimes \Sigma^{-1})/r(\Sigma) = r(W).
\end{aligned}$$

□

With some matrix operations, the following sufficient condition is easily derived from Corollary 3.1.9.

Corollary 3.1.10. *In Theorem 3.1.3, if Σ_Y is nonsingular and $Y'WY \sim \mathcal{W}_p(m, \Sigma)$, then Σ is a nonsingular covariance matrix and the W is a nonnegative definite matrix.*

Corollary 3.1.10 tells us that the algebraic conditions obtained in Theorem 3.1.3 do determine not only the distribution of a matrix quadratic form but also the property of the underlying matrix W being nonnegative definite in the case of nonsingular Σ_Y. So when Σ_Y is nonsingular and W is symmetric rather than nonnegative definite, the matrix quadratic form $Y'WY$ does not have any Wishart distribution. When W is symmetric rather than nonnegative definite and the Wishartness of $Y'WY$ holds, then Σ_Y must be a singular matrix. In addition, when Σ is singular and $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$, then Σ_Y must be a singular matrix.

If we use the set $\mathbb{S}_p$ to replace the similar base $\mathbb{H}_p$ in Theorem 3.1.3, we can easily obtain the following result.

Theorem 3.1.11. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(0, \Sigma_Y)$ and W be a symmetric matrix of order n. Then matrix quadratic form $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$ for some $m \in \{0, 1, 2, \dots\}$ if and only if there exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,*

$$\Sigma_Y(W \otimes \mathbf{s}\Sigma\mathbf{s})\Sigma_Y = F(\mathbf{s}, \mathbf{s}, W, \Sigma_Y) \tag{3.25}$$

with

$$\{\mathbf{s} : \ \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y = 0\} = \{\mathbf{s} : \ \Sigma\mathbf{s}\Sigma = \mathbf{0}\} \tag{3.26}$$

and

$$m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma).$$

Proof. By Lemma 2.1.7, (3.25) is equivalent to that for any $\mathbf{s}, \tilde{\mathbf{s}} \in \mathbb{S}_p$,

$$\Sigma_Y[W \otimes (s\Sigma\tilde{s} + \tilde{s}\Sigma s)]\Sigma_Y = F(\mathbf{s}, \tilde{\mathbf{s}}, W, \Sigma_Y) + F(\tilde{\mathbf{s}}, \mathbf{s}, W, \Sigma_Y). \tag{3.27}$$

Since it is obvious to prove Eq. (3.8) from Eq. (3.27), it suffices to show that Eq. (3.27) follows from Eq. (3.8) and condition (3.26) is equivalent to condition (3.9).

Assume that Eq. (3.8) holds. For any $\mathbf{s}$, $\tilde{\mathbf{s}}$ in set $\mathbb{S}_p$, $\mathbf{s}$ and $\tilde{\mathbf{s}}$ can be expressed as the linear combinations of $\mathbf{h}_{ij} \in \mathbb{H}_p$, $1 \le i \le j \le p$, i.e.

$$\mathbf{s} = \sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij}, \ \ s_{ij} \in \mathbb{R}$$

and

$$\tilde{\mathbf{s}} = \sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl}, \ \ \tilde{s}_{kl} \in \mathbb{R}.$$

Then we have

$$\begin{aligned}
&\Sigma_Y[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})]\Sigma_Y \\
&= \sum_{1\le i\le j\le p} \sum_{1\le k\le l\le p} s_{ij}\tilde{s}_{kl}\Sigma_Y[W \otimes (\mathbf{h}_{ij}\Sigma\mathbf{h}_{kl} + \mathbf{h}_{kl}\Sigma\mathbf{h}_{ij})]\Sigma_Y \\
&= \sum_{1\le i\le j\le p} \sum_{1\le k\le l\le p} s_{ij}\tilde{s}_{kl}[F(\mathbf{h}_{ij}, \mathbf{h}_{kl}, W, \Sigma_Y) + F(\mathbf{h}_{kl}, \mathbf{h}_{ij}, W, \Sigma_Y)] \\
&= F(\sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij}, \sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl}, W, \Sigma_Y) + F(\sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl}, \sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij}, W, \Sigma_Y) \\
&= F(\mathbf{s}, \tilde{\mathbf{s}}, W, \Sigma_Y) + F(\tilde{\mathbf{s}}, \mathbf{s}, W, \Sigma_Y),
\end{aligned}$$

that implies that Eq. (3.27) holds.

Note that (3.26) is equivalent to

$$\{\mathbf{s} : \Sigma_{YH}(W \otimes \mathbf{s})\Sigma_{YH} = 0\} = \{\mathbf{s} : \Lambda\mathbf{s}\Lambda = 0\} \tag{3.28}$$

where H is an orthogonal matrix such that

$$H'\Sigma H = \Lambda \equiv diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0], \; r = r(\Sigma), \text{ with } \sigma_i > 0, \; i = 1, 2, \ldots, r.$$

And (3.9) is equivalent to (3.12). So it suffices to show that (3.12) is equivalent to (3.28).

Suppose condition (3.12) holds. Let $\mathbb{T} = \{\mathbf{t}_{ij} \in \mathbb{E}_p : \Sigma_{YH}(W \otimes \mathbf{t}_{ij})\Sigma_{YH} \neq \mathbf{0}\}$, a subset of $\mathbb{E}_p$. Then the set $< \mathbb{T} >$ is a subset of $\mathbb{S}_p$ and, for any nonzero $\mathbf{s} \in < \mathbb{T} >$, we have

$$\Sigma_{YH}(W \otimes \mathbf{s})\Sigma_{YH} \neq 0.$$

It implies that

$$\{\mathbf{s} : \Sigma_{YH}(W \otimes \mathbf{s})\Sigma_{YH} = \mathbf{0}\} =< \{\mathbf{t} : \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \mathbf{0}, \; \mathbf{t} \in \mathbb{E}_p\} > .$$

Since

$$\{\mathbf{s}:\ \Lambda\mathbf{s}\Lambda = \mathbf{0}\} =< \{\mathbf{t}:\ \Lambda\mathbf{t}\Lambda = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\} >=< \mathbb{K}_0 >,$$

Eq. (3.28) holds from (3.12).

Conversely, suppose Eq. (3.28) holds. Since

$$\{\mathbf{t}:\ \Lambda\mathbf{t}\Lambda = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\} = \{\mathbf{s}:\ \Lambda\mathbf{s}\Lambda = \mathbf{0}\} \cap \mathbb{E}_p,$$

and

$$\{\mathbf{t}:\ \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \mathbf{0},\ \mathbf{t} \in \mathbb{E}_p\} = \{\mathbf{s}:\ \Sigma_{YH}(W \otimes \mathbf{s})\Sigma_{YH} = \mathbf{0}\} \cap \mathbb{E}_p,$$

condition (3.12) holds. So the proof is complete. □

Masaro and Wong (2004a) essentially obtained Theorem 3.1.11 as the special case of their main result by using Jordan algebra homomorphisms in their technical report. Their result was obtained for very general case and the proof is more mathematically involved. In this thesis, we consider the set $\mathbb{S}_p$ and use a matrix approach stated in this section to obtain the same result as Masaro and Wong. So, the result given in Theorem 3.1.11 has the advantage to be less mathematically involved while it gives the same result as in Masaro and Wong (2004a).

Putting Theorem 3.1.3, Theorem 3.1.11 and Corollary 2.3.1 of Wong *et al.* (1991) together, we obtain the following corollary.

Corollary 3.1.12. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W be a symmetric matrix of order n. Suppose P is an idempotent square matrix of order p. Then the following statements (a)-(g) are equivalent.*

(a) $Y'WY$ has a Wishart distribution $\mathcal{W}_p(m, \Sigma)$ with m degrees of freedom and covariance matrix Σ.

(b) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$|I_{np} - \Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |I_p - \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^m.$$

(c) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$|I_{np} - \Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |I_{np} - P \otimes (\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})|.$$

(d) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2} \text{ and } P \otimes (\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) \text{ are similar.}$$

(e) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$tr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2})^k = m\,tr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^k, \; k = 1, 2, \ldots.$$

(f) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$\Sigma_Y(W \otimes \mathbf{s}\Sigma\mathbf{s})\Sigma_Y = F(\mathbf{s}, \mathbf{s}, W, \Sigma_Y)$$

with $\{\mathbf{s} : \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y = \mathbf{0}\} = \{\mathbf{s} : \Sigma\mathbf{s}\Sigma = \mathbf{0}\}$ and $m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma)$;

(g) There exists a $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{h}, \tilde{\mathbf{h}}$ in the similar base $\mathbb{H}_p$ associated with Σ,

$$\Sigma_Y \left[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})\right] \Sigma_Y = F(\mathbf{h}, \tilde{\mathbf{h}}, W, \Sigma_Y) + F(\tilde{\mathbf{h}}, \mathbf{h}, W, \Sigma_Y)$$

with $\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \{\mathbf{h} : \Sigma\mathbf{h}\Sigma = \mathbf{0}\}$ and $m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma)$.

It is seen that (g) of Corollary 3.1.9 is easy to verify, compared to the rest.

3.2 Wishartness and Independence of MQFs

We have studied the Wishartness of a matrix quadratic form in a normal random matrix in Section 3.1. In this section we focus our attention on the Wishartness and independence of a set of matrix quadratic forms.

Before establishing our succinct and verifiable multivariate version of Cochran's Theorem, we give the following necessary and sufficient condition for the independence of a set of matrix quadratic forms. The more general result refers to Lemma 3.4.1 in Section 3.4.

Lemma 3.2.1. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and $W_1, W_2, \ldots, W_l$ be symmetric matrices of order n. Then the matrix quadratic forms $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are independent if and only if for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{t}_i, \mathbf{t}_j$ in the basic base $\mathbb{E}_p$,*

$$\Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_j \otimes \mathbf{t}_j)\Sigma_Y = \mathbf{0}. \tag{3.29}$$

Proof. Suppose the matrix quadratic forms $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are independent. For distinct i, j and any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{E}_p$, the trace inner products $< \mathbf{t}_i, Y'W_iY >$ and $< \mathbf{t}_j, Y'W_jY >$ are independent. Since

$$< \mathbf{t}_i, Y'W_iY) >=< vec(Y), (W_i \otimes \mathbf{t}_i)vec(Y) >= vec(Y)'(W_i \otimes \mathbf{t}_i)vec(Y)$$

and $vec(Y)$ has multivariate normal distribution $\mathcal{N}_{nxp}(\mathbf{0}, \Sigma_Y)$, we obtain

$$\Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_j \otimes \mathbf{t}_j)\Sigma_Y = \mathbf{0}$$

from Theorem 4s of Searle (1971), which (3.29) holds.

Conversely, suppose that for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{E}_p$, condition (3.29) holds. Then for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$, condition (3.29) still holds, namely. for any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$,

$$\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)\Sigma_Y = \mathbf{0}. \tag{3.30}$$

Let $Y'WY = (Y'W_iY)$. To show that the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ are independent, it suffices to show that

$$M_{Y'WY}(\mathbf{s}) = \prod_{i=1}^{l} M_{Y'W_iY}(\mathbf{s}_i)$$

for $\mathbf{s} = (\mathbf{s}_i)$ in $\mathcal{N}_0$ where $\mathcal{N}_0$ is a neighborhood of $\mathbf{0}$ in $\mathbb{S} = \mathbb{S}_p \times \mathbb{S}_p \times \ldots \times \mathbb{S}_p$ (l times). Now,

$$\begin{aligned} M_{Y'WY}(\mathbf{s}) &= E(exp < (\mathbf{s}_i), (Y'W_iY) >) = E\left(exp \sum_{i=1}^{l} < \mathbf{s}_i, Y'W_iY >\right) \\ &= E\left(exp < vec(Y), \sum_{i=1}^{l}(W_i \otimes \mathbf{s}_i)vec(Y) >\right) \\ &= E\left(exp < vec(Y)vec(Y)', \sum_{i=1}^{l}(W_i \otimes \mathbf{s}_i) >\right) \end{aligned}$$

By Corollary 2.3.3,

$$M_{Y'WY}(\mathbf{s}) = \left| I_n \otimes I_p - 2\Sigma_Y \left(\sum_{i=1}^{l}(W_i \otimes \mathbf{s}_i)\right) \Sigma_Y \right|^{-1/2}.$$

Then by condition (3.30),

$$(I_n \otimes I_p - 2\Sigma_Y \sum_{i=1}^{l}(W_i \otimes \mathbf{s}_i)\Sigma_Y) = \prod_{i=1}^{l}[I_n \otimes I_p - 2\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y].$$

Thus

$$M_{Y'WY}(\mathbf{s}) = \prod_{i=1}^{l} |I_n \otimes I_p - 2\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y|^{-1/2} = \prod_{i=1}^{l} M_{Y'WY}(\mathbf{s}_i)$$

and the proof is complete. □

If $\Sigma_Y = A \otimes \Sigma$ for some $A \in \mathbb{N}_n$ and $\Sigma \in \mathbb{N}_p$, Lemma 3.2.1 is reduced to the following well known result.

Corollary 3.2.2. *In Lemma 3.2.1, suppose $\Sigma_Y = A \otimes \Sigma$ for some $A \in \mathbb{N}_n$ and $\Sigma \in \mathbb{N}_p$, then the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ are independent if and only if $AW_iAW_jA = \mathbf{0}$ for all distinct $i, j \in \{1, 2, \ldots, l\}$.*

Based on Theorem 3.1.1, Theorem 3.1.3 and Lemma 3.2.1, we shall develop a succinct and verifiable multivariate version of Cochran's theorem concerning the Wishartness and independence of matrix quadratic forms in a normal random matrix Y with mean $\mathbf{0}$ and covariance matrix Σ_Y.

First, we focus our attention on the case where the common covariance of Wishart $\mathcal{W}_p(m_i, \Lambda)$ random matrices is a diagonal matrix Λ.

Theorem 3.2.3. *Suppose that Y is an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W_1, W_2, $\ldots$, W_l are symmetric matrices of order n. Then a set of matrix quadratic forms $Y'W_iY$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ is an independent family of Wishart $\mathcal{W}_p(m_i, \Lambda)$ random matrices for some $m_i \in \{0, 1, 2, \ldots\}$ if and only if there exists a $\Lambda \in \mathbb{N}_p$ such that the following statements (a) and (b) hold.*

(a) For any $i \in \{1, 2, \ldots, l\}$ and any elements $\mathbf{t}_i$ and $\tilde{\mathbf{t}}_i$ in the basic base $\mathbb{E}_p$,

$$\Sigma_Y \left[W_i \otimes (\mathbf{t}_i \Lambda \tilde{\mathbf{t}}_i + \tilde{\mathbf{t}}_i \Lambda \mathbf{t}_i)\right] \Sigma_Y = F(\mathbf{t}_i, \tilde{\mathbf{t}}_i, W_i, \Sigma_Y) + F(\tilde{\mathbf{t}}_i, \mathbf{t}_i, W_i, \Sigma_Y)$$

where $F(\mathbf{t}_i, \tilde{\mathbf{t}}_i, W_i, \Sigma_Y) = \Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_i \otimes \tilde{\mathbf{t}}_i)\Sigma_Y$ with $\{\mathbf{t}_i : \Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y = \mathbf{0}\} = \mathbb{K}_0$ and $m_i = tr(\Sigma_Y(W_i \otimes \Sigma^+))/r(\Sigma)$; and

(b) For any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_j \otimes \Lambda^+)\Sigma_Y = \mathbf{0}. \tag{3.31}$$

Proof. Let $\{Y'W_iY\}_{i=1}^{l}$ be an independent family of Wishart $\mathcal{W}_p(m_i, \Lambda)$ random matrices. Then statements (a) and (b) follow from Theorem 3.1.1 and Lemma 3.2.1.

Conversely, suppose (a) and (b) hold. For $i = 1, 2, \ldots, l$, the matrix quadratic form $Y'W_iY$ has a Wishart distribution $\mathcal{W}_p(m_i, \Sigma)$ from Theorem 3.1.1. To complete the proof, it suffices to show that condition (3.29) holds from statements (a) and (b).

Since (3.29) is equivalent to

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = \mathbf{0} \text{ where } L'L = \Sigma_{Y'} \text{ and } \mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p \tag{3.32}$$

and (3.31) amounts to

$$L(\Lambda^+ \otimes W_i)L'L(\Lambda^+ \otimes W_j)L' = \mathbf{0}, \tag{3.33}$$

we only need to prove (3.32) from statements (a) and (3.33).

For $\mathbf{s}_i$ in set $\mathbb{S}_p$, $\mathbf{s}_i$ can be written as

$$\mathbf{s}_i = \begin{bmatrix} \mathbf{a} & * \\ * & * \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathbb{S}_r.$$

Write

$$\mathbf{s}_i^* = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathbb{S}_r.$$

By (A1) of Theorem 3.1.1, for any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$,

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = L(\mathbf{s}_i^* \otimes W_i)L'L(\mathbf{s}_j^* \otimes W_j)L'. \tag{3.34}$$

By Lemma 2.1.6 and Lemma 2.1.7, we can obtain from statement (a) that for $\mathbf{s}_i^* \in \mathbb{S}_p$

$$L(\mathbf{s}_i^* \Lambda \mathbf{s}_i^* \otimes W)L' = \Theta(\mathbf{s}_i^*, \mathbf{s}_i^*, W_i, L)$$

and

$$\begin{aligned} L(\mathbf{s}_i^* \otimes W_i)L' =& L\left[\frac{1}{2}(\Lambda^+\Lambda\mathbf{s}_i^* + \mathbf{s}_i^*\Lambda\Lambda^+) \otimes W_i\right]L' \\ =& \frac{1}{2}[\Theta(\Lambda^+, \mathbf{s}_i^*, W_i, L) + \Theta(\mathbf{s}_i^*, \Lambda^+, W_i, L)] \end{aligned} \tag{3.35}$$

where $\Theta(\Lambda^+, \mathbf{s}_i^*, W_i, L) = L(\Lambda^+ \otimes W_i)L'L(\mathbf{s}_i^* \otimes W_i)L'$. In particular,

$$L(\Lambda^+ \otimes W_i)L' = \Theta(\Lambda^+, \Lambda^+, W_i, L). \tag{3.36}$$

With (3.35), (3.36) and by Lemma 2.1.4,

$$L(\mathbf{s}_i^* \otimes W_i)L'L(\Lambda^+ \otimes W_i)L' = \Theta(\Lambda^+, \mathbf{s}_i^*, W_i, L). \tag{3.37}$$

So, from (3.35) and (3.37)

$$L(\mathbf{s}_i^* \otimes W_i)L' = \Theta(\mathbf{s}_i^*, \Lambda^+, W_i, L). \tag{3.38}$$

Similarly,

$$L(\mathbf{s}_j^* \otimes W_j)L' = \Theta(\Lambda^+, \mathbf{s}_j^*, W_j, L). \tag{3.39}$$

Thus, by (3.34), (3.38) and (3.39), for any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$,

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = L(\mathbf{s}_i^* \otimes W_i)L'L(\Lambda^+ \otimes W_i)L'L(\Lambda^+ \otimes W_j)L'L(\mathbf{t}_j^* \otimes W_j)L' = \mathbf{0},$$

that completes the proof. □

Next, we shall extend Theorem 3.2.3 from the diagonal covariance Λ to the general nonnegative definite covariance Σ.

Theorem 3.2.4. *Suppose that Y has a normal distribution $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W_1, W_2, ..., W_l are symmetric matrices of order n. Then a set of matrix quadratic forms*

$Y'W_iY, Y'W_2Y, \ldots, Y'W_lY$ is an independent family of Wishart $\mathcal{W}_p(m_i, \Sigma)$ random matrices for some $m_i \in \{0, 1, 2, \ldots\}$ if and only if there exists a $\Sigma \in \mathbb{N}_p$ such that the following statements (a) and (b) hold.

(a) For any $i \in \{1, 2, \ldots, l\}$ and any elements $\mathbf{h}_i, \tilde{\mathbf{h}}_i$ in the similar base $\mathbb{H}_p$ associated with Σ,

$$\Sigma_Y \left[W_i \otimes (\mathbf{h}_i \Sigma \tilde{\mathbf{h}}_i + \tilde{\mathbf{h}}_i \Sigma \mathbf{h}_i)\right] \Sigma_Y = F(\mathbf{h}_i, \tilde{\mathbf{h}}_i, W_i, \Sigma_Y) + F_i(\tilde{\mathbf{h}}_i, \mathbf{h}_i, W_i, \Sigma_Y) \quad (3.40)$$

with

$$\{\mathbf{h}_i : \Sigma_Y (W_i \otimes \mathbf{h}_i) \Sigma_Y = \mathbf{0}\} = \mathbb{K} \quad (3.41)$$

and

$$m_i = tr(\Sigma_Y (W_i \otimes \Sigma^+))/r(\Sigma); \quad (3.42)$$

and

(b) For any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$\Sigma_Y (W_i \otimes \Sigma^+) \Sigma_Y (W_j \otimes \Sigma^+) \Sigma_Y = \mathbf{0}. \quad (3.43)$$

Proof. Since $\Sigma \in \mathbb{N}_p$, by Lemma 2.1.1, there is an orthogonal matrix H of order p such that $H'H = I_p$ and

$$H' \Sigma H = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0] \equiv \Lambda, \; r = r(\Sigma), \; \sigma_i > 0, \; i = 1, 2, \ldots, r.$$

And $YH \sim \mathcal{N}_{n \times p}(\mathbf{0}, \Sigma_{YH})$, where $\Sigma_{YH} = (I \otimes H') \Sigma_Y (I \otimes H)$, follows from Lemma 2.1.2 and $Y \sim \mathcal{N}_{n \times p}(\mathbf{0}, \Sigma_Y)$.

Defining

$$\mathbf{t}_i = H' \mathbf{h}_i H \;\text{ for any } \mathbf{h}_i \in \mathbb{H}_p, \; i = 1, 2, \ldots, l,$$

for any $i = 1, 2, \ldots, l$, the function $\mathbf{t}_i = H'\mathbf{h}_i H$ is a 1-1 map from the similar base $\mathbb{H}_p$ associated with Σ onto the basic base $\mathbb{E}_p$. By replacing $\mathbf{h}_i$, $\tilde{\mathbf{h}}_i$, Σ and Σ_Y, respectively, with $H\mathbf{t}_i H'$, $H\tilde{\mathbf{t}}_i H'$, $H\Lambda H'$ and $(I \otimes H)\Sigma_{YH}(I \otimes H')$ in (3.40)-(3.43), we obtain

$$\Sigma_{YH}\left[W \otimes (\mathbf{t}_i\Lambda\tilde{\mathbf{t}}_i + \tilde{\mathbf{t}}_i\Lambda\mathbf{t}_i)\right]\Sigma_{YH} = F(\mathbf{t}_i, \tilde{\mathbf{t}}_i, W_i, \Sigma_{YH}) + F(\tilde{\mathbf{t}}_i, \mathbf{t}_i, W_i, \Sigma_{YH}) \quad (3.44)$$

$$\{\mathbf{t}_i : \Sigma_{YH}(W \otimes \mathbf{t}_i)\Sigma_{YH} = \mathbf{0},\ \mathbf{t}_i \in \mathbb{E}_p\} = \mathbb{K}_0, \quad (3.45)$$

$$m_i = tr(\Sigma_{YH}(W \otimes \Lambda^+))/r(\Lambda) \quad (3.46)$$

and

$$\Sigma_{YH}(W_i \otimes \Lambda^+)\Sigma_{YH}(W_j \otimes \Lambda^+)\Sigma_{YH} = \mathbf{0} \quad (3.47)$$

which are equivalent to (3.40)-(3.43), respectively. By Theorem 3.2.3, (3.44)-(3.47) are the necessary and sufficient conditions for the matrix quadratic forms $H'Y'W_iYH$'s to be an independent family of Wishart $W_p(m_i, \Lambda)$ random matrices for some $m_i \in \{1, 2, \ldots\}$. By Lemma 2.2.1, (3.40)-(3.43) are the necessary and sufficient conditions for matrix quadratic forms $Y'W_iY$'s to be an independent family of Wishart $W_p(m_i, \Sigma)$ random matrices for some $m_i \in \{1, 2, \ldots\}$ and vice versa. So we have completed the proof of the desired result. □

Theorem 3.2.4 is the core result which we intend to establish for the model stated in this Chapter.

Now, let us present its special cases.

In Theorem 3.2.4, if the covariance Σ_Y of Y is replaced with the Kronecker product $A \otimes \Sigma$ where $A \in \mathbb{N}_n$, we have the following corollary which was obtained by Khatri (1963).

Corollary 3.2.5. *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, A\otimes\Sigma)$, $A\otimes\Sigma \neq \mathbf{0}$, $W_i \in \mathbb{S}_n$, $i = 1,2,\ldots,l$. *Then* $\{Y'W_iY\}_{i=1}^{l}$ *is an independent family of Wishart* $\mathcal{W}_p(m_i,\Sigma)$ *random matrices for some* $m_i \in \{0,1,2,\ldots\}$ *if and only if for distinct* $i,j \in \{1,2,\ldots,l\}$,

$$AW_iAW_iA = AW_iA; \tag{3.48}$$

$$AW_iA \neq \mathbf{0}; \tag{3.49}$$

$$m_i = tr(AW_i); \text{ and} \tag{3.50}$$

$$AW_iAW_jA = \mathbf{0}. \tag{3.51}$$

Proof. Use $A\otimes\Sigma$ to replace Σ_Y in (3.40)-(3.43). The desired results (3.48)-(3.51) follow from (3.40)-(3.43), respectively, in Theorem 3.2.4. □

In Theorem 3.2.4, if we replace the covariance Σ_Y of Y with the sum of special Kronecker products, Theorem 3.2.4 is reduced to the following corollary which is an extension of Corollary 3.1.7.

Corollary 3.2.6. *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *with* $\Sigma_Y = \sum_{i=1}^{r} A_i \otimes E_{ii}$, $r \leq p$, *where* $A_i \in \mathcal{N}_n$ *and* $W_i \in \mathbb{S}_n$, $i = 1,2,\ldots,l$. *Then* $\{Y'W_iY\}_{i=1}^{l}$ *is an independent family of Wishart* $\mathcal{W}_p(m_\alpha,\Sigma)$ *random matrices, where* $\Sigma = \sum_{i=1}^{r}\sigma_i E_{ii}$, *for some* $m_\alpha \in \{0,1,2,\ldots\}$ *if and only if there exist real numbers* $\sigma_1 > 0, \sigma_2 > 0, \ldots, \sigma_r > 0$ *such that for all* $i,j,k \in \{1,2,\ldots,r\}$ *and* $a,b \in \{1,2,\ldots,l\}$,

$$A_iW_\alpha A_kW_\alpha A_j = \sigma_k A_iW_\alpha A_j; \tag{3.52}$$

$$A_iW_\alpha A_j \neq \mathbf{0}; \tag{3.53}$$

$$m_a = \frac{1}{r}\sum_{i=1}^{r}\frac{1}{\sigma_i}tr(A_iW_a); \text{ and} \tag{3.54}$$

$$A_iW_aA_iW_bA_i = \mathbf{0}. \tag{3.55}$$

Proof. From (3.43), replacing Σ_Y and Σ with $\sum_{i=1}^{r} A_i \otimes E_{ii}$ and $\sum_{i=1}^{r} \sigma_i E_{ii}$, respectively, we obtain,

$$\sum_{i,j,k=1}^{r} A_i W_\alpha A_k W_\beta A_j \otimes E_{ii} \Sigma^+ E_{jj} \Sigma^+ E_{kk} = \mathbf{0}.$$

Namely, $A_i W_\alpha A_i W_\beta A_i = \mathbf{0}$. Relations (3.52)-(3.54) follow immediately from Corollary 3.1.9 and the proof is completed. □

In Theorem 3.2.4, if $\mathbf{y}$ is an $n \times 1$ random normal vector with mean vector $\mathbf{0}$ and covariance C, (3.40)-(3.43) are reduced to the following familiar result which were shown by Khatri (1963), Rayner and Livingstone (1965), Shanbhag (1968) and Styan (1970).

Corollary 3.2.7. *Let* $\mathbf{y}$ *be a random vector normally distributed as* $\mathcal{N}_n(\mathbf{0}, C)$ *and* $W_i \in \mathbb{S}_n$, $i = 1, 2, \ldots, l$. *Then* $\{\mathbf{y}' W_i \mathbf{y}\}_{i=1}^{l}$ *is an independent family of chi-square* $\chi^2(m_i)$ *random variables with* m_i *degrees of freedom for* $m_i \in \{0, 1, 2, \ldots\}$ *if and only if for any distinct* $i, j \in \{1, 2, \ldots, l\}$,

$$CW_iCW_iC = CW_iC; \tag{3.56}$$

$$m_i = r(CW_i); \text{ and} \tag{3.57}$$

$$CW_iCW_jC = \mathbf{0}. \tag{3.58}$$

Proof. In the univariate case $p = 1$, $\Sigma_Y = C$, $\Sigma = 1$ without loss of generality (if $Q \sim \mathcal{W}_1(m, \sigma)$, then $Q/\sigma \sim \chi^2(m)$). Then, (3.43) is reduced to (3.58). The rest follows from Corollary 3.1.8. □

In Theorem 3.2.4, if the covariance Σ_Y of Y is nonsingular, Theorem 3.2.4 is reduced to the following corollary.

Corollary 3.2.8. *In Theorem 3.2.4, suppose Σ_Y is nonsingular. Then $\{Y'W_iY\}_{i=1}^{l}$ is an independent family of Wishart $\mathcal{W}_p(tr(W_i), \Sigma)$ random matrices if and only if there exists a $\Sigma \in \mathbb{N}_p$ such that for any distinct $i, j \in \{1, 2, \ldots, l\}$,*

(a) $(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = W_i \otimes \Sigma$; *and*

(b) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = \mathbf{0}$.

From Corollary 3.2.8, we can infer the fact that the underlying matrices W_i's are nonnegative definite if matrix quadratic forms are Wishartness.

Putting Theorem 3.1.11 and Theorem 3.2.4 together, we can obtain the following result.

Theorem 3.2.9. *Suppose that random matrix Y is normally distributed as $\mathcal{N}_{n\times p}(0, \Sigma_Y)$ and $\{W_i\}$ is a family of symmetric matrices of order n. Then matrix quadratic forms $\{Y'W_iY\}$ is an independent family of Wishart $\mathcal{W}_p(m_i, \Sigma)$ random matrices for some $m_i \in \{0, 1, 2, \ldots\}$ if and only if there exists a $\Sigma \in \mathbb{N}_p$ such that the following statements (a) and (b) hold.*

(a) For $i \in \{1, 2, \ldots, l\}$ and $\mathbf{s}_i \in \mathbb{S}_p$,

$$\Sigma_Y \left[W_i \otimes \mathbf{s}_i\Sigma\mathbf{s}_i\right] \Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y \tag{3.59}$$

with

$$\{\mathbf{s}_i : \Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y = \mathbf{0}\} = \{\mathbf{s}_i : \Sigma\mathbf{s}_i\Sigma = \mathbf{0}\} \tag{3.60}$$

and

$$m_i = tr[\Sigma_Y(W_i \otimes \Sigma^{+})]/r(\Sigma). \tag{3.61}$$

(b) For any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = 0. \tag{3.62}$$

Theorem 3.2.9 gives a matrix presentation of a similar result obtained by Masaro and Wong (2004a) through Jordan algebra homomorphisms.

3.3 Noncentral Wishartness of a MQF

In this section, we shall use the moment generating function of $Y'WY$ to study the noncentral Wishartness of a matrix form in a normal random matrix. The following theorem is the main result of this section.

Theorem 3.3.1. *Let* $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ *and* W *be a symmetric matrix of order* n. *Then the matrix quadratic form* $Y'WY$ *has a noncentral Wishart distribution* $\mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$ *for some matrix* $\boldsymbol{\lambda} \in \mathbb{M}_{p\times p}$ *and some* $m \in \{0, 1, 2, \ldots\}$ *if and only if there exists a* $\Sigma \in \mathbb{N}_p$ *such that (a) and (b) hold.*

(a) For any elements $\mathbf{h}, \tilde{\mathbf{h}}$ *in the similar base* $\mathbb{H}_p$ *associated with* Σ,

$$\Sigma_Y\left[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})\right]\Sigma_Y = F(\mathbf{h}, \tilde{\mathbf{h}}, W, \Sigma_Y) + F(\tilde{\mathbf{h}}, \mathbf{h}, W, \Sigma_Y) \tag{3.63}$$

with

$$\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \mathbb{K} \quad and \tag{3.64}$$

$$m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma). \tag{3.65}$$

(b) For any $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0}$ *in* $\mathbb{S}_p$ *and* $n = 1, 2, \ldots$,

$$tr(\boldsymbol{\lambda}(\mathbf{s}\Sigma)^n\mathbf{s}) = tr(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'[(W \otimes \mathbf{s})\Sigma_Y]^n(W \otimes \mathbf{s})) \tag{3.66}$$

with

$$\boldsymbol{\lambda} = \boldsymbol{\mu}' W \boldsymbol{\mu}. \tag{3.67}$$

Proof. From Lemma 2.3.1, the moment generating function $M(\mathbf{s})$ of $Y'WY$ is given by

$$M(\mathbf{s}) = |I_{np} - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2} exp\{< \mathbf{s}, \boldsymbol{\mu}'W\boldsymbol{\mu} > +2\Phi_0\} \tag{3.68}$$

for any $\mathbf{s} \sim \mathbb{S}_p$ such that $sr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}) < 1/2$ where

$$\Phi_0 =< vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})',\ (W \otimes \mathbf{s})\Sigma_Y^{1/2}[I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}]^{-1}\Sigma_Y^{1/2}(W \otimes \mathbf{s}) > .$$

Comparing the moment generating function $M_1(\mathbf{s})$ of the Wishart distribution stated in Corollary 2.3.2 with $M(\mathbf{s})$ given in (3.68), we obtain that $Y'WY \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$ if and only if for any symmetric matrix $\mathbf{s}$ of order p in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$,

$$|I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2} = |I - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m/2} \tag{3.69}$$

and

$$< \boldsymbol{\lambda}, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >= \Phi_0 \tag{3.70}$$

with $\boldsymbol{\lambda} = \boldsymbol{\mu}'W\boldsymbol{\mu}$.

Since, from Lemma 2.3.1, the moment generating function, $M_0(\mathbf{s})$, of matrix quadratic form $(Y - \boldsymbol{\mu})'W(Y - \boldsymbol{\mu})$ is given by

$$M_0(\mathbf{s}) = |I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2}, \text{ for any } \mathbf{s} \in \mathbb{S}_p \cap \mathcal{N}_0, \tag{3.71}$$

(3.69) amounts to $(Y - \boldsymbol{\mu})'W(Y - \boldsymbol{\mu}) \sim \mathcal{W}_p(m, \Sigma)$. By Theorem 3.1.3, (3.69) amounts to (3.63)-(3.65).

Moreover,

$$<\boldsymbol{\lambda}, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >= tr\left(\boldsymbol{\lambda}\mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s}\right)$$

$$= tr\left(\boldsymbol{\lambda}\mathbf{s}\Sigma^{1/2}[I_p + \sum_{n=1}^{\infty}(2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^n]\Sigma^{1/2}\mathbf{s}\right) = tr\left(\boldsymbol{\lambda}[\mathbf{s}\Sigma + \frac{1}{2}\sum_{n=1}^{\infty}(2\mathbf{s}\Sigma)^{n+1}]\mathbf{s}\right)$$

and

$$\Phi_0 = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'\left((W\otimes\mathbf{s})\Sigma_Y + \frac{1}{2}\sum_{n=1}^{\infty}[2(W\otimes\mathbf{s})\Sigma_Y]^{n+1}\right)(W\otimes\mathbf{s})\right).$$

Thus, (3.70) is equivalent to

$$tr\left(\boldsymbol{\lambda}[\mathbf{s}\Sigma + \frac{1}{2}\sum_{n=1}^{\infty}(2\mathbf{s}\Sigma)^{n+1}]\mathbf{s}\right)$$
$$= tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'\left((W\otimes\mathbf{s})\Sigma_Y + \frac{1}{2}\sum_{n=1}^{\infty}[2(W\otimes\mathbf{s})\Sigma_Y]^{n+1}\right)(W\otimes\mathbf{s})\right) \quad (3.72)$$

for $\mathbf{s} \in \mathbb{S}_p \cap \mathcal{N}_0$.

We arbitrarily choose $\mathbf{s}$ in $\mathcal{N}_0$. Replacing $\mathbf{s}$ in (3.72) by $\alpha\mathbf{s}$ with very small positive number α, two sides of (3.72) are two power series with respect to α. Comparing two power series implies that (3.72) amounts to (3.66), and that proves the desired result. □

From the proof in Theorem 3.3.1, we obtain the following relation between $Y'WY$ and $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu})$.

Corollary 3.3.2. *Let* $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ *and* W *be a symmetric matrix of order* n. *Then* $Y'WY \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ *for some matrix* $\boldsymbol{\lambda} \in \mathbb{M}_{p\times p}$ *and some* $m \in \{0, 1, 2, \ldots\}$ *if and only if there exists a* $\Sigma \in \mathbb{N}_p$ *such that*

(a) $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu}) \sim \mathcal{W}_p(m, \Sigma)$ *and*

(b) for any $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0}$ *in* $\mathbb{S}_p$ *and* $n = 1, 2, \ldots,$

$$tr\,(\boldsymbol{\mu}' W \boldsymbol{\mu}(\mathbf{s}\Sigma)^n \mathbf{s}) = tr\,(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'[(W \otimes \mathbf{s})\Sigma_Y]^n (W \otimes \mathbf{s}))\,.$$

The following example gives an application of Theorem 3.3.1 and also provides an illustration of a quadratic form $Y'WY$ which is a noncentral Wishart but where W is not nonnegative definite.

Examples 3.3.3. *Let* $Y \sim N_{3\times 2}(\boldsymbol{\mu}, \Sigma_Y)$ *with*

$$\boldsymbol{\mu} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad \Sigma_Y = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$W = \begin{bmatrix} 2 & a & \sqrt{2} \\ a & b & c \\ \sqrt{2} & c & 1 \end{bmatrix} \in \mathbb{S}_3, \ a,\ b,\ c \in \mathbb{R}.$$

Then, we discuss the noncentral Wishartness of the matrix quadratic form $Y'WY$ *and determine its degrees if* $Y'WY$ *has a noncentral Wishart distribution.*

Proof. Consider $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. In this case

$$\mathbb{H}_p = \mathbb{E}_p = \left\{ E_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

With necessary matrix operations, (4.1) holds for all element pairs (E_{11}, E_{11}), (E_{11}, E_{12}), (E_{11}, E_{22}), (E_{12}, E_{12}), (E_{12}, E_{22}) and (E_{22}, E_{22}).

Obviously, $\mathbb{K} = \emptyset$. Let $\mathbf{h} = \begin{bmatrix} h_1 & h_3 \\ h_3 & h_2 \end{bmatrix} \in \mathbb{H}_2$. Then

$$\Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \begin{bmatrix} 2h_1 & 0 & 0 & 0 & 0 & \sqrt{2}h_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}h_3 & 0 & 0 & 0 & 0 & h_2 \end{bmatrix} = \mathbf{0}$$

$\Rightarrow \mathbf{h} = \mathbf{0}$, implying that (3.64) holds. And

$$m = tr\left(\begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{bmatrix}\begin{bmatrix} 2\Sigma^+ & a\Sigma^+ & \sqrt{2}\Sigma^+ \\ a\Sigma^+ & b\Sigma^+ & c\Sigma^+ \\ \sqrt{2}\Sigma^+ & c\Sigma^+ & \Sigma^+ \end{bmatrix}\right)/r(\Sigma)$$

$$= \frac{1}{2}tr\left(\begin{bmatrix} 2A\Sigma^+ & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & B\Sigma^+ \end{bmatrix}\right) = \frac{1}{2}[tr(2A\Sigma^+) + tr(B\Sigma^+)] = 1.$$

To verify (3.66), we write

$$C_* = (W \otimes \mathbf{s})vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \sqrt{2}\mathbf{s}B \\ \mathbf{0} & \mathbf{0} & c\mathbf{s}B \\ \mathbf{0} & \mathbf{0} & \mathbf{s}B \end{bmatrix}$$

and

$$D_* = (W \otimes \mathrm{s})\Sigma_Y = \begin{bmatrix} 2\mathrm{s}A & \mathbf{0} & \sqrt{2}\mathrm{s}B \\ a\mathrm{s}A & \mathbf{0} & c\mathrm{s}B \\ \sqrt{2}\mathrm{s}A & \mathbf{0} & \mathrm{s}B \end{bmatrix} \quad \text{for } \mathrm{s} \in \mathbb{S}_2.$$

Note that $\Sigma = 2A + B$. We have

$$D_*^{n+1} = \begin{bmatrix} * & \mathbf{0} & * \\ * & \mathbf{0} & * \\ \sqrt{2}(\mathrm{s}\Sigma)^n\mathrm{s}A & \mathbf{0} & (\mathrm{s}\Sigma)^n\mathrm{s}B \end{bmatrix} \quad \text{for } n = 0, 1, 2, \ldots. \tag{3.73}$$

We shall use mathematical induction to prove (3.73) as follows:

For $n = 0$, (3.73) is reduced to the trivial case. Suppose (3.73) holds for $n = k \geq 1$.

Then

$$D_*^{k+1} = D_*^k D_* = \begin{bmatrix} * & \mathbf{0} & * \\ * & \mathbf{0} & * \\ \sqrt{2}(\mathrm{s}\Sigma)^{k-1}\mathrm{s}A & \mathbf{0} & (\mathrm{s}\Sigma)^{k-1}\mathrm{s}B \end{bmatrix} \begin{bmatrix} 2\mathrm{s}A & \mathbf{0} & \sqrt{2}\mathrm{s}B \\ a\mathrm{s}A & \mathbf{0} & c\mathrm{s}B \\ \sqrt{2}\mathrm{s}A & \mathbf{0} & \mathrm{s}B \end{bmatrix}$$

$$= \begin{bmatrix} * & \mathbf{0} & * \\ * & \mathbf{0} & * \\ \sqrt{2}(\mathrm{s}\Sigma)^{k-1}\mathrm{s}(2A+B)\mathrm{s}A+ & \mathbf{0} & (\mathrm{s}\Sigma)^{k-1}\mathrm{s}(2A+B)\mathrm{s}B \end{bmatrix}$$

$$= \begin{bmatrix} * & \mathbf{0} & * \\ * & \mathbf{0} & * \\ \sqrt{2}(\mathrm{s}\Sigma)^{k}\mathrm{s}A & \mathbf{0} & (\mathrm{s}\Sigma)^{k}\mathrm{s}B \end{bmatrix}.$$

It follows from (3.73) that $n = 0, 1, 2, \ldots$

$$tr(C_* D_*^n) = tr\left(2\mathrm{s}B(\mathrm{s}\Sigma)^{n-1}\mathrm{s}A + \mathrm{s}B(\mathrm{s}\Sigma)^{n-1}\mathrm{s}B\right) = tr\left(\mathrm{s}B(\mathrm{s}\Sigma)^n\right).$$

And

$$\boldsymbol{\lambda} = \boldsymbol{\mu}' W \boldsymbol{\mu} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = B.$$

So (3.66) holds. Hence, it follows from Theorem 3.3.1 that $Y'WY$ has a Wishart distribution $\mathcal{W}_2(m, \Sigma, \boldsymbol{\lambda})$ with $m = 1$ degree of freedom, covariance $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and non-centrality matrix $\boldsymbol{\lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. □

Assume that the covariance Σ_Y of Y is a Kronecker product covariance structure $A \otimes \Sigma$ with non-negative definite matrix A, Theorem 3.3.1 is reduced to the following corollary which was essentially obtained by Khatri (1963) and De Gunst (1987).

Corollary 3.3.4. *In Theorem 3.3.1, supposed that $\Sigma_Y = A \otimes \Sigma$ for some $A \in \mathbb{N}_n$. Then, for the symmetric matrix W of order n, the matrix quadratic form $Y'WY$ has a noncentral Wishart distribution $\mathcal{W}_p(r(AW), \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ if and only if*

(a) $AWAWA = AWA \neq \mathbf{0}$, and

(b) $\boldsymbol{\mu}'WAWAW\boldsymbol{\mu} = \boldsymbol{\mu}'WAW\boldsymbol{\mu} = \boldsymbol{\mu}'W\boldsymbol{\mu}$

Proof. (a) follows from Corollary 3.1.6. Replacing Σ_Y with $A \otimes \Sigma$ in (3.66), we obtain that for any symmetric matrix s of order p in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $n = 1, 2, \ldots,$

$$\begin{aligned} tr(\boldsymbol{\lambda}(\mathrm{s}\Sigma)^n\mathrm{s}) &= tr\left(vec(\boldsymbol{\mu})'[(W \otimes \mathrm{s})(A \otimes \Sigma)]^n (W \otimes \mathrm{s}) vec(\boldsymbol{\mu})\right) \\ &= tr\left(vec(\boldsymbol{\mu})'[(WA)^n W \otimes (\mathrm{s}\Sigma)^n \mathrm{s}] vec(\boldsymbol{\mu})\right) \\ &= tr\left(vec(\boldsymbol{\mu})' vec[(WA)^n W \boldsymbol{\mu} (\mathrm{s}\Sigma)^n \mathrm{s}]\right) = tr(\boldsymbol{\mu}'(WA)^n W \boldsymbol{\mu} (\mathrm{s}\Sigma)^n \mathrm{s}). \end{aligned}$$

By Lemma 2.1.8, for n=1,2,...

$$\boldsymbol{\mu}'(WA)^n W\boldsymbol{\mu} = \boldsymbol{\lambda}. \tag{3.74}$$

(3.74) is equivalent to, under (a),

$$\boldsymbol{\mu}'WAWAW\boldsymbol{\mu} = \boldsymbol{\mu}'WAW\boldsymbol{\mu} = \boldsymbol{\mu}'W\boldsymbol{\mu},$$

so the desired result follows from Theorem 3.3.1. □

Moreover, taking $\Sigma_Y = I_n \otimes \Sigma$ in Theorem 3.3.1, we have the following corollary, which was proved earlier by Khatri (1959).

Corollary 3.3.5. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, I_n \otimes \Sigma)$ and W be a symmetric matrix of order n. Then a necessary and sufficient condition for the matrix quadratic form $Y'WY$ to have a noncentral Wishart distribution $\mathcal{W}_p(tr(W), \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ is that W is idempotent. This distribution is central when $W\boldsymbol{\mu} = \mathbf{0}$.*

The following result is due to Eaton (1983).

Corollary 3.3.6. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, A \otimes \Sigma)$ and W be a symmetric nonnegative definite matrix of order n. Write $W = V^2$, where V is nonnegative definite. If VAV is an orthogonal projection of rank m and $VAW\boldsymbol{\mu} = V\boldsymbol{\mu}$, then $Y'WY \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$.*

Assume that W is a symmetric matrix such that $tr(AW) = r(A)$ and $\Sigma = A \otimes \Sigma$, we get the following result, due to Vaish and Chaganty (2004).

Corollary 3.3.7. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, A \otimes \Sigma)$ and W be a symmetric matrix of order n. Then a necessary and sufficient condition for matrix quadratic form $Y'WY$ to have a noncentral Wishart distribution $\mathcal{W}_p(r(A), \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ is that $WAW = W$.*

In Theorem 3.3.1, if we replace the covariance Σ_Y of Y with the sum of special Kronecker products, Theorem 3.3.1 is reduced to the following corollary, another extension of Corollary 3.1.7.

Corollary 3.3.8. *Let* $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ *with* $\Sigma_Y = \sum_{i=1}^{r} A_i \otimes E_{ii}$, $r \leq p$, *where* $A_i \in \mathbb{N}_n$. *Then, for* $W \in \mathbb{S}_n$, *the matrix quadratic form* $Y'WY$ *has a noncentral Wishart distribution* $\mathcal{W}_p(m, \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ *for some nonnegative integer* m, *where*

$$\Sigma = \sum_{i=1}^{r} \sigma_i E_{ii},$$

if and only if there exist real numbers $\sigma_l > 0$, $l = 1, 2, \ldots, r$ *such that for all* $i, j, k \in \{1, 2, \ldots, r\}$,

$$A_i W A_k W A_j = \sigma_k A_i W A_j \neq \mathbf{0}, \; m = \frac{1}{r}\sum_{i=1}^{r} \frac{1}{\sigma_i} tr(A_i W) \tag{3.75}$$

and

$$\sigma_i \boldsymbol{\mu}'W\boldsymbol{\mu} = \boldsymbol{\mu}'WA_iW\boldsymbol{\mu}, \; \sigma_i\sigma_j\boldsymbol{\mu}'W\boldsymbol{\mu} = \boldsymbol{\mu}'WA_iWA_jW\boldsymbol{\mu}. \tag{3.76}$$

Proof. (3.75) follows from Corollary 3.1.7. Replacing Σ_Y and Σ with $\sum_{i=1}^{r} A_i \otimes E_{ii}$ and $\sum_{i=1}^{r} \sigma_i E_{ii}$, respectively, in (3.66), we obtain that for any symmetric matrix $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $n = 1, 2, \ldots,$

$$tr\left(\boldsymbol{\lambda}(\mathbf{s}\sum_{i=1}^{r}\sigma_i E_{ii})^n \mathbf{s}\right) = tr\left(vec(\boldsymbol{\mu})'\left[(W\otimes\mathbf{s})\sum_{i=1}^{r}(A_i\otimes E_{ii})\right]^n (W\otimes\mathbf{s})vec(\boldsymbol{\mu})\right), \tag{3.77}$$

where $\boldsymbol{\lambda} = \boldsymbol{\mu}'W\boldsymbol{\mu}$.

When $n = 1$, (3.77) is reduced to

$$\sum_{i=1}^{r} tr(\boldsymbol{\lambda}(\sigma_i \mathbf{s} E_{ii}\mathbf{s})) = \sum_{i=1}^{r} tr(\boldsymbol{\mu}'WA_iW\boldsymbol{\mu}(\mathbf{s}E_{ii}\mathbf{s})), \text{ for } \mathbf{s} \in \mathcal{N}_0 \cap \mathbb{S}_p.$$

Lemma 2.1.9 and the arbitrariness of s imply that for $i = 1, 2, \ldots, r$

$$\sigma_i \boldsymbol{\mu}' W \boldsymbol{\mu} = \boldsymbol{\mu}' W A_i W \boldsymbol{\mu}. \tag{3.78}$$

In a similar way, for $n = 2$, (3.77) is equivalent to

$$\sigma_i \sigma_j \boldsymbol{\mu}' W \boldsymbol{\mu} = \boldsymbol{\mu}' W A_i W A_j W \boldsymbol{\mu}, \text{ for any } i, j = 1, 2, \ldots, r. \tag{3.79}$$

The results of $n > 2$ in (3.77) can be obtained from (3.77)-(3.79). So the desired result follows immediately from Theorem 3.3.1. □

In Theorem 3.3.1, if $\mathbf{y}$ is $n \times 1$ random normal vector with mean vector $\boldsymbol{\mu}$ and covariance C of order n, (3.63)-(3.67) are reduced to the following familiar result which were shown in the sixties.

Corollary 3.3.9. *Let* $\mathbf{y}$ *be a random vector normally distributed as* $\mathcal{N}_n(\boldsymbol{\mu}, C)$ *and* W *be a symmetric matrix of order* n. *Then the quadratic form* $\mathbf{y}'W\mathbf{y}$ *has a noncentral Wishart* $\mathcal{W}_1(r(CW), 1, \boldsymbol{\mu}'W\boldsymbol{\mu})$ *or chi-square distribution with* $r(CW)$ *degrees of freedom and parameter* $\delta^2 = \boldsymbol{\mu}'W\boldsymbol{\mu}$ *if and only if*

$$CWCWC = CWC, \text{ and} \tag{3.80}$$

$$\boldsymbol{\mu}' WCWCW \boldsymbol{\mu} = \boldsymbol{\mu}' WCW \boldsymbol{\mu} = \boldsymbol{\mu}' W \boldsymbol{\mu}. \tag{3.81}$$

Proof. In the univariate case $p = 1$, (3.80) follows from Corollary 3.1.8. Replacing Σ_Y with C and Σ with 1 (if $Q \sim \mathcal{W}_1(m, \sigma)$, then $Q/\sigma \sim \chi^2(m)$), (3.66) is reduced to

$$\boldsymbol{\mu}' W \boldsymbol{\mu} \mathbf{s}^{n+1} = \boldsymbol{\mu}' (WC)^n W \boldsymbol{\mu} \mathbf{s}^{n+1}, \text{ for } n = 1, 2, \ldots,$$

which is equivalent to (3.81) under (3.80). So the desired result follows from Theorem 3.3.1. □

Suppose that Σ_Y is nonsingular in Theorem 3.3.1 we get the following corollary.

Corollary 3.3.10. *In Theorem 3.3.1, suppose Σ_Y is nonsingular, then the matrix quadratic form $Y'WY$ has a noncentral Wishart distribution $\mathcal{W}_p(tr(W), \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that*

(a) $W \otimes \Sigma = (W \otimes I)\Sigma_Y(W \otimes I)$; and

(b) for any symmetric matrix $\mathbf{s}$ of order p in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $n = 1, 2, \ldots,$

$$tr\left(\boldsymbol{\mu}'W\boldsymbol{\mu}(\mathbf{s}\Sigma)^n\mathbf{s}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'[(W \otimes \mathbf{s})\Sigma_Y]^n(W \otimes \mathbf{s})\right).$$

Proof. (a) follows from Corollary 3.1.9 and (b) follows from Theorem 3.3.1. □

3.4 Noncentral Wishartness and Independence of Matrix Quadratic Forms

Although the following result and its proof imitate Theorem 2.2 and its proof of Wong *et al.* (1991), some modifications have been made so that the corresponding necessary and sufficient conditions can be verified.

Lemma 3.4.1. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ and $\{W_i\}_{i=1}^{l}$ be symmetric matrices in $\mathbb{S}_n$. Then a set of matrix quadratic forms $Y'W_iY$'s is independent if and only if for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{h}_i, \mathbf{h}_j \in \mathbb{H}_p$, where $\mathbb{H}_p$ is a similar base associated with any given $\Sigma \in \mathbb{N}_p$*

(a) $\Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)\Sigma_Y = \mathbf{0}$,

(b) $\Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)vec(\boldsymbol{\mu}) = \mathbf{0}$, *and*

(c) $vec(\boldsymbol{\mu})'(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)vec(\boldsymbol{\mu}) = 0$.

Proof. Suppose $\{Y'W_iY\}_{i=1}^l$ is an independent family. Let $i \neq j$ and $\mathbf{h}_i, \mathbf{h}_j \in \mathbb{H}_p$, then $< \mathbf{h}_i, Y'W_iY >$ and $< \mathbf{h}_j, Y'W_jY >$ are independent. Since

$$< \mathbf{h}_i, Y'W_iY) >=< vec(Y), (W_i \otimes \mathbf{h}_i)vec(Y) >= vec(Y)'(W_i \otimes \mathbf{h}_i)vec(Y)$$

and $vec(Y)$ has a normal distribution $\mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$, it follows from Theorem 4s of Searle (1971) that (a), (b) and (c) hold.

Conversely, assume that for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{h}_i, \mathbf{h}_j \in \mathbb{H}_p$ (a), (b) and (c) hold. Then for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$, $\mathbf{s}_i = \sum_{1\le k\le l\le p} s_{kl}\mathbf{h}_{kl}$ ($s_{kl} \in \mathbb{R}$ and $\mathbf{h}_{kl} \in \mathbb{H}_p$) and $\mathbf{s}_j = \sum_{1\le k\le l\le p} s^*_{kl}\mathbf{h}_{kl}$ ($s^*_{kl} \in \mathbb{R}$). With simple operations, we have

(a') $\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)\Sigma_Y = \mathbf{0}$,

(b') $\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) = \mathbf{0}$, and

(c') $vec(\boldsymbol{\mu})'(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) = 0$.

Recall that the family $Y'WY = (Y'W_iY)$ is independent if and only if

$$M_{Y'WY}(\mathbf{s}) = \prod_{i=1}^{l} M_{YW_iY}(\mathbf{s}_i)$$

for $\mathbf{s} = (\mathbf{s}_i)$ in $\mathcal{N}_0$, where $\mathcal{N}_0$ is a neighborhood of $\mathbf{0}$ in $\mathbb{S} = \mathbb{S}_p \times \mathbb{S}_p \times \ldots \mathbb{S}_p$ (1 times). So by Lemma 2.3.1, the family $\{Y'W_iY\}_{i=1}^l$ is independent if and only if

(i) $|I - 2\sum_{i=1}^{l} \Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}| = \prod_{i=1}^{l} |I - 2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}|$, and

(ii) $< \sum_{i=1}^{l} vec(W_i\boldsymbol{\mu}\mathbf{s}_i), \Sigma_Y^{1/2}(I_{np} - 2\Sigma^*)^{-1}\Sigma_Y^{1/2} \sum_{j=1}^{l} vec(W_j\boldsymbol{\mu}\mathbf{s}_j) >$

$$= \sum_{i=1}^{l} < vec(W_i \boldsymbol{\mu} \mathbf{s}_i), \Sigma_Y^{1/2}[I - 2\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}]^{-1}\Sigma_Y^{1/2} vec(W_i \boldsymbol{\mu} \mathbf{s}_i) > .$$

where $\Sigma^* = \Sigma_Y^{1/2}\left(\sum_{i=1}^{l} W_i \otimes \mathbf{s}_i\right)\Sigma_Y^{1/2}$. By (a'), $\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)\Sigma_Y = \mathbf{0}$ for all distinct $i, j \in \{1, 2, \ldots, l\}$. So (i) follows. For the same reasons,

$$(I - 2\Sigma^*)^{-1} = \prod_{i=1}^{l}\left(I - 2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^{-1}.$$

Let $D_{ij} \equiv < (W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}), \Sigma_Y^{1/2}(I - 2\Sigma_*)^{-1}\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) >$, then for (ii), it suffices to show that

$$D_{ij} = 0 \quad \text{for } i \neq j \text{ and} \tag{3.82}$$

$$D_{ii} \equiv < vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\left(I - 2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^{-1}\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}) > . \tag{3.83}$$

From (a'), for $i \neq j$,

$$(I - 2\Sigma^*)^{-1} = \prod_{i=1}^{l}\left(I - 2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^{-1} = \prod_{i=1}^{l}\sum_{k=0}^{\infty}\left(2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^k$$
$$= I + \sum_{i=1}^{l}\sum_{k=1}^{\infty}\left(2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^k.$$

And for any $i, j \in \{1, 2, \ldots, l\}$

$$D_{ij} = < (W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}), \Sigma_Y^{1/2}\left(I + \sum_{i=1}^{l}\sum_{k=1}^{\infty}\left(2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^k\right)\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) >$$
$$= < (W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}), \Sigma_Y^{1/2}\left(I + \sum_{k=1}^{\infty}\left(2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^k\right)\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) >$$
$$= < vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) > +$$
$$< vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\sum_{k=1}^{\infty}\left(2\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)\Sigma_Y^{1/2}\right)^k\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) > .$$

(3.84)

By (c'), for distinct $i, j \in \{1, 2, \ldots, l\}$,

$$< vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) >= vec(\boldsymbol{\mu})'(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) = 0$$

and by (b'), for distinct $i, j \in \{1, 2, \ldots, l\}$

$$\begin{aligned} &< vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2} \sum_{k=1}^{\infty} \left(2\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)\Sigma_Y^{1/2}\right)^k \Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) > \\ &= vec(\boldsymbol{\mu})'(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2} \sum_{k=1}^{\infty} \left(2\Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)\Sigma_Y^{1/2}\right)^k \Sigma_Y^{1/2}(W_j \otimes \mathbf{s}_j)vec(\boldsymbol{\mu}) = 0. \end{aligned}$$

So $D_{ij} = 0$ for $i \neq j$, proving (3.82). Using (3.84) again

$$\begin{aligned} D_{ii} &= < vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2} \left(I + \sum_{k=1}^{\infty} \left(2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^k\right) \Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}) > \\ &= < vec(\boldsymbol{\mu}), (W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2} \left(I - 2\Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)\Sigma_Y^{1/2}\right)^{-1} \Sigma_Y^{1/2}(W_i \otimes \mathbf{s}_i)vec(\boldsymbol{\mu}) >, \end{aligned}$$

that proves (3.83). (ii) follows from (i) and the proof is completed. □

Replacing $\mathbb{H}_p$ with $\mathbb{E}_p$ is a usual usage in Lemma 3.4.1 while replacing $\mathbb{H}_p$ with $\mathbb{S}_p$ is the result of Wong *et al.* (1991). Suppose that $\boldsymbol{\mu} = \mathbf{0}$ in Theorem 3.4.2, it is reduced to Lemma 3.2.1 stated in Section 3.2.

Combining Theorem 3.3.1 with Lemma 3.4.1, we obtain the following multivariate version of Cochran's theorem.

Theorem 3.4.2. *Suppose that* $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ *and* $\{W_i\}_{i=1}^l$ *is a family of symmetric matrices of order* n. *Then a set of matrix quadratic forms* $\{Y'W_iY\}_{i=1}^l$ *is an independent family of noncentral Wishart* $\mathcal{W}_p(m_i, \Sigma, \boldsymbol{\lambda}_i)$ *random matrices for some* $m_i \in \{0, 1, 2, \ldots\}$ *and some matrix* $\boldsymbol{\lambda}_i \in \mathbb{M}_{p\times p}$ *if and only if there exists some* $\Sigma \in \mathbb{N}_p$ *such that the following statements (a), (b) and (c) hold.*

(a) For $i \in \{1, 2, \ldots, l\}$ and any elements $\mathbf{h}_i, \tilde{\mathbf{h}}_i$ in the similar base $\mathbb{H}_p$ associated with Σ,

$$\Sigma_Y \left[W_i \otimes (\mathbf{h}_i \Sigma \tilde{\mathbf{h}}_i + \tilde{\mathbf{h}}_i \Sigma \mathbf{h}_i) \right] \Sigma_Y = F(\mathbf{h}_i, \tilde{\mathbf{h}}_i, W_i, \Sigma_Y) + F(\tilde{\mathbf{h}}_i, \mathbf{h}_i, W_i, \Sigma_Y) \quad (3.85)$$

with

$$\{\mathbf{h}_i : \Sigma_Y (W_i \otimes \mathbf{h}_i) \Sigma_Y = \mathbf{0}\} = \mathbb{K} \quad and \quad m_i = tr(\Sigma_Y (W_i \otimes \Sigma^+))/r(\Sigma). \quad (3.86)$$

(b) For any $i \in \{1, 2, \ldots, l\}$, any symmetric matrix $\mathbf{s}_i$ of order p in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $n = 1, 2, \ldots$,

$$tr\,(\boldsymbol{\lambda}(\mathbf{s}_i \Sigma)^n \mathbf{s}_i) = tr\,(vec(\boldsymbol{\mu}) vec(\boldsymbol{\mu})' [(W_i \otimes \mathbf{s}_i) \Sigma_Y]^n (W_i \otimes \mathbf{s}_i)) \quad (3.87)$$

with $\boldsymbol{\lambda}_i = \boldsymbol{\mu}' W_i \boldsymbol{\mu}$.

(c) For any distinct $i, j \in \{1, 2, \ldots, l\}$ and $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{E}_p$

$$\Sigma_Y (W_i \otimes \Sigma^+) \Sigma_Y (W_j \otimes \Sigma^+) \Sigma_Y = \mathbf{0}, \quad (3.88)$$

$$\Sigma_Y (W_i \otimes \mathbf{t}_i) \Sigma_Y (W_j \otimes \mathbf{t}_j) vec(\boldsymbol{\mu}) = \mathbf{0}, \quad and \quad (3.89)$$

$$vec(\boldsymbol{\mu})' (W_i \otimes \mathbf{t}_i) \Sigma_Y (W_j \otimes \mathbf{t}_j) vec(\boldsymbol{\mu}) = 0. \quad (3.90)$$

Proof. Let $\{Y' W_i Y\}_{i=1}^l$ be an independent family of noncentral Wishart $W_p(m_i, \Sigma, \boldsymbol{\lambda}_i)$ random matrices. Then statements (a)-(c) follow from Theorem 3.3.1 and Lemma 3.4.1.

Conversely, suppose statements (a)-(c) hold. Since (a) and (b), from Theorem 3.3.1, for any $i = 1, 2, \ldots, l$, $Y' W_i Y \sim \mathcal{W}_p(m_i, \Sigma, \boldsymbol{\lambda}_i)$.

To prove the independence of matrix quadratic forms $Y'W_iY$'s, it suffices to show that for any $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{E}_p$, we have

$$\Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_j \otimes \mathbf{t}_j)\Sigma_Y = \mathbf{0}. \tag{3.91}$$

First, assume Σ is diagonal matrix, namely, $\Sigma = \Lambda = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0 \ldots, 0]$. Due to the structure of Λ, we only need to consider elements E_{ij}, $1 \le i \le j \le r$, in the basic base $\mathbb{E}_p$ for (3.91).

Exactly as in the proof of Lemma 2.1.6, we can prove that (3.88) is equivalent to

$$L(\Lambda^+ \otimes W_i)L'L(\Lambda^+ \otimes W_j)L' = \mathbf{0} \text{ where } L'L = \Sigma_{Y'} \tag{3.92}$$

and (3.91) is equivalent to

$$L(\mathbf{t}_i \otimes W_i)L'L(\mathbf{t}_j \otimes W_j)L' = \mathbf{0},\ \mathbf{t}_i, \mathbf{t}_j \in \{E_{ij} :\ 1 \le i \le j \le r\}. \tag{3.93}$$

So it suffices to show (3.93). Since $\Lambda^+ = \sum_{i=1}^{r} \frac{1}{\sigma_i} E_{ii}$, by (3.85) or its corresponding Lemma 2.1.6,

$$L(\Lambda^+ \otimes W_i)L' = L(\Lambda^+\Lambda\Lambda^+ \otimes W_i)L' = \Theta(\Lambda^+, \Lambda^+, W_i, L). \tag{3.94}$$

Also by (3.85) or its corresponding Lemma 2.1.6,

$$L(\mathbf{t}_i \otimes W_i)L' = L\left[\frac{1}{2}(\Lambda^+\Lambda\mathbf{t}_i + \mathbf{t}_i\Lambda\Lambda^+) \otimes W_i\right]L' = \frac{1}{2}[\Theta(\Lambda^+, \mathbf{t}_i, W_i, L) + \Theta(\mathbf{t}_i, \Lambda^+, W_i, L)]. \tag{3.95}$$

With (3.94) and (3.95), we obtain from Lemma 2.1.4

$$L(\mathbf{t}_i \otimes W_i)L'L(\Lambda^+ \otimes W_i)L' = \Theta(\Lambda^+, \mathbf{t}_i, W_i, L). \tag{3.96}$$

Thus, (3.95) and (3.96) give

$$L(\mathbf{t}_i \otimes W_i)L' = \Theta(\mathbf{t}_i, \Lambda^+, W_i, L). \tag{3.97}$$

In a similar way, we get

$$L(\mathbf{t}_j \otimes W_j)L' = \Theta(\Lambda^+, \mathbf{t}_j, W_j, L). \tag{3.98}$$

Hence, (3.93) follows from (3.97), (3.98) and (3.88), which equivalently proved (3.91).

Next, for the nonnegative definite Σ, there exists an orthogonal matrix H such that $H'\Sigma H = \Lambda$. Then (3.88) can be written as

$$\Sigma_{YH}(W_i \otimes \Lambda^+)\Sigma_{YH}(W_j \otimes \Lambda^+)\Sigma_{YH} = \mathbf{0}$$

where $\Sigma_{YH} = (I \otimes H')\Sigma_Y(I \otimes H)$. Also, (3.85) can be written as

$$\Sigma_{YH}\left[W_i \otimes (\mathbf{t}_i\Lambda\tilde{\mathbf{t}}_i + \tilde{\mathbf{t}}_i\Lambda\mathbf{t}_i)\right]\Sigma_{YH} = F(\mathbf{t}_i, \tilde{\mathbf{t}}_i, W_i, \Sigma_{YH}) + F(\tilde{\mathbf{t}}_i, \mathbf{t}_i, W_i, \Sigma_{YH}).$$

The first equation of (3.86) can be written as

$$\{\mathbf{t}_i : \Sigma_{YH}(W_i \otimes \mathbf{t}_i)\Sigma_{YH} = \mathbf{0}\} = \mathbb{K}_0.$$

From the proof of the previous special case, for any $\mathbf{t}_i, \mathbf{t}_j \in \mathbb{E}_p$,

$$\Sigma_{YH}(W_i \otimes \mathbf{t}_i)\Sigma_{YH}(W_j \otimes \mathbf{t}_j)\Sigma_{YH} = \mathbf{0},$$

equivalently, for any $\mathbf{h}_i, \mathbf{h}_j \in \mathbb{H}_p$, $\Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)\Sigma_Y = \mathbf{0}$. Thus, (3.91) follows from the fact that each $\mathbf{t} \in \mathbb{E}_p$ is the linear combination of elements in the similar base $\mathbb{H}_p$ associated with Σ, that completes the proof. □

Theorem 3.4.2 is reduced to Theorem 3.2.4 when $\boldsymbol{\mu} = \mathbf{0}$.

In Theorem 3.4.2, suppose Σ_Y is nonsingular, we have the following corollary.

Corollary 3.4.3. *In Theorem 3.4.2, suppose Σ_Y is nonsingular, then $\{Y'W_iY\}_{i=1}^l$ is an independent family of noncentral Wishart $\mathcal{W}_p(tr(W_i), \Sigma, \boldsymbol{\mu}'W_i\boldsymbol{\mu})$ random matrices if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that*

(a) $W_i \otimes \Sigma = (W_i \otimes I)\Sigma_Y(W_i \otimes I)$;

b) for any $i \in \{1, 2, \ldots, l\}$, any symmetric matrix $\mathbf{s}_i$ of order p in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $n = 1, 2, \ldots,$

$$tr\,(\boldsymbol{\mu}'W_i\boldsymbol{\mu}(\mathbf{s}_i\Sigma)^n\mathbf{s}_i) = tr\,(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'[(W_i \otimes \mathbf{s}_i)\Sigma_Y]^n(W_i \otimes \mathbf{s}_i))\,; \text{ and}$$

(c) for any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = \mathbf{0}.$$

Proof. The proof follows immediately from Corollary 3.3.2 and Corollary 3.2.5. □

If Σ_Y is a Kronecker product structure $A \otimes \Sigma$ for some $A \in \mathbb{N}_n$, Theorem 3.4.2 is reduced to the following familiar result.

Corollary 3.4.4. *In Theorem 3.4.2, suppose $\Sigma_Y = A \otimes \Sigma$ for some $A \in \mathbb{N}_n$, then $\{Y'W_iY\}_{i=1}^l$ is an independent family of noncentral Wishart $\mathcal{W}_p(tr(AW_i), \Sigma, \boldsymbol{\mu}'W_i\boldsymbol{\mu})$ random matrices if and only if for any distinct $i, j \in \{1, 2, \ldots, l\}$,*

(a) $AW_iAW_iA = AW_iA \neq \mathbf{0}$;

(b) $\boldsymbol{\mu}'W_iAW_iAW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iAW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W\boldsymbol{\mu}$;

(c) $AW_iAW_jA = \mathbf{0}$;

(d) $AW_iAW_j\boldsymbol{\mu} = \mathbf{0}$; and

(e) $\boldsymbol{\mu}'W_iAW_j\boldsymbol{\mu} = \mathbf{0}$.

Note that when $p = 1$, Theorem 3.4.2 is reduced to the chi-square version of Cochran's theorem obtained in the sixties.

Corollary 3.4.5. *Let* $\mathbf{y} \sim \mathcal{N}_{\mathbf{n}}(\boldsymbol{\mu}, \mathbf{C})$ *and* $\{W_i\}$ *be a set of symmetric matrices of order* n*. Then a set of quadratic forms* $\mathbf{y}'W_i\mathbf{y}$*'s is an independent family of noncentral chi-square* $\chi^2(r(CW_i), \boldsymbol{\mu}'W_i\boldsymbol{\mu})$ *random variables if and only if for any distinct* $i, j \in \{1, 2, \ldots, l\}$,

(a) $CW_iCW_iC = CW_iC \neq \mathbf{0}$*;*

(b) $\boldsymbol{\mu}'W_iCW_iCW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iCW_i\boldsymbol{\mu} = \boldsymbol{\mu}'C\boldsymbol{\mu}$*; and*

(c) $CW_iCW_jC = \mathbf{0},\ CW_iCW_j\boldsymbol{\mu} = \mathbf{0},\ \boldsymbol{\mu}'W_iCW_j\boldsymbol{\mu} = 0.$

Chapter 4

A Multivariate Version of Cochran's Theorem Concerning DINWRMs

In this Chapter we shall discuss the extended problem stated in Section 1.1. Let Y be an $n \times p$ multivariate normal random matrix with mean $\boldsymbol{\mu}$ and general covariance Σ_Y. In Section 4.3, we give a set of necessary and sufficient conditions (Theorem 4.3.1) for the matrix quadratic form $Y'WY$ with the symmetric matrix W to be distributed as *differences of independent noncentral Wishart random matrices (DINWRMs).* In Section 4.4, we consider the symmetric matrices W_1, W_2, $\ldots$, W_l. Then we develop a set of necessary and sufficient conditions (Theorem 4.4.1) for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ to be an independent family of random matrices distributed as differences of independent noncentral Wishart random matrices.

In this Chapter, as the intermediate result, we also use a matrix approach to present the proven result for the mean zero case. This case has been solved by Masaro and Wong (2004b). They used Jordan algebra homomorphisms to obtain the necessary and sufficient conditions for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of random matrices distributed as *differences of independent Wishart random matrices (DIWRMs)*. Their result and proof is more mathematically involved. Our presentation provides a discrete representation version of Cochran's theorem concerning DIWRMs. For details, in Section 4.1 a set of necessary and sufficient conditions (Theorem 4.1.2 and Theorem 4.1.1 for a special case) is established for the matrix quadratic form $Y'WY$ with the symmetric matrices W to be distributed as a *difference of independent Wishart random matrices (DIWRM)*. In Section 4.2, we consider symmetric matrices W_1, W_2, ..., W_l and develop a set of necessary and sufficient conditions (Theorem 4.2.2) for the matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of random matrices distributed as differences of independent Wishart random matrices. Some special cases are presented. Also, we use a matrix approach to present the result (Theorem 4.1.8) obtained by Masaro and Wong (2004b).

4.1 Conditions for a MQF to be Distributed as a DIWRM

The following theorem gives a set of condition for the matrix quadratic form $Y'WY$ to be distributed as a difference of independent Wishart random matrices, where the Wishart distributions $\mathcal{W}_p(m_1, \Lambda)$ and $\mathcal{W}_p(m_2, \Lambda)$ have a diagonal common covariance

Λ.

Theorem 4.1.1. *Let Y be an $n \times p$ random matrix normally distributed as $\mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W be a symmetric matrix of order n. Then $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$ for some nonnegative integers m_1 and m_2 if and only if there exists a diagonal $\Lambda \in \mathbb{N}_p$ such that for any elements $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathbb{E}_p$,*

$$\Sigma_Y \left[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})\right] \Sigma_Y = G(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, \Sigma_Y) + G(\tilde{\mathbf{t}}, \mathbf{t}, \Lambda, \Sigma_Y) \tag{4.1}$$

where $G(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, \Sigma_Y) = \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \tilde{\mathbf{t}})\Sigma_Y$ and

$$\Sigma_Y(W \otimes \Lambda^+)\Sigma(W \otimes \mathbf{t})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y \tag{4.2}$$

with

$$\{\mathbf{t} : \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \mathbf{0}\} = \mathbb{K}_0 \tag{4.3}$$

where $\mathbb{K}_0 \equiv \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}, \mathbf{t} \in \mathbb{E}_p\}$ and

$$tr(\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \mathbf{t})) + tr(\Sigma_Y(W \otimes \mathbf{t})) = 2m_1 tr(\Lambda\mathbf{t}) \tag{4.4}$$

$$tr(\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \mathbf{t})) - tr(\Sigma_Y(W \otimes \mathbf{t})) = 2m_2 tr(\Lambda\mathbf{t}). \tag{4.5}$$

Proof. By Lemma 2.1.1, decompose the nonnegative definite matrix $\Sigma_{Y'}$ as

$$\Sigma_{Y'} = L'L, \quad L = [L_1, L_2, \ldots, L_p]$$

with $L_i \in \mathbb{M}_{q\times n}$ $(i = 1, 2, \ldots, p)$ and $\mathrm{r}(\Sigma_{Y'}) \leq q \leq np$.

Exactly as in the proof of Lemma 2.1.6, we obtain the following equivalent relations of (4.1)-(4.5), respectively,

(a) $L[(\mathbf{t}\Lambda\tilde{\mathbf{t}}+\tilde{\mathbf{t}}\Lambda\mathbf{t})\otimes W]L' = \Gamma(\mathbf{t},\tilde{\mathbf{t}},\Lambda,L)+\Gamma(\tilde{\mathbf{t}},\mathbf{t},\Lambda,L)$;

(b)$L(\Lambda^+\otimes W)L'L(\mathbf{t}\otimes W)L' = L(\mathbf{t}\otimes W)L'L(\Lambda^+\otimes W)L'$;

(c) $\{\mathbf{t}: \ L(\mathbf{t}\otimes W)L' = \mathbf{0}\} = \mathbb{K}_0$;

(d) $tr(L(\Lambda^+\otimes W)L'L(\mathbf{t}\otimes W)L') + tr(L(\mathbf{t}\otimes W)L') = 2m_1 tr(\Lambda\mathbf{t})$; and

(e) $tr(L(\Lambda^+\otimes W)L'L(\mathbf{t}\otimes W)L') - tr(L(\mathbf{t}\otimes W)L') = 2m_2 tr(\Lambda\mathbf{t})$

where $\Gamma(\mathbf{t},\tilde{\mathbf{t}},\Lambda,L) = L(\mathbf{t}\otimes W)L'L(\Lambda^+\otimes W)L'L(\tilde{\mathbf{t}}\otimes W)L'$.

Let

$$B_{ij} = [L_iWL_j' + L_jWL_i']/2\sqrt{\sigma_i\sigma_j}, \ i,j\le r.$$

Then from the Theorem of the Appendix, we only show that $(a)-(e)$ are equivalent to the following conditions (C1)-(C6).

(C1) $L_iWL_j' + L_jWL_i' = \mathbf{0}$ for i or $j > r$;

(C2) $B_{ii}^3 = B_{ii}$, $tr(B_{ii}) = m_1 - m_2$, $tr(B_{ii}^2) = m_1 + m_2$;

(C3) $B_{ii}B_{jj} = \mathbf{0}$, $i\ne j$;

(C4) $4B_{ij}^2 = B_{ii}^2 + B_{jj}^2$, $i\ne j$;

(C5) $B_{ii}B_{ij} = B_{ij}B_{jj}$, $i\ne j$; and

(C6) $2(B_{ii}+B_{jj})(B_{ik}B_{jk}+B_{jk}B_{ik}) = B_{ij}$ for all distinct i,j,k.

First of all, suppose conditions (C1)-(C6) hold. We show that (4.1)-(4.5) hold.

Let

$$B = \sum_{i=1}^{r} B_{ii} \tag{4.6}$$

and use $(ij, i'j')$ to represent combination $(\mathbf{t},\tilde{\mathbf{t}})$ from the basic base $\mathbb{E}_p$. Then by (C1) we only consider these combinations $(ij,i'j')$, $1\le i\le j\le r, 1\le i'\le j'\le r$. Write $\Omega = \{(ij,i'j'): \ 1\le i\le j\le r, 1\le i'\le j'\le r\}$. Divide the index set Ω into the following seven index subsets:

$$D_1 = \{(ii, ii) : 1 \le i \le r\};$$

$$D_2 = \{(ij, ij) : 1 \le i < j \le r\};$$

$$D_3 = \{(ii, jj) : 1 \le i, j \le r; i \ne j\};$$

$$D_4 = \{(ii, ij) \cup (ij, ii) : 1 \le i < j \le r\};$$

$$D_5 = \{(ik, jk) : 1 \le i, j < k \le r; i, j \text{ distinct}\};$$

$$D_6 = \{(ii, i'j') \cup (i'j', ii) : 1 \le i, i', j' \le r; i, i', j' \text{ distinct}, i' < j'\}; \text{ and}$$

$$D_7 = \{(ij, i'j') : 1 \le i < j \le r, 1 \le i' < j' \le r; i, j, i', j' \text{ distinct}\}.$$

Note that by (C3), (C4) and Lemma 2.1.3,

$$B_{ij}B_{kk} = \mathbf{0} \quad \text{for distinct } i, j, k. \tag{4.7}$$

For $(ij, i'j') \in D_1$, (a) is reduced to $\sigma_i\sigma_j(B_{ii} + B_{jj}) = \sigma_i^2 B_{ii}BB_{ii}$, which follows from (C2) and (C3).

For $(ij, i'j') \in D_2$, (a) is reduced to $\sigma_i\sigma_j(B_{ii} + B_{jj}) = 4\sigma_i\sigma_j B_{ij}BB_{ij}$, which is derived from (C5) and (4.7).

For $(ij, i'j') \in D_3$, (a) is reduced to $\sigma_i\sigma_j(B_{ii}BB_{jj} + B_{jj}BB_{ii}) = \mathbf{0}$, which is obtained from (C3).

For $(ij, i'j') \in D_4$, (a) is reduced to

$$2\sqrt{\sigma_i\sigma_j}\sigma_i B_{ij} = 2\sqrt{\sigma_i\sigma_j}\sigma_i(B_{ii}BB_{ij} + B_{ij}BB_{ii}),$$

which follows from (C5), (C6) and (4.7).

For $(ij, i'j') \in D_5$, (a) is reduced to

$$2\sqrt{\sigma_i\sigma_j}\sigma_k B_{ij} = 4\sqrt{\sigma_i\sigma_j}\sigma_k(B_{ik}BB_{jk} + B_{jk}BB_{ik}),$$

which follows from (C5), (C6) and (4.7).

For $(ij, i'j') \in D_6 \cup D_7$, (a) is reduced to

$$4\sqrt{\sigma_i\sigma_j\sigma_{i'}\sigma_{j'}}(B_{ij}BB_{i'j'} + B_{i'j'}BB_{ij}) = \mathbf{0},$$

which follows from (4.7), that proves that (a) holds.

For $(ij, i'j') \in \Omega$, (b) or $BB_{ij} = B_{ij}B$ follows from (4.7) and (C5).

Further, let

$$\mathbf{t} = \sum_{1\le i\le j\le p} t_{ij}E_{ij},\ t_{ij} = 0 \text{ or } 1,\ 1 \le i \le j \le p.$$

Then, by (C1)

$$L(\mathbf{t}\otimes W)L' = L(\sum_{1\le i\le j\le p} t_{ij}E_{ij}\otimes W)L' = \sum_{i=1}^{r} t_{ii}\sigma_i B_{ii} + \sum_{1\le i<j\le r} t_{ij}\sqrt{\sigma_i\sigma_j}B_{ij}.$$

Since $\mathbf{t} \in \{\mathbf{t} :\ L(\mathbf{t}\otimes W)L' = \mathbf{0}\}$, we get

$$\sum_{i=1}^{r} t_{ii}\sigma_i B_{ii} + \sum_{1\le i<j\le p} t_{ij}\sqrt{\sigma_i\sigma_j}B_{ij} = \mathbf{0}.$$

From (C2) and (C5), $B_{ij} \ne \mathbf{0}$, $1 \le i \le j \le r$, $1 \le i' \le j' \le r$. So $t_{ij} = 0$ for $i, j = 1, 2, \ldots, r$, which is equivalent to (c).

Finally, taking $\mathbf{t} = E_{ii}$, by (C2) and (C3),

$$tr(L(\Lambda^+\otimes W)L'L(E_{ii}\otimes W)L') + tr(L(E_{ii}\otimes W)L')$$

$$= \sigma_i tr(BB_{ii} + B_{ii}) = \sigma_i[tr(B_{ii}^2) + tr(B_{ii})] = 2m_1\sigma_i = 2m_1 tr(\Lambda E_{ii})$$

and similarly,

$$tr(L(\Lambda^+\otimes W)L'L(E_{ii}\otimes W)L') - tr(L(E_{ii}\otimes W)L') = \sigma_i[tr(B_{ii}^2) - tr(B_{ii})] = 2m_2 tr(\Lambda E_{ii}).$$

Taking $\mathbf{t} = E_{ij}$ $i \neq j$, $tr(\Lambda E_{ij}) = 0$. By (C2) and (C3), there exists an orthogonal matrix H, which does not depend on i, such that

$$B_{ii} = H(E_{ii} \otimes A_{ii})H' \tag{4.8}$$

where $A_{ii} = diag[I_{m_1}, -I_{m_2}, \mathbf{0}]$. By (C3)-(C5) and (4.8) we obtain

$$2B_{ij} = H(\mathbf{e_{ij}} \otimes A_{ij} + \mathbf{e}_{ji} \otimes A_{ji})H' \tag{4.9}$$

where $A_{ij} = diag[U_{ij}, V_{ij}, \mathbf{0}] \in \mathbb{M}_{n \times n}$, $U_{ij} \in \mathbb{M}_{m_1 \times m_1}$, $V_{ij} \in \mathbb{M}_{m_2 \times m_2}$ and $A'_{ij} = A_{ji}$, $U_{ij}U'_{ij} = I_{m_1}$, $V_{ij}V'_{ij} = I_{m_2}$. So

$$\begin{aligned} & tr(L(\Lambda^+ \otimes W)L'L(E_{ij} \otimes W)L') \pm tr(L(E_{ij} \otimes W)L') \\ &= 2\sqrt{\sigma_i \sigma_j}[tr(BB_{ij}) \pm tr(B_{ij})] = 2\sqrt{\sigma_i \sigma_j} tr[(B_{ii} + B_{jj} \pm I)B_{ij}] \\ &= \sqrt{\sigma_i \sigma_j} tr((\mathbf{e}_{ii} \otimes A_{ii} + \mathbf{e}_{jj} \otimes A_{jj} \pm I)(\mathbf{e_{ij}} \otimes A_{ij} + \mathbf{e}_{ji} \otimes A_{ji})) = 0, \end{aligned}$$

which proves (d) and (e).

Conversely, suppose (4.1)-(4.5) hold and, equivalently, (a)-(e) hold. We show that conditions (C1)-(C6) hold.

(C1) follows from (c), i.e. $L(E_{ij} \otimes W)L' = \mathbf{0}$, for i or $j > 0$.

Fixing $(1 \leq i < j \leq r)$ and taking $\mathbf{t} = \tilde{\mathbf{t}} = E_{ii}$ in (a)-(b), we have

$$B_{ii} = B_{ii}BB_{ii}, \; B_{ii}B = BB_{ii} \tag{4.10}$$

and $tr(BB_{ii} + B_{ii}) = 2m_1$, $tr(BB_{ii} - B_{ii}) = 2m_2$ or

$$tr(B_{ii}) = m_1 - m_2, \; tr(BB_{ii}) = m_1 + m_2. \tag{4.11}$$

Taking $\mathbf{t} = E_{ii}$ and $\tilde{\mathbf{t}} = E_{jj}$ in (a) gives

$$B_{ii}BB_{jj} + B_{jj}BB_{ii} = \mathbf{0}. \tag{4.12}$$

By (4.10) and (4.12), we have $||B_{ii}B_{jj} \pm B_{jj}B_{ii}||^2 = 0$, i.e. $B_{ii}B_{jj} = \mathbf{0}$, which proves (C3). Also by (4.10) and (4.12), we obtain

$$B^3 = B, \tag{4.13}$$

and then

$$B_{ii}^3 = B_{ii}, \tag{4.14}$$

so $tr(B_{ii}^2) = tr(B_{ii}B_{ii}BB_{ii}) = tr(B_{ii}^3 B) = tr(B_{ii}B) = m_1 + m_2$, which proves (C2).

Taking $\mathbf{t} = E_{ii}$ and $\tilde{\mathbf{t}} = E_{ij}$ in (a) gives

$$B_{ij} = B_{ii}BB_{ij} + B_{ij}BB_{ii}. \tag{4.15}$$

Taking $\mathbf{t} = E_{ij}$ and $\tilde{\mathbf{t}} = E_{jj}$ in (a) and (b) gives

$$B_{ij} = B_{ij}BB_{jj} + B_{jj}BB_{ij} \tag{4.16}$$

and

$$BB_{ij} = B_{ij}B. \tag{4.17}$$

So $B_{ii}B_{ij} = B_{ii}B_{ij}BB_{jj}$ and $B_{ij}B_{jj} = B_{ii}BB_{ij}B_{jj}$, which proves (C5).

Taking $\mathbf{t} = \tilde{\mathbf{t}} = E_{ij}$ in (a) gives

$$4B_{ij}BB_{ij} = B_{ii} + B_{jj}. \tag{4.18}$$

From (C3), (C5) and (4.16)-(4.18), we obtain $4B_{ij}^2 = B_{ii}^2 + B_{jj}^2$, which proves (C4).

From (4.9), (4.15), (4.16) and (C3), we obtain that for distinct i, j, k,

$$B_{ij}B_{kk} = \mathbf{0} \tag{4.19}$$

Taking $\mathbf{t} = E_{ik}$ and $\tilde{\mathbf{t}} = E_{jk}$ for distinct i, j, k in (a) gives

$$B_{ij} = 2B_{ik}BB_{jk} + 2B_{jk}BB_{ik}. \tag{4.20}$$

So from (4.20), (4.19)and (C5),

$$B_{ij} = 2B_{ik}B_{kk}B_{jk} + 2B_{jk}B_{kk}B_{ik} = 2B_{ii}B_{ik}B_{jk} + 2B_{jj}B_{jk}B_{ik}$$
$$= 2(B_{ii} + B_{jj})(B_{ik}B_{jk} + 2B_{jk}B_{ik}),$$

which proves (C6) and then the proof is completed. □

Theorem 4.1.1 provides us the equivalent matrix algebraic conditions of $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$. Here the covariance Λ is a diagonal matrix. Based on Theorem 4.1.1, we establish the matrix algebraic conditions equivalent to $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ with a common covariance Σ. The following theorem is for the case of a common covariance matrix $\Sigma \in \mathbb{N}_p$ instead of a diagonal common covariance Λ.

Theorem 4.1.2. *Suppose that $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W is a symmetric matrix of order n. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ for nonnegative integers m_1 and m_2 if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that for any elements $\mathbf{h}, \tilde{\mathbf{h}}$ in the similar base $\mathbb{H}_p$ associated with Σ,*

$$\Sigma_Y \left[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})\right] \Sigma_Y = G(\mathbf{h}, \tilde{\mathbf{h}}, \Sigma, \Sigma_Y) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Sigma, \Sigma_Y) \tag{4.21}$$

and

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y \tag{4.22}$$

with

$$\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \mathbb{K} \tag{4.23}$$

where $\mathbb{K} \equiv \{\mathbf{h} : \Sigma\mathbf{h}\Sigma = \mathbf{0},\ \mathbf{h} \in \mathbb{H}_p\}$ *and*

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) + tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_1 tr(\Sigma\mathbf{h}) \tag{4.24}$$

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) - tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_2 tr(\Sigma\mathbf{h}). \tag{4.25}$$

Proof. Since $\Sigma \in \mathbb{N}_p$, there is an orthogonal matrix H of order p such that

$$H'\Sigma H = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0] \equiv \Lambda,\ r = r(\Sigma),\ \sigma_i > 0,\ i = 1, 2, \ldots, r,$$

and YH has a normal distribution $\mathcal{N}_{n \times p}(\mathbf{0}, \Sigma_{YH})$ where $\Sigma_{YH} = (I \otimes H')\Sigma_Y(I \otimes H))$.

Assume (4.21)-(4.25) hold. Let

$$\mathbf{t} = H'\mathbf{h}H,$$

then the function $\mathbf{t} = H'\mathbf{h}H$ is a one to one map from the similar base $\mathbb{H}_p$ associated with Σ onto the basic base E_p. By replacing $\mathbf{h}$, $\tilde{\mathbf{h}}$ and Σ, respectively, with $H\mathbf{t}H'$, $H\tilde{\mathbf{t}}H'$ and $H\Lambda H'$ in (4.21)-(4.25), with (2.5) and necessary tensor calculations, (4.21)-(4.25) are, respectively, expressed as, for any $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathbb{E}_p$.

$$\Sigma_{YH}\left[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})\right]\Sigma_{YH} = G(\mathbf{h}, \tilde{\mathbf{h}}, \Lambda, \Sigma_{YH}) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Lambda, \Sigma_{YH})$$

$$\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH},$$

$$\{\mathbf{t} : \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \mathbf{0}\} = \mathbb{K}_0,$$

$$tr(\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})) + tr(\Sigma_{YH}(W \otimes \mathbf{t})) = 2m_1 tr(\Lambda\mathbf{t}),$$

and

$$tr(\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})) - tr(\Sigma_{YH}(W \otimes \mathbf{t})) = 2m_2 tr(\Lambda\mathbf{t}).$$

By Theorem 4.1.1, $H'Y'WYH \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$. Hence $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Lambda)$ follows from Lemma 2.2.2.

The converse can be shown by following the above steps backwards. □

Remark 4.1.3. *In fact, whenever $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Lambda)$, the degrees of freedom m_1 and m_2 can be given by*

$$m_1 = \frac{1}{2r(\Sigma)}\left[tr(\Sigma_Y(W \otimes \Sigma^+))^2 + tr(\Sigma_Y(W \otimes \Sigma^+))\right],$$
$$m_2 = \frac{1}{2r(\Sigma)}\left[tr(\Sigma_Y(W \otimes \Sigma^+))^2 - tr(\Sigma_Y(W \otimes \Sigma^+))\right].$$

Next we shall discuss the applications of Theorem 4.1.2 and Theorem 4.1.1.

In Theorem 4.1.2, suppose that the covariance Σ_Y is the Kronecker product $A \otimes \Sigma$ for nonnegative definite A of order n. Theorem 4.1.2 is reduced to the following corollary, which was obtained by Tan (1975).

Corollary 4.1.4. *Let W be a symmetric matrix of order n and $Y \sim \mathcal{N}_{n \times p}(\mathbf{0}, A \otimes \Sigma)$ with $A \in \mathbb{N}_n$ and $\Sigma \in \mathbb{N}_p$. Then, $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ for nonnegative integers m_1 and m_2 if and only if*

(1)$AWA = AWAWAWA \neq \mathbf{0}$; and

(2) $tr(AW)^2 + tr(AW) = 2m_1$, $tr(AW)^2 - tr(AW) = 2m_2$.

Proof. Replace $A \otimes \Sigma$ with Σ_Y in (4.21)-(4.25). With (2.5) and some calculations, we prove (4.23). Then, (1) follows from (4.21) and (4.22); and (2) follows from (4.24) and (4.25), which proves the desired result. □

In Theorem 4.1.2, if $\mathbf{y}$ is an $n \times 1$ random normal vector with mean vector $\mathbf{0}$ and covariance C of order n, (4.21)-(4.25) are reduced to the familiar conditions which were showed by Tan (1975).

Corollary 4.1.5. *Let* $\mathbf{y} \sim \mathcal{N}_n(\mathbf{0}, C)$ *and* W *be a symmetric matrix of order* n. *Then* $\mathbf{y}'W\mathbf{y} \sim \chi^2(m_1) - \chi^2(m_2)$ *for nonnegative integers* m_1 *and* m_2 *if and only if*

$$CWC = CWCWCWC \neq \mathbf{0}; \text{ and} \tag{4.26}$$

$$tr(CW)^2 + tr(CW) = 2m_1, \ tr(CW)^2 - tr(CW) = 2m_2. \tag{4.27}$$

Proof. In the univariate case $p = 1$, $\Sigma_Y = C$. Using Theorem 4.1.2, we get Corollary 4.1.5. □

If C=I in Corollary 4.1.5, (4.26) is reduced to the well-known condition, $W^3 = W$, if and only if the quadratic form $\mathbf{y}'W\mathbf{y}$ is distributed as a difference of two independent chi-square random variables, see Luther (1965) and Graybill (1969).

In Theorem 4.1.2, if we replace the covariance Σ_Y of Y with the sum of special Kronecker products, we have the following corollary.

Corollary 4.1.6. *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *with* $\Sigma_Y = \sum_{i=1}^{r} A_i \otimes E_{ii}$, $r \leq p$, $A_i \in \mathbb{N}_n$. *Then, for* $W \in \mathbb{S}_n$, $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$, *where* $\Sigma = \sum_{i=1}^{r} \sigma_i E_{ii}$, *for nonnegative integers* m_1 *and* m_2 *if and only if there exist real numbers* $\sigma_k > 0$, $k = 1, 2, \ldots, r$ *such that for all* $i, j, k \leq r$,

(1) $A_iWA_kWA_kWA_j = \sigma_k^2 A_iWA_j \neq \mathbf{0}$;

(2) $\sigma_j A_iWA_iWA_j = \sigma_i A_iWA_jWA_j$; *and*

(3) $\frac{1}{\sigma_i^2} tr(A_iW)^2 + \frac{1}{\sigma_i} tr(A_iW) = 2m_1$, $\frac{1}{\sigma_i^2} tr(A_iW)^2 - \frac{1}{\sigma_i} tr(A_iW) = 2m_2$.

Proof. Replace Σ_Y and Σ with $\sum_{i=1}^{r} A_i \otimes E_{ii}$ and $\sum_{i=1}^{r} \sigma_i E_{ii}$, respectively, in (4.21)-(4.25). (4.21)-(4.25), respectively, become

$$\sum_{i,j,k=1}^{r} (\sigma_k A_iWA_j - \frac{1}{\sigma_k} A_iWA_kWA_kWA_j) \otimes E_{ii}(\mathbf{t}E_{kk}\tilde{\mathbf{t}} + \tilde{\mathbf{t}}E_{kk}\mathbf{t})E_{jj} = \mathbf{0},$$

$$\sum_{i,j=1}^{r}(\frac{1}{\sigma_i}A_iWA_iWA_j - \frac{1}{\sigma_j}A_iWA_jWA_j) \otimes E_{ii}\mathbf{t}E_{jj} = \mathbf{0},$$

$$\left\{\mathbf{t} : \sum_{i,j=1}^{r} A_iWA_j \otimes E_{ii}\mathbf{t}E_{jj} = \mathbf{0}\right\} = \left\{\mathbf{t} : \sum_{i,j=1}^{r} \sigma_i\sigma_j E_{ii}\mathbf{t}E_{jj} = \mathbf{0}\right\},$$

and

$$tr\left(\sum_{i=1}^{r}\frac{1}{\sigma_i}A_iWA_iW \otimes E_{ii}\mathbf{t}\right) + tr\left(\sum_{i=1}^{r} tr(A_iW)tr(E_{ii}\mathbf{t})\right) = 2m_1 tr(\sum_{i=1}^{r}\sigma_i E_{ii}\mathbf{t})$$

$$tr\left(\sum_{i=1}^{r}\frac{1}{\sigma_i}A_iWA_iW \otimes E_{ii}\mathbf{t}\right) - tr\left(\sum_{i=1}^{r} tr(A_iW)tr(E_{ii}\mathbf{t})\right) = 2m_2 tr(\sum_{i=1}^{r}\sigma_i E_{ii}\mathbf{t}).$$

The desired results follow from the above equations by the arbitrariness of $\mathbf{t} \in \mathbb{E}_p$. □

Corollary 4.1.7. *In Theorem 4.1.2, suppose Σ_Y is nonsingular. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ for nonnegative integers m_1 and m_2 if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that for any element $\mathbf{h}$ in the similar base $\mathbb{H}_p$ associated with Σ,*

$$W \otimes \Sigma = (W \otimes I)\Sigma_Y(W \otimes \Sigma^{-1})\Sigma_Y(W \otimes I) \tag{4.28}$$

and

$$(W \otimes \Sigma^{-1})\Sigma_Y(W \otimes \mathbf{h}) = (W \otimes \mathbf{h})\Sigma_Y(W \otimes \Sigma^{-1}) \tag{4.29}$$

with

$$tr(\Sigma_Y(W \otimes \Sigma^{-1})\Sigma_Y(W \otimes \mathbf{h})) + tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_1 tr(\Sigma\mathbf{h}) \tag{4.30}$$

$$tr(\Sigma_Y(W \otimes \Sigma^{-1})\Sigma_Y(W \otimes \mathbf{h})) - tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_2 tr(\Sigma\mathbf{h}). \tag{4.31}$$

Proof. Note that if Σ_Y is nonsingular, then Σ^{-1} exists from (4.23) in Theorem 4.1.2. The desired results are obtained from Theorem 4.1.2 with routine tensor operations.

□

Considering the symmetric matrix set $\mathbb{S}_p$ rather than the similar base set $\mathbb{H}_p$, the following theorem can be obtained from Theorem 4.1.2.

Theorem 4.1.8. *Let Y an $n \times p$ random matrix normally distributed with $\mathcal{N}_{n\times p}(0, \Sigma_Y)$ and W be a symmetric matrix of order n. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ for nonnegative integers m_1 and m_2 if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that for any matrix $\mathbf{s}$ in $\mathbb{S}_p$,*

$$\Sigma_Y(W \otimes \mathbf{s}\Sigma\mathbf{s})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y \tag{4.32}$$

and

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma(W \otimes \mathbf{s})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y \tag{4.33}$$

with

$$\{\mathbf{s} : \ \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y = \mathbf{0}\} = \{\mathbf{s} : \ \Sigma\mathbf{s}\Sigma = \mathbf{0}\} \tag{4.34}$$

and

$$tr[(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y + I)\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y] = 2m_1 tr(\Sigma\mathbf{s}) \tag{4.35}$$

$$tr[(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y - I)\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y] = 2m_2 tr(\Sigma\mathbf{s}). \tag{4.36}$$

Proof. (4.22) $\iff$ (4.33), (4.24) $\iff$ (4.35) and (4.25) $\iff$ (4.36) are trivial by the linearity of these conditions. And the equivalence between (4.23) and (4.34) can be shown by using the same way in the proof of Theorem 3.1.11.

Since for any elements $\mathbf{s}, \tilde{\mathbf{s}}$ in the similar base $\mathbb{S}_p$,

$$\Sigma_Y[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})]\Sigma_Y = \frac{1}{2}\Sigma_Y\left(W \otimes [(\mathbf{s}+\tilde{\mathbf{s}})\Sigma(\mathbf{s}+\tilde{\mathbf{s}}) - (\mathbf{s}-\tilde{\mathbf{s}})\Sigma(\mathbf{s}-\tilde{\mathbf{s}})]\right)\Sigma_Y, \tag{4.37}$$

(4.32) is equivalent to

$$\Sigma_Y \left[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})\right] \Sigma_Y = G(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, \Sigma_Y) + G(\tilde{\mathbf{s}}, \mathbf{s}, \Sigma, \Sigma_Y) \text{ for } \mathbf{s}, \tilde{\mathbf{s}} \in \mathbb{S}_p. \quad (4.38)$$

Obviously, (4.38) implies (4.21) and hence (4.32) $\Longrightarrow$ (4.21).

Further, assume that (4.21) holds. For any $\mathbf{s}$, $\tilde{\mathbf{s}}$ in the set $\mathbb{S}_p$, $\mathbf{s}$ and $\tilde{\mathbf{s}}$ can be expressed as the linear combinations of $\mathbf{h}_{ij} \in \mathbb{H}_p$, $1 \le i \le j \le p$. Let

$$\mathbf{s} = \sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij},\ s_{ij} \in \mathbb{R} \ \text{ and } \ \tilde{\mathbf{s}} = \sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl},\ \tilde{s}_{kl} \in \mathbb{R}.$$

Then we have

$$\begin{aligned}
&\Sigma_Y[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})]\Sigma_Y \\
&= \sum 1 \le i \le j \le p \sum_{1\le k\le l\le p} s_{ij}\tilde{s}_{kl}\Sigma_Y[W \otimes (\mathbf{h}_{ij}\Sigma\mathbf{h}_{kl} + \mathbf{h}_{kl}\Sigma\mathbf{h}_{ij})]\Sigma_Y \\
&= \sum_{1\le i\le j\le p}\sum_{1\le k\le l\le p} s_{ij}\tilde{s}_{kl}[G(\mathbf{h}_{ij}, \mathbf{h}_{kl}, \Sigma, \Sigma_Y) + G(\mathbf{h}_{kl}, \mathbf{h}_{ij}, \Sigma, \Sigma_Y)] \\
&= G(\sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij}, \sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl}, \Sigma, \Sigma_Y) + G(\sum_{1\le k\le l\le p} \tilde{s}_{kl}\mathbf{h}_{kl}, \sum_{1\le i\le j\le p} s_{ij}\mathbf{h}_{ij}, \Sigma, \Sigma_Y) \\
&= G(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, \Sigma_Y) + G(\tilde{\mathbf{s}}, \mathbf{s}, \Sigma, \Sigma_Y),
\end{aligned}$$

that proves that (4.21) implies (4.38), which implies (4.32), and that completes the proof. □

Masaro and Wong (2004b) essentially obtained Theorem 4.1.8 as the special case of their main result by using Jordan algebra homomorphisms in their technical report. Their result was obtained for very general case and its proof was quite technical. In this thesis, we use a matrix approach to obtain the same result as Masaro and Wong. Thus, our approach has advantage to be simple for applications while providing the same result as in Masaro and Wong (2004b).

Putting Theorem 4.1.2, Theorem 4.1.8 and Lemma 2.3.5 together, we have the following corollary.

Corollary 4.1.9. *Let $Y \sim \mathcal{N}_{n \times p}(0, \Sigma_Y)$ and $\Sigma \in \mathbb{N}_p$. Then the following statements are equivalent.*

(a) $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ for nonnegative integers m_1 and m_2;

(b) There exists some $\Sigma \in \mathbb{N}_p$ such that for any $\mathbf{s} \in \mathbb{S}_p$,

$$|I_{np} - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}| = |I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_1}|I_p + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{m_2};$$

(c) There exists some $\Sigma \in \mathbb{N}_p$ such that the matrix $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ and the diagonal matrix $diag[I_{m_1} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, -I_{m_2} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, \mathbf{0}] \in \mathbb{S}_{np}$ have the same characteristic polynomial for all $\mathbf{s} \in \mathbb{S}_p$; and

(d) There exists some $\Sigma \in \mathbb{N}_p$ such that for any positive integer k and any $\mathbf{s} \in \mathbb{S}_p$,

$$tr(\Sigma_Y(W \otimes \mathbf{s}))^k = [m_1 + (-1)^k m_2] tr(\Sigma\mathbf{s})^k.$$

(e) There exists some $\Sigma \in \mathbb{N}_p$ such that for any matrix $\mathbf{s}$ in $\mathbb{S}_p$,

$$\Sigma_Y(W \otimes \mathbf{s}\Sigma\mathbf{s})\Sigma_Y = G(\mathbf{s}, \mathbf{s}, \Sigma, \Sigma_Y)$$

and

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma(W \otimes \mathbf{s})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y$$

with $\{\mathbf{s} : \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y = \mathbf{0}\} = \{\mathbf{s} : \Sigma\mathbf{s}\Sigma = \mathbf{0}\}$ and

$$tr[(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y + I)\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y] = 2m_1 tr(\Sigma\mathbf{s})$$

$$tr[(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y - I)\Sigma_Y(W \otimes \mathbf{s})\Sigma_Y] = 2m_2 tr(\Sigma\mathbf{s}).$$

(f) There exists some $\Sigma \in \mathbb{N}_p$ *such that for any elements* $\mathbf{h}, \tilde{\mathbf{h}}$ *in the similar base* $\mathbb{H}_p$ *associated with* Σ,

$$\Sigma_Y \left[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})\right] \Sigma_Y = G(\mathbf{h}, \tilde{\mathbf{h}}, \Sigma, \Sigma_Y) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Sigma, \Sigma_Y)$$

and

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y$$

with $\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \mathbb{K}$ *and*

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) + tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_1 tr(\Sigma\mathbf{h})$$

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) - tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_2 tr(\Sigma\mathbf{h}).$$

It is seen that (f) of Corollary 4.1.9 is easy to verify, compared to the rest.

4.2 Conditions for MQFs to be an Independent Family of Random Matrices Distributed as DI-WRMs

Replacing $\mathbb{H}_p$ with $\mathbb{E}_p$ and applying Theorem 4.1.1 and Lemma 3.4.1, we establish a multivariate version of Cochran's theorem. Namely, we prove a result concerning differences of independent Wishart random matrices with a common diagonal covariance Λ.

Theorem 4.2.1. *Suppose that* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *and* W_i*'s are symmetric matrices of order* n*. Then* $\{Y'W_iY\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Lambda) - \mathcal{W}_p(m_{2i}, \Lambda)$ *for nonnegative integers* m_{1i} *and* m_{i2} *if and only if there* *exists some* $\Lambda \in \mathbb{N}_p$ *such that for any distinct* $i, j \in \{1, 2, \ldots, l\}$ *and* $\mathbf{t}_i$, $\tilde{t}_i$ *in the basic base* $\mathbb{E}_p$,

(a) $\Sigma_Y[W_i \otimes (\mathbf{t}_i\Lambda\tilde{\mathbf{t}}_i + \tilde{\mathbf{t}}_i\Lambda\mathbf{t}_i)]\Sigma_Y = G_i(\mathbf{t}_i, \tilde{\mathbf{t}}_i, \Lambda, \Sigma_Y) + G_i(\tilde{\mathbf{t}}_i, \mathbf{t}_i, \Lambda, \Sigma_Y)$;

(b) $\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y$;

(c) $\{\mathbf{t}_i : \Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y = \mathbf{0}\} = \{\mathbf{t}_i : \Lambda\mathbf{t}_i\Lambda = \mathbf{0}\}$;

(d) $tr(\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t}_i)) + tr(\Sigma_Y(W_i \otimes \mathbf{t}_i)) = 2m_{1i}tr(\Lambda\mathbf{t}_i)$,

$tr(\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t}_i)) - tr(\Sigma_Y(W_i \otimes \mathbf{t}_i)) = 2m_{2i}tr(\Lambda\mathbf{t}_i)$; *and*

(e) $\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_j \otimes \Lambda^+)\Sigma_Y = \mathbf{0}$

where $G_i(\mathbf{t}_i, \tilde{\mathbf{t}}_i, \Lambda, \Sigma_Y) = \Sigma_Y(W_i \otimes \mathbf{t}_i)\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \tilde{\mathbf{t}}_i)\Sigma_Y$.

Proof. Let $\{Y'W_iY\}_{i=1}^l$ be an independent family of random matrices distributed as the differences of independent Wishart random matrices. Then (a)-(e) hold by Theorem 4.1.1 and Lemma 3.2.1.

Conversely, suppose (a)-(e) hold. For $i = 1, 2, \ldots, l$,

$$Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Lambda) - \mathcal{W}_p(m_{2i}, \Lambda)$$

follows from Theorem 4.1.1. Thus, to prove the independence of the matrix quadratic family, by Lemma 3.2.1, it suffices to show condition (3.29) or condition (3.30), from conditions (a)-(e).

Exactly as in the proof of Lemma 2.1.6, (3.30) is equivalent to

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = \mathbf{0} \text{ where } L'L = \Sigma_{Y'}\ \mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p \tag{4.39}$$

and (e) amounts to

$$L(\Lambda^+ \otimes W_i)L'L(\Lambda^+ \otimes W_j)L' = \mathbf{0}. \tag{4.40}$$

Namely, we only need to obtain (4.39) from statements (a)-(e).

For any $\mathbf{s}_i$ in the set $\mathbb{S}_p$, $\mathbf{s}_i$ can be written as

$$\mathbf{s}_i = \begin{bmatrix} \mathbf{a} & * \\ * & * \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathbb{S}_r.$$

Write

$$\mathbf{s}_i^* = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathbb{S}_r.$$

By (c), for any $\mathbf{s}_i, \mathbf{s}_j \in \mathbb{S}_p$,

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = L(\mathbf{s}_i^* \otimes W_i)L'L(\mathbf{s}_j^* \otimes W_j)L'. \tag{4.41}$$

Since, by (a) and (b),

$$\begin{aligned} L(\mathbf{s}_i^* \otimes W_i)L' =& L\left[\frac{1}{2}(\Lambda^+\Lambda\mathbf{s}_i^* + \mathbf{s}_i^*\Lambda\Lambda^+) \otimes W_i\right] L' \\ =& \frac{1}{2}\left([L(\Lambda^+ \otimes W_i)L']^2 L(\mathbf{s}_i^* \otimes W_i)L' + L(\mathbf{s}_i^* \otimes W_i)L'[L(\Lambda^+ \otimes W_i)L']^2\right] \\ =& L(\mathbf{s}_i^* \otimes W_i)L'[L(\Lambda^+ \otimes W_i)L']^2, \end{aligned} \tag{4.42}$$

similarly,

$$L(\mathbf{s}_j^* \otimes W_j)L' = [L(\Lambda^+ \otimes W_j)L']^2 L(\mathbf{s}_j^* \otimes W_j)L', \tag{4.43}$$

we obtain (4.39) from (4.41), (4.42) and (4.43), so the proof is complete. □

Based on Theorem 4.2.1, we obtain a multivariate version of Cochran's theorem concerning differences of independent Wishart random matrices with a common covariance Σ rather than a diagonal common covariance Λ. Exactly as in the proof of Theorem 4.1.2, we derive the following theorem from Theorem 4.2.1 and (1) of Lemma 3.4.1.

Theorem 4.2.2. *Suppose that $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and W_i's are symmetric matrices of order n. Then $\{Y'W_iY\}_{i=1}^l$ is independent and, for $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$ for nonnegative integers m_{1i} and m_{i2} if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that for any $i \in \{1, 2, \ldots, l\}$ and any elements $\mathbf{h}_i$ and $\tilde{\mathbf{h}}_i$ in the similar base $\mathbb{H}_p$ associated with Σ,*

(a) $\Sigma_Y \left[W_i \otimes (\mathbf{h}_i\Sigma\tilde{\mathbf{h}}_i + \tilde{\mathbf{h}}_i\Sigma\mathbf{h}_i)\right] \Sigma_Y = G_i(\mathbf{h}_i, \tilde{\mathbf{h}}_i, \Sigma, \Sigma_Y) + G_i(\tilde{\mathbf{h}}_i, \mathbf{h}_i, \Sigma, \Sigma_Y)$*;*

(b) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma(W_i \otimes \mathbf{h}_i)\Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y$*;*

(c) $\{\mathbf{h}_i : \Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y = \mathbf{0}\} = \mathbb{K}$;

(d) $tr(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{h}_i)) + tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{1i}tr(\Sigma\mathbf{h}_i)$,

$tr(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{h}_i)) - tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{2i}tr(\Sigma\mathbf{h}_i)$*; and*

(e) for any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = \mathbf{0}. \tag{4.44}$$

Proof. Since $\Sigma \in \mathbb{N}_p$, by Lemma 2.1.1, there is an orthogonal matrix H of order p such that $H'H = I_p$ and

$$H'\Sigma H = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0] \equiv \Lambda, \;\; r = r(\Sigma), \;\; \sigma_i > 0, \;\; i = 1, 2, \ldots, r.$$

And $YH \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_{YH})$, where $\Sigma_{YH} = (I \otimes H')\Sigma_Y(I \otimes H)$.

Defining

$$\mathbf{t}_i = H'\mathbf{h}_iH \;\; \text{for any } \mathbf{h}_i \in \mathbb{H}_p, \; i = 1, 2, \ldots, l.$$

for any $i = 1, 2, \ldots, l$, the function $\mathbf{t}_i = H'\mathbf{h}_iH$ is a one to one map from the similar base $\mathbb{H}_p$ associated with Σ onto the basic base $\mathbb{E}_p$. By replacing $\mathbf{h}_i$, $\tilde{\mathbf{h}}_i$, Σ and Σ_Y, respectively, with $H\mathbf{t}_iH'$, $H\tilde{\mathbf{t}}_iH'$, $H\Lambda H'$ and $(I\otimes H)\Sigma_{YH}(I\otimes H')$ in (a)-(e), we obtain

$$\Sigma_{YH}\left[W_i\otimes(\mathbf{t}_i\Lambda\tilde{\mathbf{t}}_i+\tilde{\mathbf{t}}_i\Lambda\mathbf{t}_i)\right]\Sigma_{YH}=G_i(\mathbf{h}_i,\tilde{\mathbf{h}}_i,\Lambda,\Sigma_{YH})+G_i(\tilde{\mathbf{h}}_i,\mathbf{h}_i,\Lambda,\Sigma_{YH}) \quad (4.45)$$

$$\Sigma_{YH}(W_i\otimes\Lambda^+)\Sigma_{YH}(W_i\otimes\mathbf{t}_i)\Sigma_{YH}=\Sigma_{YH}(W_i\otimes\mathbf{t}_i)\Sigma_{YH}(W_i\otimes\Lambda^+)\Sigma_{YH}, \quad (4.46)$$

$$\{\mathbf{t}_i:\Sigma_{YH}(W_i\otimes\mathbf{t}_i)\Sigma_{YH}=\mathbf{0},\mathbf{t}_i\in\mathbb{E}_p\}=\mathbb{K}_0, \quad (4.47)$$

$$\begin{aligned} &tr(\Sigma_{YH}(W_i\otimes\Sigma^+)\Sigma_{YH}(W_i\otimes\mathbf{h}_i))+tr(\Sigma_{YH}(W_i\otimes\mathbf{h}_i))=2m_{1i}tr(\Sigma\mathbf{h}_i),\\ &tr(\Sigma_{YH}(W_i\otimes\Sigma^+)\Sigma_{YH}(W_i\otimes\mathbf{h}_i))-tr(\Sigma_{YH}(W_i\otimes\mathbf{h}_i))=2m_{2i}tr(\Sigma\mathbf{h}_i)\end{aligned} \quad (4.48)$$

and

$$\Sigma_{YH}(W_i\otimes\Lambda^+)\Sigma_{YH}(W_j\otimes\Lambda^+)\Sigma_{YH}=\mathbf{0}. \quad (4.49)$$

By Theorem 4.2.1, (4.50)-(4.49) are the necessary and sufficient conditions for matrix quadratic forms $H'Y'W_iYH$'s to be an independent family of random matrices distributed as differences of independent Wishart random matrices with $\mathcal{W}_p(m_{1i},\Lambda)$ and $\mathcal{W}_p(m_{2i},\Lambda)$. Then $\{Y'W_iY\}_{i=1}^l$ is an independent family of random matrices distributed as differences of independent Wishart random matrices with $\mathcal{W}_p(m_{1i},\Sigma)$ and $\mathcal{W}_p(m_{2i},\Sigma)$ from Lemma 2.2.2 and vice versa, so the proof is completed. □

Theorem 4.2.2 is the core result in this chapter. In the sequence, we discuss its special cases and applications.

Corollary 4.2.3. *In Theorem 4.2.2, suppose Σ_Y is nonsingular. Then $\{Y'W_iY\}_{i=1}^l$ is independent and, for $i=1,2,\ldots,l$, $Y'W_iY\sim\mathcal{W}_p(m_{1i},\Sigma)-\mathcal{W}_p(m_{2i},\Sigma)$ for non-negative integers m_{1i} and m_{i2} if and only if there exists some $\Sigma\in\mathbb{N}_p$ such that for any distinct $i,j\in\{1,2,\ldots,l\}$ and any element $\mathbf{h}_i\in\mathbb{H}_p$,*

(1) $W_i\otimes\Sigma=(W_i\otimes I)\Sigma_Y(W_i\otimes\Sigma^{-1})\Sigma_Y(W_i\otimes I)$;

(2) $(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes \mathbf{h}_i) = (W_i \otimes \mathbf{h}_i)\Sigma_Y(W_i \otimes \Sigma^{-1})$;

(3) $tr(\Sigma_Y(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes \mathbf{h}_i)) + tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{1i}tr(\Sigma\mathbf{h}_i)$,

$tr(\Sigma_Y(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes \mathbf{h}_i)) - tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{2i}tr(\Sigma\mathbf{h}_i)$; *and*

(4) $(W_i \otimes I)\Sigma_Y(W_j \otimes I) = \mathbf{0}$.

Proof. Note that if Σ_Y is nonsingular, then Σ^{-1} exists from (c) in Theorem 4.2.2. (4) follows from (e) of Theorem 4.2.2. With routine tensor product calculations, the rest follows from Corollary 4.1.7. □

In Theorem 4.2.2, suppose that the covariance Σ_Y of Y is the Kronecker product $A \otimes \Sigma$ for nonnegative definite A of order n. Theorem 4.2.2 is reduced to the following corollary, which was shown by Tan (1975) and Wong and Wang (1995).

Corollary 4.2.4. *Let $W_1, W_2, \ldots, W_l$ be symmetric matrices of order n and $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, A \otimes \Sigma)$ with $A \in \mathbb{N}_n$ and $\Sigma \in \mathbb{N}_p$. Then, $\{Y'W_iY\}_{i=1}^{l}$ is independent with for $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$ for nonnegative integers m_{1i} and m_{i2} if and only if any distinct $i, j \in \{1, 2, \ldots .l\}$,*

(1)$AW_iA = AW_iAW_iAW_iA \neq \mathbf{0}$;

(2) $tr(AW_i)^2 + tr(AW_i) = 2m_{1i}$, $tr(AW_i)^2 - tr(AW_i) = 2m_{2i}$; *and*

(3) $AW_iAW_jA = \mathbf{0}$.

Proof. (3) is obtained by replacing $A \otimes \Sigma$ with Σ_Y in (4.50) in Theorem 4.2.2. (1) and (2) follow from Corollary 4.1.4; that proves the desired result. □

In particular, if $\mathbf{y}$ is an $n \times 1$ random normal vector with mean vector $\mathbf{0}$ and covariance C of order n, Theorem 4.2.2 is reduced to the familiar conditions which were shown by Tan (1977).

Corollary 4.2.5. *Let* $\mathbf{y} \sim \mathcal{N}_{\mathbf{n}}(\mathbf{0}, \Sigma)$ *and* $W_1, W_2, \ldots, W_l$ *be symmetric matrices of order* n. *Then* $\{\mathbf{y}'W_i\mathbf{y}\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \ldots, l$, $\mathbf{y}'W_i\mathbf{y} \sim \chi^2(m_{1i}) - \chi^2(m_{2i})$ *for nonnegative integers* m_{1i} *and* m_{i2} *if and only if any distinct* $i, j \in \{1, 2, \ldots, l\}$

(1) $CW_iC = CW_iCW_iCW_iC \neq \mathbf{0}$;

(2) $tr(CW)^2 + tr(CW) = 2m_1$, $tr(CW)^2 - tr(CW) = 2m_2$; *and*

(3) $CW_iCW_jC = \mathbf{0}$.

In Theorem 4.2.2, if we replace the covariance Σ_Y of Y with the sum of special Kronecker products, we have the following corollary.

Corollary 4.2.6. *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *with* $\Sigma_Y = \sum_{a=1}^{r} A_a \otimes E_{aa}$, $r \leq p$, $A_a \in \mathbb{N}_n$ *and* $W_i \in \mathbb{S}_n$, $i = 1, 2, \ldots, l$. *Then* $\{Y'W_iY\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$, *where* $\Sigma = \sum_{b=1}^{r} \sigma_b E_{bb}$, *for nonnegative integers* m_{1i} *and* m_{i2} *if and only if there exist real numbers* $\sigma_c > 0$, $c = 1, 2, \ldots, r$, *such that for all* a, b, $c \in \{1, 2, \ldots, \leq r\}$ *and any distinct* $i, j \in \{1, 2, \ldots, l\}$,

(1) $A_aW_iA_cW_iA_cW_iA_b = \sigma_c^2 A_aW_iA_b$;

(2) $A_aW_iA_b \neq \mathbf{0}$;

(3) $\sigma_b A_aW_iA_aW_iA_b = \sigma_a A_aW_iA_bW_iA_b$;

(4) $A_aW_iA_aW_jA_a = \mathbf{0}$; *and*

(5) $\frac{1}{\sigma_a^2} tr(A_aW_i)^2 + \frac{1}{\sigma_a} tr(A_aW_i) = 2m_{1i}$, $\frac{1}{\sigma_a^2} tr(A_aW_i)^2 - \frac{1}{\sigma_a} tr(A_aW_i) = 2m_{2i}$.

Proof. We use (4.50) by replacing Σ_Y and Σ with $\sum_{a=1}^{r} A_a \otimes E_{ii}$ and $\sum_{b=1}^{r} \sigma_b E_{bb}$ respectively. Then, (4.50) becomes

$$\sum_{a,b,c=1}^{r} A_aW_iA_bW_jA_c \otimes E_{aa}\Sigma^+E_{bb}\Sigma^+E_{cc} = \mathbf{0},$$

hence, $A_a W_i A_a W_j A_a = \mathbf{0}$, which proves (4). The other conditions follow from Corollary 4.1.6. □

Putting Theorem 4.1.8 and Theorem 4.2.2 together, we obtain the following result.

Theorem 4.2.7. *Suppose that* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *and* W_i*'s are symmetric matrices of order* n. *Then* $\{Y'W_iY\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$ *for nonnegative integers* m_{1i} *and* m_{i2} *if and only if there exists some* $\Sigma \in \mathbb{N}_p$ *such that for any* $i \in \{1, 2, \ldots, l\}$ *and any elements* $\mathbf{s}_i$ *in* $\mathbb{S}_p$,

(a) $\Sigma_Y[W_i \otimes (\mathbf{s}_i\Sigma\mathbf{s}_i)]\Sigma_Y = G_i(\mathbf{s}_i, \mathbf{s}_i, \Sigma, \Sigma_Y)$

(b) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma(W_i \otimes \mathbf{s}_i)\Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y$;

(c) $\{\mathbf{s}_i : \Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y = \mathbf{0}\} = \{\mathbf{s} : \Sigma\Sigma = \mathbf{0}\}$;

(d) $tr(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{s}_i)) + tr(\Sigma_Y(W_i \otimes \mathbf{s}_i)) = 2m_{1i}tr(\Sigma\mathbf{s}_i)$,

$tr(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{s}_i)) - tr(\Sigma_Y(W_i \otimes \mathbf{s}_i)) = 2m_{2i}tr(\Sigma\mathbf{s}_i)$; *and*

(e) for any distinct $i, j \in \{1, 2, \ldots, l\}$,

$$\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = \mathbf{0}. \tag{4.50}$$

Theorem 4.2.7 was essentially obtained by Masaro and Wong (2004b) through Jordan algebra homomorphisms.

4.3 Conditions for a MQF to be Distributed as a DINWRM

In this section, we extend Theorem 4.1.2 to the case where $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ with nonzero mean matrix $\boldsymbol{\mu}$.

Theorem 4.3.1. *Suppose that $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ and W is a symmetric matrix of order n. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ for some matrices $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{M}_{p\times p}$ and nonnegative integers m_1, m_2 if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that the following statements (a) and (b) hold.*

(a) For any elements $\mathbf{h}, \tilde{\mathbf{h}}$ in the similar base $\mathbb{H}_p$ associated with Σ,

$$\Sigma_Y \left[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})\right] \Sigma_Y = G(\mathbf{h}, \tilde{\mathbf{h}}, \Sigma, \Sigma_Y) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Sigma, \Sigma_Y) \tag{4.51}$$

such that

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma(W \otimes \mathbf{h})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y \tag{4.52}$$

with

$$\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \mathbb{K} \tag{4.53}$$

and

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) + tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_1 tr(\Sigma\mathbf{h}) \tag{4.54}$$

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) - tr(\Sigma_Y(W \otimes \mathbf{h})) = 2m_2 tr(\Sigma\mathbf{h}); \tag{4.55}$$

(b) For any $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $k = 1, 2, \ldots,$

$$tr\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k-1}\right) \tag{4.56}$$

$$tr((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k}\right) \tag{4.57}$$

with

$$\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}. \tag{4.58}$$

Proof. By Lemma 2.3.1, the moment generating function $M(\mathbf{s})$ of $Y'WY$ is given by

$$M(\mathbf{s}) = |I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2} exp\{< \mathbf{s}, \boldsymbol{\mu}' W \boldsymbol{\mu} > +2\Phi_0\} \tag{4.59}$$

where $sr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}) < 1/2$ and $\Phi_0 =< vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})', (W \otimes \mathbf{s})\Sigma_Y^{1/2}[I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}]^{-1}\Sigma_Y^{1/2}(W \otimes \mathbf{s}) >\}$.

Recall that $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ means that $Y'WY$ can be expressed as a difference of two independent random matrices D_1 and D_2 with $D_1 \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$ and $D_2 \sim \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$. By Corollary 2.3.2, the moment generating function $M_1(\mathbf{s})$ of D_1 and the moment generating function $M_2(\mathbf{s})$ of $-D_2$, respectively, are given by

$$M_1(\mathbf{s}) = |I - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_1/2} exp\{< \mathbf{s}, \boldsymbol{\lambda}_1 > +2\Phi_1\} \tag{4.60}$$

and

$$M_2(\mathbf{s}) = |I + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_2/2} exp\{< -\mathbf{s}, \boldsymbol{\lambda}_2 > +2\Phi_2\} \tag{4.61}$$

where $\Phi_1 =< \boldsymbol{\lambda}_1, \mathbf{s}\Sigma^{1/2}(I - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >$, $\Phi_2 =< \boldsymbol{\lambda}_2, \mathbf{s}\Sigma^{1/2}(I + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2})^{-1}\Sigma^{1/2}\mathbf{s} >$ and $\mathbf{s} \in \mathbb{S}_p$ such that $sr(\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}) < 1/2$.

Independence implies that for $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$,

$$M(\mathbf{s}) = E(e^{<\mathbf{s},D>}) = E(e^{<\mathbf{s},D_1-D_2>}) = E(e^{<\mathbf{s},D_1>})E(e^{<\mathbf{s},-D_2>}) = M_1(\mathbf{s})M_2(\mathbf{s}).$$

Using (4.59)-(4.61) and comparing the same items in both sides of $M(\mathbf{s}) = M_1(\mathbf{s})M_2(\mathbf{s})$, we obtain the following conditions:

(i) $|I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2} = |I - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_1/2}|I + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}|^{-m_2/2}$;

(ii) for any symmetric matrix $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$

$$\Phi_0 = \Phi_1 + \Phi_2; \text{ and} \tag{4.62}$$

(iii) $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}' W \boldsymbol{\mu}$, which proves (4.58).

By Lemma 2.3.5, (i) is equivalent to $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu}) \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_1, \Sigma)$. (4.51)-(4.55) follow from Theorem 4.1.2.

For any $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$, we have

$$\Phi_1 = tr(\boldsymbol{\lambda}_1[\mathbf{s}\Sigma\mathbf{s} + 2\mathbf{s}(\Sigma\mathbf{s})^2 + 2^2\mathbf{s}(\Sigma\mathbf{s})^3 + \ldots]), \tag{4.63}$$

$$\Phi_2 = tr(\boldsymbol{\lambda}_2[\mathbf{s}\Sigma\mathbf{s} - 2\mathbf{s}(\Sigma\mathbf{s})^2 + 2^2\mathbf{s}(\Sigma\mathbf{s})^3 + \ldots]) \tag{4.64}$$

and

$$\Phi_0 = tr(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'((W \otimes \mathbf{s})\Sigma_Y \Upsilon + 2(W \otimes \mathbf{s})\Upsilon^2 + 2^2(W \otimes \mathbf{s})\Upsilon^3 + \ldots)) \tag{4.65}$$

where $\Upsilon = \Sigma_Y(W \otimes \mathbf{s})$.

We arbitrarily choose $\mathbf{s}$ in $\mathcal{N}_0$. Replacing $\mathbf{s}$ in (4.63)-(4.65) by $\alpha\mathbf{s}$ with very small positive number α and putting (4.63)-(4.65) into (4.62), two sides of (4.62) are two power series with respect to α. Comparing two power series implies that (4.62) amounts to (4.56) and (4.57), and that proves the desired result. □

In fact, we have obtained the following relation between $Y'WY$ and $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu})$ in the proof of Theorem 4.3.1.

Corollary 4.3.2. *Let $Y \sim \mathcal{N}_{n \times p}(\boldsymbol{\mu}, \Sigma_Y)$ and W be a symmetric matrix of order n. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ for matrices $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{M}_{p \times p}$ and nonnegative integers m_1, m_2 if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that*

(1) $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu}) \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$; and

(2) for any $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0} \in \mathbb{S}_p$ *and* $k = 1, 2, \ldots,$

$$tr\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k-1}\right),$$

$$tr\left((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k}\right);$$

with $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}$.

Suppose that Σ_Y is the Kronecker product covariance structures $A \otimes \Sigma$ for $A \in \mathbb{N}_n$ Theorem 4.3.1 is reduced to the following corollary.

Corollary 4.3.3. *Let* $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, A\otimes\Sigma)$ *with* $A \in \mathbb{N}_n$ *and* W *be a symmetric matrix of order* n. *Then* $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ *for matrices* $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{M}_{p\times p}$ *and nonnegative integers* m_1, m_2 *if and only if the following statements (1)-(4) hold.*

(1) $AWAWAWA = AWA \neq \mathbf{0}$;

(2) $tr(AW)^2 + tr(AW) = 2m_1$, $tr(AW)^2 - tr(AW) = 2m_2$;

(3) $\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'WAW\boldsymbol{\mu} = \boldsymbol{\mu}'WAWAWAW\boldsymbol{\mu}$; *and*

(4) $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu} = \boldsymbol{\mu}'WAWAW\boldsymbol{\mu}$.

Proof. (1) and (2) follow from Corollary 4.1.4. Use $A \otimes \Sigma$ to replace Σ_Y in (4.56) and (4.57) of Theorem 4.3.1. By (2.5) and (2.6), (4.56) and (4.57) are expressed as, for $k = 1, 2, \ldots,$

$$\begin{aligned} tr\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) &= tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes s)]^{2k-1}\right) \\ &= tr\left(vec(\boldsymbol{\mu})'vec(W(AW)^{2k-1}\boldsymbol{\mu}\mathbf{s}(\Sigma\mathbf{s})^{2k-1})\right) \\ &= tr\left(\boldsymbol{\mu}W(AW)^{2k-1}\boldsymbol{\mu}\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) \end{aligned}$$

and

$$tr\left((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = tr\left(\boldsymbol{\mu}W(AW)^{2k}\boldsymbol{\mu}\mathbf{s}(\Sigma\mathbf{s})^{2k}\right).$$

By Lemma 2.1.8, we have

$$\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W(AW)^{2k-1} \text{ and} \tag{4.66}$$

$$\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W(AW)^{2k} \tag{4.67}$$

for $k = 1, 2, \ldots$. (3) and (4) are equivalent to (4.66) and (4.67) from (1), and so the proof is completed. □

The following corollary is for the special case $p = 1$ of Theorem 4.3.1.

Corollary 4.3.4. *Let* $\mathbf{y} \sim \mathcal{N}_\mathbf{n}(\boldsymbol{\mu}, \mathbf{C})$ *with* $C \in \mathbb{N}_n$ *and* W *be a symmetric matrix of order* n. *Then* $\mathbf{y}'W\mathbf{y} \sim \chi^2(m_1, \delta_1^2) - \chi^2(m_2, \delta_2^2)$ *for some numbers* δ_1^2, δ_2^2 *and nonnegative integers* m_1, m_2 *if and only if the following statements (1)-(4) hold.*

(1) $CWCWCWC = CWC \neq \mathbf{0}$;

(2) $tr(CW)^2 + tr(CW) = 2m_1$, $tr(CW)^2 - tr(CW) = 2m_2$;

(3) $\delta_1^2 + \delta_2^2 = \boldsymbol{\mu}'WCW\boldsymbol{\mu} = \boldsymbol{\mu}'WCWCWCW\boldsymbol{\mu}$; *and*

(4) $\delta_1^2 - \delta_2^2 = \boldsymbol{\mu}'W\boldsymbol{\mu} = \boldsymbol{\mu}'WCWCW\boldsymbol{\mu}$.

4.4 Conditions for MQFs to be an Independent Family of Random Matrices Distributed as DINWDs

In this section, we shall extend Theorem 4.2.2 to the case where $Y \sim \mathcal{N}_{n \times p}(\boldsymbol{\mu}, \Sigma_Y)$ with nonzero mean matrix $\boldsymbol{\mu}$.

Exactly as in the proofs of Theorem 4.2.2 and Theorem 4.3.1, we only need to put them with Lemma 3.4.1 together and obtain the following multivariate version of Cochran's theorem concerning differences of independent noncentral Wishart random matrices.

Theorem 4.4.1. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ and $W_1, W_2, \ldots, W_l$ be symmetric matrices of order n. Then $\{Y'W_iY\}_{i=1}^{l}$ is independent and, for $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$ for some matrices $\boldsymbol{\lambda}_{1i}, \boldsymbol{\lambda}_{2i}$ of order p and some nonnegative integers m_{1i}, m_{2i} if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that the following statements (a), (b) and (c) hold.*

(a) For any $i \in \{1, 2, \ldots, l\}$ and any elements $\mathbf{h}_i, \tilde{\mathbf{h}}_i$ in the similar base $\mathbb{H}_p$ associated with Σ,

$$\Sigma_Y \left[W \otimes (\mathbf{h}_i\Sigma\tilde{\mathbf{h}}_i + \tilde{\mathbf{h}}_i\Sigma\mathbf{h}_i)\right] \Sigma_Y = G_i(\mathbf{h}_i, \tilde{\mathbf{h}}_i, \Sigma, \Sigma_Y) + G_i(\tilde{\mathbf{h}}_i, \mathbf{h}_i, \Sigma, \Sigma_Y) \tag{4.68}$$

such that

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma(W \otimes \mathbf{h}_i)\Sigma_Y = \Sigma_Y(W \otimes \mathbf{h}_i)\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y \tag{4.69}$$

with

$$\{\mathbf{h}_i : \ \Sigma_Y(W \otimes \mathbf{h}_i)\Sigma_Y = 0, \} = \mathbb{K} \tag{4.70}$$

and

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h}_i)) + tr(\Sigma_Y(W \otimes \mathbf{h}_i)) = 2m_{1i}tr(\Sigma\mathbf{h}_i) \tag{4.71}$$

$$tr(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h}_i)) - tr(\Sigma_Y(W \otimes \mathbf{h}_i)) = 2m_{2i}tr(\Sigma\mathbf{h}_i); \tag{4.72}$$

(b)For any distinct $i, j \in \{1, 2, \ldots, l\}$ and $\mathbf{h}_i, \mathbf{h}_j \in \mathbb{H}_p$

$$\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = \mathbf{0}, \tag{4.73}$$

$$\Sigma_Y(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)vec(\boldsymbol{\mu}) = \mathbf{0} \tag{4.74}$$

$$vec(\boldsymbol{\mu})'(W_i \otimes \mathbf{h}_i)\Sigma_Y(W_j \otimes \mathbf{h}_j)vec(\boldsymbol{\mu}) = 0; \tag{4.75}$$

and

(c) For any $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathbb{S}_p$ and $k = 1, 2, \ldots,$

$$tr\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes s)]^{2k-1}\right) \tag{4.76}$$

$$tr\left((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k}\right) \tag{4.77}$$

with

$$\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}. \tag{4.78}$$

Theorem 4.4.1 is the core result in this chapter. Now, let us discuss its special cases.

Corollary 4.4.2. *In Theorem 4.4.1, suppose Σ_Y is nonsingular. Then $\{Y'W_iY\}_{i=1}^{l}$ independent and, for $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$ for matrices $\boldsymbol{\lambda}_{1i}$, $\boldsymbol{\lambda}_{2i}$ of order p and some nonnegative integers m_{1i}, m_{2i} if and only if there exists some $\Sigma \in \mathbb{N}_p$ such that for distinct $i, j \in \{1, 2, \ldots, l\}$ and any element $\mathbf{h}_i$ in the similar base H_p associate with Σ,*

(1) $(W_i \otimes \Sigma) = (W_i \otimes I)\Sigma_Y(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes I)$;

(2) $(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes \mathbf{h}_i) = (W_i \otimes \mathbf{h}_i)\Sigma_Y(W_i \otimes \Sigma^{-1})$;

(3) $tr(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{h}_i)) + tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{1i}tr(\Sigma\mathbf{h}_i)$,

$$tr(\Sigma_Y(W_i \otimes \Sigma^{-1})\Sigma_Y(W_i \otimes \mathbf{h}_i)) - tr(\Sigma_Y(W_i \otimes \mathbf{h}_i)) = 2m_{2i}tr(\Sigma\mathbf{h}_i);$$

(4) $(W_i \otimes I)\Sigma_Y(W_j \otimes I) = 0$;

(5) for any $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0} \in \mathbb{S}_p$ *and* $k = 1, 2, \dots,$

$$tr\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k-1}\right),$$

$$tr\left((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = tr\left(vec(\boldsymbol{\mu})vec(\boldsymbol{\mu})'(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k}\right); \text{ and}$$

with $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}$.

Proof. Conditions (1)-(4) follow from Corollary 4.2.3 while conditions (5) follows from Theorem 4.4.1. □

Corollary 4.4.3. *In Theorem 4.4.1, suppose* $\Sigma_Y = A \otimes \Sigma$ *for some* $A \in \mathbb{N}_n$*. Then* $\{Y'W_iY\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \dots, l$,

$$Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$$

for matrices $\boldsymbol{\lambda}_{1i}$, $\boldsymbol{\lambda}_{2i}$ *of order* p *and nonnegative integers* m_{1i}, m_{2i} *if and only if for any distinct* $i, j \in \{1, 2, \dots, l\}$,

(1) $AW_iAW_iAW_iA = AW_iA \neq \mathbf{0}$;

(2) $tr(AW_i)^2 + tr(AW_i) = 2m_{1i}$, $tr(AW_i)^2 - tr(AW_i) = 2m_{2i}$;

(3) $\boldsymbol{\lambda}_{1i} + \boldsymbol{\lambda}_{2i} = \boldsymbol{\mu}'W_iAW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iAW_iAW_iAW_i\boldsymbol{\mu}$,

$\boldsymbol{\lambda}_{1i} - \boldsymbol{\lambda}_{2i} = \boldsymbol{\mu}'W_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iAW_iAW_i\boldsymbol{\mu}$;

(4) $AW_iAW_jA = \mathbf{0}$;

(5) $AW_iAW_j\boldsymbol{\mu} = \mathbf{0}$; *and*

(6) $\boldsymbol{\mu}'W_iAW_j\boldsymbol{\mu} = \mathbf{0}$.

Proof. (1)-(3) follow from Corollary 4.4.2. (4)-(6) follow from Corollary 3.4.4. □

Let us look the special case $p = 1$ of Theorem 4.4.1.

Corollary 4.4.4. *Let* $\mathbf{y} \sim \mathcal{N}_\mathbf{n}(\boldsymbol{\mu}, \mathbf{C})$ *with* $C \in \mathbb{N}_n$ *and* $W_1, W_2, \ldots, W_l$ *be symmetric matrices of order* n. *Then* $\{\mathbf{y}'W_i\mathbf{y}\}_{i=1}^{l}$ *is independent and, for* $i = 1, 2, \ldots, l$,

$$\mathbf{y}'W_i\mathbf{y} \sim \chi^2(m_{1i}, \delta_{1i}^2) - \chi^2(m_{2i}, \delta_{2i}^2)$$

for some numbers δ_{1i}, δ_{2i} *and nonnegative integers* m_{1i}, m_{2i} *if and only if for any distinct* $i, j \in \{1, 2, \ldots, l\}$

(1) $CW_iCW_iCW_iC = CW_iC \neq \mathbf{0}$;

(2) $tr(CW_i)^2 + tr(CW_i) = 2m_{1i}$, $tr(CW_i)^2 - tr(CW_i) = 2m_{2i}$;

(3) $\delta_{1i}^2 + \delta_{2i}^2 = \boldsymbol{\mu}'W_iCW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iCW_iCW_iCW_i\boldsymbol{\mu}$;

(4) $\delta_{1i}^2 - \delta_{2i}^2 = \boldsymbol{\mu}'W_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iCW_iCW_i\boldsymbol{\mu}$;

(5) $CW_iCW_jC = \mathbf{0}$;

(6) $CW_iCW_j\boldsymbol{\mu} = \mathbf{0}$; *and*

(7) $\boldsymbol{\mu}'W_iCW_j\boldsymbol{\mu} = 0$.

These special cases stated in Corollary 4.4.3 and Corollary 4.4.4 were also discussed in Tan (1975), Tan (1977) and Wong and Wang (1995).

Chapter 5

Conclusions and Future Research

5.1 Model One

The case discussed in Chapter 3 is called model one. In model one assume that Y is an $n \times p$ multivariate normal random matrix with nonzero mean $\boldsymbol{\mu}$ and general covariance Σ_Y and W, W_1, W_2, $\ldots$, W_l are symmetric matrices of order n. We have derived a set the general necessary and sufficient conditions (Theorem 3.3.1 and Theorem 3.1.1, Theorem 3.1.3 for special cases) for matrix quadratic form $Y'WY$ to have a noncentral Wishart distribution and then obtained a set of general necessary and sufficient conditions (Theorem 3.4.2 and Theorem 3.2.3, Theorem 3.2.4 for special cases) for matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, $\ldots$, $Y'W_lY$ to be an independent family of noncentral Wishart random matrices.

Now, let $W = \sum_{i=1}^{l} W_i$. Consider the following propositions:

(B_1) The matrix quadratic form $Y'WY$ has a $\mathcal{W}_p(m, \Sigma, \boldsymbol{\mu}'W\boldsymbol{\mu})$ distribution.

(B_2) The matrix quadratic form $Y'W_iY$ has a $\mathcal{W}_p(m_i, \Sigma, \boldsymbol{\mu}'W_i\boldsymbol{\mu})$ distribution ($i =$

$1, 2, \ldots, l$).

(B_3) $Y'W_iY$ and $Y'W_jY$ are independent ($i \neq j$; $i, j = 1, 2, \ldots, l$).

The interrelationship of propositions (B_1), (B_2) and (B_3) will be the first topic of interest. Proposition (B_1) follows from propositions (B_2) and (B_3) by Theorem 10.3.4 of Muirhead (1982). We are wondering if the other two implications hold. That is, (B_1) and (B_3) imply (B_2); (B_1) and (B_2) imply (B_3). Then, we shall study the interrelationship of propositions (B_1), (B_2) and (B_3) with some imposing conditions, for example, nonsingular covariance Σ_Y or nonnegative definite matrices W_1, W_2, ..., W_l.

The second topic will be to study new propositions or conditions, denoted as (B_4), (B_5), ..., (B_k), and then discuss the interrelationship of (B_1), (B_2), ..., (B_k), also see Vaish and Chaganty (2004) and Tian and Styan (2005).

Although we have established a general multivariate version of Cochran's theorem stated in Theorem 3.4.2, the improvement of condition (b) in Theorem 3.4.2 is required. Then whether there exists some verifiable condition equivalent to condition (b) will be one of our interested topics. Other topics of future research on this model will include the refinement, simplification of Theorem 3.4.2 and the extension of Theorem 3.4.2 to more general matrix quadratic expressions.

5.2 Model Two

The case discussed in Chapter 4 is called model two. Model two involves the problem that asks what are the equivalent conditions for matrix quadratic forms to be an independent family of random matrices distributed as differences of independent noncentral Wishart random matrices. We have established a set of general necessary and sufficient conditions (Theorem 4.3.1 and Theorem 4.1.1 or Theorem 4.1.2 for special cases) for the matrix quadratic form $Y'WY$ to be distributed as differences of independent noncentral Wishart random matrices and then we obtained a set of general necessary and sufficient conditions (Theorem 4.4.1 and Theorem 4.2.1 or Theorem 4.2.2 for special cases) for matrix quadratic forms $Y'W_1Y$, $Y'W_2Y$, ..., $Y'W_lY$ to be an independent family of random matrices distributed as differences of independent noncentral Wishart random matrices.

Now suppose $W = \sum_{i=1}^{l} W_i$. Consider the following propositions:

(D_1) $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$.

(D_2) For $i = 1, 2, \ldots, l$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$.

(D_3) $Y'W_iY$ and $Y'W_jY$ are independently $(i \neq j;\ i, j = 1, 2, \ldots, l)$.

The interrelationship of propositions (D_1), (D_2) and (D_3) will be the first topic of interest. Proposition (D_1) follows from propositions (D_2) and (D_3) by Theorem 10.3.4 of Muirhead (1982) and Lemma 2.2.4. We are wondering if the other two implications hold. That is, (D_1) and (D_3) imply (D_2); (D_1) and (D_2) imply (D_3). We may investigate the interrelationship of (A), (B) and (C) with some imposing conditions, for example, nonsingular covariance Σ_Y or nonnegative definite matrices W_1, W_2, ..., W_l.

The second topic will study new propositions or conditions, denoted as (D_4), (D_5), ..., (D_k), and study the interrelationship of (D_1), (D_2), ..., (D_k), also see Tan (1975).

Moreover, we are wondering if its condition (4.67) and (4.68) can be replaced with the following conditions

$$tr(\Sigma_Y(W_i \otimes \Sigma^+))^2 + tr(\Sigma_Y(W_i \otimes \Sigma^+)) = 2m_{1i}r(\Sigma)$$

$$tr(\Sigma_Y(W_i \otimes \Sigma^+))^2 - tr(\Sigma_Y(W_i \otimes \Sigma^+)) = 2m_{2i}r(\Sigma).$$

The examples and applications of Theorem 4.4.1 or Theorem 4.2.2 should be investigated. Other topics of future research on this model will include the refinement, simplification of Theorem 4.4.1 and the extension of Theorem 4.4.1 to more general matrix quadratic expressions. Tan (1975) and Wong and Wang (1995) obtained their results for matrix quadratic expressions.

Appendix: Necessary and Sufficient Conditions for a MQF to be Distributed as a DIWRM

The following result and its proof are due of Masaro and Wong (2004b).

Theorem *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *and* $W \in \mathbb{S}_n$. *Decompose* $\Sigma_{Y'}$ *as*

$$\Sigma_{Y'} = L'L, \quad L = [L_1, L_2, \ldots, L_p] \tag{1}$$

with $L_i \in \mathbb{M}_{q\times n}$ *and* $r(\Sigma_{Y'})\ q \leq np$. *Assume (without loss of generality) that* $L_iWL_i' \neq 0$ $(i \leq r)$. *Let* $\Lambda = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0]$ $(\sigma_i > 0,\ i \leq r)$ *and define*

$$B_{ij} = [L_iWL_j' + L_jWL_i']/2\sqrt{\sigma_i\sigma_j},\ i, j \leq r.$$

Then $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$ *with the common covariance* Λ *for nonnegative integers* m_1 *and* m_2 *if and only if*

$(C1)$ $L_iWL_j' + L_jWL_i' = \mathbf{0}$ *for* i *or* $j > r$

$(C2)$ $B_{ii}^3 = B_{ii}$, $tr(B_{ii}) = m_1 - m_2$, $tr(B_{ii}^2) = m_1 + m_2$

$(C3)$ $B_{ii}B_{jj} = \mathbf{0}$, $i \neq j$

$(C4)$ $4B_{ij}^2 = B_{ii}^2 + B_{jj}^2$, $i \neq j$

$(C5)$ $B_{ii}B_{ij} = B_{ij}B_{jj}$, $i \neq j$

$(C6)$ $2(B_{ii} + B_{jj})(B_{ik}B_{jk} + B_{jk}B_{ik}) = B_{ij}$ *for all distinct* i, j, k.

Proof. First note that in (1) we may, without loss of generality, assume $q = np$ (or just replace L' by $[L', \mathbf{0}] \in \mathbb{M}_{np\times np}$). This will simplify the notation later on in our proof (see (7)-(9)).

First assume that conditions $(C1)$-$(C6)$ hold. By Lemma 2.3.5 we must show that for $k = 1, 2, \ldots$

$$tr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2})^k = [m_1 + (-1)^k m_2] tr(\Lambda^{1/2}\mathbf{s}\Lambda^{1/2})^k \ , \ \mathbf{s} = (s_{ij}) \in \mathbb{S}_p \tag{2}$$

Using (2.7) - (2.8) we obtain

$$tr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2})^k = tr(L(\mathbf{s} \otimes W)L')^k = tr\left(\sum_{i,j=1}^{r} s_{ij} L_i W L_j'\right)^k = tr\left(\sum_{i,j=1}^{r} \sqrt{\sigma_i \sigma_j} s_{ij} B_{ij}\right)^k .$$

Note that $tr(\Lambda^{1/2}\mathbf{s}\Lambda^{1/2})^k = tr(\sqrt{\sigma_i\sigma_j} s_{ij})^k$. Thus since $\mathbf{s} \in \mathbb{S}_p$ is arbitrary we must show that

$$tr\left(\sum_{i,j=1}^{r} u_{ij} B_{ij}\right)^k = [m_1 + (-1)^k m_2] tr U^k, \quad U = [u_{ij}] \in \mathbb{S}_r \ , \ k = 1, 2, \ldots \tag{3}$$

Now from condition $(C2)$ it follows that

$$B_{ii}^{2k+1} = B_{ii}, \quad tr(B_{ii}^{2k+1}) = m_1 - m_2 \quad k = 1, 2, \ldots \text{ and} \tag{4}$$

$$B_{ii}^{2k} = B_{ii}^2, \quad tr(B_{ii}^{2k}) = m_1 + m_2, \ k = 1, 2, \ldots.$$

Thus using $(C4)$, $(C3)$ and (4) it is easily shown that for $i \neq j$, $||B_{ij} - 4B_{ij}^3||^2 = 0$ and so

$$B_{ij} = 4B_{ij}^3. \quad i \neq j \tag{5}$$

Combining $(C4)$ with (5) we get that, for $i \neq j$, $(B_{ii}^2 + B_{jj}^2)B_{ij} = 4B_{ij}^2 B_{ij} = B_{ij}$. The symmetry of B_{ij} then yields

$$B_{ij} = (B_{ii}^2 + B_{jj}^2) B_{ij} (B_{ii}^2 + B_{jj}^2). \quad i \neq j \tag{6}$$

Now by (4) and $(C3)$ we may choose an orthogonal matrix H which does not depend on i and such that

$$\tilde{B}_{ii} \equiv H' B_{ii} H = \mathbf{e}_{ii} \otimes A_{ii} \tag{7}$$

where $A_{ii} = diag[U_{ii}, V_{ii}, \mathbf{0}] \in \mathbb{M}_{n\times n}$ and $U_{ii} = I_{m_1}, V_{ii} = -I_{m_2}$.

For $i \neq j$, let $\tilde{B}_{ij} \equiv H'B_{ij}H$. Then using (6), ($C5$) and ($C4$) with B_{ii}, B_{jj} and B_{ij} replaced by $\tilde{B}_{ii}, \tilde{B}_{jj}$ and $\tilde{B}_{ij}$ we obtain

$$\tilde{B}_{ij} = \frac{1}{2}(\mathbf{e}_{ij} \otimes A_{ij} + \mathbf{e}_{ji} \otimes A_{ji}) \tag{8}$$

where $A_{ij} = diag[U_{ij}, V_{ij}, \mathbf{0}] \in \mathbb{M}_{n\times n}$, $U_{ij} \in \mathbb{M}_{m_1\times m_1}, V_{ij} \in M_{m_2\times m_2}$ and $A'_{ij} = A_{ji}$, $U_{ij}U'_{ij} = I_{m_1}$, $V_{ij}V'_{ij} = I_{m_2}$. Thus from (7) and (8) we have

$$H'\left(\sum_{i,j=1}^{r} u_{ij}B_{ij}\right)H = diag[[u_{ij}A_{ij}], \mathbf{0}]. \tag{9}$$

We now claim that for all i, j, k

$$U_{ik}U_{kj} = U_{ij}, \; V_{ik}V_{kj} = -V_{ij}. \tag{10}$$

Indeed (10) is clear if $k = i$ or j (since $U_{ii} = I_{m_1}$ and $V_{ii} = -I_{m_2}$) or if $i = j$ (since $U_{ij}U'_{ij} = I_{m_1}$ and $V_{ij}V'_{ij} = I_{m_2}$). The remaining case (i.e. when i, j and k are distinct) is obtained by substituting the matrix representations $\tilde{B}_{ij}$ given in (8) into ($C6$).

Now from (10) it easily follows that for all i, j, k, ℓ

$$A_{ik}A_{k\ell}A_{\ell j} = A_{ij}. \tag{11}$$

Finally, by (9), we have

$$tr\left(\sum_{i,j=1}^{r} u_{ij}B_{ij}\right)^k = tr(u_{ij}A_{ij})^k$$

$$= tr(\sum_{i=1}^{r} \sum_{\ell_1,\ell_2,\ldots,\ell_{k-1}=1}^{r} u_{i\ell_1}u_{\ell_1\ell_2}\ldots u_{\ell_{k-2}\ell_{k-1}}u_{\ell_{k-1}i}A_{i\ell_1}A_{\ell_1\ell_2}\ldots A_{\ell_{k-2}\ell_{k-1}}A_{\ell_{k-1}i})$$

$$= tr(\sum_{i=1}^{r} \sum_{\ell_1,\ell_2,\ldots,\ell_{k-1}=1}^{r} u_{i\ell_1}u_{\ell_1\ell_2}\ldots u_{\ell_{k-2}\ell_{k-1}}u_{\ell_{k-1}i}C)$$

where

$$C = \begin{cases} diag[I_m, \mathbf{0}], & \text{if } k \text{ is even;} \\ diag[I_{m_1}, -I_{m_2}, \mathbf{0}], & \text{if } k \text{ is odd.} \end{cases} \quad \text{(by (11))}$$

Thus $\left[\sum_{i,j=1}^{r} u_{ij}B_{ij}\right]^k = [m_1 + (-1)^k m_2]trU^k$, which proves (3) as required.

Conversely assume $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$. We show that conditions $(C1)$-$(C6)$ hold.

Using (2) and (2.7)-(2.8) and arguing as before we have that for $k = 1, 2 \ldots$

$$tr\left(\sum_{i,j=1}^{p} s_{ij}L_iWL_j'\right)^k = [m_1 + (-1)^k m_2]tr(\sqrt{\sigma_i\sigma_j}s_{ij})^k \quad , \quad \mathrm{s} = (s_{ij}) \in \mathbb{S}_p. \tag{12}$$

Fixing u and v with u or $v > r$ and letting $s_{uv} = s_{vu} = 1$ and all other $s_{ij} = 0$ we see that condition $(C1)$ then follows from (3.13) since $\sigma_i = 0$ for $i > r$.

Now setting $u_{ij} = \sqrt{\sigma_i\sigma_j}s_{ij},\ i, j \leq r$, we obtain from (12) and $(C1)$ that

$$tr\left(\sum_{i,j=1}^{r} u_{ij}B_{ij}\right)^k = [m_1 + (-1)^k m_2]trU^k \ , \ U = [u_{ij}] \ \in \ \mathbb{S}_r, \quad k = 1, 2, \tag{13}$$

Fixing i and letting $u_{ii} = 1$ and all other $u_{k\ell} = 0$ in (13) yields

$$tr(B_{ii}^k) = m_1 + (-1)^k m_2, \quad k = 1, 2, \tag{14}$$

which proves $(C2)$.

Also, fixing $i \neq j$ and letting $u_{ij} = u_{ji} = 1$ and all other $u_{kl} = 0$ in (13) gives

$$tr(2B_{ij})^k = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ 2(m_1 + m_2), & \text{if } k \text{ is even.} \end{cases} \tag{15}$$

Now from (13) we get for $i \neq j$

$$tr(aB_{ii} + bB_{jj} + 2cB_{ij})^4 = (m_1 + m_2)(a^4 + b^4 + 2c^4 + 4a^2c^2 + 4b^2c^2 + 4abc^2). \tag{16}$$

By comparing the coefficients of a^2b^2 on the left and right sides of equation (16) we find that

$$2tr(B_{ii}^2B_{jj}^2) + tr(B_{ii}B_{jj})^2 = 0\ ,\ i \neq j \tag{17}$$

Now since $|tr(B_{ii}B_{jj})^2| = |< B_{ii}B_{jj}, B_{jj}B_{ii} >| \leq ||B_{ii}B_{jj}||^2 = tr(B_{ii}^2B_{jj}^2)$, (17) implies that $0 = tr(B_{ii}^2B_{jj}^2) = ||B_{ii}B_{jj}||^2$ which proves $(C3)$.

Comparing the coefficients of abc^2 on the left and right sides of equation (16) and using $(C3)$ we obtain

$$4tr(B_{ii}B_{ij}B_{jj}B_{ij}) = m_1 + m_2,\quad i \neq j \tag{18}$$

Thus, by (C3), (14), (15)) and (18)

$$tr(4B_{ij}^2 - (B_{ii}^2 + B_{jj}^2))^2 = tr(16B_{ij}^4 + B_{ii}^4 + B_{jj}^4 - 8B_{ij}^2(B_{ii}^2 + B_{jj}^2))$$

$$= 4(m_1 + m_2) - 8[||B_{ii}B_{ij}||^2 + ||B_{ij}B_{jj}||^2]$$

$$\leq 4(m_1 + m_2) - 16||B_{ii}B_{ij}||||B_{ij}B_{jj}|| \leq 4(m_1 + m_2) - 16 < B_{ii}B_{ij}, B_{ij}B_{jj} >$$

$$= 4(m_1 + m_2) - 16(m_1 + m_2)/4 = 0$$

which proves $(C4)$.

Also using $(C4)$, $(C3)$, (14) and (18), a straightforward calculation shows that $||B_{ii}B_{ij} - B_{ij}B_{jj}||^2 = 0$ which proves $(C5)$.

Finally, to prove $(C6)$, note first that $(C6)'$ $\left(\sum_{i,j=1}^{r} B_{ij}\right)^3 = r^2 \sum_{i,j=1}^{r} B_{ij}$. Indeed this is a consequence of setting all $u_{ij} = 1$ in (13). Also note that since we have shown that $(C1)$-$(C5)$ hold, we may use the matrix representation given by (7)-(9). Using this representation in $(C6)'$ gives

$$r^2[A_{ij}] = [A_{ij}]^3 \tag{19}$$

where $A_{ij} = diag[P_{ij}, \mathbf{0}] \in \mathbb{M}_{n\times n}, P_{ij} \in M_{m\times m},\ P'_{ij} = P_{ji},\ P_{ij}P'_{ij} = I_m,\ m = m_1 + m_2.$ We claim that for all i, j, k, ℓ

$$A_{ik}A_{k\ell}A_{\ell j} = A_{ij}. \tag{20}$$

Indeed by (19) we have

$$r^2 P_{ij} = \sum_{k=1}^{r}\sum_{\ell=1}^{r} P_{ik}P_{k\ell}P_{\ell j}.$$

Hence,

$$r^2 I_m = \sum_{k=1}^{r}\sum_{\ell=1}^{r} P_{ji}P_{ik}P_{k\ell}P_{\ell j} = \sum_{k=1}^{r}\sum_{\ell=1}^{r} F_{k\ell}\ ,\quad F_{k\ell} = P_{ji}P_{ik}P_{k\ell}P_{\ell j}. \tag{21}$$

Now since $F_{k\ell}$ is orthogonal,

$$tr(F_{k\ell}) =< F_{k\ell}, I_m >\leq ||F_{k\ell}||||I_m|| = m. \tag{22}$$

Thus from (21) and (22) we have

$$mr^2 = \sum_{k=1}^{r}\sum_{\ell=1}^{r} tr(F_{k\ell}) \leq mr^2. \tag{23}$$

Now (23) shows that the Cauchy-Schwarz inequality in (22) must be an equality. Thus $F_{k\ell}$ is a scalar multiple of I_m and since $tr(F_{k\ell}) = m$, $F_{k\ell} = I_m$. Hence $P_{ji}P_{ik}P_{k\ell}P_{\ell j} = I_m$ or $P_{ik}P_{k\ell}P_{\ell j} = P_{ij}$ which proves (20).

Now taking $k = i$ in (20) yields $A_{ii}A_{i\ell}A_{\ell j} = A_{ij}$ or

$$A_{ii}A_{i\ell} = A_{ij}A_{j\ell}. \tag{24}$$

Equation $(C6)$ can now be verified by replacing the B_{uv} in that equation by their matrix representations given in (7) and (8) and using (24). □

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Vita Auctoris

Jianhua Hu was born in Hunan, China. He received his M.S. in Applied Mathematics in Changsha Railway University (currently Central South University), China, in 1989. He earned his M.A. in Statistics at University of Windsor, Canada, in 2001. He is currently a candidate for the Doctoral degree in Statistics at the University of Windsor and hopes to finish his graduate study in June of 2007.